AN INSIGHT INTO STUDENT UNDERSTANDING OF FUNCTIONS IN A
GRAPHING CALCULATOR ENVIRONMENT

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DECLARATION OF ORIGINALITY

This thesis contains no material which has been accepted for any other degree in any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by any other person, except where due reference is given in the text.

Signature: ………………………
ADDITIONAL PUBLICATIONS BY CANDIDATE ON MATTERS RELEVANT TO THESIS


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The impetus to commence this study was the interest and enthusiasm for graphing calculator technology in mathematics education of my first supervisor, Gary Asp. This thesis and the learning undertaken on its journey could not have happened or been seen to completion without the support of my second supervisor, Dr. Gloria Stillman. Gloria encouraged me to expand my knowledge base, particularly about qualitative research methods, and in addition to being my teacher became my mentor.

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In conclusion, the contribution of all the students I have taught and will teach in the future to the writing of this thesis must be acknowledged. They have made me the teacher I am and will be, and allowed me to have the insight into their behaviour and actions, and hence improve my teaching and in turn the learning of my future students. Thanks to the students and colleagues who participated in the pilot study, especially Wayne Rickard, who was also the other teacher of the students in the main study. Finally, I wish to thank the five pairs of students in the main study, they know who they are, without whom this thesis would not have been possible.
ABSTRACT

The introduction of graphing calculators into senior secondary schools and mandating of their use in high stakes assessment makes student expertise in finding a complete graph of a function essential. This thesis investigated the cognitive, metacognitive, mathematical, and technological processes senior secondary students used in seeking a complete graph of a difficult cubic function. A pretest of function knowledge was administered to two mixed ability classes in their final two years of secondary school. Five pairs of experienced users of TI-83 or 82 graphing calculators from these classes were audio and videotaped solving a problem task. Protocols were constructed and subjected to intensive qualitative macroanalysis and microanalysis using tools developed by the researcher from Schoenfeld’s work. The findings were: (1) all students demonstrated understanding of the local and global nature of functions and the synthesis of these in determining a complete graph; (2) a range of mathematical and graphing calculator knowledge was applied in seeking a global view of the function with their combined application being more efficient and effective; (3) an understanding of automatic range scaling features facilitated efficient finding of a global view; (4) all pairs demonstrated having a clear mental image of the function sought and the possible positions of the calculator output relative to this; (5) students were able to resolve situations involving unexpected views of the graph to determine a global view; (6) students displayed understanding of local linearity of a function; (7) when working in the graphical representation, students used the algebraic but not the numerical representation to facilitate and support their solution; (8) scale marks were used to produce more elegant solutions and facilitate identification of key function features to produce a sketch but some students misunderstood the effect of altering these; (9) pairs differed in the proportion of cognitive and metacognitive behaviours demonstrated with question asking during evaluation supporting decision making; (10) correct selection of
an extensive range of graphing calculator features and use of dedicated features facilitated efficient and accurate identification of coordinates of key function features.
CHAPTER ONE
THE NATURE AND PURPOSE OF THE STUDY

1.1 Rationale

With the introduction of graphing calculators occurring in the late eighties, their widespread availability in the early nineties, and their mandating in senior secondary mathematics classrooms in Victorian schools in the late nineties, the need for research in a graphing calculator learning environment became a priority. Barrett and Goebel (1990) predicted that “the presence of graphing calculators in high school mathematics classrooms [would] have a significant impact on the teaching and learning of secondary school mathematics in the 1990s” (p. 205). In particular, the introduction of graphing calculators was expected to impact on the understanding senior secondary students have of what constitutes a “complete graph” of a function, that is, a graph showing “all the relevant behaviour” (Demana & Waits, 1990, p. 216). However, Doerr and Zangor (2000), Penglase and Arnold (1996), Williams (1993), and others suggested there was a lack of detailed documentation about how students make use of graphing calculators in the classroom, especially from the students’ point of view. Whilst the types of technology potentially available in mathematics classrooms around the world have expanded to include computer algebra systems (CAS), as recently as 2002, Zbiek pointed out that “secondary mathematics teachers—like researchers and college instructors (Zbiek, in press)—may find the lure of graphing capabilities is more compelling” (p. 132). For this reason, and to add to the knowledge base against which the use of other technologies can be evaluated, it is imperative research continues to be conducted into how students use graphing calculators in their learning of functions. No matter how sophisticated other technologies may be, it is the graphing calculator that students do have and as such it is crucial that the effects of this technology in mathematics classrooms are understood by teachers. According to Burrill, Allison,
Breaux, Kastberg, Leatham, and Sanchez (2002), “By conducting rigorous studies of important questions and relating the result to classroom practice, we can ensure that handheld technology contributes in positive ways to improved mathematics education” (p. 56).

1.2 Background to the Study

Graphing calculators, or graphic calculators as they are also called, are now assumed tools in Victorian senior mathematics classrooms and assessment tasks (Board of Studies [BOS], 1999, p. 135). Whilst the term tool has various meanings in the literature, in this thesis the definition of Salomon (1993) is used. He argues that tools “need not be a real object” (p. 179), they have a purpose and require skills and knowledge for their use, “serve functions beyond themselves” (p. 179), and can be distinguished from machines in that “they need to be skilfully operated upon throughout their functioning to achieve their purpose” (p. 180). More specifically, Salomon proposes that educationally useful tools are those that “stimulate higher order thinking and guide it in a way that makes learners better and more independent thinkers” (p. 181). Hence, graphing calculators are cognitive tools or smart tools (Pea, 1993) as they provide an environment where students can transcend their cognitive limitations, access more complex thought and activity, and engage in learning which without the tool they would not have been able. The now compulsory use of graphing calculator technology in the Victorian mathematics classroom raises many issues. These include the effects of the technology on student learning and the extent of changes, if any, in teaching practice as graphing technology becomes commonplace in the classroom.

Although many studies have been undertaken in the use of graphing calculators, there are still many questions to be asked and answered. Of particular interest to this thesis is students’ understanding of a complete graph of a function in a graphing calculator environment. Prior to this study beginning, most previous research provided
statistical data, rather than descriptions or explanations of student use of this technology in their learning. Many graphing calculator studies simply “compare the use of the graphing calculator to the use of paper-and-pencil on the same tasks giving only limited insight into how and why students use calculators in the instructional context” (Doerr & Zangor, 2000, p. 144). Penglase and Arnold (1996) in their review of over one hundred papers found that much of the research, in the years 1990 to 1995 did little to inform and guide practice as it failed to distinguish the role of the tool from that of the instructional process. Burrill et al. in their 2002 review of the research literature note that the core finding is that the type and extent of gains in student learning in the presence of handheld graphing technology “are a function, not simply of the presence of the graphing technology, but of how the technology is used in the teaching of mathematics” (p. i). They concluded that “specific issues regarding the effective use of handheld graphing technology in the classroom have not yet been adequately addressed” (p. ii).

There are, however, some difficulties in observing students’ use of graphing calculators (Williams, 1993). It is not easy to see the screen as the student uses the calculator and even more difficult to identify the keystrokes used to achieve these windows. In this study, the use of a projection device, namely a View Screen, to project the graphing calculator screen and the subsequent videotaping of this projection went some way to overcoming this difficulty. This recording of the screen outputs, produced by the students, allowed new insight into students’ understandings when using such a calculator.

Penglase and Arnold (1996) suggest the need for studies that clearly define the learning environment and address graphing calculator use within this environment. While this researcher believes that the tool is an integral part of the instructional process and as such cannot be separated from it, the role of the teachers in the classroom, their
knowledge and skill in using the graphing calculator in mathematics learning, and the particular learning environments of the students in the study will be clearly described.

1.3 The Learning Environment

As recently as 2000, Waits and Demana pointed out that “the pedagogical significance to the mathematics community of the small, inexpensive, hand-held graphing calculator should not be underestimated” [italics in original] (p. 72). To take advantage of the opportunities arising, the learning environment should have changed with the introduction of the graphing calculator to the classroom. Lindsay (2002), for instance, points out that not all students will “embrace the technology” (p. 416) simply because it is available. Although their focus is on the use of more sophisticated technology, namely computer algebra systems (CAS), Pierce and Stacey (2002) concur with this view that the mere presence of technology “in a classroom does not mean its potential benefits will be realised” (p. 575).

The learning environment of the students in this study included the teacher as a fellow learner alongside the students. The two teachers of the classes in the study held an investigative, rather than rule-based (Boaler, 1997), view of mathematics learning. For example, in the study of Pythagoras’ theorem, students in these classrooms would undertake a range of activities that allowed them to explore, conjecture, test, and re-conjecture until they discovered Pythagoras’ theorem for themselves. This is in contrast to a rule-based view of learning where students would be told Pythagoras’ theorem and provided with exercises to practise using it. Collaborative group work was encouraged and the classroom was a place for discussion, both between teacher and students, and between students. Students were expected to make and test conjectures, develop as critical thinkers and effective communicators of their learning.

In addition, the students in this study were situated in a technologically-rich learning environment where they were provided with experiences that allowed a deep, rather
than a shallow, procedural approach to the learning of their mathematics (Boaler, 1997, p. 143). Tasks undertaken in the classroom included those that had the chance to be influenced by graphing calculator use (Tofaridou, 2002), rather than only tasks where the graphing calculator can be used to produce or check results but not necessarily contribute to the development of students’ understanding, and concept and skill development. Interpretive and constructive skills were expected, encouraged, and modelled. Students were expected to discuss their mathematical understanding and thinking with their teacher and peers.

The classroom was a place where the students were expected to use the graphing calculator intelligently, to consider when to use it and when its use was inappropriate, and when using it to contemplate alternative methods for any given problem. The use of the graphing calculator was integrated throughout the course. Both teachers’ views of mathematics concurred with that of Devlin who described mathematics as “the identification, abstraction, study and application of patterns, using the mental tools of logical reasoning” (2000, p. 26).

1.3.1 The Victorian Mathematics Curriculum

The mathematics curriculum in the final two years of secondary school, years 11 and 12, in Victoria is specified in the Mathematics Study Design (BOS, 1999). For this research study two aspects of the Victorian Certificate of Education [VCE] Mathematics Study Design were considered. One of these was content, that is, the expected knowledge and skills, and the other was process, that is, what teaching and learning opportunities are provided in order to allow students to acquire this knowledge and skills, that is, the nature of the learning as detailed below.

1.3.1.1 Content

Key knowledge and skills, of relevance to this study, for Outcome 1 of the VCE Mathematics Study Design include using “a variety of analytic, graphical or numerical
approaches to determine and verify solutions” (BOS, 1999, p. 71) and identifying “the effect of various transformations on the graph of a function or relation” (p. 72). For Outcome 3, these include identifying the relationship “between numerical, graphical and symbolic forms of information about functions and equations and the corresponding features of those functions or equations” (p. 72).

1.3.1.2 Process

The *Mathematics Study Design* informs Victorian secondary teachers that mathematics is “both a framework for thinking and a means of symbolic communication” (BOS, 1999, p. 7) and there are three underlying principles of the mathematics study that all students will engage in. These should underpin all mathematical activity and promote and develop key aspects of mathematical behaviour. The principles are (a) applying knowledge and skills; (b) modelling, investigating, and solving problems; and (c) using technology. The last is of particular concern in this study. It is characterised as “the effective and appropriate use of technology to produce results which support learning mathematics and its application in different contexts” (p. 7).

1.4 Graphing Calculators and Functions

Graphing calculators have particular application in the teaching and learning of functions, a key mathematical concept in senior secondary school mathematics (Ferrini-Mundy & Lauten, 1993; Heingraj & Shield, 2002; Knuth, 2000; Leinhardt, Zaslavsky, & Stein, 1990). Ferrini-Mundy and Lauten, for example, see function as a concept “central to modern mathematics” (1993, p. 155). Furthermore, Demana and Waits (1990, p. 222) suggest “deeper understanding and an intuition about functions are important by-products of a technologically rich approach to the teaching and learning of mathematics”. 
The concept of function contains many interconnected ideas. Generally, a single notation system or representation cannot adequately represent all aspects of a complex idea (Ferrini-Mundy & Lauten, 1993; Kaput, 1989). Kaput argues that “representation systems, when learned, are used by individuals to structure the creation and elaboration of their own mental representations” (p. 167, 1989). He suggests that multiple linked representations allow learners to combine understanding from different representations in such a way as to build a better understanding of complex ideas and to apply these ideas and concepts more effectively. In *Problems of Representation in the Teaching and Learning of Mathematics*, Janvier (1987a, 1987b), Kaput and others discuss the idea of representation in mathematics education, the translation process and the continuing difficulties experienced by students in changing from one representation to a second. Zaslavsky, Sela, and Leron propose that some of these difficulties are a result of confusion “between the algebraic and geometric aspects of slope, scale and angle” (2002, p. 119).

Functions require multiple linked representations for their full expression. In fact, multiple representations are common and virtually unavoidable in functions (Eisenberg & Dreyfus, 1994). These representation systems include: algebraic, numerical, and graphical (Kaput, 1989, p. 171; Lloyd & Wilson, 1998, p. 252). The algebraic representation is also known as the symbolic representation (Dick, 1992). The graphical representation is sometimes referred to as the visual (Acuña, 2002; Eisenberg & Dreyfus, 1994) or geometric (Zaslavsky, Sela, & Leron, 2002) representation. Each of these representations illuminates some, but not all, aspects of the idea of a function. Each “has its intrinsic level of complexity” according to Janvier (1987b, p. 31). The connections and links between the different representations are key aspects of much mathematics. It is very important that a learner has multiple aspects to their concept image (Lloyd & Wilson, 1998). Students need to be able to work both within and across
representations. The ability to pass from one representation to another when appropriate, the flexibility to use the most appropriate representation in solving a problem, and the ability to “see” one representation when working in another are the most essential components of function sense (Schwarz, cited in Eisenberg & Dreyfus, 1994, p. 46).

For a complete understanding of functions the various representations demand that students move from one to another. Firstly, given the algebraic representation of a function, the students should be able to generate the numerical table and the graph of that function. Secondly, given the graph of a function, the students should be able to generate the numerical table and the algebraic representation of that function. Thirdly, given the numerical table of a function, the students should be able to generate the graph and the algebraic representation of that function. Further, students need to be able to make connections across the three representations.

The use of the graphing calculator makes it much easier for students and teachers to work simultaneously with all three representations. Hence, the depth of students’ understanding of functions would be expected to change with the use of graphing technology, including increased depth of understanding, and new challenges due to misconceptions becoming apparent with graphing calculator use.

Graphing calculators and their linked representations provide students with immediate feedback as to how changes in one representation affect another. Students then have the opportunity to modify their ideas before strengthening misconceptions. However, learning tasks need to be chosen that facilitate this learning process. Teachers need to be careful not to continue past practice that focussed mainly on a single representation, as this can easily be reflected in the use of the graphing calculator.
1.5 Aims and Purposes

The aim of the research study that forms the basis of this thesis was to explore students’ solution processes as they worked in pairs to solve a function problem using a graphing calculator. The specific focus of the study was student understanding of the graphical representation of functions, particularly of cubic functions. Of key interest were the cognitive, metacognitive, mathematical, and technological processes students used to find a global view of the function and their identification of the key features of that function in a graphing calculator environment.

1.6 The Nature of the Study

By looking closely at what a small group of students understand about functions in a graphing calculator learning environment, teachers will be provided with a clearer picture of what is happening in their classrooms where graphing calculators are mandated tools. The employment of a qualitative approach will provide the researcher with a coherent picture of the interconnections between the students, the graphing calculator, and the learning.

The complex nature of the phenomena being studied necessitated the need for qualitative methods in order to gain insight into students’ actions (vom Hofe, 2001; Strauss & Corbin, 1998). A qualitative case study (Merriam, 2002c, p. 8) was used for the research described in this thesis as it was considered to be the most appropriate methodology to achieve the aims of the study as outlined previously. The intent of this study was on “understanding the dynamics present” within a single setting (Eisenhardt, 2002, p. 8) as is the intended purpose of case study research according to Eisenhardt. Given the case itself was of secondary importance as the purpose was the development of a general understanding of the issues involved as students search for a complete graph of a cubic function in a graphing calculator learning environment, an instrumental case study (Stake, 1995) was undertaken. This case study describes the results of
practice occurring in the classroom of this researcher and her colleague, through the provision of a snapshot of the students’ responses to a problem task.

Before commencing the main case study, a pilot study was undertaken, that involved two Year 11 student pairs and two teacher pairs. The pilot study explored the suggestion from research at the time (e.g., Steele, 1995a) that students would accept what they saw on the graphing calculator screen as a global view of the function. The pilot study raised more questions than it answered, producing findings that conflicted with this other research but demonstrating that the problem task was a suitable instrument to use in further research.

The main case study involved the administration of a pretest to two classes in their last two years of secondary school at an inner city, Australian, state secondary college. The results of the pretest together with additional assessment data were used in the purposive selection (Merriam, 2002a) of five student pairs who would be able to access the problem task used in this study in order to maximise what could be learned. Four students were selected from a Year 12 Mathematical Methods class and six students from a Year 11 Mathematical Methods class. Audio and video recordings were made of the attempts of the five pairs of students solving a problem task. The recordings, written scripts of the students, and observational notes made by the researcher were used to construct protocols for each student pair. These protocols were then subjected to macroscopic and microscopic analysis using analysis tools developed by the researcher from Schoenfeld’s framework for examining protocols developed for his work on mathematical problem solving (1985a, 1985b 1992a, 1992b). This protocol analysis involved activity classification, episode parsing, and the construction of episode and time-line diagrams.
1.7 Limitations and Delimitations

When considering the research detailed in this thesis the following limitations and delimitations should be taken into consideration.

1. The research is limited to a single inner city school.
2. The students in the study are non-representative of Year 11 and 12 students studying VCE Mathematical Methods in their high representation from non-English speaking backgrounds [NESB].
3. The timing of the administration of the problem task was constrained by the school program and individual student needs.
4. Teaching styles around the state and the world vary so the findings may be limited to students with teachers having a similar style.

1.8 Outline of the Thesis

This chapter details the nature and purpose of the research study and sets out the background for the study including the use of graphing calculator technology in Victorian secondary schools, the role of the graphing calculator in the learning of functions, and how this learning would be situated in the learning environment of the students in the study. Chapter two reviews the relevant literature. The context for the study, the research methodology, and analysis tools used are detailed in chapter three. Chapter four presents the macroscopic analysis of the students’ solutions, whilst the microscopic analysis of the students’ attempts at the problem task are presented in chapter five. Chapter six addresses the research questions specifically discussing the findings and implications. How each of the remaining chapters advances this thesis is explained in the following paragraphs.

Chapter two reviews the literature concerned with graphing technology and students’ understandings of functions both in a graphing technology and a non-graphing technology learning environment. This literature review reveals that use of a graphing
calculator does not by itself improve students’ understanding. Confounding factors include the teaching and learning environment, the time spent using the graphing calculator, and the specific experiences of the student with the graphing calculator. In addition, many authors suggested the use of graphing calculators and the multiple representations embedded in their use, essential to the exploration of functions, may uncover additional misconceptions held by students, that had previously remained hidden from teachers and researchers’ view. The review of the literature identifies a definite need for further research, particularly that which considers how students’ cognition of function is changed through the use of graphing calculators. Finally, the research questions arising from the literature that will be investigated in the research study are presented.

The aim of the study and its situational context, as well as details of the students involved in the study, the selection process, and learning experiences are delineated in chapter three. These are followed by details of the methodology, including reasons for a qualitative approach which incorporated the use of an instrumental case study. The data collection methods and analysis tools employed and the rationale for their selection and use are also explained. Quality control in the research is discussed, as are the limitations of the methods. The research instruments used are presented as are their administration and the history of the development of these instruments. Results of a pilot study are presented and discussed, both confirming the need for further research through conflicting findings with other research and validating the use of the problem task for further research as presented in this thesis.

In chapter four the students’ responses to the problem task are examined and analysed in order to gain a greater insight into the processes students use when using a graphing calculator to solve the given problem. This insight is gained through a variety of methods. The graphing calculator screen is used as a window into the minds of the
students. In conjunction with this, the dialogue of the students and their interactions are analysed. The coding scheme developed by this researcher is explained as are diagrammatic tools based on this. The macroscopic analysis using these tools led to a number of defining moments becoming apparent in the solution processes of the pairs of students as they worked through the problem task. A defining moment refers to an important or momentous event that may have had the potential to facilitate or impede the solution process. These defining moments are identified and documented.

In chapter five the microscopic analysis of these defining moments is presented and discussed in order to suggest possible explanations for the occurrence of each of the defining moments, their use or lack of use by different pairs, and the potential impact they had on the solution process.

Some preliminary discussion relating to the research questions is provided when detailing the results of the macroscopic and microscopic analysis in chapters four and five. However, in the final chapter these questions are specifically addressed. The final chapter also contains the conclusions and implications the findings of the study have for teaching and learning about functions in a graphing calculator environment. Future directions for research are also identified.
CHAPTER TWO
GRAPHING TECHNOLOGY AND ITS ROLE IN TEACHING AND LEARNING FUNCTIONS

2.1 Background

The use of a coordinate system, often attributed to René Descartes, has existed since before the time of Apollonius of Perga (about 300-200 B.C.). This notion was further developed by Oresme into a coordinate system close to our modern version prior to 1361. He provided “an early suggestion of what we now describe as the graphical representation of function” (Boyer, 1991, p. 264) as “he grasped the essential principle that a function of one unknown can be represented as a curve” (p. 265). This graphical representation increased the repertoire of representations available to mathematicians and allowed a greater insight into functions. However, usage of the algebraic representation still dominates the secondary school mathematics curriculum (Arnold, 1992, p. 115).

The impact of recent technology has allowed the production of graphs to change from a time consuming and often difficult activity to a simple and quick occurrence, whether one begins with an algebraic or numerical representation. The easy creation of graphs allows a large number to be observed and provides easy access to myriad function types. Graphing calculators are not merely providing an alternative way of doing the same old things; rather they allow us to do new things, including accessing and linking multiple representations almost simultaneously (Dick, 1996; Kaput, 1989, 1992; Pea, 1993).

Without doubt, functions have an important place in the secondary mathematics curriculum (Dubinsky & Harel, 1992; Ferrini-Mundy & Lauten, 1993; Knuth, 2000; Leinhardt, Zaslavsky, & Stein, 1990; Marjanovic, 1999; Yerushlamy & Shternberg, 2001; Zaslavsky, 1997). Marjanovic (1999), whilst discussing reform in the high school mathematics curriculum during the first half of the twentieth century, proposes “the
Chapter 2

most important innovation at that period was the introduction of the concept of function into the secondary school, emphasising the dominant place of that concept in contemporary mathematics” (p. 43).

2.2 Context of the Study

In its Statement on the use of calculators and computers for mathematics in Australian schools the Australian Association of Mathematics Teachers [AAMT] (1996b) signalled the need for technology to be used by all students. Simply having the technology is not enough, there is a corresponding need for teachers to actively seek to change the teaching and learning experiences to take advantage of the technology. They recommend “all students have ready access” (p. 1) to technology and students be immersed in a “technologically-rich learning environment” (p. 1) where the technology is used “both to support and extend their mathematics learning experiences” (p. 1). With regard to teachers, AAMT recommend they “be actively involved in exploring ways to take full advantage of the potential” and ensure that where students have access to technology for learning they also have access when their understanding is being assessed. They define appropriate technology as that which “places users in control of their own learning … encourages both independent and collaborative learning while extending and supporting the learning process” (p. 3).

In secondary mathematics education one major interest is the potential of graphing calculators to dramatically change the teaching and learning of functions. The use of graphing technology has the potential to move the implementation of the mathematics curriculum from the passive transfer of knowledge to students being in the position of taking an active approach to their learning (van der Kooij, 2001). Mitchelmore and Cavanagh (2000) echo the call of Penglase and Arnold (1996) and others for research to explore changes to learning functions when using graphing calculators. Cavanagh’s clinical interview with twenty five Year 10 and twenty five Year 11 high achieving
students suggested that “many difficulties in using a graphics calculator may be due to an inadequate understanding of some fundamental mathematical ideas including scale, accuracy and approximation, and the link between different representations of functions” (2001, p. 1). Cavanagh suggests that these difficulties are related to shortcomings in the curriculum and exist in both a graphing calculator and a non-graphing calculator environment. Barret and Goebel (1990) suggest access to graphing calculators that allow students to graph both functions and ordered pairs of data will allow students to “investigate and explore mathematical concepts with keystrokes” (p. 205). Furthermore, they add the graphing calculator will allow a learning environment to exist where students and teachers become partners in developing mathematical understandings and solving problems. Vonder Embse (1992) suggests the interactive graphing calculator environment allows students to explore and experiment, to integrate the numerical and graphical representations, and fosters mathematical reasoning, connections, and communication thus providing an “ideal environment for teaching and learning mathematics” (p. 65). This study is part of one teacher’s way to see better how students use graphing calculators in their learning of functions, specifically when information provided within one representation must be applied in a second representation.

This literature review addresses the relevance of multiple representations—including the importance of the graphical representation particularly in the teaching and learning of functions; differences between graphing calculators and other graphing technology; the relevance of multiple representations when using graphing technology; the various ways in which functions can be perceived in a multiple representation environment; the issues of scale including effects of scale and shape; and changes to, and implications for, teaching and learning in a graphing technology learning environment.
2.3 Multiple Representations

Graphing technology use has the potential to highlight the importance of multiple representations, their growing relevance, and impact on understanding. In the following section the *three core representations* (Kaput, 1989) are detailed, as are their differences and the links between them. The benefits or otherwise of a multiple representation learning environment are also presented and discussed.

According to the Australian Association of Mathematics Teachers (1996a):

> It is particularly important that students understand the connections between the relationships and the information (data) that are represented in the very concise symbol system of algebra and that they can interpret a variety of visual representations of that information. (p. 4)

The use of multiple representations has the potential to make learning meaningful and effective. Eisenberg and Dreyfus (1994) suggest that in the study of functions “multiple representations are so ubiquitous … they can hardly ever be avoided” (p. 46). However, in order to realise this potential the advantages and disadvantages of each representation must be known (Friedlander & Tabach, 2001; Kaput, 1989, 1992). For the *numerical representation* or table, the use of numbers allows students to use familiar objects to demonstrate relationships and consider specific cases. However, the lack of generality of this representation may result in some solutions of a problem being overlooked. The *graphical representation* or graph provides a visual display, portrays a collection of specific cases, and can always be used when an algebraic solution is beyond the current capabilities of a student, or when no algebraic method exists. Accuracy may be limited and scaling or illusions may affect interpretation. As with the numerical representation, only a portion of the domain is visible. Friedlander and Tabach (2001) describe the *algebraic representation* or formula or equation as “concise, general, and effective … sometimes the only method of justifying or proving … however, an exclusive use …
may obstruct mathematical meaning ... and cause difficulties in some students’ interpretation of their results” (p. 174).

Current views on mathematical learning include those that suggest “multiple representations of concepts yield deeper and more flexible understandings” (Keller & Hirsch, 1998, p. 1). Importantly, they stress that the use of multiple representations “is not based solely on the fact that technology permits it” (p. 1). The notion is that the use of multiple representations in teaching and learning has the potential to increase student understanding (Even, 1998; Friedlander & Tabach, 2001; Goldenberg, 1987; Leinhardt, et al., 1990).

As graphing technology becomes more common, the use of multiple representations is gaining more attention (Piez & Voxman, 1997). The interactive environment allows students to explore patterns and processes via the multiple inputs and outputs, thus providing students with an opportunity for better understanding. Immediate feedback intrinsic in the tool allows students to modify their ideas before they develop entrenched misconceptions. The strategy of trial and improvement is utilised to construct meaning (Barnes, 1995; Ruthven, 1992; Smart, 1995). Knuth (2000) agrees that the study of multiple representations is important, but suggests that many students leave secondary school “lacking an understanding of the connections between these representations” (p. 500).

### 2.3.1 The Three Core Representations

The three core representations (Kaput, 1989) used in senior secondary mathematics classrooms and of major interest in this study are the algebraic representation, the graphical representation, and the numerical representation. Kaput distinguishes between the numerical and graphical representations in the way they sample the domain. The numerical representation “displays discrete, finite samples, whereas coordinate graphs display continuous, infinite samples” (p. 172). He argues that the graphical
representation is fuller and simpler, in part due to the condensing of a pair of numbers to a single point. However, he notes graphs do have disadvantages related to misleading views and inferences and scale. These include the lack of a referential system, conflicts occurring with everyday experiences, and the effects of scaling. However, the graphical representation “leave(s) precise details unclear” according to Goldenberg (1987, p. 197). The shape of a graph is dependent upon the viewing window and this contributes to the confusion between transformations and rescaled views of graphs (Dunham & Osborne, 1991; Goldenberg, 1987). The finite viewing screen of graphing technology “creates a need for scaling and positioning skills” (Dick, 1992, p. 152) as the graphical behaviour of the function under consideration may be outside the current viewing window or distorted by the scale as “zooming in obscures global information and zooming out obscures local information” (p. 153). Advantages of tables include their specific nature and the fact that they are often ordered and evenly spaced. This allows changes in consecutive values to be explicitly determined. Tables provide examples of the relationship but not its exact nature (Goldenberg, 1987) although students need an understanding of the accuracy of numerical values provided by graphing technology. In contrast, the algebraic representation specifies the exact nature of a relationship but provides neither specific examples nor visual display. Notwithstanding these differences and difficulties, students need all three representations as different features apply differently in different situations (Lloyd & Williams, 1998). Each representation, whilst conveying some aspects of the function well, leaves other aspects unclear.

Moving from one representation to another is a translation according to Janvier (1987a, 1987b) and Kaput (1987). The graphing calculator has the potential to change teaching and learning with regard to the translation of functions. It allows the technology to translate actions across representations and provides the learner time to observe and explain the consequences. “The cognitive linking of representations creates
a whole that is more than the sum of its parts [italics in original]” (Kaput, 1989, p. 179). The graphing calculator allows us “to make fuller use of numerical and graphical techniques, techniques typically of greater simplicity and generality” (Kaput, 1989, p. 192) and hence has the potential to allow improved understanding.

2.3.2 Benefits of Multiple Representations

The review of claims made for learning with multiple representations by Ainsworth, Bibby, and Wood (1997) suggest several benefits. Firstly, multiple representations provide support for different ideas and processes as properties from multiple representations can only enhance the view provided by a single representation. This contrasts with Even’s (1990) view. She suggests it does not follow that understanding a concept in one representation implies understanding the same concept in a second representation. Secondly, they clarify concepts and ideas by providing different views of the same idea. Thirdly, they provide a rich source of views of a concept and create the opportunity to make links across the representations. Ainsworth et al. (1997) conclude that if development of better understanding is the desired outcome of the teaching program, then two things need occur. Firstly, unfamiliar representations should be presented alongside familiar ones, and, secondly, students need to experiment with constraints available in one representation in order to control outcomes in another. So while multiple representations can improve student understanding they do not necessarily do so. This concurs with the claims of Even (1990, 1998).

“Working simultaneously with at least two linked representations is more manageable” with graphing technology according to Leinhardt et al. (1990. p. 7) in their substantial review of the research on functions and graphs. Although each of the representations, and its advantages and disadvantages, should be considered regardless of the medium, it is only with graphing technology that this is truly possible. Demana
and Waits (1992) suggest that a global understanding is enhanced as students can see both their algebraic input and the solution or graphical representation simultaneously.

2.3.2.1 Importance of the Graphical Representation: Visualisation

Many authors argue that it is through access to the graphical representation wherein lies the power of the graphing calculator. The benefits of visualisation have been recognised by many including Brown (1998), Kaput (1992), Smart (1995), Tall (1996) and Underwood (1997). These benefits include the immediate visual feedback provided by the graphing calculator (Selinger & Pratt, 1997; Williams, 1993), its interactive nature, and the opportunity to understand the links between the graphical and other representations (Hector, 1992; Hollar & Norwood, 1999; Kaput, 1992; van der Kooij, 2001; Wilson & Krapfl, 1994).

Graphical visualisation allows students to develop an understanding of the relationship between functions and their graphs. “The global nature of the graphical image means that information may be extracted quickly and easily” (Arnold, 1998, p. 182). Spatial visualisation skills are improved through the use of graphing technology according to Ruthven (1996, p. 448) and Vasquez (1991, as cited in Shoaff-Grubbs, 1994). The quantity of graphs that a student can generate allows them to construct knowledge of a collection of functions and their graphical representation, and provides them with the ability to link these and make generalisations as a basis for future work. Movshovitz-Hadar (1993) suggests that an increased emphasis on a graphical approach provides students with an understanding of the parameters of a function and the effect these have on the graph. She suggests that the more common algebraic approach does not result in this important understanding (p. 391).

On the other hand, Goldenberg (1987) contends that the graphical representation may cause students to alter correct beliefs and he proposes several reasons for this. It is not because the graph is incorrect but rather the observation skills of students need
development. Students often focus on a particular feature of a view of a function and ignore others. Sometimes multiple features are considered but one is treated as dominant. To help overcome these difficulties teachers need to provide students with situations that allow them to focus on all the important aspects. Illusions can also be created as a result of several factors including the infinite size of a graph, the position of the viewing window on the function, and the effect of the scale on the view visible.

Not only does the graphical representation allow for an expanded understanding of the concept of function, but also it improves spatial skills. However, visual images of functions do not necessarily equate with being able to make connections between algebraic and graphical representations. Again, it is the curriculum and instructional environment that affect both learning outcomes and difficulties experienced.

2.3.2.2 Is More Better?

Underwood (1997) questions whether technology that uses multiple representations “always means better or more efficient (learning) or does the use of multiple representations place new learning demands on the child” (p. 3). She poses the question as to “whether new technologies offer significantly different ways of representing ideas and knowledge” (p. 5). Wilson and Krapfl (1994) suggest that graphing calculators do just this. The interactive nature of the tool allows students to “experiment and explore, thus fundamentally changing the way they learn important properties of functions” (p. 253). By toggling between “the three most common functional representations … [students are able to] … build conceptual links among these representations” (p. 254). Van der Kooij (2001) supports this view suggesting it as one of the reasons for the fluency attributed to students in a graphical tool environment by Hollar and Norwood (1999). By combining representations students are no longer restricted by the weaknesses of one particular representation. Clearly the graphing calculator learning
environment is one where interactions with more than one representation is almost inevitable.

2.3.3 The Multiple Representations of Functions

The easy access to multiple representations has significant implications for the teaching and learning of functions in senior secondary mathematics. Functions are “multi-faceted” (Lloyd & Wilson, 1998, p. 250) and cannot be fully understood within a single representation environment as each representation illustrates only some of their complexity. The need for multiple representations to express complex ideas, the links between the representations, and difficulties encountered in relating the local and global views of functions will be considered now.

2.3.3.1 Complex Ideas Require Multiple Representations

According to Coulombe and Berenson a “fluency with multiple representations of mathematical relationships plays a significant role in the successful development of algebraic thinking” (2001, p. 168). This fluency is enhanced for students who learn the algebra of functions in a graphical tool environment according to Hollar and Norwood (1999).

Complex mathematical ideas frequently cannot be expressed with a single representation system according to Asp, Dowsey, and Stacey (1993), Kaput (1989), Markovits, Evelon, and Bruckheimer (1986), Norman (1993), and others. “The idea may require multiple, linked representations for its full expression and these different representations may aid the learner’s understanding of the idea” (Asp, et al., 1993, p. 51). The concept of function is one such complex idea. Ferrini-Mundy and Lauten (1993) suggest students’ interactions with this concept are very complex. Understanding the relationship between different representations of a function is essential for a fully developed concept of function, thus contributing to this complexity.
Being able to make links between representations is crucial to the underlying concept of functions (Even, 1998; Kaput, 1992; Yerushalmy, 1991). Graphing technology provides students with the opportunities to make these links. Functions can be represented in a variety of ways including verbal descriptions, ordered pairs, tables, equations, and graphs. Students need to understand and be able to work within each of these representations. They also need to be able to translate freely between them, to understand that “the same function can be represented by each of the above representations” (Markovits et al., 1986, p. 19), and deal with two representations simultaneously. To fully understand the concept of function students must be able to treat the different representations as different systems that by themselves cannot completely describe a function.

The graphing technology provides the opportunity for students to build a deeper understanding of functions and so explore the complexity of the concept. Understanding the concept of solution and the solution process is also enhanced by graphing technology. It provides the opportunity to make connections between the algebraic and graphical representation of functions (Duren, 1991). Lessons that take advantage of the power of the graphing technologies not only generate more data and ideas, but also encourage discussion and negotiation of the meaning of student observations and findings. This enhanced involvement allows students to construct their own understanding of mathematical concepts (p. 24).

Heid (1988) undertook a study comparing a small experimental group of 39 students using graphing technology with a larger group of 122 students following a traditional lecture course in a USA college applied calculus course. As well as using the technology, the experimental group placed greater emphasis on graphs and used a wider range of representations to explore the meanings of concepts. Heid reported that the experimental group “spoke about the concepts ... in more detail, with greater clarity, and
with more flexibility” (p. 21), were more able to translate from one representation to another, and performed almost as well on the final examination (with no significant differences being reported). Interviews with the students showed they believed graphing technology “reduced the manipulative aspects, ... gave them more confidence in their results, ... and focused their attention on more global aspects of the problem” (p. 22). Limitations of the study, as noted by the author, included different class sizes and the multiple roles of Heid as interviewer, instructor, and investigator. However, this study raised issues that deserve attention and further research—the positive view of students using the technology who felt the focus changed from manipulation to analysis, the students’ increased confidence, and their increased attention to more global aspects of functions.

2.3.3.2 Difficulties Working in Multiple Representations

Students experience difficulties working with functions in the algebraic and graphical representations (Billings & Klanderman, 2000; Even, 1993; Kaput, 1989; Piez & Voxman, 1997; Selinger & Pratt, 1997; Tall, 1996). Many students in Even’s study, for example, experienced problems when the scale was changed. Further, they demonstrated a limited concept of function, as they believed that all functions could be represented by an equation and have nice graphs.

Even (1998) collected data via an open-ended questionnaire from 152 college mathematics students who were prospective secondary mathematics teachers and in the second phase of the study an additional 10 students completed the same questionnaire and were interviewed. She suggests that not enough attention has been given, by students, teachers, and researchers, to the flexibility to move between representations and other knowledge and understandings. This study of prospective mathematics teachers, like many others, holds a two-fold interest in that it informs about current understandings of students as well as informs about the knowledge of our future
teachers. The students had difficulties linking different representations. Even suggests that students who could “easily and freely use a global analysis of changes in the graphic representation” (p. 119) had a better, more powerful understanding of the links between the graphical and algebraic representations of functions, than those students who preferred to use a more local, specific view. However, she warns against concluding that the global view leads to a better understanding of functions and their representations. She points to several items in her study where a point-wise perspective was more powerful. Further, she notes that a global approach does not equate with understanding. She suggests that a combination of a global and point-wise approach to functions, as well as being critical and flexible is the most powerful, concurring with the findings of Leinhardt et al. (1990) and Moschkovich, Schoenfeld, and Arcavi (1993). Even’s warnings provide a timely reminder that the opportunities opened up by the use of graphing calculators to increasingly focus on the global view of functions should not be undertaken at the expense of local, specific views. This tool allows both of these important aspects of functions to be viewed concurrently.

2.3.4 Summary: Multiple Representations

Access to multiple representations and the connections between them are important for student understanding. Whilst multiple representations allow ideas to be explored more broadly, the additional information they provide can hinder as well as help learning. However, each representation provides only some aspects of a concept and multiple representations needs to be used not just because this is easily possible with graphing technology but because it is needed to increase student understanding. The algebraic, numerical, and graphical representations used by secondary mathematics students are the three core representations (Kaput, 1989). These are of major interest in this thesis. Multiple representations support the clarification of ideas, processes, and concepts. Hence, students are able to develop and deepen their understandings.
However, teaching must allow students to develop an understanding of the links between the representations, as understanding a concept in one representation does not necessarily imply understanding it in a second. In addition, students need to understand the limitations of each representation and have misconceptions related to shape, the effect of scale on the graphical view, and the infinite size of a graph challenged.

Functions in particular require multiple representations for a complete understanding. The research suggests that students (and prospective teachers) experience difficulties working in a graphing technology multiple representation environment. In addition, it has been shown that both the global and local aspects of a function are required for understanding. The graphing technology allows both of these aspects to be considered and explored concurrently with multiple representations.

Questions arising from the literature reviewed in this section include (a) what understandings do students demonstrate of multiple representations of functions and (b) what understandings do students demonstrate of the local and global nature of functions in a graphing calculator learning environment?

2.4  Graphing Calculators versus Other Graphing Technology

Graphing calculators are not the only technological tool available to the classroom teacher to facilitate students’ development of a rich knowledge base about functions in a multi-representational environment. The focus of this section is on comparing the graphing calculator with other graphing technology.

2.4.1  Advantages of Graphing Calculators over Other Graphing Technology

While there is much overlap between the graphing calculator and computer graphing applications for example, ANUGraph (Ward & Smythe, 1986) and Graphmatica (Hertzer, 1991), there are some major differences. These differences include ownership, cost, and availability. The last of these includes their portability, access via class sets, and ease of sharing. In addition, the fact that the use of the graphing calculator is now
mandated in mathematics assessment tasks in the final year of secondary schooling in several Australian states adds another key difference. Although other graphing technologies may be used in school based assessment tasks in both Victoria and Western Australia, for instance, the only electronic graphing technology that is allowed to be used by students during examinations is the graphing calculator.

In her discussion of the range of technology now used across the mathematics curriculum, Heid (1997) suggested “perhaps the single most important technological influence on high school and early college mathematics classrooms has been the graphics calculator” (p. 24). Graphing calculators are a personal, user-friendly, and portable technology (Brown, 1998; Dick, 1996). They are available and inexpensive (Demana & Waits, 1992; Waits & Demana, 2000), able to be accessed at all times, and enable “connections to be made between different representations of mathematical ideas” (Goos, 1998, p. 103). According to Vonder Embse (1992) the graphing calculator provides a unique environment in which to link different representations. “The graphing calculator bestows a sense of personal ownership on graphs, and that phenomenon alone can make a tremendous difference in the dynamics of the classroom” (Dick, 1996, p. 33). For these reasons, the use of graphing calculators can be considered as distinct from the use of other graphing technology.

