Mortality and Disability Modelling with an Application to the Pricing of a Reverse Mortgage Contract

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Abstract

This thesis investigates models for mortality rate forecasting and estimating probabilities of disablement. In addition, it also presents an application of the resulting forecasts of mortality rates and the estimated probabilities of disablement to the pricing of a reverse mortgage contract.

In Chapter 2, we investigate the suitability of the Gaussian and Poisson Lee-Carter (LC) models for forecasting mortality rates at ages 60 and above in Australia. We restrict our investigation on one-term and two-term variants of the LC model. We find that the two-term Poisson LC model is not suitable for our application, while the one-term Poisson LC model is preferable over the Gaussian LC models. In this chapter, we also investigate 95% prediction intervals (PIs) of future mortality rates estimated under a semi-parametric bootstrap. As compared to the 95% PIs of future mortality rates estimated under the traditional method, the 95% PIs estimated under a semi-parametric bootstrap are wider only at ages (around) 90 and above. In addition, from an assessment of the out of sample forecasts, the 95% PIs of future mortality rates estimated under a semi-parametric bootstrap are likely to be too narrow.

In Chapters 3 and 4, we estimate disability transition probabilities in Australia. In Chapter 3, we apply the multi-state model adopted by Rickayzen and Walsh (2002) and Leung (2004) to estimate the disability transition probabilities using the 2003 Survey of Disability, Ageing and Carers (SDAC) data. In Chapter 4, we present a method to improve the accuracy of the disability transition probabilities estimated in Chapter 3. This method employs the Iterative Proportional Fitting procedure and uses the disability transition probabilities estimated in Chapter 3 as an input of the estima-
tion. Fundamental to the implementation of this method is the apparent stability of disability prevalence rates over the past two decades. Despite the uncertainties in the estimation, reasonable estimates are obtained with several informative observations emerging. In Chapter 4, we also estimate the conditional probabilities of admission into an aged care home (ACH) which are estimated from the final estimates of disability transition probabilities and the ACH prevalence rates estimated from the 2003 SDAC data.

In Chapter 5, we present a methodology for measurement of crossover and liquidity risks for a reverse mortgage contract in Australia. The house prices and the interest rates are modelled and simulated under a house price inflation linked LIBOR market model, while the termination probabilities of a reverse mortgage contract are estimated under a multi-state modelling framework to take into account a variety of modes of termination. There are uncertainties in our estimation which are, in particular, due to the basis risk and the issues in estimating the termination probabilities. Despite these uncertainties, several insights are obtained, in particular, we find that the liquidity risk is significant and the loss risk is insignificant.
Declaration

This is to certify that:

1. the thesis comprises only my original work towards the PhD except where indicated in the Preface,

2. due acknowledgment has been made in the text to all other material used,

3. the thesis is less than 100,000 words in length, exclusive of tables, bibliographies and appendices.

Signed,

Evan A. Hariyanto
Preface

This thesis was completed under the supervision of Professor David Dickson in the Centre for Actuarial Studies, The University of Melbourne and Associate Professor David Pitt in the Department of Applied Finance and Actuarial Studies, Macquarie University. Chapters 2 to 5 contain the original research of this thesis, except as otherwise noted.

Chapters 3 and 4 have been written up as two papers “Estimation of Disability Transition Probabilities in Australia I: Preliminary” and “Estimation of Disability Transition Probabilities in Australia II: Implementation”, co-authored by David Dickson and David Pitt. These papers will appear in Annals of Actuarial Science in 2014.

None of the work appearing here has been submitted for any other qualifications, nor was it carried out prior to PhD candidature enrollment.
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Chapter 1

Introduction

Australia and many other developed nations have been witnessing an increase in life expectancy and, at the same time, declining birth rates. Both of these factors will lead to an increase, over time, in the proportion of the population who are classified as elderly. Based on the Intergenerational Report 2010 (Treasury, 2010), produced by the Commonwealth Government of Australia, the proportion of the population aged 85 and above is expected to increase from 1.8% in 2010 to 5.1% in 2050. This ageing presents a challenge for the provision of aged care services and for formulating suitable funding mechanisms to support the increased need for such care. Several commentators suggest that the current funding system is unlikely to be sustainable and the current aged care services will not be able to deliver the quantity and quality of aged care needed in the future (Australian Institute of Health and Welfare (AIHW), 2007a; Access Economics, 2010).

Fundamental to addressing the issues associated with the ageing population are reliable estimates of how long people will live and how long they will remain active. Therefore, estimation of future mortality rates and probabilities of disablement is important to address these issues. This thesis contributes to the existing literature on the subjects of mortality rate forecasting and estimation of probabilities of disablement in Australia. In exploring these topics, we keep in mind of our third objective: pricing analysis of a reverse mortgage contract.
The first part of this thesis (Chapter 2) investigates the suitability of the Lee-Carter (LC) mortality rate forecasting model to forecast mortality rates at ages 60 and above in Australia. Originally proposed by Lee and Carter (1992), the LC model has gained a wide acceptance in North America. In addition, the LC model has been adopted to forecast mortality rates of other developed countries with promising results (Tuljapurkar et al., 2000; Lee and Miller, 2001). In this chapter, we investigate the suitability of the Gaussian and Poisson LC models for forecasting mortality rates. We restrict our investigation to one-term and two-term variants of the LC model. We consider two fitting periods: 1921–2004 and 1975–2004, for the purpose of capturing the long-term trend and the recent short-term trend of mortality rates. In this chapter, we also investigate the 95% PIs of future mortality rates estimated under a semi-parametric bootstrap.

The second part of this thesis (Chapters 3 and 4) estimates disability transition probabilities in Australia. In Australia, estimating these transition probabilities is difficult since, at a national scale, we only have cross sectional disability data (which are reported in the Survey of Disability, Ageing and Carers (SDAC)). A sophisticated multi-state model to estimate disability transition probabilities from cross sectional data has been developed by Rickayzen and Walsh (2002). This multi-state model was adopted by Leung (2004) to project the cost of long-term care in Australia. In Chapter 3, we apply the multi-state model adopted by Leung (2004) to estimate the disability transition probabilities using the more recent disability data. In Chapter 4, we improve the accuracy of the disability transition probabilities estimated in Chapter 3. The method presented in Chapter 4 employs the Iterative Proportional Fitting procedure and uses the disability transition probabilities estimated in Chapter 3 (initial transition probabilities) as an input of the estimation. Fundamental to the implementation of this method is the apparent stability of disability prevalence rates over the past two decades. Practically, the procedure involves projecting the disabled population from one year to the next year (using the initial transition probabilities) and adjusting the projection such that it satisfies the age structure of the disabled population at the following year. The results of the procedure are the estimates of
one year disability longitudinal data. From these estimates of longitudinal data, the refined estimates of one-year disability transition probabilities are calculated. Following that, we graduate the refined estimates of disability transition probabilities to obtain the final estimates of disability transition probabilities. In Chapter 4, we also estimate the conditional probabilities of admission into an aged care home (ACH). The conditional ACH admission probabilities are estimated from the final estimates of disability transition probabilities and the ACH prevalence rates estimated from the 2003 SDAC data.

The insurance and finance industries have developed products which might be beneficial in addressing the issues associated with the ageing population. These include long-term care insurance and equity release products. A methodology for pricing and reserving a long-term care insurance contract in Australia has been developed by Leung (2006). In the last part of this thesis (Chapter 5), we conduct a pricing analysis of a reverse mortgage contract, which is the most popular type of equity release product in Australia. Specifically, we present a methodology for measurement of crossover and liquidity risks, which are the most significant risks for a reverse mortgage contract. The house prices and the interest rates are modelled and simulated under a house price inflation linked LIBOR market model, while the termination probabilities of a reverse mortgage contract are estimated under a multi-state modelling framework to take into account a variety of modes of termination. The differences of pricing results between gender and marital status of the borrowers are analysed, and the sensitivity of pricing results for couple borrowers to the spread, loan to value ratio, origination age and termination decrements are investigated. The work in this chapter is part of a wider project on reverse mortgages and can only be viewed as a subset of this project. In this chapter, we use the work of other people. Specifically, we use a program which has been developed to simulate future house price index and LIBOR rates under the developed house price inflation linked LIBOR market model.
2.1 Introduction

In Australia, for the period 1921 to 2004, mortality rates at ages 60 and above have decreased most rapidly for the past three decades. From 1970 to 2004, the life expectancy at age 60 had increased by 6.96 years for males and by 6.12 years for females. In comparison, from 1921 to 1970, the corresponding life expectancy had only increased by 1.9 years for females, and for males it had even decreased by 0.34 years. This finding is in line with Booth et al. (2002) who found that in Australia, deaths due to chronic diseases at older ages have been decreasing from about 1970. Australian experience is in line with the international experience. According to Wilmoth (2000), in industrialized countries, the most rapid improvement in death rates since about 1970 has been at older ages. In forecasting mortality rates at old ages, it is important to consider the degree of the continuity of this trend in the future.

For lenders of reverse mortgage contracts, continuation of the current rapid improvements in mortality rates at old ages will increase the risk of holding (and issuing) these contracts. This is exacerbated by the recent
global financial crisis (GFC) which increases the volatility of house prices (Lee, 2009) and the cost of funds (Brown et al., 2010). In addition, the recent GFC might also increase the demand for equity release products as the superannuation saving is adversely impacted (Kelly, 2009). Note that notwithstanding the effect of the GFC, there has been a continuing debate on the sufficiency of the current level of superannuation contributions to finance retirement (for example, Rice Warner Actuaries (2010)). Therefore, risk management of this product, which requires a forecast of mortality rates at the relevant ages (60 and above), is important especially in the current financial and demographic environment.

There are three approaches to forecast mortality rates: extrapolation, expectation and theory-based structural modelling involving exogenous variables (Booth, 2006). Under the extrapolation method, we extrapolate historical patterns of mortality rates into the future, while under the expectation method, the forecast of mortality rates is based upon the expectation or view of a group of experts. Lastly, under the structural modelling method, the mortality rates are believed to be strongly linked with some exogenous variables (for example, socio-economic, biological and scientific factors) and we forecast mortality rates from the forecast (or future expectation) of these exogenous variables.

We choose to forecast mortality rates under the extrapolation method for several reasons. Firstly, extrapolation is the most common approach to forecast demographic variables (Booth, 2006). Secondly, in economically advanced societies, the human life span has been increasing steadily for over a century and does not seem to be approaching a fixed limit (Wilmoth, 2000). Thirdly, our understanding of the complex interactions of exogenous variables which determine the levels of mortality is still imprecise (Wilmoth, 2000). Lastly, in low mortality countries (including Australia), incorporation of subjective views (for example, expert expectation) in a forecast often results in an under-prediction of the future improvement of mortality rates (Lee and Miller, 2001; Booth and Tickle, 2003).

Under the extrapolation approach, one of the most well-known mortality rates forecasting models is the Lee-Carter (henceforth LC) model (Booth,
2.1. INTRODUCTION

Originally proposed by Lee and Carter (1992), the model has gained a wide acceptance in North America. For example, in the population projections of the U.S. for the period 1999 to 2100, the U.S. Census Bureau used a forecast under the LC model as a benchmark for their long-run forecast of life expectancy (Hollmann et al., 2000). In addition, the LC model has been adopted to forecast mortality rates of other developed countries with promising results (Tuljapurkar et al., 2000; Lee and Miller, 2001). For example, from hypothetical projections from various historical jump-off dates, Lee and Miller (2001) found that the LC model would have performed very well for Canada, Sweden, France and Japan.

Several advantages of the LC model are: simple to implement, its application involves only minimal subjective judgement, and a probabilistic confidence interval is provided in its forecast (Lee and Carter, 1992). In addition, since the model was originally proposed in 1992, a variety of improvements have been suggested which strengthened the theoretical basis of it, improved its applicability and increased its forecast performance (Lee, 2000). Therefore, for practical purposes, the LC model is a reasonable choice of mortality rates forecasting method.

In recent years, several other stochastic mortality models have been developed. For example, the Cairns-Blake-Dowd (CBD) model (Cairns et al., 2006), Age-Period-Cohort (APC) model (Currie, 2006) and Poisson GLM model (Renshaw and Haberman, 2003a). For a comprehensive discussion on these stochastic mortality models see Cairns et al. (2007). In this Chapter, we focus on the Lee-Carter model.

The LC model has been applied in Australia by Booth et al. (2000), Booth et al. (2002) and Booth and Tickle (2003). Booth et al. (2000) found a departure from linearity of the estimated mortality indexes (i.e. parameters \( \{k_t\} \) of (2.1)) in the application of the LC model in Australia. This issue is addressed by Booth et al. (2002) by proposing a methodology to identify the optimal fitting period at which the linearity assumption of the mortality indexes is satisfied. The proposed methodology is adopted by Booth and Tickle (2003) in their application of the LC model to project the size of the elderly population in Australia in the future.
There are several differences between our implementation of the LC model and the previous implementations of this model in Australia as we have a different orientation (i.e. pricing of a reverse mortgage contract). Firstly, we focus our analysis on ages 60 and above. Secondly, we consider both the least squares and the maximum likelihood fitting methods. Lastly, the prediction interval of future mortality rates is estimated under a semi-parametric bootstrap.

2.2 Data source

The data for our analysis are obtained from the Human Mortality Database (HMD) (www.mortality.org).

2.2.1 Overview of the Human Mortality Database

The HMD provides comprehensive mortality and population data for researchers, policy analysts, journalists, students and others interested in human longevity. The HMD is sponsored by the University of California at Berkeley (U.S.) and the Max Planck Institute for Demographic Research (Rostock, Germany) (Wilmoth et al., 2007). It receives financial support from the U.S. National Institute on Aging and technical assistance from many international collaborators. Data from the HMD have been used in many research projects (for example, Li and Chan (2007), Wang (2007), Koissi et al. (2006), Pedroza (2006) and Haberman and Russolillo (2005)).

From the HMD, we obtained the mortality rates, death counts and exposure to risk data for Australia from 1921 to 2004. At the time when this work started, in the HMD, the earliest record of Australian data is in 1921 and the latest is in 2004. The Australian data in the HMD are sourced from the Australian Bureau of Statistics (ABS), the Australian Centre for Population Research (ACPR) and the Australian Institute of Health and Welfare (AIHW) (Wilmoth et al., 2007).
2.3 Mortality trend at ages 60 and above over the twentieth century

Figure 2.1 presents the life expectancy at age 60 ($\hat{e}_{60}$) from 1921 to 2004. Overall, there were significant increases in $\hat{e}_{60}$ for both males and females over this period. The total increase in $\hat{e}_{60}$ for males was 6.62 years and for females was 8.02 years. Note that most of the increase occurred from 1970 to 2004.

The plot of $\hat{e}_{60}$ for females was higher than for males and the gap was initially increasing until 1980 (the gap at this year reached 4.89 years) before decreasing over time. In 2004, the difference of $\hat{e}_{60}$ between genders decreased to 3.56 years.

Table 2.1 presents the proportion of deaths caused by each of the four major diseases to total deaths in 2000 at age groups 65–84 and 85+ for males
and females. Note that at ages 60 and above, diseases are the main causes of deaths. In 2000, the four major diseases contributed to more than 80% of total deaths at ages 65 and above (Table 2.1).

From 1921 to 1970, there was no increase in $\hat{e}_{60}$ for males and a small increase for females. Note that during this period, Australia experienced a major adverse economic shock (the Great Depression which started around 1929), followed by the second World War. These two events seemed to have more effect on the mortality of males than females.

Table 2.2 presents the death rates (per 100,000 population) at ages 60 and above for males and females in 1970 and 2004. Note that from 1970 onwards, the $\hat{e}_{60}$ values have been increasing rapidly due to the decreasing death rates of the majority of diseases, in particular the circulatory diseases (Table 2.2). Despite the rapid fall of its death rate, circulatory diseases remained the leading cause of deaths by the end of the century (Table 2.1). While the death rates of the majority of diseases have been falling over this period, the death rate of cancer has increased, albeit marginally. In 2000, cancer was the second leading cause of death.

Table 2.1: Proportion of deaths caused by each of the four major diseases to total deaths, year 2000, age groups 65–84 and 85+.

<table>
<thead>
<tr>
<th></th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>65–84  85+</td>
<td>65–84  85+</td>
</tr>
<tr>
<td>Circulatory</td>
<td>38.2%  47.9%</td>
<td>40.7%  55.6%</td>
</tr>
<tr>
<td>Cancer</td>
<td>34.6%  18.3%</td>
<td>29.9%  11.4%</td>
</tr>
<tr>
<td>Respiratory</td>
<td>10.1%  12.7%</td>
<td>8.7%  8.9%</td>
</tr>
<tr>
<td>Endocrine</td>
<td>3.4%   3.4%</td>
<td>3.9%  4.3%</td>
</tr>
</tbody>
</table>

*Source: Australian Institute of Health and Welfare (2005b)*

Table 2.2: Death rates at ages 60 and above (per 100,000 population).

