Initial capital and margins required to secure a Japanese life insurance policy portfolio under stochastic interest rates

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# Table of contents

Abstract................................................................................................................................................ IV

1 Background....................................................................................................................................... 1
   1.1 Recent life insurance company failures.................................................................................... 1
   1.2 The Japanese life insurance industry ....................................................................................... 3
   1.3 Regulatory requirements ........................................................................................................... 4

2 Introduction.................................................................................................................................... 7

3 The nature of the risk profile of life insurance policy portfolios ............................................. 9
   3.1 General features of life insurance policies ............................................................................... 9
   3.2 The importance of interest rate risk...................................................................................... 10
   3.3 Interest rate risk .................................................................................................................... 12

4 General issues in interest rate modelling ................................................................................... 14
   4.1 Notation ..................................................................................................................................... 14
   4.2 General arbitrage relationship ............................................................................................... 15
   4.3 Bond price formula ................................................................................................................ 17
   4.4 Treatment of market price of risk .......................................................................................... 20

5 Overview of interest rate models ................................................................................................. 22
   5.1 Commonly used models ......................................................................................................... 22
   5.2 The classical approach and the HJM approach .................................................................... 24
   5.3 Models based on the classical approach .............................................................................. 24
   5.4 The Heath-Jarrow-Morton approach ...................................................................................... 29
   5.5 Descriptive Models ............................................................................................................... 32

6 Yield curves in a very low interest rate environment............................................................... 34
10 Simulation results ........................................................................................................... 74

10.1 Sample scenarios ....................................................................................................... 74
10.2 Distributions ............................................................................................................. 82
10.3 Initial capital and margin .......................................................................................... 85

11 Conclusion ..................................................................................................................... 89

REFERENCES .................................................................................................................... 92

APPENDICIES .................................................................................................................... 95

Appendix 1 · Yen fitted Yield Curve .............................................................................. 95
Appendix 2 · Model Volatility ......................................................................................... 97
Appendix 3 · Mortality Table, Male (1999) .................................................................. 98
Appendix 4 · Expected Cash Flow .................................................................................. 99
Abstract

During the last decade several Japanese life insurance companies failed mainly due to interest losses. In fact, interest rate risk dominates mortality risk for a portfolio of business in force.

When the interest rates are modelled as random variables, the yields on bonds are the sum of expected short spot rates and a risk premium for random bond prices. However, in our study, we assume a risk-neutral environment, i.e. zero risk premiums.

As tools to deal with stochastic interest rates, various interest rate term structure models are considered. The Vasicek model, the Heath-Jarrow-Morton (hereafter “HJM”) approach and Cairns’ model are explained in detail.

The history and nature of the very low interest rate environment in Japan is described in line with the monetary policy framework of the central bank. An unusual interest rate movement in the very low interest rate environment is identified.

A modified HJM approach and Cairns’ model are chosen in our study. Cairns’ model is used to graduate the initial yield curve. The HJM approach with a specific volatility function and modified to deal with very low interest rates is used for simulating subsequent developments of the initial yield curve.

After the introduction of various concepts needed to investigate a life insurance policy portfolio, we prepare for simulation by collecting information and by fitting parameters to market observations. The Yen swap curve is chosen as a base yield curve.

The simulation results show how much initial capital and/or margins are needed in order to avoid the ruin of a portfolio.
1 Background

1.1 Recent life insurance company failures

In March 2001, the Tokyo Life Mutual Company made an application under the Reorganization and Rehabilitation Act. It was the seventh successive life insurance company failure during the current lingering stagnant economy in Japan. The data in Table 1-1 illustrate the magnitude of these failures. The total gross assets held by the failed companies far exceeded Yen 10 trillion (AUD 160 billion). Even though this amount is a small fraction of the total assets held by the Japanese life insurance industry, these failures led to serious social problems.

<table>
<thead>
<tr>
<th>Company Name</th>
<th>Failure Date</th>
<th>Gross Asset Value (Yen trillion)</th>
<th>(In AUD(^1) billion)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tokyo Life</td>
<td>March 2001</td>
<td>1.1</td>
<td>17</td>
</tr>
<tr>
<td>Kyoei Life</td>
<td>October 2000</td>
<td>4.6</td>
<td>73</td>
</tr>
<tr>
<td>Chiyoda Life</td>
<td>October 2000</td>
<td>3.5</td>
<td>56</td>
</tr>
<tr>
<td>Taisyo Life</td>
<td>August 2000</td>
<td>0.2</td>
<td>3</td>
</tr>
<tr>
<td>Daihyaku Life</td>
<td>May 2000</td>
<td>2.2</td>
<td>35</td>
</tr>
<tr>
<td>Toho Life</td>
<td>June 1999</td>
<td>2.8</td>
<td>44</td>
</tr>
<tr>
<td>Nissan Life</td>
<td>April 1997</td>
<td>Unavailable</td>
<td></td>
</tr>
</tbody>
</table>


The Japanese economy has been stagnant during the last decade after the stock market declined sharply from its peak in 1989 – an event known as the “collapse of the bubble”. The “collapse of the bubble” resulted in falling asset values and

\(^1\) Using the exchange rate 63 YEN/AUD
very low interest rates. As shown in the Table 1-2 the overnight call rate and TOPIX\textsuperscript{2} stock price index were 6.66\% per annum and 2,881.37 respectively at the end of 1989. They have fallen to 0.2\% per annum and 1283.67 at the end of 2000. Since 1991 land prices have also been declining continually.

Table 1-2 - History of Japanese economic indices (1988 – 2000)

<table>
<thead>
<tr>
<th>Fiscal Year End\textsuperscript{3}</th>
<th>GDP Growth (Real %)</th>
<th>GDP Deflator</th>
<th>Urban Land Price index\textsuperscript{4}</th>
<th>Stock Year End</th>
<th>Stock Price (TOPIX)</th>
<th>Stock Overnight Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988</td>
<td>6.3</td>
<td>91.3</td>
<td>107.7</td>
<td>1988</td>
<td>2357.03</td>
<td>4.38</td>
</tr>
<tr>
<td>1989</td>
<td>4.9</td>
<td>93.6</td>
<td>139.9</td>
<td>1989</td>
<td>2881.37</td>
<td>6.66</td>
</tr>
<tr>
<td>1990</td>
<td>5.5</td>
<td>95.9</td>
<td>144.1</td>
<td>1990</td>
<td>1733.83</td>
<td>8.34</td>
</tr>
<tr>
<td>1991</td>
<td>2.5</td>
<td>98.5</td>
<td>121.8</td>
<td>1991</td>
<td>1714.68</td>
<td>5.56</td>
</tr>
<tr>
<td>1992</td>
<td>0.4</td>
<td>100.0</td>
<td>100.0</td>
<td>1992</td>
<td>1307.66</td>
<td>3.91</td>
</tr>
<tr>
<td>1993</td>
<td>0.4</td>
<td>100.4</td>
<td>88.5</td>
<td>1993</td>
<td>1439.31</td>
<td>2.44</td>
</tr>
<tr>
<td>1994</td>
<td>1.1</td>
<td>100.3</td>
<td>76.6</td>
<td>1994</td>
<td>1559.09</td>
<td>2.28</td>
</tr>
<tr>
<td>1995</td>
<td>2.5</td>
<td>99.8</td>
<td>68.1</td>
<td>1995</td>
<td>1577.70</td>
<td>0.46</td>
</tr>
<tr>
<td>1996</td>
<td>3.4</td>
<td>99.1</td>
<td>62.9</td>
<td>1996</td>
<td>1470.94</td>
<td>0.44</td>
</tr>
<tr>
<td>1997</td>
<td>0.2</td>
<td>99.8</td>
<td>59.7</td>
<td>1997</td>
<td>1175.03</td>
<td>0.47</td>
</tr>
<tr>
<td>1998</td>
<td>-0.6</td>
<td>99.2</td>
<td>55.3</td>
<td>1998</td>
<td>1086.99</td>
<td>0.32</td>
</tr>
<tr>
<td>1999</td>
<td>1.4</td>
<td>97.7</td>
<td>50.6</td>
<td>1999</td>
<td>1722.20</td>
<td>0.05</td>
</tr>
<tr>
<td>2000</td>
<td>1.0</td>
<td>96.3</td>
<td>46.4</td>
<td>2000</td>
<td>1283.67</td>
<td>0.20</td>
</tr>
</tbody>
</table>


\textsuperscript{2} The widely used Nikkei225 index is a simple weighted average of selected 225 shares, while the TOPIX index is the capital-weighted average of all shares listed on the Tokyo Stock Exchange Section 1.

\textsuperscript{3} The end of March

\textsuperscript{4} 6 Large City Areas, average of commercial, residential and industrial, inversely calculated from the yearly change setting the index to 100 at the end of fiscal year 1992
The failed companies had been suffering from interest losses, a phenomenon well known as “Gyakuzaya” in Japan. The interest loss arises when the actual rates of return on paid premiums are lower than the “premium basis rate”, which is the interest rate used to calculate premiums. The deterioration of their loan portfolios amid the stagnant economy and falling prices of land held as collateral caused them to fail. The Daily Mainichi (1999) reports:

“As the result of an avalanche of selling of high-yield saving products during the ‘bubble era’, the average premium basis rate on 2.6 million individual and pension policies in force has remained at 5.06%. This is one of the causes of the net interest income loss during the very low interest rate period.”

In other words, the actual return on the investments in respect of saving products sold during the “bubble era” turned out to be much lower than the premium basis rate, which is called the “expected interest rate” in Japan.

Nobuo Uemori (1999), a senior analyst at the Rating and Investment Information Inc., stated:

“Kyoei Life aggressively sold saving products such as endowment assurances and individual pensions with high expected interest rates aiming at expanding its balance sheet during the 1980s. That strategy ended up damaging the company critically. The company has refrained from selling saving products recently for several years. However, the company has been suffering from the loss arising from net negative interest income in the very low interest rate environment every fiscal year.”

1.2 The Japanese life insurance industry

The surviving Japanese life insurance companies are far from being in a healthy
financial condition. A recent investigation conducted by the FSA revealed that all life insurance companies were still being affected by severe interest losses. In The Daily Mainichi (2001), it was reported that:

“According to the summary by the FSA, the total interest loss of the whole life insurance industry amounted to Yen 1.4 ~ 2.7 trillion (AUD 22~43 billion) between the fiscal periods ending March 1998 and March 2000. However, that amount of loss was fully covered by the expense profit and mortality profit. The net operating profit amounted to Yen 1.2 ~ 2.2 trillion (AUD 19~35 billion).”

These figures clearly show that the industry has been exposed to a severe interest rate risk. The resulting interest loss was covered by other income arising from margins for mortality and expenses, which were allowed for when the premiums were calculated. However, structural problems common to all Japanese life insurance companies are eroding their financial condition, as illustrated by the failure of several companies.

In an environment of global competition with a small possibility of longer life expectancy in the Japanese population, continually relying on mortality and expense margins may not be a realistic option. Competition will probably put some pressure on margins assumed in premium calculations, forcing them to marginal levels. Therefore, management of interest rate risk is an urgent task for the Japanese life insurance industry.

1.3 Regulatory requirements

In Japan, there are regulatory reserve requirements for life insurance companies. In addition to policy values and unearned premiums, which are premiums paid in advance for the periods that the insurance policies have not covered yet, two risk reserves are required. One reserve is for insurance risk
and another is for interest rate risk, i.e. the risk of the interest rate's departure from the premium basis rate. The secretary general of the FSA determines the method of calculating and accounting for the risk reserves. However, a life insurance company can use other methods if the method prescribed by the FSA is impracticable.5

According to these regulations, the interest rate risk reserve is required to be calculated as follows. First, divide the premium basis rate for each policy into bands according to Table 1-3. Next, for each band, multiply the difference between the highest and lowest rate by the Risk Coefficient in the table. Then sum up figures over the portions to obtain a total risk coefficient for each policy. Finally, add up the product of the total risk coefficient and the policy value over all policies.

Table 1-3  - Risk coefficients

<table>
<thead>
<tr>
<th>Portion of Expected Interest Rate</th>
<th>Risk Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>From 0.0% to 2.0% inclusive</td>
<td>0.01</td>
</tr>
<tr>
<td>From 2.0% to 3.0% inclusive</td>
<td>0.2</td>
</tr>
<tr>
<td>From 3.0% to 4.0% inclusive</td>
<td>0.4</td>
</tr>
<tr>
<td>From 4.0% to 5.0% inclusive</td>
<td>0.6</td>
</tr>
<tr>
<td>From 5.0% to 6.0% inclusive</td>
<td>0.8</td>
</tr>
<tr>
<td>From 6.0%</td>
<td>1.0</td>
</tr>
</tbody>
</table>

For example, if the premium basis rate is 2.2% for a policy, the total risk coefficient is calculated as 2%×0.01+(2.2%-2.0%)×0.2=0.06(%) . Then we get the interest risk reserve for the policy by multiplying the policy value by 0.06%. Adding up each reserve for each policy over all policies in force, we get the total interest rate reserve.

The reserve required for the interest rate risk is based solely on the premium basis rates and policy values, whatever the terms of the policies are. However,

5 The Insurance Law Implementation Rule Article 69, Clauses 1,6 and 7
6 The Ministry Of Finance Decree 50, Article 2
each life insurance company is ultimately responsible for setting aside enough reserves to support its own operation.
2 Introduction

Our study is aimed at measuring interest rate risks and initial capital and/or profit margins associated with a life insurance portfolio. Several Japanese insurance companies failed mainly due to interest losses. In Chapter 3, we indicate how interest rate risk dominates mortality risk for a unit of business in force when a policy portfolio is composed of many independent and identical policies.

In a stochastic interest rate environment, we investigate how economic values of the policy portfolio develop as time elapses. Tools for this investigation are stochastic interest rate term structure models.

In Chapter 4, basic concepts and notation used in interest rate term structure models are introduced. In addition, general bond pricing formulae under a stochastic interest rate environment are shown. The market price of risk associated with random bond prices is discussed at the same time.

Chapter 5 reviews models commonly used in actuarial and financial communities. In particular, the Vasicek model, the Heath-Jarrow-Morton (“HJM” hereafter) approach and Cairns’ model are explained in detail. The Vasicek model is the most fundamental classical model, which models instantaneous spot rates. The HJM approach is essentially different from classical models in that it models forward rates. The reason the HJM approach is adopted in our study will be explained in Chapter 5. Cairns’ model was considered appropriate to describe the initial yield curve.

We observe unusual behaviour of interest rates when they are very low. In Chapter 6, in line with the policy stance of the BOJ, while overviewing the history of “zero rates” in Japan, we characterize the unusual behaviour of interest rates in a very low interest rate environment.

In Chapter 7, the model used in our study is explained. The HJM approach is used with a specific volatility function and a modification to deal with very low interest rates. In the HJM approach, a volatility function completely determines
the development of yield curves. The volatility function used in our study will be shown and discussed.

Several important concepts needed for investigating developments of the economic values of a policy portfolio are introduced in Chapter 8. Concepts such as rates of return, discount factors, surplus, initial capital and profit margins are defined in the context of our study. In particular, discounted surplus is introduced as a value representing a development of the economic value of a policy portfolio.

Chapter 9 starts with an explanation of why the Yen swap curve was selected as a base yield curve. Then, we fix the model parameters by fitting them to market observations. For the purpose of fitting volatilities, the quotations of the Yen market for options on swaps are used.

Simulation results are shown in Chapter 10. Several sample realizations of the developments of yield curves, in addition to sample realizations of the surplus process, are shown. Required initial capital and profit margins are found by looking into the empirical distribution of the discounted surplus. In the conclusion in Chapter 11, it is shown that a significant amount of initial capital and/or profit margin is needed to keep a portfolio’s ruin probability within a certain level.
3 The nature of the risk profile of life insurance policy portfolios

3.1 General features of life insurance policies

Under normal social and economic circumstances major risks life insurance companies are exposed to are mortality risk, interest rate risk and expense risk. Our study will not be involved in expense risks. We consider the interest rate risk and mortality risk under the situation where life insurance companies have large numbers of policies, which share similar terms and conditions. In practice, Japanese life insurance companies have been marketing standardized products such as whole-life-with-term assurances. Terms and conditions of these policies are similar except additional riders such as sickness benefits and rights to increase sum assured without health checks.

Mortality causes financial losses or profits in two cases. The first case is that mortality rates change due to social, economic or medical developments, etc., even without random fluctuation of mortality experience. The second case is that the insurer experiences mortality different from expected mortality rates due to random fluctuation even when using an appropriate mortality table.

The first type of mortality risk, risk of mortality basis change, is a problem that must be borne by insurers. It cannot be eliminated even though they transfer it to reinsurers. However, insurers can usually mitigate this risk by building margins for adverse mortality into their premium calculation. Assuming that enough cushion exists and the mortality basis is stable, we do not deal with this type of mortality risk in our study.

As shown in Section 3.2, we can also ignore the second type of mortality risk, the risk where an insurer experiences mortality rates that are different from the mortality basis because of random fluctuation, in our study. This type of mortality risk is small when there are large numbers of similar policies, assuming independent mortality risks. When there are thousands of policies with independent lives insured, the number of deaths is likely to closely follow...
the expectation since the variance for a unit amount of business in force decreases in proportion to the number of independent lives at risk.