2.4.2 Graphing Calculators: The Available Technology

Computer graphing applications are much less likely to be acquired by schools than graphing calculators for the mathematics classroom (Kissane, 1996; Waits & Demana, 2000). The growing number of computers available in schools has had minimal impact in the teaching of mathematics (Barret & Goebel, 1990; Harskamp, Suhre, & Van Struen, 2000; Waits & Demana, 2000). Kissane (1996) and Waits and Demana (2000) argue that the graphing calculator is a much more realistic alternative to computer technology. In fact, given their mandating in assessment tasks (in Victoria since 1997,
West Australia since 1998, and recently in South Australia), students in senior secondary mathematics classes are quite likely to own their own calculators, allowing student control over how they use the technology.

Graphing calculators are much more accessible to students, being available for a very low cost (Faragher, 1999; Ruthven, 1995; Waits & Demana, 2000). Where students cannot afford to buy their own, schools can provide class sets of graphing calculators for less than the price of a computer (Keiran, 1993; Kissane, 1996; Ruthven, 1992) through favourable purchasing and leasing agreements with graphing calculator companies. Almost every student has access to the tool at school and at home. Whilst equity questions with regard to cost are frequently raised in the literature (Kissane, 1996), cost does not seem to be a major issue as evidenced by the reported widespread use of graphing calculators in Australian (Kissane, 1996), British (Ruthven, 1995), and United States (Doerr & Zangor, 1999; Kissane, 1996; Waits & Demana, 2000) schools.

2.4.3 A Multi-purpose Tool

In addition, graphing calculators provide teachers and students with an extensive range of functionalities. Up until recently, if using a computer, several packages were needed to provide the same range. The recently released Texas Instruments (TI) Interactive appears to provide all the functionality of the TI-83 graphing calculator at low cost on a computer (only for PC users, however). Whilst this and other future packages may be equivalent or better than the TI-83, one still needs to consider the cost of and access to the computer and the increased likelihood of off task behaviour, technological and security problems, and time cost (e.g., enabling the whole class to start up their computer and open the relevant application). The functions of graphing calculators are broader than their uses in the area of graphing. They also contain statistical functions, for example, and this is further argument for the superiority of the cheaper graphing calculator over the more expensive computer.
2.4.4 A Summary of the Comparison between Graphing Calculators and other Graphing Technologies

After comparing graphing calculators and other graphing technologies it appears the graphing calculators have many advantages over other technologies. Graphing calculators have an “influence and impact on mathematics education that has far exceeded that of computers” (Dick, 1996, p. 31). Their use beyond that of a function grapher further suggests the need to consider them as different teaching and learning tools. Their nature as personal technology, ease of use in the classroom, portability, relative cheapness, relevance to the learning of function, widespread use, and recent mandated usage during assessment in secondary schools in several Australian states all suggest they are different to other graphing technologies and further suggest the need for research in this area. For these reasons, graphing calculators will be the technology used in the study presented in this thesis.

2.5 Gender Equity Using Graphing Technology

With regard to gender, the small number of studies, particularly those addressing secondary mathematics students, give conflicting results. A closer analysis, however, suggests that girls are not disadvantaged in mathematics, as often suggested, where the use of graphing calculators is an integral and important part of the teaching and learning and when assessment questions and tasks are completed using graphing calculators. Some studies (Forster & Mueller, 2001; Hollar & Norwood, 1999) suggest the graphing technology results in no disadvantage although more recent work by Forster and Mueller (2002) found small differences in performance, not always significant, on function application examination questions when students used graphing calculators. Other research provides some evidence that the use of graphing calculators resulted in improved learning for females. For example, in her 1993 study, Shoaf-Grubbs (1994) found (a) increased understanding by females of elementary graphing and algebra
concepts and (b) females using graphing calculators outperforming those without where visual spatial competencies were required. Ruthven (1990) found that females experienced decreased anxiety and increased confidence with regular use. However, others including Faragher (1999) warn that not all students, female or male, find using a graphing calculator a positive experience.

The study by Shoaf-Grubbs (1994) explored the spatial visualisation and mathematical understanding of female students, described as having weak mathematical abilities, spatial visualisation skills, and mathematical understanding. She found that the level of understanding of linear equations and spatial visualisation abilities related to linear equations and parabolas of those students in the class using graphing calculators was clearly enhanced, compared to those students in the non-graphing calculator class who were presented with as identical as possible curriculum materials. Shoaf-Grubbs (1994, p. 189) concluded that students’ enhanced spatial visualisation skills during graphing calculator use “strengthened [their] abilities to mentally move or alter all or part of a function representation”. As some authors such as Tartre (1993) claim “spatial skill level does seem to be more related to mathematics performance for females than for males” (p. 57) and girls with low skills in this area have previously been said to be unable to compensate for this as do their male counterparts, Shoaf-Grubb’s findings that spatial visualisation skills can be enhanced for girls by graphing calculator use is encouraging.

Ruthven (1990) found the performance of upper secondary female students, using graphing calculators, was clearly superior to that of their male counterparts on items that required visual-spatial abilities. He attributed his finding to the extent of exposure the experimental group had previously to regular use of the graphing calculator. He further suggested that this extensive experience led to decreased anxiety and increased confidence amongst female students, as they were not working under conditions of
uncertainty. In the control group, where graphing calculators were not used, females under performed in the interpretation of graphs compared to their male peers.

Faragher (1999, p. 236) who had used graphing calculators in her secondary mathematics classes since 1993, found that although most students enjoyed and were motivated in lessons with resultant enhanced learning outcomes, it was not so for all students. Faragher identified both male and female students who had a negative or indifferent attitude to the use of graphing calculators. It was apparent from her report that the role of the calculator in these Queensland secondary mathematics classrooms at the time of her study was not as pivotal as for the students in Ruthven’s 1990 study and in current classrooms in Victoria.

Whilst there is certainly a need for further research in the area of gender and technology including graphing calculators, some of the literature suggests that with regular use females show significant improvement in their mathematical understanding and spatial visualisation skills when dealing with linear and quadratic functions; however, this is dependent on the nature of their experiences, including the classroom culture and the teaching and learning activities experienced. In addition, extensive experience can lead to decreased anxiety and increased confidence amongst female students.

2.6 Issues of Scale

Aspects of scale impact on students’ understanding of functions in many ways. Two aspects that have received considerable research attention are (a) the effect that changing scale or scale marks or the lack thereof, has on a graph (Billings & Klanderman, 2000; Cavanagh & Mitchelmore, 2000; Demana & Waits, 1990; Dunham & Osborne, 1991; Mitchelmore & Cavanagh, 2000; Steele, 1995a; van der Kooij, 2001; Williams 1993) and (b) the fact that shape is an artefact of scale (Demana & Waits, 1990; Dreyfus & Halevi, 1991; Dunham & Osborne, 1991; Goldenberg, 1987;
Zaslavasky, Sela, & Leron, 2002). The lack of an annotated scale on the axes on a graphing calculator adds to already existing misconceptions and difficulties. An understanding of the effect of changes of scale is essential for successful graphing calculator use.

Leinhardt et al. (1990) define scale as “the assignment of values to intervals between the lines on the Cartesian system. In two dimensional graphs, there are two decisions, one for the x-axis and one for the y-axis” (p. 4). Furthermore, Mitchelmore and Cavanagh (2000) distinguish relative scale from absolute, the former being “correctly regarding scale as a ratio of distance to value” and the latter interpreted “solely as either the distance between adjacent marks or the value represented by this distance” (p. 262). Difficulties with scale are not confined to, or an artefact of, the use of graphing technology. Kerslake (1981) found that scale was problematic for secondary students with students frequently choosing unsuitable scales for a given problem and being unable to identify similarities when the scale, but not the function, was altered. Demana and Waits (1990) have noted the crucial nature of scale and the increased importance of understanding its effects when working in a graphing calculator environment.

### 2.6.1 Effect of Scale Marks

It is essential that students appreciate that scale marks have no effect on the section of the graph portrayed in the window of the graphing calculator. Studies including those by Dunham and Osborne (1991), Williams (1993), and more recently, Mitchelmore and Cavanagh (2000), suggest that students do not have a good understanding of this concept.

Students often focus on adjusting the scale marks, not realising this has no effect on the view they see, and actually ignore the function (Williams 1993). In two studies, when drawing a graph by hand, students always marked the axes at unit intervals (Mitchelmore & Cavanagh, 2000) or when translating from the graphical representation
always assumed the axes had unit intervals (Dunham & Osborne, 1991). This practice can lead to errors when other than unit intervals are present. Moreover, students fail to understand the differences in representation when comparing the graphical view of a function where both axes have the same scale marks (i.e., a *homogeneous coordinate system*) to the situation where the scale marks are different (a *non-homogeneous coordinate system*) (Zaslavsky et al., 2002). These difficulties with understanding scale marks and their effects may be related to paper and pencil graphing where marking in the scale is one of the first things undertaken by students (van der Kooij, 2001; Williams, 1993). In addition, the overuse by textbooks of homogeneous coordinate systems continues to contribute to the difficulties faced by students (van der Kooij, 2001).

Teaching is clearly needed to address this problem. Suggestions include students’ experimenting with editing the scale marks and determining their effect when no graph is entered on the graphing calculator, changes to the traditional approach of text book writing, and providing experiences with graphs having realistic non-homogeneous coordinate systems that force students’ attention to the critical aspects (Cavanagh & Mitchelmore, 2000; Dunham & Osborne, 1991; Hector, 1992; van der Kooij, 2001; Mitchelmore & Cavanagh, 2000; Williams, 1993).

### 2.6.2 Scaling of the Axes

The literature reports that students’ experiences continue to include mainly homogeneous scaling systems for axes as evidenced by most textbooks using equally scaled axes most, if not all, of the time (Cavanagh & Mitchelmore, 2000; van der Kooij, 2001; Mitchelmore & Cavanagh, 2000). Students pay little attention to, and place minimal importance on, scale and often treat graphs and axes as separate and independent entities (Cavanagh & Mitchelmore, 2000; Dunham & Osborne, 1991; Mitchelmore & Cavanagh 2000). Graphing calculators “allow students to specify a
viewing window or rectangle when drawing graphs” (Demana & Waits, 1990, p. 215). Both students and teachers must accept and understand that the shape of a graph is dependent on the viewing rectangle within which the graph is viewed and they “must learn to choose viewing rectangles that give good pictures as they explore” functions and search for a “‘complete’ picture of the behaviour of graphs … a graph that shows all the relevant behaviour is called a complete graph [italics added]” (Demana & Waits, p. 216).

Students have difficulties when confronted with non-homogeneous scaling of the axes and often ignore the effects of scale (Billings & Klanderman, 2000; Hector, 1992; Steele, 1995b). The continual use of identical scales can lead to students making incorrect generalisations about functions. Van der Kooij (2001) and others suggest that these difficulties are not caused by the use of graphing calculators per se, rather that the increased repertoire that students are now expected to experience due to the use of graphing technologies includes more realistic graphs. Suggestions from the literature to overcome student difficulties include: where textbooks with graphs having predominately homogeneous coordinate systems are used, teachers need to ensure that these are supplemented by graphs having non-homogeneous coordinate systems; ensuring that students record the scale whenever sketching a graph and appreciate that a graph can not be interpreted unless the scale is known; and allowing students to experience the effect of altering equal scaling to unequal scaling on a range of graphs.

2.6.3 The Effect of Scale on Shape

Students’ confusion of scale and shape when using graphing technology has been noted by several researchers (e.g., Dunham & Osborne, 1991; Goldenberg, 1987, Goldenberg, Harvey, Lewis, Umiker, West, & Zodhiates, 1988). This confusion is compounded by many factors including the fact that only a portion of the graph can be seen, the infinite size of the graph, the fixed size of the viewing rectangle, and the fact
that no distinct points are observable. When observing the graphical output on a graphing calculator screen, only a portion of the shape appears. When trying to interpret these objects of infinite size and few distinguishable features, our everyday strategies fail according to Goldenberg (1987). The overall shape, magnitude of translation, direction of movement, and scale all become ambiguous. In addition, “interactions between the position and orientation of the graph and the shape of its window, and the interaction between the scale of the graph and the scale of the window” (p. 12) create illusions.

As only a portion of a graph can be seen, our interpretation of changes in this view is not the same as for familiar discrete objects. The infinite size of a graph and the lack of distinct points to observe can create illusions of changes that are incorrect. As a result, shifting a linear function in the vertical direction, for example, may appear to have caused a shift in the horizontal direction, as shown in Figure 2.1. If we consider the two pairs of linear transformations shown in Figure 2.1 where $y_b = y_a - 10$ and $y_d = y_c - 20$, students with some experience in the transformation of functions would in both cases see the expected, that is, the second function of each pair has been shifted down as indicated by the algebraic representation. However, without previous experience the alternate view that the second function of each pair has moved to the right may be accepted in one or both cases. It is not that the second interpretation is incorrect per se, rather it is not an interpretation that can be applied to all other function types nor does it as directly inform the student of the magnitude of the transformation.

![Figure 2.1. Illusions in the transformation of linear functions.](image)

$\begin{align*}
(a) \quad y_a &= 0.5x + 5 \\
(b) \quad y_b &= 0.5x - 5 \\
(c) \quad y_c &= 2x + 10 \\
(d) \quad y_d &= 2x - 10
\end{align*}$
Students can develop “visual” misconceptions between scale and shape and these misconceptions are related to the function type. Zooming in on a quadratic function appears to change its shape, as the viewing window is a fixed size. This is not the case for linear functions, or for everyday experiences, where perspective comes into play. Goldenberg (1987) argues that the confusion between transformations and rescaled views is further complicated by the fact that the shape of the graph depends on the viewing window being used. Zaslavsky et al. (2002) suggest there is much confusion about “the connection between the algebraic and geometric aspects of slope, scale and angle” (p. 119). They question the often stated belief that zooming in and out on the graphical representation of a function “doesn’t change the behaviour of the functions under investigation” (Zaslavsky et al., p. 119) and question whether the very different views seen make it possible for students to observe common behaviour. This view concurs with that of Goldenberg et al. (1988) who distinguished between “zoom” operations where the horizontal and vertical scale changes are equal as “in the real-world” (p. 36) and other scale changes where the horizontal and vertical scale can be changed independently. This latter scale change conflicts with our intuition explaining why “our almost automatic approach is to change both scales by the same factor, preserving the ratio of the two resulting scales” (p. 36).

For functions defined over the domain of real numbers a graph can only ever show a portion of the shape. The fact that the object being viewed is infinite in size and has few identifiable features brings illusions into play. The “overall shape, magnitude of translation … direction of movement … and scale may all become ambiguous when the object viewed is infinite in size and has the kinds of regularities of shape and lack of readily recognizable sub-elements inherent in lines and parabolas” (Goldenberg, 1987, p. 200).
To develop improved mental images of functions student need to make connections and resolve the confusion between transformations and changes of scale. Dunham and Osborne (1991) suggest that students treat graphs and axes as separate and independent entities, they “confuse the idea of a geometric transformation with a scale change” (p. 45). These authors suggest that students need to recognise that a transformation described as stretching or shrinking of a function transforms the location of the graph in the plane and its associated algebraic equation. For example, in Figure 2.2 the first two graphical representations are of the same function with a different viewing window, whereas, the third graphical representation is a geometric transformation of the first function having undergone a dilation in the vertical (or horizontal direction). The viewing windows for the first and third graphical representation are identical, however, the graphical representations in the second (not the first) and the third windows appear identical.

Figure 2.2. The effect of a (a) change of scale and (b) transformation on $y = 0.5x^2$

Scale has other effects on shape. Zooming in on a graph may make it appear closer in the case of a linear function or in the case of a quadratic function may make it appear to change shape, as shown in Figure 2.3. Students need to understand that shape is an artefact of scale and not confuse geometric transformations with a scale change. (Demana & Waits, 1990; Dunham & Osborne, 1991; Goldenberg 1987).
Dreyfus and Halevi (1991) believe that in the past students had a limited view of shape when it came to graphs. Hector (1992), on the other hand, suggests that students using graphing technology will have an increased *item bank of graphs*, both across and within function types, to fall back on and, hence, a better understanding of shape.

### 2.6.3.1 Misconceptions of Point and Curve

Students, and teachers, often describe and think of axial intercepts as a single value rather than a related pair of values (Dunham & Osborne, 1991; Zaslavsky, 1997) and a curve as a single entity rather than a collection of an infinite number of points (Dunham & Osborne, 1991). (It should be noted that the definition of axial intercepts is unclear even after referring to mathematics dictionaries where the axial intercept is at times defined as a distance!) This further exacerbates the difficulties already presented by the infinite size of a graph and issues related to homogeneous and non-homogeneous coordinate systems. Similarly to Dunham and Osborne, Zaslavsky found students assuming just one value is needed to define axial coordinates as they overlooked the fact that one ordinate (zero) is already known. He suggests this may exacerbate difficulties in making connections between the representations.

Lack of understanding is further demonstrated by students as to the effect of zooming in to a turning point. Misconceptions reported by Cavanagh and Mitchelmore (2000) included the expectation, by all students in their study, that zooming in to a turning point, although initially resulting in a view of a horizontal section, would eventually result in one pixel being seen at the vertex.

*Figure 2.3. Using Zoom Box to show the apparent linearity of a quadratic function.*
2.6.4 Summary: Scale Issues

Scale continues to be problematic for students and future teachers alike. Some of these difficulties occur with or without the technology, whereas others only occur with its use, however this is often due to the broader range of graphs and scales encountered; not the use of the technology per se. Both curriculum and pedagogical changes are necessary to overcome these difficulties. Suggestions include focussing on the effects of each ‘window’ element, increasing understanding that only a portion of the graph is shown in any given window, access to realistic graphs where students need to search for an appropriate viewing window, and comparison of the effects of transformations in different viewing windows. The difficulties students experience with scale signal the need for careful interpretation of the results of research studies.

An entire graph of a function is not representable on any media used, including graph paper, graphing calculator screens, or with graphing software. The definition of a complete graph for the purpose of this study follows that of Demana and Waits (1990, p. 216). It is a sketch of the graphical representation of a function that shows both the global nature, or view, and identifies all key features (i.e., local aspects), This can be produced with the aid of graphing technology.

Questions arising from this section of the literature review include (a) what understandings do students have of the effect of the scale and scale marks and (b) what understandings do students demonstrate of the fact that the graphing calculator screen presents only a portion of the graphical representation of a function?

2.7 Functions and Graphing Technology

The search for an appropriate viewing window teaches students about functional behaviour over various domains. Working with a variety of function types and their graphs provides students with the ability to classify graphs according to function type (Hector, 1992). Algebraic methods can be clarified and checked graphically, improving
the algebraic understanding of students. All of these situations can be facilitated by the use of electronic graphing technologies especially graphing calculators.

Leinhardt et al. (1990), in their extensive review of the literature regarding students’ understanding of functions and their graphs concur with Hector (1992) that students who are introduced to graphs via the traditional approach have a point to point focus that causes them to overlook the global characteristics of functions. They suggest that electronic graphing technology should allow this difficulty to be overcome by providing ready access to a large number of graphs. In addition, the teaching approach must be taken into account and an approach taken, regardless of the type of graphing technology, that encourages students to consider both global and local aspects as being crucial to a well developed understanding of functions.

Access to a large number of graphs allows students to gain valuable experiences. With guidance, students can develop correct understandings about functions, their graphs, and the links between the algebraic and graphical representation of functions (Vonder Embse, 1992). The graphing calculator clarifies the algebraic approach; it does not replace it according to Hector (1992) although many, including this researcher, would argue with the second part of this statement.

Wilson and Krapfl (1994) state that because students can easily view families of functions, when using a graphing calculator, they are more likely to see the links between the algebraic representation and the family of graphs. It could, therefore, be argued that any improved learning is the result of increased experience with graphs rather than the use of multiple representations. The use of the technology does allow students to spend more time on the task of learning and less on the mechanics of producing graphs. Graphing calculators allow students to “quickly and accurately represent the Cartesian graphs of algebraically defined functions [to] easily adjust the scale of axes [and to] easily link between” representations (Wilson & Krapfl, 1994, p. 41).
It is not just regular use of a graphing calculator that is important, according to Ruthven (1990, p. 447), but the way the technology is used. In particular both students and teachers need to be encouraged to make greater use of graphical approaches. Graphing calculators allow students access to functions whose complete graphs are not visible in the standard viewing window, and scales and windows other than those presented to them via their text books (van der Kooij, 2001). Some of these researchers (e.g., van der Kooij, 2001; Ruthven, 1990) but not all, acknowledge that the access to the graphing technology by itself will not necessarily enable all students to make the links between the representations. Specific teaching is required, in addition to access to the tool, if all students are to make these links.

The use of technology not only increases students’ experiences of graphs, it provides immediate links between the graphs and their algebraic representations and allows more time to be spent on the task of learning. This can result in a shift from answering low level questions to qualitative reasoning supported by the graphing calculator—a shift from an algebraic and algorithmic method to a more conceptual approach (van der Kooij, 2001). With graphing calculators as tools “extensive, very specific algebraic training is no longer needed” by students, argues van der Kooij (2001, p. 609), rather students need to focus on the “functional and language-based aspects of algebra” (p. 606) as they explore more complex (and real) problems. Again, this focus must be made explicit through teaching. It will not automatically happen with the use of graphing calculators in the classroom. Teaching needs to refocus to enable students to focus on these visual language aspects of algebra and maximise the opportunities provided by access to graphing calculators.

### 2.7.1 Effects of Experience and the Particular Graphing Calculator Model

Mitchelmore and Cavanagh (2000) undertook clinical interviews with 25 Grade 10-11 students as they used graphing calculators to study graphs of linear and quadratic
functions. The importance of the students’ experiences and also the type of graphing calculator used are highlighted by this study. When asked to find the coordinates of the vertex of a particular function, 84% of the students believed that the \( x \) ordinate of “the vertex of the parabola could be found by averaging the \( x \)-values of the two centre pixels in the row at the base of the graph” (p. 260) and 88% of the students zoomed in repeatedly in the apparent belief that they would eventually see only one pixel at the vertex. Clearly the method of identifying the coordinates of a turning point of a graph, using a Casio \( fx-7400G \) is not the same as for the Texas Instruments TI-82 or TI-83, where a menu item (Calculate Minimum, or Calculate Maximum) can be selected, a domain specified for the graphing calculator to search within, with the result being that the calculator determines and displays the coordinates having the minimum (or maximum) value of the function within the stated domain.

2.7.2 Acceptance of the Initial View

In another study, Steele (1995a) found that the combination of daily use of graphing calculators in conjunction with a teaching emphasis on scale and obtaining a complete view of a graph lead to significant improvement in students’ ability to find a complete graph of a function, particularly where the initial view was potentially misleading. In addition students’ attitudes to using the graphing calculator were very positive. Prior to the study, the experience of Steele was that students tended to accept a given view as the complete graph. Students often accepted the view in the standard viewing window as providing a complete view of the graphical representation of a function. In addition, they also appeared to believe that a large window always provided a good view of a function and ignored the effects of scale. This study involved 180 Year 11 students in eight mathematics classes at two campuses of a large co-educational Melbourne independent school. At each campus there were four groups, each experiencing a different combination of access and emphasis. With regard to usage of the graphing
calculators the daily use groups used the graphing calculators daily and the less frequent use groups used the graphing calculators every second mathematics lesson. With regard to emphasis the strong emphasis groups experienced strong teacher emphasis on issues of scaling and obtaining a complete view of the function and the low emphasis groups experienced little emphasis on zooming out to obtain a complete view of the function. The groups with the strong emphasis received additional teaching material that involved an emphasis on zooming out over the fifteen lessons of graph sketching. Steele assessed students’ understanding via a pre-test and post-test in two parts, the first of which was completed without a TI-81 graphing calculator and the second section was completed with the aid of a graphing calculator.

He found both teachers and students were very positive about the use of the graphing calculators. Daily use of the calculator led to significantly more improvement on general graphing questions than less frequent use and a teaching emphasis on scaling led to significantly improved ability to cope with functions whose key features were not all visible in the standard viewing window. He found that as students searched for appropriate windows on a graphing calculator they were continually exposed to the effect of changed scale and the different views a single function could have.

2.7.3 Summary of the Implications of Graphing Calculators for the Learning of Functions

The literature reviewed has shown that the use of graphing calculators has the potential to increase student understanding of functions, particularly as they search for an appropriate window, access a large number of graphs, spend less time on low level tasks, move freely between representations, and experience a broad range of scales and windows. This potential is just that—potential—teaching must explicitly present and maximise the learning opportunities offered by the technology.
The study of Steele (1995a) showed teachers and students working together in a graphing calculator environment developed a very positive attitude toward using graphing calculators, daily calculator use led to significantly improved understanding of general graphing questions, and where the teaching emphasised the effects of scaling students demonstrated greater ability in finding global views of functions whose key features were not visible in the standard window. In addition, the research of Mitchelmore and Cavanagh (2000) highlights the need for all studies to detail both the previous experiences of the students with the graphing technology and the type of graphing technology used, as is the case in the study presented in this thesis. The major questions arising from this section are (a) can students find an appropriate window for a function and (b) are students accepting of an other than global view as being the complete graph of a function?

2.8 Changes to Teaching and Learning in a Graphing Calculator Environment

In their discussion of research related to graphing calculators, Dunham and Dick (1994) cite many studies including those by Rich (1991), Ruthven (1990), and Shoaf-Grubbs (1992) showing increased conceptual understanding by students using graphing calculators. The results are mixed with regard to achievement, however. Studies such as those by Ruthven (1990), Shoaf-Grubbs (1992), and Quesada and Maxwell (1992), for example, found significant differences between experimental and control groups, however, no differences were found by Rich (1991) whereas Giamati (1991) found significant differences in favour of the control group. In all these cases the “mere presence of graphing technology cannot account for the results” (Dunham & Dick, p. 442). Dunham and Dick concur with the words of others in that the combination of technology use, curriculum, and instruction must all be taken into account when researchers interpret the findings of research studies in this area.
Ainsworth et al. (1997), Alexander (1993), Devantier (1993), van der Kooij (2001), Rich (1991), Ruthven (1990), and Shoaf-Grubbs (1994) share the view that understanding is enhanced by the use of graphing calculators. Their findings suggest that the links between representations allowed the students in their studies to develop a better understanding of the relationship between different aspects of functions. In addition, Alexander and Shoaf-Grubbs support the view that concrete visualisation is essential for students to fully understand the concept of functions.

The literature suggests that experienced users of graphing calculators are better able to describe a given graph in algebraic terms (the reverse of what the calculator does), consider more of the important features of the graph (Ruthven, 1990), demonstrate an understanding of the link between the graphical and algebraic representation (Alexander, 1993; Devantier, 1993; Rich, 1991; Ruthven, 1990; Shoaf-Grubbs, 1994), relate a graph to its equation (Ruthven, 1990), and to find the algebraic representation of the graph (Ruthven, 1990). Although the results of Alexander’s study cannot distinguish between the tool and the changed instruction, it is important to note that the pre- and post-test items used were not designed specifically for his study nor any other technology assisted instruction. Consequently, the findings of Alexander would be expected in other situations where technology is used to enhance instruction. Many of these findings reinforce those of Goos, Galbraith, Renshaw, and Geiger (2000) in that the mere use of technology is not enough—the teaching and learning must change if the tool is to be used effectively and exploited to its fullest.

Dunham and Osborne (1991), however, did not find student understanding was improved with graphing calculator use. Dunham and Osborne (1991) and Underwood (1997) go further suggesting that the use of the graphing technology makes the task harder and students understand less when using this tool. Dunham and Osborne analysed the responses of 400 Ohio State University students to two particular questions.
in their pre-calculus course examination. They found that when information was presented graphically students found the question considerably difficult. Although the examination was undertaken after intensive instruction concerning functions and their graphs in an environment stressing the importance of graphing technology, the proportion of time (in their schooling) these students had spent focussing on algebraic approaches to mathematics compared to a short, intensive, graphical approach could have been a confounding factor.

The questions raised by the previous papers concern how teaching and learning change through sustained graphing calculator use. With the use of graphing calculators Dick (1996) argues that “graphs generated by the technology can be used to effectively communicate and discuss the meaning … graphing has become a means rather than an end” (p 38). Given that this tool is available should what is currently taught, and how we teach, change? How should teaching change in order to maximise the learning opportunities available with the use of the graphing calculator? Clearly, some concepts become accessible to students earlier, but equally the question needs to be asked whether some of the mathematics currently taught is now less relevant, less important, or perhaps even redundant. In addition, links that may be obvious to teachers are not automatically obvious to students and even where the link between representations is visual, students will not necessarily “see” the links in a meaningful way. In the study to be presented in this thesis, the students were experienced users of graphing calculators and hence it was expected that they would be able to demonstrate their understanding of the links between the algebraic and graphical representations of functions.

**2.8.1 Implications for Changes to Teaching and Learning**

“Graphing is a conceptually simple procedure that is very tedious to carry out in practice without technology” according to Stacey, Kendal, and Pierce (2002, p. 123). In addition, with graphing technology the “graph has more functionality than a paper
graph: one can zoom in and out to change the picture” (p. 123) enabling students to view a function and its graph globally rather than as just a collection of points (Dreyfus & Halevi, 1991). The ease with which graphing calculators produce complete graphs for functions provides students with a broad base to compare and contrast functions. The search for an appropriate viewing domain and viewing range to determine a global view of a given function focuses students on the global characteristics of a particular function (Hector, 1992) because when students can easily view families of graphs, they are more likely to see the links between the algebraic representation and the family of graphs (Wilson & Krapfl, 1994).

Students using graphing technology are better able to develop their understanding of quadratic functions by building on their existing knowledge and understanding of linear functions (Movshovitz-Hadar, 1993). Movshovitz-Hadar argues that graphing technology improves the learning of concepts of quadratic functions by allowing students to make connections between linear and quadratic functions. The tool allows investigations that focus on the links between these two types of functions and hence reduces the compartmentalisation of mathematics that inhibits learning. Students’ knowledge of linear functions, their algebraic and graphical representations, and the link between these provide the foundations for meaningful construction of the concept of quadratic functions. The tool enables students to expand their concept of function from linear to include quadratic. Movshovitz-Hadar proposes changes to the way quadratic functions are taught. He suggests they be introduced as the product of two linear functions, thus allowing the tool to be effectively exploited, consequently links are explicitly made for students and learning improved as a result. This advice can be extended to cubic functions. These can be introduced as the product of three linear functions, thus allowing the links between linear, quadratic, and cubic functions to be determined.
Van der Kooij (2001) suggests that the tools support the learning process by allowing students to check answers or ideas in multiple ways. The use of the graphing calculator makes it possible to learn mathematical concepts in ways completely different from the traditional approach. Moreover, the availability of the graphing calculator and its multiple ways of representing functions “stimulates students’ flexibility in solving problems” (p. 611).

Rich (1991) and Ruthven (1992) found students in their studies undertook more conjecturing and analysis, asked a greater number of higher level questions, and saw the importance of graphs and approximate solutions. Rich suggested that teachers using graphing calculators stressed the importance of the graphical representation and the value of approximate solutions in mathematics and actually used examples differently, again showing that teaching, and hence learning, changes with the use of graphing calculators.

When considering what teaching should occur in an eleventh grade class using graphing calculators to prepare for calculus, Lagrange argues “these calculators make the traditional balance of symbolic and graphico-numerical representations redundant, but a new balance is not yet clearly established” (1999, p. 70). He proposes, as does Asp et al. (1993) and Piez and Voxman (1997), that traditional techniques including work with paper and pencil still need to be used in order for students to make sense of the algebraic calculations as well as the graphical and numerical approaches.

2.8.1.1 Communication

With regard to communication the research is varied. Some researchers, van der Kooij (2001) for example, suggest that graphing calculators appear to provide potential for support of classroom communication. Goos et al.’s (2000) study provided evidence that the use of the graphing calculator “can facilitate social interaction and sharing of knowledge” (p. 318). Lloyd and Wilson (1998) found that an experienced teacher’s
well-articulated concepts of functions and use of multi-representations in reform materials, based on the 1989 National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics*, supported meaningful student discussion. Given the graphing calculator’s potential in the dynamic linking of multiple representations, it could be argued that its use would only enhance the potential for meaningful discussion in such a context.

Doerr and Zangor (2000), however, reported mixed results. They found the use of the calculator, as a personal device, inhibited communication in a small group setting, although when used as a shared device by the whole class it supported mathematical learning. The students in their study, although working as part of a group, tended not to share the view on their calculator leading to students progressing in different directions, often resulting in the group breaking apart into individuals.

Farrell (1996) explored changes in student and teacher behaviour when graphing calculators were situated in the learning environment. She suggests that students using graphing calculators demonstrate a wider range of roles. She found that teachers also took on different roles when using technology including spending less time as a task setter and explainer and more time as a consultant and fellow investigator. There was, however, a large variability from teacher to teacher in the time spent on the last of these roles.

Goos et al. (2000) undertook a three-year (1998-2000) longitudinal study examining the roles for technology in facilitating exploration and in student-student and student-teacher interactions. They identify four roles for interaction between teachers or students and technology. These are as master, servant, partner, and extension of self. These roles are hierarchical but not necessarily related to the level of mathematics used and teachers and students may be in transition between the different roles.
In an earlier study Goos, when working with Geiger (Geiger & Goos, 1996), used Kumpulainen’s (1994) framework for analysing functions of student talk when working collaboratively with technology. The functions of student talk according to this framework are informative, organisational, argumentational, exploratory, and metacognitive. These are described in Table 2.1.

Table 2.1

Functions of Student Talk

<table>
<thead>
<tr>
<th>Classification</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Informative</td>
<td>Providing information</td>
</tr>
<tr>
<td>Organisational</td>
<td>Organising the task or the learning process, or controlling behaviour</td>
</tr>
<tr>
<td>Argumentational</td>
<td>Seeking and providing clarification, explanation and justification</td>
</tr>
<tr>
<td>Exploratory</td>
<td>Speculating, predicting, discovering, hypothesising</td>
</tr>
<tr>
<td>Metacognitive</td>
<td>Planning, monitoring progress, evaluating outcomes</td>
</tr>
</tbody>
</table>


Combining this framework with a two-dimensional framework of student interaction developed by Granott (1993), Geiger and Goos (1996) were able to show the relationship between talk, interaction, and task in a computer environment.

Differences in the social organization of students’ work, identified in the function of their talk and the structure of their interaction, were associated with differences in task focus, with a focus on process, rather than product or means, producing collaborative discussion. (p. 235)
They associated informative or organisational talk with low-collaborative tasks and argumentational, exploratory, and metacognitive talk with high-collaborative tasks.

Analysis of videotape of class discussion held after pairs of primary school children had been working on mathematical tasks resulted in Wood and Turner-Vorbeck (Wood, 2002) developing a theoretical framework for teaching and learning. From this framework, they were able to identify “generalizable patterns of interactive and communicative exchanges” (Wood, 2002, p. 64) that form the basis of different classroom cultures. Wood (2002) suggests that differences in classroom culture, particularly with regard to interaction and discussion, occur due to different expectations for participation established by the teacher and the cognitive demand placed by the teacher’s questions on students’ mathematical reasoning. The questions that the teacher and listening students asked of the explainers were test questions which merely required the explainers to tell answers and procedures; inquiry questions that required the explainers to clarify their solution and give reasons for their actions; and argumentational questions where the questioner disagreed or challenged the explainer who had to justify and defend the solution. Thus, the types of questions asked were positioning the explainers in the class discussion in different roles.

All studies confirm the often stated belief that the use of the tool must be situated in a particular learning environment, namely one encouraging discussion and sharing of graphing calculator views. Both Kumpulainen’s framework for the function of children’s talk in a collaborative environment and Wood and Turner-Vorbeck’s question and role types in classroom discussion may prove fruitful in developing tools for the analysis of student discussion later in this thesis.

2.8.1.2 Increased number of strategies

Rather than simply relying on algebraic techniques, in a graphing calculator environment, students are faced with presenting solutions involving qualitative
reasoning supported by sketches and explanations as to how the graphing calculator was used to provide key information (van der Kooij, 2001). A year long study of twelve year 10 classes in the Netherlands by Harskamp et al. (2000) found that students who used the calculator throughout the year used graphical strategies more often, and tended to gain higher test scores, than those who used them for a short time. These findings mirror those of Ruthven (1990) in that weaker students appear to gain most from graphing calculator use. The reason suggested is that graphing calculator use provides weaker students (who tend to prefer the numerical representation and trial and error) the opportunity to expand their repertoire.

### 2.8.2 Summary: Changes to Teaching and Learning

Graphing calculator use can facilitate less teacher directed learning and increased communication both between students and between teacher and students and enables increased understanding as students are presented with opportunities to explore ideas in a variety of connected representations. This may allow students to move away from the view of mathematics learning as rule-following (Boaler, 1997) where students believe “in the need to remember rules” (p. 36) or the demonstration of cue-based behaviour where “students (base) their mathematical thinking on what they thought was expected of them, rather than on the mathematics within a question” (p. 37). Whilst it can be argued that pedagogy necessarily changes with the use of graphing calculators in the classroom, researchers and teachers alike should realise that the greatest benefits of the tool are effected through deliberate changes to teaching and learning (Goos at al., 2000).

### 2.9 Student Competencies in the Various Representations

Students often prefer one representation. Piez and Voxman (1997) found this preference is difficult to change. Harskamp et al. (2000) suggest above average students have a preference for graphical solutions. Keller and Hirsch (1998) explored students’ preferences for certain representations, the extent these preferences were contextually...
related, and if the availability of various representations affected the preference. Fifty-two first semester calculus students at Western Michigan University participated in the study, twenty-four of these in a graphing calculator class. The results concurred with those of Piez and Voxman (1997). Students using graphing calculators “were more likely to have a graphical preference on both contextualized and non-contextualized settings than students not using technology” (Piez & Voxman, p. 14). The use of technology changed the use of a representation as difficulties eased and students’ real access to all representations increased. The use of the technology allowed students “to view the graph as an easily manipulated representation” (Piez & Voxman, p. 16) and thus it became the representation of choice when it was appropriate.

Clearly, students must be comfortable using all representations and teachers must provide activities that facilitate this. There is little benefit if graphing calculators simply change preferences from algebraic to graphical; students need the facility to work in all (three) representations equally well and choose the most appropriate for any given situation. With the graphing calculator students can experience different representations of functions at a much earlier stage of their education. If teachers (and curriculum writers) are prepared to introduce functions before students have the “necessary” algebraic support, there may be subsequent improved changes in students’ ability to understand and manipulate graphs, reduced difficulties with scale, and greater ability to appreciate links between representations.

2.10 Conclusion

The literature reviewed has presented evidence that access to multiple representations is important for students’ understanding of functions. This needs to include the three core representations, each of which contribute some, but not all, aspects of function. Access to multiple representations in a graphing calculator learning environment necessarily increases the information presented to students. Without
explicit teaching that focuses on both local and global aspects of functions, in this environment, students will continue to experience difficulties. In addition, teaching needs to address the information portrayed by each of the representations, including their limitations and aspects linked with scale and shape.

Clearly, greater understanding is required as to the advantages and disadvantages of the interactive nature of graphing calculators, student (mis)understanding of scale, and the effects of graphing technology on students’ understanding of functions, especially how students determine a complete graph of a function. The use of graphing technology does not guarantee improved understanding; however, “access seems to make a difference” according to Burrill et al. (2002, p. v) in their meta-analysis. In addition they note the way technology is used is a factor in differences in student learning.

The study which follows will need to clearly describe the teaching and learning environment, as tool effects cannot be distinguished from this. It is impossible to distinguish between tool use and simultaneous change in teaching. Graphing technology use cannot be considered in isolation. The contribution of the tool and the changed teaching and learning must be considered together (Dunham & Dick, 1994). As is prudent to expect, access to graphing technology will not, in itself, change students’ understanding. The effect of the technology and simultaneous change in teaching and learning is complex.

Whilst some research has been undertaken in the area of student understanding of functions in a graphing calculator environment, there is a definite need for more which considers how students’ cognition of function (and other concepts) is changed through the use of graphing calculators. During the literature review a number of questions surfaced and after synthesis the following research questions were developed.
• Research Question 1: How do students perceive what constitutes a complete graph of a function?
• Research Question 2: How do students apply their mathematical knowledge and graphing calculator knowledge to determine a global view of a difficult function?
• Research Question 3: What are student behaviour and actions when confronted with unexpected views of a function? Do they apply their mathematical knowledge and graphing calculator knowledge to resolve the situation?
• Research Question 4: When working mainly in the graphical representation, do students use the algebraic and numerical representations to facilitate and support their solution process?
• Research Question 5: What understandings do students have of the effect of the scale marks?
• Research Question 6: What cognitive and metacognitive processes are used by students to support their use of mathematical and technological knowledge in the solving of the problem task?
• Research Question 7: What particular features of the graphing calculator do students use to identify key features of the function?

These questions will be addressed in the research study which is the basis of this thesis.

The next chapter presents the aim of the study, describes the students, the setting, and their learning experiences, and explains the research methods used. The research instrument used is presented and the mathematical and graphing calculator knowledge required for its solution detailed, as are the methods by which the results were analysed.
CHAPTER THREE
METHODOLOGY

3.1 Overview

A qualitative case study (Merriam, 2002b, p. 8) was used for the research described in this thesis as it was considered to be the most appropriate methodology to achieve the goals of the study. Raw data were collected in the form of audiotape recorded sessions of pairs of students undertaking a problem task using a graphing calculator. Videotaping of the graphing calculator screen output via a viewscreen allowed a permanent record of the results of students’ interactions with the graphing calculator to be made. A protocol of each pair’s efforts was produced using the combined recordings. These protocols were then analysed to explore the nature of student interactions with the graphing calculator as they attempted a problem task.

This chapter includes details of the methodology, the situational context, details of the students involved in the study and why they were selected, the setting of the study, and the learning experiences of the students in the study, followed by details of the pilot study. Quality control in the research is discussed as are the limitations of the methodology. The research instrument used is presented as are its administration and the history of the development of this instrument. Where appropriate, links are made to relevant literature. The ways in which the results were analysed are also presented.

3.2 Methodology

3.2.1 Qualitative Research Methods

The phenomena being studied are quite complex, thus necessitating the use of qualitative methods. “Meaning is socially constructed by individuals in interactions with their world. The world, or reality, is not the fixed, single, agreed upon, or measurable phenomenon that it is assumed to be in positivist quantitative research” (Merriam, 2002c, p. 3). “Qualitative methods can be used to obtain the intricate details about
phenomena such as … thought processes … that are difficult to extract or learn about through more conventional research methods” (Strauss & Corbin, 1998, p. 11). Qualitative methods are particularly useful where the research is focussed on areas that cannot be described by directly measurable data, where the task is to “document, to grasp and to explain individual processes of mathematical thinking and acting” (vom Hofe, 2001, p. 109) as is the intent of this study.