<table>
<thead>
<tr>
<th></th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circulatory</td>
<td>3,805.1  1,258.6</td>
<td>3,054.9  1,261.0</td>
</tr>
<tr>
<td>Cancer</td>
<td>1,054.2  1,128.7</td>
<td>640.7    729.2</td>
</tr>
</tbody>
</table>

*Source: Australian Institute of Health and Welfare (2005a)*
At ages 65–84, without a breakthrough in the treatment of cancer, the recent rapid improvement in mortality is unlikely to continue in the future. In the past three decades, the rapid improvement in mortality at these ages is mainly due to the reduction in the death rate of circulatory diseases (see Figures 6.32 and 6.33 of AIHW (2005b)). However, at the current level of death rates from circulatory diseases at these ages, future improvement is unlikely to result in a similar degree of improvement of the overall mortality rate as in the past three decades. Note that the improvement of death rates of other major diseases (i.e. other than circulatory and cancer) is likely to have only a minimal impact on the overall mortality rate since their current contribution to total deaths is already quite low at these ages.

At ages 85 and over, the recent improvement of mortality is likely to continue in the future. The reduction in mortality rates at these ages from 1970 is mainly due to the reduction in death rates from circulatory diseases (see Figures 6.39 and 6.40 of AIHW (2005b)). In 2000, deaths due to circulatory diseases still represent a significant proportion of total deaths at these ages. However, as the deaths due to circulatory diseases represent a decreasing proportion of total deaths, the improvement of mortality at these ages is likely to be diminishing.

We have made a subjective opinion on the likely future mortality level. However, as discussed in Section 2.1, such an opinion often under-predicts the future improvement of mortality. In the following, we present a mortality rates forecasting model which is purely extrapolative.

### 2.4 Lee-Carter model

The basic LC model for mortality rate forecasting is (Lee and Carter, 1992):

\[
\log(m_{x,t}) = a_x + b_x k_t + \varepsilon_{x,t}
\]  

(2.1)

where

\(m_{x,t}\) is the central mortality rate at age \(x\) in year \(t\),
$a_x$ is an age-specific component of $\log(m_{x,t})$ which is independent of time (it is regarded as the general level of $\log(m_x)$),

$k_t$ is a time-varying parameter which is regarded as the mortality index in year $t$,

$b_x$ is a parameter which represents how rapidly (or slowly) the mortality rate at age $x$ varies when the mortality index changes,

$\varepsilon_{x,t}$ is the residual at age $x$ in year $t$. These residuals are assumed to be independent and identically distributed $\mathcal{N}(0, \sigma^2_{\varepsilon})$.

Note that by modelling the log of the (central) mortality rate, the model does not result in negative values of fitted and forecast mortality rates.

The LC model decomposes $\log(m_{x,t})$ into two components: the general level of $\log(m_x)$, $a_x$, which is adjusted to the level of mortality index in year $t$ through the second component, $b_x k_t$. Therefore, the parameter $b_x$ determines the adjustment of $a_x$ to the level of $k_t$. Positive values of $b_x$ represent a positive correlation between the mortality rate at age $x$ and the mortality index (and vice versa), while the level of $b_x$ represents the degree of sensitivity of the mortality rate at age $x$ to the mortality index. While $b_x$ is allowed to have a different sign (i.e. can be positive or negative) across ages, fitting of the LC model over a fairly long period of time generally results in the same sign of $b_x$ across ages (Lee and Carter, 1992; Haberman and Russolillo, 2005). Note that under the LC model, the ratio of the change of the log of mortality rates between ages is constant over time.

The trend of $k_t$ captures the main time trend of $\log(m_{x,t})$ across ages (Booth et al., 2002). This is a convenient feature of the model for forecasting purposes. By summarising the progression of mortality rates over time into a single index, the task of forecasting mortality rates is simplified into the forecasting of one variable. Therefore, if the estimates of $\{k_t\}$ accurately capture the main time trend of $\{\log(\hat{m}_{x,t})\}$ (where $\hat{m}_{x,t}$ is the observed value of $m_{x,t}$) and exhibit a linear trend over time (which indicates that, overall, mortality has been declining (or increasing) at a constant rate), the forecasts of mortality rates under the LC model are likely to be accurate.
There are three main methods to fit the LC model: Singular Value Decomposition (SVD), Weighted Least Squares (WLS) and Maximum Likelihood Estimation (MLE) (Koissi et al., 2006). Note that the LC model is under-determined. For example, suppose that \( a, b \) and \( k \) are solutions for \( a_x, b_x \) and \( k_t \) respectively; then for any scalar \( c \); \( a - bc, b \) and \( k + c \) are also solutions for \( a_x, b_x \) and \( k_t \) each, as well as \( a, bc \) and \( k/c \) (Lee and Carter, 1992). Therefore, in order for the LC model to be identifiable (i.e. to have a unique set of parameter estimates), we need constraints for the parameters. The constraints should be chosen such that the trend of mortality rates across ages over time is accurately reflected in parameters \( \{k_t\} \) (Wilmoth, 1993). There are two types of constraints generally used:

(a) \( \sum_t k_t = 0 \) and \( \sum_x b_x = 1 \) (Lee and Carter, 1992; Koissi et al., 2006).

(b) \( \sum_t k_t = 0 \) and \( \sum_x (b_x)^2 = 1 \) (Wilmoth, 1993; Girosi and King, 2007).

Note that both constraints result in \( a_x \) as the arithmetic mean of \( \log (m_{x,t}) \) across \( t \) (Lee and Carter, 1992). While constraints (a) and (b) result in different estimates of \( \{b_x\} \) and \( \{k_t\} \), the qualities of fit and the resulting forecasts of mortality rates under these constraints are similar (from our experiments).

In this chapter, we fit the LC model under SVD and MLE methods. These methods are described in the following.

### 2.4.1 The singular value decomposition (Gaussian Lee-Carter)

Under the SVD method, we find the ordinary least squares (OLS) solution to model (2.1) (Lee and Carter, 1992). Suppose that the number of ages is \( X \) and the number of years is \( T \) (i.e. we consider ages \( x_1, x_2, \ldots, x_X \) and years \( t_1, t_2, \ldots, t_T \)). Note that \( a_x = \frac{1}{T} \sum_{i=1}^{T} \log(m_{x,t_i}) \) (due to constraints on the parameters). Let \( z_{x,t} = \log (m_{x,t}) - a_x \) and denote:

\( Z \): matrix with entry \( z_{x,t} \) with age \( x \) in rows and year \( t \) in columns.
Applying SVD to $Z$ gives $Z = UDV^*$, where $U$ is an $X \times X$ matrix, $D$ is an $X \times T$ matrix, $V$ is an $T \times T$ matrix and $V^*$ is the conjugate transpose of $V$. Let $z_{i,j}$ denote the $(i, j)^{th}$ entry of $Z$ and similarly for $u_{i,j}, d_{i,j}$ and $v_{i,j}$. $D$ is a diagonal matrix with diagonal entries that are the singular values of $Z$ being ordered from the highest to the lowest (i.e. $d_{1,1} \geq d_{2,2} \geq \ldots \geq d_{r,r}$, where $r$ is the rank of $Z$). The $i$th columns of $U$ and $V$ are respectively the left and right singular vectors for $d_{i,i}$ ($i \leq r$). Further details of SVD are described in Lawson and Hanson (1974).

For clarity, we replace the index of row $(i)$ and column $(j)$ respectively with the age $(x)$ and year $(t)$ that they represent. Hence, the entries of $Z$ can be written as:

$$z_{x,t} = \sum_{j=1}^{r} u_{x,j} d_{j,j} v_{t,j} \quad x = x_1, x_2, \ldots, x_X; t = t_1, t_2, \ldots, t_T \quad (2.2)$$

where $r$ is the rank of $Z$.

The SVD approximation of $z_{x,t}$ at rank $h$ is (Koissi et al., 2006):

$$z_{x,t} = \sum_{j=1}^{h} u_{x,j} d_{j,j} v_{t,j} + \varepsilon_{x,t} \quad x = x_1, x_2, \ldots, x_X; t = t_1, t_2, \ldots, t_T; h \leq r \quad (2.3)$$

where $\varepsilon_{x,t}$ are the SVD rank $h$ residuals and given by (Renshaw and Haberman, 2003b):

$$\varepsilon_{x,t} = \begin{cases} \sum_{j=h+1}^{r} u_{x,j} d_{j,j} v_{t,j} & : h < r, \\ 0 & : \text{otherwise}. \end{cases} \quad (2.4)$$

The SVD residuals $(\varepsilon_{x,t})$ are assumed to be independent and identically distributed $\mathcal{N}(0, \sigma^2_\varepsilon)$ (homoscedastic error structure).

Let $b^{(j)} = u_{x,j}$ and $k^{(j)} = d_{j,j} v_{t,j}$. Then (2.3) can be written as:

$$\log(m_{x,t}) = a_x + \sum_{j=1}^{h} b^{(j)} k^{(j)} + \varepsilon_{x,t} \quad x = x_1, x_2, \ldots, x_X; t = t_1, t_2, \ldots, t_T; h \leq r \quad (2.5)$$
with
\[ \sum_{t} k_{t}^{(j)} = 0, \sum_{x} b_{x}^{(j)} = 1 \text{ or } \sum_{x} \left( b_{x}^{(j)} \right)^{2} = 1 \quad \forall j. \] (2.6)

Model (2.5) is the general form of model (2.1) and referred to as LC(h).

We implement the SVD procedure by using Excel (utilizing the Biplot package (Lipkovich and Smith, 2002)) and R.

We fitted model (2.5) to the dataset of crude (central) mortality rates \( \{ \hat{m}_{x,t} : x = x_{1}, x_{2}, \ldots, x_{X}, t = t_{1}, t_{2}, \ldots, t_{T} \} \) to obtain the sets of parameters estimates \( \{ \hat{a}_{x} \}, \{ \hat{b}_{x}^{(j)} \} \) and \( \{ \hat{k}_{t}^{(j)} \} \) (where \( \hat{a}_{x} \) is the estimate of \( a_{x} \), and similarly for \( \hat{b}_{x}^{(j)} \) and \( \hat{k}_{t}^{(j)} \)). Since we apply the log function to crude mortality rates, the SVD method requires positive observed numbers of deaths for all ages and years. Let \( \hat{m}_{x,t} = \exp \left( \hat{a}_{x} + \sum_{j=1}^{h} \hat{b}_{x}^{(j)} \hat{k}_{t}^{(j)} \right) \) denote the fitted mortality rate at age \( x \) and time \( t \), and \( \hat{\varepsilon}_{x,t} = \log (\hat{m}_{x,t}) - \log (\hat{m}_{x,t}) \) denote the estimates of SVD residuals. To measure the quality of fit, we calculate the following ratio:
\[ R^{2} = 1 - \frac{\sum_{x,t} \left( \log (\hat{m}_{x,t}) - \log (\hat{\varepsilon}_{x,t}) \right)^{2}}{\sum_{x,t} \left( \log (\hat{m}_{x,t}) - \log (\hat{m}_{x,t}) \right)^{2}} \] (2.7)

where
\[ \log (\hat{m}_{x,t}) = \frac{1}{T} \sum_{t} \log (\hat{m}_{x,t}). \]

\( R^{2} \) (the coefficient of determination) measures the proportion of variability in a dataset that is accounted for (explained) by the model. Generally, for countries with low mortality, a significant proportion of the variance in \( \log (\hat{m}_{x,t}) \) is explained by the first term of the SVD (Tuljapurkar et al., 2000; Booth et al., 2002; Renshaw and Haberman, 2003a; Koissi et al., 2006). For example, from their application of the LC model to the G7 countries, Tuljapurkar et al. (2000) found that the first term of the SVD explained over 94% of the variance in \( \log (\hat{m}_{x,t}) \).

To measure the quality of fit, in addition to \( R^{2} \), it is also necessary to examine the pattern of the estimates of residuals (Renshaw and Haberman, 2003b). A good fit is achieved if the estimates of residuals exhibit a pattern which indicates that the residuals are independent and identically distributed (Koissi et al., 2006). This is indicated by the absence of systematic pattern
Inclusion of additional terms from the SVD will improve the fit of the LC model. However, the patterns of the estimates of time components of these additional terms (i.e. $\hat{k}_{jt}^{(j)}, j \geq 2$) are often too erratic to forecast. In such a case, the inclusion of these additional terms in the forecasts of mortality rates might adversely impact the forecast performance. Although Booth et al. (2002) found that the second and third terms of the SVD capture the cohort-period effect, they refrained from incorporating these additional terms in their forecasts due to the difficulty in forecasting the time component of these additional terms. In most applications of the LC model, the forecasts only incorporate the first term of the SVD (Lee and Carter, 1992; Tuljapurkar et al., 2000; Lee and Miller, 2001; Booth et al., 2002; Booth and Tickle, 2003; Renshaw and Haberman, 2003a; Haberman and Russolillo, 2005; Koissi et al., 2006; Li and Chan, 2007).

The main shortcoming of the SVD method is the homoscedasticity assumption of $\varepsilon_{x,t}$. This assumption is unrealistic; since the exposures at older ages are lower than at younger ages, the variances of $\log(m_{x,t})$ at older ages should be higher (Brouhns et al., 2002b). The implication of this is that under the SVD method, the same weight is given to $\log(\tilde{m}_{x,t})$ for all $x$ and all $t$ in the fitting of the LC model and, therefore, differences in exposure are not taken into account. To correct this, Lee and Carter (1992) recommended an adjustment to $\{\hat{k}_{t}^{(1)}\}$ such that in a given year, the estimate of the number of deaths matches the total observed number of deaths. Specifically, $\{\hat{k}_{t}^{(1)}\}$ are recalculated such that the following equation holds (Renshaw and Haberman, 2003b):

$$\sum_{x} d_{x,t} = \sum_{x} E_{x,t} \exp \left\{ \hat{a}_{x} + \hat{b}_{x}^{(1)} \hat{k}_{t}^{(1)} + \sum_{j=2}^{h} \hat{b}_{x}^{(j)} \hat{k}_{t}^{(j)} \right\} = \sum_{x} E_{x,t} \tilde{m}_{x,t}$$

for $t = t_1, t_2, \ldots, t_T$ (2.8)

where

$d_{x,t}$ is the observed number of deaths at age $x$ in year $t$,
2.4. LEE-CARTER MODEL

$E_{x,t}$ is the exposure at age $x$ in year $t$,

$\tilde{k}_t^{(1)}$ is the recalculated $\hat{k}_t^{(1)}$,

$\hat{m}_{x,t}$ is the fitted mortality rate at age $x$ and year $t$ after recalculation of $\hat{k}_t^{(1)}$.

After recalculation of $\{\hat{k}_t^{(1)}\}$, $R^2$ is calculated according to (2.7) with $\hat{m}_{x,t}$ replaced by $\tilde{m}_{x,t}$, while the estimates of the SVD residuals ($\hat{\varepsilon}_{x,t}$) are calculated according to the following

$$\hat{\varepsilon}_{x,t} = \log(\hat{m}_{x,t}) - \log(\tilde{m}_{x,t}).$$

(2.9)

Booth et al. (2002) found that although adjustment (2.8) generally gives greater weight to ages with higher observed numbers of deaths (and therefore, higher exposures), the weighting is complex and does not conform to standard statistical criteria. Lee and Miller (2001) proposed an alternative adjustment procedure by matching the observed and expected life expectancy. However, the weighting under this alternative adjustment method is more complex and even less clear than adjustment (2.8) (Booth et al., 2002).

The homoscedasticity assumption of $\varepsilon_{x,t}$ also results in an unsuitable property of the prediction interval of future mortality rates estimated under a semi-parametric bootstrap (discussed in Section 2.6.5).

As alternatives to SVD method, the LC model can be fitted under WLS or MLE methods (Wilmoth, 1993). There are two advantages of these methods over the SVD method. Firstly, these methods can be used with data with zero observed number of deaths at some ages. Secondly, these methods already allocate more weight to ages and years with higher observed numbers of deaths and hence, an adjustment procedure is not necessary. The MLE method has better statistical properties and is computationally easier to fit than the WLS method (Girosi and King, 2007) and, therefore, we choose it over the WLS method. In the following, we describe the MLE method under a Poisson formulation for the number of deaths.
2.4.2 The maximum likelihood estimation (Poisson Lee-Carter)

Under this method, we model the number of deaths as a Poisson random variable (for a discussion on the suitability of the Poisson formulation for the number of deaths, see Brillinger (1986)). Let $D_{x,t}$ denote the random variable for the number of deaths at age $x$ in year $t$. We model $D_{x,t}$ as (Wilmoth, 1993; Booth et al., 2002; Brouhns et al., 2002b; Renshaw and Haberman, 2003b):

$$D_{x,t} \sim \text{Poisson} \left( E_{x,t} m_{x,t} \right) \text{ with } m_{x,t} = \exp \{ a_x + b_x k_t \},$$

(2.10)

where the parameters $\{b_x\}$ and $\{k_t\}$ are subject to constraints (2.6).