Interest loss or profit may arise in two forms in financial reports. We may see profits or losses when the actual interest rate earned on the investment portfolio differs from the expected interest rate that was used to calculate the premium. The change of policy value caused by the change of interest rate also results in losses or profits. We call the former risk an investment risk, the latter risk a valuation risk. However, regardless of the reporting titles, both of them represent profits or losses caused by interest rate fluctuation, originating from the same source of risk.

The interest rate risk cannot be reduced with the number of policies since every policy has the same or similar risk with respect to interest rates.

3.2 The importance of interest rate risk

The following simple example shows that interest rate risk is far more important than random mortality fluctuations.

Suppose there are n independent and identical pure endowment policies paying a benefit of 1/n in 30 years time if a policyholder is alive then. Note that the total sum assured of this policy portfolio is always 1, regardless of the number of policies. In other words, the risk is being measured relative to the unit amount of business in force.

Let us assume the \( \{ \delta_t \}_{t=1}^{30} \) is a sequence of multivariate normal random variables with the correlation matrix:

\[
\begin{pmatrix}
1 & \rho & \rho^2 & \ldots & \rho^9 \\
1 & \rho & \rho^2 & \ldots & \\
1 & \rho & \rho^2 & \\
1 & \rho & \\
1 & \\
\end{pmatrix}
\]

where \( \delta_t \) represents the force of interest in year \( t \), with mean \( \mu \) and variance
Now let a random variable \( N \) denote the number of claims, and assume that the survival probability for 30 years is \( q=0.5 \). Then \( N \) follows the binomial distribution \( B(n,q) \). Here we assume that \( N \) is independent of \( \{\delta_i\}_{i=1}^{30} \). The total claim amount \( Y \) is a random variable \( Y = N/n \).

The expected value and variance of \( Y \) are:

\[
E[Y] = nq/n = q = 0.5 \\
V[Y] = V[N]/n^2 = nq(1-q)/n^2 = 0.25/n
\]

Note that no other choice of survival probability gives a larger value of \( V[Y] \) for fixed \( n \). The discount factor \( X \) is: \( X = e^{-\delta} \), where \( \delta = \sum_{i=1}^{30} \delta_i \). Now:

\[
E[\delta] = 30\mu, \\
V[\delta] = \sigma^2 \left[ 30 + 2 \left\{ \rho(30-1) + \rho^2(30-2) + \cdots + \rho^{30-2} + \rho^{30-1} \right\} \right] \\
= \sigma^2 \left[ 30 + 2 \rho^{30} \left\{ \rho^{-1} + 2(\rho^{-1})^2 + \cdots + (30-1)(\rho^{-1})^{30-1} \right\} \right] \\
= \sigma^2 \left[ 30 + 2 \left( \frac{\rho^{30-1}-1}{(1-\rho^{-1})^2} - \frac{(30-1)}{1-\rho^{-1}} \right) \right] \text{ (when } \rho \neq 0,1). \]

Suppose we set \( \mu \) to 0.04, \( \sigma \) to 0.01 and \( \rho \) to 0.98, we have:

\[
E[\delta] = 1.2 \text{ and } V[\delta] = 742.87 \sigma^2 = 0.07428.
\]

\( X \) is lognormally distributed. Then we get:

\[
E[X] = 0.3126 \text{ and } V[X] = 0.007529.
\]

The present value of the benefits is:

\[
PV = \text{Discount Factor} \times \text{Claim Amount} = XY.
\]

The variance of the present value can be calculated as:

\[
V[XY] = E[X^2Y^2] - \{E[XY]\}^2 \\
= E[X^2]E[Y^2] - \{E[X]\}^2 \{E[Y]\}^2 \text{ (By independence)} \\
= 0.01882/n + 0.02443/n + 0.01882 \\
= 0.04325/n + 0.01882
\]
When $n=100$, the variance is 0.01925. It is obvious that when $n$ becomes large (for example, 10,000), the variance becomes dominated by the term \( \{E[Y]\}^2 \text{Var}[X] \).

This contains the variance of the discount factor, which does not depend on $n$.

Therefore, when similar types of policy are held by large numbers of independent policyholders, interest rate risk is far more important than the risk of random mortality fluctuation. Assuming an appropriate mortality table, a portfolio of such policies can effectively be regarded as a fixed income portfolio exposed to interest rate risk with random fluctuations in mortality ignored for the purpose of practical risk investigation provided the number of policies is large and policies are independent with respect to mortality risk.

### 3.3 Interest rate risk

The example in the previous section has shown how the present value of claims from a policy portfolio is exposed to interest rate risk. To analyse this risk we need to consider two components of interest rate risks, a change in overall interest rate level and a change in the shape of the yield curve.

The financial profits or losses due to interest rate risks can be attributed to the above two components of interest rate risks. For example, if the yield curve moves up in parallel, we may make profits from high investment returns on short-ends, lower policy values and the reduced reserve required for interest rate risks. If the yield curve becomes inverted, i.e. high short-term interest rates and low long-term interest rates, we may still make profits from high investment return on short-ends. However we may suffer losses from higher policy values and the increased reserve required for interest rate risks caused by low long-term interest rates.

In a portfolio of life insurance policies, cash flows are spread over the term of the policies. Even assuming that mortality is exactly as expected, i.e. assuming cash flows occur exactly as expected in a non-random fashion, the shape of cash flows
is not as simple as that of a standard financial product such as fixed coupon bonds. In addition, the duration of the contract is usually very long. Therefore, the shape of the interest rate term structure affects the present value of the portfolio considerably as well as the overall interest rate level.

In a life insurance contract, the net cash flows are usually positive in its early stages followed by net negative cash flows. This implies that the contract is exposed to a significant investment risk. In fact, the investment risk is critical since the financial result caused by investment is a realized loss or profit while a valuation loss or profit is on an unrealised basis. Moreover, the larger the duration of the policy is, the more significant the accumulated investment loss or profit is.
4 General issues in interest rate modelling

In the next chapter, we show that there are a number of models that describe the stochastic behaviour of interest rates and bond prices. Following the common approach taken by financial economists, we model interest rates and bond prices by stochastic differential equations involving a Wiener process. Under this modelling framework, it is well known that there is an important arbitrage relationship between bonds. The relationship is obtained under a no-arbitrage hypothesis applied to bonds, which are tradable assets. Vasicek (1977) derived a “term structure equation”, which described the relationship among bonds with different maturities and the instantaneous spot rates in the form of a differential equation. Here, we introduce the basic idea of the relationship when the process of interest rates or bond prices is driven by one Wiener process, i.e. we focus on one-factor model. A model, which contains \( n \) Wiener processes, regardless of whether they are independent or correlated, is called an \( n \)-factor model.

4.1 Notation

Following notation commonly used by financial economists, for example Neftci (2000), we introduce the following definitions. The time unit is always a year.

1. Let \( B(t,T) \) denote the price of a zero coupon bond at time \( t \), that is redeemed at time \( T \) with redemption amount 1.
2. \( Y(t,T) \) is the continuously compounding spot yield at time \( t \) up to the time \( T \), or for the duration \( T-t \), given by:

\[
Y(t,T) = - \frac{\log(B(t,T))}{T-t} \tag{4-1}
\]

Note that the word “yield” is used here in order to explicitly distinguish it from the instantaneous spot rate to be mentioned below.
3. $F(t,S,T) \ (t \leq S<T)$ is the continuously compounding forward rate at time $t$ for the period $[S,T]$, given by:

$$F(t,S,T) = \frac{\log(B(t,T)) - \log(B(t,S))}{T-S}$$

Thus $F(t,S,T)$ represents the yield of a zero coupon bond at time $t$, which will be settled at time $S$ and will mature at time $T$.

4. $f(t,T)$ is the instantaneous forward rate with maturity time $T$ at time $t$, given by:

$$f(t,T) = \lim_{S \to T} F(t,S,T)$$

5. $r_t$ is the instantaneous spot rate at time $t$, given by:

$$r_t = \lim_{T \to t} f(t,T)$$

Thus, $r_t$ represents the instantaneous continuously compounding growth rate of money market accounts at time $t$.

6. A set $Z_t$ whose elements are identified by non-negative real numbers is the yield curve at time $t$, given by $Z_t = \{Y(t,T), T \in [t, \infty)\}$.

All these quantities are modelled as random variables. Now, suppose we start a process from $t_0$. One objective of interest rate modelling is to obtain the yield curve at time $t$, $Z_t$, as a set of non-random numbers, given the model realisation of interest rates from the time $t_0$ to time $t$. In the next section, we will show the general formula to obtain bond prices at time $t$ when bond prices follow a general one-factor model. Once bond prices are obtained, other values can be obtained by definitions. However, note that for $s>t$:

$$r_s, B(s,T), F(s,S,T), Y(s,T), f(s,T)$$

are still random variables at time $t$, even when the yield curve $Z_t$ has been obtained by the general formulas.

### 4.2 General arbitrage relationship

The most of the idea of the following arguments come from Vasicek (1977).
Suppose the bond prices can be described by:
\[
\begin{align*}
\frac{dB(t_i, B(t_i))}{dB(t_i)} / \frac{dB(t_j, B(t_j))}{dB(t_j)} &= \mu(t_i, B(t_i)) \cdot dt + \sigma(t_i, B(t_i)) \cdot dW_i, \\
\frac{dB(t_i, B(t_i))}{dB(t_i)} / \frac{dB(t_j, B(t_j))}{dB(t_j)} &= \mu(t_j, B(t_j)) \cdot dt + \sigma(t_j, B(t_j)) \cdot dW_i, \quad (4.2)
\end{align*}
\]
where \( \mu(t_i, B(t_i)) \) \((i = 1, 2)\) is the drift coefficient of bonds maturing at \( T_i \), \( \sigma(t_i, B(t_i)) \) \((i = 1, 2)\) is the volatility of bonds maturing at \( T_i \), and \( \{W_i\} \) is a standard Wiener process.

Here we have set up two bonds with arbitrary maturity dates \( T_1 \) and \( T_2 \), and we call the bonds as BOND1 and BOND2. The same Wiener process denoted by \( \{W_i\} \) drives both bonds. Thus the Wiener process is not indexed with \( T_i \). Note that we have not assumed risk-neutrality. In a risk neutral market, all assets traded in the market have the same instantaneous expected rate of return regardless of their price risks. See, for example, Hull (1999, p205). In other words, the expected rate of return is the risk-free rate in a risk neutral market.

Since risk neutrality is not assumed, drift coefficients are not the same as the instantaneous spot rate, i.e., in general:
\[
\mu(t_i, B(t_i)) \neq r_i \quad \text{and} \quad \mu(t_j, B(t_j)) \neq r_j.
\]

Again following Vasicek (1977), we consider the portfolio composed of \( +\sigma_2B_2 \) units of BOND1 (for simplicity, we write \( B_i = B(t_i, B_i) \), \( \mu_i = \mu(t_i, B_i) \), \( \sigma_i = \sigma(t_i, B_i) \) \((i = 1, 2)\)) and \( -\sigma_iB_i \) units of BOND2. Then the value of the portfolio at time \( t \) is:
\[
\Psi = (\sigma_2B_2)B_1 - (\sigma_iB_i)B_2 = (\sigma_2 - \sigma_1)B_1B_2, \quad (4.3)
\]

The investment gain of this portfolio for the period from time \( t \) to \( t+dt \) is, using (4.2):
\[
\sigma_2B_2(dB_2) - \sigma_1B_1(dB_2) = (\sigma_2\mu_1 - \sigma_1\mu_2)B_1B_2dt
\]

This amount does not contain any random term. The random terms, \( \sigma_i dW_i \), were cancelled by each other. Assuming no arbitrage, the portfolio should earn the risk-free return over the period \((t, t+dt)\). Therefore:
\[
(\sigma_2\mu_1 - \sigma_1\mu_2)B_1B_2dt = \Psi r_t dt, \quad (4.4)
\]
where $r_t$ is the instantaneous spot rate at $t$.

Substituting (4-3) into (4-4), we get:

$$\frac{\mu_1 - r_t}{\sigma_1} = \frac{\mu_2 - r_t}{\sigma_2} \quad (4-5)$$

which we call $\lambda(t, r_t)$, or equivalently:

$$\mu_1 = r_t + \sigma_1 \lambda(t, r_t) \text{ and } \mu_2 = r_t + \sigma_2 \lambda(t, r_t) \quad (4-6)$$

Since equation (4-5) is valid for arbitrary maturity dates $T_1$ and $T_2$, it follows that the ratio $\frac{\mu_i - r_t}{\sigma_i}$ ($i=1,2$) is independent of $T$ and of $B(t, T)$. This quantity $\lambda(t, r_t)$ is called the market price of risk, and is widely used by financial economists. See, for example, Hull (1999, p498). Equation (4-5) shows that the real-world drift coefficients cannot be determined unless we know the market price of risk. This argument can be extended to the case when the bond prices are driven by $n$ Wiener processes. Suppose the bond prices follow the process:

$$dB(t, T)/B(t, T) = \mu(t, T, B) \cdot dt + \sum_{i=1}^{n} \sigma_{(i)}(t, T, B) \cdot dW^{(i)}_t$$

where $\mu(t, T, B)$ is the drift coefficient of the bond, $\sigma_{(i)}(t, T, B)$ is the volatility of the bond for $i$-th Wiener process, and $\{W^{(i)}_t\}$ is the $i$-th standard Wiener process.

The no arbitrage condition among bonds leads to:

$$\mu(t, T, B) = r_t + \sum_{i=1}^{n} \sigma_{(i)} \lambda_{(i)}(t, W^{(i)}_t) \quad (4-7)$$

The amount $\lambda_{(i)}(t, W^{(i)}_t)$ is the market price of risk for the $i$-th factor. See Hull (1999, Section 19.2) for explanations and proofs.

4.3 Bond price formula

At time $t$, the zero coupon bond prices can be obtained by the following formulae depending on whether we know forward rates or future instantaneous spot rates.
Then we get the yield curve at time $t$ using formula (4-1). If we know forward rates, $f(t,u)$ ($t \leq u$), by definition, the zero coupon bond price is given by:

$$B(t,T) = \exp\left[-\int_t^T f(t,u) du\right]$$

(4-8)

This formula tells us that forward rates are what is known as the force of interest in actuarial circles.

Or, if we know the current instantaneous spot rate $r_t$ and future random instantaneous spot rates $r_u$ ($t < u$):

$$B(t,T) = E\left[\exp\left[-\int_t^T r_u du - \frac{1}{2} \int_t^T \lambda(u, r_u)^2 du - \int_t^T \lambda(u, r_u) dW_u\right]\right],$$

(4-9)

assuming that the bond prices at time $t$ are determined by the assessment of $r_u$ ($t < u$). See Vasicek (1977). To prove this, define for $s > t$:

$$V(s) = \exp\left[-\int_t^s r_u du - \frac{1}{2} \int_t^s \lambda(u, r_u)^2 du - \int_t^s \lambda(u, r_u) dW_u\right]$$

Using Ito’s lemma:

$$dV(s) = V(s)\left(-r_s ds - \frac{1}{2} \lambda(s, r_s)^2 ds - \lambda(s, r_s) dW_s\right) + \frac{1}{2} V(s)\lambda(s, r_s)^2 ds$$

$$= V(s)\left(-r_s ds - \lambda(s, r_s) dW_s\right)$$

Now using (4-2) (for simplicity, the argument $(t, T, B)$ is omitted):

$$dB(s, T) = B(s, T)(\mu ds + \sigma dW_s).$$

We get,

$$d(V(s)B(s, T)) = dB(s, T) + V(s)dB(s, T) + dB(s, T)B(s, T)$$

$$= B(s, T)V(s)\left(-r_s ds - \lambda(s, r_s) dW_s\right) + V(s)B(s, T)(\mu ds + \sigma dW_s) - B(s, T)V(s)\lambda(s, r_s)\sigma ds$$

$$= B(s, T)V(s)\left((\mu - r_s - \lambda(s, r_s)\sigma)ds + (\sigma - \lambda(s, r_s))dW_s\right)$$

$$= B(s, T)V(s)(\sigma - \lambda(s, r_s))dW_s$$
by the definition of \( \lambda(s, r_s) \). Taking expectation after integrating over \((t, T)\), and using \( B(T, T) = 1 \) and \( V(t) = 1 \), we get formula (4-9).

The risk-neutral world is the world where the market price of risk is zero, whereas in the real world the market price of risk is given by formula (4-5). See Hull (1999, p508). Therefore, the bond price in the risk-neutral world is given by setting \( \lambda(u, r_u) = 0 \) in formula (4-9):

\[
B(t, T) = E \left[ \exp \left( - \int_t^T r_u \, du \right) \right] \tag{4-10}
\]

The above formula is applicable only in a risk-neutral environment. In general, the exact market price of risk is needed to price a bond as well as the stochastic representation of the instantaneous spot rate. If \( \lambda(u, r_u) > 0 \) for all \( u \) and \( r_u \), the bond price would be smaller than in the case when it is calculated in a risk-neutral environment. The lower bond price is equivalent to a higher yield. This additional yield can be interpreted as a compensation for the price risk borne by investors, implied by a positive market price of risk. This additional return is called a risk premium, consistent with our general understanding of the term “compensation for risks”.