3.2.2 Case Study

Case study research is to be distinguished from statistical sampling research. The latter is concerned with establishing a relationship between the sample and the population, from which the sample is drawn. The intent is to generalise the findings from the sample to the population. In a “case study the relationship between a case … and any population … is essentially a matter of judgement” (Stenhouse, 1985, p. 265). Case studies are important as evidence of what is occurring in a particular situation. The term case study has many interpretations. Yin (1994) defines case study as "a comprehensive research strategy" (p. 13) allowing the examination of contemporary events where relevant behaviours cannot be manipulated, whereas Stake (1995) suggests it is less of a methodological choice and more the choice of what is to be studied. Case study allows an “intensive description and analysis of a phenomenon or social unit” (Merriam, 2002c, p. 8), the social unit including both individuals or groups. According to Stake (1995), Smith defines the case as a bounded system allowing focus on the case as an object rather than a process. Both Stake (1995) and Merriam (2002c) ascribe to this idea that the case is a "bounded, integrated system". However, others such as Wolcott see it more as “a format for reporting” (2002, p. 101) and less as an adequate explanation of the research methodology employed. Even though the present researcher still holds that the term is useful as a description of the overarching methodology involved, she will heed Wolcott’s advice to “be sure to provide adequate
detail about the specific research techniques” (p. 207) employed.

According to Eisenhardt, and the interpretation that will be used in this study, case study is a research approach “which focuses on understanding the dynamics present within single settings” (2002, p. 8). Stake (1995) distinguishes between *intrinsic* and *instrumental* case study, the former being where “the case itself is of primary, not secondary, interest” (p. 171) and the later where, rather than the focus being on understanding the case itself, “the case study serves to help us understand the phenomena or relationships within it” as is the situation in this study. The focus here is the development of a general understanding of the issues involved as students search for a complete graph of a cubic function in a graphing calculator learning environment. This case study is describing the results of practice in the classroom of the researcher and her colleague as evidenced by a snapshot of the students’ responses to a problem task.

Cases are relevant both to the researcher and others, including the participants, as only by increasing one’s understanding of what occurs in the classroom is the teacher able to become more effective. Case study also allows interpretation and evaluation of current practice with a view to enhancing understanding and providing a meaningful guide to action (Gillham, 2000; Merriam, 2002c; Stake, 1995).

By concentrating on the case, the object studied in a case study (Stake, 1995), this approach seeks to describe the phenomenon in depth. However, defining "what the unit of 'case' is - [is] a problem that has plagued many investigators" (Yin, 1994, p. 21). The unit of analysis characterises the case study, the case may be an individual, or an event, or an entity that is less well definable than a single individual (Yin, 1994). Merriam (2002b) states that the case can be a person, site, program, process, or community. The case studied in this thesis is the community of students studying functions in a graphing calculator learning environment as part of their study in Mathematical Methods in their
final two years at a particular Victorian secondary school. This case was selected "because it [exhibited] characteristics of interest to the researcher” (Merriam, 2002b, p. 179).

Within this case student pairs were *purposely selected* (Merriam, 2002a, p. 20). The "sample is selected on purpose to yield the most information about the phenomena of interest" (Merriam, 2002a, p. 20). Given the limited number which can be studied in depth, it makes sense to select pairs where the process of interest is "transparently observable" (Huberman & Miles, 2002, p. 13). By concentrating on the case, this approach seeks to describe the phenomenon in depth. The case and the student pairs were selected to maximise what could be learned.

Any generalisation, from this case study, would be to a theoretical proposition (Yin, 1994) (or consideration of a substantive theory as termed by Strauss and Corbin, 1998), rather than to a population. “A theory is usually more than a set of findings; it offers an explanation about the phenomena” (Strauss & Corbin, 1998 p. 22). It enables users to explain and predict events, thereby providing guides to action. Yin (1994) points out that issues of generalisation are relevant for other research strategies, “that case studies, like experiments, are generalisable to theoretical propositions and not to populations” (p. 10). Miles and Huberman (1994, p. 28) expand the argument of Firestone (1993), that the most useful generalisations from qualitative data are analytic (theory connected), rather than sample to population. “The real business of case study is particularization, not generalization” according to Stake (1995, p. 8). In this thesis the intention is to come to know this case well and as new issues and questions arise during the study of the case these become the focus of the research in accordance with the approach adopted by Stake.

The purpose of this study is to look at what is occurring in two particular classrooms at the same school. Specifically, the purpose is (a) to investigate processes undertaken
by senior secondary students as they search for a complete graph of a particular function using a graphing calculator, (b) to identify the mathematical knowledge and features of the graphing calculator used by students in their response to the problem, and (c) to identify which actions of the students best facilitate the solution process. As the use of graphing calculators is mandated in Victorian schools, the use of a control group, not taught with a graphing calculator, is not feasible. The emphasis is not on comparing the work of students with or without a graphing calculator, rather it is on exploring students’ understanding of function developed in classrooms where the teachers value a graphing calculator environment. The aim is to make observations of students in a situation as close as practicable to their normal classroom setting. To this end, qualitative data were collected in order to best provide a snapshot of the practice in the two classrooms in the study.

3.3 Situational Context of the Study

3.3.1 The School

The school is an inner city, Australian, state secondary college. It was formed in 1993 through a reorganisation of three high schools and a technical school. Although the College originally had two campuses, it has since become a single campus school. An English Language Centre (ELC) is part of the college. A very high proportion of the students come from a Non-English Speaking Background (NESB). Many of these students are very recent arrivals from overseas and a significant number are refugees, including a number from famine stricken and/or war torn countries.

3.3.2 The Students in the Study

The students in the study were in their last two years of secondary school—a time when graphing calculators can be used in most areas of mathematics studied in Victoria. In particular, students are expanding their knowledge from simple polynomials to include power and transcendental functions and beginning to learn calculus in which
algebraic, numerical, and graphical representations all play an important role.

The study was undertaken in semester 2, 2000. Four students were selected from a class of fifteen students studying Mathematical Methods Unit 4. An additional six students were selected from a class of twenty-four students studying Mathematical Methods Unit 2. For the purpose of this study those students undertaking Units 1 and 2 of Mathematical Methods are referred to as Year 11 students. Those students undertaking Units 3 and 4 of Mathematical Methods are referred to as Year 12 students. Both classes were of mixed ability.

The Unit 2 class was taught by the researcher, whilst the other class was taught by a colleague. These two teachers worked closely together and have similar teaching styles. They both use a variety of methods emphasising understanding through exploration, discussion, and collaboration. These classes were selected on the basis that firstly, the students were studying Mathematical Methods where the mathematical topic of interest forms a foundational part of the course of study and secondly, the classes were either taught by the researcher or a colleague with a similar teaching style. It must be acknowledged that the way in which the students in the study were prepared to participate in the study and expound their ideas may be a result of the situational context established by the teachers, a context in which “knowing occurs, and ... the learner is participating” (Barab & Kirshner, 2001) and students’ voices are expected to be heard.

The students in the two classes were relatively typical of students in Victorian senior mathematics classes at the time in their choice of mathematics units (46% and 47% undertaking two mathematics subjects at Year 11 and 12, respectively) and calculator ownership (86% and 73% at Year 11 and 12, respectively). They were less typical in the state in their language background with a high proportion of students having English as a Second Language (ESL, defined as having been in an English speaking school for seven years or less) or having Non-English Speaking Backgrounds (NESB). Their
slightly higher ages than the norm is an artefact of their ESL or NESB status. In the Year 11 class, 75% of students had ESL or NES Backgrounds compared to 93% of the Year 12 class.

3.3.3 Participant Selection

Purposive sampling (Merriam, 2002a, p. 20) is used where the intent is to “generate information-rich cases that [illuminate] the study and [elucidate] variation as well as significant common patterns within that variation” (Brott & Myers, 2002, p. 148). To this end, five pairs of students were selected to work through a problem task on the basis of being pairs who often worked with each other in the mathematics classroom, and who could be expected both to solve the problem and to articulate their ideas as they proceeded with this task. Before this selection a pretest was administered to both classes to ensure that the students selected had the necessary skills and conceptual knowledge to complete the task. In addition the results of Year 11 students on a previous supervised assessment task dealing with the graphing of a cubic function were taken into consideration.

All Year 12 students in the study were ESL students with Ahmed, Linh, and Hao having been in Australia for less than two years at the time of the study. Abdi had been in Australia for four years at the time of the study. Two of the Year 11 students, Reem and Ali, were ESL students having been in Australia for two and a half years. Jing was born overseas and had a Non-English Speaking Background but had been in Australia for nine years. The remaining students, Kate, Pete, and Susan, were born in Australia and had an English Speaking Background.

3.3.4 The Participants

3.3.4.1 Pair 1: Linh and Ahmed

These students were in Year 12 and were both recent arrivals to Australia. Linh had not used a graphing calculator prior to her arrival in an Australian school, in the middle
of Year 11. Her previous schooling in China included little emphasis on the use of technology in mathematics, self guided exploration, or the importance of explanations of her mathematics. Ahmed, originally from Egypt, had been at an Australian school since late Year 10. He was a keen problem solver and user of technology, who wrote and creatively modified many programs for the graphing calculator. Both students, although English is their second language, were capable and experienced articulators of their mathematics. Both enjoyed mathematics, had high expectations of their studies, and intended to pursue tertiary studies.

3.3.4.2 Pair 2: Abdi and Hao

These young men were also in Year 12. Abdi had been in an Australian school since Year 9. He was introduced to the graphing calculator during that year in his study of linear functions. He used the graphing calculator extensively in Year 10, particularly as an investigative tool in units of work focusing on linear and quadratic functions and in the study of probability. The second student, Hao, first entered an Australian school just after the beginning of Year 11. His introduction to graphing calculators and the Victorian curriculum was thus very recent at the time of the study. Both of these students enjoyed mathematics and were keen to articulate their ideas when asked.

3.3.4.3 Pair 3: Kate and Pete

These students were in Year 11. Both were born in Australia and have an English speaking background. They had attended the same secondary college for their previous four years of schooling. Kate lacked confidence in her mathematical abilities. She often became sidetracked when solving problems and became worried when her methods or solutions were different from those of other students whose work she valued. She sought regular help from her peers and teachers and enjoyed toying with mathematical ideas particularly when using technology. Pete, on the other hand, was a hard working student who was very confident in his mathematical abilities. He was a dedicated
student who intended to achieve very highly in his secondary and tertiary studies. Whilst he often worked independently and at a faster pace than the rest of the class, he was generally willing to discuss his ideas and findings with other students in a constructive manner. Both of these students had used the graphing calculator in their studies since Year 9. They often worked together both in and out of the classroom.

3.3.4.4  **Pair 4: Reem and Ali**

The fourth pair, also in Year 11, are sister and brother. Born in Kuwait they had been in Australia for about two and a half years at the time of this study, enrolling in an English Language Centre before transferring to the secondary college, where the study was undertaken. The young woman, Reem, was a keen student of mathematics although, possibly partially related to her English skills, she sometimes lacked confidence when offering her ideas or solutions to the class, however, this was not the case in a one to one or small group situation. She generally sat next to her brother in class although she also worked with another female student sitting on the other side of her brother. Her younger brother, Ali, although a reasonably capable student, did not have the same educational aspirations as Reem, either at the secondary or tertiary level. This pair was selected on the basis of Reem’s interest in mathematics, demonstrated ability to articulate her ideas, and the fact that they worked together in class. Unfortunately, when situated as a pair out of the classroom for this study and given a problem to solve, they neither worked very cooperatively (tending to take turns with the task) nor articulated their thoughts. Their experience of functions and graphing calculators was similar to that of Abdi, excluding his Year 9 experiences.

3.3.4.5  **Pair 5: Jing and Susan**

The third pair of Year 11 students had been at the secondary college throughout their secondary schooling. Jing arrived from China when she was in Grade 3. She is a very capable student of mathematics and well able to articulate her ideas both orally and in
written form. Susan had a reduced short-term memory resulting in her being able to process only single pieces of information at a time. Aural instructions needed to be provided in single steps for her to follow. Although some of her teachers believed she would be unable to succeed in the higher levels of secondary schooling, in her study of mathematics she always presented herself as a keen and interested student who enjoyed the problem solving aspects. This pair had worked together in mathematics since Year 7. Their experience of functions and graphing calculators was similar to that of Abdi.

3.3.5 Graphing Calculator Ownership of Students in the Study

With one exception, all students participating in the study owned their own graphing calculator, being a TI-83 or a TI-83Plus. The remaining student rented a TI-82 graphing calculator from the school. The siblings shared ownership of their calculator. This level of graphing calculator ownership was relatively typical of the classes at the school. Across the two Mathematical Methods classes from which the students were drawn, the majority, 87% of students, either owned a Texas Instruments (TI) graphing calculator, TI-83 or TI-83Plus, or hired a TI-82 from the school for the year. The remaining 13% had access during class time to TI-83 graphing calculators belonging to the school.

3.3.6 Conventions used for Writing Graphing Calculator Functions

The descriptions of the graphing calculator buttons and the menu items that follow are deliberately stated in terms of the model of graphing calculator used in the study. To do otherwise may allow incorrect assumptions to be made, as some of the behaviour of interest to this researcher or others may be an artefact of the type of calculator used. By using the alternative of generic descriptions of the graphing calculator functions, the possibility that this is the case becomes impossible to distinguish. Without specific details readers may be unable to compare the model and its features as used by students in this study with other makes and models and their features. Without this detail, unless the reader is experienced with the specific type of calculator (and sometimes the model)
being used, they can (and probably cannot help but) misinterpret the environment in which the research is situated.

When specific buttons on the graphing calculator are pressed, the convention will be followed that this is written with uppercase letters, for instance, GRAPH, WINDOW, and ZOOM. When selection is made of a menu option this is represented exactly as shown on the graphing calculator, for example, Zoom Out, or where the Zoom has been abbreviated on the calculator to the single letter Z, as in ZStandard, then the option is represented as the experienced user would be expected to describe it, that is, as Zoom Standard or the standard window. Similarly, items in the WINDOW such as Xmin, Xmax, Ymin, Ymax, are referred to as x minimum, x maximum, y minimum, and y maximum, respectively. To facilitate a better understanding of the uses made of the graphing calculator by students in this study a detailed explanation of the functions of the Texas Instruments TI-83 graphing calculator is provided in Appendix A.

3.3.7 Previous Use of the Graphing Calculator by Students in the Study

In class, the use of the graphing calculator was integrated throughout the course. The graphing calculators were used for a range of learning purposes including exploration and the finding and checking of solutions. Many of the advanced features such as entering and graphing functions, selecting various items from the ZOOM menu to facilitate finding an appropriate view of the function, and the selection of items from the CALCULATE menu to identify specific coordinates, or the approximate gradient at a point, were used. In addition, students were familiar with the TABLE function including editing both its initial value and increment, and with using the LIST and STAT PLOT functions to enter and plot data. Students were expected to use their graphing calculator both when directed by the teacher and at their own instigation. A viewscreen (linking the graphing calculator to an overhead projector) was used by both teachers to
demonstrate the use of particular features and solution methods, and to facilitate effective student use of the calculators during the learning process.

3.3.8 Graphing Calculator Skills Required by the Students

Students need skills in entering a function on the calculator and selecting an appropriate viewing window for the graph. Further, they should be able to access specific features (including the TABLE) to determine the coordinates of the key features of the graph.

Both the TABLE and the CALCULATE menu can be used to assist students to determine key features of the graph. For the table, both the initial values and the increment can be set by the user. Whilst accessing the table, the user can scroll up or down to access as many pairs of coordinates as required. Using the CALCULATE menu allows students to access Value (which calculates the function value for a specified $x$ value), Zero (which finds a zero of a function within a specified range), Minimum (which finds the minimum value of a function within a specified range), and Maximum (which finds the maximum value of a function within a specified range). TRACE can also be used to approximate these values.

Students also need an understanding of the relationship between the complete graph and the viewing window. The viewing window can be altered either via the ZOOM or the WINDOW menus. The ZOOM menu has ten options - the most useful in the solving of the problem task being Zoom Box, (which zooms in on an area specified by the user), Zoom In (which zooms in by a [default] factor of four on a point specified by the user), Zoom Out (which zooms out by a [default] factor of four on a point specified by the user), Zoom Standard (which sets the window at $-10 \leq x \leq 10$ and $-10 \leq y \leq 10$) and Zoom Fit (which selects $y$ values that include the maximum and minimum $y$ values of the function in the domain specified). The viewing window allows the user to specify the viewing domain and range. In addition, the scale marks on the axes can be
independently specified. Depending on the changes made, using either the ZOOM or WINDOW menu, the view of the function displayed in the viewing window may or may not change.

The key to effective and efficient use of the calculator features is understanding how selection of each alters the window settings and the resultant effect on the view of the graph. Determining which menu item to select to adjust the view of the graph in any required way is neither simple nor intuitive. To become expert in the correct selection, most students require significant experience in observing the effect of their use. In addition, students need to be observant in their use of the menu items in the CALCULATE menu. The incorrect use of any item will result in an incorrect answer, and unless students take notice of the calculator display they may draw incorrect conclusions about coordinates of the function.

3.3.9 Mathematical Background Knowledge of Students

At the time of the study, two mathematics subjects were available for Year 11 and three mathematics subjects at Year 12 as part of the offerings of the Victorian Certificate of Education. At Year 11 the subjects offered were General Mathematics and Mathematical Methods, the content of the first having a broad mathematical content and the later focussing on the study of functions and pre-calculus mathematics. At the Year 12 level the subjects were Further Mathematics, broad in content and including a substantial amount of statistics, Mathematical Methods extending the mathematics studied at Year 11 and considering the derivatives and anti-derivatives of a range of functions and applications of these, and Specialist Mathematics which must be undertaken in conjunction with Mathematical Methods and includes content suitable for students intending to study mathematics or engineering at a tertiary level. All five mathematics subjects require students to use technology including graphing calculators to support and enhance their learning throughout the subject.
Both pairs of Year 12 students were undertaking two Year 12 mathematics subjects. In addition, all four students had studied Mathematical Methods and General Mathematics in Year 11. Linh, Ahmed, and Hao were studying Mathematical Methods and Specialist Mathematics Units 3 and 4, whereas, Abdi, of Pair 2, was studying Mathematical Methods and Further Mathematics.

Of the Year 11 students, half were studying only one mathematics subject that being Mathematical Methods. The studying of only one mathematics subject should not be taken as an indicator of the abilities or aspirations of the students. All three of the Year 11 students studying only one mathematics subject at Year 11 continued on to study Specialist Mathematics during Year 12. The remaining students were also undertaking General Mathematics. Pete and Kate, of Pair 3, were both studying only one mathematics subject. Reem and Ali (Pair 4) were both studying two mathematics subjects. Of the last pair, Jing was studying only Mathematical Methods whereas Susan was undertaking both Year 11 mathematics subjects.

### 3.4 Student Skill Level

#### 3.4.1 The Pretest

Prior to the selection of the student pairs to undertake the problem task a test was administered to the Mathematical Methods classes (Year 11, \( n = 19 \), Year 12, \( n = 13 \)) from which the students were to be selected. The test was administered under examination conditions, with all students having access to a graphing calculator. The test items are shown in Table 3.1. The maximum score for each sub-part of a question was 3, consisting of 1 for a correct response and 2 for the explanation and /or reasoning shown. These test data provided additional information as to the understanding of both the participating students and the classes from which they were selected allowing inferences to be made about the typicality of students in the sample in relation to other members of the classes in the case study. Initially, the overall results of the classes are
Table 3.1

**Pretest Items**

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>Use the data in the table below. What information can you add to the graph shown in the standard window? Explain your reasoning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
<th>Y2</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1b</td>
<td>Use the data in the table below. What information can you add to the graph shown in the standard window? Explain your reasoning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>X</th>
<th>Y1</th>
<th>Y2</th>
</tr>
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<tr>
<td>4</td>
<td>16</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>2a</td>
<td>Look at the function shown below.</td>
</tr>
</tbody>
</table>

Is this the graph of a linear function?  
Yes  No  Maybe

Give an example to explain your answer.

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>2b</td>
<td>The graph of  ( y = -2x^2 + 12 ) is shown below.</td>
</tr>
</tbody>
</table>

Find a window that will make the function appear as shown above. Explain each step.

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>2c</td>
<td>Look at the function shown below.</td>
</tr>
</tbody>
</table>

Is this the graph of a quadratic function?  
Yes  No  Maybe

Give an example to explain your answer.
presented (see Table 3.2) and then the results of those students participating in the study are discussed. The test consisted of 5 items written by the researcher. The first two items, Questions 1a and 1b, involved translating numerical information to a graphical representation of a function. Questions 2a and 2c presented students with a graphical representation of a function and asked them to consider if they could determine if the function was linear (Question 2a), quadratic (Question 2c), or it was impossible to determine from the view presented. In Question 2b a graphical view of a function was provided and students asked to determine window values that would provide a similar view. Table 3.2 shows that each pair except Pair 4 had at least one student who scored at least 2 on both sub-parts of Question 1 and all pairs had at least one student who scored at least 2 on all three sub-parts of Question 2.

The mean and standard deviation of the students’ total score, and by year level, for each item of the test are presented in Table 3.3. The greater success occurred on Question 2 where the three sub-parts focused on the effect of scale on shape and the role the viewing window plays in this. Question 1 required use of a table of values to add information to a graph, that is, a translation from a numerical to graphical representation. Both questions were intended to direct students’ attention to local aspects of the functions. The Year 12 students achieved a higher mean score on all sub-parts except 2a (where students had to specify if a given graph was definitely linear or not).

For Question 2 the classes had very close, and relatively high mean scores on all sub-parts at both Year levels. The question focused on local aspects of the function in a translation from the graphical to the algebraic representation.

3.4.1.1 Pretest Question 1

For Question 1 overall performance by the Year 11 and 12 classes showed no significant difference $\chi^2(1, 32) = 2.281, p > .05$ (see Appendix B). No differences were
Table 3.2

Students' Raw Scores on Pretest

<table>
<thead>
<tr>
<th>Student (Name - Pair)</th>
<th>Pretest Item Number</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1a</td>
</tr>
<tr>
<td>1101</td>
<td>2</td>
</tr>
<tr>
<td>1102</td>
<td>0</td>
</tr>
<tr>
<td>1103</td>
<td>0</td>
</tr>
<tr>
<td>1104</td>
<td>3</td>
</tr>
<tr>
<td>1105</td>
<td>0</td>
</tr>
<tr>
<td>1106 (Ali - 4)</td>
<td>NA</td>
</tr>
<tr>
<td>1107 (Reem - 4)</td>
<td>1</td>
</tr>
<tr>
<td>1108</td>
<td>2</td>
</tr>
<tr>
<td>1109</td>
<td>1</td>
</tr>
<tr>
<td>1110</td>
<td>2</td>
</tr>
<tr>
<td>1111 (Jing - 5)</td>
<td>2</td>
</tr>
<tr>
<td>1112</td>
<td>0</td>
</tr>
<tr>
<td>1113</td>
<td>3</td>
</tr>
<tr>
<td>1114</td>
<td>3</td>
</tr>
<tr>
<td>1115 (Pete - 3)</td>
<td>3</td>
</tr>
<tr>
<td>1116 (Kate - 3)</td>
<td>3</td>
</tr>
<tr>
<td>1117</td>
<td>0</td>
</tr>
<tr>
<td>1118</td>
<td>0</td>
</tr>
<tr>
<td>1119</td>
<td>2</td>
</tr>
<tr>
<td>1201 (Abdi - 2)</td>
<td>2</td>
</tr>
<tr>
<td>1202</td>
<td>3</td>
</tr>
<tr>
<td>1203</td>
<td>2</td>
</tr>
<tr>
<td>1204 (Ahmed - 1)</td>
<td>2</td>
</tr>
<tr>
<td>1205</td>
<td>1</td>
</tr>
<tr>
<td>1206</td>
<td>1</td>
</tr>
<tr>
<td>1207 (Hao - 2)</td>
<td>2</td>
</tr>
<tr>
<td>1208</td>
<td>0</td>
</tr>
<tr>
<td>1209</td>
<td>1</td>
</tr>
<tr>
<td>1210</td>
<td>2</td>
</tr>
<tr>
<td>1211</td>
<td>2</td>
</tr>
<tr>
<td>1212</td>
<td>3</td>
</tr>
<tr>
<td>1213 (Linh - 1)</td>
<td>2</td>
</tr>
</tbody>
</table>

Note. The values represent the sub-part score, each sub-part of a question had a maximum possible score of 3, consisting of 1 for a correct response and 2 for the explanation and/or reasoning shown. A student code of 11XY indicates a year 11 student, whereas a student coded 12XY indicates a year 12 student. Results of students in sample are highlighted. NA = No attempt. Susan (Pair 5) was absent.
Table 3.3

Mean and Standard Deviation per Pretest Item

<table>
<thead>
<tr>
<th></th>
<th>Q1a</th>
<th></th>
<th>Q1b</th>
<th></th>
<th>Q2a</th>
<th></th>
<th>Q2b</th>
<th></th>
<th>Q2c</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>µ</td>
<td>SD</td>
<td>µ</td>
<td>SD</td>
<td>µ</td>
<td>SD</td>
<td>µ</td>
<td>SD</td>
<td>µ</td>
</tr>
<tr>
<td>Year 11</td>
<td>1.42</td>
<td>1.26</td>
<td>1.47</td>
<td>1.12</td>
<td>2.53</td>
<td>0.70</td>
<td>2.42</td>
<td>0.90</td>
<td>2.47</td>
</tr>
<tr>
<td>Year 12</td>
<td>1.77</td>
<td>0.83</td>
<td>1.77</td>
<td>0.93</td>
<td>2.46</td>
<td>0.66</td>
<td>2.69</td>
<td>0.75</td>
<td>2.62</td>
</tr>
<tr>
<td>Total</td>
<td>1.56</td>
<td>1.11</td>
<td>1.59</td>
<td>1.04</td>
<td>2.50</td>
<td>0.67</td>
<td>2.53</td>
<td>0.84</td>
<td>2.53</td>
</tr>
</tbody>
</table>

found in identifying the various local features of a function, namely axial intercepts or coordinates of turning points. It appears that students were competent in interpreting information presented in a numerical representation to be applied to a graphical representation of a function and these competencies were achieved at the Year 11 level. Of the four students obtaining a maximum score of 6 on this question, three were Year 11 students, one being Pete of Pair 3.

3.4.1.2 Pretest Question 2

There was no difference between the year levels for sub-part 2a, however the Year 12 class performed better on the other two sub-parts. Year 12 students performed best on 2b, whereas Year 11 students performed equal best on sub-parts 2a and c. For Question 2, overall 13 students, or 41% of students achieved the maximum possible score of 9, and 21 students or 66% of students achieved 8 or more, including at least one from each pair. Fisher’s exact test (see Appendix B) for Question 2 showed no statistically significant difference between the Year 11 and 12 class means (p = .655).

Given students achieved most successfully on Question 2, it appears that when information was provided graphically, the students in the classes in this case study had a good understanding that shape is an artefact of scale. Moreover, they had developed this understanding during Year 11. The responses to sub-part 2b, addressing the second question, further suggested that when information was presented in an algebraic
representation, students, in this study, had a good understanding that shape is an artefact of scale.

3.4.1.3 The Responses of the Students in the Sample

All students, except Ali, were able to identify information from the table and interpret this in terms of the graphical representation of the function shown. Ali did not attempt sub-part 1a, however, his partner, Reem was able to identify the coordinates of both \( x \) intercepts from the table of values, recognising that one of these intercepts was identified exactly but the other only approximately. Pete went beyond what was asked as he used Quadratic Regression (which is an in-built function of the TI-83 graphing calculator) to determine the algebraic representation of the function and then proceeded to identify the coordinates of all the key features of the function.

For sub-part 1b, the students were expected to identify the exact \( y \) intercept and the approximate coordinates of the turning point, from the numerical data provided. All students, except Ali, were able to identify information from the table and interpret this in terms of the graphical representation of the function shown. Ali did not attempt item 1b, however, his partner Reem was able to identify the coordinates of the \( y \) intercept.

For Question 2a all students in the sample responded that the graph maybe that of a linear function except for Abdi who said it was not. Abdi reasoned that “if you zoomed in to \( x^2 \) you get [an] almost straight line” and provided accompanying graphs to support this reasoning. The other students provided reasons to support their argument such as “it might be a curve that has been zoomed in to”, “it may be a linear function or it may be part of a curve”, “we can’t know which kind of function it is unless we can see the whole graph”, “this may be the close up of a curve”, or “a linear function with the axes turned off”. Ali and Reem, of Pair 4, for example, demonstrated their understanding with Ali stating, “if we Zoom In many times it [a curve] will look like a linear function, as we did in class” and whilst Reem reasoned “because this graph may be for a
quadratic function and after doing Zoom In, [what] we can see is a linear but is not”. Both Hao and Kate used appropriate diagrams to support their reasoning showing a good understanding of the effect of scale on shape. These responses indicate that all students, except Susan who was absent, understood the local linearity of functions having curved graphical representations given particular scales on the axes.

For Question 2b all students except Kate (who did not attempt this question) successfully produced window settings to display the parabola shown. For Question 2c all but two of the students, Ali and Reem, thought that the graph could be something other than a quadratic function. Pete, for example, stated that “it could be the minimum turning point of a quadratic or, it could be the bottom of a cubic function, or any other polynomial. It could even be a section of a periodic function (e.g., sine or cosine)”. In addition, he provided a sketch to support his final suggestion. Both Ali and Reem who believed it was a quadratic function stated that it was translated right by 2.5 and shifted up, suggesting their further misunderstanding of the scale marks as displayed.

The responses to Question 2c, in conjunction with those to Question 2a, indicate that all students, except Susan who was absent, had a good understanding of the effect of scale on shape and the role the viewing window has on this effect. Furthermore, each pair demonstrated their understanding that shape is an artefact of scale when information is presented in a graphical or an algebraic representation.

3.4.2 Additional Data on the Skills and Conceptual Knowledge of the Students

The results of a previous supervised assessment task undertaken by the Year 11 students are discussed for Susan who was absent for the pretest and for Ali and Reem, of Pair 4, who scored less well than the other pairs on Question 1 of the pretest, and for Kate who made no attempt at Question 2b of the pretest. These results were considered to ensure that all students in the sample had the conceptual understanding required for a successful attempt of the problem task to be feasible. The test items were administered
under examination conditions and are presented in Table 3.4. Question 1a required students to find an appropriate window for the function and identify the coordinates of the key features. Ali, Reem, and Susan were able to find an appropriate window in order to identify the correct shape of the function, the number of $x$ intercepts, and the number of stationary points. In addition, all students demonstrated their ability to determine the coordinates of the axial intercepts. Furthermore, Susan was able to correctly identify the coordinates of both stationary points. Question 1b required the students to identify the coordinates of the points of intersection between the function from question 1a and a specified quadratic function. Neither Ali nor Reem correctly addressed this item, both misinterpreting or misreading the question and providing the coordinates of the three axial intercepts of the specified quadratic function. This may have been a result of their ESL status or the fact that this question was unusual given their previous experiences. On the other hand, Susan was able to correctly identify both points of intersection and clearly explain the method she used.

Table 3.4

*Additional Examination Questions*

<table>
<thead>
<tr>
<th>Item Number</th>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>Use the graphing calculator to help you sketch a complete graph of $y_1 = 0.1x^3 - 0.4x^2 - 16x - 31$. Label all key features, accurate to 1 decimal place.</td>
</tr>
<tr>
<td>1b</td>
<td>Consider a second function $y_2 = -x^2 + 5$. Use the graphing calculator to identify and hence label all coordinates where the two curves intersect. Explain your method.</td>
</tr>
<tr>
<td>2</td>
<td>For the function $y = 2x^2 - 15$, find a window so the graph appears as shown below. (You may have any scale marks you like.)</td>
</tr>
</tbody>
</table>
Question 2, will be discussed only for Susan and Kate as this was mathematically identical to question 2b on the pretest where Reem and Ali scored the maximum possible marks. On this item, both Susan and Kate scored equivalent to a score of 3 on the pretest, indicating that all students in the sample could successfully link the effect of altering the window settings on the view of the graph shown.

### 3.4.3 Discussion

From their responses, to both the pretest and the additional assessment task (for Reem and Ali) it appears that all the sample students were competent in interpreting information presented in a numerical representation to be applied to a graphical representation of a function. Both Kate and Susan demonstrated on the additional assessment task that they could successfully produce window settings to display a given quadratic function. These responses, together with the pretest results indicate that all students were able to successfully adjust the WINDOW settings to determine an appropriate viewing window that allowed a global view of a quadratic function to be seen. Furthermore, each pair demonstrated their understanding that shape is an artefact of scale when information is presented in a graphical or an algebraic representation. Hence, taking into account the students responses’ to the pretest, the additional assessment task, where appropriate, and the observations of this researcher of the students working together in class, it was determined that each of the five pairs of students had the necessary skills, conceptual knowledge, and a good working relationship to complete the problem task.

### 3.5 Data Collection Methods

#### 3.5.1 The Use of Verbal Methods

Participants’ behaviour in experimental settings can be shaped by many factors. These include their response to being recorded (as they may be speaking merely
because they are being recorded, their beliefs about the nature of the experimental setting (as evidenced by one student in this study who claimed that using the table feature of the graphing calculator was cheating in the circumstances), and their beliefs about the nature of mathematics itself (Schoenfeld, 1985b). In this study, the safeguards taken included the purposive selection (Merriam, 2002a) of pairs who (a) worked and talked together in class, so providing an experimental setting paralleling the classroom situation, and (b) were confident in mathematics, even if not always in public.

Moreover, the use of two person protocols helps reduce the pressure on the individual and is more likely to capture students’ typical thinking thus reducing some of the limitations of verbal data (Goos & Galbraith, 1996; Schoenfeld, 1985b). The novel method of recording all graphing calculator screens on the videotape of the output from the viewscreen allowed the researcher to assemble a more complete record than merely ordinary videotaping using two students would have. The screens appear to capture more of the students’ immediate thought processes than the words being uttered and recorded. Thus, not only do graphing calculators increase the learning opportunities for students, they provide the opportunity for teachers and researchers to witness more closely the understandings students have as inferred by the results of the actions of students represented by the graphing calculator screens.

In addition, the use of video recording supplemented the dialogue and a combination of the two was used to determine the behaviour of the students. However, this researcher acknowledges that there are no guarantees that the participants acted accordingly and care must be taken in interpreting the behaviour so inferred.

3.5.2 The Recording of “Naturally Occurring” Data

Exchanges between the students are what would be expected in a classroom setting. This is not to deny or ignore the experimental setting. In making interpretations of the data the researcher has been careful to consider any obvious intrusions of the setting,
that is, the situational context (Wedge, 1999), into the interactions between pairs of students in the study. Observational data are of value here because they focus on naturally occurring activities and have been used to supplement and verify data collected by other methods.

The detail found in transcripts of recorded talk is a result of “close, repeated listenings to recordings” (Silverman, 1993, p. 117). The permanency of the recordings allows repeated and detailed examination of the raw data. Furthermore, it allows other researchers direct access to the data. Any analysis can be, therefore, thoroughly scrutinised. In addition, the data can be used in other research or re-examined as a result of later findings.

“The goal of data collection is to capture learning in a way that allows researchers to fully appreciate its complexity and make it accessible to future analysis”, according to Barab, Hay, and Yamagata-Lynch (2001, p. 72), and the use of both audio and video recording goes some way toward meeting this goal. The audio taping focuses the researcher on the words of the participants, whereas, the videotaping allows the researcher to also focus on the action, or results of those actions, that occurred concurrently as the words were spoken. Often the reactions of the students on the graphing calculator, as captured on the video of the viewscreen output, are to their immediate thoughts and do not merely reflect what they are saying. The combination of two forms of taping allows better quality data to be collected. Rather than just a verbal dialogue being recorded, the graphing calculator screens add clarity and depth to the student-student and student-calculator interactions. The action of the students is apparent rather than having to be inferred by the researcher, adding validity to the case record and any interpretation of it (Peräkylä, 1997). Although dialogue is not thinking, the speed of the dialogue and key actions provide some insight into the trace of the participants’ thinking. The “rich source of data” (Barab et al., 2001) provided by the
video and audio recordings of the problem task sessions were reviewed many times by the researcher to check the accuracy (Peräkylä, 1997) of the transcripts and to thoroughly familiarise herself with the data. Further, the use of video and audio taping allowed the researcher to revisit the raw data as often as necessary (Peräkylä, 1997; Silverman, 1993).

In addition to the audio and video recording of the students attempting the problem task and their final written solution, the researcher made observational notes during the students’ solving of the problem task. These observational notes were used to assist in interpretation of student behaviour. In addition, where technical difficulties occurred, they were used to supplement the production of the protocols of the students’ efforts in undertaking the problem task. The observational notes also allowed “descriptions of non-verbal aspects of interaction” to be included in the protocols (Silverman, 1993, p. 115).

3.6 The Protocols

Using both the audio and video recordings, protocols were produced for each pair of students (see Appendix C). The protocol of each pair consists of any verbal exchanges and screen shots of the graphing calculator, and the written scripts where students recorded their solution. These have been augmented, where necessary, by the observational notes recorded by the researcher. The protocols were coded and then divided into “macroscopic chunks” (Schoenfeld, 1985b, p. 292) that were then classified according to the particular behaviours of interest associated with working with functions in a graphing calculator environment. These are referred to as episodes and “represent periods of time during which the problem solvers are engaged in a particular activity” (Schoenfeld, 1992b, p. 189). The "juncture between episodes" (Schoenfeld, 1985b, p. 300) is referred to as a transition point and these “identify points where the solution attempt changed direction” (1992b, p. 189).
3.6.1 Schoenfeld’s Coding Scheme and its Adaptation in this Study

The coding of the protocols was adapted from a scheme used by Schoenfeld (1985b, 1992b) in his identification of metacognitive behaviour during mathematical problem solving. Schoenfeld used the following categories to code both activities and episodes: reading, analysis, exploration, planning, implementation, and verification. His coding categories and their definitions are shown in Table 3.5.

<table>
<thead>
<tr>
<th>Coding category</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reading</td>
<td>Reading or rereading the problem.</td>
</tr>
<tr>
<td>Analysis</td>
<td>Analysing the problem, in a coherent and structured way.</td>
</tr>
<tr>
<td>Exploration</td>
<td>Exploring aspects of the problem, in a much less structured way than in analysis.</td>
</tr>
<tr>
<td>Planning</td>
<td>Planning all or part of a solution.</td>
</tr>
<tr>
<td>Implementation</td>
<td>Implementing a plan.</td>
</tr>
<tr>
<td>Verification</td>
<td>Verifying a solution.</td>
</tr>
</tbody>
</table>

All coding schedules are open to interpretation. Schoenfeld (1985b) suggests that “almost by definition, protocol coding schemes and subsequent analysis of them focus on overt actions” (p. 288). He developed his framework to ensure that major decisions of a covert nature were also recorded and hence analysed. Scott (1996), however, when using this scheme to detect and compare control-based aspects of metacognitive behaviour concluded that “Schoenfeld’s parsing process does not necessarily produce an accurate narrative of a solution - the episodes reflect ‘surface level’ decisions and actions and often cannot indicate the quality and depth of thinking” (p. viii). In his attempt to duplicate some of Schoenfeld’s results, Scott produced and analysed
protocols for novice and expert problem solvers undertaking thirty minute think aloud problem solving sessions. Interestingly, the experts in Scott’s study attempted the ‘staircase numbers’ problem (Stacey & Groves, 1985, pp. 125-130) on their own, whereas, the novice subjects worked in pairs “because a think aloud methodology was central to [his] study” (Scott, 1994, p. 532). Scott claims that with Schoenfeld’s parsing categories “fundamental ambiguity remains regarding the meaning” (Scott, 1994, p. 537) of these. The fact that not all Scott’s subjects worked in pairs, where verbalisation of thoughts happens more naturally, may have contributed to his difficulty in coding and discovery of the quality and depth of thinking. Even though both Schoenfeld and Scott believe that there is no need for a partner to help experts verbalise their thoughts, this is not necessarily the case.

The analysis in this thesis goes beyond the macroscopic analysis of Schoenfeld and is used by this researcher as a way of looking for defining moments in the solution process which are then explored using microscopic analysis to search for explanations. The use of the graphing calculator screen data to supplement and enhance the dialogue data in the current study allowed a more finely grained classification than was possible by Schoenfeld’s (1985b) scheme. The codes devised by this researcher which classify actions according to distinct behaviours specific to solving the particular problem task used in this study ensure clear distinctions between each category.

3.6.2 A Framework for Analysing the Protocols

The framework for the analysis of the protocols is adapted from that used by Schoenfeld (1985a, 1985b, 1992b) and others (e.g., Goos & Galbraith, 1996; Scott, 1994; Stillman, 1989) to analyse cognitive behaviour and metacognitive processes occurring during problem solving. Time-line diagrams were adapted by Schoenfeld from Woods (1983, cited in Schoenfeld, 1985b), and used by him in his work on metacognition and problem solving (1985b, 1992a) and others to extend his work (e.g.,
Scott, 1994, 1996; Stacey & Scott, 2000; Stillman, 1989). In this study they are used to provide a graphical representation of the flow of events over the period of the task solution. They have been modified by the researcher to include the type of representation of function being used at any point throughout the task solving. This was usually the representation being displayed by the graphing calculator although at times the dialogue of the students made it clear they were actually working in another representation, and they may well have been operating in two representations simultaneously. Schoenfeld (1985b, 1992a) was interested in the presence and absence of decision making and part of the analysis here is also interested in this aspect of solving a problem task. However, the modified framework used in this analysis codes according to local or specific behaviour, such as reading the problem statement or adjusting the scale marks, whereas, the parsing of the protocols into “macroscopic chunks of consistent behaviour called episodes” (Schoenfeld, 1985b, p. 292) is classified according to global categories of behaviour such as searching for a global view of the function or identifying a key feature.

The framework enabled the data to be divided into macroscopic chunks (represented diagrammatically as episode diagrams and graphically as time-line diagrams) that can then be considered abstractly on a broader scale, including by comparison across the pairs of students. When behaviour of interest was observed this was then considered more microscopically, enabling the researcher to write an analytical story about points of interest; for example, one pair of the students’ lack of use of the dedicated functions of the graphing calculator such as Calculate: Zero or Calculate: Maximum, even though they these were familiar to them, when identifying key features of the function.