The MLE method (under a Poisson formulation) gives greater weight to ages and years with higher observed numbers of deaths. The weighting under this method is better understood and more accurate than the adjustment (2.8) (Booth et al., 2002). Although the annual total observed number of deaths might not be reproduced exactly, this disadvantage is outweighed by the greater accuracy of the weighting in the fitting of the model. In addition, under this model, the variance of $m_{x,t}$ is negatively related to $E_{x,t}$ (the $\{D_{x,t}\}$, or equivalently the $\{m_{x,t}\}$ are heterocedastic). This property is suitable for the estimation of a prediction interval of future mortality rates under a semi-parametric bootstrap (see Section 2.6.5).

The parameters of model (2.10) (Poisson LC model) are estimated such that the following log-likelihood function is maximised:

$$l(\mathbf{a}, \mathbf{b}, \mathbf{k}) = \sum_{x,t} \left\{ d_{x,t} (a_x + b_x k_t) - E_{x,t} \exp \{ a_x + b_x k_t \} \right\} + \text{constant}. \quad (2.11)$$

The iterative algorithm to maximise the log-likelihood function (2.11) is described in Brouhns et al. (2002b). Under this algorithm, we update a single set of parameters simultaneously (for example, $a_x \forall x$) while fixing other sets of parameters at their current estimates using the following updating scheme
(suppose we update parameter $\theta$):

$$\hat{\theta}^{(v+1)} = \hat{\theta}^{(v)} - \frac{f(\hat{\theta}^{(v)})}{g(\hat{\theta}^{(v)})}$$

where

$\hat{\theta}^{(v)}$ is the estimate of parameter $\theta$ at iteration step $v$,

$$f(\hat{\theta}^{(v)}) = \frac{\partial l(\theta)}{\partial \theta}|_{\hat{\theta}^{(v)}}$$

$$g(\hat{\theta}^{(v)}) = \frac{\partial^2 l(\theta)}{\partial \theta^2}|_{\hat{\theta}^{(v)}}$$

Note that we have three sets of parameters: $\{\hat{a}_x\}$, $\{\hat{k}_t\}$, and $\{\hat{b}_x\}$. Updating the sets of parameters $\{\hat{a}_x\}$, $\{\hat{k}_t\}$ and $\{\hat{b}_x\}$ in sequence has been found to be efficient (quickly converge to the required estimates). After updating the $\hat{k}_t$ parameters, we need to adjust the updated $\hat{k}_t$ parameters such that $\sum_t \hat{k}_t = 0$. An adjustment is also necessary for the updated $\hat{b}_x$ parameters such that $\sum_x (\hat{b}_x)^2 = 1$ or $\sum_x \hat{b}_x = 1$.

Define one updating cycle as the updating of the sets of parameters $\{\hat{a}_x\}$, $\{\hat{k}_t\}$ and $\{\hat{b}_x\}$ in sequence. We decide that convergence has been achieved if an updating cycle results in an increase of the log-likelihood function (2.11) of less than $10^{-10}$.

The general form of the Poisson LC model is:

$$D_{x,t} \sim \text{Poisson} \left( E_{x,t} \exp \left( a_x + \sum_{j=1}^{h} b_x^{(j)} k_t^{(j)} \right) \right). \quad (2.12)$$

Model (2.12) is referred to as PLC($h$).

The algorithm to maximise the log-likelihood function of model PLC($h$) is similar to the algorithm for model PLC(1) (model (2.10)) with a simple extension to the updating cycle: after updating the set of parameters $\{\hat{a}_x\}$, we update (and adjust according to the chosen constraints) the sets of parameters $\{\hat{k}_t\}$ and $\{\hat{b}_x\}$ in sequence starting from $j = 1, 2, \ldots, h$.

Further details of the fitting method of the Poisson LC model are described in Brouhns et al. (2002b) and Renshaw and Haberman (2003b).
We only consider models PLC(1) and PLC(2). A program was written in R to fit these models.

To assess the quality of fit of model PLC\((h)\), we examine the following deviance residual (Dobson, 2002):

\[
r_{x,t} = \text{sign} (d_{x,t} - \hat{d}_{x,t}) \sqrt{2 \left( d_{x,t} \log \left( \frac{d_{x,t}}{\hat{d}_{x,t}} \right) - (d_{x,t} - \hat{d}_{x,t}) \right)}
\]

(2.13)

where

\[r_{x,t}\] is the deviance residual at age \(x\) in year \(t\), and

\[
\hat{d}_{x,t} = E_{x,t} \exp \left\{ \hat{a}_x + \sum_{j=1}^h \hat{\beta}_x (\hat{k}_t^{(j)}) \right\}
\]

is the expected (fitted) number of deaths at age \(x\) in year \(t\).

\[d_{x,t} \] is the expected number of deaths at age \(x\) in year \(t\).

\[
\hat{a}_x = E_{x,t} \exp \left\{ \sum_{j=1}^h \hat{\beta}_x (\hat{k}_t^{(j)}) \right\}
\]

is the expected (fitted) number of deaths at age \(x\) in year \(t\).

## 2.5 Forecasting

### 2.5.1 Forecasting procedure

In this section, we describe the mortality rates forecasting procedure for the LC models. For convenience, in this section, the recalculated \(\hat{k}_t^{(1)}\) is also denoted by \(\hat{k}_t^{(1)}\).

Forecasting mortality rates requires forecasting each of the mortality indexes under a suitable ARIMA time series model. Following Renshaw and Haberman (2003b), we model \(\{\hat{k}_t^{(j)} : j = 1, 2, \ldots, h\}\) as separate ARIMA processes. In practice, ARIMA \((0, 1, 0)\) has been used almost exclusively (Girosi and King, 2007). Forecasts of mortality rates are then computed from the forecasts of mortality indexes. There are two forecasting equations generally used:

\[
\hat{m}_{x,t+s} = \hat{m}_{x,t} \times \exp \left\{ \sum_{j=1}^h \hat{\beta}_x (\hat{k}_{t+s}^{(j)} - \hat{k}_t^{(j)}) \right\}
\]

\[
= \exp \left\{ \hat{a}_x + \sum_{j=1}^h \hat{\beta}_x \hat{k}_{t+s}^{(j)} \right\}, \quad s > 0
\]

(2.14)
and

\[
\tilde{m}_{x,t_T+s} = \hat{m}_{x,t_T} \times \exp \left\{ \sum_{j=1}^{h} \tilde{b}_{x}^{(j)} \left( \hat{k}_{t_T+s} - \hat{k}_{t_T}^{(j)} \right) \right\}, \quad s > 0
\]

(2.15)

where

- \( t_T \) is the last observation year,
- \( \tilde{m}_{x,t_T+s} \) is the forecast of \( m_{x,t_T+s} \),
- \( \hat{k}_{t_T+s}^{(j)} \) is the forecast of \( k_{t_T+s}^{(j)} \).

For model LC(h), we replace the term \( \hat{m}_{x,t_T} \) in forecasting equation (2.14) with \( \tilde{m}_{x,t_T} \).

Note that under both forecasting equations, \( \tilde{m}_{x,t_T+s} \) is the product of the assumed baseline mortality rate in year \( t_T \) and the forecast mortality rate improvement factor from year \( t_T \) to \( t_T + s \) which is calculated as \( \exp \left\{ \sum_{j=1}^{h} \tilde{b}_{x}^{(j)} \left( \hat{k}_{t_T+s} - \hat{k}_{t_T}^{(j)} \right) \right\} \). Under forecasting equation (2.14), the assumed baseline mortality rate in year \( t_T \) is \( \hat{m}_{x,t_T} \), while under forecasting equation (2.15), it is \( \tilde{m}_{x,t_T} \).

Application of forecasting equation (2.14) to U.S. mortality data resulted in discontinuities between the observed mortality rates in year \( t_T \) and the forecasts of mortality rates in year \( t_T + 1 \) (discontinuity problem) at ages with very low observed mortality rates (Lee and Carter, 1992). The discontinuities are due to the discrepancies between the observed and fitted mortality rates in year \( t_T \) (the forecasts of mortality rates themself continue smoothly from the fitted rates). The discrepancies are generally higher at ages with lower observed numbers of deaths due to the weighting in the fitting of the model. In the analysis of the U.S. data, the discontinuity problem resulted in a persistent bias of the forecasts of life expectancies at birth (Lee and Miller, 2001). In our experiment with Australian data (where we fitted the LC model to all ages), forecasting equation (2.14) also resulted in a discontinuity problem at some young ages. To address the discontinuity problem, Lee (2000) recommended forecasting under equation (2.15). This forecasting equation generally results in forecasts of mortality rates which continue
smoothly from the most recent observed rates. However, if there is a large discrepancy between \( \hat{k}^{(j)}_{tT} \) and \( \tilde{k}^{(j)}_{tT+1} \), forecasting equation (2.15) also results in a discontinuity problem. In addition, forecasting equation (2.15) also exposes the forecasts of mortality rates to any peculiarities of the observed mortality rates in year \( t_T \).

The discontinuity problem can also be addressed by altering the fitting method as described in Lee and Carter (1992): set \( \hat{a}_x \) equal to \( \log(\hat{m}_{x,tT}) \) and fit the model according to the SVD method as described previously without implementing adjustment (2.8). Under this alternative fitting method, the fitted mortality rates in year \( t_T \) are constrained to be similar to the observed rates and, hence, forecasting equation (2.14) will result in forecasts which continue smoothly from the most recent observed rates (Lee, 2000). However, as compared to the original fitting method, this alternative fitting method exposes the estimated parameters of the LC model to any peculiarities of the observed mortality rates in year \( t_T \) and, hence, is rarely used in practice.

Forecasting equations (2.14) and (2.15) are both widely used in practice. In this chapter, we examine the suitability of these forecasting equations in our application.

### 2.5.2 ARIMA model

Consider a sequence of random variables \( \{Y_t\} \) and let \( X_t = \nabla^d Y_t \). \( \{Y_t\} \) follows an ARIMA \((p, d, q)\) process if \( \{X_t\} \) is stationary and if for all \( t \),

\[
\phi(B) \nabla^d Y_t = \mu \left( 1 - \sum_{j=1}^{p} \phi_j \right) + \theta(B) w_t
\]

where

- \( B \) is the backshift operator,
- \( \nabla \) is the backward differencing operator,
- \( \mu = E(\nabla^d Y_t) \),
- \( \phi(B) = 1 - B\phi_1 - B^2\phi_2 - \ldots - B^p\phi_p \) is the autoregressive operator,
\[
\theta (B) = 1 + B\theta_1 + B^2\theta_2 + \ldots + B^q\theta_q
\]
is the moving average operator, and
\[
w_t
\]
is the random shock at time \( t \) (the random shocks are assumed to be independent and identically distributed \( N(0, \sigma_w^2) \)).

The estimates of \( \mu, \phi_i, \theta_i \) and \( w_t \) are denoted by \( \hat{\mu}, \hat{\phi}_i, \hat{\theta}_i \) and \( \hat{w}_t \) respectively.

Let \( y_t \) denote the realisation of \( Y_t \). The forecast of \( y_{t+l} \) given the information up to time \( t \) is given by:

\[
\hat{y}_{t+l} = E(P_{t+l}|F_t)
\]

where

\[
\hat{y}_{t+l}
\]
is the forecast of \( y_{t+l} \) given the information up to time \( t \),

\( P_{t+l} \) is the estimator of \( E(Y_{t+l}) \),

\( F_t \) is the information up to and including time \( t \), and

\( E(X) \) is the expected value of a random variable \( X \).

The 95\% prediction interval of \( y_{t+l} \) is given by:

\[
\left[ \hat{y}_{t+l} - 1.96 (\text{Var}(e_{t+l}|F_t))^{0.5}, \hat{y}_{t+l} + 1.96 (\text{Var}(e_{t+l}|F_t))^{0.5} \right]
\]

where \( e_{t+l} = Y_{t+l} - P_{t+l} \) and \( \text{Var}(X) \) is the variance of a random variable \( X \).

The fitting, forecasting and estimation of the prediction interval under an ARIMA model are described in Box et al. (1994). We estimate the parameters of the ARIMA models using EViews under a least squares method.

### 2.5.3 Prediction interval

Assuming that model specification and data are correct, there are two sources of uncertainty of the forecasts of mortality rates under the LC model: sampling errors in the parameters (first source) and forecast errors in the forecasted mortality indexes (second source) (Brouhns et al., 2002a). These
sources of uncertainty can be taken into account in the prediction interval (PI) of future mortality rates under a simulation method. We estimate the PI of future mortality rates under a semi-parametric bootstrap with Poisson LC formulation for the number of deaths.

### 2.6 Applications of the Lee-Carter models

As discussed before, at ages 60 and above, there are two trends of mortality over the period 1921 to 2004: slow improvement from 1921 to 1970 and rapid improvement from 1970 onwards. Therefore, we forecast mortality rates from two fitting periods: 1921–2004 and 1975–2004. The forecasts from the fitting period 1921–2004 are intended to capture the long term trend of mortality while the forecasts from the fitting period 1975–2004 are intended to capture the recent rapid improvement in mortality.

We fit the LC model to mortality data at ages 60, 61, . . . , 109 and 110+. Note that the mortality data at age category 110+ can be assumed to be applicable to age 110. We fit the LC model only to ages 60 and above (60+) since for pricing a reverse mortgage contract, we are only interested in the forecasts of mortality rates at these ages. While, as compared to fitting the LC model to all ages (i.e. ages 0+), limiting the implementation of the LC model to ages 60+ results only in a marginally better fit, the trend of mortality rates over time at ages 60+ is better captured by the main mortality indexes (i.e. parameters $\{k(1)\}$) estimated from fitting the LC model only to ages 60+. For example, fitting the LC model to ages 0+ in the period 1921–2004 results in $\hat{k}(1)$ which decrease linearly over time (due to the improvement of mortality at young ages in the beginning of this period and the improvement of mortality at older ages in the later part of this period); however, mortality at ages 60+ does not decrease linearly over this period. Note that the accuracy of the forecasts of mortality rates significantly depends on the accuracy of the estimated mortality indexes in capturing the trend of mortality rates across ages over time.

We restrict our analysis to models LC(1), LC(2), PLC(1) and PLC(2); and forecasting equations (2.14) and (2.15). For identification of the models,
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we adopt the constraints $\sum t \hat{\kappa}_t = 0$ and $\sum x (\hat{b}_x)^2 = 1$. The $\{ \hat{\kappa}_t^{(1)} \}$ of models LC($h$) are recalculated according to (2.8).

2.6.1 Fitting

In general, there is a complexity associated with forecasting mortality rates at very high ages. Usually this is because the mortality data are spurious and incomplete (not available for all years) at these ages. This is also true in our case. For example, in 1921–2004, the last ages at which the observed mortality rates are available for all years are 102 for males and 103 for females. This problem can be addressed using methods proposed by Currie (2011) and Li et al. (2008).

Currie (2011) forecasted mortality rates at very high ages using an extension of the two-factor CBD model (which was originally proposed by Cairns et al. (2006)). This model, unlike the LC model, does not contain any age specific parameters and, therefore, avoids issues associated with poor data quality at very high ages. This poor data quality issue in forecasting mortality rates under the LC model can be addressed using methods described in Li et al. (2008). Li et al. (2008) applied a model under which survival up to some threshold age followed Gompertz’ law, and thereafter followed a generalized Pareto distribution. Using this result, we are able to fill the missing observation of mortality rates at very high ages and, therefore, the LC models can be fitted over these ages. However, we choose to adopt alternative approaches which are described below. Despite its simplicity, our approaches produce satisfying results.

To fit models LC($h$), we estimate the missing observations of mortality rates under an extrapolation using a Gompertz’ formula. Each year is considered separately in the extrapolation.

The incompleteness of mortality data (i.e. the exposure and observed number of deaths), which only occur at very high ages, does not hinder the fitting of models PLC($h$). However, we need to examine whether the incompleteness of mortality data might adversely impact the quality of fit and hence the quality of forecasts of these models. To investigate this, we vary
the range of ages at which we fit model PLC(1). Specifically, we consider age ranges of 60 to 100, 60 to 101, \ldots, 60 to 109, and 60 to 110. We investigate for fitting period 1921–2004. We find that, although fitting model PLC(1) to ages 60 to 110 results in slightly different estimates of the parameters (as compared to fitting this model to other age ranges), the qualities of fit and the resulting forecasts of mortality rates are very similar under all of the age ranges considered. We also perform a similar investigation for model PLC(2); however, for this model, we only experiment with age ranges of 60 to 100 and 60 to 110. The similarity of the qualities of fit and the resulting forecasts of mortality rates under fitting to these age ranges is also observed for this model. Therefore, models PLC(1) and PLC(2) are not sensitive to the incompleteness of mortality data at very high ages and hence, we can fit these models to ages 60 to 110.

The spuriousness of mortality data at very high ages results in peculiar forecasts of mortality rates at these ages. To address this problem, we adopt a “smoothing” method. The adopted “smoothing” methods are different between models LC(h) and PLC(h) and these are described below.