Among a number of different theories of interest rate term structure, expectation theory (sometimes called pure expectation hypothesis), segmentation theory and liquidity preference theory are well known. See Hull (1999, p97). Pricing bonds in a risk-neutral world means we price bonds assuming zero risk premiums. This is equivalent to pricing bonds using expectation theory. Pricing bonds under \( \lambda(u, r_u) > 0 \) for all \( u \) and \( r_u \) is equivalent to assuming the liquidity preference hypothesis, where yields of bonds can be interpreted as the sum of risk-neutral bond yields and risk premiums as compensations for price risks.
4.4 Treatment of market price of risk

In order to obtain exact prices of zero coupon bonds at time t and developments of the prices afterwards, formulae (4-9) indicates that we need to know the exact market prices of risk $\lambda(u, r_u)$ for all $u \geq t$ and $r_u$. In other words, by the relationship (4-6), we need the exact drift coefficient $\mu(u, T, B(u, T))$ for all $u \geq t$ and $B(u, T)$. However, we conduct our study under the risk-neutral assumption, i.e. the market price of risk is assumed to always be zero. This assumption seems reasonable for the following reasons:

1. The drift coefficient represents instantaneous expected return, while the volatility represents instantaneous deviation from the expected value, i.e. “error”. We are interested in the deviation from the expectation since the deviation causes profits or losses when simulated interest rate term structures are applied to policy portfolios. In other words, one primary concern is the errors, which are largely determined by volatilities, rather than the expected returns.

2. Given the price of a bond or a yield on a bond, it is almost impossible to separate the risk premium portion from the risk-neutral bond price. For example, suppose we know that the yield of a 10-year Government bond is 2%. We could assume that the risk-neutral bond yield is 1.8% and the risk premium is 0.2%. However, we could also assume that the risk-neutral bond yield is 1.6% and the risk premium is 0.4%. We cannot tell which decomposition is correct, i.e. the market the price of risk is unobservable.

3. Under the pure expectation hypothesis of the interest rate term structure, any bond can be priced assuming risk-neutrality.

4. As seen in the definition of the market price of risk, the market price of risk defines the additional return required for a unit of risk. The market price of risk, in other words an investor’s return requirement for compensation for taking the risk of price fluctuation, changes over time. The market price of risk under a steeply increasing yield curve environment might be quite different from that under an inverted curve environment. In addition, the market price of risk can be affected by other economic factors. Finding out the correct market price of risk in a large number of circumstances is impossible in practice.
Therefore, we assume that we are in risk-neutral environment in a bond market. This approach is exactly the same as used for pricing derivatives whose underlying assets are tradable. Hull (1999, p502) states that one approach to valuing a derivative is to reduce the expected growth rate, and then behave as though the world is risk neutral.
5 Overview of interest rate models

Many kinds of “term-structure models” of interest rates have been established with numerous contributions from financial economists. A summary appears in Chaplin (1998). These models provide a theoretical basis for valuing interest rate derivatives such as American-style swap options, callable bonds and bond prices (Hull (1999)), as well as describing the stochastic behaviour of interest rates and bond prices. There are strengths and weaknesses of each model. Chaplin (1998) concluded that all models might be useful for specific purposes at specific times. In this Chapter, models that are commonly used by market practitioners and academics are briefly introduced. The Vasicek model, Cairns’ model and the HJM approach, which are used in our study, are described in more detail.

5.1 Commonly used models

Chaplin (1998) assigned a model to one of three broad categories: equilibrium, evolutionary and descriptive. At first, we provide an overview of interest rate term structure models following this classification.

5.1.1 Equilibrium models

Equilibrium models usually start with assumptions about economic variables and derive a process for the instantaneous spot rate (Hull (1999, p564)). Once the process for the instantaneous spot rate is described, bond prices, option prices and other derivative prices are obtained in a risk-neutral environment. Even though a wide variety of interest rate term structures can be represented using equilibrium models, the exact description of the initial term structure and of its evolution in real world, where risk premiums do exist, is not a major concern.

Models included in this category are the one-factor models of Randleman and Bartter (1980), Vasicek (1977) and Cox, Ingersoll, and Ross (1985), and the
two-factor models of Longstaff-Schwartz (1992) and Brennan-Schwartz (1979, 1982).

5.1.2 Evolutionary models
Equilibrium models are driven by one or two factors, with several parameters. The number of these parameters is not enough to produce a variety of yield curve shapes observed in the market. This drawback leads to arbitrage opportunities between synthetic bonds, constructed from options, and actual bonds when these models are put into practical use. As Chaplin (1998) states, the development of derivatives, especially fixed income options, required a model which eliminates the possibility of arbitrage. Evolutionary models emerged to meet this need. Evolutionary models were developed mostly by adding additional parameters to equilibrium models. These models formulate the time series development of a yield curve in a way that is consistent with its initial shape. Models included in this category are the one-factor models of Ho and Lee (1986) and Hull and White (1990), and multi-factor models based on the HJM approach.

5.1.3 Descriptive models
Descriptive models are used when the objective is a reasonably accurate description of the yield curve and, by implication, an accurate indication of yields which can be obtained in the market (Chaplin (1998)). The yield curve is expressed in terms of a spanning set of elementary functions. Models included in this category are those of Dobbie and Wilkie (1978) and Cairns (1998).

5.1.4 Wilkie's model
Wilkie's model (1986, 1995) is widely used in actuarial circles. This model was primarily developed to deal with stochastic returns on ordinary shares with inflation being the driving factor. This model deals with four economic variables: retail price index, share yield, share dividends and yield on consols, an irredeemable British Government security. They are related in a “cascade
structure” where the retail price index is the most fundamental variable influencing the other three variables. The variables are modelled using time-series.

Wilkie’s model has an emphasis on returns on equities rather than the stochastic behaviour of interest rates and their term structure. In our study, we are interested in how interest rate term structure develops away from the initial one and how much profits or losses are incurred as the result of the development. In other words, we are interested in bonds rather than equities. Therefore, Wilkie’s model is not the most appropriate modelling tool for our study.

5.2 The classical approach and the HJM approach
The equilibrium and evolutionary models discussed above, except the HJM approach, deal with the development of an instantaneous spot rate. Focusing on this point, Neftci (2000) categorizes equilibrium and evolutionary models differently. Based on its underlying methodology, a model is assigned to either the classical approach or the HJM approach.

Evolutionary models are in most cases developments of equilibrium models. Thus, the mathematical features and economic aspects of the evolutionary models (except HJM) are similar to those of equilibrium models, which are driven by the process of instantaneous spot rates. On the other hand, the HJM approach is quite different from other equilibrium and evolutionary models because it deals with forward rates. We now introduce some models commonly used by financial economists following this categorization.

5.3 Models based on the classical approach
According to Neftci (2000), the one-factor classical approach involves modelling instantaneous spot rates. Two-factor models use another random variable in addition to instantaneous spot rates, for example, yield on consols. Regardless of the number of factors, this approach has several strengths. Firstly, this approach is intuitively understandable, since the random variable driving the process usually represents an observable variable like the instantaneous spot
rate. Secondly, a great deal of material is already available since significant resources have been invested in research into these models. Both academics and practitioners have published a large amount of papers from both theoretical and empirical studies. Thirdly, most models using the classical approach are mathematically tractable. Some models yield an analytical solution for prices of bonds and plain European options.

However, there are some drawbacks as well. Firstly, in general, objects as the random variables such as interest rates are not tradable. Because interest rates are not tradable, there are no prices for interest rates. In contrast, prices of bonds, which are tradable assets, do exist. We can assume risk neutrality, or even assume some reasonable risk premiums, for tradable assets. However, we cannot directly apply the same argument to non-tradable “assets” like interest rates. Thus, the market price of risk for each modelled object needs to be worked out. This may cause difficulty since the risk premium for each random variable is not observable. See general arguments in Hull (1999, Section 19.2). Secondly, in order to obtain bond prices, we need to calculate expected values as shown in formula (4·10), even under one-factor models in a risk-neutral environment. On the other hand, formula (4·8) shows that bond prices are given by simple integration when we know forward rates. Thirdly, in the case of a one-factor model, the process is Markovian with regards to the instantaneous spot rate. This means that an identical term structure is produced by the same current instantaneous spot rate whatever the elapsed time is. Fourthly, in the case of multi-factor models (including two-factor), the numerical calculation is sometimes complex (Chaplin (1998)). Finally, only evolutionary models guarantee consistency with the initial term-structure. Other equilibrium models allow theoretical arbitrage opportunities, i.e. they cannot produce prices of financial instruments which are consistent over the whole maturity spectrum. In fact, the equilibrium models are likely to produce off-market bond option prices for some maturities.
5.3.1 The Vasicek Model

Vasicek (1977) proposed one of the most fundamental models of interest rate term structure. In the Vasicek model, the instantaneous spot rate is described by:

\[ dr_t = a(b - r_t)dt + \sigma dW_t, \]  

(5-1)

where \( r_t \) is the instantaneous spot rate at time \( t \), \( a, b, \sigma \) are constants and \( \{W_t\} \) is a standard Wiener process.

This formula says that the deviation from the long-term convergence level, \( b \), is reverted by the “force”, \( a \), with an accompanying random fluctuation whose magnitude is defined by \( \sigma \). Not only is this model mathematically tractable, but also the philosophy and approach behind this model still remain of significant importance. The most prominent feature of this model is mean reversion, a phenomenon by which interest rates revert to some long-run convergence level over time. It is also known that a variety of shapes of term-structure can occur under this model. See, for example, Hull (1999). However, the interest rate \( r_t \) can become negative.

Formula (5-1) says the instantaneous spot rate \( r_t \) follows an Ornstein-Uhlenbeck process. It is well known (see, for example, Rolksi et al. (1998, p562)) that, for given \( r_0 \):

\[ r_t = b + (r_0 - b)e^{-at} + \sigma \int_0^t e^{-a(t-s)}dW_s \]  

(5-2)

It follows that:

\[ E[r_t] = b + (r_0 - b)e^{-at} \quad \text{and} \quad Var[r_t] = \sigma^2 \frac{1 - e^{-2at}}{2a} \]

The variance per time unit, \( Var[r_t]/t \), is given by:

\[ \sigma^2 \frac{1 - e^{-2at}}{2at} \]  

(5-3)

As Neftci (2000, p416) states, \( e^{-\int_0^tr_tds} \) can be interpreted as the random price of
a zero coupon bond with maturity \( t \) at time zero in a risk-neutral environment.

The amount \( \int_0^t r_s ds \), which we denote by \( I \), represents “aggregated interest on a money market account” from time zero to time \( t \). This value is obtained by integrating (5-1) over (0,t). Using (5-1):

\[
I = \int_0^t r_s ds = \frac{1}{a} \int_0^t dr_s + b \int_0^t ds + \frac{\sigma}{a} \int_0^t dW_s \\
= bt + \frac{r_0 - b}{a} (1 - e^{-at}) + \frac{\sigma}{a} \int_0^t (1 - e^{-a(t-s)}) dW_s
\]

Then by the formula (4-1), \( I/t \) is the random spot yield at time \( t \) for maturity at \( T \). Assuming risk neutrality, the fair price of the zero coupon bond is obtained as the expectation \( E[e^{-I}] \) using formula (4-10).

5.3.2 One-factor models

In many established models, the stochastic process for instantaneous spot rates involves only one source of random fluctuation, as seen in the Vasicek model. The instantaneous spot rate is described by an Ito process of the general form:

\[
dr_t = \{ \theta(t) + a[b - r_t] \} dt + \sigma r_t dB_t,
\]

where \( a, b, \beta \) and \( \sigma \) are constants, \( \theta(t) \) is a deterministic function of \( t \) and \( \{ W_t \} \) is a standard Wiener process.

The mathematical formulae for the following models are obtained by assigning values to parameters in formula (5-4).

Setting, \( \theta(t) = 0 \) and \( \beta = 0 \) we obtain the Vasicek model:

\[
dr_t = a[b - r_t] dt + \sigma r_t dW_t.
\]

Setting, \( \theta(t) = 0, \beta = 1 \) and \( b = 0 \) we get the Rendleman and Bartter model where the instantaneous spot rate follows geometric Brownian motion (see, for example, Hull (1999)):

\[
dr_t = -ar_t dt + \sigma r_t dW_t.
\]

This model has a significant drawback in that the instantaneous spot rate
increases exponentially since the volatility of the instantaneous spot rate increases in proportion to the value of the instantaneous spot rate. Setting $\theta(t) = 0$ and $\beta = 0.5$, we obtain the Cox, Ingersoll, and Ross model:
\[
dr_t = a[b - r_t]dt + \sigma \sqrt{r_t} dW_t.
\]
By replacing the constant volatility with one proportional to the square root of the interest rate level, the possibility of negative spot rates, a feature of the Vasicek model, is eliminated. However, the model suffers from an inability to produce a humped yield curve (Chaplin (1998)). Setting $a = 0$ and $\beta = 0$, we obtain the Ho and Lee model:
\[
dr_t = \theta(t) dt + \sigma dW_t.
\]
We can choose the parameter $\theta(t)$ so that the initial term structure generated by this model can fit to a market yield curve. Thus this model is an evolutionary (i.e. no-arbitrage) model. This model has no mean-reversion feature since the model formula does not include the term $a[b - r_t]dt$. It is also possible for the instantaneous spot rate to become negative.

Setting $b = 0$ and $\beta = 0$, we obtain the Hull and White model:
\[
dr_t = \{\theta(t) - ar_t\} dt + \sigma dW_t.
\]
This model is an extension of the Vasicek model, with the features of the Ho and Lee model. The Hull-White model is also an evolutionary (i.e. no-arbitrage) model. It is a development of the Ho and Lee model in that a mean-reversion feature is included. Negative spot rates, however, are still possible.

5.3.3 Two-factor models
According to Feldman (1998), empirical analysis using principal component analysis showed that 95% of the observed variability of a yield curve is explained by two factors, while 85% can be explained by one factor. This conclusion implies that a two-factor model may well explain the variability of the interest rate term structure. We introduce the Longstaff and Schwartz model and the Brennan and Schwartz model, both of which are well known by financial economists.

The Longstaff and Schwartz model is described by the joint dynamics of two
factors of the form:

\[ dr_i = \mu_r(r_i, V, t)dt + V(r_i, t)dW'_i, \]

\[ dV = \mu_r(r_i, V, t)dt + \sigma_r dW'_i \]

where \( \{W'_i\} \) and \( \{W'_V\} \) are independent standard Wiener processes.

In this model, both instantaneous spot rates and volatilities of the instantaneous spot rates are random variables. According to Rebonato (1996), the model is mathematically tractable, with relatively easy calibration of the parameters. It can generate a variety of yield curves. As implied in formula (4-7), the no arbitrage argument requires two market prices of risk for each random variable.

The Brennan and Schwartz model has the form:

\[ dr_i = \mu_r(r_i, L, t)dt + \sigma_r(r_i, L, t)dW'_i, \]

\[ dL = \mu_L(r_i, L, t)dt + \sigma_L(r_i, L, t)dW'_i \]

where \( \{W'_i\} \) and \( \{W'_L\} \) are independent standard Wiener processes with:

\[ E[dW'_i dW'_L] = \rho dt. \]

This model has two random variables, the instantaneous spot rate and the yield on a consol. Rebonato (1996) states that only one market price of risk is required even though the model has two random variables. However, he also states that there is difficulty with the inter-relationship between these variables.

### 5.4 The Heath-Jarrow-Morton approach

The HJM approach models forward rates as random variables that drive an arbitrage-free stochastic development of yield curves. It is quite different from the classical approach, which models instantaneous spot rates.

At time \( t \), there is only one instantaneous spot rate \( r_i \), while we have a series of forward rates \( f(t, T) \) for all \( T \geq t \). Once forward rates over an entire maturity spectrum are obtained, we can calculate zero coupon bond prices and yields of these bonds using formula (4-8) and (4-1) without calculating any expected
values. Therefore, we can characterize that the HJM approach deals with the term structure $Z_t$ as a whole, while models using the classical approach model only instantaneous spot rates $r_s$. In the classical approach, we need to know “future instantaneous spot rates”, i.e. $r_s$ for all $s>t$ in order to attain the interest rate term structure at time $t$, since the classical approach requires calculating expected values to obtain the zero coupon bond prices as shown in formula (4.10).

In the HJM approach, instantaneous forward rates are modelled: i.e. we need to specify both the drift coefficients and volatilities of forward rates. However, in the risk-neutral environment, the drift coefficients are obtained as functions of volatilities of forward rates as, for example, Neftci (2000) showed. In other words, the development of forward rates, i.e. the development of yield curves, is dominated solely by volatility parameters in the risk-neutral environment.

Another notable feature of the HJM approach is that the process of the instantaneous spot rates under the HJM approach is, in general, path-dependent. We observe different shapes of market yield curves even when short-term interest rates stay at the same level. In contrast, under one-factor classical models, we observe the identical yield curves for the same instantaneous spot rates.

The features of the HJM approach described above lead to several advantages. One advantage is that there is no problem in choosing parameters that conform to the initial yield curve. This is because the development of a yield curve starts from the yield curve at present, which is known. Another advantage is that there is great flexibility in specifying volatility parameters. In contrast, most of the models using the classical approach assume constant volatility. A great variety of implementation is feasible in the HJM approach by choosing different volatility parameters. Moreover, the HJM approach can be extended to as many factors as required.
On the other hand, there is no obvious way to add a factor to a classical-approach model without introducing another market price of risk and solving the resulting estimation problems (Rebonato (1996)).