Clearly, the framework, the episode diagrams, and time-line diagrams do not tell the whole story. Their function is to highlight behaviours of interest to the purposes of the
research as outlined previously and to allow comparisons to be made across the pairs of students.

3.6.3 Coding the Protocols

The type of analysis afforded by the framework allowed sense to be made of the mass of data present, and enabled the researcher to focus on behaviours of specific interest to this study. To facilitate this process, a coding scheme was developed by this researcher for classifying macroscopic chunks of behaviour. Codes are “tags or labels for assigning units of meaning to the descriptive or inferential information compiled during a study” (Miles & Huberman, 1994, p. 56). Miles and Huberman define coding as “conceptualizing, reducing, elaborating, and relating” data (p. 12). The use of the coding scheme is the first step in the process “through which data are fractured” (Strauss & Corbin, 1998, p. 3) so that it can be viewed more conceptually.

Brief descriptions of the activity codes used in this thesis are presented in Table 3.6. The development of codes specifically relates to the description of activities when using a graphing calculator to solve a specific class of problem—finding a complete graph of a given function. More detailed descriptions with illustrations of each type of activity are included in the next chapter to alert the reader to the interpretation by this researcher of her coding scheme.

The codes used in the analysis of the protocols describe the following behaviours which are referred to as activities: reading (R), organising or planning (O), selecting a viewing window (VW), searching for or identifying a local feature (ID), considering the global view (GV), adjusting scale marks (SM), evaluating (E) (either the graphing calculator output or the solution state), and recording (Re) as shown in Table 3.6. Coding these activities is detailed in the next chapter. The representations that were available to the students or were being used or considered by the students were also
Table 3.6

_Coding Categories for the Activities_

<table>
<thead>
<tr>
<th>Code</th>
<th>Coding category</th>
<th>Activity Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>Reading</td>
<td>Reading or rereading the problem</td>
</tr>
<tr>
<td>O</td>
<td>Organising</td>
<td>Organising and planning</td>
</tr>
<tr>
<td>VW</td>
<td>Viewing window</td>
<td>Selecting a viewing window</td>
</tr>
<tr>
<td>ID</td>
<td>Identification</td>
<td>Searching for or identifying a local feature of the function</td>
</tr>
<tr>
<td>GV</td>
<td>Global view</td>
<td>Considering the global view of the function</td>
</tr>
<tr>
<td>SM</td>
<td>Scale marks</td>
<td>Adjusting scale marks</td>
</tr>
<tr>
<td>E</td>
<td>Evaluation</td>
<td>Evaluating either the graphing calculator output or the solution state</td>
</tr>
<tr>
<td>Re</td>
<td>Recording</td>
<td>Recording the solution</td>
</tr>
</tbody>
</table>

coded (i.e., as A = algebraic, G = graphical, or N = numerical). The coding is dependent on what is seen, as well as what is heard by the researcher or written by students.

In addition, the more global consistent behaviours of the students, over a given period of time, related to the production of the complete graph of a function were chunked macroscopically into episodes. The protocols were parsed into episodes by examining the behaviours of the students when they were engaged in one large task. The codes used to describe the episodes can be the same as those used for the line by line analysis, for instance reading, or can be different, for example, observing the function in the standard window. This method of coding is distinct from that of Schoenfeld (1985b, 1992b) who uses identical codes for both activities and episodes, although he allows an episode to consist of one or more activities and then gives it a compound name.

The purpose of the coding was (a) to organise the data in order to allow the researcher to make connections between student actions and understandings, in the
context of solving a problem task, and (b) to allow comparisons to be made. The gaining of a global, as well as, detailed sense of the data was facilitated by this process. It allowed the emergence of questions from the data and highlighted differing processes used by students in their solution of the given problem. It is acknowledged that what any individual researcher sees in a protocol is unavoidably selective. As with any methodology, some aspects of behaviour will be highlighted and others obscured or even distorted by the analysis process (Miles & Huberman, 1994; Schoenfeld, 1985b). What is important, however, is that the analysis methods allow the researcher to gain the view of the data that best matches the behaviours from which insight is sought.

An example of the coding is shown in Table 3.7. The protocols include both dialogue and graphing calculator screens. Lines of dialogue, considered with their concurrent graphing calculator screen(s), were categorised as one of eight possible activities as identified previously. The initial exchange is an example of organisation and planning (O) followed by selection of a viewing window (VW). These four exchanges of dialogue occurred over 25 seconds, from 0.20 - 0.45 from the beginning of the tape, and constitute a single episode in which the students considered the view of the function in the standard window. The first exchange involved students operating in the algebraic representation whilst in the remaining exchanges in this episode the students were working in the graphical representation.

3.6.4 Parsing the Protocols into Episodes

Following the coding of the protocols using the activity categories just described, the protocols were divided into episodes. According to Barab et al. (2001), the size of chunk that is important when undertaking “parsing of rich learning experiences” (p. 65) is determined by the researcher’s purpose. It is “the minimal amount of information that needs to be described so that the researcher can derive a useful interpretation” (pp. 65-66). Accordingly, an *episode* is taken by this researcher to mean a period of time where
Table 3.7

An Example of the Coding of the Activities for a Single Episode

<table>
<thead>
<tr>
<th>Time</th>
<th>Rep</th>
<th>Activity</th>
<th>Dialogue</th>
<th>Graphing calculator screen</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.20</td>
<td>A</td>
<td>O</td>
<td>Ahmed: Firstly what we should do is get an idea of what the graph looks like [they enter the function].</td>
<td></td>
</tr>
<tr>
<td>0.30</td>
<td>G</td>
<td>VW</td>
<td>Ahmed: Graph it, I don’t think we should use Zoom Standard. [They press GRAPH.] Linh: What’s our window. Ahmed: It's the standard window.</td>
<td></td>
</tr>
</tbody>
</table>

Note: Rep = Representation, A = Algebraic, G = Graphical, O = organising or planning, VW = selecting a viewing window.

the students are engaged in one large task or a series of closely related tasks in the pursuit of some goal. An example of an episode for Pair 2, as they search for key features follows:

Episode 3 begins as the pair use their graph to search for key features. They begin with the y intercept and then the left x intercept. The episode ends upon finding the left x intercept and the students being dissatisfied with the clarity of the feature shown.

This episode took 80 seconds (beginning 2 minutes into the task and concluding 3 minutes and 20 seconds into their solution process). The episode consisted of 9 items of dialogue.

The end of an episode occurs when there is a clear change in direction. The protocol is parsed into these distinct sections by considering the behaviour of the students as evidenced by their words, observed actions, and the viewscreen output that resulted from their thinking. The points between episodes are classified as transition points.
These may be statements by the students indicating a change of direction (e.g., “We should change the domain so we can look at it more clearly”) or simply a point in time marking the end of one episode and the beginning of the next. Transition points where statements or overt actions occur are recorded.

Episode 3 illustrates that the classification of an episode in this analysis is distinct from that of Schoenfeld (1985b, 1992a, 1992b). His episodes were classified according to the same categories as the activities and consist of one, or possibly two, types of activities. Given the nature of the problem task and the purpose of the analysis, it was more appropriate to classify episodes according to their specific purpose in the solution of the problem task, for example, observing the function in the standard window, searching for key features, and, searching for a complete graph. However, each episode, as per Schoenfeld’s classification (see Table 3.4), describes consistent behaviour, the difference is that in this case they are defined in terms of the solution process for a specific class of problem rather than classified according to problem solving activity.

3.6.5 Depth of the Coding Scheme

At first glance the coding scheme appears to be what has been described as broad grained (Schoenfeld, 1992b) or only coding surface features (Scott, 1996). However, the use of the video recording of the graphing calculator screens as part of the protocols allowed the coding scheme to be far more fine grained than that which would have been produced if the protocol consisted of dialogue and actions alone. In addition, the use of microscopic analysis to explain the findings of the macroscopic analysis adds to the “thick description” (Merriam, 2002a, p. 29) that is essential for validity and reliability of the study and ensures “enough description and information is provided that readers will be able to determine how closely their situations match” (p. 29).
3.7 Analytical Tools

Diagrams are a valuable tool at the qualitative researcher’s disposal as they enable the researcher “to gain distance from the data, forcing him or her to work with concepts rather than with the details of the data” (Strauss & Corbin, 1998, p. 153). Visual displays present information systematically, so the user can draw valid conclusions to answer the research question of interest. They condense and distil the data to improve the chances of drawing and verifying valid conclusions. They allow careful comparison and the emergence of trends and patterns from the data (Miles & Huberman, 1994). The types of diagrams used in this analysis include: episode diagrams, time-line diagrams of the protocols (showing the sequencing of episodes and the use of various representations), balancing matrices, contingency tables, and flow diagrams.

3.7.1 Episode Diagrams

An episode diagram (Schoenfeld, 1985b) provides a visual representation of the macroscopic chunks of the protocol that can be helpful in identifying relationships between student behaviour both within and between the pairs. The episode diagram summarises the behaviour of the students by identification of the major directions followed by the student pairs in their solution process. It provides an illustration of some aspects of the data, including for instance, illuminating the students’ initial action, their response to the result of this initial and following actions, the underlying planning that allowed the pairs of students to focus at length on a broad aspect of the problem, the use of particular dedicated functions of the graphing calculator, and the application of mathematical knowledge in their solution of the problem.

Figure 3.1 shows part of the episode diagram for Pair 1. Each episode is labelled with the consistent behaviour undertaken by the students. The length of time and the number of items that constitute the episode are also recorded. Figure 3.1 also shows a
Figure 3.1. Part of the episode diagram of Linh and Ahmed.

transition point, consisting of one item which occurred between Episodes 3 and 4. This item signalled the end of the students’ specific search for an \( x \) intercept and the beginning of their search for a global view of the function. A comparison of the episode diagrams for all pairs will facilitate a comparison of the approaches undertaken by students in the study.

### 3.7.2 Time-Line Diagrams

A time-line diagram is used to provide a visual representation of the parsing of the protocols into both activities and episodes. The time-line diagrams record the detail of the episodes and show the progress of the solution as the students proceed. Figure 3.2 shows the first five minutes of the time-line diagram for Pair 1.

The first column of the time-line diagram identifies the various types of activities as listed in Table 3.5, namely, reading (R), organising or planning (O), selecting a viewing window (VW), searching for or identifying a local feature (ID), considering the global view (GV), adjusting scale marks (SM), evaluating (either the graphing calculator
output or the solution state) (E), and recording (Re). It also can include transition points (T). The horizontal bar labelled REP shows the representation of the function being used by the pair of students. A time-line is given at the bottom of the diagram showing alternating black and white rectangles each representing 10 seconds. Episodes, which are periods of consistent global behaviour, are identifiable by the alternating use of colour for the activities that were undertaken by the students. In the example, the first episode for this pair took 20 seconds and consisted solely of reading with an algebraic representation being used, whereas the second episode consisted of 10 seconds of organising and planning followed by 15 seconds of selecting a viewing window, all of which occurred in a graphical representation. The fourth episode provides an example of two activities occurring simultaneously, namely, selecting a viewing window and evaluating the graphing calculator output. A total of five episodes are identifiable in the section of the time-line diagram shown. The fifth episode represents ninety five seconds where the students were working simultaneously in an algebraic and graphical representation. An overt transition also occurs between episodes 3 and 4.

3.7.3 Balancing Matrix

A balancing matrix is used to consider actions or conditions of interest and their consequences (Strauss & Corbin, 1998, p. 237). These diagrams can help the researcher
discover relationships in data. The balancing matrix in Table 3.8, for example, allows the consequences of various actions for (a) the view of the graph seen on the graphing calculator screen, (b) the routineness of the solution, and (c) the time to reach a global view of the function to be observed, compared, and contrasted.

Table 3.8

*Initial Actions and Consequences in Accessing the Problem Task for Pairs 1 and 2*

<table>
<thead>
<tr>
<th>Pair</th>
<th>Actions</th>
<th>Consequences for View</th>
<th>Solution Process</th>
<th>Time till Global View was seen</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Using GRAPH followed by ZoomFit</td>
<td>Becomes routine</td>
<td>2:20 mins (18 windows)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Setting WINDOW using y intercept followed by ZoomFit</td>
<td>Becomes routine</td>
<td>1:00 mins (7 windows)</td>
<td></td>
</tr>
</tbody>
</table>

3.7.4 *Flow Diagrams*

Flow diagrams or activity records are network diagrams (Miles & Huberman, 1994), consisting of a series of nodes with links between them. They allow the display of (a) a specific number of relevant actions, (b) who undertook these, and (c) the particular sequence of actions undertaken. They consist of rectangles which show actions and series of arrows indicating the sequence of action. In addition, the pairs of students undertaking the actions and action sequences are identifiable. The flow diagram in Figure 3.3 clearly shows that each pair of students undertook a different sequence and/or selection of actions in accessing the task in an effort to observe a global view of the function. The flow diagram highlights actions and the sequences undertaken by each pair and allows these to be compared and contrasted. The six actions are represented by
Figure 3.3. Task access: Beginning steps for the five pairs.

the rectangles and the identification of a pair inside the rectangle indicates the initial action of a pair of students. A series of arrows indicates the sequence of actions undertaken by each pair. It can thus be inferred which action finally resulted in the pair achieving a global view of the function.

3.8 Quality Control in the Research

3.8.1 Reliability / Auditability

As Silverman (1993) suggests the checking of reliability is closely associated with assuring consistency in the collection of data and preparation of transcripts allowing access to the data by other researchers. Working with videotapes, audiotapes, and transcripts “eliminates at one stroke many of the problems … with the unspecified accuracy of field notes and with the limited public access to them” (Peräklyä, 1997, p. 203). These tapes form part of the primary data source from which the case record (Stenhouse, 1978) is compiled and these data can be reviewed and reanalysed by other researchers.

In addition, when collecting the data, reliability was ensured by following a standard protocol, providing the same information to each pair of students, and making the settings on the graphing calculator and the physical environment as identical as possible. When producing the protocols for coding, the audio and video tapes were
reviewed many times. This researcher acknowledges, however, that transcripts can always be improved. In addition to consideration of validity of procedures, interpretive validity must also be considered (Maxwell, 2002). Not only does the data collection and analysis need to be thorough, alternate explanations need to be considered as “you do not necessarily rely on the consensus of others who are looking at the same data, because you acknowledge that each might bring a legitimate but different perspective to the data” (Worthen, 2002, p. 141).

When considering the reliability of the coding, questions need to be asked as to the consistency of assigning codes by the same researcher on different occasions or across different researchers. “Reliability in parsing protocols is quite high”, according to Schoenfeld (1985b, p. 293) when discussing his coding scheme. In later work (1992b), however, he admitted that for this to be achievable “some experience coding tapes jointly appears to be necessary to achieve the consensus that produces reliability” (p. 194).

For the coding scheme used in this study, the scheme was designed by the researcher who produced a coding schedule as shown in Table 3.5. This is explained further in chapter four. The initial coding was redone until code-recode consistencies were consistent. The schedule was then used by the researcher to code the protocols. It was essential that the coding could be reliably done by the researcher over time. Six months later a sample of the protocols was recoded by the researcher and there was a high level of consistency with 97% agreement. The researcher’s supervisor also used the coding scheme at this point to code a sample of the protocols. This person also researches in a similar area, that is, analysing tasks undertaken by senior secondary students in mathematics education. In addition, she was provided with clear code descriptions as found in chapter 4 of this thesis, had substantial experience as a secondary teacher of mathematics, and clear understandings of the mathematical and graphing calculator
knowledge required by the problem task and the interplay between these. Using the following formula: 
\[
\text{reliability} = \frac{\text{number of agreements}}{\text{total number of agreements} + \text{disagreements}}
\]
(Miles & Huberman, 1994, p. 64), intercoder-reliability between this researcher and a more experienced coder, albeit in a different but similar context, was calculated to be 86%. This level of reliability was acceptable. Discrepancies did not appear to be systematic.

Processes of the study were consistent as evidenced by their documentation. Purposefully looking for evidence of contrasting cases further supports the validity of the descriptions, understandings, and findings. The provision of comprehensive and “thick” descriptions in the portrayal of the case in chapters four and five allows context rich and meaningful descriptions to be provided for others who can then decide if they would draw the same conclusions from the data. This is not to say that analysis and interpretation are not present, on the contrary. As Eisner (1997) points out “good description, even very good description, is not likely to be enough” (p. 267). Analysis and theoretical interpretations of the situations being described are necessary to provide the “distinctly educational added value of our work” (Eisner, 1997, p. 267) as researchers. By explicitly detailing the research process, the researcher is laying down an audit trail that any other researcher could follow if they wanted to undertake a similar analysis (Merriam, 2002a). The data, the analytical tools, the reasons for decisions, and interpretations are clearly detailed in order that the research is auditable by others (Miles & Huberman, 1994).

### 3.8.2 Internal Validity / Credibility

The validity of the case is derived from the thoroughness of the analysis, not from the representativeness of the sample as occurs in quantitative methodology. Validity is taken to mean “truth: interpreted as the extent to which an account accurately represents the social phenomena to which it refers” (Hammersley, 1990, p. 57 as cited in Silverman, 1993, p. 149). Any patterns found in diagrams and tables were checked by
referring back to the protocols, to ensure the same patterns could be seen in the raw data. Use of the video recordings, in particular, should help ensure that the account provided by the protocols credibly represents the phenomena to which it refers. In addition, the data were seen firsthand by the researcher, consist of observed behaviour and activities, and were collected in a relatively informal setting by a researcher trusted by the participants with no other participants present. The setting used was not contrived, although different from that of the classroom to the extent that there were only two students present and it was being videotaped. The behaviours of the students, in the opinion of the researcher, were unmodified by the researcher’s presence. The setting was closer to their everyday classroom than to an experimental setting. During the data collection care was taken to ensure that the raw data were free from bias. Taken together these measures should help ensure that data quality is strong and hence meaning emerging from the data is credible (Miles & Huberman, 1994, p. 268).

3.8.3 Multiple Roles of the Researcher

As a teacher researcher my role is at times unclear. How much should I be the teacher? How much should I be the researcher? Stake (1995) argues that the roles overlap, as the role of the case researcher is “to inform, to sophisticate, to assist the increase of competence and maturity, to socialise, and to liberate … (and these) are responsibilities of the teacher” (pp. 91-2). As the teacher of the participants some researchers (Gay, 1987, p. 208; White, 1998) may suggest that I create interference in the experimental scene merely by being in the room. Alternatively, one can accept the view that in the classroom environment, I am a constant factor so I have the advantage of blending in. In the analysis, I have the advantage over other researchers of seeing the actions of the students with the eyes of a teacher, their teacher. As Eisner argues, “qualitative research designed to illuminate, for example, the quality of [mathematics] teaching needs researchers who know something not only about [mathematics] but
about their teaching, … expertise does matter” (1997, p. 269). The view of this researcher concurs with that of Eisner, that as a teacher researcher in my own classroom I bring a richness of experience and expertise to the experimental/observational setting. I bring emic or “insider knowledge” (Eisner, p. 265) to the research process. I have “theoretical sensitivity” (Strauss & Corbin, 1998), as termed in grounded theory, merely by having been in the classroom and also as a reader of academic papers in my studies. “Stake has argued that the best understandings of educational phenomena are likely to be held by those closest to the educational process” (MacDonald, 1982, p. 26).

3.8.4 The Need for Theoretical Sensitivity

The relationship between the research and the researcher undertaking qualitative analysis results in the researcher becoming an instrument of the analysis, no research is untouched by human hands (Silverman, 1993, p. 26, adapted from Hammersley, 1992). This requires the researcher to be sensitive to “the issues and problems of the persons or places being investigated” (Strauss & Corbin, 1998, p. 42) and “to theoretical issues when scrutinizing the data” (Strauss, 1987, p. 300). Theoretical sensitivity (Strauss, 1987) enables the researcher to have insight into the data and give meaning to the events in the data. Sensitivity requires the researcher to be open-minded, to immerse herself in the data, and, to use professional knowledge and experience to reflect on the data. This researcher believes she has this theoretical sensitivity. She uses her experiences not to put them into the data, rather to assist in seeing what the participants in the research are seeing, in order to become “sensitive to meaning without forcing [her] explanations on the data” (Strauss & Corbin, p. 47).

3.8.5 Trustworthiness

Another important issue relates to trustworthiness. The extent to which the findings “are reasonable and justifiable given the researcher’s interests and concerns” determines the trustworthiness of the findings (Cobb & Whitenack, 1996, p. 225). By the writer
illustrating general claims and assertions, with examples from the data, as is the convention in qualitative analysis, the reader is often unable to make the same inferences and interpretations as these examples are considered in isolation from the rest of the data. The inclusion of the protocols in this thesis allows the reader to refer to the case record and determine if the findings are reasonable and grounded in the data. The fact that this researcher is a teacher of the participants and brings her first hand experience of observing and interacting with the students over an extended period of time “constitutes a crucial source of insight when attempting to account for their activity” (Cobb & Whitenack, 1996, p. 225). Expertise in the research context is an added advantage as the content being taught and the pedagogical practices make a difference for researchers trying to explore something about their teaching. Finally, the reviewing of the analysis by the second teacher of the students, who is familiar with the participants in the study, allowed him to make a positive judgement as to whether the analysis conformed with his observations and interpretations of the actions of the students. This researcher acknowledges that other interpretations are possible for the data collected in this study, but claims those that are presented are reasonable and justifiable (Cobb & Whitenack).

3.8.6 Transferability versus Generalisability

The purpose of this study is not to make generalisations from the case studied to a larger population. The focus is more on specifying the conditions under which the observed phenomena exist so that others can be informed of these conditions and related phenomena. As Schoenfeld notes “any methodology (protocol parsing included) may highlight some aspects of behaviour and may obscure or distort others. It is thus prudent to examine particular instances of … behaviour from as many perspectives as possible” (1985b, p. 316). This also applies when considering behaviour other than problem solving and is relevant here. The methodology used is looking at only some aspects of
student behaviour when working with functions in a graphing calculator environment. The purpose of the analysis is to consider those aspects of the students’ behaviour which facilitate or hinder their solution of the problem, and it is acknowledged that other behaviours are being ignored as they are not the focus of this particular research. Generalisations are not being made for this reason.

More importantly we need to know if the findings of a study are transferable to other contexts (Miles & Huberman, 1994). The provision of “thick” descriptions allows readers to assess the potential transferability from the study and its context to their own setting (Merriam, 2002a; Miles & Huberman, 1994). The goal of qualitative research is to ensure that the findings are consistent with the data and that what we learn from the study can then be applied to other situations. “What qualitative research yields is a set of observations or images that facilitate the search and discovery process when examining other situations, including classrooms and schools” (Eisner, 1997, p. 270). No classroom in the world will have students with the same experiences and expertise in functions and the use of graphing calculators as the students in this study, neither could we find a classroom where teachers have exactly the same pedagogical ideas, experiences, expertise, and practices as those of the teachers of the students in this study. However, enough similarities will be found, and differences explicitly stated, to consider that the actions of students in this study are likely to be found in other groups of students. The intent of the study is to provide usable knowledge about a particular case that can guide teachers in their teaching in a graphing calculator learning environment.

3.8.7 Limitations of the Methods

In considering the transferability of the study its limitations must be considered. The researcher, although not teaching all the students in the year of the study, had taught all of them at some stage of their secondary schooling. The three pairs of Year 11 students
were selected from the class of the researcher. The two pairs of Year 12 students were selected from the class of a second teacher, however, three of these students were also undertaking a second mathematics subject with the researcher as the teacher. In all classes with this researcher, or the second teacher, students received a similar emphasis on the way they learned mathematics and the ways graphing calculators were used to enhance this. This included an emphasis on understanding, explanations by students of their work, and, use of the graphing calculator as a tool to both explore and confirm ideas and to enhance student learning. The teachers were allocated to the particular classes by the Principal. Teaching style may be a confounding factor in the study, however, both teachers have a similar philosophy of teaching and learning. The two teachers met both formally and informally on a regular basis to discuss planning and what occurred in their classrooms. A second study, similar to this one, should therefore have similar findings if the teachers have a similar teaching style and philosophy.

A second possible confounding factor in the study is the different year levels of the students involved. Some differences in the study will be due to this factor. This will be taken into account in the analysis. A comparison of the Year 11 and Year 12 students may provide some insight into process defined by Strauss and Corbin (1998) as “the dynamic and evolving nature of action/interaction” (p. 179). Barab et al. (2001) use a similar notion when they speak of detailing the “evolving trajectory of the phenomena of interest” (p. 63). In the present study an example is the students’ conception of a complete graph and how they can best find it in a graphing calculator environment. This trajectory will be illuminated by comparing the differences between the Year 11 and Year 12 students.

### 3.9 The History of the Study

The study detailed in this thesis had its basis in an earlier pilot study of this researcher. Some of the research (e.g., Smart, 1995; Ruthven, 1995) at the time reported
that users of function graphers did not recognise that the viewing screen showed only a window of the graph. This suggested that more information, and possibly a different domain and range, were needed in order to determine if the window was providing the viewer with all the essential information, for example, shape, intercepts, and stationary points.

Smart (1995, p. 208) suggested that students were “accepting what they [see] on the screen as the whole graph rather than a window onto part of the graph.” Ruthven (1995) also pointed out the need to recognise this window onto the graph:

The role of the user is even more central when a (graphing) calculator is used to graph symbolic expressions ... more is required than simply transcribing the expression: it has to be translated…. Equally the resulting graphic display needs to be interpreted. Here, whichever default setting is used for the range of the axes, the user needs to recognise that only a restricted portion of the graph is shown, and then appropriate rescaling of the axes [needs to be undertaken]. (p. 241)

3.9.1 Pilot Study

Since the idea that students would accept what they saw on the graphing calculator screen as the global view of the graph was a new idea, it was decided to investigate this further. A case study was undertaken by this researcher to obtain greater information and insight into how particular types of users worked cooperatively to produce a complete graph of a particular function.

The task set was that of graphing a specific cubic function, namely, $y = x^3 - 19x^2 - 1992x - 92$. Binder (1995) had used this particular function previously to investigate a graphical approach to solving an algebraic problem. The task was used in the study in a substantially different way from how Binder used it, and was
selected on the basis that no part of the function appears to be visible in the standard viewing window, as shown in Figure 3.4.

![Figure 3.4](image)

*Figure 3.4. The view of the function (a) in the standard window, (b) having window settings $-10 \leq x \leq 10, -10 \leq y \leq 10$.*

This pilot study, in the early days of graphing calculators, involved two Year 11 student pairs undertaking Mathematical Methods Units 1 and 2 and two teacher pairs. Of the student pairs one was considered as having the opportunity to be more experienced with use of a graphing calculator from their ownership of such a calculator, with the less experienced pair using the school owned graphing calculators during class. All four students were participating in the trial of graphing calculator use in their Mathematical Methods class which was taught by this researcher. The trial involved the frequent use of TI-82 graphing calculators in the area of functions. Of the teacher pairs, the more experienced users were both teachers of senior secondary mathematics with some use of the graphing calculators in their classrooms. These teachers had previously used graphing software, such as ANUGraph, in their teaching practice. The less experienced pair taught only junior mathematics and had used graphing calculators only during faculty based professional development. Each pair was asked to work cooperatively to produce a graph of a given function whilst verbalising their thoughts. Their final graph was to be produced only when both agreed that the graph showed all important features. Only one graphing calculator was available for use by each pair as this was considered to encourage cooperation. Each session involving the pairs undertaking the problem task was audiotaped. In addition, observational notes were made by this researcher.
3.9.2 Findings of the Pilot study

The findings of the pilot study were:

1. *The greater the mathematical knowledge of the users, the greater the amount of thought that went into the task of finding an appropriate window. The less this mathematical knowledge, the greater the haphazardness of the approach.*

The experienced teacher pair, for example, initially worked in both the algebraic and graphical representation. They demonstrated their mathematical knowledge by recognising the y intercept of the function from the algebraic representation and considered the effect of the coefficient of $x^3$ on the graphical representation of the function. After entering the function, they used their mathematical and graphing calculator knowledge to alter the WINDOW settings to $-100 < x < 100$ and $-100 < y < 100$. The resulting view had three apparently vertical lines (see Figure 3.5(a)). After selecting Zoom Out several times the graph appeared coincident with the y axis (Figure 3.5(b)). At this point they demonstrated their awareness of how the tool operated as they discussed how Zoom Out had an identical effect in both the horizontal and vertical directions. Their discussion continued as they considered why the y intercept was not visible in the viewing window and further agreed that the shape of the graph was a ‘positive’ cubic with two turning points. They used TRACE to try to identify the turning points, noticing that the TRACE cursor, whilst disappearing from the screen, continued to display coordinates of the function. Using these values they adjusted the WINDOW settings, to $-40 < x < 100$ and

![Figure 3.5](image-url)  
*Figure 3.5. Finding of a global view of the function by the teacher expert pair in the Pilot Study.*
-50000 < y < 25000 and immediately found an almost global view of the function as shown in Figure 3.5(e).

This pair clearly had a well established understanding of the possible shapes of a cubic function and appreciated that the graphing calculator provided a window onto a section of the graph. In addition, the linking of their mathematical knowledge and their knowledge of various features of the graphing calculator in conjunction with the result of their mathematical knowledge being portrayed in a particular way given the representation being used, allowed them to both estimate and verify the coordinates of key features of the function. Their use of TRACE, for example, to identify approximate coordinates of the maximum turning point, by noting the point where the y ordinate changed from increasing in value to decreasing in value, as the x ordinate increased, preceded their use of the TABLE, which with the increment set to 0.01 allowed a more accurate pair of coordinates to be determined. They used the TABLE in a similar way to identify x intercepts, searching for the x value closest to the point where the y ordinate changed sign (see Figure 3.6) and reducing the TABLE increment to improve the accuracy of the coordinates so found.

![Figure 3.6. Using the TABLE to identify the central x intercept.](image)

In contrast, the less experienced teacher pair demonstrated that less mathematical knowledge appeared to result in less thought being used in the search for an appropriate window. Although this pair initially discussed the existence of two possible shapes for a graph of a cubic function, one immediately stated on seeing the graph that it was linear. Their lack of understanding of the effect of altering WINDOW settings, in particular the relationship between the altering of the minimum and maximum x values, for example, and the scale marks was apparent by their confusion on viewing the graphing calculator
output. Although one teacher suggested the use of Zoom Out and TRACE to inform their alteration to the WINDOW settings, early in their solution process, their lack of understanding seemed to halt the pursuit of this path. Their lesser mathematical understanding was demonstrated in the number of solution paths attempted with little evaluation of the solution state or results of their actions as they proceeded. When an action did not have the expected result, this pair changed strategy and pursued a different action. This was demonstrated by their initial ideas to use TRACE to inform their adjustments to the WINDOW settings, followed by deliberate alterations to the WINDOW settings one value at a time, repeated Zooming Out, use of the algebraic representation to inform the WINDOW settings in conjunction with Zooming Out as they focused on the y intercept and finally a combination of Zooming Out focusing on the viewing range, in conjunction with TRACE, and the shape of the function. Being less certain of the application of their mathematical knowledge appeared to result in this pair taking a haphazard approach to their solution process.

2. *The more cooperative the pair, the greater their ability to make inroads into the problem.*

Both the experienced teacher pair and the less experienced student pair demonstrated a cooperative approach to the solving of the problem task. Both these pairs worked together throughout their attempt at the problem task, sharing and discussing their mathematical ideas, their graphing calculator knowledge, and their responses to the output of the graphing calculator. It can be inferred that the sharing of their knowledge and ideas contributed to their focussed approach to the solving of the problem. Although, due to circumstances beyond the control of this researcher, the solution of the less experienced student pair was stopped prematurely, their cooperative approach allowed them to access the task and make substantial progress efficiently. The experienced teacher pair, on the other hand, found a global view of the function and
demonstrated their understanding of how to use the graphing calculator features to identify each of the key features of the function.

In contrast, both the less experienced teacher pair and the more experienced student pair demonstrated less ability to work cooperatively and displayed a disjointed approach to solving the problem task. Although individuals in each pair made several sensible suggestions, it appeared that these were ignored by the other member of the pair. For example, one of the less experienced teacher pair suggested they Zoom Out and use TRACE, a method successfully followed by the experienced teacher pair, however the lack of response by her partner saw this pair follow a different and less successful solution path. The sensible suggestions or observations made by one member of this pair were generally ignored by the second teacher, who made few contributions to the dialogue, and little real discussion occurred. The experienced student pair undertook a similar path in that the ideas of one tended to be ignored by the other, however, both students contributed their ideas and observations. For instance, after pressing GRAPH followed by Zoom Out, this pair saw apparent vertical sections of the graph of the function in the viewing window, one section being coincident with the y axis, as shown in Figure 3.7(a). One of the students stated that the graph was a straight line, however, the other traced out the shape of a cubic function, having two turning points with his finger and suggested that the section they could see would “continue down”. After Zooming Out again they saw a view similar to that seen by the experienced teacher pair as shown in Figure 3.7(b). One of the students restated his belief that the sections of the graph were straight while the other student continued to insist that the visible sections would join. After the second student acted upon his own suggestion that they

![Figure 3.7. Apparently vertical sections of the graph, after Zooming Out once.](image)
should TRACE along one of the sections and “see where it ends”, they moved up a section of the graph and Zoomed Out repeatedly in order to keep the TRACE cursor in the viewing window. Both students then identified the three sections of the function that became visible in the viewing window. This, however, did not alter the stated beliefs of the respective students, one believing that there were three vertical lines and the other believing that the graph had two turning points even though these were still not visible in the viewing window. Their approach continued with each student taking turns to use the graphing calculator to pursue individual ideas.

The behaviour of both of these less cooperative pairs tended to be typified by one pair pursuing a particular solution path, followed by the second member trying an idea of their own. Little reflection or real discussion occurred as actions tended not to be informed by the results of their own or their partner’s previous actions.

3. Not one pair or individual believed that the standard window showed a global view of the graph of the function.

The experienced teacher pair was successful in their search for a global view and used a variety of calculator features to identify the key features of the function. It can be inferred from this that they understood what it meant to sketch completely. The less experienced teacher pair also did not accept the view shown in the standard window and found an almost global view (see Figure 3.8). Although the time available to complete the task expired before they could determine window settings that would allow a global view of the function to be seen, it can be inferred from their comments and actions that this pair also did not accept the initial view as being that of a global view of the function and they understood the terms global view and complete graph.

Figure 3.8. The almost global view found by the less experienced teacher pair.
It was inferred that none of the students believed that the initial view seen on the graphing calculator was a global view of the function. Neither pair accepted the initial view as they continued on to search for a global view of the function. For the experienced student pair, although unsuccessful in the time available in finding a global view of the function, it was apparent by their comments and actions that neither believed that the view seen on the graphing calculator was a global view of the function. For the less experienced student pair, behaviour that supports this inference included their linking of the algebraic representation of the function with the graphical view and their non-acceptance of the sections of the graph visible as being of vertical lines.

As this small study raised more questions than it answered, it was decided to develop this same problem task for the larger study described in this thesis.

### 3.10 The Problem Task and its Administration

The problem task involved producing a complete graph for a cubic function and is shown in Figure 3.9. The task solving session was audiotaped, with students asked to articulate their ideas as much as possible. Students used a graphing calculator, set to the standard window, attached to a viewscreen and overhead projector. The graphing calculator screen was then videotaped via the overhead projector and an accurate record of the students’ working recorded. The sessions were also audiotaped. In addition, the researcher took observational notes during the session. Students were provided with a written copy of the task (see Figure 3.9) and plain paper on which to record any working out, or notes, in order to complete a hand-sketched solution to the problem.

Work cooperatively, using the graphing calculator, to sketch completely the graph of
\[ y = x^3 - 19x^2 - 1992x - 92. \]
Show all important features.

*Figure 3.9. The problem task as presented to the student groups.*
3.10.1 Knowledge and Skills Required to Solve the Problem Task

To successfully undertake the problem task students need to understand the relationship between the three possible representations of a function, namely, algebraic, graphical, and numerical, and what information each provides about this function. They need to understand what constitutes a complete graphical representation of any function and the possible shapes of cubic functions. They also will be required to have a good working knowledge of the graphing calculator, in particular its features dedicated to altering the viewing window and those dedicated to identification of key features of a function in order to successfully complete the task. Any degree of success would demonstrate some of the students' understanding about functions and their representations. A solution to the Problem Task and further details of the mathematics involved is provided in Appendix D.

3.11 Summary

This chapter has detailed the purpose of the study as to explore how students use the graphing calculator whilst working together to solve a problem task and the rationale for the choice of methodology. The situational context, the students participating, the reason for their selection and their mathematical and graphing calculator knowledge and experience have been detailed. The pretest and the understandings demonstrated by the students participating in the study were detailed, as were the results of the students of the classes from which the participants were selected. The pretest results allowed (a) a clear picture of the students’ understandings to be determined, (b) comparisons to be made between the students participating in the study and the students in their classes, and (c) a determination to be made that the students selected would be able to access the problem task used in this study.

The reasons for the choice of a qualitative research methodology in order to study the complex phenomena of interest and specifically the choice of an instrumental case study
(Stake, 1995) are outlined. The data collection methods and analysis tools employed and the rationale for their selection and use are explained. The techniques used to ensure quality control aspects including reliability and validity are detailed and the positioning of this researcher in the research is discussed. The results of the pilot study are presented and discussed, showing that the area being researched was needing further investigation, particularly as some of the findings were at odds with the assertions of other researchers. In addition, it showed that the problem task used was appropriate for further study.

The next two chapters report the analysis of the students’ attempts at the problem task. Chapter 4 explains the macroscopic analysis including the construction of the case record from protocols for each student pair. The coding scheme developed by this researcher is illustrated and the episode diagrams and time-line diagrams based on this analysis are presented for each pair. In addition, the macroscopic analysis using these diagrammatic tools led to a number of defining moments becoming apparent in the solution processes of the pairs of students as they worked through the problem task. These defining moments are identified and documented. In chapter five the microscopic analysis of these defining moments is presented and discussed as this researcher sought to explain each of the defining moments, their use or lack of use by different pairs, and the potential impact they had on the solution process.
CHAPTER FOUR

RESULTS AND MACROSCOPIC ANALYSIS OF THE PROBLEM TASK

4.1. Overview

In this chapter the students’ responses to the problem task are examined and analysed at the macroscopic level in order to gain a greater insight into the processes students used when using a graphing calculator to solve a given problem. This insight was gained through a variety of methods. The graphing calculator screen was used as a window into the minds of the students. In conjunction with this, the dialogue of the students and their interactions were analysed. The focus was on understanding the kind of knowledge gained from learning about functions in the classroom in a graphing calculator environment and the extent to which this knowledge was exhibited in solving the particular problem task used in the study.

4.2. Task Solving Sessions

The five pairs of year 11 and 12 Mathematical Methods students attempted a problem task in videotaped sessions. The problem task involved producing a complete graph for a cubic function and is shown in Figure 4.1.

Work cooperatively, using the graphing calculator, to sketch completely the graph of
\[ y = x^3 - 19x^2 - 1992x - 92. \]
Show all important features.

*Figure 4.1.* The problem task as presented to the student pairs.

The task was undertaken in pairs with students expected to work collaboratively to solve the problem. One Texas Instruments TI-83 graphing calculator was provided for students to use. The teacher researcher selected the type of technology and the students were told to use the calculator as much as needed to best solve the problem.
4.3. Production of a Case Record: The Protocols

Using both audio and video recordings, the written responses of the students, supplemented by the researcher’s observational notes made during the problem task sessions, a case record consisting of protocols for each pair of students was produced (see Appendix C). The protocols consisted of transcripts of verbal exchanges augmented by observational notes, where necessary, matched with the screen shots of the graphing calculator and the written scripts of the students. Due to technical difficulties the video data for Pair 5 were only partly usable. As a result, some of the windows presented as a record of the students’ efforts are not their actual windows but rather similar windows based on their dialogue and the researcher’s notes. In all other transcripts the windows are the actual windows recorded.

4.4. Coding at the Macroscopic Level

4.4.1 The Activity Coding Scheme Used in this Study

Brief descriptions of the activity codes used in this thesis were presented in Table 3.6. More detailed descriptions with illustrations of each type are presented here to inform the reader of the interpretation by this researcher of her coding scheme. The following categories of activities were used to code the protocols: reading (R), organising or planning (O), selecting a viewing window (VW), searching for or identifying a local feature (ID), considering the global view (GV), adjusting scale marks (SM), evaluating (E), and recording (Re). Detailed descriptions of these categories follow. These activities and their codes are summarised in Table 4.1. In addition, the representations that were available to the students or were being used or considered by them were coded (i.e., as A = algebraic, G = graphical, or N = numerical). The information presented in brackets in the dialogue in the following sub-sections describes observed behaviour or adds detail necessary for the reader to make sense of the dialogue. Information presented in parentheses in sections of dialogue indicates missing information.
Table 4.1

Coding Categories for the Activities

<table>
<thead>
<tr>
<th>Code</th>
<th>Coding category</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>Reading</td>
<td>Reading or rereading the problem</td>
</tr>
<tr>
<td>O</td>
<td>Organising</td>
<td>Organising and planning</td>
</tr>
<tr>
<td>VW</td>
<td>Viewing window</td>
<td>Selecting a viewing window</td>
</tr>
<tr>
<td>ID</td>
<td>Identification</td>
<td>Searching for or identifying a local feature of the function</td>
</tr>
<tr>
<td>GV</td>
<td>Global view</td>
<td>Considering the global view of the function</td>
</tr>
<tr>
<td>SM</td>
<td>Scale marks</td>
<td>Adjusting scale marks</td>
</tr>
<tr>
<td>E</td>
<td>Evaluation</td>
<td>Evaluating either the graphing calculator output or the solution state</td>
</tr>
<tr>
<td>Re</td>
<td>Recording</td>
<td>Recording the solution</td>
</tr>
</tbody>
</table>

4.4.1.1 Reading (R)

This activity begins when the students begin to read the problem task. It includes reading the problem aloud, asking the meaning of words, stating important words, silent time following a reading of the problem, re-reading, and students reminding themselves about the task. It can result from consideration of where they are in the task solution path (as distinct from evaluation of their solution process). For example, Hao, in response to Abdi’s summation that they had everything they needed to sketch the graph, rereads the problem statement one more time to check that they have done what was required.

Hao (Pair 2): Yes, we can sketch.
What do we need to find [rereads problem statement]?