For models LC(h), we address the peculiarity of forecasts by smoothing the observed mortality rates at very high ages. Specifically, we replace the observed mortality rates at an appropriate age range with the rates derived from the fitted Gompertz’ formula. We find that, before recalculation of \( \hat{k}_{it}^{(1)} \) (under (2.8)), smoothing of the observed mortality rates at very high ages improves the fit at younger ages (below 100). Therefore, we use this observation to guide us in determining the optimal age range for smoothing. Specifically, we vary the starting age at which we smooth the observed rates (the upper end of the age range is set at age 110) and fit the LC(1) model. We perform the investigation for fitting period 1921–2004. The starting age is chosen such that, before recalculation of \( \hat{k}_{it}^{(1)} \), younger starting ages do not result in a materially better fit at ages below 100. The chosen starting age is 102 for males and 103 for females. Note that the chosen starting age coincides with the last age at which the observed mortality rates are available for all years. For the implementation of model LC(2), the starting age is chosen to be the same as in model LC(1). The starting age for fitting
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in period 1975–2004 is chosen to be the same as in fitting period 1921–2004.

For models PLC(h), we fix the peculiarity of forecasts by smoothing $\tilde{a}_x$ and $\tilde{b}_x^{(j)}$ ($j = 1, 2, \ldots, h$) at very high ages (around 100 and above) under a suitable polynomial function.

Figure 2.2 presents the estimates of parameters $\{a_x\}$ of the LC models for males and females in fitting periods 1921–2004 and 1975–2004. For comparison, we include the estimates $\{\hat{a}_x\}$ for models LC(1)* (model LC(1) fitted to the unsmoothed observed mortality rates) and PLC(1)* (the estimates of parameters of this model are the unsmoothed estimates of the parameters of model PLC(1)). The estimates $\{\hat{a}_x\}$ for model LC(2) are the same as for model LC(1).

Overall, the pattern of $\{\hat{a}_x\}$ is sensible: increasing with age, higher for males (than females) and lower for fitting period 1975–2004 (than fitting period 1921–2004).

At ages below 100, the $\{\hat{a}_x\}$ are very similar for all of the models considered. At ages above 100, the $\{\hat{a}_x\}$ are spurious for models LC(1)* and PLC(1)*. The “smoothing” method (to fix the peculiarity of forecasts of mortality rates at very high ages) results in smoother $\{\hat{a}_x\}$ for both models LC(h) (i.e. LC(1) and LC(2)) and PLC(h) (i.e. PLC(1) and PLC(2)). However, for models LC(h), the adopted “smoothing” method results in a “jump” in $\{\hat{a}_x\}$ at (around) the age at which we begin to smooth the observed rates. To deal with the “jump” problem, we could extend the smoothing of observed mortality rates to younger ages; however, this will result in more information from the observed data being sacrificed.

Note that the $\{\hat{a}_x\}$ of models LC(h) are higher than for models PLC(h) at ages above 100. This is not an artefact of the adopted “smoothing” method since this feature is also shared between models LC(1)* and PLC(1)*. This is due to the fact that, under models LC(h), all of the parameters (other than $\{k_t^{(1)}\}$) are estimated by giving equal weight to the log of observed mortality rates. Note that the adjustment (2.8) for models LC(h) only affects the $\{k_t^{(1)}\}$ while leaving the estimates of other parameters unchanged. However, under models PLC(h), the weighting applies in the estimation of all parameters of the model (since all parameters are estimated under the MLE method, ...
Figure 2.2: Estimates of parameters \{a_x\} of the LC models, males and females, fitting periods 1921–2004 and 1975–2004.

(Section 2.4.2)). The relatively lower weight for the mortality data at very high ages (and greater weight for the mortality data at younger ages) under models PLC(\(h\)) results in, as compared to models LC(\(h\)), lower \{\(\hat{a}_x\)\} at these high ages.

The \{\(\hat{a}_x\)\} of model PLC(2) are similar to model PLC(1) (for females in the fitting period 1975–2004, there are only minor differences between \{\(\hat{a}_x\)\} between these models at ages 105+).

From the pattern of \{\(\hat{a}_x\)\} and the appropriateness of the weighting in their estimation, the \{\(\hat{a}_x\)\} estimated under models PLC(\(h\)) seem to be more suitable than the \{\(\hat{a}_x\)\} estimated under models LC(\(h\)).

Figure 2.3 presents the estimates of parameters \(\{b^{(1)}_x\}\) and \(\{b^{(2)}_x\}\) of the LC models for males and females in fitting periods 1921–2004 and 1975–2004.
Note that the \( \hat{b}_x^{(1)} \) of model LC(2) are the same as for model LC(1). We make several observations about this figure.

- The \( \hat{b}_x^{(1)} \) of models LC(1)* and PLC(1)* are very spurious at ages above 100. The “smoothing” method (to fix the peculiarity of forecasts of mortality rates at very high ages) results in smoother \( \hat{b}_x^{(1)} \) at these ages for models LC(\( h \)) and PLC(\( h \)). However, for models LC(\( h \)), the adopted “smoothing” method results in a “jump” problem for \( \hat{b}_x^{(1)} \) (similar to the case of \( \hat{a}_x \)). For females in fitting period 1921–2004, the “jump” problem results in a peculiar pattern of \( \hat{b}_x^{(1)} \) at ages above 100. Note that for fitting period 1975–2004, the \( \hat{b}_x^{(1)} \) at ages 100+ are negative for most LC models. This reflects the fact that on average, the mortality rates at these ages had been deteriorating over this period.

- In some cases, the \( \hat{b}_x^{(1)} \) are quite different between models LC(\( h \)) and PLC(1). This is mainly due to the differences in the way we deal with the missing mortality data in the fitting of each of these models (fitting only to ages 60 to 100 results in similar \( \hat{b}_x^{(1)} \) under these models). In addition, differences in the adopted “smoothing” methods between models LC(\( h \)) and PLC(1) also result in further differences of \( \hat{b}_x^{(1)} \) between these models.

- Model PLC(2), as compared to other models, results in different \( \hat{b}_x^{(1)} \).

- The \( \hat{b}_x^{(2)} \) are different between models LC(2) and PLC(2) (possibly due to the same reasons for the differences of \( \hat{b}_x^{(1)} \) between models LC(\( h \)) and PLC(1)), and do not show a consistent pattern across genders and fitting periods. The \( \hat{b}_x^{(2)} \) and \( \hat{k}_t^{(2)} \) are supposed to capture information from the observed data missed by the \( \hat{b}_x^{(1)} \) and \( \hat{k}_t^{(1)} \).
Figure 2.3: Estimates of parameters \( \{\hat{b}_x^{(1)}\} \) and \( \{\hat{b}_x^{(2)}\} \) of the LC models, males and females, fitting periods 1921–2004 and 1975–2004.
For models LC($h$) and PLC(1), the $\{\hat{b}_x^{(1)}\}$ are generally decreasing with age which indicates that mortality at younger ages improves at a faster rate than mortality at older ages (since $\{\hat{b}_t^{(1)}\}$ are decreasing over time as discussed later). Figure 2.4 presents the average of annual improvement of mortality rates at a given age (for ages 60, 61, ..., 100) relative to the average of annual improvement of mortality rates at age 60 for males and females in fitting periods 1921–2004 and 1975–2004 (we only present up to age 100 since the mortality rates at ages above 100 are very spurious). Generally, there is lower mortality improvement at older ages, as the $\{\hat{b}_x^{(1)}\}$ suggest. For both genders, the $\{\hat{b}_x^{(1)}\}$ of models LC($h$) and PLC(1) seem to capture the overall pattern of relative mortality improvement across ages. For example, it captures the (roughly) linear decline of relative mortality improvement across ages for males (note that for males in fitting period 1921–2004, the non-linearity of decline of relative mortality improvement at ages 80+ seems to be captured by a (slight) change of slope of $\{\hat{b}_x^{(1)}\}$) and the similar mortality improvement at ages 60 to 75 for females.

In fitting period 1975–2004, the $\{\hat{b}_x^{(1)}\}$ of the LC models are negative at ages 100+. This is because the observed mortality rates at these ages are, overall, deteriorating in this period.

Note that the $\{\hat{b}_x^{(1)}\}$ are time invariant. This implies that, under the LC model, the rates of decline of mortality rates always maintain the same ratios between different ages over time (Lee, 2000). This modelling assumption does not seem to be satisfied in our application. From Figure 2.4, we note that the rates of mortality improvement at ages 80+ are closer to the rate of improvement at age 60 in period 1975–2004 than in period 1921–2004. Booth et al. (2002) also found a changing age pattern of mortality improvement in their analysis of Australian mortality data. While it might be possible to modify the model to capture the time variation of $\{\hat{b}_x^{(1)}\}$, however, it is unclear whether such modification will increase the performance of the forecast (Lee, 2000). In addition, despite this shortcoming, in our application, the qualities of fit of the LC models are good and the resulting forecasts of mortality rates are reasonable (which is partly because we only consider ages 60 and above). Therefore, we opt to keep the time invariant feature.
Figure 2.4: Average of annual improvement of mortality rates at a given age relative to the average of annual improvement of mortality rates at age 60, males and females, fitting periods 1921–2004 and 1975–2004.

Figure 2.5 presents the estimates of parameters \( \{ \hat{k}_{t}^{(1)} \} \) and \( \{ \hat{k}_{t}^{(2)} \} \) of the LC models for males and females in fitting periods 1921–2004 and 1975–2004. For convenience, in this section, the recalculated \( \{ \hat{k}_{t}^{(1)} \} \) are also denoted by \( \{ \hat{k}_{t}^{(1)} \} \). We make several observations about this figure.

- Adjustment (2.8) only results in minor differences between \( \{ \hat{k}_{t}^{(1)} \} \) between models LC(1) and LC(2).
- The \( \{ \hat{k}_{t}^{(1)} \} \) are similar between models LC(1)* and LC(1) (i.e. the adopted “smoothing” method has only a minor impact on \( \{ \hat{k}_{t}^{(1)} \} \)).
- In most cases, the \( \{ \hat{k}_{t}^{(1)} \} \) are similar between models LC(1) and PLC(1). However, for males in fitting period 1975–2004, the \( \{ \hat{k}_{t}^{(1)} \} \) are different between these models. In addition, for males in fitting period 1921–2004, as compared to model LC(1), the \( \{ \hat{k}_{t}^{(1)} \} \) of model PLC(1) decrease slightly faster over time from 1990.
- In fitting period 1921–2004, as compared to other models, model PLC(2)
results in different pattern of \( \{ \hat{k}_t^{(1)} \} \). In this fitting period, the \( \{ \hat{k}_t^{(1)} \} \) of this model are distorted by the oscillating pattern of observed mortality rates at ages 85+ (discussed in Section 2.6.2).

- The \( \{ \hat{k}_t^{(2)} \} \) are different between models LC(2) and PLC(2), and do not show a consistent pattern across genders and fitting periods. In fitting period 1921–2004, the oscillating pattern of \( \{ \hat{k}_t^{(2)} \} \) in 1921 to 1960 is due to the oscillating pattern of observed mortality rates at ages 85+.

For males in fitting period 1975–2004, although the \( \{ \hat{k}_t^{(1)} \} \) of model PLC(1) decrease faster over time than the \( \{ \hat{k}_t^{(1)} \} \) of models LC(\( h \)), these models suggest a similar trend of overall mortality level in this period (as indicated by the similarity of the fitted and forecasts of \( \hat{e}_{60} \) under these models (Sections 2.6.2 and 2.6.3)). In inferring the trend of mortality level under the LC model, the \( \{ \hat{k}_t^{(1)} \} \) need to be assessed together with the \( \{ \hat{b}_x^{(1)} \} \). Note that in this case, the \( \{ \hat{b}_x^{(1)} \} \) of model PLC(1) are lower at most ages than the \( \{ \hat{b}_x^{(1)} \} \) of models LC(\( h \)) (Figure 2.3).

In both fitting periods, the pattern of \( \{ \hat{k}_t^{(1)} \} \) for models LC(\( h \)) and PLC(1) reflects the pattern of empirical values of \( \hat{e}_{60} \). For example, in fitting period 1921–2004, the \( \{ \hat{k}_t^{(1)} \} \) of these models exhibit the following pattern which is similar to the pattern of empirical values of \( \hat{e}_{60} \) in this period: relatively stagnant for males (or slowly decreasing for females) in 1921 to 1970 and rapidly decreasing over time from 1970 onwards for both genders. This indicates that, in both fitting periods, the \( \{ \hat{k}_t^{(1)} \} \) of models LC(\( h \)) and PLC(1) accurately capture the main time trend of \( \log(\hat{m}_{x,t}) \) at ages 60+.

In fitting period 1921–2004, due to the distortion caused by the oscillating pattern of observed mortality rates at ages 85+ (discussed in Section 2.6.2), the \( \{ \hat{k}_t^{(1)} \} \) of model PLC(2) fail to accurately capture the main time trend of \( \log(\hat{m}_{x,t}) \). The implication of this on the forecasts of mortality rates is discussed in Section 2.6.3.
Figure 2.5: Estimates of parameters \( \hat{k}_t^{(1)} \) and \( \hat{k}_t^{(2)} \) of the LC models, males and females, fitting periods 1921–2004 and 1975–2004.
2.6. APPLICATIONS OF THE LC MODELS

2.6.2 Quality of fit assessment

Table 2.3 presents the values of $R^2$ (calculated according to (2.7)) of the LC models in fitting periods 1921–2004 and 1975–2004. Since $R^2$ is calculated by applying a logarithmic function to the mortality rates, the differences in variability of the death rates at different ages are taken into account (i.e. the qualities of fit at ages with low mortality rates affect the calculation equally as the qualities of fit at ages with high mortality rates). Several observations from this table follow.

- In all cases, the values of $R^2$ are above 90% for most LC models. This indicates that generally, the overall qualities of fit of these models are good.

- The “smoothing” method (to fix the peculiarity of forecasts of mortality rates at very high ages (Section 2.6.1)) adopted for model LC(1) has only minimal impact on the quality of fit of this model (as the values of $R^2$ are similar between models LC(1)* and LC(1)). This is also the case for model LC(2).

- The “smoothing” method adopted for models PLC($h$) improves the qualities of fit of these models. For males in fitting period 1975–2004, the low value of $R^2$ of model PLC(1)* is due to the poor fit of this model at very high ages in which the smoothing of the estimates of parameters at these ages significantly improves the overall fit (note that the value of $R^2$ of model PLC(1) is much higher than model PLC(1)*).

- The fits of models LC($h$) are slightly better than models PLC($h$). This is because $R^2$ is calculated by giving equal weight to the log of observed mortality rates (differences of exposure are not taken into account), which is in line with the fitting method of models LC($h$).

- The fits of models LC(2) and PLC(2) are generally slightly better than models LC(1) and PLC(1) respectively.

- The fits are generally better for females (than males) and for fitting period 1975–2004 (than fitting period 1921–2004).
Table 2.3: Values of $R^2$ for the LC models.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>LC(1)*</td>
<td>96.22%</td>
<td>97.74%</td>
<td>97.92%</td>
<td>98.77%</td>
</tr>
<tr>
<td>LC(1)</td>
<td>96.13%</td>
<td>97.65%</td>
<td>97.31%</td>
<td>98.41%</td>
</tr>
<tr>
<td>LC(2)</td>
<td>97.50%</td>
<td>98.61%</td>
<td>97.64%</td>
<td>98.51%</td>
</tr>
<tr>
<td>PLC(1)*</td>
<td>89.90%</td>
<td>95.10%</td>
<td>69.46%</td>
<td>97.90%</td>
</tr>
<tr>
<td>PLC(1)</td>
<td>93.26%</td>
<td>96.33%</td>
<td>96.16%</td>
<td>98.10%</td>
</tr>
<tr>
<td>PLC(2)</td>
<td>90.91%</td>
<td>96.61%</td>
<td>96.22%</td>
<td>98.27%</td>
</tr>
</tbody>
</table>

$R^2$ measures the overall quality of fit. In the following we examine the quality of fit at an individual age.

Suppose we consider ages $x_1, x_2, \ldots, x_X$ and years $t_1, t_2, \ldots, t_T$; then the arithmetic mean of the absolute value of relative fitting error of mortality rates at age $x$ across years is defined as

$$EFA_x = \frac{1}{T} \sum_{i=1}^{T} \left| \frac{\hat{m}_{x,t_i} - \tilde{m}_{x,t_i}}{\hat{m}_{x,t_i}} \right|. \quad (2.19)$$

Similarly, the arithmetic mean of the absolute value of relative fitting error of mortality rates in year $t$ across ages is defined as

$$EFY_t = \frac{1}{X} \sum_{i=1}^{X} \left| \frac{\hat{m}_{x_i,t} - \tilde{m}_{x_i,t}}{\hat{m}_{x_i,t}} \right|. \quad (2.20)$$

Figure 2.6 presents $EFA_x$ for $x = 60, 61, \ldots, 110$ for males in fitting period 1921–2004. We make several observations about this figure.

- The qualities of fit of one-term LC models (i.e. LC(1)*, LC(1), PLC(1)* and PLC(1)) are similar at ages 60 to 100. The fits of these models are good at ages 60 to 85. However, the fits of these models are worsening from age 85 onwards.

- The fits of two-term LC models (i.e. LC(2) and PLC(2)) are better than one-term LC models at ages 85 to (around) 100. This feature is only observed in fitting period 1921–2004; in fitting period 1975–2004,
the fits of one-term LC models are as good as two-term LC models at these ages. The reason for this is discussed later.