However there are some drawbacks in the HJM approach. In general, no analytical solution exists for prices of zero coupon bonds, or of plain options. Simulation is the common approach. In particular, when a multi-factor approach is adopted, analytical solutions are difficult to find. The resulting lack of computational speed can be a big disadvantage in a trading environment (Chaplin (1998)). Another drawback is that the behaviour of instantaneous spot rates, depending on the choice of volatility parameters, is hard to understand intuitively as opposed to the classical approach. In addition, instantaneous spot rates may diverge or become negative in finite time, depending on the volatility function (Chaplin (1998)).

There are several ways to represent the HJM approach (Rebonato (1996)), since zero coupon bond prices, spot yields and forward rates are related to each other through simple functions. Hull (1999) starts his treatment of the HJM approach from

\[
\frac{dB(t, T)}{B(t, T)} = r_t dt + \sigma(t, T, B(t, T)) dW_t^T
\]  

(5-5)

where \( B(t, T) \) is the price of the zero coupon bond maturing at \( T \) at \( t \), \( r_t \) is the instantaneous spot rate at \( t \), \( \{W_t^T\} \) is the standard Wiener process driving the bond maturing at \( T \) and \( \sigma(t, T, B(t, T)) \) is the volatility of the price at \( t \) of the bond maturing at \( T \) when the bond price is \( B(t, T) \).

In the above notation, the volatility is the instantaneous percentage change of the zero coupon bond price per annum. Here, the drift coefficient is set to \( r_t \), the instantaneous spot rate. This setting is equivalent to assuming risk neutrality, where all bonds share the same instantaneous risk-free return, regardless of their maturities and associated price risks. It should also be noted that the Wiener process is indexed by \( T \). It means, in principle, bonds with different
maturities are allowed to be influenced by different shocks. This is the reason why the HJM approach is sometimes referred to as a multi-factor model.

5.5 Descriptive Models

The Dobbie and Wilkie model (1978) had been in use for the purpose of the construction of published indices (FTSEAGS Yield Indices) for many years. Cairns (1997) replaced the Dobbie and Wilkie model to cope with the tax reform introduced by the British Government in April 1996 under which capital and income gains are taxed on the same basis. Another objective of creating Cairns’ model is to solve the problem known as “catastrophic jumps” that appears in parameter estimation.

The mathematical formula for the Dobbie and Wilkie model is:

\[ y(t) = A + Be^{-ct} + De^{-Ft} \]

where \( y(t) \) is the bond-equivalent gross redemption yield of gilts (British Government securities) whose duration is \( t \), and \( A, B, C, D \) and \( F \) are constants.

The formula for Cairns’ (1997) model is:

\[
f(t, t+s) = b_0 + \sum_{i=1}^{4} b_i e^{-c_i t}
\]

where \( f(t, t+s) \) is the continuously compounding instantaneous forward rate observed at \( t \) for \( s > 0 \) and \( \{b_i\}_{i=0}^{4} \) and \( \{c_i\}_{i=1}^{4} \) are constants.

Cairns developed the Dobbie and Wilkie model by introducing a forward-rate curve instead of a gross redemption yield curve, and by proposing a more flexible model with nine parameters.

The choice of \( \mathbf{c} = (c_1, c_2, c_3, c_4) = (0.2, 0.4, 0.8, 1.6) \) was found to be appropriate for the UK gilt market for the period 1992 to 1996. According to Cairns (1997), with this set of values for \( \mathbf{c} \), a wide variety of shapes for the forward rate curve
could be generated by using different combinations of values for \( \{b_i\}_{i=0}^4 \).

The use of these fixed exponential parameters \( c \) eliminated the risk of multiple solutions, which had sometimes caused serious problems known as “catastrophic jumps” in parameter estimation for the Dobbie and Wilkie model. The catastrophic jump is a sudden unnoticed jump from one minimum of a weighted square error sum to another, which happens in parameter estimation.

By straightforward integration using (4-8), the zero coupon bond price with maturity \( T \) at time \( t \) is given by:

\[
B(t,T) = \exp \left[ - \left( b_0(T-t) + \sum_{i=1}^{4} b_i \left( 1 - e^{-\zeta_i(T-t)} \right) / c_i \right) \right].
\]  

(5-7)
6 Yield curves in a very low interest rate environment

Since the BOJ adopted a “zero rate policy” on 9 September 1998, Japanese short-term interest rates have been hovering just above zero. The behaviour of the interest rate term structure under this “zero rate” environment is to some extent different from what we observe in an environment where rates are in a more traditional range, for example from 2% to 10%. The fact that nominal interest rates cannot be negative implies that the movement of short rates is greatly restricted by the lower boundary of 0%. For example, suppose the short-term rate is 0.1%. The rate may move up by 0.3% to 0.4%. However, the rate may not move down by 0.3% to –0.2% symmetrically. In addition, long term rates seem to have a non zero lower bound, as will be shown later in this chapter. These phenomena imply that the behaviour of interest rates in very low interest rate environments may not be handled by established models. Therefore, allowing for this feature is necessary to model Yen interest rates satisfactorily.

The monetary authorities generally have the greatest influence on interest rates. Therefore very low interest rates will be considered in line with the monetary policy framework of the BOJ, paying particular attention to the behaviour of long rates.

Most of the arguments in this chapter follow the study by Oda and Okina (2001).

6.1 A history of Bank Of Japan monetary policy

The BOJ controls money market rates by supplying liquidity through purchasing securities from the market, or by withdrawing liquidity through selling back these securities to the market. Supplied liquidity sometimes remains as banks’ deposit to the BOJ beyond legal requirement. The excess portion is called “excess reserve”, which the BOJ was using as one of guidance figure in its money market operation. Oda and Okina (2001) state that the BOJ
started to provide a massive amount of excess reserves and subsequently took the following steps:

1. On 9 September 1998, the uncollateralized overnight call rate guideline\(^7\) was lowered from below 0.50% to 0.25% on average.
2. On 12 February 1999, the zero interest rate policy was introduced.
3. On 13 April 1999, the BOJ Governor Hayami announced: “We will continue the zero interest rate policy until we reach a situation where deflationary concerns are dispelled.”

Figure 6-1 shows that short-term rates stayed under 0.5% except for the disturbance period at the end of 1999 (i.e. the so called Y2K period) and the end of 2000, while long-term rates fluctuated showing a downward trend. In general we observe much greater liquidity demand at a calendar year end and a fiscal year end, usually the end of March, than usual. At the end of the year 1999 (Y2K), the demand for liquidity was significant since huge numbers of institutions and individuals put aside liquidity on hand in case failures of computer programs caused unexpected incidents.

\(^7\) The “uncollateralized overnight call rate guideline” is the overnight cash rate targeted by the BOJ.
6.2 Behaviour of long-term interest rates

Nominal long-term interest rates can be regarded as a combination of a sequence of expected short-term rates and a risk premium. Oda and Okina (2001) pointed out two main reasons why the long term rate had declined. Firstly, reflecting the anticipated worsening of the economy, the expected duration of the zero interest rate policy was prolonged. This change in anticipation resulted in lower expected short-term rates. Secondly, the risk premium decreased through the BOJ’s effort to eliminate uncertainty, in relation to its monetary policy stance. The announcement of Governor Hayami on 13 April 1999 was particularly important in this respect.

6.3 A model of long-term interest rates

Oda and Okina (2001) pointed out that Japanese monetary policy is effectively transmitted through the interest rate mechanism or through the foreign
exchange mechanism. In their explanation of the interest rate mechanism, Oda and Okina assumed the following model as a process where factors determine long-term interest rates by affecting the formation of market expectations:

\[ R_t = \sum_{j=0}^{n} \alpha_j E[r_{t+j}] + \theta_t, \]

where \( R_t \) is the long-term risk-free interest rate at time \( t \),
\( n \) is the maximum maturity term,
\( r_t \) is the short-term risk-free interest rate at time \( t \),
\( \theta_t \) is the risk premium, and
\( \alpha_j \) is a weight (constant).

Under this model, long-term interest rates are determined by the weighted average of future expected short-term interest rates and a risk premium. The risk premium \( \theta_t \) can be considered to be composed of the following three components. The first component is uncertainty with respect to future short-term interest rates stemming from an unexpected economic shock (demand/supply shock). The second component is uncertainty with respect to future short-term interest rates due to the non-transparency of monetary policy, supposing economic shocks are well forecast. The third component is the effect of bond prices (long-term interest rates) being affected according to the supply of long-term government bonds. The excess supply affects the long-end of the yield curve even when expected short-rates remain at the same level, because of market segmentation. The long-term bond market is, at least in part, driven by capital transactions and government finance rather than monetary policy and short-term interest rates.

6.4 Forecasting long-term interest rates

In analysing likely options for further monetary policy, Oda and Okina (2001) refer to the consequences for long-term rates. The first option is to influence the expected value of future short-term interest rates by properly conveying the BOJ’s future monetary policy stance to the market. The BOJ has repeatedly announced it would continue its zero interest
rate policy until deflationary concerns are dispelled. Therefore, the effect of further detailed announcements and/or commitment of specific policies would be small. In addition, such announcements may deprive the BOJ of flexibility in future policy action.

The second option is to reduce risk premiums by diminishing the uncertainty pertaining to future monetary policy. This option would require the BOJ to further clarify its guidelines on monetary policy. However, it cannot reduce uncertainty arising from any unexpected shocks, and in this sense any commitment to future policy is limited.

The third option is to reduce risk premiums by tightening the supply-demand relationship in the bond market through the outright purchase of large quantities of medium to long-term government bonds. When the integrity of government debt is in doubt, the increased purchase of medium to long-term government bonds could be interpreted as a loss of fiscal discipline, resulting in an increased risk premium for government bonds. Even when the integrity of government debt is not in doubt, there are other issues to be considered. For example, in order for this policy to have any practical effects on the expectations of market participants, such an operation would require continuity. The unhedged long bond position accumulated through the purchase operation would erode the financial conditions of the central bank in a recovery phase of the economy. In the recovery phase, a significant decline in the prices of long-term government securities is anticipated since market interest rates would rise a great deal. As a result, if the BOJ continues outright purchase operations, the BOJ could lose control over inflation since monetary policy would no longer be effective.

6.5 The prospect for long-term rates

In any case, the extent of any further reduction in expected short-term rates and/or risk premium is limited. Therefore, we can conclude that there is a lower boundary to long-term rates in a zero-rate policy environment.
7 The interest rate models used in our study

7.1 Model selection

In order to analyse the interest rate risk of a life insurance policy portfolio, interest rate term-structure models appropriate for this specific purpose need to be selected. The models were selected according to the following criteria.

First, the models should allow accurate fitting of initial term structures, which are observed in Yen money markets, fixed income markets or interest rate swap markets. Premiums and margins for our portfolio are calculated using the initial term structure produced using the models. Therefore, in order for the premiums and margins to be calculated based on then prevailing market interest rates, the initial term structure produced using the models should reasonably fit to the then prevailing interest rate term structure.

Second, developments of an interest rate term structure, as well as a development of instantaneous short rates, should be well modelled over the period where policies are in force. Japanese life insurance contracts tend to have terms of a few decades. Thus the shape of a yield curve, in addition to the overall interest rate level, significantly affects present values of the expected cash flows arising from the policy portfolio over time.

Third, the model should be flexible enough to be applicable to any interest rate situation such as very low yields, high yields, inverted yield curves, etc. In particular, it should be able to handle the unique behaviour of interest rates under the current very low rate environment in Japan.

Considering these points, Cairns’ model and the HJM approach were chosen for our study. Cairns’ model is used to graduate the initial yield curve. The HJM approach (modified for handling zero rates) is used for obtaining developments of the initial term structure. These models have common features even though they were originally developed for different purposes. Firstly, both models deal with forward rates over the whole maturity spectrum. Secondly, for both models, a sufficient number of parameters can be incorporated easily. Thirdly, both
models are relatively easily implemented on computers. Cairns’ model is a descriptive model. This model was developed to obtain a complete yield curve using several market observation points. Since the model is designed for this purpose, this model is most appropriate for setting up an initial yield curve when several market observation points are given.

The HJM approach takes the initial yield curve as an input, i.e. any shape of initial yield curve is acceptable as a starting set. In addition, we can simulate a variety of development patterns using the HJM approach.

### 7.2 Initial term structure

Several kinds of market yield curves are publicly available in Yen fixed-income markets, for example, the Government bond curve, bank debenture curves, swap curves, etc. The market rates are available in various financial news services including newspapers. These fixed income financial instruments are usually traded over the counter; hence money brokers or securities companies provide market rates. No matter what market yield curve is used, rates are quoted only for a limited number of maturity dates, e.g. quoted only for 2,3,4,5,7,10 and 20 year maturities. Therefore the first step in modelling the development of the yield curve is to describe the initial yield curve as a mathematical representation by which we can specify the interest rate corresponding to any maturity time. This act of creating a smooth yield curve from several points observed in the market is a form of graduation. For the graduation of the initial yield curve, we used Cairns’ model with parameters \( c = (0.2, 0.4, 0.8, 1.6) \) in formulae (5-6) and (5-7).

### 7.3 Evolution of the term structure

The HJM approach was selected as the most appropriate method for modelling developments of yield curves. Computation time due to no closed analytical solutions is a serious problem for real-time application of the HJM approach. Using a classical model, pricing financial products can be achieved numerically using a recombining binominal tree approach (see, for example, Hull (1999,
(Chapter 16)), even when analytical solutions for the pricing do not exist. However, we cannot use the recombining binominal tree approach for models based on the HJM approach since instantaneous short rates (more generally, state variables) are not in general Markovian under the HJM approach. The recombining binominal tree approach is available only when state variables are Markovian. We usually use Monte Carlo simulation methods to implement models based on the HJM approach, causing much longer computation time than when models based on a classical approach are used. Practitioners who need real-time pricing may not be prepared to accept such computation time. However, for research purposes, computation time required under the Monte Carlo simulation method is acceptable. In the Monte Carlo simulation environment, mathematical difficulties where there is no analytical closed formula for zero coupon bond prices are not important at all.

Another advantage of the HJM approach is that its non-Markovian feature allows various shapes of yield curve under similar overall interest rate levels. This is desirable since we observe various shapes of yield curve under the same overall interest rate levels, e.g. 5%, in markets. In addition, the HJM approach eliminates concerns about initial yield curve fitting, since the initial yield curve is given as a starting set in the HJM approach, as opposed to the case of classical models, which require fitting of this curve. In other words, there cannot be any discrepancy between the model yield curve and the market yield curve at the starting point in the HJM approach.

The HJM approach provides only the framework for modelling. We need to formulate a specific model for our study under the general HJM framework. Here, we make the following assumptions:

(1) The same Wiener process drives the prices of bonds with any maturity, i.e. we are using a one-factor model.

(2) The fixed-income market operates in a risk-neutral environment, for reasons discussed in Section 4-4.

(3) During the period \((t, t+ \Delta t)\) where \(\Delta t\) is small, the volatility of a bond price remains constant.
(4) The volatility of the price of a bond depends only on its time to maturity and its yield to maturity, irrespective of what time has elapsed since the starting time.

In a one-factor case, the HJM stochastic differential equation (5·5) becomes

\[ dB(t, T) / B(t, T) = r_t \, dt + \sigma(t, T, B(t, T)) \, dW_t, \]

dropping the superscript \( T \) from the Wiener process increment \( dW_t \), since only one Wiener process drives the process.

Simulation was carried out in discrete time intervals. In a discrete time interval and under these assumptions, the HJM stochastic differential equation (7·1) leads to, using Ito's lemma and under the above assumption of constant volatility during the period \((t, t + \Delta t)\):

\[
B(t + \Delta t, T) = B(t, T) \cdot \exp \left( r_t \, \Delta t - \frac{1}{2} \sigma(t, T, B)^2 \Delta t + \sigma(t, T, B) \Delta W_t \right)
\]

\[
= B(t, T) \cdot \exp [r_t \cdot \Delta t] \cdot \varepsilon_t,
\]

where \( \varepsilon_t \) is a random error:

\[
\varepsilon_t = \exp \left( -\frac{1}{2} \sigma^2 \cdot \Delta t + \sigma \cdot \Delta W_t \right)
\]

and \( \Delta W_t = W_{t+\Delta t} - W_t \).

Suppose \( B(t, T) \) is given. Since \( \Delta W_t \sim N(0, \Delta t) \), \( E[\varepsilon_t] = 1 \). Then, taking expectation in (7·2):

\[
E[B(t + \Delta t, T)] = B(t, T) \cdot \exp [r_t \cdot \Delta t]
\]

(7·3)

Note that the bond price \( B(t + \Delta t, T) \) is still a random variable at time \( t \). However, we know the expected value of the random bond price: \( \hat{E}[B(t + \Delta t, T)] \).