4.4.1.2 Organising and Planning (O)

This activity includes overt planning, however, it is noted that absence of such overt planning actions “does not necessarily indicate the absence of a plan” (Schoenfeld, 1985b, p. 300). Organising and planning include students’ verbalising their thinking
about what to do to solve the problem, consideration of what constitutes a solution, how to use the graphing calculator to do the current subtask, and suggesting strategies. This planning can be of a local or global nature. Local plans may have as their focus specific subtasks that are used to implement the global plan as shown in the following.

Ahmed (Pair 1): First of all we have to find the \( x \) intercepts. We should use Equation Solver [MATH: Solver, a function of the graphing calculator being used that allows the identification of zeroes of a given function].

In other instances, local plans form part of an exploratory phase in search of a global plan.

Ahmed: Firstly, what we should do is get an idea of what the graph looks like.

Sometimes, a statement of planning may be incomplete and/or will not always be followed.

Linh (Pair 1): Can we use…

4.4.1.3 Selecting a Viewing Window (VW)

This activity involves using, or suggesting the use of, the ZOOM menu items, or adjusting any of the items in the WINDOW settings, or any discussion or statement about the range or domain of the viewing window. This activity occurs in the search for local or key features and during the search for a global view of the function. The first option in the ZOOM menu is Zoom Standard which selects the WINDOW settings of \(-10 \leq x \leq 10, x \text{ scale } = 1, -10 \leq y \leq 10, \text{ and } y \text{ scale } = 1\). For instance, Ahmed suggests to Linh, from his initial inspection of the equation of the function, that the standard window will not be appropriate for a global view of the function. However, when their use of the GRAPH facility produces the standard window, no comment is made about the lack of any visible part of the function, perhaps, because that is what was expected.

Ahmed: Graph it, I don’t think we should use Zoom Standard. [They press GRAPH.]
Linh: What's our window?
Ahmed: It's the standard window.

Also included in the ZOOM menu is Zoom Out, where the default setting of a factor of four in both the horizontal and vertical direction occurs. Zoom In reverses the effects
of Zoom Out. Alternatively, Zoom Fit can be selected, in order to have the calculator
determine an appropriate viewing range given the current viewing domain. The effects
of Ahmed and Linh’s use of Zoom Fit on the view of the graph and the settings of the
viewing window are shown in Figure 4.2. In addition, their behaviour of altering only
two values at a time reflects the behaviour modelled by their teachers.

Ahmed: I think we have to Zoom Out.
Linh: What’s that?
Ahmed: Zoom Fit, it fits all the points.
Ahmed: Yes, we’ll Zoom Fit.

Figure 4.2. The effect of Zoom Fit on the standard window view of the function.

Alternatively, values in the WINDOW menu, namely, x minimum, x maximum, y
minimum, and y maximum, can be altered to find a suitable viewing window. Adjusting
these values directly alters the viewing domain and/or range displayed by the graphing
calculator giving the user more control of the screen output. Abdi and Hao, for example,
alter the viewing range in order to locate the turning points in the viewing window as
seen in Figure 4.3. They consider several values before they actually press GRAPH.

Abdi (Pair 2): Just change two at a time. Make it...
Hao: We just need to find the turning points.
Abdi: The turning points.
Hao: It’s okay now, still …
Abdi: Change the minimum again, make it 500.
Hao: 600?
Abdi: 20000, not 2000.

Figure 4.3. Altering the viewing range via the WINDOW settings in order to view the
turning points of the function.
Altering the viewing window also occurs when students strive to improve their global view of the function (demonstrated by Pair 1 in Figure 4.4), or when they are focussing their attention on a specific key feature (demonstrated by Pair 5 in Figure 4.5).

![Figure 4.4. The effect of altering WINDOW values on the function.](image)

![Figure 4.5. Using Zoom Box (twice) to select a viewing window in order to view an x intercept.](image)

**4.4.1.4 Searching for, or Identifying, a Local Feature (ID)**

This activity involved students specifically searching for and/or identifying the coordinates of a key feature of the function. This includes stating, using, or attempting to use dedicated features such as Zero from the CALCULATE menu, or using TRACE or the free cursor or, lastly, stating a local feature exists. In this first example, Ahmed uses the in-built equation solver (MATH: Solver) to find the central $x$ intercept.

Ahmed: First of all we have to find the $x$ intercepts. We should use Equation Solver.
Linh: What?
Ahmed: Equation solver. Use Solve(r), …
Must be greater than 10, [enters 10 as estimate of an $x$ intercept] left bound, …
right bound. That is one of our solutions.

They later recalculate this central $x$ intercept using the Zero feature from the CALCULATE menu and then continue to use it to find the left $x$ intercept:

Ahmed: Left bound, right bound, guess [enters $x = 0$].
Linh: .046.
Ahmed: Negative .046. Another $x$ intercept would be at something around (negative) 30.

Other dedicated features from the CALCULATE menu that can prove useful in searching for a local feature of the graph are the Value feature for finding the $y$ intercept and the Maximum or Minimum features for finding turning points.
4.4.1.5 Considering the Global View of the Graph of the Function (GV)

An activity could only be classified in this way where this could be clearly distinguished from selecting a viewing window. The purpose of this activity was to obtain a view of what the global view of the graph of the function looks like, and, includes discussion of the global view of the graph and physically identifying what the possible shape of the graph is. As a result, the activity is a cognitive, rather than physical, interaction with the calculator. It is defined as discussing or considering the shape of the graph. This was undertaken in order to facilitate students’ search for a global view. Only one pair, Kate and Pete, undertook behaviour that could be clearly classified as this.

Pete (Pair 3): It will be one of these ones I reckon, like that [tracing a cubic with two turning points in the air].

And shortly after:

Kate (Pair 3): Yeah.
Pete: No, it’s still straight, … that can’t be linear though, it’s got …
Kate: No, it’s not linear. It’s cubic.
Pete: That’s what I mean.
Kate: But no, wait, wait, wait, wait, … we’ve got …
Pete: Remember how we did… those graphs where we got cubic?
Kate: Yeah.
Pete: Squared and that.
Pete: Yeah.
Kate: What do they look like?
Pete: A cubic is like that [At this point, Pete refers to his previous sketch from their earlier discussion of the possible shape of the graph ], I reckon it’s one of those ones. But that looks wrong because …
Kate: One of them ones, all right.

4.4.1.6 Adjusting Scale Marks (SM)

An activity was categorised as adjusting scale marks if it involved adjusting the scale values in the WINDOW settings. In this example, Linh and Ahmed adjust the scale marks on both axes in order to facilitate their finding of the x intercepts. This dialogue follows overt planning by Ahmed as noted by his statement that they still have to fix up the scale.

Linh: 100 [talking about the y scale].
Ahmed: 5000 because of the large x values, and x scale of 10.
Ahmed: (Inaudible) then we have to find the x intercepts.
4.4.1.7 Evaluating (E)

Evaluating involves students comparing the graphing calculator output to what they were expecting, evaluating the state or completeness of their solution, or evaluating their recording. In this first example, Ahmed and Linh focus on the adequacy of the view shown by the graphing calculator.

Ahmed: That is one of our solutions.
Ahmed: Still we can’t see the whole thing.
Linh: What's our x scale?
Ahmed: I think if I was going to sketch it I’d make a bigger scale on the x axis.

Alternatively, the focus could be on an evaluation of the state of the solution. This next example shows the difficulties Ahmed and Linh had producing a sketch of the graphical representation of the solution with which they were satisfied.

Ahmed: The x intercept [(-0.046,0)] … on the graph it would look close to the origin.
Ahmed: Because the x scale is too big, you won't really be able to see that x intercept.
Linh: Just put it on the same point.
Ahmed: The graph will appear to pass through the origin.

4.4.1.8 Recording (Re)

The final type of activity is the recording of the solution. This activity is often not apparent merely from a reading of the protocols. The observational notes were also used to decide when this occurred, being when students were observed sketching or writing on the paper provided or making comments as to the actual recording of the solution. This activity involves such actions as sketching the graph, writing down key features, and discussing the recording of the students’ solution. The following example clearly indicates the recording of Ahmed and Linh’s solution including their discussion of scale to facilitate the recording of their solution.

Ahmed: Use the scale. The scale is ten thousand, so twenty thousand is then two units.
Linh: Is it ten thousand?
Ahmed: Yes.
Linh: Minimum, -32, so around here somewhere.
Ahmed: Five divisions, the same scale. It’s too much. Just five steps.

4.4.2 Parsing the Protocols into Episodes

Following the coding of the protocols using the activity categories just described, the protocols were divided into episodes and transition points between these involving overt
statements or actions by the students were noted. An episode is taken by this researcher to mean a period of time where the students are engaged in one large task or a series of closely related tasks in the pursuit of some goal. The end of an episode occurs when there is a clear change in direction. The protocol is parsed into these distinct sections by considering the behaviour of the students as evidenced by their words, observed actions, and the view screen output that resulted from their thinking. The points between episodes are classified as transition points. These may be statements or actions by the students indicating a change of direction or simply a point in time marking the end of one episode and the beginning of the next. Episodes are, hence, classified according to categories related to their specific purpose in the solution of the problem task such as observing the function in the standard window, searching for key features, or searching for a complete graph.

4.5. **Macroscopic analysis of the Case Record**

4.5.1 *The Episode Diagrams*

The parsing of the protocols into episodes provides an overview of the activities undertaken by each pair as they proceeded to solve the problem task. Episode diagrams present a diagrammatic representation of this analysis, and include the number of episodes undertaken, the length of each episode (by time and item number, that is, number of dialogue exchanges), and provide a description of the focus of the activity of the pairs during each episode. In addition, the labels of the episodes include the main features of the graphing calculator used by the students during the episode. A diagram of the episode parsing is presented and discussed for each pair of students followed by a comparison of these for all pairs.

4.5.1.1 *Pair 1 and their Episode Structure*

The protocol of Linh and Ahmed was divided into six main parts or episodes, each of different length and with a different focus, as shown in Figure 4.6. Initially, they read
the problem. Secondly, they inspected the function in the standard window, contrary to what might be expected from their stated opinion of this view as not being appropriate. Thirdly, they returned to the algebraic representation to use MATH: Solver to find the one $x$ intercept that they knew must exist. Next, they undertook a successful and efficient search for a global view of the function using their understanding of the scale values to enhance this. This was followed by their organised and accurate identification of all key features. Finally, they recorded their successful solution to the problem task.

Linh and Ahmed took the shortest time of the pairs to solve the task (10 minutes) and engaged in the least number of dialogue exchanges (92) of the four pairs for which this was recorded.

**Figure 4.6.** Parsing of the protocol for Pair 1 into episodes of consistent behaviour.

### 4.5.1.2 Pair 2 and their Episode Structure

The protocol of Abdi and Hao was divided into six episodes where episodes 4 and 5 are a re-iteration of the focus activities of episodes 2 and 3, respectively, as shown in Figure 4.7. They found a global view of the function within two minutes and then proceeded to identify key features. When beginning to identify the $x$ intercepts they
Figure 4.7. Parsing of the protocol for Pair 2 into episodes of consistent behaviour.

decided they needed a better global view of the function and hence they made a series of WINDOW settings adjustments, all but one of these altering only the viewing range. When the view of the graph of the function was to their satisfaction, they returned to identifying key features. Finally, they recorded their solution. Pair 2, although taking four minutes and 39 dialogue exchanges more than Pair 1, took 12 minutes less time than the quickest Year 11 pair.

4.5.1.3 Pair 3 and their Episode Structure

Kate and Pete undertook nine different episodes and took the longest time overall (29 minutes 30 seconds) to complete the problem task, as shown in Figure 4.8. They took about five minutes longer than the other Year 11 pairs and double the time required by the slower of the Year 12 pairs. They were the only pair who had an episode entirely of discussion. The level of discussion by this pair throughout the task was illuminating. In addition, they were also the only pair to have an evaluation episode. Whilst other pairs did undertake evaluation this occurred simultaneously with other actions, whereas Pair
3 spent a period of time where the major focus was on discussion of their solution. Kate and Pete followed a similar pattern to Pair 2 in that, after beginning with an episode of planning and reading, they had three episodes where they searched for a complete graph, (i.e., episodes 2, 4, and 7), followed by three episodes of searching for, and/or, identifying specific features. Similarly to Abdi and Hao’s solution, there was a distinct episode of recording of the solution. Finally, they concluded with an episode where they undertook an evaluation of their solution.

### 4.5.1.4 Pair 4 and their Episode Structure

The solution activity of Reem and Ali was divided into fourteen episodes, the largest number of the five pairs. It took twenty four and a half minutes, as shown in Figure 4.9. As a result of the limited conversation of Pair 4 and technical difficulties, the number of dialogue exchanges was not recorded. Six of their episodes involved using WINDOW to search, in four cases for a global view and in one case each for the two

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**Figure 4.8. Parsing of the protocol for Pair 3 into episodes of consistent behaviour.**
Figure 4.9. Parsing of the protocol for Pair 4 into episodes of consistent behaviour.

turning points of the function. All searching episodes, except one, were followed by the identification or attempted identification of a key feature of the function. The one exception was followed by an adjustment of the scale marks. This pair did not have a distinct episode of recording their solution. This is partly accounted for by their solution to the task not including a sketch showing identified key features. Instead, their solution consisted of a sketch with key values listed next to it. These key values were recorded during episodes 7, 9, 11, and 13 where the major focus was on identifying key features rather than on the recording of their solution. As a convention of the coding scheme used, the episodes are labelled according to the major focus. The much larger number of
episodes undertaken by Reem and Ali is a reflection of their lack of discussion, their apparent reluctance to use graphing calculator features other than TRACE to identify key features of the function, and their reliance on making adjustments to the WINDOW settings in their search for a global view of the function. All of these contributed to their viewing a much larger number of windows than the other four pairs including more possible challenging views of the function.

4.5.1.5 Pair 5 and their Episode Structure

Figure 4.10 shows that Jing and Susan undertook 12 episodes in their solution of the problem task. They took a similar time to complete the task to Pair 4, being approximately double the time taken by the Year 12 pairs. This pair recorded their solution throughout the solution path, although mainly in the last episode. They did not begin the process of identifying key features until they had a complete view of the graph of the function. Pair 5 undertook three episodes (2, 4, and 6) of searching for a complete graph. Whilst the first of these resulted in an almost global view, the potential to achieve a global view was not fulfilled during this episode as Jing and Susan decided to change approaches and search for a factor of the function as an alternative solution path. When they returned to searching for a global view of the function, their first action resulted in their almost global view of the function being lost. Following this second episode of searching for a global view of the function (i.e., episode 4), Pair 5 returned to the algebraic representation in order to check they had correctly entered the function and proceeded to look for inconsistencies between the two representations under consideration, specifically the graphical and algebraic, as they considered the apparent linearity of the section of the graph visible in the viewing window. Episode 5 saw these students return to the task of searching for a global view of the function, and they succeeded within five minutes after making a series of considered alterations to the
**Figure 4.10.** Parsing of the protocol for Pair 5 into episodes of consistent behaviour.

**WINDOW settings.** Next, Jing and Susan began recording their solution script, using the CALCULATE menu and Value to identify the y intercept of the function. Episode 7 began with an attempt to identify the coordinates of the minimum turning point. Initially, they correctly used CALCULATE: Minimum to do so, however, the overlapping of the graph with the solution output of the calculator meant this could not be read clearly. They sensibly adjusted the WINDOW settings to overcome this situation and repeated their use of CALCULATE: Minimum. Unfortunately, on this occasion, failure to include the turning point in the narrow search domain resulted in one endpoint of this domain being recorded as the minimum turning point. The narrowness of their search domain prevented Jing and Susan making a visual check of their solution as being an actual turning point of the function. They then proceeded to
make a second inaccurate identification of a key feature as they identified the coordinates of the maximum turning point, for some inexplicable reason, using TRACE. During Episode 10, Pair 5 returned to the CALCULATE menu and accurately determined the coordinates of the left and right x intercepts using Zero. For the central x intercept, Jing and Susan followed a plan discussed during the previous episode using Zoom Box to zoom in to the central x intercept and accurately determine its coordinates using Zero. Finally, in their concluding episode they finished their solution sketch and checked that they had recorded a complete graph of the function.

### 4.5.1.6 Comparing the Episode Structures of the Five Pairs

From the number and length of the episodes it can be inferred that Pair 1 was the most decisive and purposeful of the five pairs. Although undertaking the same number of episodes as Pair 1, albeit over a longer period of time, Pair 2 was less purposeful as illustrated by their behaviour in moving from a global view of the function to a search for key features that resulted in their needing to re-find a global view of the function. For both Pairs 1 and 2 the longest episode involved searching for and identifying key features of the function. Pair 3 took the longest time to complete the problem task. Similarly to Pairs 1 and 2, the longest episode for Pair 3 involved determining key features of the function, which was undertaken in conjunction with recording of the solution. In addition to their longest episode of almost seven minutes, this pair undertook three other episodes over five minutes in length, all of which involved searching for a global view of the function. Pair 3 exhibited behaviours that fortuitously spiralled into a solution rather than being purposeful, as evidenced by their three episodes of searching for a global view interspersed with behaviour and actions having a more local or specific focus. Pairs 1, 2, 3, and 5 all had their longest episode as their penultimate episode. Pair 4, with the largest number of episodes, exhibited the least purposeful behaviour of the five pairs, as suggested by their limited use of the dedicated
features of the graphing calculator, the use of which would have enabled a more
effective and efficient solution process. The longest episode of Pair 4 involved the
search for a global view of the function, and this occurred during their fourth episode.
They undertook four episodes in total that had their focus on finding a global view of
the function. Similarly to Pair 3, Pair 5 also demonstrated behaviour spiralling into a
solution as they undertook three episodes involving searching for a global view of the
function. Pair 5, unlike Pair 3, however, had an almost global view within five minutes
although they took an additional ten minutes to find an actual global view of the
function on the graphing calculator screen. The penultimate episode of Pair 5 involved
using Zoom Box (the only pair to use this menu item) to identify the coordinates of the
central x intercept, demonstrating their need (similarly to Pair 2) for a view of this local
feature, clearly distinguishable from the origin, before they were satisfied and prepared
to proceed to identify its coordinates.

4.5.2 The Time-line Diagrams

A time-line diagram is used to provide a visual representation of the parsing of a
protocol, both into activities and episodes. The time-line diagrams record the detail of
the episodes and show the progress of the solution as the students proceed. The first
column of the time-line diagram identifies the various types of activities as listed in
Table 4.1, and also includes overt transition points and the representation of the function
being used by the pair of students. A time-line is given at the bottom of the diagram as
alternating black and white rectangles each representing 10 seconds. The time-line
diagrams for each pair are presented followed by a comparison of these for all pairs.

4.5.2.1 The Time-line Diagram for Pair 1

Pair 1 began the problem task working in the algebraic representation undertaking a
single focus episode of reading, as shown in Figure 4.11. The second episode saw Linh
and Ahmed begin working in the graphical representation as they observed the function
in the standard window as they undertook planning in conjunction with selection of a viewing window. Returning to the algebraic representation, Pair 1 engaged in both planning and identification of key features during their third episode as they specifically used the calculator features to identify the one \( x \) intercept they knew must exist. Linh and Ahmed began their fourth episode by switching back to the graphical representation. They selected viewing windows, simultaneously evaluating the output of the calculator as they successfully searched for a global view of the function. Once this view was found, brief overt organisation in the form of discussion as to the fact that only part of the graph was currently visible was followed by their adjustment of the scale marks. This facilitated their later sketching, identification, and recording of the key features of the graph. During the first part of their penultimate episode, Pair 1 focused on the identification of key features, working solely in the graphical representation. During the second part of this episode they worked simultaneously in the

**Figure 4.11.** The time-line diagram for Linh and Ahmed.
algebraic and graphical representations continuing to identify key features however, this was interspersed with other activities, namely, organising, recording, reading, and evaluating. The final episode focused on their written recording of their solution to the problem task and making modifications to their viewing window which were preceded and followed by evaluation of the window allowing the students to more easily record their solution. The final episode and the problem task concluded with Linh and Ahmed evaluating their solution state.

4.5.2.2 The Time-line Diagram for Pair 2

Abdi and Hao began their first episode with the simultaneous activities of reading and organising as they worked in the algebraic representation and then continued reading as they began working in the graphical representation, as shown in Figure 4.12. Episode 2 saw Abdi and Hao continue to work mainly in the graphical representation alternating between the activities of organisation, identification of key features, selection of viewing windows, and evaluation as they undertook a successful search for a global view of the function. During the third episode this pair continued working solely in the graphical representation as they began their identification of key features. As the third episode ended, a transition point occurred between episodes when Hao suggested they “should change the domain”. During episode 4 this pair continued in the graphical representation as they made a series of adjustments to the viewing window in their second search for a global view of the function. Twice during this episode they undertook an evaluation of the state of their solution. The episode concluded when they (re)found a global view of the function, on this occasion, however, one that better enabled them to identify key features of the function. Episode 5 saw Abdi and Hao begin the identification of the key features of the graph, initially with the y intercept and using both the algebraic and graphical representations to do so. From this point forward
they remained working solely in the graphical representation. During this final episode the students undertook evaluation several times as they identified all the key features of the function. The focus of this episode was on the recording of their solution but this was substantially undertaken simultaneously with the evaluation of both the graphing calculator output and their solution.

4.5.2.3 The Time-line Diagram for Pair 3

Figure 4.13 indicates that Pete and Kate began in the algebraic representation planning their approach to the task and then moved to the algebraic and graphical representation as they alternated between reading and organising. Pete and Kate’s identification of the y intercept brought their first episode to a conclusion. All this was completed without the use of the graphing calculator. A transition point occurred as
Figure 4.13. Time-line diagram for Kate and Pete.
they decided to make use of the graphing calculator. They began their second episode by entering the function into the calculator and then continued to work in both the algebraic and graphical representations as they searched for a complete view of the graph of the function. The majority of this episode consisted of the students undertaking short activities including organising, evaluating, selecting, and adjusting the viewing window, and searching for a global view, all of which contributed to their overall focus of finding a global view of the function. A second distinct transition point occurred when after almost 6 minutes and little success occurring in their search for a global view of the function, Pete and Kate decided to change direction and zoom in using Zoom Box on the one key feature they had finally found. Their third episode was the first of three episodes to consist of a single activity, in this instance being the identification of a key feature. Episode 4 began with a brief consideration of working in the numerical representation, however this idea was rejected, and then the pair mainly worked in the graphical representation until in the latter half of this episode they began to work simultaneously in the algebraic and graphical representations followed by a concluding period in the graphical representation. The focus during this episode was again the search for a global view of the function, mainly undertaken by adjusting the viewing window; however, the episode began with planning and as the episode progressed Kate and Pete also undertook evaluation in conjunction with adjusting the viewing window. This episode concluded as they determined the coordinates of the minimum turning point they had found in their viewing window using TRACE. Their short fifth episode was their second single activity episode consisting of organisation and planning as they discussed the possibility of the function having a point of inflection. During this episode the students worked simultaneously in the algebraic and graphical representations. Episode 6 saw a return to the graphical representation, where the students remained working until the conclusion of the task. During this episode, after
some adjustment to the viewing window, they used the dedicated features of the graphing calculator to identify the coordinates of the minimum turning point, the values of which they estimated with TRACE at the conclusion of episode 4. Episode 7 saw Pete and Kate return for the third time to search for a global view of the function. This time, however, their dialogue clearly showed that their discussion and reflection on the section of the graph visible in the viewing window and its possible position relative to their mental image of the complete graph allowed them to successfully find the global view of the function. This episode included an adjustment to the scale marks, with both $x$ scale and $y$ scale being set to zero. During this episode Pair 3 used TRACE to identify the coordinates of the maximum turning point. Episode 8 began with Pete and Kate simultaneously evaluating and identifying a key feature of the function, namely the $y$ intercept. They then began the careful process of recording their written solution. Once they were satisfied with the shape of their sketch and the labelling of the $y$ intercept and both turning points, they then set about determining the coordinates of the remaining key features, recording these as they were identified. The final episode, the third single activity episode, involved a global evaluation of their solution.

**4.5.2.4 The Time-line Diagram for Pair 4**

No organisation and planning could be inferred from the actions of Reem and Ali. In addition, their minimal dialogue as compared to all other pairs made it difficult to infer any planning from their discussion. Figure 4.14 shows they began in the algebraic representation by reading. This single activity episode was typical of the actions of this pair as all except one of their episodes followed this pattern. During their second episode, they began searching for a global view of the function by altering the WINDOW settings. This episode began with the students using both the algebraic and graphical representations and then moving to only the graphical representation which they continued in for the remainder of the problem task.
Figure 4.14. Time-line diagram for Reem and Ali.
During episode 3, Reem and Ali made several alterations to the scale marks, none of which assisted in their search for a global view of the function. In addition, their adjustment to the scale marks suggested their lack of understanding of the effect of this action. Episode 4 involved a relatively unsuccessful sequence of alterations to the WINDOW settings.

A brief change of direction occurred during episode 5 as the pair used an inappropriate dedicated feature of the graphing calculator to attempt to determine a key feature of the function. The use of Calculate Intersect was never going to eventuate in the successful determination of the coordinates of the $x$ intercept. As Reem pointed out “Don’t, there are not two graphs”.

Episode 6 was a repeat of episode 4 as Reem and Ali continued to make adjustments to the WINDOW settings in their search for a global view of the function. During the brief episode 7, Pair 4 used TRACE to identify the $y$ intercept of the function. This was followed by episode 8 which saw behaviour similar to that of episodes 2, 4, and 6 as they altered the WINDOW values in their search for a global view of the function. The ninth episode saw Reem and Ali again use TRACE to identify key features, on this occasion the left and right $x$ intercepts. During episode 10 the pair demonstrated purposeful adjustment of the WINDOW settings as they searched successfully for the minimum turning point. This was followed by an episode where TRACE was used, on this occasion to identify the minimum turning point. Episode 12 was a repeat of episode 10, this time purposefully searching for the maximum turning point, followed by episode 13, where TRACE was used to identify the maximum turning point. Finally, Reem and Ali concluded their problem task as they made last adjustments to the viewing window and recorded the final details of their solution.
4.5.2.5 The Time-line Diagram for Pair 5

Jing and Susan began the problem task with an episode of reading undertaken in the algebraic representation (see Figure 4.15). For the remainder of the task, they operated in either the graphical representation or a combination of the algebraic and graphical representations, with the graphical representation being the case in the majority of episodes. The second episode saw Jing and Susan almost successful in searching for a complete graph. They began by planning and then undertook various adjustments to the viewing window interspersed with evaluation of the effects of their actions. Their third episode began with a change of direction as they attempted to identify zeroes of the function in order to determine the \( x \) intercepts of the graphical representation of the function. After evaluating their efforts, episode 4 began with a return to the graphical representation as they resumed their search for a global view of the function. During this episode a conflict between the graphical representation and the prior mathematical knowledge of the students occurred as apparent vertical sections of the cubic function became visible in the viewing window. This resulted in their rereading of the problem task as they began episode 5 and reflection upon apparent inconsistencies between the algebraic and graphical representations of the function. Episode 6 saw the students return for the third time to search, this time successfully, for a global view of the function. During episode 7, Jing and Susan began their recording of the solution. Unlike with the other four pairs, recording followed immediately finding a global view of the function. The identification of the minimum turning point, the subsequent adjustment of the viewing window to facilitate this, and the addition of this information to their written record occurred during episode 8. The coordinates of the maximum turning point were identified in episode 9. Episode 10 involved over five minutes spent identifying the right \( x \) intercept of the function and concluded with evaluation of the graphing calculator output. Before identifying the final key feature, this pair used Zoom
Figure 4.15. Time-line diagram for Jing and Susan.
Box to determine an improved view of the central intercept during their eleventh episode. Finally, Jing and Susan undertook their only episode where two activities were undertaken simultaneously, these being evaluation and recording.

4.5.2.6 Comparing the Time-line Diagrams of the Five Pairs

The episode diagrams draw the reader’s attention to the individual activities undertaken by each pair and specifically focussed on the relationship between a given episode and the preceding and following episodes. The time-line diagrams, however, allow the reader to focus more globally on the episodes collectively. Obvious differences included the length of time to solution of the task and the directness of the route to the final solution. The Year 11 students engaged in many more episodes where they were engaged in one large task or a series of closely related tasks in the pursuit of some goal in attempting to find a complete graph of the function. This behaviour was markedly different from that of the Year 12 students who were more likely to undertake multiple activities within a single episode. Some of these differences could be attributable to differences in cognitive and metacognitive actions by the pairs in supporting their use of mathematical and technological knowledge to solve the problem task. Other differences occurred in use of particular representations, use of scale marks, and how activities were sequenced.

4.5.2.6(a) Cognitive and metacognitive action during the solution process

There were several places where cognitive action could have been enhanced by metacognitive activity. All pairs began with overt reading and/or planning, however, only some returned to these activities during the remainder of their solution. Only a brief time was devoted to reading ranging from 15 to 45 seconds in total as shown in Table 4.2 where the time spent by each pair on each activity or simultaneously in two activities is recorded. Reading was predominately cognitive. However, when rereading
Table 4.2

*Amount And Proportion of Time Spent Undertaking each Type of Activity by Pair*

<table>
<thead>
<tr>
<th>Activity</th>
<th>Time in minutes and seconds per activity (% of total time taken)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pair 1</td>
</tr>
<tr>
<td>R</td>
<td>0:25 (4)</td>
</tr>
<tr>
<td>O</td>
<td>1:05 (11)</td>
</tr>
<tr>
<td>VW</td>
<td>1:50 (18)</td>
</tr>
<tr>
<td>GV</td>
<td>0:00 (0)</td>
</tr>
<tr>
<td>SM</td>
<td>0:30 (5)</td>
</tr>
<tr>
<td>ID</td>
<td>3:40 (37)</td>
</tr>
<tr>
<td>E</td>
<td>2:10 (22)</td>
</tr>
<tr>
<td>Re</td>
<td>1:35 (16)</td>
</tr>
<tr>
<td>Total</td>
<td>10:00</td>
</tr>
</tbody>
</table>

*Note.* R = reading; O = organising or planning; VW = selecting a viewing window; ID = searching for or identifying a local feature; GV = considering the global view; SM = adjusting scale marks; E = evaluating; Re = recording. a Includes 10 s of simultaneous R and O. b Includes 10 s of simultaneous O and E. c Includes 5 s of simultaneous O and E. d Includes 10 s of simultaneous V and E. e Includes 1 min of simultaneous VW and E. f Includes 1:50 min of simultaneous VW and E. g Includes 2:00 min of simultaneous Re and VW. h Includes 5 s of simultaneous ID and E. i Includes 35 s of simultaneous ID and E. j Includes 10 s of simultaneous ID and E. k Includes 1:15 min of simultaneous Re and E. l Includes 10 s of simultaneous transition and E. m Includes 1:50 min of simultaneous Re and E. n Includes 55 s of simultaneous E and Re.

occurred there was an opportunity for metacognitive activity to occur as rereading could be accompanied by evaluation of the current or final solution state. Rereading by Pair 1, for example, was followed by the question: “All the important features, is that all?” which initiated a short period of evaluation of organisation and planning as the adequacy of interim results was assessed against the problem parameters. Even though this type of metacognitive activity was a possibility, not much time was allocated to it by the pairs.

The activity classified as organisation and planning was predominantly metacognitive. As shown in Table 4.3, it involved identification of the overall goal and subgoals for subtasks, local planning to implement a global plan, local
Table 4.3

*Number of Protocol Events Coded as Organisation per Pair*

<table>
<thead>
<tr>
<th>Organisation activities</th>
<th>Pair 1</th>
<th>Pair 2</th>
<th>Pair 3</th>
<th>Pair 4</th>
<th>Pair 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Explicit identification of goal of task</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Identification of subgoals</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>Local planning to implement global plans</td>
<td>3</td>
<td>1</td>
<td>4</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>Local planning in search of a global plan (exploratory phase)</td>
<td>0</td>
<td>0</td>
<td>7</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Allocating responsibilities</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>7</strong></td>
<td><strong>3</strong></td>
<td><strong>14</strong></td>
<td><strong>1</strong></td>
<td><strong>14</strong></td>
</tr>
</tbody>
</table>

planning in search of a global plan (during an exploratory phase), and allocating responsibilities. Only one pair, Pair 2, explicitly identified the goal, however, all pairs, except Pair 4, identified subgoals. Ahmed, of Pair 1, for example, after his partner reread the problem statement and questioned whether they had already reached their goal, pointed out that they had still to determine the turning points, a subgoal in pursuit of the overall goal of finding a complete graph. Global planning were not verbalised by any pair although it could be inferred by the directness and orderliness of students’ execution of their solution that they were following a global plan, if not for the whole solution then at least for those parts where they had a clear idea of how to proceed. All pairs, except Pair 4, undertook local planning to implement global plans, however, the actions of the Year 11 pairs differed from those of the Year 12 pairs in that they all appeared to engage in periods of exploration. Pair 3 in particular, formulated several local plans in search of a global plan during various periods of exploration. Only Pairs 3 and 5, explicitly allocated responsibilities such as who would record.

Pair 4 engaged in little discussion, and consequently minimal organisation and planning behaviour. There was a distinct difference between the Year 12 pairs and the...
remaining Year 11 pairs. The Year 12 pairs engaged in fewer organisational activities, seven and three for Pairs 1 and 2 as opposed to fourteen each for Pairs 3 and 5, respectively. However, as shown in Table 4.3, Pair 1 and Pair 5 used a similar pattern of concentration of activities, with identification of subgoals (4 and 5, respectively) and local planning to implement global plans (3 and 6, respectively) being predominant. On the other hand, Pair 2 undertook little organisation and planning.

Evaluation activities involved evaluation of organisation and planning, evaluation of execution, evaluation of the state of the final solution, and verification that the final solution was complete, all of which are metacognitive. Where an evaluative activity was undertaken simultaneously with another activity its purpose could be inferred as evaluating the particular activity, however, where the activity of evaluation was undertaken in isolation then the preceding and following activities were used to identify the specific evaluation activity being undertaken. In addition, the dialogue undertaken by the students during the evaluation activities further facilitated this analysis.

There were seventeen occasions where evaluation of organising and planning occurred. These evaluations were almost evenly divided amongst the pairs (see Table 4.3), with Pairs 1, 2, and 3 accounting for four each, Pair 5 for three, and Pair 4 for two. These evaluations consisted of choosing a particular strategy or local plan after evaluating the calculator output or solution so far (7), evaluation of their current position in their solution path (4), evaluating the adequacy of results so far against the problem parameters (3), evaluating usefulness of a chosen strategy for the current plan (2), checking progress of a local plan (1), and stating the current subgoal (1). Evaluation of execution occurred on twenty occasions. This activity was predominately engaged in by Pair 3 (7), Pair 5 (6), and Pair 2 (4). These evaluations involved evaluation of results of local actions (18) and evaluation of procedures (2). Evaluation of the state of the solution occurred on six occasions and there were three instances of verification of the
completeness of a solution. Pair 4 and Pair 5 did not evaluate the state of their solution or verify its completeness. Pair 3, on the other hand, evaluated the elegance of the state of their solution three times and verified its completeness twice.

The undertaking of evaluation of verification activities, which involved mainly metacognitive actions, varied both in the proportion undertaken and in their purpose across the pairs (see Table 4.4). The Year 12 pairs did not differ markedly with Pair 1 engaging in eight evaluations whereas Pair 2 used ten. Of the Year 11 pairs, Pair 3 engaged in by far the most evaluations (17) with nearly half of these being evaluation of execution of local actions often involving correction of errors related to the viewing window settings or the identification of key features. These errors were detected and rectified. They also undertook several evaluations of the state of their final solution and its completeness. This was in stark contrast to the other pairs. Pair 4 undertook little evaluative activity whereas Pair 5 followed a pattern of frequent sharp evaluative bursts rather than extended periods of time with the exception of their final evaluative period in conjunction with recording of their solution. Most of these evaluations engaged in by Pair 5 were evaluations of execution.

The activities searching for a viewing window, considering the global view, adjusting the scale marks, searching for or identifying key features, and recording the Table 4.4

<table>
<thead>
<tr>
<th>Evaluation activities</th>
<th>Pair 1</th>
<th>Pair 2</th>
<th>Pair 3</th>
<th>Pair 4</th>
<th>Pair 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluation of organisation and planning</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Evaluation of execution</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>Evaluation of state of solution</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Verification that solution is correct/complete</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>8</strong></td>
<td><strong>10</strong></td>
<td><strong>17</strong></td>
<td><strong>3</strong></td>
<td><strong>9</strong></td>
</tr>
</tbody>
</table>
solution were classified as execution activities for this problem task. These execution activities were predominantly cognitive. There were opportunities, however, for metacognitive activity to be associated with execution. As can be seen in Table 4.2 on several occasions execution occurred concurrently with evaluation. At other times brief evaluative interludes were interspersed throughout the execution activities.

4.5.2.6(b) Use of the representations by the pairs

A comparison of the time-line diagrams also revealed that, as expected, all pairs mainly used the graphical representation. The algebraic representation was used by all pairs, usually in the earlier part of their solution either by itself or in conjunction with the graphical representation. Only one pair used or considered using the numerical representation (unlike the participants in the pilot study). The differences between the use of the numerical representation by the students in the main study and the participants in the pilot study will be discussed in chapter six.

The percentage of time spent by each pair in the various representations, as recorded in the time-line diagrams, is shown in Table 4.5. Clearly the graphical representation was favoured by students in their solution of the problem task used in this study. This preference is more than an artefact of the nature of the task as this same preference was not demonstrated by the pairs in the pilot study.

Table 4.5

<table>
<thead>
<tr>
<th>Pair</th>
<th>Percentage of Time spent in the Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Algebraic</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

*Note:* Kate suggests the table be used but this is initially rejected by Pete as cheating and then after she repeats the suggestion he argues that the graphical representation is better given the situation.
The most successful pair as determined by the “routineness”, efficiency, and effectiveness of their solution process, Pair 1, spent 8% of their time working in the algebraic representation alone and almost one-quarter of the time working in either the algebraic representation alone or simultaneously working in the algebraic and graphical representations. Of the time Pair 1 spent working in the algebraic representation, two-thirds of this time was spent simultaneously working in the graphical representation. In contrast, Pair 3 who also spent a significant amount of time working in an algebraic representation spent almost 90% of this time working simultaneously in the graphical representation. All pairs spent at least 93% of the total solution time working in a graphical representation, either solely or in conjunction with an algebraic representation, with Pairs 2 to 5 spending 97% or more of their time undertaking such activity.

4.5.2.6(c) Use of the scale marks

Furthermore, the time-line diagrams reveal that only three of the pairs adjusted the scale marks. As described previously, these were used in different ways and this impacted on the effect of this alteration as well as highlighting the misunderstanding of their effect by Pair 4. The pairs who did understand the effect of altering the scale marks used these in distinctly different ways. Pair 1 undertook carefully considered changes to facilitate their sketching, identification, and recording of the key features of the graph of the function. Pair 3, by setting the scale marks to zero, used their understanding of the effect of the scale marks to improve the graphing calculator output of a global view of the graph of the function by eliminating the thick axes.

4.5.2.6(d) Sequencing of activities

Differences are also apparent in the sequencing of the subtasks required to achieve the goal of sketching a complete graph. There were two major subtasks (a) to determine a global view of the function and (b) the identification of each of the key features. These
subtasks could be undertaken sequentially, (a) then (b), or concurrently. The latter of these saw the second subtask being partially undertaken as soon as a portion of the global view became visible in the viewing window. This observation will be explored in more detail later in this thesis.

4.5.3 Overview of the Results of the Macroscopic Analysis

Initial comparison of the episode diagrams and the time-line diagrams revealed that there was a distinct difference between the ability of the Year 12 students and that of the Year 11 students to access the task and produce a global view of the function. Other obvious differences were the length of time to complete the solution of the task, the directness of the route to the final solution, the sequencing of activities to achieve the goal of the problem task, and the use made of the various representations. Differences were also seen between the pairs in the proportion of time spent in each activity type within the categories reading, planning, execution, and evaluation. The subtasks undertaken within the categories organisation and evaluation also differed. Some of these differences occurred across the year levels; for instance, the number of events coded as organisation was noticeably greater for the Year 11 pairs, excluding Pair 4, whereas when considering the subtasks within organisation one of the Year 12 and one of the Year 11 pairs demonstrated similar patterns of behaviour. Also, the Year 11 students engaged in many more episodes in their pursuit of a complete graph of the function and these episodes were qualitatively different from those used by Year 12 students. The Year 11 students took longer, both in time and the number of screens viewed, to access a global view of the function; and approached the finding of the solution in a more ‘stop-start’ manner. With regard to approach, Year 11 pairs tended to follow a cycle of behaviour whereby they made some use of the calculator, often adjusting the viewing window, considered the resulting graphical view, often for some time, made another adjustment to the calculator, and so on. Year 12 students, in...
contrast, tended to move from screen to screen very quickly with their conversation at times lagging behind their actions.

4.5.4 Defining Moments

After close analysis of the episode and time-line diagrams and scrutiny of the case records where this facilitated macroscopic analysis, a number of defining moments became apparent in the solution processes of the pairs of students working through the problem task. The term, defining moment (DM), is being used to refer to an important or momentous event rather than a particular instant in time. A defining moment was interpreted as being a circumstance where some action, cognitive or physical, or a decision (i.e., a metacognitive action), or a series of these may have had the potential to facilitate or impede the solution process. Defining moments, therefore, occurred at critical points in the solution process. Similar circumstances may have occurred in the solution processes of other pairs but it was the responses of the pairs to these circumstances that determined whether or not they became defining moments for a particular pair’s solution. Each circumstance can be described in terms of the situation, the condition, and the response action, as shown in Figure 4.16. For every circumstance the question that must be answered is: What situation gave rise to this condition which, in turn, led to this response?

![Figure 4.16. Identification of a Defining Moment.](image)

A situation was the context in which the students found themselves. Usually it was a specific stage in the solution process; however, it could be a general situation rather than one that occurred at a specific point in time. For example, the ongoing search for a
global view of the function was a situation some pairs found themselves in for a substantial amount of time throughout their attempt at the problem task, whereas the situation, global view found, occurred at a particular point in the solution process, albeit during the solution process or at the end of the solution process. One situation could lead to a number of conditions, as was seen with the situation, ongoing search for a global view, as this occurred in conjunction with both conditions, sighting of apparent no view of the function and sighting of apparent vertical lines. The condition was the particular state that one or more pairs was observed experiencing or considering. Differing situations may be seen in conjunction with the same condition. The condition, sighting of apparent no view of the function, for example, occurred simultaneously with the situation, initial search for a global view, and the situation, ongoing search for a global view. In turn, several conditions can give rise to the same response action as seen by the action use of Zoom Fit, which was undertaken in response to both the conditions, sighting of apparent no view of the function and sighting of apparent vertical lines. The same combination of situation and condition may have occurred in conjunction with differing response actions from different pairs of students. For instance, given the situation, initial search for a global view, and the condition, sighting of apparent no view of the function, the response action of Pair 1 was the use of Zoom Fit, whereas the response of Pair 5 was the use of Zoom Out.