- For all LC models, the fitting errors are high and spurious at ages above 100. In addition, there is no model which consistently has superior fit at these ages. Note that the observed mortality rates at these ages fluctuate significantly from one year to the next. It is difficult to obtain a good fit to observed mortality rates which exhibit such a pattern.

- The fits of models LC\((h)\) (including LC\((1)^*\)) seem to improve at ages above 105. This is because at these ages, there are many years with missing observed mortality rates in which these missing observed mortality rates are estimated under a Gompertz’ formula (Section 2.6.1). This results in a less spurious pattern of “observed” mortality rates at these ages than at ages 100 to 105 and hence better fits of models LC\((h)\) at these ages. In fitting period 1975–2004, an improvement of fit at these ages is also observed for models PLC\((h)\).

- The fit of model PLC\((1)\) is better than model PLC\((1)^*\) at ages above 105.

- The fit of model PLC\((2)\) is consistently worsening from age 100 onwards. This feature is specific for males in fitting period 1921–2004; in other cases, the fit of model PLC\((2)\) is similar to model PLC\((1)\) at these high ages.

In other cases (i.e. for females and for fitting period 1975–2004), generally, we have similar observations. The main exception is that in fitting period 1975–2004, the fits of one-term LC models are only measurably worsening from age 100 onwards.

Figure 2.7 presents \(EFY_t\) for males in fitting period 1921–2004. Note that in 1921 to 1960, the fitting errors of LC models (excluding LC\((2)\)) are high and exhibit a systematic fluctuation. Similar systematic fluctuation of fitting errors during this period is also observed for females. This systematic fluctuation is because the observed mortality rates at ages 85+ exhibit the
following oscillating pattern: increasing from 1921 to early 1930s, then decreasing up to 1945 and subsequently increasing until 1960. The oscillating pattern of observed mortality rates is stronger at older ages. The impact of the oscillating pattern of observed mortality rates on the qualities of fit of the LC models (i.e. how this oscillating pattern of observed mortality rates causes fluctuation of fitting errors) can be traced from Figure 2.8 which presents the observed and fitted mortality rates of models LC(2) and PLC(1) at age 94.

Model LC(2) captures the oscillating pattern of observed mortality rates at ages 85+ (illustrated in Figure 2.8 at age 94). This results in low fitting errors (and absence of its systematic fluctuation) of model LC(2) in 1921 to 1960 (Figure 2.7). Model PLC(2) also captures the oscillating pattern (of observed mortality rates), however, only until age 100. The oscillating pattern is captured by \( \hat{\theta}_x^{(2)} \) and \( \hat{\gamma}_t^{(2)} \). In Figure 2.5, note that in fitting
period 1921–2004, the pattern of \( \hat{k}^{(2)}_t \) in 1921 to 1960 resembles the oscillating pattern of observed mortality rates in this period. However, for model PLC(2), the oscillating pattern distorts \( \hat{k}^{(1)}_t \). This indicates that model PLC(2) is less stable as compared to model LC(2). The reduced stability of model PLC(2) (as compared to model LC(2)) is not due to the missing mortality data; fitting this model only to ages 60 to 100 still results in distorted \( \hat{k}^{(1)}_t \). The capability of two-term LC models in capturing the oscillating pattern of observed mortality rates results in better fits of these models over one-term LC models at ages 85 to 100 (Figure 2.6). Although model PLC(1) does not capture the oscillating pattern, the fitted mortality rates of this model are at the middle of the oscillating observed rates (as illustrated in Figure 2.8). This is also the case for other one-term LC models.

In 1960 to 2004, the fitting errors of the LC models appear to be random. The relative fitting errors (2.20) of these models are between 5% and 20% in
Figure 2.8: Observed and fitted mortality rates at age 94.

In this period. The fitting performance for females is similar in this period.

The fitting performances of the LC models in fitting period 1975–2004 are similar to the fitting performances of these models in fitting period 1921–2004 in the corresponding years.

In estimating the eventual likelihood of death, mortality rates at younger ages are more important than at older ages. The fitting error of life expectancy presents the overall quality of fit of the model by taking into account the differences of importance of mortality rates at different ages. Figure 2.9 presents the fitting error (empirical values minus fitted) of ε60 (age 60 is the youngest eligible age to enter a reverse mortgage contract) of the LC models for males and females in fitting periods 1921–2004 and 1975–2004. Note that in all cases, the fitting error of ε60 is less than 0.15 in absolute value. This indicates that the fits of the LC models are good in general. In addition, the best fit is generally achieved by model PLC(2) and followed by model PLC(1). This indicates that the weighting under models PLC(h) is more accurate than under models LC(h). Note that this comparison takes into account the adjustment (2.8) for models LC(h).

Figure 2.10 presents the residuals of the LC models by age and year separately, for males in fitting period 1921–2004. For models LC(h), we examine the SVD residuals (εx,t (2.9)), while for models PLC(h), we examine the deviance residuals (rx,t (2.13)). A good fit is indicated by the absence of systematic pattern of the corresponding residuals across ages and years. In
2.6. APPLICATIONS OF THE LC MODELS

Figure 2.9: Fitting error of $\hat{\epsilon}_{60}$ of the LC models, males and females, fitting periods 1921–2004 and 1975–2004.

In the following we present the analysis of the residuals of the LC models.

For all LC models, the central values of the residuals across ages and years are around zero.

The first column of Figure 2.10 indicates that the dispersion of SVD residuals of models LC($h$) are larger at ages 90+ than at ages below 90. Comparison of the SVD residuals before and after adjustment (2.8) suggests that this is not a feature of the adjustment. This shows that the structure of models PLC($h$), in particular the homoscedasticity assumption of $\epsilon_{x,t}$, is not correct. Incorporation of further terms from the SVD does not solve this error of modelling structure. The implication of this error of modelling structure is that the higher variability of the log of observed mortality rates at older ages is not taken into account under models LC($h$). Note that across
ages, the dispersions of the deviance residuals of models PLC($h$) are more homogeneous than the dispersions of SVD residuals of models LC($h$). This indicates that the structures of models PLC($h$) are more appropriate than models LC($h$). This is in line with the better fit of $\hat{e}_{60}$ under models PLC($h$) than under models LC($h$) (Figure 2.9).

The second column of Figure 2.10 indicates that there are larger dispersions of the deviance residuals of models PLC($h$) from (around) 1965 onwards. This suggests that the pattern of observed mortality rates in 1965–2004 is different to in 1921–1964. Indeed, there is a greater variability of observed mortality rates at ages 60 to 90 from 1970 onwards. The differences in the pattern of observed mortality rates between these periods might be due to the differences in the rate of mortality improvement in these periods. There are also differences in the dispersion of SVD residuals in these periods, however, this is due to the peculiarity of observed mortality rates at ages above 100.

Lastly, the residuals of one-term LC models exhibit an oscillating pattern in 1921–1960. This is due to the oscillating pattern of observed mortality rates at ages 85+ in this period (which is not captured by one-term LC models) as discussed before. Note that for two-term LC models, the oscillating pattern of the residuals in this period is much less observed.

In other cases, generally, we have similar observations. The main exception is that in fitting period 1975–2004, the homogeneity of the dispersions of residuals by year are better satisfied (illustrated in Figure 2.11 for females under model PLC(1)).
Figure 2.10: Residuals of the LC models, males, fitting period 1921–2004.
Figure 2.11: Residuals of model PLC(1), females, fitting period 1975–2004.

2.6.3 Comparison of forecasts

For each \( \hat{k}_t^{(j)}: j = 1, 2, \ldots, h \) of each LC model, a time series model selection procedure is performed and an optimal ARIMA model is chosen. We follow the ARIMA model selection procedure described in Box et al. (1994). Generally, similar patterns of \( \{\hat{k}_t^{(j)}\} \) result in similar ARIMA models. For example, for females in fitting period 1975–2004, ARIMA (0, 1, 1) is optimal to forecast \( \{\hat{k}_s^{(1)}: s > 2004\} \) for models LC(1)*, LC(1), LC(2) and PLC(1) (the estimate \( \hat{\mu} \) of ARIMA (0, 1, 1) for these models are \(-0.14, -0.13, -0.13\) and \(-0.14\) respectively), while for model PLC(2), ARIMA (0, 1, 0) is suitable (with \( \hat{\mu} \) of \(-0.13\)).

In fitting period 1975–2004, time series analysis suggests that the \( \{\hat{k}_t^{(2)}\} \) of model LC(2) fluctuates randomly around zero (i.e. the estimates \( \hat{\mu}, \{\hat{\phi}_t\} \) and \( \{\hat{\theta}_t\} \) of the optimal ARIMA model are not statistically different from zero). This is in line with the pattern of \( \{\hat{k}_t^{(2)}\} \) in this fitting period (Figure 2.5). As a result, the forecasts of \( \{\hat{k}_s^{(2)}: s > 2004\} \) are zero for model LC(2). For model PLC(2), ARIMA (0, 1, 1) without a drift (i.e. \( \hat{\mu} = 0 \)) is chosen to forecast \( \{\hat{k}_s^{(2)}: s > 2004\} \).

Table 2.4 presents the parameter estimates of the chosen ARIMA model for model PLC(1) (PLC(1) is our chosen model as discussed later). For this model, ARIMA (0, 1, 1) is found to be optimal in all cases.
Table 2.4: ARIMA (0, 1, 1) parameter estimates, with (standard errors), model PLC(1).

<table>
<thead>
<tr>
<th>Fitting period</th>
<th>( \hat{\mu} )</th>
<th>( \hat{\theta}_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1921–2004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Males</td>
<td>-0.0797 (0.0233)</td>
<td>-0.2222 (0.1093)</td>
</tr>
<tr>
<td>Females</td>
<td>-0.0783 (0.0148)</td>
<td>-0.3515 (0.1043)</td>
</tr>
<tr>
<td>1975–2004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Males</td>
<td>-0.4650 (0.0488)</td>
<td>-0.4523 (0.1725)</td>
</tr>
<tr>
<td>Females</td>
<td>-0.1430 (0.0157)</td>
<td>-0.5546 (0.1627)</td>
</tr>
</tbody>
</table>

From the forecasts of \( \{ k^{(j)}_s : s > 2004, j = 1, 2, \ldots, h \} \), the forecasts of mortality rates are computed. We examine forecasting equations (2.14) (referred to as forecasting method 1) and (2.15) (referred to as forecasting method 2). Note that forecasting method 1 results in forecasts of mortality rates which continue smoothly from the fitted rates, while forecasts under forecasting method 2 generally continue smoothly from the most recent observed mortality rates (unless there is a large discrepancy between \( \hat{k}^{(j)}_{2004} \) and \( \hat{k}^{(j)}_{2005} \)). In our application, forecasting method 1 results in a significant discontinuity problem (Section 2.5.1) only at ages 100+. Note that the observed mortality rates at ages 100+ fluctuate significantly and hence, continuation of the forecasts from the most recent observed rates might result in peculiar forecasts at these ages.

Figure 2.12 presents the forecasts of mortality rates in 2040 of the LC models for males in fitting period 1921–2004. Note that for both forecasting methods 1 and 2, excluding model PLC(2), forecasts under all LC models are similar at ages 60 to 90. Forecasts under model PLC(2) are different to forecasts under other models at all ages. Under models LC(1), LC(2) and PLC(1), forecasting methods 1 and 2 result in similar forecasts at ages 60 to 90, while at ages 90 to 100, there are only minor differences in the forecasts under these forecasting methods.

Under forecasting method 1, forecasts under models LC(1)* and PLC(1)* are peculiar at ages 100+. The “smoothing” method (to fix the peculiarity of forecasts of mortality rates at very high ages (Section 2.6.1)) adopted for models LC(\( h \)) and PLC(\( h \)) (Section 2.6.1) fixes this peculiarity. Note that at
Figure 2.12: Forecasts of mortality rates in 2040 of the LC models, males, fitting period 1921–2004.

At ages 90+, as compared to forecasting method 1, forecasting method 2 results in more similar forecasts between the LC models (excluding model PLC(2)). However, under forecasting method 2, forecasts at ages 100+ are peculiar. This is due to the continuity of the forecasts from the last year’s observed rates under forecasting method 2.

In other cases, observations from the forecasts of mortality rates are similar.

Figure 2.13 presents the empirical values and forecasts of $\hat{e}_{60}$ of the LC models under forecasting method 1 for males and females in fitting periods 1921–2004 and 1975–2004. Excluding model PLC(2), the forecasts of $\hat{e}_{60}$ under all LC models are similar in all cases. This is also true for forecasting method 2. This is in line with observation from Figure 2.12 (i.e. at ages 60 to 90, excluding model PLC(2), the forecasts of mortality rates are similar between the LC models). Forecasts of $\hat{e}_{60}$ are similar between forecasting methods 1 and 2. Note that fitting period 1975–2004 results in a greater increase in $\hat{e}_{60}$ than fitting period 1921–2004.

Figure 2.14 presents the empirical values and forecasts of $\hat{e}_{90}$ (life expectancy at age 90) under forecasting methods 1 and 2 for males in fitting
2.6. APPLICATIONS OF THE LC MODELS

Figure 2.13: Empirical values and forecasts of $\hat{e}_{60}$ of the LC models under forecasting method 1, males and females, fitting periods 1921–2004 and 1975–2004.

periods 1921–2004 and 1975–2004. Forecasts under both forecasting methods are included for comparison. Note that model PLC(2) results in very different forecasts as compared to other models. Under model PLC(1), forecasting method 2 results in higher forecasts than forecasting method 1.

In fitting period 1975–2004, model PLC(2) results in a discontinuity problem of the forecasts of $\hat{e}_{90}$ under both forecasting methods. Therefore, under this model, not only is there a discrepancy between the observed and fitted mortality rates at ages 90+ in 2004, but also there is a (large) discrepancy between $\hat{k}^{(2)}_{2004}$ and $\hat{k}^{(2)}_{2005}$ (which is due to the high fluctuation of $\{\hat{k}^{(2)}_t\}$ under this model (Figure 2.5)). In this fitting period, under model PLC(2), there is also a discontinuity problem of the forecasts of $\hat{e}_{60}$ (Figure 2.13), although
Figure 2.14: Empirical values and forecasts of $\hat{e}_{90}$ of the LC models under forecasting methods 1 and 2, males, fitting periods 1921–2004 and 1975–2004.

In the following, we present a comparison of the forecasts of $\hat{e}_{90}$ under forecasting method 2 between the LC models and followed by a comparison of the forecasts under forecasting method 1. Note that under forecasting method 2, since the assumed baseline mortality rates (Section 2.5.1) in 2004 are the same between the LC models, differences in the forecasts of $\hat{e}_{90}$ reflect only the differences of the forecasts of mortality improvement at ages 90+ between these models.

Under forecasting method 2, the forecasts of $\hat{e}_{90}$ are similar under models LC(1) and LC(2). Indeed, the forecasts of mortality improvement under these models are similar at ages 60+ in both fitting periods. This indicates
that the second terms of model LC(2) (i.e. \( \hat{b}_x^{(2)} \) and \( \hat{k}_t^{(2)} \)), which capture the oscillating pattern of observed mortality rates at ages 85+ in 1921–1960 (Section 2.6.2), have only minimal impact on the forecasts of mortality improvement.

As compared to models LC(\( h \)), model PLC(1) results in lower forecasts of \( \hat{e}_{90} \) in fitting period 1921–2004 and higher forecasts in fitting period 1975–2004. This is due to the differences of the forecasts of mortality improvement at ages 90+ (95+ for females) between these models. The differences of the forecasts at ages 90 to 100 are genuine features of these models. The reason for the differences at ages 100+ is described in the following.

In fitting period 1921–2004, the “jump” problem (Section 2.6.1) causes an upward bias in \( \hat{b}_x^{(1)} \) of models LC(\( h \)) at ages 100+ and hence, as compared to model PLC(1), forecasts of mortality improvement at these ages are higher under models LC(\( h \)). In fitting period 1975–2004, due to the smoothing of \( \hat{b}_x^{(1)} \) under model PLC(1), in which we choose not to fully reflect the rapid decrease of raw \( \hat{b}_x^{(1)} \) at ages 100+, \( \hat{b}_x^{(1)} \) at these ages decrease at a slower rate for model PLC(1) as compared to models LC(\( h \)). As a result, the forecasts of mortality improvement (deterioration) under models LC(\( h \)) are lower (higher) than under model PLC(1) at these ages in this fitting period.

For females, observations from fitting period 1921–2004 are similar, while in fitting period 1975–2004, the forecasts of \( \hat{e}_{90} \) under models LC(\( h \)) and PLC(1) are similar. The similarity of the forecasts in fitting period 1975–2004 is due to the “jump” problem of \( \hat{b}_x^{(1)} \) of models LC(\( h \)) which offsets the rapid decrease of \( \hat{b}_x^{(1)} \) at ages 100+ (in this fitting period, the “jump” problem of \( \hat{b}_x^{(1)} \) is more pronounced for females than males (Figure 2.3)). Model PLC(2) also results in very different forecasts as compared to other models and a discontinuity problem in fitting period 1975–2004.