Then we can calculate the yield of the bond when its price happens to be the expected value. We denote this yield as \( \hat{Y}(t + \Delta t, T) \). Using (7·3):
\[ \tilde{Y}(t+\Delta t,T) = -\log(E[B(t+\Delta t,T)])/\{T-(t+\Delta t)\} \]
\[ = \left\{Y(t,T) \cdot (T-t)-r \cdot \Delta t\right\}/\{T-(t+\Delta t)\}, \quad (7-4) \]

where \( Y(t,T) \) is the spot yield at \( t \) of a bond maturing at \( T \).

Note that this “yield of the expected value of random bond prices” differs from (though, is close to) the “expected value of random spot yields”, which is:
\[ E[Y(t+\Delta t,T)] = E[-\log(B(t+\Delta t,T))]/\{T-(t+\Delta t)\}. \]

In order to show that they are different, consider the function \( y=\log(x) \). Since the function is concave, using Jensen’s inequality:
\[ \log[E[B(t+\Delta t,T)]] > E[\log[B(t+\Delta t,T)]] \]
leading to:
\[ \tilde{Y}(t+\Delta t,T) < E[Y(t+\Delta t,T)], \]
which is equivalent to:
\[ E[Y(t+\Delta t,T)] - \tilde{Y}(t+\Delta t,T) > 0. \]
This difference arises because bond prices are a convex function of their yield to maturity. The approximation of the difference up to second order is called a “convexity adjustment” (see, for example, Hull (1999, Section 20.6)).

By definition, \( B(t+\Delta t, t+\Delta t) \) is always 1, i.e. the price movement from time \( t \) to \( t+\Delta t \) of the bond with the maturity \( t+\Delta t \) is deterministic. Hence \( \sigma(t, t+\Delta t, B(t, t+\Delta t)) \) is 0. Substituting this into (7-2), the instantaneous (strictly, ‘short’ in this case) spot rate is:
\[ r = -\log(B(t,t+\Delta t))/\Delta t \quad (7-5) \]

Note that \( B(t,T) \) for any \( t \) and \( T \) should be always less than 1. However, formula (7-2) does not guarantee that \( B(t+\Delta t, T) \) is less than 1 since the random error could be any positive number. In order to avoid the occurrence of a negative interest rate, the maximum price of any bond was set to 0.999999999 in our simulations.
7.4 The volatility function

In the HJM approach, the volatility function \( \sigma(t,T,B) \) completely determines the behaviour of bond prices in a risk-neutral environment. In our study, volatilities are calculated depending only on the duration of the bond and yield to maturity. Now, suppose the yield curve at time \( t \) is given. In order to obtain the volatility of the bond with the maturity \( T \) at time \( t \), we first calculate the yield to maturity by formula (4-1):

\[
Y(t,T) = -\log(B(t,T))/(T-t).
\]

Next, we obtain the yield volatility, denoted by \( \sigma_y \), assuming the yield volatility to be a function of the yield to maturity and duration \( T-t \) of the bond:

\[
\sigma_y = \sigma_y(Y(t,T),T-t).
\]

Next, convert the yield volatility to price volatility, denoted by \( \sigma_p \), by multiplying \( \sigma_y \) by the bond duration:

\[
\sigma_p = \sigma_p(t,T,B) = \sigma_y \cdot (T-t).
\]

Refer to Hull (1999, p536) for a description of how to convert from yield volatility to price volatility.

7.4.1 The yield volatility formula

Even under the simple assumption that the yield volatility depends only on its duration and yield, finding an appropriate formula is not an easy task. In our study, we use the following form of volatility function:

\[
\sigma_y(y,t) = \sigma_o \cdot (1-\exp(-\alpha y)) \cdot (t+0.01)^\beta \cdot (1-\exp(-\beta t))/(\beta t), \tag{7-6}
\]

where \( y \) is the yield to maturity,
\( t \) is the bond duration,
\( \sigma_o \) is the maximum volatility, i.e. \( \sigma_y(\infty, \infty) \) and
\( \alpha, \beta \) are constants.

The use of this formula is justified by the following considerations.

First, in the Vasicek model, as shown in formula (5-3), the variance of the spot rate per time unit for the period from time 0 to \( t \) is proportional to \( (1-e^{-\alpha t})/\alpha t \).

Second, the yield volatility decreases dramatically when the yield to maturity
becomes low. This feature is captured by the term $1 - \exp(-\alpha y)$ in formula (7.6). Third, the term $(t + 0.01)^\beta$ creates the well known “humped” shape in the volatility curve observed in the market. Hull (1999) states that the peak of the hump is at about the two or three year term. This term, in conjunction with the exponential function, produces a function shaped similarly to the probability density function of the gamma distribution.

We then choose the constants so that the volatility term structure fits the “base yield curve” using a numerical least square method. The “base yield curve” is a market yield curve, which is explained in Chapter 9.

7.5 Special treatment of very low interest rates

If we apply the established models discussed in Chapter 5 without any modification in a very low interest rate environment, we encounter the unrealistic situation where the entire term structure is unable to break away from close to zero. Such a term structure implies that the short rate will continue to be close to zero at least until the end of the latest maturity of bonds. Once the entire yield curve becomes flat around zero (we refer to this as a ‘zero flat curve’), the yield curve tends to keep this shape in the developments thereafter. Under a zero flat curve, all spot rates are almost zero. This implies that expected zero coupon bond prices are nearly par over all maturity spectrums. Since negative interest rates are not allowed, zero coupon bond prices cannot exceed par, i.e. the size of possible upside movements is very small. In order for the expectations to be around par, the size of possible downside movements should also be almost zero. As a result, bond prices move with little volatility, i.e. almost deterministically at par.

The zero flat curve would never be seen in markets unless all market participants are sure that current zero short rates will continue indefinitely. It is unrealistic to imagine that there is a situation where all market participants without exception are so surely pessimistic, believing that Japanese macroeconomic growth rates will remain at zero for decades. As explained in Chapter 6, yields on bonds with long duration should have lower bounds. In other words,
yield curves demonstrate a unique development pattern in a very low interest rate environment so that the zero flat curve cannot be realized. Thus the model should have the capability of producing the unusual development pattern of yield curves in a very low interest rate environment.

7.5.1 The basic idea
We modify the general HJM approach in order to make it work reasonably even in a very low interest rate environment as follows.
First, we define a “very low interest environment” mathematically. In order to determine whether a yield curve represents a very low interest environment or not, we introduce three threshold constants, $r_1, r_2$ and $r_3$. If all of the following three conditions are satisfied, the yield curve is defined to be in a very low rate interest rate environment.

1. A spot yield on any bond with any maturity is below $r_1$.
2. The difference between the spot yield of the bond with a two-year duration and the yield with a 10-year duration (we refer to this as 2-10 spread) is below $r_2$.
3. The 10-20 spread is below $r_3$, where the 10-20 spread is defined in the same way as the 2-10 spread. Figure 7-1 illustrates the 2-10 spread, 10-20 spread, $k$ and $T^*$, which are explained below.

Figure 7-1 - Illustration of $k$, $T^*$ and spreads
Second, we assume that yield curves move in parallel in a very low interest rate environment. In a very low interest rate environment, both long and short rates fluctuate only by a very small amount since they are already close to lower boundaries. We model this movement as a parallel fluctuation of the yield curve. The movement amount is determined as follows.

First, we calculate an expected yield curve at time \( t + \Delta t \) using bond prices expected at \( t \). Then we find the lowest spot yield on the yield curve, denoted by \( k \), and let \( k \) be the yield of the bond with maturity \( T^* \). The yield of the bond with maturity \( T^* \) during the time interval \( (t, t + \Delta t) \) becomes either of \( 2^{|n|} \cdot k \) or \( (1/2)^{|n|} \cdot k \). Probabilities are assigned so that the expected value follows formula \((7-3)\). Now we have obtained the upside movement amount and downside movement amount with some probabilities for the bond with maturity \( T^* \). For other bonds, in principle, their spot yields move by the same amounts with the same probabilities. Then we get a parallel shift of the yield curve. However, convexities (which were described in Section 7.3) are different between bonds with different maturities. Therefore if other bonds followed exactly the same movement as the bond with maturity \( T^* \), the expected values of these bonds would be slightly different from the correct expected values shown in formula \((7-3)\). Thus, we incorporate small adjustments to the amount of upside and downside movements for other bonds.

### 7.5.2 Model description

Suppose bond prices for all durations are given at time \( t \). Now we consider bonds with maturity dates \( T_i, \quad i = 0,1,2,\ldots,n \) where \( T_i = t + (i + 1)\Delta t \).

We know \( \tilde{Y}(t + \Delta t, T_i), \quad i = 0,1,2,\ldots,n \) defined in formula \((7-4)\), the expected price of the bond with maturity date \( T_i \) at \( t + \Delta t \) expressed in the form of a spot yield. Note that \( Y(t + \Delta t, T_i) \) and \( B(t + \Delta t, T_i), \quad i = 0,1,2,\ldots,n \) are still random variables at this stage even though \( \tilde{Y}(t + \Delta t, T_i), \quad i = 0,1,2,\ldots,n \) are known.

Thus, we define \( k \) as:
\[ k = \min[\tilde{Y}(t + \Delta t, T_i)], \quad i = 0, 1, 2, 3, 4, \ldots, n \]

or, equivalently, \( k = \tilde{Y}(t + \Delta t, T^*) \).

Now we obtain the 2-year, 10-year and 20-year spot yields:

- 2-year spot yield = \( \tilde{Y}(t + \Delta t, t + \Delta t + 2) \),
- 10-year spot yield = \( \tilde{Y}(t + \Delta t, t + \Delta t + 10) \) and
- 20-year spot yield = \( \tilde{Y}(t + \Delta t, t + \Delta t + 20) \).

Then, we can check the conditions required to be in a very low interest rate environment. In the case when the yield curve satisfies all the conditions, movement of the yield curve is restricted so that only a parallel-shift movement can occur. The minimum random spot yield, i.e. \( \tilde{Y}(t + \Delta t, T^*) \), is assumed to follow a discrete stochastic movement over the time period \( (t, t + \Delta t) \):

\[
\tilde{Y}(t + \Delta t, T^*) = \begin{cases} 
  k \cdot 0.5^\nu & \text{with probability } \frac{1}{2} \\
  k & \text{with probability } 0.5 \\
  k \cdot 2.0^\nu & \text{with probability } 0.5 - p 
\end{cases}
\]

where \( 0 < p < 0.5 \).

Note that \( \tilde{Y}(t + \Delta t, T^*) \) is still a random variable while \( \tilde{Y}(t + \Delta t, T^*) \) is given.

The probability \( p \) that the spot yield goes down (bond price goes up) can be calculated by equating expected bond prices:

\[
e^{-\theta} = p \cdot e^{-k \cdot 0.5^\nu \theta} + 0.5 \cdot e^{-k \theta} + (0.5 - p) \cdot e^{-k \cdot 2.0^\nu \theta}
\]

where \( \theta = T^* - (t + \Delta t) \).

Now define \( \Delta d = k \cdot 0.5^\nu - k \), and \( \Delta u = k \cdot 2.0^\nu - k \).

These values represent upside and downside movement amounts of the yield curve. For bonds with maturity \( T_i \) other than maturity \( T^* \), the stochastic movements of the spot yields are described as:

\[
\tilde{Y}(t + \Delta t, T_i) = \begin{cases} 
  \tilde{Y}(t + \Delta t, T_i) + \Delta d + \xi(t, T_i) & \text{with probability } \frac{1}{2} \\
  \tilde{Y}(t + \Delta t, T_i) & \text{with probability } 0.5 \\
  \tilde{Y}(t + \Delta t, T_i) + \Delta u + \xi(t, T_i) & \text{with probability } 0.5 - p 
\end{cases}
\]

where \( \xi(t, T_i) \) is an adjustment term which allows a correct expected bond price to be obtained. The adjustment term is also calculated by equating expected bond prices:
\[ e^{-\tilde{Y}_t} = p \cdot e^{-\left[\tilde{Y} + \Delta t + \tilde{\xi}(t, T_i)\right] / \theta_i} + 0.5 \cdot e^{-\tilde{Y}_t / \theta_i} + (0.5 - p) \cdot e^{-\left[\tilde{Y} + \Delta t + \tilde{\xi}(t, T_i)\right] / \theta_i} \]

where \( \tilde{Y}_t = \tilde{Y}(t + \Delta t, T_i), \quad \theta_i = T_i - (t + \Delta t). \)

Note the probability \( p \) is independent of \( i \).

\( \tilde{\xi}(t, T_i) \) is given by:

\[ 1 = p \cdot e^{-\tilde{\xi}(t, T_i) / \theta_i} + 0.5 + (0.5 - p) \cdot e^{-\Delta t / \theta_i} \]
\[ \Rightarrow 0.5 = e^{-\tilde{\xi}(t, T_i) / \theta_i} \left[ p \cdot e^{-\Delta t / \theta_i} + (0.5 - p) \cdot e^{-\Delta t / \theta_i} \right] \]
\[ \Rightarrow \tilde{\xi}(t, T_i) = -\log\left[ 0.5 / \left[ p \cdot e^{-\Delta t / \theta_i} + (0.5 - p) \cdot e^{-\Delta t / \theta_i} \right] \right] / \theta_i \]

However, in practice, this adjustment is fairly small since the time interval \( \Delta t \) is small. Once spot yields are given, bond prices can be calculated using formula (4.1).
8 Concepts for simulation

We are interested in how much loss a life insurer suffers from a policy portfolio when interest rates are random. First, we assume that the losses and profits occur only because yield curves behave differently from those initially expected. Developments of yield curves are simulated by the Monte Carlo simulation method. Several concepts needed for the Monte Carlo simulation, such as yield curves, rate of return, discount factors, surpluses and initial capital, are introduced here. In particular, the assumptions about expected and actual rate of return on assets are explained.

8.1 Yield curve trajectories

Suppose we start from time 0 with a given yield curve, calling it the initial yield curve. We define the time interval where each simulation step is taken, denoted by $\Delta t$ (years). Starting from the initial yield curve, the yield curve at time $\Delta t$ is obtained by the Monte Carlo simulation method using a random sample that follows the Normal distribution. Once the yield curve at time $\Delta t$ is obtained, we next simulate the yield curve at time $2\Delta t$ as an evolution from the yield curve at $\Delta t$. We repeat this step until the time $t_{\text{max}}$, where $t_{\text{max}}/\Delta t$ is an integer. When we have reached time $t_{\text{max}}$, we have a sample realization of the development of the initial yield curve as a set of yield curves. We call this set a scenario and the process to get a scenario a simulation. We repeat the simulation $N$ times to get $N$ scenarios.

Now, we define $M = t_{\text{max}}/\Delta t$. Consider bonds whose maturities, denoted by $T_i$, are $T_i = t_i, i = 0, 1, 2...M$, calling each bond $\text{BOND}_0, \text{BOND}_1, \ldots, \text{BOND}_M$. Let $B(i; t, T_j)$ denote the present value of bond $\text{BOND}_j$ at time $t$ in the $i$th simulation. We need to simulate $B(i; t, T_j)$ for $j = 0, 1, 2, \ldots, M$ at $t = t_0, t_1, t_2, \ldots, t_M$. These values are calculated as follows.

At $t=0(= t_0)$, we know all prices of $\text{BOND}_0, \text{BOND}_1, \ldots, \text{BOND}_M$ since the initial yield curve is given. Note $B(i; t_0, t_0) = 1$, and in general $B(i; T_p, T_j) = 1$ since $\text{BOND}_j$ is redeemed at 1 at $T_j$. 

50
At $t=t_1$, i.e. at the next simulation step, $B(i;t_1,T_j)=1$. We already know $B(i;t_0,T_j)$ at time $t_0$. Thus the behaviour of the present value of $BOND_j$ during the time from $t_0$ to $t_1$ is deterministic, not random. Therefore we can calculate the deterministic short spot rate at time $t_0$ up to $t_1$ in the $i$-th simulation, denoted by $r(i;t_0)$, as:

$$r(i;t_0) = - \frac{\log(B(i;t_0,T_j))}{\Delta t}.$$ 

Similarly, $r(i;t_j)$ denotes the short rate during $t_j$ to $t_{j+1}$ at time $t$.

The procedure to generate bond prices at $t_1$, i.e. $B(i;t_1,T_j)$ ($j=2,\ldots,M$), is as follows. Note that prices for all bonds at time $t_0$, i.e. $B(i; t_0,T_j)$, are given.

1. Calculate the expected value of bond prices:
   $$E[B(i; t_1,T_j)] = B(i;t_0,T_j) \cdot \exp[r(i;t_0) \cdot \Delta t].$$

2. For $BOND_j$ (except $BOND_0$, $BOND_j$), calculate the spot yield of the expected bond price (denoted by $\bar{Y}(i;t_1,T_j)$):
   $$\bar{Y}(i;t_1,T_j) = -\log(E[B(i; t_1,T_j)]) / (T_j - t_1).$$

3. Determine whether we are in a very low interest rate environment.

4. Generate a random number $U \sim U(0,1)$ for very low interest rate environment and $\Phi \sim N(0,1)$ for other, then $\Delta W = \sqrt{\Delta t} \cdot \Phi$.