Questions asked of the data to identify defining moments included:

- What was the situation that led to the defining moment?
- What condition resulted from the situation?
- What response action resulted from the condition?
- Did this response action establish this set of circumstances as a defining moment for a particular pair’s solution?
The defining moments identified during the macroscopic analysis were related to the following:

- How students responded to particular views of the function, including apparent no view, apparent vertical lines, and other unusual or unexpected views. This also included the non-acceptance by the students of an other than global view of the function.
- Use of the numerical representation.
- How students made use of the scale marks, including their misuse and interactions between scale marks and the view.
- Use of opportunistic planning.
- Engagement in discussion.
- Identification of key features of the function.

Observing a prospective defining moment in one or more pairs’ solution processes resulted in the researcher scanning across all pairs’ solutions for the presence or absence of instances of the particular defining moment. After initial identification, further consideration was given to the set of defining moments as a whole and the situation, conditions, and response actions identified. These are detailed in the next chapter. In addition, the response may be no action at all as the pair may not have realised the importance resolving the situation had for their solution attempt. The response action determines whether or not the set of circumstances becomes a defining moment for the particular pair. The next chapter details and considers each of the sets of circumstances for the defining moments in turn.

Seven research questions were presented in chapter two, and the analysis in this chapter provides partial answers to all these except for the first research question which can now be addressed in full.
Research Question 1: How do students perceive what constitutes a complete graph of a function?

It is apparent from the macroscopic analysis that all pairs demonstrated correct understanding of what constitutes a complete graph of a function. This is inferred from the actions of each pair as they undertook a search for a global view of the function, not accepting the initial view as being the global view, and in addition they set about identifying key features of the function.

Further analysis is required in order to answer the remaining research questions fully. Microscopic analysis will be undertaken in the next chapter in order to answer these, specifically by analysing the defining moments identified in the macroscopic analysis.

4.6. Conclusion

In this chapter, the problem task and the conditions of the task solving sessions were presented. The contribution of data from the various data collection methods to the production of protocols for all pairs was outlined. These collectively form the case record. The activity coding scheme adapted by this researcher from that of Schoenfeld (1985b, 1992b) was explained and each coding category explicitly detailed as was the parsing of the coded protocols into episodes of distinct behaviour. Episode diagrams for each pair were presented and the structure outlined. Time-line diagrams were constructed and these included the activity being undertaken, representation being used, and the identification of each episode at any point in time. The episode structure and time-line diagrams for the five pairs were compared and contrasted. This macroscopic analysis allowed Research Question 1 to be answered. Students in this study did demonstrate their understanding of a complete view of a graph, as being both the global view and identification of all key features, as inferred from their actions and written scripts. Finally, the macroscopic analysis led to the identification of the existence of a number of defining moments in the solution process. Microscopic analysis will be
necessary to explain these defining moments and more specifically address the remaining research questions. This analysis is presented in chapter five before remaining questions are addressed in chapter six where the implications of these findings for teaching practice and further research are also discussed.
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CHAPTER FIVE
MICROSCOPIC ANALYSIS

5.1. Overview

As detailed at the end of the last chapter, the defining moments identified during the macroscopic analysis related to the following:

- How students responded to particular views of the function, including apparent no view, apparent vertical lines, and other unusual or unexpected views. This also included the non-acceptance by the students of an other than global view of the function.
- Use of the numerical representation.
- How students made use of the scale marks, including their misuse, and interactions between scale marks and the view.
- Use of opportunistic planning.
- Engagement in discussion.
- Identification of key features of the function.

Microscopic analysis of the data began by considering the data at a macroscopic level then progressed through increasing levels of detail. Going from episode diagrams to the timeline diagrams to in-depth protocol analysis, the data were scrutinised for explanations of why particular situations became defining moments in the solution process for some pairs and not others, and how these defining moments impacted on the students’ solutions.

To this end, the case record for each pair of students was then analysed separately for each set of defining moments. The following questions were considered, where relevant.
• Did the students link their mathematical knowledge and understandings with the graphing calculator output? Did the students use these links to inform their solution process? Did the students use their knowledge of (cubic) functions to inform their use of the graphing calculator?

• Did the students use their graphing calculator skills (i.e., knowledge of the tool’s features) to inform the solution process?

• Did the students use their graphing calculator skills to enhance the graphing calculator output?

• When a conflict occurred between the ideas of the students, as inferred from their statements and actions, and the graphing calculator output, was this noted?

• When questions were asked, was the purpose comprehension, organisation, argumentation, inquiry, or verification?

5.2. **Defining Moment 1: Initial Search for a Global View**

The first set of defining moments to be subjected to more in-depth analysis were those related to students’ initial *search for a global view*. These included how students responded to particular views of the function and their non-acceptance of an other than global view of the function. The views seen included that of *apparent no view, apparent vertical lines, a section of the function coincident with the y axis,* and other *unusual or unexpected views*. (These views will be explained below.) Three response actions were observed in this set of circumstances, namely, the *use of Zoom Fit*, the *use of Zoom Out*, and the making of *adjustments to the WINDOW settings*. The sets of situation, condition, and response action are shown in Figure 5.1.

![Figure 5.1](image-url)

*Figure 5.1. The circumstances related to the first set of defining moments.*
In attempting to access the task, the students used a variety of processes and their choice of these seemed to affect how successful this access attempt was initially and how quickly they found a complete view of the graph. The balancing matrix (Strauss & Corbin, 1998) in Table 5.1 shows the actions the students used and the consequences of these initial actions for (a) the view of the graph, (b) the routineness of the solution, and (c) how long it took to produce a global view of the graph which would allow them to begin to search for key features. The length of time to the global view is presented both as time and the number of windows viewed by the pairs of students, however, it must be noted that the latter, whilst informative, is approximate only in that the windows presented in the case record are a static representation of a dynamic solution process. Hence, some windows presented may have been viewed statically for several minutes whereas others were observed for a fleeting moment in time.

Table 5.1

*Initial Actions and Consequences in Accessing the Problem Task*

<table>
<thead>
<tr>
<th>Pair</th>
<th>Actions</th>
<th>Consequences for</th>
<th>Time till Global View was seen</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>View</td>
<td>Solution Process</td>
</tr>
<tr>
<td>1</td>
<td>Using GRAPH followed by Zoom Fit</td>
<td>Becomes routine</td>
<td>2:20 mins (18 windows)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Setting WINDOW using y intercept followed by</td>
<td>Becomes routine</td>
<td>1:00 mins (7 windows)</td>
</tr>
<tr>
<td></td>
<td>Zoom Fit</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(Table continues)
Table 5.1 (continued)

<table>
<thead>
<tr>
<th>Pair</th>
<th>Actions</th>
<th>Consequences for View</th>
<th>Solution Process</th>
<th>Time till Global View was seen</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Using GRAPH &amp; setting the WINDOW using the y intercept</td>
<td>![Graph Image]</td>
<td>“all over the place”</td>
<td>17:10 mins (74 windows)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Graph Image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Graph Image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Setting the WINDOW using y intercept &amp; adjusting the WINDOW setting &amp; scale marks</td>
<td>![Graph Image]</td>
<td>“stuck in the loop for a time” making innumerable WINDOW adjustments</td>
<td>19:00 mins (76 windows)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Graph Image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Graph Image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>Using Zoom Standard &amp; Zoom Out followed by adjusting the WINDOW</td>
<td>![Graph Image]</td>
<td>Potentially routine</td>
<td>12:30 mins (30 windows)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Graph Image]</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>![Graph Image]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In three instances (Pairs 1, 2, and 5), the results of these actions were tempered by interactions with the quality of scale chosen or the quality of the students’ planning. At times the students undertook sensible actions based on their mathematical and graphing calculator knowledge; however, given the nature of the task (i.e., the particular function involved) these actions did not result in the expected view of the graph. In particular, a view showing two vertical lines in addition to a section of the graph coincident with the y axis proved confronting. The students’ responses to these views led to consideration by the researcher of the initial actions and the consequences these had in accessing the task for the solution process becoming routine or problematic, as shown in Table 5.1. How students were able to draw on their mathematical and graphing calculator knowledge in these novel circumstances proved critical to a relatively efficient solution to the task and/or the curtailment of inappropriate solution pathways.

Students selected a combination of actions as shown in the second column of Table 5.1 in attempting to access the task. These actions were either physical (e.g., pressing calculator buttons such as GRAPH, selecting menu items which included Zoom Standard, Zoom Fit, Zoom Out, or a combination of button pressing and menu selection when adjusting WINDOW settings), or cognitive (e.g., using the y intercept to inform WINDOW setting selection). Contrary to what may be likely to happen in usual graphing calculator use, GRAPH and Zoom Standard had the same effect as the graphing calculator had been set to the standard window before students began the task, in order to ensure a common starting point. The consequences of these initial actions on the view of the graph seen on the graphing calculator are shown in the third column of the table as screen shots from the protocol of the particular pair of students. Next, in column 4, the consequences of these initial actions of the students on the solution process are given. These were categorised as the solution becoming routine, potentially routine, “all over the place”, or “stuck in the loop for a time”. The last category, “stuck
in the loop for a time”, saw students undertake minimal discussion but make innumerable WINDOW adjustments which often had little effect on the view of the graph. Their actions were neither assessed nor altered and none of the other dedicated functions of the graphing calculator were used. The penultimate category, “all over the place”, saw students spend a significant amount of time and discussion pursuing one particular solution strategy before they changed approaches and spent a significant amount of time and discussion pursuing another solution strategy. This was repeated until they finally completed the task. The last column in the table shows the consequences of the initial actions of the students with respect to the length of time needed and the approximate number of windows accessed in order for students to observe a global view of the function.

The balancing matrix highlights the apparent difference between the Year 11 and Year 12 students. The initial actions of the latter led to routineness of their solution process. The initial actions of the Year 11 students in contrast became problematic, although for one pair (Pair 5) the solution did become potentially routine with the fourth window shown being seen 1 minute 30 seconds into their solution. However, this potential was not realised, with a global view finally displaying after 12 minutes 30 seconds. The interaction between the nature of their planning and their discovery of a global view of the function is discussed later in this chapter. In looking for an explanation of these differences the action sequences, as detailed in the next section, were identified. The microscopic analysis of task access revealed that this set of defining moments, initial search for a global view, occurred in conjunction with other defining moments, namely, those associated with use of scale marks, the level of discussion, interactions between planning and other actions, and the type of dedicated graphing calculator features used.
5.2.1 The Action Sequences

Microanalysis of the situation, *initial search for a global view*, revealed that four different conditions (indicated in Figure 5.1) arose in this situation. A closer examination of Table 5.1 revealed that there were six different actions or processes undertaken by the students as they attempted to access the task. Of these six actions three, namely, pressing GRAPH, selecting Zoom Standard, and setting the WINDOW informed by the y intercept, describe the initial physical (and possibly cognitive) action of the students whereas the remaining three are their response actions (*use of Zoom Fit, use of Zoom Out, and adjustment of the WINDOW settings*) to the condition presenting itself as a consequence of their initial actions which immediately resulted in three of the conditions for the first set of defining moments. The remaining condition (function coincident with y axis) arose as students’ viewed output from subsequent actions in this initial search for a global view.

Although two of the actions involved the WINDOW settings they were distinctly different. The first of these involved both cognitive and physical actions when the viewing window was adjusted as the initial action of the pair of students and their knowledge of the y intercept clearly informed their adjustments of the WINDOW values. In the second, the changing of the WINDOW setting can be either merely a physical action, or both physical and cognitive. Actions categorised in this way can reflect the students’ using either the algebraic or graphical representation of the function to inform their WINDOW adjustments, or they can appear to be making adjustments in a relatively haphazard way.

Each pair of students undertook a different sequence and/or selection of these actions in an effort to observe a global view of the function. The sequences chosen affected how successful they were in accessing the task. Figure 5.2 shows the actions and the
sequences of these undertaken by each pair. The six actions are represented by the rectangles and the identification of a pair inside a rectangle indicates this was their initial action. The annotated arrows allow the sequence of actions undertaken by each pair to be determined. It can thus be inferred which actions finally resulted in the pair achieving a global view of the function. Actions may or may not have been informed by mathematical knowledge. The latter occurred in the protocols of Pairs 1, 3, and 5, all of whom explicitly stated that the standard window would not be appropriate to view the complete graph of the function, yet immediately proceeded to access the standard window.

5.2.2 A Closer Examination of the Six Actions

In this section, the various actions, initial and response, undertaken by the student pairs during the beginning of their solution process are considered. These include pressing GRAPH, selecting the standard window, adjusting the WINDOW settings and selecting items from the ZOOM menu. Similarities and differences between their actions and the consequences of these are identified, and possible reasons for their actions discussed. Successful and unsuccessful action sequences are also discussed after the analysis of the specific actions.
5.2.2.1 Initial Actions

5.2.2.1(a) Initial action: Beginning with GRAPH

Two pairs, 1 and 3, indicated that they did not think the graph would be visible in the standard window of \(-10 \leq x \leq 10\) and \(-10 \leq y \leq 10\), and so began by pressing GRAPH. As the calculator had been set to the standard window, pressing GRAPH had an identical effect to the selection of Zoom Standard. From their conversations, it was inferred that neither pair believed the standard window would provide them with a complete view of the graph. However, their action did not reflect their words although they would not necessarily be aware this would produce the standard window.

Kate, for example, signalled her intention to edit the WINDOW settings rather than use the standard window by stating, “Play with the WINDOW”. This suggests that she wanted to edit the viewing window rather than select GRAPH or use the ZOOM menu items. Furthermore, Kate appeared to be using the algebraic representation of the function to inform her ideas about appropriate WINDOW values when she continued, “I guarantee you can’t see it, … (be)cause if this is 1992 …” Her partner, however, decided to press GRAPH anyway, possibly suggesting he was too impatient to begin by considering the viewing window.

Similarly, Pair 1 began discussing an appropriate viewing window but, after rejecting Zoom Standard which would have resulted in the standard window, they realised they had inadvertently arrived at the standard window via GRAPH (see Figure 5.3).

Ahmed (Pair 1): GRAPH it, I don’t think we should use Zoom Standard. [They press GRAPH.]
Linh: What’s our window?
Ahmed: It’s the standard window.

Figure 5.3. The view of the graph in the standard window.
They presumably used the position of the origin in the window and the number of scale marks in their identification of this window as the standard window, and, discounted (reasonably in the circumstances) the fact that other WINDOW settings can produce the same view. One interpretation of the dialogue and actions of Pair 1 is similar to that for Pair 3, that is, an apparent contradiction between their words and actions. Alternatively, Pair 1 may have been trying to determine the current WINDOW settings from the graphical view, rather than directly by selection of the viewing window and observing the settings in use.

After their initial selection of GRAPH, the two pairs followed very different solution paths, both in process and success. Pair 1 decided to move to an algebraic representation and find an \( x \) intercept before they proceeded to use Zoom Fit. This pair had the graphing calculator knowledge to select processes that they knew would provide them with a greater insight into the solution. Their knowledge and choices ensured that their solution process quickly became routine. Pair 3, on the other hand, with less graphing calculator knowledge than their older peers, was unable to make the same choice. They did not have the option of selecting Zoom Fit as this calculator feature had not been shown to them by their teachers or peers, nor had it been discovered by themselves in their own use of the graphing calculator. They did, however, have choices and these included *adjustment of the WINDOW settings* and the *use of Zoom Out*. They chose to *adjust the WINDOW settings*. Their rejection of the more salient choice of Zoom Out, possibly added eight minutes to their solution time. This is inferred from the fact that in the tenth minute of their solution process they returned to the standard window, eight minutes after first seeing this view, and on this occasion *selected Zoom Out*, leading to a global view after a further ten minutes. The selection of GRAPH as their initial action, on the other hand, did not necessarily determine whether the solution process became routine or problematic as evidenced by the fact that for one of these pairs the solution
process became routine whereas for the other pair the solution process became problematic.

5.2.2.1(b) Initial action: Deliberate selection of the standard window

Pair 5, alone, made a deliberate selection of the standard window for their initial view of the function. Their conversation suggests that like Pairs 1 and 3, they did not believe the function would be visible in the standard window.

Susan: You won’t (see the graph), … but just to see [selects Zoom Standard].

If they truly did not believe the function would be visible in the standard window, their selection of Zoom Standard is perplexing. Possible reasons for this action could be (a) to provide them with more thinking time, (b) to be seen to be doing something, or (c) less confidence in their conviction that the graph would not be visible than their words imply. After seeing no part of the graph in the standard window, Pair 5 proceeded to use Zoom Out leading to a section of the function becoming visible in the viewing window. The initial action of this pair, however, was the first in a series that allowed the solution process to become potentially routine for them.

5.2.2.1(c) Initial action: Adjusting the WINDOW settings

Three pairs adjusted the viewing window informed by the y intercept, two of these were as their initial action and one pair subsequent to their initial action. Pairs 2 and 4, began by selecting WINDOW. This allowed them to observe exactly where in the Cartesian plane the graphing calculator settings were selected to display. In addition, both pairs were then able to adjust the viewing window informed, if they chose, by their mathematical and graphing calculator knowledge. Both pairs used the y intercept to inform their choice of WINDOW settings, however, each pair altered a different number of the WINDOW settings. Pair 2 altered four values—the x minimum and maximum, and the y minimum and maximum values. Pair 4 altered two values—the x minimum and the y minimum values. The resulting views and WINDOW settings are
shown in Figure 5.4. The classroom experiences of the students in the study included their teachers’ modelling the altering of no more than two values in WINDOW at any one time and then using the resulting graphical view to inform any further adjustments that were required. In addition, students had been instructed to make larger rather than smaller alterations to values in the WINDOW settings on the understanding that it is more efficient to reduce the viewing domain and range once the complete view of the graph is seen rather than make a series of small adjustments to the WINDOW settings that eventually allow the complete view of the function to be seen.

![Figure 5.4](image)

*Figure 5.4.* The views and WINDOW settings of pairs 2 and 4, respectively, after their initial adjustment of the viewing window.

Both pairs altered the $y$ minimum, on the surface suggesting this decision was informed by the $y$ intercept. However, given the viewing domain of Pair 2 did not include the $y$ ordinate of the $y$ intercept, this interpretation must be questioned. Pair 2 noted that the $y$ intercept occurred at $(0, -92)$ and let $x$ minimum in the WINDOW setting be -100. In addition, they set $x$ maximum to be +100 and the corresponding $y$ values to be -50 and +50, as shown in Figure 5.4(a). For the vertical and horizontal axes, they selected values with symmetry about the axes that would result in the axes remaining centred in the viewing screen. Whether the reason for this is related to the pairs of numbers or the resulting window cannot be inferred from these data, however, it is noted that this behaviour was common amongst the group of students in the study.
The use of the $y$ intercept to inform the decision of Pair 4 is more evident. After reading the problem, Pair 4 stated “-92 is the $y$ intercept”. They set $x$ minimum at -100. In addition, they set $y$ minimum to be -100 also, as shown in Figure 5.4(b). So, although this pair altered more than one value, clearly symmetry of the axes in the window was not a consideration. The use of identical values on both axes, rather than symmetry, appears to have been foremost in their minds. By selecting a viewing range that included the $y$ ordinate of the $y$ intercept, Reem and Ali could reasonably expect to see a view of the graph intersecting the $y$ axis in the resulting viewing window. The resulting view, however, showed one part of the graph coincident with the $y$ axis, a fact the students may not have realised, as this was most probably their first experience of this occurrence.

The resulting viewing windows, and the WINDOW settings selected for both pairs were shown in Figure 5.4. For both pairs the solution process became problematic. With this particular cubic function, a sensible action, informed by their mathematical knowledge and past experiences, did not result in progress along the solution path as could have been expected from their previous classroom experiences graphing functions.

Although the pairs appeared to be following the same initial solution path, it is apparent that the adjustment by Pair 2 of four of the WINDOW settings resulted in a view of more of the function than the other pair saw. Pair 2 could not see the $y$ intercept as their selected viewing range did not include -92, however, the other pair did not appear to recognise the presence of the $y$ intercept in their resulting view. Neither pair explicitly commented on the lack of a visible $y$ intercept initially. Their responses proved critical to achieving a potentially routine solution.
5.2.2.2 **Response Actions**

5.2.2.2(a) **Response action: Use of Zoom Fit**

Zoom Fit was an action only undertaken by the Year 12 students. This dedicated feature of the calculator had not been formally taught or informally used in the classroom by either of the teachers of the students in the study. Both pairs made judicious *use of Zoom Fit* which they selected after their initial WINDOW selection, as described previously. In Pair 2, both students appeared to be familiar with Zoom Fit.

Hao: Zoom, Zoom Fit, we do it in CALC [referring to the CALCULATE menu].  
Abdi: Yeah, Zoom Fit.

In Pair 1, however, Ahmed explained the use of the function to Linh.

Ahmed: Can’t see anything. I think we have to zoom out.  
Linh: What’s that?  
Ahmed: Zoom Fit, it fits all the points.

The resulting view of the graph, and the view that precipitated the use of Zoom Fit, for each pair is shown in Figure 5.5.

![Selecting Zoom Fit](image1)

**The Standard Window**  
Selecting Zoom Fit

![Resulting view for Pair 1](image2)

**Resulting view for Pair 1**  
-10<X<10,  
-20912<Y<16928

![Selecting Zoom Fit](image3)

-100<X<100,  
-50<Y<50

![Resulting view for Pair 2](image4)

**Resulting view for Pair 2**  
-100<X<100,  
-990892<Y<810708

*Figure 5.5. The views of the graph for Pairs 1 and 2, respectively, after their use of Zoom Fit.*

For Pair 1, the *use of Zoom Fit* did not result in a global view of the function. However, the resulting view, integrated with their knowledge of cubic functions and the Zoom Fit function, quickly allowed them to obtain a global view of the function.
Ahmed: Still we can’t see the whole thing.
Linh: No.
Ahmed: We can see some of the graph.

The combination of mathematical knowledge and *use of Zoom Fit* proved to be a major factor for this pair quickly accessing the task and resulted in the solution process becoming routine.

The response of Abdi and Hao of Pair 2, upon not seeing the $y$ intercept as expected was to comment “it’s still not good” and to *select Zoom Fit* which resulted in a global view of the graph. This action proved critical to the solution process becoming routine for this pair. As this view allowed them to begin the process of identifying key features of the function.

Pairs 1 and 2 undertook sensible actions, using Zoom Fit, knowing that the graphing calculator would determine $y$ maximum and minimum values that would provide an improved view of the function. Their knowledge of the graphing calculator and their appropriate application of this knowledge immediately resulted in a non-challenging view of the function and this resulted in the solution process becoming routine for these students.

5.2.2.2(b) **Response action: Use of Zoom Out**

Two of the pairs made *use of Zoom Out*—Pair 5 in their initial search and Pair 3 in their *ongoing search for a global view* of the function as described later. After observing the view in the standard window, Pair 5 selected Zoom Out and immediately saw their first view of the graph of the function. Upon seeing a perplexing view of the graph of the function as shown in Figure 5.6, they proceeded to alter the WINDOW values in their initially successful ongoing search for a global view of the function. The response action, *use of Zoom Out*, identified a defining moment for Pair 5. Not only did it result in a section of the graph of the function to be clearly visible in the viewing
Figure 5.6. The resulting view after selecting Zoom Standard followed by Zoom Out.

window but also the resulting view used in conjunction with their mathematical and graphing calculator knowledge allowed this pair to undertake physical and cognitive actions that led to their finding of an almost global view of the graph of the function within minutes.

5.2.2.2(c) **Response action: Adjusting the WINDOW settings**

All of the pairs made alterations to the WINDOW settings in their initial search for a global view of the graph. Two pairs began with this process, as described previously. Four of the pairs adjusted the viewing window soon after their initial selection. Pair 1 adjusted the WINDOW settings in conjunction with their mathematical knowledge and understanding of Zoom Fit, being that given the domain of the function is all real numbers then a section of the function must become visible given an appropriate viewing range, irrelevant of the viewing domain. Pairs 3 and 5 adjusted the WINDOW settings in a variety of ways, ranging from those informed by, or in contradiction to, the algebraic or graphical representation to an adjustment of the scale marks. Pair 4 began by adjusting the WINDOW settings and this response action was repeated many times in their solution process. It appears that, although the action of adjusting the WINDOW settings played a role for all pairs, it was other actions and a combination of mathematical knowledge and graphing calculator knowledge that determined if the initial actions resulted in the solution process becoming routine or problematic.
5.2.2.2(c)i Adjusting WINDOW settings informed by graphical representation

The graphical representation of the function can be used to inform the choice of WINDOW values to be altered. After three apparently vertical sections of the graph appeared in the window, as shown in Figure 5.7, Pair 5 focused on the graphical representation of the function and their mathematical knowledge of cubic functions.

Susan: Now we just have one, no two big lines okay ... change the WINDOW ...
Jing: We need to see the turning points.
   Make it bigger.
Susan: Yes, try that.
Jing: Which way is the turning point?

![Screenshot](image)

*Figure 5.7.* The vertical line effect as seen by Pair 5 and the almost global view resulting from their informed alterations to the WINDOW settings.

From their dialogue and from their adjustment of both the y minimum and the y maximum values, it can be inferred that Susan and Jing were unaware of the section of the graph coincident with the y axis. The steepness of the lines did not stop their using their mathematical knowledge to infer that there must be a turning point in the viewing domain under observation, however. As they were unsure of the type of turning point, they sensibly adjusted both endpoints of their viewing range. These changes resulted in a maximum turning point becoming visible in the viewing window, albeit probably not quite where they expected.

The response of Pair 4, to the lack of an expected view of the y intercept was to further adjust their x minimum and maximum values, multiplying both by twenty. A third change to the WINDOW settings produced a view similar to that seen by Pair 2 in their initial window as seen in Figure 5.4(a). The response of Pair 4 was, however, very different to that of Pair 2 and their continued adjustment of the WINDOW setting was relatively ineffectual.
5.2.2.2(c)ii  Adjusting WINDOW settings informed by multiple representations

A combination of representations could also have been used to inform changes to the viewing window but this was not always helpful. After Zooming Out once from the standard window, Pair 5 observed the WINDOW settings of \(-40 \leq x \leq 40\) and \(-40 \leq y \leq 40\), and decided to adjust the viewing window directly using the value of the constant in the function to inform their adjustment of the WINDOW settings.

Susan:  Maybe go to WINDOW seeing it’s a large number [meaning the constant which was -92]. I’d probably go up a bit.
That’s a bit extreme, \([-100 \leq x \leq 100 \text{ and } -30000 \leq y \leq 30000]\) but then ...

Jing:  (mumbles) Make the scale a bit bigger.

When Pair 3 used the algebraic representation of the function to inform their actions, they appeared to become more confused or indecisive. They experienced conflict between their ideas and the output of the graphing calculator. Not only did they comment on this conflict, they proceeded to discuss their understandings and the conflict between these and the calculator output. The lack of a visual view of the graph followed by an unexpected view dominated their conversation.

Pair 3 noted the constant term of the function, with Pete stating, “… the y intercept is definitely -92”, and entered this value for the y minimum value. Even though -92 is not the value their teachers would have encouraged or expected them to enter as the y minimum value, they should have been able to reasonably expect to see a section of the function in the viewing window. Pete, from Pair 3, questioned the lack of a visible y intercept. Pete and Kate appeared to still believe the y intercept was at \((0, -92)\), however, similarly to Pair 2, they felt the need to check this and altered the y minimum to -1992!

Pete:  -92.
Kate:  I don’t think it’s that far, I reckon it’s further.
Pete:  Doesn’t that show the y intercept?
Kate:  Nn, mnn, 1992, so, take it down to about two, … see.

It could be inferred from their actions that they were unsure which coefficient in the equation provides information about the y intercept. The resulting view allowed them to
see a clear view of the y intercept, however, Leong, and Malone (2002) would classify this incident as a “misperception” rather than a “misconception”, meaning that students were perceiving something that is different from reality. Their experience of altering the y minimum from -92 to -1992 resulting in the y intercept becoming visible, possibly reinforced their use of the coefficient of x as a sensible choice for the y minimum.

Abdi and Hao, the pair who altered four WINDOW settings (not what was expected by their teachers), had the greatest success as this action resulted in their being close to a global view of the function. The choice of the setting to alter and the values selected may provide some insight into the thinking of the students. Assuming their actions were logical, a reasonable assumption in this case, the students appeared to be linking their mathematical knowledge (i.e., the global view of the function) with their previous graphing calculator output. The students appeared to be positioning the current output of the graphing calculator screen with their mental image of an archetypal cubic function.

The most successful use of the WINDOW settings was by Pair 1 after their selection of Zoom Fit. They observed the resulting WINDOW settings, changed the domain from $-10 \leq x \leq 10$ to $-20 \leq x \leq 20$, and then observed the effect on the graph. From their next actions it can be inferred that they believed this viewing domain was too small. Their behaviour suggests that using the graphing calculator output as a stimulus, in conjunction with their previous mathematical knowledge, they had a clear mental image of exactly which section of the function was the focus for the viewing window. Their next steps were to alter the values of both the x minimum and x maximum to give $-50 \leq x \leq 50$. From this it may be inferred they had a global view of the function “in their heads”. This is confirmed by the fact that once the horizontal position of the minimum turning point was identified ($-50 \leq x \leq 80$) only the x maximum value was altered.
5.2.2.2(c)iii Adjusting the WINDOW settings in apparent contradiction

At times the pairs adjusted the WINDOW settings in apparent contradiction to the graphical representation shown, or to the algebraic representation and their mathematical knowledge. The actions of Pairs 3 and 4, for example, reduced the number of quadrants visible in the viewing window, a result that it can reasonably be inferred that these students would not expect to improve their chances of seeing a global view. These actions contributed to the solution process becoming problematic.

Pete from Pair 3, for example, drew attention to the domain of the function being all real numbers but then they proceeded, inexplicably, to edit the WINDOW settings, reducing the viewing range, as shown in Figure 5.8.

Kate: We’ve got two, four, five on the x, how about we make it, … (be)cause it’s all on the right side of the y axis.
Pete: Yeah, no, it’s not the graph, it’s a cubic. It goes forever!

Figure 5.8. Kate and Pete’s altering the viewing window to focus on the fourth quadrant.

The changes to the WINDOW settings by Kate and Pete, shown in Figure 5.8, are puzzling as it would be expected that this pair would deduce that the graph should be clearly visible in the second quadrant, particularly given Pete’s previous correct statement regarding the domain of the function (“It goes forever!”). However, they proceeded to set the WINDOW so that only the fourth quadrant was visible. It is impossible to infer if they believed they were seeing the same section of the graph in the two views of the graph shown in Figure 5.8.
5.2.2.2(c)iv Adjusting the WINDOW settings in response to unusual views

The categorisation of a view as most unusual or unexpected is subjective, however, it was clear from the protocols that in undertaking the problem task many of the students were exposed to views of the function that left them confused or pondering. This was particularly true for the Year 11 pairs. Figure 5.9 shows views seen by Pairs 3 to 5 that may have been unexpected and shows their consequential actions and the resulting views. In this diagram, all arrows, unless specifically noted otherwise, represent a response action involving adjustment of some or all values in the WINDOW settings.

![Diagram showing adjustment of WINDOW settings](image)

*Figure 5.9. Varying consequences for the Year 11 Pairs’ action, adjusting the viewing window.*
As discussed previously, upon seeing their first window shown in Figure 5.9, Pair 3 reduced the number of quadrants that were visible to one. It is impossible to determine what relationship this pair held between the sections of the graph visible in their resulting windows. There was no indication, for example, that they were aware there were two sections of the graph visible in this initial window nor the three sections in the third and final window shown. For Pair 4 the sight of an apparently vertical line led to many adjustments of the viewing window, the results being views of two apparently vertical lines. For Pair 5, the initial WINDOW adjustments were to all values of both the viewing domain and range but subsequently after viewing the resulting window, they only adjusted the values affecting the viewing range. From their response actions, it can be inferred that the apparently vertical lines fitted within their mental images of possible views of cubic functions as widening the range allowed the two turning points to become visible or clearly inferred.

All the Year 11 pairs *adjusted the WINDOW settings*; however, this had very different consequences for different pairs, suggesting that of itself this was not an inappropriate action to take. Mathematical knowledge and differences in the graphical view of the function visible may have been confounding factors in the differences between the results of the actions of the Year 11 pairs. The apparent reluctance of some Year 11 students to enter numbers having a large magnitude for values in the viewing window contributed to the number of adjustments required before a view of the function became visible in the viewing window. The findings here are in contrast to those of Steele (1995a, 1995b) who suggested that students might have accepted these initial views of the function as the complete graph. This behaviour was not displayed by any of the pairs of students in this study.
5.2.3 Routineness versus Problematic Nature of Solution Process

Figures 5.10 and 5.11 show the sections of the task access diagram for the students for whom the solution process became at least potentially routine and those for whom it became problematic, respectively. The notable difference between the students appears to be their response actions to the results of their initial actions. The use of Zoom Fit and the use of Zoom Out became defining response actions in the particular sets of circumstances as these actions allowed the students to avoid repetition of the vertical line view of the function or avoid it altogether. The use of these two defining response actions appears to have been essential for the task to become quickly routine.

**Figure 5.10.** Successful beginning task access.

**Figure 5.11.** Task access resulting in the solution process becoming problematic.
It has been shown that the response actions of the Year 11 and the Year 12 students in their initial search for a global view were different and the consequences of these affected the progress of the students toward a solution, as indicated in Table 5.2. The use of Zoom Fit by both Year 12 pairs immediately provided a familiar view for these students. However, the use of Zoom Out followed by adjustment of the viewing WINDOW in conjunction with mathematical knowledge also led to a familiar view of the graphical representation being observed.

Table 5.2

<table>
<thead>
<tr>
<th>Response Action</th>
<th>Consequences for progress toward solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Not Effective</td>
</tr>
<tr>
<td>Using Zoom Fit</td>
<td>Pair 1 &amp; 2</td>
</tr>
<tr>
<td>Using Zoom Out</td>
<td>Pair 5</td>
</tr>
<tr>
<td>Adjusting the WINDOW settings</td>
<td>Pair 3 &amp; 4</td>
</tr>
<tr>
<td></td>
<td>Pair 3</td>
</tr>
</tbody>
</table>

5.2.4 The Effect of the Action Sequences

In attempting to access the task, the students used a variety of processes and their choice of these affected how successful this access attempt was initially and how quickly they found a complete view of the graph. The selection and sequencing of the six actions undertaken by the students affected how successful they were in accessing the task. These actions may have been both cognitive and physical, and their use may have been informed by previous experience, mathematical knowledge, graphing calculator knowledge, and reflection on either or both the graphical and algebraic representations of the function. Each pair of students undertook a different sequence of these actions and this affected how efficient they were in accessing the task. The
knowledge and choices of Pair 1 enabled them to select a sequence of actions that ensured that their solution process quickly became routine. None of the other pairs of students consistently applied their knowledge in conjunction with sensible choices of actions.

All of the pairs adjusted the WINDOW settings in their initial search for a global view of the graphical representation of the function. When combined with their mathematical knowledge and the use of other features of the graphing calculator the use of WINDOW lead to the solution process becoming routine or potentially routine. All pairs undertook actions that demonstrated they had a clear mental image of the function for which they were searching, however, the actions of some pairs on some occasions suggested that they did not have a clear understanding of which section of the function was currently the focus for the viewing window.

The actions of several pairs did not reflect their words. Reasons for this may include the opportunity for more thinking time, to be seen to be doing something, or the students were less confident in their beliefs than their words implied. In solving the problem task undertaking sensible actions, as modelled by their classroom teachers, such as altering no more than two values in WINDOW at any one time, were at times less successful than could reasonably be expected. In contrast, apparently less sensible actions at times resulted in greater success.

The view of the function used in the problem task coincident with the y axis was a sight most unlikely to have been previously experienced by the students in the study and was confronting for them. Few of the pairs, however, who had this experience spent any apparent time reflecting on what in fact must be present in the viewing window given their previously demonstrated mathematical knowledge. This lack of reflection resulted in the solution process being less efficient than may have been expected if an evaluation of the output of the graphing calculator had been undertaken at this point in time.
At various stages of their solution, but particularly at the beginning, some students were confronted with a viewing window where the graph was absent or appeared to be absent. The use of Zoom Fit or use of Zoom Out in response to an apparent no view of the function facilitated the solution process becoming routine. This occurred either as the view in the standard window, occurring in this way for Pairs 1, 3, and 5, or occurring in other than the standard window as experienced by Pairs 3 and 4.

The actions and resulting windows of the first experience of the pairs sighting an apparent no view of the function is shown in Figure 5.12. The arrows represent the actions undertaken by the students and these consist of either selecting Zoom Fit or Zoom Out or adjusting the WINDOW settings. The graphing calculator screens on the left of Figure 5.12 display the view prior to their response action and the graphing calculator screens on the right show the result of the action taken.

![Diagram showing actions and resulting windows]

_Figure 5.12. Actions following the pairs’ first sight of apparent no view._

Figure 5.12 shows that three of the pairs (1, 4, and 5) undertook successful initial actions that immediately resulted in the function becoming visible in the viewing
window. In contrast, the action of Pair 3, Kate and Pete, in adjusting only the $y$ minimum value of the WINDOW settings, from -10 to -92, resulted in no portion of the graph of the function becoming visible.

There were three other occurrences of sightings of an apparent no view of the function, one in the initial search for a global view and the others occurring in the ongoing search for a global view, discussed later in this chapter. When Pete and Kate’s adjustment of the WINDOW settings, did not change this output, they repeated their actions and adjusted the WINDOW setting again altering only the $y$ minimum value, this time from -92 to -1992 as shown in Figure 5.13. This time their action was successful and a portion of the graph appeared in the viewing window.

![Image](image.png)

**Figure 5.13.** Actions following the additional sightings of apparent no view.

Thus, different response actions to the same situation and condition produced varied consequential conditions and this in turn may or may not have facilitated the solution process. From the sequences, shown in Figure 5.12 and 5.13, it is apparent that some actions were more effective than others, given the situation, apparent no view. The most effective response action was the use of Zoom Fit. The various response actions to the condition, apparent no view of the function, are discussed further in the next section where they again occur but on this occasion during the situation, ongoing search for a global view.

Initial comparison revealed an apparent distinct difference between the abilities of Year 12 and Year 11 students to access the task efficiently and produce a global view of
the function. The response action: *use of Zoom Fit*, was undertaken only by the Year 12 pairs in the study and their judicious use of this action allowed the solution process to become routine for them. All pairs made alterations to the WINDOW settings, however, it appeared that it was other actions and a combination of mathematical and graphing calculator knowledge that determined if the initial actions of the students resulted in the solution process becoming routine or problematic.

The actions the students used and the consequences of these for the view of the graph, routineness of the solution, and how long it took to produce a global view of the graph sufficient to begin a search for key features appeared to be significant in explaining the differences seen. The response action of either *use of Zoom Fit* or *use of Zoom Out* in particular situations and circumstances identified defining moments in the solution process as these actions both determined initial success in accessing the problem task and allowed the students to avoid repetition of the vertical line view of the function or avoid it altogether. On the other hand, the response action, *adjustment of the WINDOW settings*, when used in particular situations and conditions may have been taken in conjunction with the application of mathematical and/or graphing calculator knowledge, or previous views of the function seen in the viewing window. Any of these factors may have contributed to the circumstance became a defining moment or not.

5.3. **Defining Moment 2: Ongoing Search for a Global View of the Function**

The second set of defining moments to be subjected to more in-depth analysis related to the situation where students moved beyond their initial search and undertook an *ongoing search for a global view* (see Figure 5.14). This ongoing search may have

Figure 5.14. The circumstances related to the second set of defining moments.

involved the students experiencing difficulties finding a global view of the function after a period of exploration. This second set of defining moments relates to how students responded to particular views of the function during their ongoing search for a global view. The views seen included an apparent no view and the apparent vertical lines. The response actions observed were only the use of Zoom Out and the making of adjustments to the WINDOW settings in contrast to the first set of defining moments when use of Zoom Fit was also observed.

5.3.1 Response to Apparent No View of the Function

At various stages of their solution some students were confronted with a viewing window where the graph was absent or appeared to be absent. The use of Zoom Out in response to an apparent no view of the function by students in this study facilitated the solution process becoming routine. This occurred in the standard window (Pair 3) and in a view other than the standard window as experienced by Pair 4. In addition, Pairs 3 and 4 had previously experienced similar situations in their initial search for a global view.

In their ongoing search for a global view, Pair 3, Pete and Kate, returned to the standard window, however unlike during their initial search, on this occasion they undertook the response action, use of Zoom Out, resulting in part of the function becoming immediately visible in the viewing window, as shown in Figure 5.15. However, they did not appear to use this view of the graph to inform their selection of values in the viewing window as evidenced by their consequential actions of altering the WINDOW settings to view only the first quadrant. This was despite their response action, use of Zoom Out, clearly showing a section of the graph visible in the second and third quadrants as shown in Figure 5.15.
When Pair 4, Reem and Ali, initially found a viewing window with apparently no visible graph, they *adjusted the WINDOW settings* (Figure 5.15). It was not the standard window they began in, nor early in their solution process, and, they clearly used previous views to inform their changes as they reduced both the viewing domain and range, resulting in an apparent “thickening” of the $y$ axis as a section of the graph began to become distinguishable. It is inferred from their actions that they were aware that part of the graph, coincident with the $y$ axis, was visible in the first and subsequent windows.

*Figure 5.15.* Response actions to apparent no view during ongoing search for a global view.

Again, students undertook different actions upon seeing an apparent no view of the function and it can be inferred from this that their mathematical, as well as their graphing calculator, knowledge may have been a contributing factor in their action selection. The fact that Pair 3 followed different actions on different occasions suggests that their confidence (or lack of confidence) in the correctness of their mathematical knowledge may also have been a contributing factor in their action selection. Alternatively, they may have been experimenting with different actions. With regard to graphing calculator knowledge, it is apparent that Reem and Ali of Pair 4 knew that the
degree of magnification coupled with the resolution of the graphing calculator screen may not have allowed the y axis and a branch of the graph to be displayed separately. By reducing the viewing domain they were effectively increasing the magnification of the screen to facilitate this separation.