At ages 100+, model PLC(1) is likely to result in more accurate forecasts of mortality improvement than models LC(\( h \)). In fitting period 1921–2004, due to the “jump” problem of \( \hat{b}_x^{(1)} \), models LC(\( h \)) result in forecasts which are counterintuitive: older ages experience higher mortality improvement than younger ages (for females, this is also true for fitting period 1975–2004). In addition, for males in fitting period 1975–2004, models LC(\( h \)) results in
forecasts of mortality rates which are too rapidly deteriorating at these ages (which is not in line with the pattern of the observed mortality rates in the past).

In the following we discuss the forecasts of $\hat{e}_{90}$ under forecasting method 1. Note that under forecasting method 1, the differences in the forecasts of $\hat{e}_{90}$ under the LC models are due to differences in the the fitted mortality rates in 2004 and the forecasts of mortality improvement at ages 90+ between these models (Section 2.5.1).

In fitting period 1921–2004, the forecasts of $\hat{e}_{90}$ under model LC(1) are higher than the forecasts under models LC(2) and PLC(1). The differences of the forecasts between models LC(1) and LC(2) are mainly due to differences in the fitted mortality rates in 2004 under these models. As compared to model PLC(1), model LC(2) results in slightly lower forecasts of $\hat{e}_{90}$ in early forecast years and similar forecasts in later forecast years. This is because model LC(2) results in higher forecasts of mortality improvement at ages 90+ than model PLC(1). The observations for females are similar.

In fitting period 1975–2004, observations from the comparisons of the forecasts of $\hat{e}_{90}$ under forecasting method 1 are similar to observations under forecasting method 2 for males and females.

The differences of the forecasts of mortality improvement between the LC models have been discussed above. In the following we discuss the differences of the fitted mortality rates in 2004 ($\hat{m}_{x,2004}$) between these models.

At ages 90 to 100, $\{\hat{m}_{x,2004}\}$ are similar between the LC models. At these ages, the qualities of fit of $\{\hat{m}_{x,2004}\}$ of these models are good. At ages 100+, model PLC(1) results in lower $\{\hat{m}_{x,2004}\}$ than models LC($h$). Specifically, at these ages, under model PLC(1), $\{\hat{m}_{x,2004}\}$ increase with age at a decreasing rate, while under models LC($h$), they increase with age at an increasing rate. This is due to differences in $\{\hat{a}_x\}$ under these models (Figure 2.2). At these ages, the qualities of fit of $\{\hat{m}_{x,2004}\}$ are poor under all of the LC models. However, note that the observed mortality rates at these ages are very spurious. The poor qualities of fit of $\{\hat{m}_{x,2004}\}$ at ages 100+ result in a discontinuity of the forecasts of $\hat{e}_{90}$ under these models (Figure 2.14).

We believe that model PLC(1) results in more suitable $\{\hat{m}_{x,2004}\}$ than
models LC(h) due to the following reasons. Firstly, \( \{\hat{a}_x\} \) and \( \{\hat{b}_x^{(1)}\} \) of models LC(h) suffer from a “jump” problem. Secondly, as compared to models LC(h), the weighting in the fitting of model PLC(1) is more accurate (Section 2.4.2) and hence, model PLC(1) results in more appropriate estimates of the parameters (i.e. \( \{\hat{a}_x\} \), \( \{\hat{b}_x^{(1)}\} \) and \( \{\hat{k}_t^{(1)}\} \)) as compared to models LC(h).

Lastly, model PLC(1) results in a more appropriate pattern of \( \{\hat{\mu}_{x, 2004}\} \) at ages 100+ than models LC(h). Therefore, assuming model PLC(1) results in more accurate forecasts of mortality improvement at ages 90 to 100 (which is likely, given this model results in more suitable \( \{\hat{b}_x^{(1)}\} \) and \( \{\hat{k}_t^{(1)}\} \)), under forecasting method 1, this model is likely to result in more accurate forecasts of mortality rates at ages 90+ than models LC(h).

The discontinuity problem of forecasting method 1 is addressed in Section 2.6.6.

Discussion on model PLC(2)

Model PLC(2) results in different forecasts of mortality rates than other models in all cases. We believe that the forecasts under model PLC(2) are inferior compared to forecasts under other models. This is due to the following two reasons. Firstly, the forecasts under this model do not accurately reflect the pattern of the observed rates. For example, for males in fitting period 1921–2004, the forecasts of improvement of \( \hat{\epsilon}_{90} \) are too high in comparison with the pattern of empirical values of \( \hat{\epsilon}_{90} \) (Figure 2.14). Secondly, there are indications that the estimates of mortality indexes of this model are distorted or exhibit an artificial pattern which are described in the following.

In fitting period 1921–2004, \( \{\hat{k}_t^{(1)}\} \) of model PLC(2) are distorted by the oscillating pattern of the observed mortality rates at ages 85+ and therefore, it fails to accurately capture the main time trend of \( \{\log (\hat{\mu}_{x, t})\} \) (Section 2.6.1 and 2.6.2).

In fitting period 1975–2004, \( \{\hat{k}_t^{(2)}\} \) of model PLC(2) seems to exhibit an artificial pattern. In this fitting period, which is relatively short, the pattern of mortality improvement is generally similar across ages, and the second terms of the LC model (i.e. \( \{\hat{b}_x^{(2)}\} \) and \( \{\hat{k}_t^{(2)}\} \)) are unlikely to be necessary. As
compared to model PLC(1), relative to the number of additional parameters, the improvement in the maximum likelihood value of model PLC(2) is too small in this fitting period. For example, for males, the value of the log-likelihood ratio statistic \( L^2 \) of model PLC(1) is 1,985, while the value of \( L^2 \) of model PLC(2) is 1,639 (the \( L^2 \) values are calculated before the smoothing of \( \{\hat{\alpha}_x\} \) and \( \{\hat{\beta}_x^{(j)}\} \)). Therefore, the systematic pattern of \( \{\hat{k}_t^{(2)}\} \) of model PLC(2) in this fitting period (which results in ARIMA \((0, 1, 1)\)) is likely to be artificial. Note that for model LC(2), from the time series analysis, \( \{\hat{k}_t^{(2)}\} \) are found to be insignificant in this fitting period. In addition, for males, the fluctuation of \( \{\hat{k}_t^{(2)}\} \) under model PLC(2) in this fitting period is very high (Figure 2.5) which indicates its peculiarity.

As compared to model LC(2), model PLC(2) appears to be less stable. This is because in the fitting of model PLC(2), maximisation of the likelihood function relies on a numerical method, while in the fitting of model LC(2), the exact solution to the least squares problem can be found under the SVD method.

To conclude, model PLC(2) is not suitable for forecasting mortality rates in our application. Excluding model PLC(2), the forecasts of mortality rates are different between the LC models only at ages 90+. At ages 100+, model PLC(1) is likely to result in more accurate forecasts than other models. Lastly, forecasting method 2 is unsuitable for forecasting at ages 100+.

In the following, model PLC(2) is not considered any further.

### 2.6.4 Out of sample forecasts

The purposes of performing out of sample forecasts of mortality rates are to have an indication of the accuracy of forecasts and to provide a further comparison of the forecasts between the LC models. Given the length of available data, we are only able to assess the short term performance of the out of sample forecasts.

In performing out of sample forecasts, the fitting periods are chosen to resemble the fitting periods of our main forecasts. Specifically, we consider fitting periods 1931–1994 and 1975–1994 which are chosen to resemble fitting
periods 1921–2004 and 1975–2004 respectively. Note that the weight of the recent trend of mortality (i.e. the recent rapid improvement of mortality from 1970) is roughly similar in the chosen fitting period and in the fitting period it resembles.

In the out of sample forecasts, to fix the peculiarity of forecasts of mortality rates at very high ages, we also apply the “smoothing” method described in Section 2.6.1. Note that the adopted “smoothing” method is different between models LC\( (h) \) and PLC\((1)\).

For each \( \{ \hat{k}_t^{(j)} : j = 1, 2, \ldots, h \} \) of each LC model, time series model selection procedure is performed and an optimal ARIMA model is chosen. Forecasting methods 1 and 2 are examined. The forecasts of mortality rates in 1995 to 2004 are compared with the observed mortality rates in this period.

We calculate the arithmetic mean of absolute value of relative forecast error of mortality rates at age \( x \) across forecast years according to

\[
EF_{Ax} = \frac{1}{10} \sum_{t=1995}^{2004} \frac{\hat{m}_{x,t} - \tilde{m}_{x,t}}{\tilde{m}_{x,t}},
\]

and we also calculate the arithmetic mean of absolute value of relative forecast error of mortality rates in year \( t \) across ages according to

\[
EF_{Y_t} = \frac{1}{51} \sum_{x=60}^{110} \frac{\hat{m}_{x,t} - \tilde{m}_{x,t}}{\tilde{m}_{x,t}}.
\]

Figure 2.15 presents the forecast errors (2.21) for females in fitting period 1931–1994 with the forecasts calculated under forecasting method 1, while Figure 2.16 presents the forecast errors (2.21) for females in fitting period 1975–1994 with the forecasts calculated under forecasting method 2 (except the forecasts under models LC\((2)\) and PLC\((1)\), where they are calculated under forecasting method 1). These two cases are chosen to illustrate the general observations. In the following, we present a comparison of forecast performance between the LC models.

The “smoothing” method adopted for models LC\((h)\) and PLC\((1)\) improves the forecast performance at very high ages. Note that at ages 100+,
the forecast performance of models LC(1) and PLC(1) is better than models LC(1)* and PLC(1)* respectively. In addition, generally, forecasting method 1 results in more accurate forecasts at ages 100+ as compared to forecasting method 2. This is because forecasting method 2 results in peculiar forecasts at these ages (due to the reason described in Section 2.6.3). At ages 60 to 100, the forecast performance of forecasting methods 1 and 2 is similar (there are only minor differences of forecast performance at ages 90 to 100 between these forecasting methods in which forecasting method 1 generally performs better). Therefore, in the following, we restrict the comparison to models LC(1), LC(2) and PLC(1) under forecasting method 1.

In fitting period 1931–1994, the forecast performances at ages 60 to 90 are similar between the LC models. At ages 90+, the forecast performances of models LC(2) and PLC(1) are better than model LC(1). This is because of the better fit of \( \hat{m}_{x,1994} \) of models LC(2) and PLC(1) over model LC(1) at
Figure 2.16: The $EFrA_x$ under forecasting method 2 (except models LC(2) and PLC(1)), females, fitting period 1975–1994.

In fitting period 1975–1994, the forecast performances at ages 60 to 100 are similar between the LC models. At ages 100+, the forecast performance of model PLC(1) is better than that of models LC(1) and LC(2). This is likely due to the “jump” problem of $\{\hat{a}_x\}$ and $\{\hat{b}^{(1)}_x\}$ of models LC($h$). This indicates the unsuitability of the “smoothing” method adopted for models LC($h$) (Section 2.6.1).

In our main forecasts, differences of the forecasts of mortality rates at ages 90+ between models PLC(1) and LC($h$) are quite pronounced in later forecast years (Figure 2.14). Due to limited data, we are not able to assess this in the out of sample forecasts.
In the following we present the comparison of forecast performance between fitting periods and genders.

Forecasts from fitting period 1975–1994 are more accurate than forecasts from fitting period 1931–1994. For example, for females, the forecast errors (2.21) at ages 60 to 100 are mostly less than 20% in fitting period 1931–1994 (Figure 2.15), while in fitting period 1975–1994, they are mostly less than 10% (Figure 2.16).

The forecast errors (2.21) at ages 100+ are quite high. However, note that the observed mortality rates at these ages fluctuate significantly.

The forecast performance of females is better than males with the differences being more pronounced in fitting period 1931–1994 (than in fitting period 1975–1994). In fitting period 1931–1994, as compared to females, the forecast error (2.21) for males is higher by 9% on average, while in fitting period 1975–1994, it is higher by 5% on average. This is due to differences in the pattern of observed mortality rates between males and females in these fitting periods.

Figure 2.17 presents the forecast errors (2.22) under model PLC(1) with forecasting method 1 for males and females in fitting periods 1931–1994 and 1975–1994. Note that the forecast performance is deteriorating over time. The deterioration is more rapid for males (than females) and for fitting period 1931–1994 (than fitting period 1975–1994).

To conclude, from the out of sample forecasts, model PLC(1) with forecasting method 1 is preferable for forecasting mortality rates. The forecasts from fitting period 1975–2004 are likely to be more accurate than the forecasts from fitting period 1921–2004 (as indicated by the superiority of forecast performance of fitting period 1975–1994 over fitting period 1931–1994) and the forecasts for females are likely to be more accurate than the forecasts for males.

In the out sample forecasts, we only assess the first 10 years of forecasts in a period in which the rapid improvement of mortality is still observed. Therefore, the findings from the assessment of out of sample forecasts are likely to be valid only in the short term future (where the rapid improvement of mortality is still likely to continue). Over the long run, it is likely that
there will be changes in the trend of mortality.

2.6.5 Model selection

To forecast mortality rates, we consider models LC(1), LC(2), PLC(1) and PLC(2). In addition, we also consider forecasting methods 1 and 2.

From the assessment of the resulting forecasts, model PLC(2) is found to be unsuitable. In the following, we only consider models LC(1), LC(2) and PLC(1).

In fitting period 1921–2004, model LC(2) captures the oscillating pattern of the observed mortality rates at ages 85+ in 1921–1960. However, this has only minimal impact on the resulting forecasts of mortality improvement (as indicated by the similarity of the forecasts of mortality improvement between models LC(1) and LC(2)).

As compared to other LC models, model PLC(1) results in the most accurate weighting in the fitting of the model. In addition, the “smoothing” method (Section 2.6.1) adopted for model PLC(1) is more suitable than the “smoothing” method adopted for models LC(h). The “smoothing” method adopted for models LC(h) results in a “jump” problem for the estimated parameters. This “jump” problem results in some undesirable properties of the forecasts of mortality rates at ages 100+ (Section 2.6.3). From the assessment
of the out of sample forecasts, the “jump” problem is found to adversely impact the accuracy of the forecasts. Lastly, as compared to the homoscedastic error structure of models LC(h), the heteroscedasticity of \(\{D_{x,t}\}\) under model PLC(1) is more suitable for a semi-parametric bootstrap (in the estimation of the prediction interval of future mortality rates). The heteroscedasticity of \(\{D_{x,t}\}\) under model PLC(1) results in a higher variability of \(D_{x,t}\) at ages in which the exposure are lower and, therefore, in the implementation of semi-parametric bootstrap, a wider prediction interval of future mortality rates will be estimated at these ages (as compared to ages with lower exposure).

Forecasting method 2 results in peculiar forecasts of mortality rates at ages 100+. From the assessment of the out of sample forecasts, this peculiarity is found to adversely impact the accuracy of the forecasts. In the out of sample forecasts, as compared to forecasting method 2, forecasting method 1 results in similar forecast performance at ages 60 to 100 and better forecast performance at ages 100+. Therefore, forecasting method 1 is preferable.

To conclude, due to the reasons above, the forecasts of mortality rates under model PLC(1) with forecasting method 1 are chosen. Note that these forecasts suffer from a discontinuity problem at ages 100+ (Section 2.6.3). This discontinuity problem is addressed in Section 2.6.6.

### 2.6.6 Forecast adjustments

In this section, we smooth the forecasts of mortality rates under model PLC(1) (with the forecasts calculated under forecasting method 1). From the smoothed forecasts of mortality rates, the forecasts of \(\{q_x\}\) (one-year death probability from exact age \(x\)) are calculated. Following that, the forecasts of \(\{q_x\}\) are refined by basing them on \(\{q_x\}\) presented in the Australian life tables (ALT) for the period 2005–2007 published by the ABS (ABS, 2008b).
2.6. APPLICATIONS OF THE LC MODELS

Smoothing

There are several methods proposed in literature to incorporate smoothing into the LC model. This includes D’Amato et al. (2011) and Delwarde et al. (2007). D’Amato et al. (2011) incorporate smoothing into the LC model by smoothing the observed mortality rates under a Functional Demographic Model (FDM) and penalized splines. Although, for their dataset, this results in improvements in both quality of fit and forecast performance, the resulting estimates of the parameters of the LC model are not smooth (see Figure 6 of D’Amato et al. (2011)). Therefore, this method is unlikely to result in smooth forecasts of mortality rates.

Delwarde et al. (2007) smoothed the estimates of parameters under a penalized least squares method for the LC model and penalized log-likelihood method for the PLC model. The smoothing process is done by adding a term, which is a function of a smoothing parameter, in the objective function that we need to minimize in the estimation process. This additional term penalizes the irregular shape of \( \hat{b}_x \). Delwarde et al. (2007) further proposed a cross validation method to select an optimal value for the smoothing parameter. Note that under this method, the smoothing is done within the estimation process. The advantage of this method is that it allows an explicit trade off between the fidelity to data and the smoothness. The disadvantage of this method is that improper choice in the value of the smoothing parameter might result in inaccurate forecasts since the estimated parameters might not accurately reflect the trend of the data (although the difficulty in determining a suitable value of the smoothing parameter might be reduced by the proposed cross validation method).