5. (a). If we are in very low interest rate environment, calculate the spot yield of the bond as:
   $$Y(i;t_1,T_j) = \bar{Y}(i;t_1,T_j) + Err_j(t_0,T_j) \quad \text{for} \ j > 1$$

   $Err_j(t_0,T_j)$ is the upside or downside amount whose value and associated probabilities are defined by formula (7-7):

   If $U < p$ the spot yield goes down ($Err_j(t_0,T_j) = \Delta d + \xi(t_0,T_j)$), and if $U > \frac{1}{2} - p$ the spot yield goes up ($Err_j(t_0,T_j) = \Delta u + \xi(t_0,T_j)$).

   Otherwise: it remains at the expected value ($Err_j(t_0,T_j) = 0$).

(b). Otherwise, we calculate prices of bonds:

$$B(i;t_1,T_j) = E[B(i; t_1,T_j)] \cdot \epsilon(t_0,T_j)$$

where $\epsilon(t_0,T_j) = \exp\left\{-\frac{1}{2} \sigma_p(j)^2 \cdot \Delta t + \sigma_p(j) \Delta W\right\}$.

Here the price volatility $\sigma_p(j) = \sigma_p(t_0, T_j, B(i;t_0,T_j))$ is obtained.
according to the volatility function defined in Section 7.4.

The yield curve at \( t=t_1 \) in the \( i \)-th simulation has now been obtained. Moving on to time \( t_2 \), \text{BOND}_2\) is redeemed at \( t_2 \) this time. Then, we get the deterministic short spot rate at time \( t_1 \) up to \( t_2 \) as:

\[
r(i; t_1) = -\log(\frac{\text{B}(i; t_1, T)}{\text{B}(i; t_2, T)}) / \Delta t.
\]

Then all bond prices can be calculated following the same steps as described above (steps 1-5). By repeating the same procedure until \( t = t_{\text{max}} \) the evolution of the term-structure is obtained.

The following points are to be noticed. First, for fixed \( t_j \) and \( i \) \{\text{B}(i; t_j, T_k)\}_k(j=1,2,3,...,M)\) gives the yield curve at time \( t_j \) in the \( i \)-th scenario. The time to maturity of \text{BOND}_k\) is \((T_k - t_j)\). Second, for fixed \( t_j \) and \( T_k \) \{\text{B}(i; t_j, T_k)\}_i(i=1,2,3,...,N)\) gives \( N \) sample prices of \text{BOND}_k\) at \( t_j \). Third, for fixed \( i \) and \( T_k \) \{\text{B}(i; t_j, T_k)\} (j=0,1,2,3,...,M) shows how the price of \text{BOND}_k\) develops over time in the \( i \)-th scenario. Lastly, for fixed \( k \) and any \( i \) \{\text{B}(i; t_0, T_k)\} (i=1,2,3,...,N) are identical. This is the initial yield curve.

### 8.1.1 Accumulation

When \( j>k \), we let \( \text{B}(i; t_j, T_k) \) denote the accumulated value of the bond with maturity \( T_k \) at time \( t_j \) in the \( i \)-th scenario. \text{BOND}_k\) is redeemed at 1 at time \( T_k \). The redemption amount is assumed to be successively invested at short spot rates, which are produced with the evolution of the initial yield curve along the \( i \)-th scenario. For example, consider \text{BOND}_2\). At time 0, the price of \text{BOND}_2\) is given. At this stage, the price of \text{BOND}_2\) at time \( t_1 \) is unknown, i.e. random. After the first simulation step, we obtain the price of \text{BOND}_2\) at time \( t_1 \). The price of \text{BOND}_2\) at time \( t_2 \) is 1 by definition. Moving over to time \( t_2 \), we can calculate an accumulation factor as \( e^{r(i; t_2)\Delta t} \) using the short rate \( r(i; t_2) \), which is already known at time \( t_2 \). The redemption amount of \text{BOND}_2\) was assumed to grow according to this accumulation factor. We denote this accumulated value of \text{BOND}_2\) at \( t_2 \) as \( \text{B}(i; t_2, T_k) = e^{r(i; t_2)\Delta t} \cdot \text{B}(i; t_1, T_k) \). Repeating this operation, we get accumulated values of \text{BOND}_2\) at times \( t_4 \) and \( t_M \) as:
8.2 Rate of return and discount factors

8.2.1 Rate of return

In our study, it is assumed that all cash flows derived from our policy portfolio are invested in one fund jointly managed by the insurance company. This assumption also means that any cash flow has the same rate of return during the same calendar period regardless of its source. A fund in an insurance company would be composed of various classes of assets, so that forecasting the rate of return is not an easy task. However, it is observed in Figure 8-1 that the premium basis rate, which can be deemed as the average long-term rate of return expected on assets held by all Japanese insurance companies, has been staying along the market long-term rate (10-year swap rate). Then we assume that the expected rate of return up to time \( t \) is the sum of the then-prevailing spot yield for duration \( t \) and the constant \( \alpha \). We call \( \alpha \) the extra return.

\[
B(i; t_1, T_1) = B(i; t_1, T_2) \cdot e^{r(i; t_1) \Delta t} = e^{[r(i; t_2) + r(i; t_3)] \Delta t}
\]

\[
B(i; t_M, T_2) = e^{[r(i; t_M) + r(i; t_{M-1})] \Delta t}.
\]

\(^8\) The premium basis rate is determined by the FSA, according to Ministry Of Finance Decree 48.
In fact, the FSA adopted a new rule\(^9\) in April 1999 in which the premium basis rate is determined based on 10-year market government bond yields. Data in FSA (2000 b) show that government, municipal and corporate bonds and loans, account for 56% of total assets that are managed by all Japanese life insurance companies as of the end of March 2000. This fact also supports our assumption. Most of the remaining portion of the assets is invested in other financial assets, such as foreign securities (14.4%) and equities (14.1%). The expected rate of return on financial assets may well be expressed as the sum of risk-free rates and extra returns, which are also called risk premiums. Hence, as long as the composition of the assets is stable, our assumption on the expected rate of return on the policy portfolio seems reasonable. The then-prevailing interest rates largely determine the rate of return on the fund. Extra returns are different in each asset class. Therefore \(\alpha\) could be regarded as average risk.

\(^9\) Ministry Of Finance Decree 48, article 4
premiums weighted by the proportion of each asset class to the total assets.

The second assumption we make is that the rate of return for the immediate future is realized as expected. At time \( t \), we already know the short spot rate for the next immediate time interval of length \( \Delta t \). We assume that the actual rate of return on the fund during the time interval is the sum of the short spot rate and the constant \( \alpha \).

This assumption is similar to those made by Coppola et al. (2000) who assumed the rate of return is the sum of a fixed portion and a stochastic portion in their research. In our case, \( \alpha \) is fixed and interest rates are random.

In reality, in addition to the uncertainty for interest rates, each asset is exposed to uncertainties specific to the nature of the asset. However, in order to have our study focus on the financial effect caused only by the change of interest rates, we ignore uncertainties other than those of interest rates. This treatment implies that uncertainties observed in reality are bigger than the figures our study is showing, i.e. our study more or less underestimates real uncertainties.

### 8.2.2 Discount Factors

Given rates of return on the fund into which a cash flow is invested, we know the future value of the cash flow since the cash flow grows at the given rates of return. Equivalently, the present value can be obtained by discounting a future cash flow by the rates of return. Therefore the discount factors used for calculating the present value of non-random cash flows arising from a policy portfolio are obtained based on the expected rates of return on the fund managed by the life insurance company. Under the assumed extra return of \( \alpha \), the discount factor for \( BOND_k \) at time \( t_j \) in the \( i \)-th simulation, denoted by \( v(i; t_j, T_k) \), is obtained by adding the extra return on top on the zero coupon bond price on the base yield curve, i.e.:

\[
v(i; t_j, T_k) = e^{-\left(Y(i; t_j, T_k)+\alpha(T_k-t_j)\right)} = B(i; t_j, T_k) \cdot e^{-\alpha(T_k-t_j)} \quad (8-1)
\]

Note that this formula is valid even when \( k<j \) (i.e. when \( T_k-t_j<0 \)) by the second
assumption on the realized rate of return of assets.

8.3 Expected cash flow

Suppose we consider a policy portfolio composed of a large number of independent and identical policies. Then we can assume that the cash flows are going to be realized as initially expected since random fluctuation in mortality becomes immaterial. Given a mortality table, the expected cash flows of the policy portfolio can be deterministically obtained.

Now we consider a life insurance policy portfolio composed of large number of identical life insurance policies with \(T\)-year term assurance and a whole life assurance sold to males aged exactly \(x\) at \(t=0\). Benefits are paid at the end of the year of death, and premiums are paid annually as long as the term assurance is in force. Let \(ST\), \(SW\), \(I\), \(e\) and \(P\) denote the sum assured of the term assurance, the sum assured of the whole-life assurance, initial expense, annual expense and annual premium per policy. Now define \(P(t)\) and \(CF(t)\) to be the expected premium income and the expected cash flow at time \(t\) per policy. Note that we made the assumption that cash flows are realized as expected as time elapses. Then \(CF(t)\) also represents the expected cash flow at time \(t\) arising from the policy portfolio per policy that was originally in force. For example, suppose we have 10,000 policies. We get the expected cash flow at time \(t\) arising from the policy portfolio by multiplying \(CF(t)\) by 10,000 without calculating the number of policies remaining in force at time \(t\). Then, we call \(CF(t)\) the cash flow of the policy portfolio, ignoring the constant multiplier 10,000. For integer \(t \geq 0\),

\[
P(t) = \begin{cases} 
P \cdot l_{x+t}/l_x & \text{for } t < T \\
0 & \text{for } t \geq T, \text{ and} \\
P - I & \text{for } t = 0 \\
-(ST + SW) \cdot (l_{x+t-1} - l_{x+t})/l_x + P(t) - e \cdot l_{x+t}/l_x, & \text{for } t \leq T - 1 \\
-SW \cdot (l_{x+t-1} - l_{x+t})/l_x - e \cdot l_{x+t}/l_x, & \text{for } t = T \\
0 & \text{if } x + t > w, \\
\end{cases}
\]

where \(w\) is the maximum age on a mortality table.
8.4 Present value of the benefit at time $t$

Using the $\{CF(t)\}$ and $v(i; t, T)$, the present value of the future cash flows of the policy portfolio at time $t$ in the $i$th simulation (denoted by $PVB(i; t)$) can be calculated by the formula:

$$PVB(i; t) = \sum_{j=1}^{n-x} v(i; t, t+j) \cdot CF(t+j)$$

Note that here we exclude cash flows that occur exactly at time $t$.

8.5 Process of surpluses

8.5.1 Net policy value

We define the net policy value of our policy portfolio as the present value of net future benefits arising from the policies in force using the then prevailing yield curve. In the case of Japan, the conventional policy value is calculated based on the premium basis rate, which was used for premium calculation when the policy was issued. Therefore, net policy values are different from conventional policy values. The conventional policy values do not consider any changes in interest rates after the policies have been issued, while the net policy values always reflect the market conditions at the date of calculation. Unless the forward rates implied in the initial yield curve are immunized by hedging transactions at time 0, the net policy values represent more accurate economical values of net future benefits than conventional policy values since the yield curve prevailing then allows market expectation of future interest rates. In practice, in order to cover this defect of conventional policy values, reserves for interest rate risk are set aside separately. The reserves should be taken so that the sum of the conventional policy value and the reserve for the interest rate risk amount to the net policy value, which is calculated using the best estimates of the interest rates then. In other words, the net policy value is the conventional policy value net of reserves for interest rate risk. The reserve really needed for our unhedged policy portfolio against interest rate risks is the difference between the conventional policy value and the net policy value.
8.5.2 Surpluses

A most important figure in our study is the “surplus amount”. The surplus amount at time $t$ is defined as the accumulated realized cash amount arising from the policy portfolio less the total net policy value of the portfolio at time $t$. Assuming no dividend payout, this amount is the real economical net value of the portfolio. The surplus amount at time $t$ represents the profit (or loss) of the insurance company arising from this portfolio up to time $t$. If interest rates were realized as expected initially, the surplus at time $t$ would be the accumulated surplus at time 0 at the rates implied in the initial yield curve. This is because the premium and margins are calculated using the initial yield curve, and we assumed that the initial yield curve evolves as expected as time elapses.

However, the surplus amount does not behave as expected over the lifetime of the policy portfolio when yield curves are subject to random fluctuation. Let $\text{SUR}(i;t)$, $\text{AC}(i;t)$ and $V(i;t)$ denote the surplus, the accumulated cash and the net policy value of the portfolio at time $t$ in the $i$-th scenario. In compliance with actuarial convention, the premium income at time $t$ is included in $V(i;t)$, while the benefit payments are included in $\text{AC}(i;t)$. Then:

$$\text{SUR}(i;t) = \text{AC}(i;t) - V(i;t).$$

The accumulated cash at time $t$, $\text{AC}(i;t)$, is the sum of the accumulation at time $t-1$ at the rate $v(i;t,t-1)$, plus the new cash flow at time $t$. Then:

$$AC(i;t) = \{ AC(i;t-1) + P(t-1) \} \cdot v(i;t,t-1) + (CF(t-1) - P(t)),$$

Hence:

$$AC(i,t-1) = \{ AC(i,t-2) + P(t-2) \} \cdot v(i;t-1,t-2) + (CF(t-1) - P(t-1)).$$

Then:

$$AC(i;t) = \left[ \{ AC(i;t-2) + P(t-2) \} \cdot v(i;t-1,t-2) + CF(t-1) \right] \cdot v(i;t,t-1) + (CF(t) - P(t))$$

$$= \{ AC(i;t-2) + P(t-2) \} \cdot v(i;t-1,t-2) \cdot v(i;t,t-1) + CF(t-1) \cdot v(i;t,t-1) + CF(t) - P(t)$$

$$= \sum_{j=0}^{t} CF(t-j) \cdot v(i;t,j) - P(t) \quad \text{(using } P(0) = CF(0), \ AC(0) = 0)$$

$$= \sum_{j=0}^{t} CF(j) \cdot v(i;t,j) - P(t)$$
\begin{align*}
V(i;t) &= - \left[ \sum_{j=1}^{w-x} CF(t + j) \cdot v(i;t,t + j) + P(t) \right] \\
&= - \sum_{j=t+1}^{w-x} CF(j) \cdot v(i;t,j) - P(t)
\end{align*}

Then:

\[ SUR(i;t) = \sum_{j=0}^{w-x} CF(j) \cdot v(i;t,j) \]

For each scenario \( i \), \( SUR(i;t) \) \((t=0,1, \ldots, w-x)\) shows a stochastic development of the surplus. When \( N \) simulations are completed, \( N \) sample paths of the surplus are obtained.

### 8.5.3 Discounted surpluses

The discounted surplus at time \( t \) in the \( r \)th scenario (denoted by \( DSUR(i;t) \)) is defined as:

\[ DSUR(i;t) = SUR(i,t)/v(i;t,0) \]

The quantity \( v(i;t,0) \) \((t>0)\) is the future value of a cash flow of 1 at time 0 accumulated up to time \( t \). Thus its inverse, \( 1/v(i;t,0) \), is the discount factor at time 0 for the duration \( t \) along the specific \( r \)th scenario. Note that \( 1/v(i;t,0) \) is different from \( v(i;0,t) \), which is the discount factor for the duration \( t \) at time 0.

The value of \( v(i;0,t) \) is given in the initial yield curve, while \( v(i;t,0) \) is a random variable depending on the development of a yield curve. They are the same only when the realization of short spot rates happens to be the same as the forward rates implied in the initial yield curve.

It can be shown that \( \{DSUR(i;t)\} \) is a martingale. Therefore the expected value of the discounted surplus is always the initial surplus amount, i.e.:

\[ E[DSUR(i;t)] = DSUR(i;0) = SUR(i;0) \]

In particular, when the premium is set so that the initial surplus is zero, the expected value of the discounted surplus is zero.

The discounted surplus process is a more suitable measure for us to keep track of the development of the value of the policy portfolio than the surplus process.
without discounting. In general, when we compare two values at different times, we need to compare present values of these. One dollar on hand now is worth more than one dollar that is settled in one year. If cash is settled in the future, compensation for renouncing immediate consumption should be paid. In other words, the money amount at later times includes some time value. This time value is quoted as an interest on the cash to be settled at the predetermined future time. Therefore, the present value of a cash flow is economically equivalent to the cash on hand whose amount is the same as the present value. For example, Yen 1 at time 0 is equivalent to Yen 1.04 at time 1 (year) when interest rates are 4%. These arguments tell us that present values should be used when we compare cash flows that occur at different times on the same economical basis. The discounted surplus is always expressed as its value at time 0. This standardization enables us to track developments of surplus consistently.

8.6 Capital requirement

8.6.1 Definition of initial capital
Here, let initial capital be defined as the cash injected at time 0 to prevent the surplus of the policy portfolio from becoming negative over the lifetime of the portfolio with some given probability. In other words, initial capital provides a cushion by which the surplus stays positive with some probability. Initial capital depends on the given probability, i.e. on the extent to which an insurer wants to prevent the surplus of the policy portfolio from becoming negative. The event where the surplus becomes negative is referred to as ‘portfolio ruin’ hereafter.

8.6.2 Assumptions
We make several assumptions on initial capital. First, initial capital is injected in the form of cash when the policy portfolio is set up at time 0. Second, the injected capital is jointly managed in a fund with the premium income arising from the portfolio. Thus the return on the injected cash is the same as what is assumed on the cash flows of the portfolio. Third, we can retain the interest on
the capital in the fund until all the policies in our portfolio expire.