5.3.2 Response to Sighting of Apparent Vertical Lines

Three of the pairs (2, 4, and 5) saw apparent vertical lines in their initial search for a global view as described earlier. Their actions varied from the use of Zoom Fit (Pair 2), adjustment of the WINDOW settings (Pair 4), and use of Zoom Out (Pair 5). For Pairs 2 and 5 these response actions resulted in a global, or almost global, view of the function becoming visible in the viewing window. For Pair 4, however, their repeated adjustment of viewing domain and range was unsuccessful and led to a view of a thickened y axis. However, in their ongoing search for a global view, on sighting the apparent vertical line view of the function, all pairs except Pair 1, who did not encounter this view at any time, adjusted the WINDOW settings. This adjustment to the WINDOW settings involved altering one or both of the viewing domain and range.

Pairs 2 and 5 both adjusted the WINDOW settings, altering the viewing range only, an action they repeated twice which led to success with an almost global view of the function becoming visible in the viewing window. Pair 3, in contrast, initially altered both the viewing domain and range. This was followed by a further adjustment to the viewing range, whereby the right x intercept became clearly visible. This pair then identified the coordinates of this key feature. Later in their solution process Pair 3 again sighted the apparent vertical lines and again adjusted the WINDOW setting, altering both the viewing domain and range. On this occasion a curved section of the graph became visible, the response action was repeated and an almost global view of the function resulted. Pair 4, who had little success when undertaking the response action, adjusting the WINDOW setting, in their initial search for a global view undertook the
same action in their *ongoing search for a global view*. However, on this occasion they experienced greater success with their repeated actions resulting in an almost global view of the function.

To sum up, the same pairs of students did not always undertake the same response action when confronted with identical outputs of the graphing calculator. Whilst the response action, *use of Zoom Out*, was used infrequently it was effective. However, the more common response action, *adjusting the WINDOW settings*, became an appropriate action when undertaken in conjunction with informed choices based on previous views and mathematical knowledge.

### 5.4. Defining Moment 3: Consideration of the Numerical Representation

The third set of defining moments (see Figure 5.16) relates to the situation where students were experiencing difficulties finding a global view of the function. This occurred after the pairs of students had undertaken a period of exploration using only the graphical and algebraic representations. Whilst this defining moment may appear to be similar to, or a subset of the previous set of defining moments, it is distinct in that the pair of students who experienced it, Pair 3, were focussed holistically on the solution they were attempting to determine rather than on any one particular view seen at a given time in the viewing window, as was the case for the second set of defining moments.

![Figure 5.16](image-url)

*Figure 5.16. The circumstances related to the third set of defining moments.*

Only Kate of Pair 3 considered using the numerical representation, however, when Pete suggested this was cheating, after an initial challenge to this statement, she allowed her partner to follow other pathways. In contrast, as shown in chapter 4, all pairs of
students spent some time, even if briefly, making simultaneous use of the algebraic and the graphical representations.

The behaviour and response actions of the students in this study, as they experienced difficulties in determining a global view of the function, in either ignoring the numerical representation or in considering and rejecting its use has implications for teaching and learning as will be discussed in chapter six. The additional information provided by this representation may well have facilitated the solution process for those pairs of students experiencing difficulties or provided a more efficient and effective solution process.

5.5. **Defining Moment 4: Use of Scale Marks**

The fourth set of defining moments (see Figure 5.17) is related to the response action *use of scale marks*. This occurred in three situations, namely, the *need to produce a sketch* of the function, the *need to determine the coordinates of the key features* of the function, and the *need to find a global view* of the function. The four conditions in this set of defining moments were the *facilitation of the solution sketch*, the *need to eliminate “ugly axes”*, the *need to obtain estimates of the key features* of the function in order to check the values determined by other more accurate methods, and a *lack of a global view of the function*.

![Figure 5.17](image_url)

*Figure 5.17. The circumstances related to the fourth set of defining moments.*

The use of scale marks varied in students’ intent, consequences, and effectiveness. Pair 4 demonstrated a poor understanding of the effect of the scale marks as they adjusted these in an apparent attempt to alter the portion of the graph being viewed. Pairs 1 and 5, in contrast, made different but effective use of the scale marks as they
adjusted these to improve their view of the graphical representation of the function. In addition, the less efficient and effective solution process of Pair 2 compared to Pair 1 was due in part to the lack of use of scale marks by Pair 2.

5.5.1 Misunderstanding of the Effect of the Scale Marks

In addition to setting the viewing domain and range, the WINDOW settings also include the options to adjust the scale marks on either axis. One pair undertook this action in their search for a global view of the function, demonstrating their lack of understanding of the effect of this on the graph. Reem and Ali adjusted the $x$ and $y$ scale marks from the homogeneous default settings of one to another homogeneous scaling system with the unit being altered to two, as shown in Figure 5.18 after they were confronted with a thickened $y$ axis indicating the possibility of a section of the graphical representation of the function being present in the viewing window but indistinguishable from the $y$ axis. This adjustment of the scale marks suggests they believed that the scale settings may have an effect on the view of the graph but it did not as they soon saw. However, they were undeterred by this, possibly believing that the size of the increase was not large enough to result in a section of the graph being differentiated from the $y$ axis. They made similar adjustments to the $x$ scale shortly afterwards, this time editing only the $x$ scale value, as shown in Figure 5.19.

![Figure 5.18](image1.png)  
**Figure 5.18.** Exploring the effect of the scale marks.

![Figure 5.19](image2.png)  
**Figure 5.19.** Continued exploration of the effect of the scale marks.
Obviously, in neither case *altering the scale marks* had any visible effect on the graphical representation of the function. The explicit *use of scale marks* had not been taught to this pair of students in class. Perhaps, given their lack of success using other items in the WINDOW menu and their total reluctance to use any other features of the graphing calculator their foray into exploring the scale marks was the only untried option left open to them.

### 5.5.2 Interactions between View and Scale Marks

Judicious *use of scale marks* facilitated students’ identification of key features of the function. Table 5.3 allows the effect of various uses made of the scale marks, by students in the study, on the global view of the function to be examined. Students may or may not have adjusted the scale marks, in either or both of the horizontal directions and this may or may not have improved the view of the graph. The use of scale marks is classified as poor (not adjusted or adjusted with minimal or no impact on the view of the graph) or good (effective use of scale marks or removing of the scale marks). Two pairs made good *use of scale marks*.

Of the three pairs, Pairs 1, 2, and 5, for whom the solution process was routine, only Pair 1, one of the year 12 pairs, *used the scale marks* to effectively produce a complete view of the function that would also most readily facilitate their identification of the key

Table 5.3

*The Use of Scale Marks*

<table>
<thead>
<tr>
<th>Scale Marks</th>
<th>Poor</th>
<th>Good</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pair 5</td>
<td><img src="image1" alt="Graph" /></td>
<td><img src="image2" alt="Graph" /></td>
</tr>
<tr>
<td>Pair 3</td>
<td><img src="image3" alt="Graph" /></td>
<td><img src="image4" alt="Graph" /></td>
</tr>
<tr>
<td>Pair 1</td>
<td><img src="image5" alt="Graph" /></td>
<td><img src="image6" alt="Graph" /></td>
</tr>
</tbody>
</table>
features. The other two pairs made no alteration to the scale marks throughout their solution of the problem task. Of the two pairs, 3 and 4, for whom the solution process did not become routine, one of these pairs altered the scale marks, the other did not. Unlike Pair 1 who adjusted the scale marks several times until they were satisfied with the result, Pair 3 simply set the scale marks to zero. Whilst this does nothing to facilitate the solution process, it eliminates the “thick axes” that result from the large number of scale marks in the viewing window.

With the Year 12 pairs it is apparent that the good use of scale marks in conjunction with good planning by Pair 1 resulted in their producing a better solution than Pair 2 who made no alteration to the scale marks throughout their solution. The good use of scale marks by Pair 1 resulted in a more effective graphical representation of the function, as shown in Figure 5.20. The view produced by Pair 1 was more effective in that it enabled them to work more easily with the function, allowed them to recognise approximate coordinates of the key features, readily facilitated their identification of the key features, and allowed them to confidently produce a written solution to the problem task.

![Figure 5.20. Interactions between scale and view, Pairs 1 and 2 respectively.](image)

To sum up, defining moments involving the defining action: use of scale marks, had either a positive or negative impact on the solution process. Pair 4 demonstrated that they failed to understand the effect of adjusting the scale marks on the visible section of the graph and their use of scale marks resulted in the length of time to solution being extended. In contrast, the mathematical and graphing calculator knowledge of Pairs 1 and 3 was applied to their use of scale marks to facilitate their solutions.
5.6. Defining Moment 5: Use of Opportunistic Planning

The fifth set of circumstances related to defining moments (see Figure 5.21) that occurred when the students were considering options, for example, the finding of the global view first followed by identification of the key features or the undertaking of these concurrently. Two situations gave rise to these circumstances, namely, the need to establish global goals and subgoals and the need to formulate local and global plans to achieve subgoals and goals. The response action in these circumstances was opportunistic planning.

Figure 5.21. The circumstances related to the fifth set of defining moments.

In reviewing the students’ planning behaviours it must be borne in mind that the absence of overt behaviour of the students that was categorised as organising and planning (O) in the time-line diagrams does not exclude the possibility of covert planning. The presence of good overt planning can be inferred from students making the most judicious choices at opportune times in their solution as events unfold and being able to proceed immediately with the task without an obvious need to stop and plan as they have the mathematical skills to efficiently solve the problem task in a routine manner even when they meet unexpected obstacles such as confronting views. Planning can also be inferred from behaviour when difficulties arise. In these circumstances good planning occurs when the students take a step back in their solution path and make an alternate choice, whereas bad planning is demonstrated when a current pathway is abandoned without evaluation and the students return to their initial choices without reflection on what they have just done.
Good planning can be inferred from Linh and Ahmed’s actions in that they found a perfect global view from their perspective before proceeding to identify any of the key features of the graphical representation of the function. They were the only pair whose behaviour falls into two distinct, time-wise categories, being the finding of a good global view followed by the identification of key features. This pair demonstrated their integration of graphing calculator skills seamlessly into their mathematical routines. They undertook six instances of overt planning taking a total of 55 seconds spread over their solution process which took ten minutes in total. Thus, although the length of time spent on overt planning was short, it was effective ensuring their solution evolved smoothly.

On the other hand, Hao and Abdi demonstrated poor planning early into their solution process. Although they found a global view of the graph of the function after 1 minute, compared to 2 minutes 20 seconds for Linh and Ahmed, as they proceeded to identify the key features they became dissatisfied with this view, as some of the key features were barely distinguishable from the axis. Perhaps the fact that they needed to produce a sketch contributed to their dissatisfaction with their global view of the function, alternatively, they may have believed the view would reduce the accuracy of the coordinates of the key features of the function. As they proceeded to improve the view they demonstrated little planning to facilitate their progress. Clearly, they had knowledge of Zoom Fit and using this in conjunction with some sensible adjustments to the viewing domain, even if this took several attempts, would have more efficiently produced an improved global view of the function than what they did. However, Hao and Abdi took several backward steps in their solution process and then proceeded to alter only WINDOW settings as they sought an improved view of the function. They then proceeded to make nine adjustments to the WINDOW settings until they had a
global view of the function with which they were satisfied. This took an additional 5 minutes and 20 seconds.

The next illustration demonstrates how poor choices of local plans in pursuing a subgoal (namely, the display of an entire global view of the function) proved very costly. After finding themselves very close to a global view of the function early on and the solution process having the potential to become routine, Susan and Jing exhibited poor planning in attempting to reach a subgoal in their solution process. After seeing a view of the function that included all key features except the minimum turning point, this pair turned to the algebraic representation as they searched for zeroes of the function. Being unsuccessful they returned to the WINDOW settings where they inexplicably reduced the viewing range thus presenting themselves with three apparently vertical lines, one being coincident with the y axis. Instead of immediately backtracking to their almost global view, they decided to try manipulating this rather perplexing view of the function in order to produce a global view. From their almost global view they took 10 minutes before they returned to an identical view. However, on this occasion they required only 10 seconds to obtain a global view of the function in their viewing window.

Good planning had a positive impact on the solution process, enabling an efficient and effective solution to be produced, one that the students were confident was correct even if they were dissatisfied with some aspects of their solution. Alternatively, poor planning lengthened the time required to produce a solution and may have resulted in the students being exposed to confronting views. Poor planning, however, sometimes resulted in increased opportunities for learning during the problem task as the resulting confronting views needed to be considered and a place found for them in the students’ previous understanding of cubic functions. The type of planning involved is described as opportunistic planning (Hayes-Roth & Hayes-Roth, 1979) as decision making is
essentially event-driven. The students made decisions in response to events that occurred rather than proactively pre-planning what they intended to implement.

In this section, it has been seen that the circumstances which were responded to by good planning were instrumental in the solution of Pair 1, for example, ensuring their solution was the most efficient and effective and was the major difference between their solution and that of the second pair of Year 12 students. Alternatively, when similar circumstances were responded to by poor planning whilst this increased both the time required to produce a solution and the number of confronting views seen, it allowed the opportunity for learning to occur in the process of solving the problem task.

5.7. Defining Moment 6: Engagement in Discussion

The sixth set of defining moments (see Figure 5.22) relate to the response action engagement in discussion. This set of defining moments occurred in response to four different situations, being (a) the difficult nature of the particular function being used; (b) the use of a feature of the graphing calculator with which one of the pairs was unfamiliar, or where one member of the pair anticipated this may be the case; (c) either or both students were stuck or unsure how to proceed; or (d) when several options presented themselves. Four different conditions were observed during these sets of circumstances, namely, (a) the need for the students to be confident that their mathematical understanding was correct and/or shared by their partner, (b) the need to explain their use of the graphing calculator to their partner, (c) choices needed to be made, and (d) options needed to be considered.

5.7.1 Development of a Framework for Analysing Discussion

In analysing the discussion, the focus was on understanding the kind of knowledge gained from learning about functions in the classroom in a graphing calculator
environment, the knowledge of the technology shown, the extent to which this knowledge was able to be exhibited in solving the particular problem task used in the study, and the cognitive and metacognitive processes that the students were using to achieve this. Some researchers, for example Shoaf-Grubbs (1994), have raised the notion that the graphing calculator supports a dialogue of communicating with oneself as hypotheses are tested and developed. This notion can be extended to support dialogue between students when working in pairs, as in this study, and the dialogue can either be with oneself or a partner as ideas are articulated, tested, and developed orally. An example of this occurred during the following dialogue of Kate (Pair 4):

No, because what if the graph? Actually, it’s too … it’s a linear at the moment, but I don’t know … if we start there … how about, just to check it out, to see if it’s anything else, we go down 4?

The dialogue of the students and their interactions were initially analysed using Kumpulainen’s (1994) framework for children’s talk during a collaborative task involving technology. Neither this framework nor its adaptation by Goos et al. (2000) for student roles as they work with technology proved adequate in the situation occurring in this study. A framework developed by Wood and Turner-Vorbeck (Wood, 2002, p. 64) from a microanalysis of class discussion after students had attempted a task, albeit in pairs, proved more suitable. It was felt that the categories described by Wood and her focus on the type of questions asked and the role expected of the responder could be adapted to analyse interactions occurring in this study. In transferring this notion to interaction and discussion between pairs of students working with one graphing calculator, it was decided to look at the type of questions students

Figure 5.22. The circumstances related to the sixth set of defining moments.
asked of others and themselves during the solution process. Also considered was the role the questioner expected of the partner (or themself) in responding to the question for each question type.

5.7.1.1 Framework for Analysing Discussion Questions

In the modified framework, the first three categories of questions, comprehension, inquiry, and argumentation, were adapted from Wood (2002). Two further question categories, organisation and verification, were considered necessary given the different context, which was focussing on students displaying their knowledge of functions in a graphing calculator environment, whilst working on a problem task not just discussing their solution at the end of the task. These latter categories were expected to highlight student use of cognitive and metacognitive processes not encompassed by the first three categories.

*Comprehension questions* ask for information about a strategy being used or delve into a partner’s (or a student’s own) knowledge. This knowledge can be of the cubic function or graphing calculator knowledge. The role of the partner is telling, for example, “Does it have a [stationary] point of inflection?” No reasoning is required by the partner in the response to the question. In this case the partner used his mathematical knowledge and merely replied, “No, it wouldn’t”.

*Organisation questions* are related to planning and decision making. They include those that assign tasks to the respective students (e.g., “Can I do something?”) or offer or ask for strategies (e.g., “Why don’t we just try the y?”). Organisation questions in this framework cover more than organising the task or process, or controlling behaviour as described by Kumpulainen (1994), and are closer to those described by Garofalo and Lester (1985) in their metacognitive framework for studying mathematical performance. The latter authors describe organisation as “planning of behaviour and the choice of
actions” (p. 171) including identification of goals and subgoals, global planning, and local planning (to implement global plans). The role of the partner when such questions are asked is to agree or disagree and perhaps offer alternatives.

*Inquiry questions* are “why” questions, for example, “Oooh, what sort of graph is it?” The role of the partner is to clarify actions, explain the contribution of a particular action to the solution process, or give reasons for his or her actions. For example, in response to the previous question the partner answered, “Aah, it’s one of these …. So it’s been shifted down … say that’s the graph, without that, it’s like, whatever it is, it’s like that, but it’s shifted down, that’s [why] when the WINDOW was at [-]92 you couldn’t see it”.

*Argumentation questions* ask for justification or are a challenge to the partner (or oneself). The role of the responder is to justify or defend his or her actions or suggestions based on mathematical and/or graphing calculator knowledge. For example, when trying to distinguish the central $x$ intercept from the $y$ intercept Pete asks rhetorically, “… or is that a $y$ intercept?”, Kate’s response of “What?” challenges his authority and calls on him to justify his belief.

*Verification questions* involve the “evaluation of decisions made and of outcomes of executed plans” (Garofalo & Lester, 1985, p. 171). They involve the monitoring of progress (e.g., “Is that finished?”), ask for verification (e.g., “Are we sure the equation is right?” “All right, I’ll read it and you check it”), or question the partner’s degree of confidence in an action that has been undertaken or suggested (e.g., “Are you sure?” “Yes”). The role of the partner is to agree or disagree and suggest a verification strategy or give a reason for the response. Verification questions include the checking and evaluation of local and global plans and the expression of satisfaction, “dissatisfaction, and/or frustration with the solution” (Stillman & Galbraith, 1998, p. 193).
In undertaking the categorisation of questions asked by the students, the role the partner took in the interaction was often required for the question to be unambiguously categorised. At times a question offered the opportunity for the set of circumstances which gave rise to it to become a defining moment in the solution for a particular pair of students, and equally leaving the question unanswered led to the occurrence of a defining moment as the opportunity was lost. The consequences of these defining moments were thus very different. The questions and the classification of these are documented in Appendix E.

5.7.2 Question Types Used and Roles of the Responders during Discussion

The number of questions asked by the pairs varied and this depended in part on the total number of exchanges. Both the percentage of dialogue that consisted of questions and the categories of question types asked varied from pair to pair (see Table 5.4). However, when the rate of asking questions is considered with respect to time, it is seen that Pair 3, as well as asking the most questions had the highest rate, asking on average 2.8 questions per minute during their solution. With regard to question type, Pairs 1 and 3 mainly used comprehension questions, Pair 2 asked organisation and verification questions more often than other types of questions, whereas Pair 5 asked a much higher proportion of organisation questions than any other pair. Pair 4 asked so few questions that little can be inferred with regard to question type.

As noted earlier, Pairs 1 and 3 mainly asked comprehension type questions. In addition, both these pairs asked comprehension questions that sought mathematical and graphing calculator knowledge in equal proportion. In contrast, Pairs 2 and 5 asked far fewer questions of this type. Those that they did ask were almost exclusively seeking information of a mathematical nature. Pair 4 asked only one question in this category and it sought both mathematical and graphing calculator knowledge.
Table 5.4

The Use of Question Types

<table>
<thead>
<tr>
<th>Pair</th>
<th>No. of Questions (% of total exchanges)</th>
<th>No. of Questions per min.</th>
<th>Comprehension</th>
<th>Organisation</th>
<th>Inquiry</th>
<th>Argumentation</th>
<th>Verification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>17 (18%)</td>
<td>1.6</td>
<td>10 (59%)</td>
<td>3 (18%)</td>
<td>1 (6%)</td>
<td>-</td>
<td>3 (18%)</td>
</tr>
<tr>
<td>2</td>
<td>15 (11%)</td>
<td>1.1</td>
<td>4 (27%)</td>
<td>4 (27%)</td>
<td>2 (13%)</td>
<td>-</td>
<td>5 (33%)</td>
</tr>
<tr>
<td>3</td>
<td>84 (22%)</td>
<td>2.8</td>
<td>46 (55%)</td>
<td>17 (20%)</td>
<td>10 (12%)</td>
<td>3 (4%)</td>
<td>8 (10%)</td>
</tr>
<tr>
<td>4</td>
<td>3 (30%)</td>
<td>0.1</td>
<td>1 (33%)</td>
<td>1 (33%)</td>
<td>-</td>
<td>-</td>
<td>1 (33%)</td>
</tr>
<tr>
<td>5</td>
<td>27 (18%)</td>
<td>1.0</td>
<td>6 (22%)</td>
<td>12 (44%)</td>
<td>6 (22%)</td>
<td>-</td>
<td>3 (11%)</td>
</tr>
</tbody>
</table>

Note. Percentages for the total are a comparison of the number of questions and number of dialogue exchanges, all others are as a percentage of the total number of questions for that pair.

For Pair 1, Linh asked almost all of the comprehension questions, some of these were a result of her interaction with a more competent peer with regard to useful features of the graphing calculator, namely Zoom Fit and Equation Solver. In both instances the result was a better understanding by Linh of the role of the graphing calculator as a tool to find a complete view of the function. The later comprehension questions (half of the total of this category, see Table 5.5) in this pair’s solution process were asked by Linh in her role of recorder with her partner mainly taking the responsibility for operating the graphing calculator. If their roles had been reversed, at this point, Ahmed could well have asked the same questions.

Pair 3 asked comprehension type questions, throughout the solution process, as confirmed by Table 5.5, with questions of this type being asked in six of the eight activity categories. This ensured that both students were aware of strategies being undertaken and available mathematical knowledge and graphing calculator knowledge were shared at all times. Their behaviour was in contrast to most other pairs who tended to be very quick to use the graphing calculator functions, seeking little clarification of
Table 5.5

*Number of Comprehension Questions by Activity Category*

<table>
<thead>
<tr>
<th>Activity</th>
<th>Pair 1</th>
<th>Pair 2</th>
<th>Pair 3</th>
<th>Pair 4</th>
<th>Pair 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>1</td>
<td>8</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>VW</td>
<td>1</td>
<td>3</td>
<td>14&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>SM</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ID</td>
<td>2</td>
<td>11</td>
<td>1&lt;sup&gt;b&lt;/sup&gt;</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>1</td>
<td>10&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1&lt;sup&gt;b&lt;/sup&gt;</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Re</td>
<td>5</td>
<td>4</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>10</td>
<td>4</td>
<td>47</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

Note. R = reading; O = organising or planning; VW = selecting a viewing window; ID = searching for or identifying a local feature; GV = considering the global view; SM = adjusting scale marks; E = evaluating; Re = recording. <sup>a</sup>Includes one question when simultaneously searching for a viewing window and evaluating. <sup>b</sup>Includes one question when simultaneously searching for or identifying a key feature and evaluating.

their own ideas. Again, the questions asked provided an opportunity for learning to occur, specifically the use of Zoom Box. In addition, mathematical knowledge was articulated and developed throughout the task. Similarly to Pair 1, Pair 3 undertook a series of comprehension questions as they produced a sketch during their recording activity. In this case the non-recorder, Pete, asked the questions as he made sure all the key information was recorded to his satisfaction. Whilst at times their engagement in discussion may have hindered the development of an efficient solution, it laid the groundwork for facilitating the co-construction of knowledge relating to their understanding of functions and the relevant mathematical and graphing calculator knowledge.
In comparing Pairs 1 and 3 who had their major question type as comprehension, the task was mainly routine for Pair 1, although the opportunity for Linh to learn from her more technologically expert partner was taken up and responded to. For Pair 3, the task was far from routine, with the majority of questions being comprehension reflecting the teaching of each other and the clarifying of their own understanding through the articulation of their thoughts.

Of the four comprehension questions asked by Pair 2, two of these occurred early in their solution process and consisted of Hao delving into his own mathematical knowledge. Their latter two comprehension questions, asked by Abdi, occurred later and asked for information about a strategy being used. For Pair 4, the solitary comprehension question, eliciting the response of “I don’t know”, did not facilitate further discussion and hindered the progress of this pair, whereas the use of a different question type, requiring a response involving more than merely telling may have resulted in greater progress. The comprehension questions asked by Pair 5 included two seeking mathematical and graphing calculator knowledge and four as they produced their hand drawn sketch asking for information that the action undertaken by the drawer was correct. The smaller proportion of comprehension questions by these three pairs are a reflection, in part, of the lack of teaching occurring during the task, either of mathematical or graphing calculator knowledge and minimal articulation of the students’ own ideas.

Pair 5 asked mainly organisation questions and these constituted a much higher proportion of their question asking than for the other pairs (see Table 5.4). These questions offered and asked for strategies and delved into the students’ mathematical and graphing calculator knowledge. The majority occurred whilst searching for a viewing window (see Table 5.6). This pair found the task a challenge. However, from
Table 5.6

*Number of Organisation Questions by Activity Category*

<table>
<thead>
<tr>
<th>Activity</th>
<th>Pair 1</th>
<th>Pair 2</th>
<th>Pair 3</th>
<th>Pair 4</th>
<th>Pair 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>1</td>
<td>2(^a)</td>
<td></td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>VW</td>
<td>1</td>
<td>1</td>
<td>4(^b)</td>
<td></td>
<td>7</td>
</tr>
<tr>
<td>GV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ID</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td>5(^c,b)</td>
<td></td>
</tr>
<tr>
<td>Re</td>
<td>1</td>
<td></td>
<td></td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>3</td>
<td>4</td>
<td>17</td>
<td>1</td>
<td>12</td>
</tr>
</tbody>
</table>

*Note.* R = reading; O = organising or planning; VW = selecting a viewing window; ID = searching for or identifying a local feature; GV = considering the global view; SM = adjusting scale marks; E = evaluating; Re = recording. \(^a\)Includes one question when simultaneously organising and evaluating. \(^b\)Includes one question when simultaneously searching for a viewing window and evaluating.

their emphasis on organisation questions it can be inferred that both valued the ideas of their partner, wanted to work cooperatively, and valued the opportunity to articulate their own knowledge or strategies. Both the decision making and planning by this pair were jointly undertaken.

None of the organisation questions asked by Pair 5 occurred whilst they were undertaking evaluation activities (see Table 5.6). This is in contrast to Pair 3 who also asked a high number of organisation questions but three of these were during evaluation and another two occurred when evaluation was being undertaken concurrently with other activities, namely organising and searching for a viewing window. Most of this pair's remaining organisation questions were asked during the activities searching for a
viewing window and identifying key features. The only question of any type asked during reading was an organisation question asked by Pair 2.

At least one inquiry question was asked by each of the pairs other than Pair 4, as shown in Table 5.4. For Pairs 1 and 2, this was the least used question type excluding argumentation questions and only one and two questions were asked, respectively. However, Pair 3 asked ten of this question type. For Pair 5, inquiry questions were the second most common question type used. One of the students in each of Pairs 3 and 5 asked all except one of the inquiry questions, being Kate and Jing, respectively. When used, inquiry type questions tended to be asked early in the first half of the solution process or throughout the second half of the solution process. All pairs who asked inquiry questions did so at least once during evaluation which was the activity category where they were most frequent, as shown in Table 5.7. As for the two previous question types, Pair 3 asked questions in the majority of the activity categories.

Table 5.7

<table>
<thead>
<tr>
<th>Activity</th>
<th>Pair 1</th>
<th>Pair 2</th>
<th>Pair 3</th>
<th>Pair 4</th>
<th>Pair 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td></td>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VW</td>
<td>1(^a)</td>
<td>1(^a)</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ID</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>1(^a)</td>
<td>1</td>
<td>4(^a)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Re</td>
<td></td>
<td></td>
<td></td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>0</td>
<td>6</td>
</tr>
</tbody>
</table>

*Note.* R = reading; O = organising or planning; VW = selecting a viewing window; ID = searching for or identifying a local feature; GV = considering the global view; SM = adjusting scale marks; E = evaluating; Re = recording. *Includes one question simultaneously searching for a viewing window and evaluating.
Argumentation type questions were conspicuous by their virtual absence. Only Kate of Pair 3 asked this type of question asking her partner to justify his suggestions and actions. One of these occurred whilst searching for a viewing window and the remaining two as they undertook evaluation. The reason for the lack of use of argumentations by the students varied. For Pair 1, for example, the ease with which they collaboratively solved the problem task resulted in little need for questions of this type. For Pairs 2, 3, and 5 the students tended to ask inquiry questions rather than argumentation questions possibly suggesting they valued the expertise of their partner and wanted the mathematical knowledge, graphing calculator knowledge, and strategies explained and shared rather than being placed in a position of disagreement where their role would be to ask a challenging or justifying question of their partner.

Verification questions constituted a greater proportion of the questions for Pair 2 than for the other pairs as shown in Table 5.4. All except two of their organisation and verification questions were asked by Hao, with five of their fifteen questions being verification questions, for example, “Did you put the correct window?” Given the collaborative approach undertaken by this pair, Hao’s questioning suggests that he valued the role of planning and decision making to the extent of believing that this was essential to articulate when working in pairs.

The use of verification questions, by Pair 2, may have been the distinguishing factor between their solution and that of Pair 5 in terms of efficiency and effectiveness. Both pairs found a global or almost global view of the function early in their solution process, however for Pair 5 their early success did not see them effectively and efficiently find a solution. This was possibly due in part to their low reliance on verification questions and hence the undertaking of little monitoring of their slower progress, than early plans indicated, toward their global goal.
As may be expected, the majority of verification questions were asked during an evaluation activity (see Table 5.8). On three of these occasions students were simultaneously engaging in other activities, namely, organising, adjusting the viewing window, and recording. The lack of verification questions in some categories should not be taken as being as an indication that students’ solution processes were inferior to those of others. Pairs 2 and 5, for example, did not alter the scale marks so verification questions were unnecessary in this activity category.

The defining action, *engagement in discussion*, through the analysis of question type used in that discussion has been shown to allow the solution process to be facilitated. However, this facilitation was not automatic, the response of the partner to any question was crucial in determining whether the situation became a defining moment or not. The

Table 5.8

*Number of Verification Questions by Activity Category*

<table>
<thead>
<tr>
<th>Activity</th>
<th>Pair 1</th>
<th>Pair 2</th>
<th>Pair 3</th>
<th>Pair 4</th>
<th>Pair 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O</td>
<td>1&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VW</td>
<td>2</td>
<td>3&lt;sup&gt;b&lt;/sup&gt;</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>GV</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SM</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ID</td>
<td>1&lt;sup&gt;a&lt;/sup&gt;</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>E</td>
<td>1&lt;sup&gt;a&lt;/sup&gt;</td>
<td>3&lt;sup&gt;c&lt;/sup&gt;</td>
<td>3&lt;sup&gt;b&lt;/sup&gt;</td>
<td>1</td>
<td></td>
</tr>
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<td>1&lt;sup&gt;c&lt;/sup&gt;</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

*Note.* R = reading; O = organising or planning; VW = selecting a viewing window; ID = searching for or identifying a local feature; GV = considering the global view; SM = adjusting scale marks; E = evaluating; Re = recording. <sup>a</sup>Includes one question when simultaneously organising and evaluating. <sup>b</sup>Includes one question when simultaneously searching for a viewing window and evaluating. <sup>c</sup>Includes one question when simultaneously recording and evaluating.
number, rate, and types of questions asked varied across the pairs. The use of comprehension questions provided the opportunity for learning to occur through the sharing of mathematical and graphing calculator knowledge, strategies used being determined by both students, and agreement to be reached as to what was required by the task and what should be recorded. Organisational questions allowed the students to work cooperatively to solve the problem task, whereas, an efficient and effective solution process was shown to be facilitated by the use of verification questions. Inquiry questions enabled one partner of a pair to gain knowledge from the other and thus led to shared ownership of the solution process. Argumentation questions in this study did little to assist the solution process as only three were asked and these were used by only one of the pairs. However, changes to the number, rate, and types of questions asked by the pairs may well have led to a more seamless solution process being undertaken.

As seen in chapter four, particular activity categories can be classified by their predominant cognitive or metacognitive level (Artzt & Armour-Thomas, 1992, p. 142). In this study, reading (R), selecting a viewing window (VW), searching for or identifying a local feature (ID), considering the global view (GV), adjusting scale marks (SM), and recording (Re) were classified as predominantly cognitive and organising or planning (O) and evaluating (E) were classified as predominantly metacognitive. It is possible to classify the question types according to their predominant cognitive level as well. However, regardless of the question type and activity category both cognitive and metacognitive behaviour could occur, and did, across all activity categories and the range of question types.

The analysis of question type by activity category has demonstrated that whilst the latter may be classified by their predominant cognitive or metacognitive level, question types are not category specific and may give rise to metacognitive activity in any category. Each question type was asked over the full range of activities except for
reading. The only question type asked during reading was one comprehension question. Other than Pair 4, all pairs regardless of the dominant question type demonstrated both cognitive and metacognitive behaviour. In contrast, for Pair 4 who engaged in minimal question asking and dialogue in general, little metacognitive activity can be inferred.

5.8. Defining Moment 7: Identification of Key Features of the Function

The seventh and final set of defining moments (see Figure 5.23) relate to the condition the need to identify key features of the function and the response actions, (a) use of dedicated features of the graphing calculator, (b) use of TRACE, and (c) use of the free cursor. This set of defining moments occurred in two different situations, being the global view found and the observing of key features in the viewing window. The situation, global view found, indicates that the shape and all key features of the function were visible in the viewing window as shown in Figure 5.24(a). In contrast, the situation, observing of key features in the viewing window, indicates that at least one key feature, but not all, was visible in the viewing window as in Figure 5.24(b) where only an x intercept and a y intercept are visible.

Figure 5.23. The circumstances related to the seventh set of defining moments.

Figure 5.24. The two situations resulting in the seventh defining moment.

The graphing calculator global view, together with the WINDOW settings that produced this view, and the hand sketch of Pair 1 identifying key features of the
function are shown in Figures 5.25 and 5.26, respectively. (See Appendix F for these for

![Window settings](image)

**Figure 5.25.** The global view of Pair 1 and the WINDOW settings used to produce this.

![Solution script](image)

**Figure 5.26.** Solution script of Pair 1.

the remaining pairs.) The major difference in the recording of the five pairs was that
Pair 4 listed the key features next to a rather cursory sketch (see Figure 5.27), rather than labelling the sketch as both modelled and expected by their teachers. Such labelling allows the students to check the reasonableness and accuracy of their solution more easily. The coordinates identified by each pair for the key features are shown in Table 5.9. The final row of the table identifies the coordinates of the key features correct to two decimal places.

Both the choice of calculator feature selected and the correct use of the feature impacted on the accuracy with which the pairs recorded the coordinates of the key
features. The effects of the choice of calculator feature selected and the consequences of

\[
\frac{\text{gradient}}{y} = x^3 - 19x^2 - 1992x - 92
\]

\[
= 3x^2 - 1912x - 1992
\]

\[
y_{\min} \left( x^0, -92 \right) 
\]

\[
T, P. \left( 34, -5833.87 \right) 
\]

\[
\max T, P. \left( -17.62, -50583.38 \right) 
\]

Figure 5.27. Solution script of Pair 4.

Table 5.9

<table>
<thead>
<tr>
<th>Pair</th>
<th>y intercept</th>
<th>( x_{\text{intercept}} ) (left)</th>
<th>( x_{\text{intercept}} ) (central)</th>
<th>( x_{\text{intercept}} ) (right)</th>
<th>Maximum Turning Point</th>
<th>Minimum Turning Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0, -92)</td>
<td>(-36.1, 0)</td>
<td>(-0.046, 0)</td>
<td>(55.15, 0)</td>
<td>(-20.2, 24151)</td>
<td>(32.87, -50583.4)</td>
</tr>
<tr>
<td>2</td>
<td>(0, -92)</td>
<td>(-36.10, 0)</td>
<td>(-0.05, 0)</td>
<td>(55.15, 0)</td>
<td>(-20.20, 24151.23)</td>
<td>(32.87, -50583.38)</td>
</tr>
<tr>
<td>3</td>
<td>(0, -92)</td>
<td>(-36.1, 0)</td>
<td>(-0.05, 0)</td>
<td>(55.15, 0)</td>
<td>(-20.201, 24151.23)</td>
<td>(32.87, -50583.38)</td>
</tr>
<tr>
<td>4</td>
<td>(0, -92)</td>
<td>(-34, 0) (^a)</td>
<td>(^c)</td>
<td>(55, 0) (^b)</td>
<td>(-17.02, 23354.8)(^b)</td>
<td>(34.042, -50338.71)(^b)</td>
</tr>
<tr>
<td>5</td>
<td>(0, -92)</td>
<td>(-36.1, 0)</td>
<td>(-0.046, 0)</td>
<td>(55.15, 0)</td>
<td>(-19.89, 24143.7)(^b)</td>
<td>(32.66, -50579.92)(^b)</td>
</tr>
<tr>
<td></td>
<td>(0, -92)</td>
<td>(-36.10, 0)</td>
<td>(-0.05, 0)</td>
<td>(55.15, 0)</td>
<td>(-20.20, 24151.23)</td>
<td>(32.87, -50583.38)</td>
</tr>
</tbody>
</table>

Note. The final row of the table presents the coordinates accurate to 2 decimal places. \(^a\) Inaccurate result. \(^b\) TRACE used. \(^c\) No coordinates identified. \(^d\) Inaccurate use of dedicated feature.

this choice for accuracy on the type of key feature being identified are shown in Table 5.10. When correctly selected and used the use of dedicated features (e.g., Calculate Minimum or Calculate Zero) had a direct effect on the accuracy of the coordinates of the key features of the graphical representation of the function. In contrast, when other
features of the calculator such as the *use of free cursor* or *TRACE* were used for

Table 5.10

*Methods of Determining Key Features of the Function with Consequences for Accuracy*

<table>
<thead>
<tr>
<th>Actions</th>
<th>Consequences for accuracy of</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>y intercept</td>
</tr>
<tr>
<td>Using dedicated functions&quot;</td>
<td>Accurate (^{b})</td>
</tr>
<tr>
<td>TRACE</td>
<td>Accurate (^{e})</td>
</tr>
<tr>
<td>Free cursor</td>
<td>Not Accurate</td>
</tr>
</tbody>
</table>

*Note.* "Assuming correct use of the dedicated feature. \(^{b}\)Calculate Value. \(^{c}\)Calculate Zero or Equation Solver. \(^{d}\)Calculate Minimum or Calculate Maximum depending on the type of turning point. \(^{e}\)Assuming symmetric viewing domain. \(^{f}\)Pair 4 used TRACE to determine the coordinates of the right x intercept accurate only to the nearest integer.

identification of the coordinates of the key features this was usually inaccurate. The only exception was the *use of TRACE* to determine the coordinates of the y intercept. When this occurred the y axis was centred in the viewing window, that is, a symmetric viewing domain was being used, and the resulting coordinates were able to be accurately identified by the graphing calculator.

Pairs 1, 2, and 3 correctly identified the coordinates of all key features. All pairs correctly identified the coordinates of the y intercept and the right x intercept. Pair 4 failed to identify the coordinates of the central x intercept and the accuracy of their identification of the remaining features was tempered by their method of using TRACE rather than a more accurate method such as Calculate Zero or Calculate Minimum.

The identification of the coordinates of the turning points for Pair 5 was inaccurate for two different reasons. When identifying the coordinates of the minimum turning point, they initially successfully used the Calculate Minimum feature to determine the
coordinates. However, the WINDOW settings resulted in superimposed output of the
graphical representation of the function and the coordinates of the minimum turning
point by the graphing calculator and the students could not read the values produced due
to the position of the graph. After adjusting the WINDOW settings appropriately, they
reslected Calculate Minimum, however, on this occasion poor selection of a search
domain resulted in the required point not being included. As a result, the minimum
point determined by the graphing calculator was not the minimum turning point but the
minimum value of the function in the specified domain. This was not noticed by the
students. The second inaccuracy occurred when, for some inexplicable reason, they did
not use the dedicated function Calculate Maximum instead using TRACE in their
identification of the maximum turning point. They clearly had knowledge of Calculate
Maximum as they had used its counterpart previously in the identification of the
coordinates of the minimum turning point.

The defining action: use of dedicated features of the graphing calculator, has been
shown to facilitate the finding of a complete graph of the function, however, the
inaccurate use of the dedicated features, the use of TRACE, or the use of the free cursor
to identify the coordinates of the key features tended to hinder the solution process.
Contrary to implications of research by others (e.g., Mitchelmore & Cavanagh, 2000;
Steele, 1995a, 1995b), who suggested students would stop after finding any graphical
representation of the function in the viewing window, all the students in this study
successfully solved the problem task. All five pairs of students found a global view of
the function and identified most, or all, of the key features, albeit to varying degrees of
accuracy. No matter how circuitous the route they undertook, each pair persisted until
they produced a complete view of the graph of the function.
5.9. Conclusion

This chapter has presented microscopic analysis of the seven defining moments that became apparent in the solution process of the pairs, as identified in chapter four. This analysis has been discussed as this researcher sought to explain possible reasons for the occurrence of each of the defining moments, what factors led to their use, or lack of use, by each of the pairs, and the implications this had for the solution process.

Some differences in the occurrences of the defining moments were as a result of experience; including greater time spent in a graphing technology learning environment, greater mathematical knowledge, greater graphing calculator knowledge, and greater ownership of the graphing calculator as a personal tool. A defining moment where these differences between the Year 12 and Year 11 students became apparent and may have been as a result of the aforementioned factors was Defining Moment 1: Initial Search for a Global View, where, in particular, the defining action; Use of Zoom Fit (a feature not used by the teachers of the students in the study) allowed the solution process to become routine, or potentially routine. Whilst other defining moments revealed differences between the year levels, it was only in Defining Moment 1 that the Year 12 students undertook a defining action, that none of the Year 11 pairs undertook, and that, in addition, had a significant impact on the solution process.