To smooth the forecasts of mortality rates under model PLC(1) (with forecasting method 1), we choose to adopt a simple method. We smooth the forecasts by extending the smoothing of \( \hat{b}_x^{(1)} \) (as described in Section 2.6.1) to all ages. Figure 2.18 presents the smoothed (by smoothing the \( \hat{b}_x^{(1)} \) at all ages, and denoted “smoothed” in the figure) and original forecasts of mortality rates (denoted “original” in the figure) in a selection of forecast years for males in fitting period 1921–2004. Delwarde et al. (2007) stated
that smoothing the \( \{\hat{b}_x\} \) outside of the estimation process might result in large discrepancies between observations and model predictions. However, in our case, this method results in a very minor impact on the quality of fit and as can be seen from Figure 2.18, produces smooth forecasts which are close to the original (unsmoothed) ones.

Figure 2.18: The smoothed and original forecasts of mortality rates, males, fitting period 1921–2004.
Basing the forecasts of $q_x$ on the $q_x$ published in ALT for the period 2005–2007

The most recent ALT (at the time when this work started) is the ALT for the period 2005–2007 published by the ABS (ABS, 2008b). In this ALT, $q_x$ is provided for $x = 0, 1, \ldots, 60, 61, \ldots, 100$. Therefore, firstly, we need to extrapolate \{$q_x$\} in this ALT to age 109. The \{$q_x$\} are extrapolated by applying the \{$q_x$\} presented in the Australian Life Tables 2000-02 published by the Australian Government Actuary (AGA) (AGA, 2004). Specifically:

$$\tilde{q}_{\text{ABS}}^{05-07} \times q_{100}^{AGA \ 00-02} \quad \text{for} \ x = 101, 102, \ldots, 109 \ (2.23)$$

where

$q_{\text{ABS}}^{05-07}$ is the $q_x$ value presented in ALT for the period 2005–2007 published by the ABS (ABS, 2008b),

$\tilde{q}_{\text{ABS}}^{05-07}$ is the estimate of $q_{\text{ABS}}^{05-07}$ obtained using extrapolation formula (2.23), and

$q_{\text{AGA}}^{00-02}$ is the $q_x$ value presented in the ALT 2000-02 published by the AGA (AGA, 2004).

To improve the forecast performance, we base our forecasts of \{${q}_x$\} on \{${q}_x^{\text{ABS} \ 05-07}$\} (for $x = 60, 61, \ldots, 100$) and \{${\tilde{q}}_x^{\text{ABS} \ 05-07}$\} (for $x = 101, 102, \ldots, 109$). The \{${q}_x^{\text{ABS} \ 05-07}$\} and \{${\tilde{q}}_x^{\text{ABS} \ 05-07}$\} are assumed to be applicable in 2006 and therefore, our final (improved) forecasts of \{${q}_x$\} begin from 2007. Our final forecasts of \{${q}_x$\} are obtained by applying the forecasts of mortality improvement factors obtained from our original forecasts of \{${q}_x$\} (which are calculated from the smoothed forecasts of mortality rates) to \{${q}_x^{\text{ABS} \ 05-07}$\} and \{${\tilde{q}}_x^{\text{ABS} \ 05-07}$\}. Specifically (for $s = 1, 2, \ldots, 74$):

$$q_{\text{Fin}}^{\text{2006}+s} = \begin{cases} 
q_{\text{ ABS} \ 05-07} \times \frac{q_{x,2006+s}}{q_{x,2006}} & : 60 \leq x \leq 100, \\
\tilde{q}_{\text{ ABS} \ 05-07} \times \frac{\tilde{q}_{x,2006+s}}{q_{x,2006}} & : 101 \leq x \leq 109 
\end{cases} \quad (2.24)$$
where

$q_{x,t+s}$ is the $q_x$ applicable in year $t_T + s$,

$\tilde{q}_{x,t+s}$ is the forecast of $q_{x,t+s}$ calculated from the smoothed $\tilde{m}_{x,t+s}$, and

$\tilde{q}_{x,t+s}^{Fin}$ is the final (improved) forecast of $q_{x,t+s}$.

By basing our forecasts of $\{q_x\}$ on the $\{q_x\}$ published in the most recent ALT, not only are our forecasts likely to be more accurate, but this also solves the discontinuity problem of the forecasts at ages 100+ (Section 2.6.3). Figure 2.19 presents the $\{\tilde{q}_{x,2006+s}^{Fin}\}$ in a selection of forecast years for males in fitting period 1921–2004.

Figure 2.20 presents the $\{\tilde{q}_{x,2040}^{Fin}\}$ values for males and females in fitting periods 1921–2004 and 1975–2004. For fitting period 1975–2004, the $\{\tilde{q}_{x,2040}^{Fin}\}$ for females are higher than for males at ages 100+ which indicates that the forecasts of mortality improvement (deterioration) for females are lower (higher) than males at these ages. This is due to differences of the pattern of observed mortality rates between males and females at these ages in this fitting period.
2.6. APPLICATIONS OF THE LC MODELS

Figure 2.19: \( \{q_{x,2006+s}\} \) in a selection of forecast years, males, fitting period 1921–2004.

![Males (1921-2004)](image)

Final forecast of \( q_{x} \), ages 60 to 108, for various forecast years.

Figure 2.20: \( \{q_{x,2040}^{F}\} \), males and females, fitting periods 1921–2004 and 1975–2004.

![1921-2004](image)

Final forecast of \( q_{x} \), ages 60 to 108, for males and females, 1921-2004.

![1975-2004](image)

Final forecast of \( q_{x} \), ages 60 to 108, for males and females, 1975-2004.
2.7 Estimation of prediction intervals under a semi-parametric bootstrap

In this section, we estimate the 95% prediction intervals (PIs) of future mortality rates (i.e. \{m_{x,2004+s}\}) under a semi-parametric bootstrap. The semi-parametric bootstrap is performed under model PLC(1) (note that the forecasts under model PLC(1) are chosen as described in Section 2.6). The 95% PIs of future one-year death probabilities (i.e. \{q_{x,2004+s}\}) are then calculated from the 95% PIs of \{m_{x,2004+s}\}. We also compare the 95% PI estimated under a semi-parametric bootstrap with the 95% PI estimated under the traditional method (i.e. only by considering the uncertainty of future \{k_t\}).

2.7.1 Background

Note that under model PLC(1) with forecasting method 1, the log of forecast of \(m_{x,tT+s}\) is given by (for convenience, the \(\{k_t^{(1)}\}\) is simply denoted by \(\{k_t\}\):

\[
\log (\hat{m}_{x,tT+s}) = \hat{a}_x + \hat{b}_x \hat{k}_tT+s.
\]

Assuming that model specification and data are correct, the true value of \(\log (m_{x,tT+s})\) is given by:

\[
\log (m_{x,tT+s}) = (\hat{a}_x + \alpha_x) + \left(\hat{b}_x + \beta_x\right) (\hat{k}_T + \kappa_TT+s) + \epsilon_{x,tT+s} \quad (2.25)
\]

where \(\alpha_x\) and \(\beta_x\) are the errors in estimating \(a_x\) and \(b_x\) respectively, \(\kappa_{TT+s}\) is the error in forecasting \(k_{TT+s}\) (the error \(\kappa_{TT+s}\) itself actually consists of two sources of error: the errors in estimating the ARIMA parameters and the error due to future random shocks \(w_{t+v}\) (for \(v = 1, \ldots, s\)) and \(\epsilon_{x,tT+s}\) is the random fluctuation of \(\log (m_{x,tT+s})\) from its mean. Therefore, the forecast error of \(\log (m_{x,tT+s})\) is given by:

\[
FE_{x,tT+s} = \log (m_{x,tT+s}) - \log (\hat{m}_{x,tT+s}) = \alpha_x + \left(\hat{b}_x + \beta_x\right) \kappa_{TT+s} + \beta_x \hat{k}_{TT+s} + \epsilon_{x,tT+s}. \quad (2.26)
\]
By assuming the distribution of \( \log(m_{x,t+s}) \), its PI can be estimated from \( \text{Var}(FE_{x,t+s}) \).

In most applications, \( \text{Var}(FE_{x,t+s}) \) is evaluated only by considering \( \kappa_{t+s} \). By assuming that different sources of forecast errors are independent and \( \log(m_{x,t+s}) \) is Normally distributed, Lee and Carter (1992) showed that this approach significantly underestimates the width of the PI of future mortality rates over short forecast horizons (less than 15 years).

Evaluating \( \text{Var}(FE_{x,t+s}) \) analytically is difficult since we do not know the correlation between different sources of forecast error. In addition, the measures of interest (for example, life expectancy) are often complicated non-linear functions of \( \{a_x\}, \{b_x\} \) and \( \{k_{t+s}\} \). Furthermore, since the estimation of a PI as a measure of interest relies on an assumption of its distribution, the estimated PI is exposed to the possible error of the assumed distribution.

All sources of forecast error and their correlation can be taken into account in the estimated PI of future mortality rates (or other measures of interest) under a simulation method. In addition, under a simulation method, an assumption of the distribution of the measure of interest is not required.

Under the Poisson LC model, there are three simulation methods suggested in the literature: parametric bootstrap (Monte-Carlo simulation) (Brouhns et al., 2002a), semi-parametric bootstrap (Brouhns et al., 2005) and residual bootstrap (Koissi et al., 2006). Renshaw and Haberman (2008) conducted a comparison study of these simulation methods. Two of several findings from this study are: the Monte-Carlo simulation method should not be used for risk assessment purposes since the estimated PI under this simulation method is sensitive to the choice of constraints for the parameters (note that we need to apply constraints to the parameters in order for the LC model to be identifiable) and, under Poisson LC modelling with a constant scale parameter, semi-parametric and residual bootstraps result in a similar PI of the forecasts of mortality rates (Renshaw and Haberman, 2008).

Recently, simulations under Negative Binomial (Li et al., 2009) and Binomial (Haberman and Renshaw, 2008) LC models have also been investigated.

We chose to adopt the semi-parametric bootstrap proposed by Brouhns et al. (2005) since this bootstrap method is simpler to implement than the
residual bootstrap (implementation of a residual bootstrap requires an extra step of mapping the bootstrap residuals to the simulated number of deaths (Renshaw and Haberman, 2008)). We limit our analysis to model PLC(1).

In the following we describe the procedure of semi-parametric bootstrapping to estimate the PI of a measure of interest.

### 2.7.2 Procedure

Suppose we have the observed number of deaths \(d_{x,t}\) and exposure \(E_{x,t}\) at ages \(x_1, x_2, \ldots, x_X\) and years \(t_1, t_2, \ldots, t_T\), and we want to estimate the PI of a measure of interest \(\psi\) which is related to the set of future mortality rates \(\{m_{x,t+s}: x = x_1, x_2, \ldots, x_X, s = 1, 2, \ldots, S\}\). The measure of interest can be life expectancy, present value of a death benefit or even the future mortality rate itself. Implementation of a semi-parametric bootstrap under model PLC(1) to estimate the PI of \(\psi\) involves the following steps (assuming we perform \(N\) simulations) (Brouhns et al., 2005):

1. **Fit model PLC(1) to the dataset**
   
   \(\{d_{x,t}, E_{x,t}: x = x_1, x_2, \ldots, x_X, t = t_1, t_2, \ldots, t_T\}\) to obtain \(\{\hat{a}_x\}\), \(\{\hat{b}_x\}\) and \(\{\hat{k}_t\}\). Then identify the type of ARIMA \((p, d, q)\) model (Section 2.5.2) which is suitable to model \(\{\hat{k}_t\}\).

2. **Simulate a dataset**

   \(\{d_{x,t}: x = x_1, x_2, \ldots, x_X, t = t_1, t_2, \ldots, t_T\}\) from \(D_{x,t} \sim \text{Poisson} \left( E_{x,t} \exp \left( \hat{a}_x + \hat{b}_x \hat{k}_t \right) \right)\) (where \(\hat{a}_x\), \(\hat{b}_x\) and \(\hat{k}_t\) are estimated in step (1)) \(N\) times. Let \(\{d_{x,t}\}^n\) (for \(n = 1, 2, \ldots, N\)) denote the \(n\)th simulated value of this dataset.

3. **For each \(n\) of \(\{d_{x,t}\}^n\):**

   a. **Fit model PLC(1) to obtain** \(\{\hat{a}_x\}^n\) (i.e. the \(n\)th simulated value of \(\{\hat{a}_x\}\)), \(\{\hat{b}_x\}^n\) and \(\{\hat{k}_t\}^n\).

   b. **Fit the type of ARIMA \((p, d, q)\) model identified in step (1) to** \(\{\hat{k}_t\}^n\) (for example, if ARIMA \((0, 1, 1)\) is found to be suitable in step (1), we fit ARIMA \((0, 1, 1)\) to \(\{\hat{k}_t\}^n\) to obtain the simulated ARIMA parameters \(\hat{\mu}^n\), \(\{\hat{\phi}_1\}^n\), \(\{\hat{\theta}_1\}^n\) and \(\hat{\sigma}_w^2^n\) (the variance of...
Then simulate \( \hat{w}_{t+s} \) for \( s = 1, 2, \ldots, S \) by sampling from a univariate Normal distribution with mean 0 and variance \( \hat{\sigma}_w^2 \), and let \( \{\hat{w}_{t+s}\}^n \) denote this set of simulated values. Lastly, calculate the simulated realisations of future values of \( k_t \) (i.e. \( k_{t+s} \) for \( s = 1, 2, \ldots, S \)) from \( \hat{\mu}^n, \{\hat{\phi}_i\}^n, \{\hat{\theta}_i\}^n \) and \( \{\hat{w}_{t+s}\}^n \); and denote this set of simulated values by \( \{\hat{k}_{t+s}\}^n \).

c) Calculate the simulated (future) realisation of \( m_{x,t+s} \) for \( x = x_1, x_2, \ldots, x_X \) and \( s = 1, 2, \ldots, S \) from \( \hat{m}_{x,t+s} = \exp\{\hat{a}_x + \hat{b}_x \hat{k}_{t+s}\} \); and denote this set of simulated values by \( \{\hat{m}_{x,t+s}\}^n \).

d) Calculate the simulated realisation of \( \psi \) from \( \{\hat{m}_{x,t+s}\}^n \) and denote this simulated value by \( \hat{\psi}^n \). Therefore, \( \hat{\psi}^n \) is the \( n \)th simulated value of \( \hat{\psi} \).

From step (3), we obtain \( \hat{\psi}^n \) for \( n = 1, 2, \ldots, N \).

(4) Order the simulated values of \( \hat{\psi} \) such that \( \hat{\psi}^1 \leq \hat{\psi}^2 \leq \ldots \leq \hat{\psi}^N \); then the \( (1 - 2\gamma) \) PI of \( \psi \) is estimated by (Efron and Tibshirani, 1998):

\[
\left[ \hat{\psi}^{\gamma N}, \hat{\psi}^{(1-\gamma)N} \right].
\] (2.27)

For example, suppose \( N = 10,000 \). Then the 95% PI of \( \psi \) is estimated by \( \left[ \hat{\psi}^{250}, \hat{\psi}^{9,750} \right] \) where \( \hat{\psi}^1 \leq \hat{\psi}^2 \leq \ldots \leq \hat{\psi}^{10,000} \).

Note that in each simulation, we simulate the realisations of future \( k_t \) (i.e. \( \hat{k}_{t+s} \) for \( s = 1, 2, \ldots, S \)) and therefore, we take into account the possible realisations of future random shocks \( w_{t+s} \) (for \( s = 1, 2, \ldots, S \)). This approach is different compared to the approach in forecasting mortality rates, where the forecasts of future \( k_t \) are taken as expected values (not realisations) and, therefore, the future random shocks \( w_{t+s} \) are evaluated at their expected values (i.e. zero).

The uncertainties of ARIMA parameters are estimated differently under a semi-parametric bootstrap and under the standard time series method.
Under a semi-parametric bootstrap, the uncertainties of the ARIMA parameters are estimated by simulating the \( \{ \hat{k}_t \} \) (from the true generating process of \( \{ k_t \} \)), while under the standard time series method, these uncertainties are estimated using the large-sample properties of the least squares estimators (Cryer and Chan, 2008). The estimation of uncertainties of ARIMA parameters under a semi-parametric bootstrap is more realistic (since the bootstrap method uses the true generating process of \( \{ k_t \} \)) and, therefore, is likely to result in more accurate estimates than the standard time series method.

Lastly, note that the above simulation algorithm takes into account the sampling fluctuations in the LC model parameters and stochastic error in the forecast of future values of \( k_t \). In addition, since in each set of simulations, all of the parameters of the LC model and ARIMA model are simulated simultaneously, their correlation is taken into account in the estimated PI of \( \psi \). Furthermore, the computation of the PI of \( \psi \) under this simulation algorithm does not require an assumption of the distribution of \( \psi \).

### 2.7.3 Implementation

We apply the semi-parametric bootstrap procedure described in Section 2.7.2 to estimate the 95% PI of future mortality rates (i.e. the measures of interest are future mortality rates) for males and females in fitting periods 1921–2004 and 1975–2004. In each case, 10,000 simulations are performed (i.e. \( N = 10,000 \)) resulting in 10,000 simulated values of \( \hat{m}_{x,2004+s} \) for \( x = 60, 61, \ldots, 100 \) and \( s = 1, 2, \ldots, 76 \). From these simulated values, the 95% PIs of \( m_{x,2004+s} \) are estimated. The simulation was done in R. Repetition of the simulation results in similar 95% PIs which indicates the sufficiency of the number of simulations.