We do not consider costs of capital. Practically, the actual cost of capital may be higher than the rates of return, i.e. there may be a difference between the actual cost of capital and the expected rates of return on the fund. We assume that the difference is levied to customers as a profit margin, and immediately paid to the capital provider. Then we can ignore costs of capital since the net cash flow arising from the difference is zero, i.e. the difference does not affect any cash flows under this assumption.

8.6.3 Initial capital requirement

Let $MLD(i)$ be defined as the minimum value of $DSUR(i;t)$ in the $i$-th scenario, so that $|MLD(i)|$ is the maximum discounted loss. Simulated values of $MLD(i)$ ($i=1,2,3,...N$) give an estimate of the distribution of the minimum value of the discounted surplus. Let $MLD$ denote the random variable with this distribution. Suppose $Pr[MLD>-C] = 1-\varepsilon$, where $0<\varepsilon<1$. Then the initial capital requirement under the given probability is $C$. The amount of $C$ injected at time 0 is accumulated to $C \cdot v(i;t,0)$ at time $t$. Discounting this amount to time 0, we get $C$. Thus, the cash injected at time 0 shifts up $DSUR(i;t)$ for all $t$ by exactly the same amount. Therefore the probability that the minimum value of the discounted surplus with the initial capital $C$ stays positive over time is:

$$Pr[MLD+C>0] = Pr[MLD>-C] = 1-\varepsilon$$

The term $MLD+C$ is the minimum value of the discounted surplus including initial capital $C$ injected at time 0. The distribution of $MLD$ determines the initial capital requirements. If the initial capital that makes the portfolio ruin probability 5% is required, the point of $MLD$ beyond which the minimum value of the discounted surplus falls with 5% probability shows the required initial capital.

8.6.4 Margins

Now suppose that initial capital of $C$ is required. Instead of injecting initial capital at time 0, we may obtain similar effects by levying profit margins on top
of office premiums whose present value at time 0 is $C$. In other words, we obtain
the same present value in the form of an annuity stream. However, in this case
the shape of cash flows changes, i.e. the minimum value of the discounted
surplus with the profit margins cannot be described by $MLD+C$ any more. Then
it is expected that the distribution of the minimum value of the discounted
surplus with the profit margins has a (slightly) different shape to the
distribution of $MLD+C$, which assumes no profit margins but initial capital
injection.
9 Initial settings for the simulation

9.1 Base yield curve.

In order to calculate present values of the expected cash flows arising from the policy portfolio under current assumptions, we should choose a representative yield curve from various yield curves in Japanese financial markets. This curve is used to obtain discount factors and rates of return in our simulations. We call this yield curve the “base yield curve”. We have chosen the curve for Yen swap rates as a base yield curve for the following reasons.

First, the Yen swap market is quite liquid with an enormous trading volume. The outstanding notional amount of interest rate swaps held by Japanese primary dealers at the end of December 2001 was USD 9,043,659 million (BOJ (2001 b)).

Second, the curve is smooth, even compared to the yield curve for government securities. Swaps are off-balance transactions. There are no accounting, tax or issue-specific supply and demand issues, which sometimes distort yield curves for bonds including Government securities. For example, high coupon bonds tend to have lower yields due to the high demand for the bonds by long-term investors in Japan.

Third, the swap curve is already being used as a standard for other fixed-income instruments. Fixed-income instruments, including Government securities, are sometimes quoted at a spread over the swap rates for determining prices. Spreads over swap rates regularly appear in market reports published by large financial institutions. For example, a bond on which the yield is 0.8% is quoted as “swap rate – 0.2%” when the swap rate with the same maturity is 1.0%.

An interest rate swap contract is a bilateral agreement between two parties where one party pays interest of a predetermined notional amount while the other party pays a different kind of interest. In standard Yen swap contracts namely “variable vs. fixed”, the Yen London Inter-Bank Offer Rates (“Yen Libor”),
which reflect the average credit standing of quoting banks, are paid against fixed interest rates. One party pays the 6-month Yen Libor rate reset semi-annually, while receiving a semi-annual fixed rate on the same notional amount. Each payment stream, payment of Libor rates or receipt of fixed rates, is called a “leg”. The fixed rates are called “swap rates” and are quoted in the market.

Each leg contains only a stream of interest rate payments. There are no principal payments in each leg. However, we could think of a hypothetical initial and final exchange of principal on each side, i.e. treat the contract as if each party receives initial principal at the start and pays final principal on redemption. This is acceptable since the hypothetical exchange of principal creates no net cash flow. Each leg can then be regarded as a synthetic bond. We regard one leg as a Libor floating rate bond, and the other leg as a fixed-coupon bond. Since the swap rates refer to fixed interest rates and par values are received at the start and paid on redemption, we can regard the swap rate as a par rate where the fixed rate is paid semi-annually. This approximation is reasonable even though the actual payment of the fixed leg is not exactly half of the fixed rate. The actual payment amount is calculated by multiplying the coupon rate, notional amount and number of days from the previous payment date, and dividing it by 365. This amount is quite close to the half of the coupon rate multiplied by notional amount, except for fractional amounts paid in advance or in arrears.

9.2 Obtaining the graduated yield curve

Market quotations of swap rates are readily available to the public. However, we can observe only a limited number of rates for several maturities. A graduation is needed to obtain a complete yield curve from which we can calculate spot yields with any duration. We use Cairns’ model for this purpose (see Section 5.5). In Cairns’ model, we know the parameters \( c = (0.2, 0.4, 0.8, 1.6) \) while there exist unknown parameters \( b = (b_0, b_1, b_2, b_3, b_4) \).

Once a specific \( b \) is fixed, the discount factors for any maturity date can be obtained using the model formula. Then, if \( b \) is given at time 0, the semi-annual
par rates can be obtained by solving the simple equation:

\[ 1 = \sum_{i=1}^{2T} B(0, \frac{i}{2}) \cdot \frac{pr_T}{2} + B(0, T) \]

where \( B(0, t) \) is the obtained zero coupon bond price with maturity date \( t \) at time 0, and \( pr_T \) is the par rate with maturity date \( T \) at 0.

Then, by choosing some \( b \) we know par rates with any maturities. In other words, the mapping \( b \in R^5 \xrightarrow{f} \{ pr_T(b) \} (T = 0.5, 1, 1.5, ..., M/2) \) is defined.

Then, when we have \( n > 5 \) observations of market par rates, i.e. \( \{ pr_T \} \in R^n \) \( n > 5 \), \( i = 1, 2, ..., n \) is given, \( b \) can be obtained by minimizing:

\[ Q(b) = \sum_{i=1}^{n} (pr_T(b) - \bar{pr}_T)^2 \]

9.2.1 Observed market rates

Table 9-1 shows market interest rates observed on 1 May 2001, omitting unnecessary points:

<table>
<thead>
<tr>
<th>Term</th>
<th>Deposit Rates</th>
<th>Swap Rates</th>
<th>Government Bond Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deposit Rates</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bid</td>
<td>Offer</td>
<td>Bid</td>
<td>Offer</td>
</tr>
<tr>
<td>3 (Months)</td>
<td>0.035%</td>
<td>0.05%</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>0.14%</td>
<td>0.18%</td>
</tr>
<tr>
<td>2 (Years)</td>
<td></td>
<td>0.24%</td>
<td>0.28%</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>0.39%</td>
<td>0.43%</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>0.57%</td>
<td>0.61%</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>0.95%</td>
<td>0.99%</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>1.40%</td>
<td>1.44%</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td>1.361%</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
<td>1.970%</td>
</tr>
</tbody>
</table>

Source: Reuter, Tokyo Market Summary dated 1 May 2001, and Bloomberg.co.jp
We need a graduated yield curve, which is consistent with the above market observations, in other words we estimate swap rates for some maturities for which swap rates are not quoted from the above crude data. We use mid-market values (the average of bid and offer) for swap rates in order to assign one rate to each maturity. For three-month maturity, we estimate the rate as 0.04% using the mid-point of the deposit rates. We see almost the same rates between the three-month Government bond yield and the six-month yield, so we estimate the six-month rate to be the same as the three-month rate, i.e. 0.04%. Next, we look at 10-year and 20-year Government bond yields. We find the spread between the two rates is 0.61%. Assuming the spread between the 10-year swap rate and 20-year swap rate is the same as this “Government spread”, we estimate the 20-year swap rate to be 2.03% (=1.42%+0.61%). Table 9.2 shows the base yields obtained by the above calculations:

Table 9.2  - Obtained base yields

<table>
<thead>
<tr>
<th>Term</th>
<th>Base Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 (Months)</td>
<td>0.04%</td>
</tr>
<tr>
<td>2 (Years)</td>
<td>0.16%</td>
</tr>
<tr>
<td>3</td>
<td>0.26%</td>
</tr>
<tr>
<td>4</td>
<td>0.41%</td>
</tr>
<tr>
<td>5</td>
<td>0.59%</td>
</tr>
<tr>
<td>7</td>
<td>0.97%</td>
</tr>
<tr>
<td>10</td>
<td>1.42%</td>
</tr>
<tr>
<td>20</td>
<td>2.03%</td>
</tr>
</tbody>
</table>

9.2.2 Graduation of the base yield curve

The parameter set: \( b^* = (0.0283, 0.0044, -0.1216, 0.1386, -0.0525) \) was found to minimize the sum of squares:
\[ Q(b) = \sum_{i=1}^{8} \left( pr_{T_i}(b) - pr_{T_i} \right)^2, \]

where \( T_1 = 0.5, T_2 = 2, T_3 = 3, \ldots, T_8 = 20. \)

Using this parameter set, zero coupon bond prices over the whole maturity spectrum can now be generated. The discount factors, par rates and spot rates calculated over the maturity spectrum at half-yearly intervals up to 80 years are shown in Appendix 1. These are plotted below in Figure 9-1. By inspection, we see a quite steep curve except for the one-year to three-year sector. The hollow of forward rates observed around the 1.5-year maturity creates a flat spot curve for the 1 to 3-year maturity band. Also we see that rates converge to 2.83% as maturity increases, which is the value of \( b_0 \) in Cairns’ model.

Figure 9-1 - Yen fitted yield curve

9.3 Finding parameter for very low interest rates

The next step is to fix the parameters \( r1, r2 \) and \( r3 \) introduced in Section 7.5, which are used to determine whether we are in a very low interest rate environment or not. From 12 February 1999, the day on which the zero interest
rate policy was introduced, to 11 September 2001, the 2Year-10Year spread of
swap rates hit the lowest points on 22 March 2001 and 27 June 2001. Rates are
given in Table 9-3 below.

<table>
<thead>
<tr>
<th></th>
<th>2-Year rate</th>
<th>10-year rate</th>
<th>20-year rate</th>
<th>2-10 spread</th>
<th>10-20 spread</th>
</tr>
</thead>
<tbody>
<tr>
<td>22 March 2001</td>
<td>0.17%</td>
<td>1.29%</td>
<td>2.16%</td>
<td>1.11%</td>
<td>0.87%</td>
</tr>
<tr>
<td>27 June 2001</td>
<td>0.13%</td>
<td>1.19%</td>
<td>2.44%</td>
<td>1.06%</td>
<td>1.24%</td>
</tr>
</tbody>
</table>

Note that the 20-Year rate was estimated by Cairns’ model. Based on the above
observation, we set\(^{10}\)\( r_1 \), \( r_2 \) and \( r_3 \) to 0.25%, 0.95% and 0.80% respectively.

9.4 Fitting the volatility curve

In the option-on-swap (swaption) market, quotations are made on yield volatilities. On 19 April 2001, the volatilities in Table 9-4 were quoted in the
swaption market. For example, the volatility of a 3-month–into–2-year
swaption, whose strike rate is 0.22%, is 0.16%. A strike rate is the
predetermined rate that would become a fixed rate on a swap contract when the
swaption is exercised. The “Model Volatility” are volatilities using the formula
declared in Chapter 7 using parameters obtained below.

<table>
<thead>
<tr>
<th></th>
<th>2 Year</th>
<th>5 Year</th>
<th>10 Year</th>
<th>20 Year</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strike Rate</td>
<td>0.22%</td>
<td>0.75%</td>
<td>1.60%</td>
<td>2.35%</td>
</tr>
<tr>
<td>Market Quotation(^{11})</td>
<td>0.16%</td>
<td>0.45%</td>
<td>0.65%</td>
<td>0.71%</td>
</tr>
<tr>
<td>Model Volatility</td>
<td>0.14%</td>
<td>0.43%</td>
<td>0.68%</td>
<td>0.69%</td>
</tr>
</tbody>
</table>

In the formula (7-6):

\(^{10}\) We ignore the difference between par rates and zero rates.
\(^{11}\) At-the-money volatility for options whose maturities are 3 months.
\[ \sigma(y, t) = \sigma_0 \cdot (1 - \exp(-\alpha y)) \cdot (t + 0.01)^\beta (1 - \exp(-\beta t))/\beta t \]

We set \( \alpha \) to 45.0 as an appropriate value representing the decay in proportion to interest rates. The extent to which volatilities decrease, which is determined by \( \alpha \), seems consistent with market observations. Then we obtained \( \sigma_0 = 1.53\% \) and \( \beta = 0.061 \) by minimizing the square error sum between the market volatility and model volatility.

We know the following market quotations, denoted \( \sigma \):

\[
\begin{align*}
\sigma(0.22\%, 2) &= 0.16\%, & \sigma(0.75\%, 5) &= 0.45\%, \\
\sigma(1.6\%, 10) &= 0.65\%, & \sigma(2.35\%, 20) &= 0.71\%.
\end{align*}
\]

Then, we minimized:

\[
\begin{align*}
&\left[ \sigma(0.22\%, 2) - 0.16\% \right]^2 + \left[ \sigma(0.75\%, 2) - 0.45\% \right]^2 \\
&+ \left[ \sigma(1.6\%, 10) - 0.65\% \right]^2 + \left[ \sigma(2.35\%, 20) - 0.71\% \right]^2
\end{align*}
\]

by regarding \( \sigma_0 \) and \( \beta \) as variables.

The model volatilities calculated under these parameters are shown in Appendix 2. The model volatilities are reasonably consistent with the market quotes. Note that swap rates can be deemed as par rates, not zero rates. Thus, strictly speaking, these volatilities are for par rates not for zero rates. However, we assume that these volatilities are for zero rates in order to avoid extreme complications.

Figure 9-2 shows volatilities for a fixed interest rate level, for example, when the interest rate is 3\%. We clearly see that the volatilities peak at around the two-year bond maturity and decrease gradually afterwards. The volatilities are higher for higher interest rates.
Figure 9-2 Volatility curves with maturity years

Figure 9-3 shows how volatility for a 10-year bond maturity changes according to interest rate levels. We observe that the volatility rapidly decreases when the interest rate falls below 2%, while the volatility remains almost constant for interest rates beyond 8%.

Figure 9-3 - Volatility curve with interest rate level
9.5 The mortality basis

The mortality table published by the Ministry of Health, Labor and Welfare in Japan was used in our study. Mortality after age 100 was obtained by extrapolation assuming Gompertz’s Law, and \( l_{105} = 0 \). The mortality table obtained is shown in Appendix 3. Note that this mortality basis gives a margin to the insurer assuming insured lives have lower mortality rates than the population at large.

9.6 Life insurance contracts

According to statistics of the Life Insurance Association of Japan (2000), the amount of in force whole-life with term assurance products accounts for 63% of the total amount of in force life products for individuals. Referring to product illustrations set out in that publication, we investigate a whole-life with term assurance policy with the following terms.

The policy is issued to a male aged exactly 27 with a 30-year term assurance with sum assured Yen 35,000,000, and a whole-life assurance with sum assured of Yen 5,000,000. The benefits are paid at the end of the year of death and premiums are paid in advance for 30 years. We assume our life insurance policy portfolio consists of 100,000 independent and identical policies. For convenience, we ignore the constant multiplier 100,000 when the total value of the portfolio is considered, i.e. we deal with cash flows per policy in force at the outset of the policy. For each policy, we assume the following expenses and margins. Initial expenses are 1\% of the sum assured for term assurance and 3\% of the sum assured for the whole-life assurance. Renewal expenses are 0.01\% of the sum assured for the term assurance, 0.01\% of the sum assured for the whole-life assurance and 3\% of annual net premium. These renewal expenses are paid every time premiums are paid.

The expenses are covered by levying margins on top of net premiums. Further margins will be levied to earn profits. We define the margin ratio \( m(u) \) such that we earn a profit whose present value is \( u \) at time 0 by levying the margin, whose amount is \( m(u) \) of net annual premium, on top of annual net premiums.
9.7 Obtaining discount factors

Recall formula (8-1):

\[ v(i; t_j, T_k) = B(i; t_j, T_k) \cdot e^{-\alpha(T_k - t_j)}. \]

We choose \( \alpha \) as 1.5\%, i.e. we assume that there is always an extra return of 1.5\% above the swap rate on our fund. By applying this value of \( \alpha \) to the above table, we now know all discount factors at time 0.