It appeared that the use of in-built features of the calculator better facilitated an effective and efficient solution process. This was demonstrated in Defining Moment 2: Ongoing search for a global view where the use of Zoom Out appeared to be a better choice at times than adjusting the WINDOW settings, and again in Defining Moment 7: Identification of the key features where the use of dedicated features when used appropriately and correctly enabled the key features of the function to be accurately identified.
Defining Moment 3: Consideration of the numerical representation was observed for only one of the pairs in the study, who rejected use of this representation, and this rejection contributed at least in part to their solution process being inefficient with respect to time. This was in direct contrast to the pilot study, where for example, the teacher expert pair used the numerical representation to facilitate their determination of an appropriate viewing window in which to observe a global view of the function, and to identify the key features of the function. The less experienced teacher pair at least considered the TABLE but again neither of the student pairs suggested its use.

Differences were also seen in the use of scale marks (Defining Moment 4) with Pair 1 maximising their effect in both the facilitation of their sketching and supporting their identification of key features. In contrast, Pair 3 by setting the scale marks to zero, and hence, eliminating the “ugly axes”, greatly improved (in their minds) the view of the function, a common student view as can be attested to by many secondary mathematics teachers working in a graphing calculator environment. At the other extreme, Pair 4 demonstrated their misunderstanding of the effect of the scale marks as they adjusted these in an attempt to alter the portion of the graph seen in the viewing window.

Overt planning took the form of opportunistic planning (Defining Moment 5). Good planning was instrumental in the most efficient and effective solution of Pair 1, and was the major difference between their solution and that of the second pair of Year 12 students who demonstrated little planning early in their solution process. However, it was seen that lack of opportunistic planning, whilst increasing both the time required to produce a solution and the number of confronting views seen, allowed the opportunity in this study for learning to occur in the process of solving the problem task.

The amount and rate of discussion and the types of questions asked by the pairs varied as was shown when analysing Defining Moment 6. Pairs having greater mathematical and graphing calculator skills required little discussion of alternative ideas
or questioning either student’s action at length. On the other hand, less expert students had greater opportunity to engage in discussion about possible choices of action and their responses to particular views of the function. Hence, the expertise or perceived expertise of one member of the pair may have influenced the type of questions asked.

Two different situations led to the response action for the pairs whose questions were mainly organisational. For Pair 1 it was the greater graphing calculator expertise of one student that led to a large proportion of organisation questions asked by his partner. In contrast, for Pair 3 the difficult nature of the situation and multiple options presenting themselves in the solution process led to a large number of comprehension questions being asked. The proportion of verification questions (33%) and organisation questions (27%) asked by Pair 2 distinguished their solution process from Pairs 3, 4, and 5. In particular, when compared to Pair 5 (verification questions, 11%, and organisation questions, 44%) whose initial actions led to the solution process becoming potentially routine—a potential that was not realised—the greater proportion of verification questions asked by Pair 2 contributed to their solution process proceeding smoothly, whereas, for Pair 5 the lower proportion of verification questions contributed to the difficulties they faced in solving the problem task. The lack of questions and dialogue undertaken by Pair 4 severely hindered their solution process.

The analysis of the question types across the range of activity categories showed that, other than for Pair 4, all pairs engaged in both cognitive and metacognitive behaviour across the range of activity categories, except reading, regardless of the classification of the category by its predominant cognitive or metacognitive level.

In the following chapter the findings of the macroscopic and microscopic analysis are used to address each of the research questions as listed in chapter two. Implications of the study and recommendations for mathematics teachers and textbook writers are outlined and discussed. In addition, directions for further research are addressed.
CHAPTER SIX
DISCUSSION AND RECOMMENDATIONS

6.1 Introduction

In this last chapter, the place of this study in the research area of interest is established and the outcomes and the links to relevant research are drawn together and discussed. The implications of the findings for teaching are presented, as are suggestions for further research arising from this study.

An understanding of what constitutes a complete graph of a function has always been important for senior secondary students (Leinhardt et al., 1990; van der Kooij, 2001, Teese, 2000; Zaslavsky, 1997). The examiner’s report for the first Pure Mathematics Paper in The University of Melbourne Matriculation Examination for 1948, for example, lamented that “the graphing of functions was rarely done correctly” (Teese, 2000, p. 123). The recent introduction of graphing technology and, in particular, the more accessible and personal graphing calculator has had the potential to impact on this understanding according to several authors (e.g., Barret & Goebel, 1990; van der Kooij, 2001; Leinhardt et al., 1990; Penglase & Arnold, 1996). Some researchers suggest that the use of graphing calculators adds to student difficulties as they tend to accept the view presented by the graphing calculator (Cavanagh & Mitchelmore, 2000; Smart, 1995; Steele, 1995a), need to interpret the graphing output (Ruthven, 1995), need to recognise that only a portion of the graph is displayed in the viewing window (Goldenberg, 1987; Ruthven, 1995; Steele, 1995a), need to understand the links between the various representations of functions (Cavanagh & Mitchelmore, 2000), and, often for the first time, experience graphs with different scaling on the axes (Goldenberg, 1987; van der Kooij, 2001). According to Mitchelmore and Cavanagh (2000), these difficulties in student understanding are related to shortcomings in the curriculum that exist in both a graphing calculator and non-graphing calculator
environment, a claim echoed by van der Kooij (2001) who suggests these difficulties are simply more obvious in a graphing calculator environment which highlights different areas of student misunderstanding due to its focus and the need to understand the differences between local aspects and the global nature of a function.

Of course, a graphing calculator environment implies more than simply having access to the technology. If understanding is to improve, teaching must change simultaneously to take advantage of the alternate ways of doing things and most importantly access to new opportunities, for example, actively observing “local linearity” (Dick, 1996, p. 35), “allowing manipulation of the viewing window” (Dick, 1996, p. 32), “provid[ing] the power of zooming” (Dick, 1996, p. 44), “visual[ising] the effects of scale changes”, understanding the independence of the scale of the two axes and the effects of scale marks (Goldenberg et al., 1988, p. 36), and using the graphing calculator in the learning process “to promote peer interaction and discussion” (Geiger & Goos, 1996, p. 229). However, research as to how students best learn in a graphing calculator environment must occur simultaneously with these changes to teaching. The study described in this thesis is one such study, responding to the entreaty of Arnold (1998) and others (e.g., Zbeik, 2003) that “if we are to learn to use these tools effectively, it becomes vitally important that we study the ways in which individuals make use of them within mathematical learning situations” (Arnold, p. 174).

Although as we begin the twenty-first century, the technology is evolving and now some secondary classrooms see students using computer algebra systems (CAS), Zbiek (2003) points out that “graphical representations seem to abound in CAS-using classrooms” (p. 208) and “many [research] studies of the use of CAS depend highly on the graphing component of the tool” (p. 210). Whilst current users of CAS calculators are utilising the graphing capabilities, this may change in the future; however, currently many more students have access to graphing calculators than CAS calculators. Hence,
the graphing calculator continues to be a technology needing further research as to how it can be used to enhance and extend students’ learning, including the understanding of functions in their various representations.

With the mandating of graphing calculator use in schools, expertise in using a graphing calculator to find a global view of the graphical representation is essential for senior secondary mathematics students studying functions (Anderson, Bloom, Mueller, & Pedler, 1999). The study which is the basis for this thesis investigated the approaches student pairs undertook in a problem task that required them to find a complete graph of a cubic function in a non-routine situation. Students’ and teachers’ understanding of function is receiving international attention in research studies according to Zbiek (2003) and the findings of the present study will add to this literature.

The study is different from other studies in that the students were experienced users of the graphing calculators and this is documented unlike other studies where the study was undertaken in conjunction with the students learning to use the technology or the many studies that fail to report students’ experiences and expertise in using graphing calculators. Much previous research exploring individual student understanding with graphing calculators has been comparative either using a control group (e.g., Ruthven, 1995; Shoaf-Grubbs, 1994), or simply comparing the use of graphing calculators to the use of pen and paper methods (e.g., Doerr & Zangor, 2000). This study has explored student understanding from an alternate perspective, namely how students use graphing calculators during a task to demonstrate their understanding of functions. In addition, the students in this study were selected on the basis that they were expected to be able to solve the problem task. The reason for this purposive selection (Merriam, 2002a) being that it would allow behaviour and actions undertaken, pertinent to informing future teaching, to be observed. Observing the behaviour and actions of students unable to solve the problem task would add little information to this. In addition, the use of “two-
person protocols provid[es] the richest data” (Schoenfeld, 1985a, p. 177). Many studies do not describe the specific type of graphing calculator used (Farrell, 1996; Goos et al., 2000; Keller & Hirsch, 1998), and do not describe the learning environment of the students who are the subject of the research nor the role of the tool within this environment (Burrill et al., 2002; Penglase & Arnold, 1996). The research reported in this thesis has answered in part the call of Doerr and Zangor (2000), Penglase and Arnold (1996), and others who noted the lack of detailed documentation about how students make use of graphing calculators, especially from the students’ point of view.

The methodology used has allowed a view of pairs of students as they work through a problem task to be explicitly recorded, in contrast to many studies that merely consider the aggregated results of students on tasks undertaken individually under test conditions (e.g., Shoaf-Grubbs, 1994).

The methodology used in this study is also different from that reported in other studies. In order to obtain an improved record of the results of students’ actions, videotaping of graphing calculator screen outputs supplemented audio recordings, observational notes, and students’ scripts allowing new insight into student understandings to be gained. The use of the viewscreen thus goes some way to overcome what Lochhead refers to as students’ “natural inclination towards ‘stealth thinking’—the act of keeping the thought process covert” (Yoon, 2002, p. 410). A case study was undertaken and qualitative research methods employed given the complex nature of the phenomena being studied (Merriam, 2002c). Schoenfeld’s scheme (1985b, 1992b) for coding problem solving protocols was adapted, by this researcher, to cope with the level of detail that the use of the videotaping of graphing calculator screen outputs allowed. Minor adaptations were also made to how his episode diagrams were coded and time-line diagrams constructed to make these more specific to, and of use in, analysing the task undertaken.
6.2 Research Findings

In this section each of the research questions developed after the literature review and presented at the end of chapter two will be specifically addressed. Implications of the findings for specific questions for teaching and future research will also be outlined. 

*Research Question 1: How do students perceive what constitutes a complete graph of a function?*

In undertaking the problem task all students in this study demonstrated their understanding of the local and global nature of functions, essential understandings according to Even (1993), Leinhardt et al. (1990), and Moschkovich et al. (1993). Further, all pairs displayed their knowledge of the synthesis of these local and global aspects in their determination of a complete graph. For all pairs, it was not until the graphing calculator displayed the graphical representation of the function as a global view that they accepted the view as such, no matter what difficulties were encountered in determining this view. Moreover, the identification of key features of the function was undertaken by each pair in their production of a complete graph.

Whilst some literature suggests that students, even those at the senior secondary levels, are often accepting of what their initial view is (for instance, Mitchelmore & Cavanagh, 2000; Smart, 1995; Steele, 1995a, 1995b) or an other than global view (Mitchelmore & Cavanagh, 2000; Steele, 1995a, 1995b) of the graphical representation, this was not the case in the context of this study. All pairs had a clear understanding of the requirements of the task and in particular the meaning of a complete graph (Demana & Waits (1990). Whilst Steele’s (1995a, 1995b) findings suggest that students may have accepted their initial view as that of a complete view of the graph, no student in this study did so. Nor were any students prepared to accept an other than global view even after experiencing a number of problematic views. All students in this study demonstrated they had an archetypal image of a cubic function for which they were
searching, a necessary requirement for the confident use of the calculator to graph functions according to Anderson et al. (1999, pp. 490-491). Their previous mathematical experiences allowed the students to include the only possible shapes of a cubic function in their toolbox and hold onto this view even when their ideas were challenged by unexpected views of the function on the graphing calculator screen. From their actions, dialogue, and solution it can be inferred that all students in the study correctly understood that both a global view and the identification of all key features are required for a complete graph of a function to be determined.

Previous experiences are a confounding factor here. All the students in this study had experienced a similar approach to the teaching of functions, albeit the Year 12 students for a longer period of time. Clearly, the teaching and learning experiences of the students in this study allowed them to develop a correct concept of the notion of a complete graph. Students with different teaching experiences may not demonstrate the same understandings of what constitutes a complete graph of a function. Students’ lack of understanding of a complete view of a graph could merely be a reflection of the teacher’s own lack of understanding of this aspect of function or a lack of emphasis on it in his/her teaching. In reviewing research studies such as those mentioned above, readers need to ask: What effect do the teaching and learning experiences of students have on students’ perception of what constitutes a complete graph of a function?

**Research Question 2: How do students apply their mathematical knowledge and graphing calculator knowledge to determine a global view of a difficult function?**

Students applied a range of mathematical and graphing calculator knowledge in their search for a global view of the function. This mathematical knowledge included the possible shapes of cubic functions, more specific knowledge informed by the algebraic representations of the functions including information regarding the shape (e.g., consideration of the effect of the coefficient of \(x^3\)), the coordinates of the \(y\) intercept, the
existence of at least one $x$ intercept, and the domain of the function being all real numbers. Graphing calculator knowledge included altering the WINDOW settings; use of ZOOM menu items including Zoom Standard, Zoom Out, and Zoom Fit; and use of the Equation Solver. The use of graphing calculator features may have involved physical or both physical and cognitive actions, for example, pressing GRAPH or selecting Zoom 6: Standard Window, respectively. The cognitive nature of the latter action is apparent in that the students selected a known viewing domain and range.

At times a combination of mathematical and graphing calculator knowledge was applied. This included using knowledge of the coordinates of the $y$ intercept to inform the WINDOW settings to find an appropriate view efficiently, knowledge of the existence of an $x$ intercept and use of Solver, knowledge of the domain of the function in the selection of Zoom Fit or Zoom Out, or using the current graphing calculator view linked with a range of mathematical knowledge to inform subsequent actions.

The knowledge and choices of one of the Year 12 pairs enabled them to apply their mathematical and graphing calculator knowledge in a way that ensured that their solution process quickly became routine. None of the other pairs of students consistently applied their mathematical and graphing calculator knowledge in conjunction with sensible choices of actions. One aspect of graphing calculator knowledge that was applied by all pairs was the defining action, *adjusting the WINDOW settings*. All pairs undertook this action in their initial search for a global view of the graphical representation of the function, however only when combined with mathematical knowledge and additional graphing calculator knowledge did this action lead to the solution process becoming routine or potentially routine.

All pairs undertook actions that demonstrated they had a clear mental image of the function for which they were searching and the possible position of the output of the graphing calculator relative to this (see Figure 6.1). However, the actions of some pairs
on some occasions, notably when confronted with *difficult* views of the function, suggested that they did not have a clear understanding which section of the function was currently the focus for the viewing window.

The linking of mathematical and graphing calculator knowledge to efficiently find a global view of the function was also apparent in the pilot study. Both the teacher experts and the student experts used TRACE to determine window settings where the key features were visible. The student experts, after observing the apparent parallel line view of the function, discussed the positioning of this output with their mental image of the complete graph of the function and proceeded to use TRACE, first to confirm there were three visible sections of the function and then to move up the curve and Zoom Out repeatedly in their search for a global view. Similarly, the teacher experts traced up the curve, where they were aware of the existence of a turning point and then by altering the window they were able to quickly determine WINDOW settings that showed an almost global view of the function.

Evidence has been presented in this thesis to support the view that the combined application of mathematical knowledge and graphing calculator knowledge is more efficient and effective in the determination of a global view of a difficult function. Such a finding is in keeping with the views expressed by Anderson et al. (1999, p. 491). The solution process of all students in this study unequivocally demonstrated their

*Figure 6.1. A selection of calculator outputs positioned relative to a global view.*
understanding that the graphing calculator screen presents only a portion of the graphical representation of a function, confused by many according to Dunham and Osborne (1991), Goldenberg (1987), and others. In addition they all had the ability to adjust this view, albeit varied in efficiency and effectiveness, to find an appropriate window for the function.

Two of the implications for teaching which arise from these findings involve students’ developing a correspondence between the output of the calculator and their mental image of the archetypical function under consideration and the use of the automatic range scaling feature for a given domain. Firstly, in order for students to apply their mathematical and graphing calculator knowledge to determine a global view of a difficult function, they need learning experiences that develop their understanding of positioning the specific output of the graphing calculator with the global view of the function being considered including the effect of scale and shape (Billings & Klanderman, 2000; Goldenberg, 1987; van der Kooij, 2001). Mathematical knowledge is essential for this understanding. Secondly, the use of an automatic range scaling feature for a given domain, for instance, Zoom Fit on the TI-83 or Autoscale on the HP38G, may facilitate the determining of a global view of a difficult function, however, its use does not necessarily imply mathematical understanding. Teachers and other mathematics educators need to consider the importance of students developing a mental image of a function and being able to position various outputs with this image. They may also need to consider the question: Is this mental image, and the positioning relative to this image, important for mathematical understanding, or is it simply a means to an end, an end that, in this case, can be facilitated with the use of an automatic range scaling feature for a given domain?

For students, an interpretation of these findings is that learning activities are required whereby the opportunity is provided for them to develop and improve their mental
images of functions, to make connections between transformations and changes of scale, and to resolve the confusion between these two. This understanding includes shape being an artefact of scale and challenging confusion between geometric transformations and a scale change. Both of these have already been identified as matters of concern by researchers such as Demana and Waits (1990), Dunham and Osborne (1991), and Goldenberg (1987).

Future research questions arising from these findings include: (a) If students are provided with learning opportunities to develop an understanding of automatic range scaling features for a given domain and its applications, does this help, hinder, or have no impact on their mathematical understanding of the positioning of a particular graphing calculator output relative to the complete graph and (b) What learning experiences best facilitate the development by students of how a particular graphing calculator viewing window output may relate to the global view of the function?

Further investigation is needed in order to determine the mental images held by students. A figure such as that shown in Figure 6.1 could be used to help determine this. For each WINDOW shown in Figure 6.1, what would a sensible response action be, both from the ZOOM menu, or if restricted to the adjustment of the WINDOW settings? This could begin to address the research question: What understandings do students demonstrate of the position of any particular graphing calculator view of a function relative to the global view of that function?

Research Question 3: (a) What are student behaviour and actions when confronted with unexpected views of a function? (b) Do they apply their mathematical knowledge and graphing calculator knowledge to resolve the situation?

As discussed previously, all pairs undertook actions that demonstrated they had a clear mental image of the function for which they were searching. This does not, however, imply that they did not encounter difficulties in their determination of a global
view of the function, with the actions of some pairs at times suggesting that they did not have a clear understanding which section of the function was currently the focus for the viewing window. Conditions sighted in the search for a global view included: the apparent no view of the function, apparent vertical lines, the function coincident with the y axis and unexpected views which may have included the previous conditions. Not all students had difficulties with these conditions.

Students in this study were not deterred when confronted with unexpected views of the graphical representation of the function. Their mathematical knowledge included the notion of the possible shapes of the graphs of cubic functions. When this was linked with their graphing calculator knowledge that for particular viewing domains and ranges a curve can appear linear, they were able to proceed to resolve the situation. This notion of understanding the local linearity of a function, although displayed by the students in this study, has been reported in the literature as an area where students experience difficulties (Cavanagh & Mitchelmore, 2000). The type and model of graphing calculator used could have been confounding factors in Cavanagh’s study. Only 20% of students in his study using Casio fx-7400G graphing calculators sketched a parabola, given a quadratic function that appeared linear in the default window. These findings are in direct contrast to the study presented in this thesis where when information was provided graphically, the students in the classes in this case study using TI-83 graphing calculators, demonstrated a good understanding that shape is an artefact of scale as shown in the pre-test, specifically Question 2a where no information was provided in the algebraic representation as was the case in Cavanagh’s study. Alternatively, as noted by Mitchelmore and Cavanagh (2000), students may simply “accept whatever is displayed” by the graphing calculator screen (p. 264), however, it appears this was not the most common reason for the lack of understanding of local linearity in Cavanagh’s study as of those who displayed this misunderstanding, 28% copied the graph directly
from the default window, 33% zoomed out, and 39% referred to the algebraic representation to determine the y intercept and add this to their sketch.

In interpreting research results, readers should consider if the type and model of graphing calculator used are confounding factors. Equally researchers need to ensure that this information is reported. These conflicting findings with regard to local linearity of non-linear functions raise the research question: What learning experiences support student understanding of local linearity of non-linear functions?

In answering part (b) of research question 3, the fact that all students in this study who were confronted with unexpected views of a function were able to resolve the situation and hence determine a global view of the function, even if not immediately, demonstrated that they did apply their mathematical and graphing calculator knowledge successfully. Several of the pairs in both the main and the pilot studies displayed their use of mathematical and graphing calculator knowledge in these circumstances. The teacher experts in the pilot study, for example, when presented with an unexpected view of the function shared knowledge with each other as they began planning their next action. They discussed the fact that Zoom Out has an identical effect in both the horizontal and vertical direction, questioned why the y intercept was not apparent, and verbalised their mental image of the global view of the function.

*Research Question 4: When working mainly in the graphical representation, do students use the algebraic and numerical representations to facilitate and support their solution process?*

Multiple representations are used in teaching in order to maximise student understanding of functions, each representation contributing only some aspects of a function (Ainsworth, Bibby, & Wood, 1997; Kaput, 1989). Multiple representations are not used merely because graphing calculators allow access (Keller & Hirsch, 1998),
rather through their use, students are presented with a more informed notion of a function.

In this study, students were expected to use the graphical representation, however, not to the exclusion of the other representations. The apparent preference of students in the main study for the graphical representation is more than an artefact of the problem task as this same preference was not demonstrated by the pairs in the pilot study. This difference may be explained in part by the different graphing calculator knowledge held by the pairs, specifically the lack of knowledge by the pairs in the pilot study of the CALCULATE menu items.

The algebraic representation was used by all pairs, both in the main study and in the pilot study, and this was hardly avoidable given the function was presented in its algebraic representation. However, in addition to entering the function in its algebraic form, the pairs all used this representation to some extent to inform their solution process. This included the index of the highest power identifying the function as cubic with a particular set of possible shapes and the value of the constant term allowing the coordinates of the $y$ intercept to be stated.

A different situation was observed with respect to the use of the numerical representation. Only one pair in the main study considered the use of the numerical representation. However, this consideration did not progress to its actual use due to the reluctance and opposition of one student to his partner’s suggestion of its use to the extent that he suggested that would be cheating. This is an example of what Boaler (1997, p. 37) calls “cue-based behaviour” where decisions are not made on a mathematical basis but on what the student thinks is expected of him or her. The behaviour and actions of the students in the main study were different from those of the pairs in the pilot study, with respect to the use of the numerical representation. In the pilot study, the expert teacher pair, for example, made good use of the numerical
representation in their identification of key features of the function. They used TABLE, for example, to determine the coordinates of a turning point after their initial estimation from the viewing window. However, neither student pair used the TABLE.

With regard to the time spent in each of the representations, it was shown in chapter four that the most successful pair, in the main study, as determined by the routineness, efficiency, and effectiveness of their solution process spent almost one-quarter of their time working in the algebraic representation either alone or in conjunction with the graphical representation. However, both the least successful pair and the second most successful pair spent a very large proportion of their time working solely in the graphical representation. Hence, the lack of use of the multiple representations cannot in itself be presented as the reason for a less than efficient and effective solution of the problem task used in this study. It could just be a manifestation of students’ preference for working in particular representations as reported by Keller and Hirsch (1998) and Piez and Voxman (1997) with the numerical representation being their least preferred working environment. Some researchers, for instance Kaput (1989), suggest that the use of the graphing calculator increased students’ strategies in this representation and hence, it would be expected that experienced users of graphing calculators would increase their preference for the graphical representation (Harskamp et al., 2000).

It is apparent that changes are needed to teaching practice in order for students to better appreciate the numerical representation. These include its links to the graphical and algebraic representation, and the type of information that can be gleaned from this representation (Even, 1998; Goldenberg, 1987; Kaput, 1989; Leinhardt et al., 1990). This information includes the identification of the y intercept, information regarding the range in a specified viewing domain, the identification of x intercepts, the identification of the coordinates of turning points, whether a function is increasing or decreasing overall in a particular viewing domain, and the possible domain of the function. This
information needs to be synthesised by students in such a way as to facilitate their finding a global view of a function and, hence, a complete graph. This study reinforces the views of Friedlander and Tabach (2001) and Kaput (1989, 1992) that the advantages and disadvantages of each representation must be known.

The numerical representation can be used to assist students to develop their understanding of the behaviour of a function both at, and close to, key features and the more global behaviour of a curve (Lloyd & Wilson, 1998). This should be a representation familiar and useful to students in their everyday mathematics experiences with functions. This is particularly pertinent when students are faced with a difficult situation where the graphical and algebraic representation, in conjunction with mathematical and graphing calculator knowledge, fail to provide enough information to allow students to efficiently and effectively determine a complete graph of a function. In this instance, the numerical representation provides a third alternative. A series of activities that support students’ understandings of the numerical representation and local and global aspects of a function is presented in Appendix G.

Further research in this area could consider students undertaking a set of different tasks that appear to give more weight to particular representations and explore what use is made by students of the numerical representation in the various circumstances. Research question 4 could also be explored further by the examination of students’ responses to a range of tasks.

*Research Question 5: What understandings do students have of the effect of the scale marks?*

Not all students in this study demonstrated their understandings of the effect of the scale marks. Three pairs in the main study and three of the pairs in the pilot study altered the scale marks. These adjustments were made in very different ways and these differed in intent, consequence, and effectiveness.
One Year 11 pair in the main study behaved as expected by Williams (1993) as they focused on adjusting the scale marks, not realising this has no effect on the view seen. A similar finding was recently reported by Mitchelmore and Cavanagh (2000) who found that students in their study did not have a good understanding of the essential concept of the nil effect of scale marks on the section of the graph portrayed in the window of the graphing calculator.

In contrast, the setting of the scale marks to zero by another Year 11 pair from the main study had a positive impact from their perspective on the solution process. Whilst the action undertaken may seem trivial, anecdotal evidence from teachers confirms that the removal of the “ugly axes” from the viewing window is perceived as desirable by many students as it is congruent with their notion of elegance of a solution. Actions that result in positive feedback for students are likely to boost their confidence and this in turn may result in a more effective and efficient solution path being followed. Thus, this student initiated action contributes to improving the affective component of learning.

The most effective and efficient of the Year 12 pairs demonstrated a good understanding of the effect of scale marks as they adjusted these several times until satisfied with the result. Their intention was to best facilitate their identification of the key features of the function in order to produce a pen and paper sketch of the function. The consequences were a more effective graphical representation of the function that enabled them to determine the coordinates of the key features of the function easily and confidently to produce their written solution to the problem task.

In the pilot study, both teacher pairs and the less expert student pair altered the scale marks in order to enable their estimation of key features and possibly to facilitate their solution sketch. All three pairs altered both the $x$ and $y$ scale marks. The discussion of the teacher expert pair demonstrated that they were unclear as to the effect of altering the scale marks as they pondered “Set the scale bigger, 20, or is this smaller?” This pair,
although unsure as to the effect of the scale marks, did not have the same misunderstanding as the Year 11 pair in the main study whose actions suggested that adjusting the scale marks would affect the portion of the graph seen. A possible reason for the different use of the scale marks exhibited by the participants in the pilot study may be their lack of knowledge of the CALCULATE menu items. As a result of having to identify key features using TRACE or TABLE there was a greater need for a visual check of their estimates and therefore a greater need to alter the scale marks to do this effectively.

If students are to appreciate that scale marks have no effect on the portion of the graph visible in the viewing window then this implies explicit teaching to overcome the possible misconception (Cavanagh & Mitchelmore, 2000; van der Kooij, 2001; Zaslavsky et al., 2002). Learning activities need to be presented that allow students to consider the effect of altering the scale marks in a given viewing window, for a range of window settings, and a range of functions. Students need to consider the effect of altering the scale marks on the axes and on the view of the function seen. This “misconception” with the function of scale marks is probably transient in that it should occur less as teachers become more familiar with how to teach well with graphing calculators.

*Research Question 6: What cognitive and metacognitive processes are used by students to support their use of mathematical and technological knowledge in the solving of the problem task?*

Garofalo and Lester (1985) believe that metacognition is central to intellectual activity and even though the possession of relevant knowledge, in this instance mathematical and graphing calculator knowledge relevant to the solution of the task, affects performance this only happens by the linking of that knowledge, its recall, and application during the solution of a task to the metacomponents of intellectual
functioning. “Metacomponents are higher order control processes used for decision making and executive planning” (p. 170). In solving the problem task students in this study used cognitive processes and the linking of metacognitive processes in this fashion to support their use of mathematical and technological knowledge. This was evident after an examination of the activities the students used in their interaction with the graphing calculator and through the analysis of their question asking. In order to detail which processes were involved it is necessary to link the findings from the macroanalysis of the time-line diagrams in chapter four to the microanalysis of defining moments 5 and 6, opportunistic planning and engagement in discussion, respectively, in chapter five.

Examination of the time-line diagrams revealed several places where cognitive action could have been enhanced by metacognitive activity. Reading was predominately cognitive, as expected by Artzt and Armour-Thomas (1992, p. 142), other than when rereading occurred when there was an opportunity for metacognitive activity to occur as it could be accompanied by evaluation of the current or final solution state. Even though this was a possibility, not much time was allocated by the pairs to such metacognitive activity. Only a brief time was devoted to reading as a whole, from 15-45 seconds in total. Similarly, only one question was asked during reading, albeit, an evaluation of organisation and planning. Rather than interpret this lack of metacognitive activity as an opportunity lost, perhaps the familiarity of the task requirements and the ease with which students could determine visually if they had a complete graph, once a global view was found on the graphing calculator, contributed to this lack of metacognitive activity even though the degree of difficulty of producing a complete graph for this particular function was high.

The activity of organising and planning was predominately metacognitive. The type of planning adopted by the students “followed a highly opportunistic event-driven
approach” (Stillman & Galbraith, 1998, p. 181). This is not unusual at the senior secondary level and is consistent with the findings of Stillman and Galbraith (1998). Not a large proportion of students’ total solution time was devoted to organisation and planning, ranging from 1-12%.

Organisation and planning activities involved identification of subgoals (13), local planning to implement global plans (14), local planning in search of a global plan during exploration (10), allocating responsibilities (1) and explicit identification of the goal of the task (1). However, local planning during exploration and allocating responsibilities was only undertaken by the Year 11 pairs. This was reflected in the lower overall use of organisation and planning activities by the Year 12 pairs. None of the pairs explicitly verbalised global plans but for one pair in particular the systematic and deliberate approach to the task showed they were following a global plan. For others, particularly the Year 11 pairs, periods of systematic execution were interspersed with forays into alternative pathways as they followed false leads and then returned to where they could move confidently forward in pursuit of an identified subgoal. Mainly comprehension questions (10) asking for information about a strategy or mathematical or graphing calculator knowledge and organisation questions (5) usually offering or asking for strategies were asked during organisation and planning. Few questions were asked during periods of simultaneous evaluation and organisation and planning.

Execution activities mainly involved selecting a viewing window, searching for or identifying a local feature, recording, and to a lesser extent adjusting scale marks and considering the global view. From a time perspective these were predominately cognitive, however, for some pairs they were interspersed by several short periods of evaluative activities and on several occasions execution occurred concurrently with evaluation.
By far the majority of questions were asked during execution activities (94). There were also a further seven questions asked during simultaneous evaluation and execution. The majority of the questions asked were comprehension questions (49) and organisation questions (27). It can be inferred from the relatively high rate of organisation questions that metacognitive activity was certainly not absent during execution.

Evaluation activities involved evaluation of execution (20), evaluation of organisation and planning (17), evaluation of the state of the final solution (6), and verification that the final solution was complete (3), all of which are metacognitive. Evaluation of organising and planning consisted of choosing a particular strategy or local plan after evaluating the calculator output or solution so far (7), evaluation of their current position in their solution process (4), evaluating the adequacy of results so far against the problem parameters (3), evaluating usefulness of a chosen strategy for the current plan (2), checking progress of a local plan (1), and stating the current subgoal (1). Evaluation of execution involved evaluation of results of local actions (18) and evaluation of procedures (2). Evaluation of the solution state involved evaluation of the elegance of the state of their final solution (3), consideration of how the graph would look when recorded (2), and evaluation of the consistency of the final results with the problem parameters (1). The asking of questions during evaluation was relatively high with 34 questions across all question types being asked. A high rate of organisation and verification questions were asked and this in turn supported the decision making of the pairs.

The undertaking of evaluation activities, which involved mainly metacognitive actions, varied both in the proportion undertaken and in their purpose across the pairs. The Year 12 pairs did not differ markedly. In contrast, over half of all the questions in this category were asked by the Year 11 pairs during evaluation and these in turn were
mainly asked by one pair whose major focus was evaluation of execution of local actions. Of the other Year 11 pairs, one undertook little evaluative activity whereas the third mainly engaged in evaluations of execution however these tended to be in short bursts interspersed throughout their solution.

In order to better facilitate student problem solving activity, particularly in non-standard mathematical situations, teachers need to facilitate student understanding of the use of both cognitive and metacognitive behaviours in successful problem solving particularly in situations when technology such as graphing calculators can be used. The use of questions when working with a partner over the full range of question types can be one strategy to assist in addressing this. Teaching needs to specifically address the provision of “opportunities to develop systematically [students’] metacognitive knowledge about strategies” (Stillman & Galbraith, 1998, p. 185) including through the careful selection of problems and time spent reflecting on and discussing strategies that had, or had not, been employed in the solution of these problems. By allowing students to better understand how learning in non-routine situations best occurs, the students will have improved opportunities to become successful in such situations.

**Research Question 7: What particular features of the graphing calculator do students use to identify key features of the function?**

In the main study, students used an extensive range of features of the graphing calculator to determine the coordinates of key features of the function. These included *use of dedicated features of the graphing calculator*: Calculate Value, Calculate Zero, Calculate Minimum, Calculate Maximum, Solver; *use of TRACE*; and *use of the free cursor* as detailed in chapter five. The accuracy of the identification of a particular key feature depended both on the method used and where a dedicated feature was used whether or not this was used correctly. When correctly selected and used the *use of dedicated features* had a direct effect on the accuracy of the coordinates of the key
feature so identified. In contrast when the actions, use of free cursor or use of TRACE, were employed, except in the case of the identification of the y intercept centred in the viewing domain, this was usually less accurate (as shown in Table 5.5).

*Use of dedicated features* does not necessarily imply accurate identification. The dedicated features were used un成功fully in the identification of key features on four occasions. These were (1) deliberate selection of a feature for the wrong purpose (i.e., Calculate Intersect to find the point of intersection between the graph and the y axis when its purpose is to identify points of intersection between two functions); (2) inadvertent selection of the opposite dedicated feature to identify local extrema (i.e., Calculate Minimum for a maximum turning point); (3) use of an appropriate dedicated feature for identifying local extrema but with a view that resulted in unreadable superimposed output from the calculator; and (4) use of a search domain that did not contain the key feature of the function with an appropriate calculator feature for identifying local extrema (i.e., Calculate Minimum).

The first occurrence of unsuccessful use of dedicated features was reasonable. The action of selecting Calculate Intersect to identify the coordinates of the point of intersection between the curve and the line $y = 0$ was logical. The misuse of the dedicated feature Calculate Intersect raises the importance of students being familiar with and understanding the differences between mathematical language and graphing calculator language. In this case, clearly the term, “intersect”, has a slightly different meaning on the graphing calculator to when it is used mathematically. Students thinking about the mathematical meaning of this term could reasonably assume that this feature when selected on the calculator would allow them to determine the coordinates of the point of intersection between a curve and the line $y = 0$ (i.e. the x axis). However, on the model of calculator used in this study, this is not the case. This calculator treats the axes as embedded objects rather than lines and hence, Calculate Intersect only allows the
user to determine the coordinates of the point of intersection between functions entered by the user. Teachers and students need to be aware of the specific declarative knowledge related to the use of the calculator features. In fact the graphing calculator placed limitations on the term, “intersect”, that excludes this pair of students’ seemingly logical action. As Heibert and Leferve (1986) comment “procedures … may or may not be learned with meaning” (p. 8). Where the latter of these occurs students will be unable to distinguish between the conceptual or declarative knowledge and procedural knowledge and hence, as in this case, apply procedural knowledge that is external to the domain of the conceptual knowledge.

In the pilot study, the participants used none of the dedicated features to identify key features of the function. In contrast they used a different range of graphing calculator features to those selected by the student pairs in the main study, being TRACE and TABLE. In both the pilot study and the main study TRACE was used to identify key features of the function. Whereas, some of the student pairs in the main study used the Free Cursor and ZOOM menu items, the use or knowledge of these features was not demonstrated in the pilot study. In contrast, in the pilot study the TABLE feature was used by one pair to determine coordinates of key features of the function, a feature notably absent in its use during the main study.

Implications for teaching that became apparent from these findings include a range of teaching and learning activities that include an emphasis on mathematical language, graphing calculator language, and the differences between these. Students need to experience using a variety of graphing calculator methods to find a complete graph of a function including the dedicated features, Free Cursor, TRACE, and TABLE, to identify key features and compare and contrast the results. This will enable them to make sensible decisions about when a particular selection may be pertinent. In addition, changing the viewing window by altering the WINDOW settings, Zoom In and Zoom
Box and determining their effect on accuracy in the identification of the key features, using a variety of methods, are further experiences that all students learning about function should undertake.

6.3 Conclusions

“Facilitating the acquisition of effective approaches for tackling knowledge construction is a perennial goal of educational research” (Yoon, 2002, p. 412). An understanding of what constitutes a complete graph of a function has always been important for senior secondary mathematics students. The study outlined in this thesis has shown that the use of graphing calculators has impacted on this understanding and the approaches used. All students successfully produced a complete graph of the function. However, the routes to this differed in their directness, duration, and accuracy. The following had the potential to facilitate the sketching of a complete graph: graphing calculator features dedicated to transforming the viewing window, directly altering the window settings, graphing calculator features enabling key features of the function to be identified, the linking of mathematical and graphing calculator knowledge, planning, students engaging in discussion when they ask questions of themselves and of their partner, and use of the table facility.

It has been shown in this study that the mere presence of a graphing calculator does not imply or ensure improved learning, it merely provides the opportunities. Teaching must change to ensure that the learning opportunities that arise with these tools are taken up. This must include new ways of thinking about mathematics. With regard to functions, this implies the use of all the representations. The findings of this study show the importance of students’ understanding what each representation shows, what is not shown, and the links between these. This knowledge needs to be linked to their knowledge of graphing calculator features and their use. Further, to use graphing calculators effectively in their learning process, students need to develop understanding
of the effects of changing each of the window values, including the scale marks, and how this affects the section of the graph seen. The effect of the window being rectangular also needs to be understood as to its effect on the shape of the graph. To use these features effectively, students need an understanding of the relationship between the complete graph of a function and the viewing window displaying it.

The various methods to determine key features of the graphical representation of a function and global behaviour should be explored by all students and consideration given to the advantages and disadvantages of each method, including with regard to accuracy and efficiency. Students who are expected to use one method and check with a second will quickly develop expertise in using the variety of possible methods as well as being able to make judgements as to which method is the best in a particular situation. The complexity of the types of functions used is also an area where teachers must exert their influence. If a text book is used as a major teaching tool, then teachers have the responsibility to ensure that where only nice functions in nice windows are presented by the text book, then additional materials need to be presented to students. Students must have significant experience of functions whose global view is not immediately visible in the default window of the graphing calculator. Only in this way will they understand the need to use mathematical and graphing calculator knowledge, including all representations, in order to locate a global view of the function. In addition, students need to view portions of the function, both close up and apparently linear, including horizontal, and with non-homogeneous axes to observe apparently vertical sections of a function, including those coincident with the y axis.

The current study has provided further evidence that “the acquisition of specific content knowledge greatly influences one’s ability to solve sophisticated problems” (Yoon, 2002, p. 411). In this instance both mathematical knowledge and graphing calculator knowledge have proved critical to the undertaking of a time efficient and
accurate solution. This involves mathematical knowledge including an understanding of the notion of a complete graph, an understanding of both the local and global nature of functions, and general knowledge of functions, that is informed by the algebraic, graphical and numerical representations, in addition to specific knowledge of the functions being explored, for instance an archetypal image of a cubic function. Graphing calculator knowledge includes an understanding of the various views that may be portrayed of a single function, features that allow the efficient identification of key features, adjustment of window settings and automatic range scaling. Where the graphing calculator allows it, multiple methods of identifying key features, albeit, to different degrees of accuracy, and of supporting the finding of an appropriate viewing window should be known and understood. Furthermore, students not only need the knowledge of specific features of the graphing calculator and their functions but also they need to understand the mathematics underpinning these features. Students need the ability to link their mathematical and graphing calculator knowledge and to have confidence in their mathematical knowledge in order to hold on to this when confronted with challenging or unexpected output from the calculator. All of these imply that students need to be immersed in a graphing calculator teaching and learning environment, an environment where students can experience the benefits of each of the representations to a specific type of task. In the study of functions students should experience what each representation brings to the understanding of a function. Misconceptions need to be challenged, particularly those that relate to the use of scale marks, local linearity of functions, and the effects of scale on shape in general.

In addition to mathematical knowledge and graphing calculator knowledge, the use of metacognitive activity to facilitate students’ cognitive activity requires specific teaching about learning and how best to succeed when attempting non-routine problems. Contrary to the findings of Doerr and Zangor (2000), who found that use of
the calculator as a personal device inhibited communication even when working as part of a group and in fact appeared to lead to lack of cooperation, in the study described in this thesis the use of one calculator between each pair encouraged communication. In most cases, this arrangement allowed the opportunity for co-construction of knowledge and metacognitive behaviours to occur. The question asking engaged in by the pairs facilitated joint decision making. Not only did the task and its conditions of implementation facilitate metacognitive activity, but also the context in which the students in the study were taught contributed to their willingness to engage in fruitful discussion and question asking. Even though many of the students in the study were ESL students the mathematics teaching and learning environment of their experience was to share both mathematical and graphing calculator knowledge, to view mathematical learning as an exploratory activity, and to undertake joint decision making to expedite their understanding.

The research tools developed and used in this study, although modifications to existing tools, allowed new ways of analysing data in a graphing calculator learning environment. The construction of this set of tools and the way they have been used have proved useful in completing the analysis of the study data. When used in conjunction with the novel methods of data collection they allowed greater opportunity for insight into student understanding.
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