9.8 Determining the net level premium

The expected present value of a unit temporary annuity due for \( T_{T,A} \) years at time 0 is:

\[ A(T_{T,A}) = \sum_{t=0}^{T_{T,A}-1} l_{x+t} / l_x \cdot v(0; 0, t) \]

The net premium \( \Pi \) is calculated by the principle of equivalence. As shown in Sections 8.3 and 8.4, we can calculate cash flows and expected present values of benefits. We calculate the expected present values by discounting each cash flow with the above discount factors. Dividing the expected present value of benefits by \( A(T_{T,A}) \), we get the net premium \( \Pi \). Applying figures, we obtain the annual net premium of Yen 103,963.

9.9 Expected cash flow

The expected present value of expenses can also be calculated. Table 9-5 shows those expected values.

<table>
<thead>
<tr>
<th></th>
<th>Initial Term Assurance</th>
<th>350</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Term Assurance</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Whole-Life Assurance</td>
<td>-150</td>
</tr>
<tr>
<td></td>
<td>Term Assurance</td>
<td>-66</td>
</tr>
<tr>
<td></td>
<td>Whole-Life Assurance</td>
<td>-12</td>
</tr>
<tr>
<td></td>
<td>Premium Income</td>
<td>-59</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>-637</td>
</tr>
</tbody>
</table>

Table 9-5 - Expected values
The expected present value of margin income is calculated by:

$$\Pi \cdot m(u) \cdot A(T_{T,d})$$.

By equating this value to the expected present value of expenses with $u=0$, we can obtain the margin ratio, which covers the expected expenses. Applying figures, we obtain the margin ratio of 32%. Thus the gross premium is $1.32 \Pi$, which is Yen 137,596.

Appendix 3 shows cash flows per policy expected at the outset of the policy. These cash flows include expected benefits, expenses and gross premiums, plotted in Figure 9-4. For the first 20 years, cash flows are positive due to lower mortality rates except the first cash flow, which includes initial expenses. The large negative cash flow at 30 years is due to the last expected benefit payment for the term assurance without premium income.
10 Simulation results

We obtained 1,000 sample scenarios, i.e. $N=1000$, by the Monte Carlo simulation method under the setting described in the previous chapter with $\Delta t = 1/8$ year. The number of scenarios was chosen as a compromise between computer run time on a personal computer and a large enough number to achieve meaningful results. We note that Wilkie (1995) also created 1,000 samples in his Monte Carlo simulation.

10.1 Sample scenarios

Choosing from the 1,000 sample scenarios, we show some illustrative examples below. Overall statistics are given in following sections.
Scenario number 534:

Figure 10-1

Figure A - Half-year forward rates

Figure B - Realized Short Rates

Figure C - Accumulated Surplus (in Yen 1000)

Figure D - Discounted Accumulated Surplus (in Yen 1000)
Scenario number 886:

Figure 10-2

Figure A - Half-year forward rates

Figure B - Realized Short Rates

Figure C - Accumulated Surplus (in Yen 1000)

Figure D - Discounted Accumulated Surplus (in Yen 1000)
Scenario number 905:

Figure 10-3

Figure A - Half-year forward rates

Figure B - Realized Short Rates

Figure C - Accumulated Surplus (in Yen 1000)

Figure D - Discounted Accumulated Surplus (in Yen 1000)
Scenario number 514:

Figure 10-4

Figure A - Half-year forward rates

Figure B - Realized Short Rates

Figure C - Accumulated Surplus (in Yen 1000)

Figure D - Discounted Accumulated Surplus (in Yen 1000)
Scenario number 998:

Figure 10-5

Figure A - Half-year forward rates

Figure B - Realized Short Rates

Figure C - Accumulated Surplus (in Yen 1000)

Figure D - Discounted Accumulated Surplus (in Yen 1000)
In Figure 10-1 A, simulated yield curves at times \( t=0,20,40,60 \) (years) are plotted. The yield curves stayed at low levels for the whole 78-year lifetime of the policy. As a result, realized short rates were low (less than 0.80%) for 78 years. These short rates are plotted in Figure B. These low interest rates brought persistently negative surpluses over the lifetime of the policy. In Figure C, the development of the surplus is shown in units of 1000 (Yen). The negative surplus (i.e. loss) continued to increase to more than Yen –5,000,000 at time 78. However, these values are expressed in future currency value. The development of surplus converted to present value at time 0, i.e. the discounted surplus, is shown in Figure D. Even though negative absolute figures are smaller than the future values that appears in Figure C, we see that the insurer continued to incur losses.

In the case of scenario 886 shown in Figure 10-2, the interest rates started to rise after a 45-year low interest period due to random fluctuations even though the yield curve at \( t=40 \) did not suggest any future increase of interest rates. The levels of the discounted surplus process therefore started to increase after 45 years. However, the surplus did not rise above zero to a level of profit. Note that the high interest rates at later stages compounded the loss. The final loss amount without discounting was considerable.

Scenario 905 is similar to scenario 886. However, the low interest rate period had ended before 40 years in this case. The discounted surplus recovered around time 35 years. Due to this earlier ending of the low interest rate period, the final discounted surplus (loss) turned out to be half of the loss in scenario 886.

Interest rates fluctuated considerably in scenario 514. It can be seen that the surplus evolved largely in line with the development of short rates in the early stages of life of the policy. The low interest rates that occurred from 30 to 40 years caused the final surplus to be negative.
Interest rates were more than 4% during most of the lifetime of the portfolio in scenario 998. The yield curve at 20 years suggested such future development. These favorable interest rates resulted in a constantly positive surplus. Note that the high interest rates after 65 years compounded the profit producing an effect opposite to that of scenario 886. The final profit measure without discounting was considerable.
10.2 Distributions

10.2.1 Distribution of the discounted surplus at the end of the policy term

Figure 10-6 shows the empirical distribution of the discounted surplus at the end of the policy term, i.e. 78 years after the issue the policies. Recall that these values were obtained by the Monte Carlo simulation. Table 10-1 shows summary statistics. Units are Yen 1000s.

![Empirical distribution of the discounted surplus](image)

Table 10-1 - Summary statistics of the outcomes of DSUR

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-22.4</td>
<td>177.4</td>
<td>-1371.0</td>
<td>790.5</td>
<td>565.7</td>
</tr>
</tbody>
</table>

We started from an initial surplus of zero. Hence the expected value of the discounted surplus is also zero (see Section 8.5.3). In fact, the sample mean is almost zero. The distribution is positively skewed with a median of 117.4. This means that we are more likely to obtain positive surplus than negative surplus. However, the absolute amounts of negative discounted surpluses are larger than
positive discounted surpluses once they occur. This can be inferred by the minimum discounted surplus of $-1371.0$, whose absolute amount is larger than the maximum of $790.5$.

### 10.2.2 Distribution of minimum discounted surplus

Figure 10-7 shows the empirical distribution of minimum discounted surplus (MLD), which is the minimum value of the discounted surplus, with summary statistics in Table 10-2. Units are again Yen 1000s.

![The empirical distribution of MLD](image)

**Figure 10-7 • The empirical distribution of MLD**

<table>
<thead>
<tr>
<th>MLD</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>-0.001</td>
<td></td>
</tr>
<tr>
<td>-0.002</td>
<td></td>
</tr>
<tr>
<td>-0.003</td>
<td></td>
</tr>
<tr>
<td>-0.004</td>
<td></td>
</tr>
<tr>
<td>-0.005</td>
<td></td>
</tr>
<tr>
<td>-1371.036</td>
<td></td>
</tr>
<tr>
<td>-1096.829</td>
<td></td>
</tr>
<tr>
<td>-822.621</td>
<td></td>
</tr>
<tr>
<td>-548.414</td>
<td></td>
</tr>
<tr>
<td>-274.207</td>
<td></td>
</tr>
</tbody>
</table>

**Table 10-2 • Summary statistics of the outcomes of MLD**

<table>
<thead>
<tr>
<th>Mean</th>
<th>Median</th>
<th>Minimum</th>
<th>Maximum</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>-454.7</td>
<td>-340.8</td>
<td>-1371.0</td>
<td>0</td>
<td>407.7</td>
</tr>
</tbody>
</table>

The empirical distribution has a probability mass at 0, which is 14.7%. Since we started from a surplus of zero, the MLD is zero when subsequent surpluses are positive over the whole lifetime of the policy portfolio. In other words, with 85.3% probability, the surplus falls below zero at some time during the lifetime.
of the policy portfolio. Mainly due to the probability mass at 0, the probability is skewed to right side with the median of –340.8, which is larger than the mean of –454.7. The mean –454.7 indicates that the portfolio on average suffers a loss of 454.7, which is 3.3 times the annual gross premiums, at some time during the life of the policy portfolio. The maximum amount of 1371 in our 1,000 simulations is 10 times the annual gross premiums.

10.2.3 Distribution of the time when the minimum surplus occurs

The $MLD$ takes its minimum at some time during the lifetime of the policy. Figure 10·8 shows the sample distribution of the time when the $MLD$ takes the minimum.

As explained, 14.7% of samples of the minimum of the discounted surplus process were zero, i.e. the surplus of zero occurred at $t=0$. Hence, there is a probability mass, which is 14.7%, at time zero. Including this probability mass, the discounted surplus is likely to reach its minimum in the early stages of the lifetime of the policy portfolio. However, the minimum discounted surplus could
occur any time during the lifetime, even though the probabilities that it reaches its minimum at that time are decreasing with elapsed time. In Figure 10-9, the relationship between the MLD and the time when the discounted surplus reached its minimum is shown.

Figure 10-9  · MLD and the time when DSUR reached the minimum

Generally, the later the discounted surplus reached its minimum, the larger the present value of the loss. This result is reasonable because, as time elapses, interest rates on average tend to move further away from what were expected at time 0.

10.3 Initial capital and margin

10.3.1 Initial capital requirement

In Section 8.6, we defined initial capital as the cash injected at time 0 to prevent the surplus of the policy portfolio from becoming negative over the lifetime of the
policy portfolio with some given probability. In the same section, we showed that
the initial capital shifts the distribution of the discounted surplus by the amount
of initial capital amount. Now let us choose the probability of 1%, i.e. we want to
contain ruin within 1% probability. Since 990 samples of the discounted surplus
process are more than -1351, an initial capital injection of 1351 shifts the
distribution to the right so that 1% of samples fall below zero. In other words, an
initial capital requirement to achieve a 1% ruin probability is estimated to be
1351 according to the empirical distribution. Now, our simulation result tells us
that the $MLD$ falls below $-1246$ with 5% probability, namely in 50 out of 1000
scenarios $MLDs$ were below $-1246$. Hence, the initial capital injection of 1246 is
estimated to make the portfolio ruin probability 5%, i.e. the initial capital
requirement to achieve a 5% ruin probability is estimated to be 1246. Table 10-3
summarizes these estimated initial capital requirements.

<table>
<thead>
<tr>
<th>Empirical ruin probability</th>
<th>Estimated required capital</th>
<th>Required Capital / Annual Gross Premium</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>1351</td>
<td>9.8</td>
</tr>
<tr>
<td>5%</td>
<td>1246</td>
<td>9.1</td>
</tr>
<tr>
<td>10%</td>
<td>1105</td>
<td>8.0</td>
</tr>
</tbody>
</table>

10.3.2 Profit margin

Consider an initial capital injection of 1105. A similar effect to an initial capital
injection may be obtained by imposing a profit margin whose present value is
1105 at time 0. In Section 9.5 we defined a margin ratio $m(u)$ as the margin that
is paid on top of the net premium which gives the present value $u$ at time 0.
Given $u$, $m(u)$ is calculated by the formula:

$$m(u) = (u + E)/(\Pi \cdot A(T_{r,k}))$$

where $E$ is the present value of expenses at time 0.

Now we want to obtain the margin ratio that gives a present value of 1105 at
time 0. Setting \( u \) to 1105 in the above formula, \( m(u) \) is calculated as 88%. Note that this margin ratio includes the portion allocated to expenses, which is 32%. Thus the profit margin ratio is 56% of the net premium and 42% of the gross premium.

An initial capital injection of 1105 has given a 10% empirical portfolio ruin probability. Thus, we expect the 88% margin will result in a 10% empirical ruin probability since this margin also gives an initial surplus of 1105 as a present value. However, an 88% margin has resulted in a 1% ruin probability in our simulation based on new cash flows, which include the margins. Unlike the case where capital is injected at time 0, as a result of adding the profit margin, the shape of cash flows changed in the way which interest rate risk is mitigated. The margin incomes, which spread over 30 years, are more likely to offset benefit payments without time lags than the initial capital injection. For various values of \( m(u) \), we carried out simulations, i.e. re-created 1,000 scenarios, in order to estimate portfolio ruin probabilities associated with new cash flows redefined by \( m(u) \). Table 10-4 shows calculated sample ruin probabilities for each \( m(u) \). The monetary units are again Yen 1000s.

<table>
<thead>
<tr>
<th>( m(u) )</th>
<th>Initial Surplus (u)</th>
<th>Profit Margin</th>
<th>Empirical Ruin Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>70%</td>
<td>741</td>
<td>31%</td>
<td>20%</td>
</tr>
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We defined the profit margin as the margin ratio levied on top of the gross premium as a percentage of gross premiums. We can see that the profit margin that gives a smaller present value than initial capital injection leads to the same
empirical ruin probability. For example, we can achieve a 10% empirical ruin probability by levying the profit margin of 36% or an initial capital injection of 1105. However, the profit margin of 36% gives an initial surplus of 938, which is smaller than 1105.

We could estimate the profit margin required to achieve given probabilities by locating the probability in the last column in Table 10-4. Table 10-5 shows the required profit margin for each probability.

Table 10-5 - Required ruin probabilities and estimated profit margins

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88
11 Conclusion

Several Japanese life insurance companies failed during the stagnant economic period Japan has experienced since 1991. One major cause of these failures has been interest losses, which arise from the difference between high interest rates used to calculate premiums and very low market interest rates. This fact indicates that life insurance companies have been exposed to substantial interest rate risks. In fact, it was shown that interest rate risk dominates the mortality risk for a unit of business in force as the number of policyholders increases.

In order to investigate the financial effect on life insurance policies of fluctuations in interest rates, we selected the HJM approach as the most appropriate tool for modelling developments of interest rate term structures. For the purpose of describing the initial yield curve, the Cairns’ model was chosen to establish a graduated initial yield curve. We then assumed we were in a risk-neutral environment with respect to interest rate term structure.

The HJM approach requires volatility parameters. In addition, we needed to modify the model so that it could handle very low interest rates. Referring to the monetary policy employed by the BOJ, long-term rates were assumed to have lower boundaries in a very low interest rate environment. The model was formulated incorporating this feature together with a volatility function. The volatility function, which completely determines the development of yield curves in the HJM approach, was defined so that the humped shape of volatility curves was represented.

Under the assumptions discussed in Chapter 8, present values and accumulated values of a life insurance policy portfolio were then calculated. First, the cash flows arising from the portfolio were assumed to be realized as initially expected. Second, the expected rates of return were assumed to be the sum of the base yield and a constant extra return. Third, actual rates of return for the immediate future were assumed to be realized as expected. Under these
assumptions, we defined a surplus process, which shows the development of the economic value of a life insurance policy portfolio in a stochastic interest rate environment. The net value of the policy portfolio was investigated by considering the development of the discounted surplus, which shows present values of the portfolio at the date of issue.

The model parameters for the initial yield curve and volatility function were found by reference to market observations. Simulation was conducted with these parameters on a life insurance policy portfolio typical of the Japanese market. The simulation results estimated the distribution of the discounted surplus, which enabled us to calculate the initial capital and/or margins required to prevent portfolio ruin at a given level of probability.

According to the empirical distribution we obtained, without any profit margin, the initial capital required for a 10% probability of ruin is eight times the annual gross premium. Allowing a 1% portfolio ruin requires initial capital 10 times as much as annual gross premium. When a 36% profit margin is added to the gross premium, the empirical ruin probability is also 10%. The empirical ruin probability became 1% by charging a profit margin of 42%.

Note that these margins or initial capital are reserved only for interest rate risks. In practice, more margins may be needed for asset price fluctuation not attributable to interest rate fluctuations, mortality experience different from the mortality basis used for premium calculations, and fluctuations in expenses. In addition to the fact that more reserves are needed, dividend payments are undermining the financial base of life insurance companies.

In a competitive environment where various financial products, valuation tools and accurate information about these are available to the public, high profit margins may not be achievable. Therefore, interest rate risks need to be correctly measured, managed and then reduced. There may be many options to achieve this: asset-liability matching, shortening policy duration or linking premiums to interest rates, and linking policy values to asset values, etc. These measures are especially important as the entire life insurance industry shares the same interest rate risk, i.e. low interest rates would damage portfolios of all
life insurance companies. This is an important problem for Japan, in general, because the soundness of the Japanese life insurance industry has social implications that extend well beyond the financial status of individual companies.
interest rate contingent claims”. Journal of Finance, 41, 1011-29
Wilkie, A.D. (1986) “A stochastic investment model for actuarial use”. Transaction of the Faculty of Actuaries, 39, 341-403
British Actuarial Journal, 1, 777-945
## APPENDICIES

### Appendix 1 - Yen fitted Yield Curve

*(See Section 9.2)*

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### Appendix 2 - Model Volatility

(See Section 9.4)

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Appendix 3 - Mortality Table, Male (1999)

(See Section 9.5)

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Source: Ministry of Health, Labour and Welfare
## Appendix 4 - Expected Cash Flow

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