We only consider ages 60 to 100 since the mortality data at ages 100+ are spurious and incomplete. Although model PLC(1) can still be fitted in such a situation, it will take a longer time to fit the model and hence, a large number of simulations will not be feasible (we could reduce the number of simulations to accommodate the estimation at ages 100+, however, the resulting estimates will be less reliable). The 95% PIs of future mortality
rates at ages 100+ are estimated under an alternative method as described later.

In this case, where we only consider ages 60 to 100, the resulting forecasts of future mortality rates and their 95% PIs (estimated only from the uncertainty of \{k_{2004+s}\}) are similar to the case when we consider ages 60 to 110 (i.e. our main forecasts).

Details of the results of the simulation for females in fitting period 1921–2004 (this case is chosen to illustrate) are presented in Appendix A. The results are presented for the \{a_x\}, \{b_x\}, \{k_t\} and \{k_{2004+s}\}.

The average of the simulated realisations of \{a_x\}, \{b_x\} and \{k_t\} respectively are similar to the original estimates of these parameters obtained from the fitting of model PLC(1) to the observed data. In addition, the average of the simulated values of \hat{k}_{2004+s} (for \(s = 1, 2, \ldots, 76\)) and the 95% PI of \(k_{2004+s}\) estimated from the simulation are similar to the \(\hat{k}_{2004+s}\) (forecast of \(k_{2004+s}\)) and the 95% PI of \(k_{2004+s}\) estimated under the standard time series method respectively. Lastly, the average of the simulated values of \(\hat{m}_{x,2004+s}\) is similar to \(\hat{m}_{x,2004+s}\).

The reason for the similarity of our 95% PIs of \(k_{2004+s}\) estimated under a semi-parametric bootstrap and under the standard time series method is because in our application, the uncertainty of \(k_{2004+s}\) is dominated by \{w_{2004+s}\}. This is usually the case if the volume of data used in the fitting of an ARIMA model is sufficiently large (Tsay, 2010). Note that in such a situation, the differences in the estimated 95% PIs of \(k_{2004+s}\) between a semi-parametric bootstrap and the standard time series method could still arise if there is a significant fluctuation of the simulated values of \(\hat{\sigma}_w^2\) (which fortunately is not the case, since it indicates that the \(d\)th difference of \{k_t\} does not follow a stationary process).

Evaluating the width of a 95% PI of future mortality rates from the uncertainty of \{a_x\}, \{b_x\} and \{k_{2004+s}\} separately indicates that at ages 60 to 90, the uncertainty of \(k_{2004+s}\) is the most important. At ages 90+, the uncertainties of \{b_x\} and \{k_{2004+s}\} are important. Amongst the \{a_x\}, \{b_x\} and \{k_{2004+s}\}, the uncertainty of \{a_x\} is the least important. This is because the variability of \{a_x\} is very low (note that the constraints for the
parameters (Section 2.4) result in $a_x$ as the average of log of mortality rates in the fitting period).

Figure 2.21 presents the increase in width of the 95% PI of mortality rates in 2040 under a semi-parametric bootstrap relative to the width of the 95% PI estimated only from the uncertainty of $\{k_{2004+s}\}$ for females in fitting periods 1921–2004 and 1975–2004 (we choose females to illustrate the observations; the observations for males are similar). In the following we present the observations from this figure.

The increase in width of the 95% PI of mortality rates in 2040 under a semi-parametric bootstrap is larger for fitting period 1975–2004 than for fitting period 1921–2004. This is because the shorter fitting period (i.e. 1975–2004) results in a more linear pattern of $\{\hat{k}_t\}$ which results in a lower
\( \hat{\sigma}_w^2 \) (i.e. the simpler the pattern of \( \{\hat{k}_t\} \) is, the more confident we are with our forecast of \( \{k_{2004+s}\} \)) and, therefore, as compared to fitting period 1921–2004, the uncertainty of \( \{k_{2004+s}\} \) represents a lower proportion of the total uncertainties of future mortality rates. In addition, as compared to fitting period 1921–2004, fitting period 1975–2004 results in higher variability of \( \{b_x\} \).

At ages 60 to 95 (90 for fitting period 1975–2004), a semi-parametric bootstrap results only in a marginally wider 95% PI of mortality rates in 2040 as compared to the estimation based only on the uncertainty of \( \{k_{2004+s}\} \). This is because at these ages, the uncertainties of future mortality rates are dominated by the uncertainty of \( \{k_{2004+s}\} \). At ages 95+ (90+ for fitting period 1975–2004), a semi-parametric bootstrap results in a significantly wider 95% PI with the relative differences in 95% PI exponentially increasing with age. This is because at these ages, in addition to the uncertainty of \( \{k_{2004+s}\} \), the uncertainty of \( \{b_x\} \) (which is exponentially increasing with age at these ages) also represents a significant proportion of the total uncertainties of future mortality rates. The very high increase in the width of the 95% PI at ages 90+ under a semi-parametric bootstrap is due to the peculiarity of the PI estimated only from the uncertainty of \( \{k_{2004+s}\} \) and this is discussed in the following.

Figure 2.22 presents the width of the 95% PI of mortality rates in 2040 estimated under a semi-parametric bootstrap and only from the uncertainty of \( \{k_{2004+s}\} \). At ages 60 to 95 (90 for fitting period 1975–2004), the widths are quite similar between these methods, while at ages 95+ (90+ for fitting period 1975–2004), the width estimated only from the uncertainty of \( \{k_{2004+s}\} \) is lower than the width estimated under a semi-parametric bootstrap. This is in line with our observation from Figure 2.21. Note that at ages 90+, the width estimated only from the uncertainty of \( \{k_{2004+s}\} \) is not increasing with age (even decreasing at very high ages). This indicates the unsuitability of the PI estimated under this method. In the following we describe the reason for this.

Note that the width of the 95% PIs of \( \{m_{x,2004+s}\} \) estimated only from
Figure 2.22: The width of the 95% PI of mortality rates in 2040, females, fitting periods 1921–2004 and 1975–2004.

\[ \exp \{ \hat{a}_x \} \times \left( \exp \{ \hat{b}_x \hat{\kappa}_{0.975}^{2004+s} \} - \exp \{ \hat{b}_x \hat{\kappa}_{0.025}^{2004+s} \} \right), \]  

(2.28)

where \( \left[ \hat{\kappa}_{0.025}^{2004+s}, \hat{\kappa}_{0.975}^{2004+s} \right] \) is the estimate of the 95% PI of \( k_{2004+s} \). Note that the first term, \( \exp \{ \hat{a}_x \} \), is increasing with age since \( \{ \hat{a}_x \} \) increase with age, while the second term, \( \left( \exp \{ \hat{b}_x \hat{\kappa}_{0.975}^{2004+s} \} - \exp \{ \hat{b}_x \hat{\kappa}_{0.025}^{2004+s} \} \right) \), is decreasing with age since \( \{ \hat{b}_x \} \) decrease with age. At very high ages, as \( \{ \hat{b}_x \} \) is approaching 0, the value of the second term is rapidly decreasing with age and this results in a non-increasing (even decreasing) width of the 95% PI of \( m_{x,2004+s} \). Note that for females in fitting period 1975–2004, the value of \( \hat{b}_{100} \) is close to 0 (Figure 2.3) and this results in a significant decrease in the width of the 95% PI at this age (with the 95% PI estimated only from the uncertainty of \( \{ k_{2004+s} \} \) (Figure 2.22).

Figure 2.23 is similar to Figure 2.21 where the observations are presented for females in fitting period 1975–2004 for forecast years 2025, 2045, 2065 and 2080. Note that at ages 90+, the relative increase in width of the 95% PI under a semi-parametric bootstrap is increasing with forecast horizons. This is also true for fitting period 1921–2004 although the relative increase in width over forecast horizons is much less pronounced in this fitting period.
2.7. SEMI-PARAMETRIC BOOTSTRAP

Figure 2.23: The increase in width of the 95% PI estimated under a semi-parametric bootstrap relative to the width of the 95% PI estimated only from the uncertainty of future mortality indexes, females, fitting period 1975–2004, forecast years 2025, 2045, 2065 and 2080.

2.7.4 Adjustment for the estimated 95% prediction interval (PI) under a semi-parametric bootstrap

In this section, we smooth and extend to age 109 the estimated 95% PIs of \( \{m_{x,2004+s}\} \) under a semi-parametric bootstrap. From the smoothed estimate of the 95% PIs of \( \{m_{x,2004+s}\} \), the 95% PIs of \( \{q_{x,2004+s}\} \) are calculated. Lastly, we refine the estimated 95% PIs of \( \{q_{x,2004+s}\} \) by basing them on \( \{q_x^{ABS\ 05-07}\} \) (for \( x = 60, 61, \ldots, 100 \)) and \( \{\tilde{q}_x^{ABS\ 05-07}\} \) (for \( x = 101, 102, \ldots, 109 \)) (\( q_x^{ABS\ 05-07} \) and \( \tilde{q}_x^{ABS\ 05-07} \) are described in Section 2.6.6).
Smoothing and extending to age 109 the estimated 95% PI of future mortality rates

Denote:

\[ \hat{m}^{(\gamma)}_{x,2004+s} \]: the unsmoothed estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \(m_{x,2004+s}\) under the traditional method (i.e. only by considering the uncertainty of \(k_{2004+s}\)).

\[ \hat{m}^{(\gamma)}_{B,x,2004+s} \]: the unsmoothed (original) estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \(m_{x,2004+s}\) under a semi-parametric bootstrap.

\[ \hat{m}^{(\gamma)}_{T,x,2004+s} \]: the smoothed estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \(m_{x,2004+s}\) under the traditional method.

\[ \hat{m}^{(\gamma)}_{B,x,2004+s} \]: the smoothed estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \(m_{x,2004+s}\) under a semi-parametric bootstrap.

Note that \([\hat{m}^{(0.025)}_{T,x,2004+s}, \hat{m}^{(0.975)}_{T,x,2004+s}]\) is the smoothed estimate of the 95% PI of \(m_{x,2004+s}\) under the traditional method. The \(\{\hat{m}^{(\gamma)}_{T,x,2004+s}\}\) are obtained from:

\[ \hat{m}^{(\gamma)}_{T,x,2004+s} = \exp\{\hat{a}_x + \hat{b}_x \hat{k}_{2004+s}^{\gamma}\} \]  

(2.29)

where \(\{\hat{b}_x\}\) are the smoothed estimates of \(\{\hat{b}_x\}\) (as described in Section 2.6.6) and \(\hat{k}_{2004+s}^{\gamma}\) is the estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \(k_{2004+s}\).

The \(\{\hat{m}^{(\gamma)}_{B,x,2004+s}\}\) are estimated from \(\{\hat{m}^{(\gamma)}_{T,x,2004+s}\}\) according to:

\[ \hat{m}^{(\gamma)}_{B,x,2004+s} = \left\{ \begin{array}{ll}
\hat{m}^{(\gamma)}_{x,2004+s} \times \frac{\hat{m}^{(\gamma)}_{x,2004+s}}{\hat{m}^{(\gamma)}_{T,x,2004+s}} & : 60 \leq x \leq 100, \\
\hat{m}^{(\gamma)}_{T,x,2004+s} \times \frac{\hat{m}^{(\gamma)}_{y,2004+s}}{\hat{m}^{(\gamma)}_{T,y,2004+s}} & : 100 < x \leq 109.
\end{array} \right. \]  

(2.30)

From (2.30), we obtain the \(\hat{m}^{(\gamma)}_{B,x,2004+s}\) for \(x = 60, 61, \ldots, 109\) and \(\gamma = 0.025, 0.975\). The smoothed estimate of the 95% PI of \(m_{x,2004+s}\) under a
2.7. SEMI-PARAMETRIC BOOTSTRAP

Figure 2.24: The smoothed and original estimates of the 2.5th percentile point of the distribution of \( m_{x,2004+s} \) under a semi-parametric bootstrap.

The semi-parametric bootstrap is given by \( \left[ \hat{m}_{x,2004+s}^{(0.025)} B, \hat{m}_{x,2004+s}^{(0.975)} B \right] \).

In (2.30), the age \( y \) is chosen to be either 99 or 100. Note that since the variability of \( \hat{b}_x \) is increasing with age, it is desirable to set \( y \) to be 100. However, in some cases, \( \hat{b}_{100} \) is close to 0 and, hence, \( \hat{m}_{100,2004+s}^{(\gamma)} B \left/ \hat{m}_{100,2004+s}^{(\gamma)} T \right. \) is not suitable to estimate \( \hat{m}_{x,2004+s}^{(\gamma)} B \) for \( x > 100 \) (it seems too high (low) for the estimation for \( \gamma = 0.975 \) (\( \gamma = 0.025 \))). In determining the suitable age \( y \), we assess the reasonableness of the pattern of \( \hat{m}_{x,2004+s}^{(\gamma)} B \) across ages (for each of \( \gamma = 0.025 \) and \( \gamma = 0.975 \)).

Figure 2.24 presents \( \{ \hat{m}_{x,2004+s}^{(\gamma)} B \} \) (denoted “smoothed” in the figure) and \( \{ \hat{m}_{x,2004+s}^{(\gamma)} B \} \) (denoted “original” in the figure) in a selection of forecast years for females in fitting period 1921–2004.

At ages above 100, the variability of \( \hat{b}_x \) is increasing very rapidly with age and, therefore, at these ages, it is likely that (2.30) results in a higher (lower)
value of $\tilde{m}^{(0.025)}_{x,2004+s}$ than under a semi-parametric bootstrap (assuming that we are able to perform a semi-parametric bootstrap at these ages). Note that a semi-parametric bootstrap is likely to more accurately capture the variability of $\{m_{x,2004+s}\}$ than (2.30). Therefore, (2.30) is likely to result in an underestimation of the width of the 95% PIs of $\{m_{x,2004+s}\}$ at these ages. Note that in Figure 2.24, there is an indication (although minor) that $\{\tilde{m}^{(0.025)}_{x,2004+s}\}$ at ages 100+ are too high. Nevertheless, at these ages, the resulting estimates of the 95% PIs under (2.30) are wider than under the traditional method and, therefore, they capture the variability of $\{m_{x,2004+s}\}$ more accurately than the 95% PIs estimated under the traditional method. Note that, given the incompleteness of data, inclusion of ages above 100 in the semi-parametric bootstrap is not feasible and, hence, an alternative method is needed for the estimation of the PIs at these ages. Lastly, given that the average age of borrowers of reverse mortgages in Australia is 74 (Hickey et al., 2009), this possible underestimation of the 95% PIs of future mortality rates at ages above 100 will not be a problem for the pricing in the majority of reverse mortgage contracts.

**Basing the 95% PI of $q_x$ on the $q_x$ published in ALT for the period 2005–2007**

Let $q^{(\gamma)}_{x,t+s}$ denote the $q_{x,t+s}$ value calculated from $\tilde{m}^{(\gamma)}_{x,t+s}$. Note that since $q_{x,t+s}$ is increasing with $m_{x,t+s}$, $q^{(\gamma)}_{x,t+s}$ is effectively the smoothed estimate of the $(100 \times \gamma)$th percentile point of the distribution of $q_{x,t+s}$ under a semi-parametric bootstrap. Since $q^{ABS\ 05–07}_{x}$ (Section 2.6.6) is the most recent estimate of $q_{x,2006}$, we can improve $\{q^{(\gamma)}_{x,2006+s}\}$ using $\{q^{ABS\ 05–07}_{x}\}$ and $\{\tilde{q}^{ABS\ 05–07}_{x}\}$ according to the following:

$$
q^{Fin\ (\gamma)}_{x,2006+s} = \begin{cases} 
q^{ABS\ 05–07}_{x} \times q^{(\gamma)}_{x,2006+s} / q^{(\gamma)}_{x,2006} & : 60 \leq x \leq 100, \\
\tilde{q}^{ABS\ 05–07}_{x} \times \tilde{q}^{(\gamma)}_{x,2006+s} / \tilde{q}^{(\gamma)}_{x,2006} & : 101 \leq x \leq 109
\end{cases}
$$

(2.31)
where

\[ q_{x,2006+s}^{\text{Fin}}(\gamma) B \] is the final (improved) smoothed estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \(q_{x,2006+s}\) under a semi-parametric bootstrap, and

\[ q_x^{\text{ABS} 05-07}, q_x^{\text{ABS} 05-07} \] and \(\bar{q}_{x,2006}\) are as described in Section 2.6.6.

The final estimates of the 95% PIs for \(\{q_{x,2006+s}\}\) are given by

\[ [q_{x,2006+s}^{\text{Fin}}(0.025) B, q_{x,2006+s}^{\text{Fin}}(0.975) B]. \]

Note that the interval \([q_{x,2006+s}^{\text{Fin}}(0.025) B, q_{x,2006+s}^{\text{Fin}}(0.975) B]\) is effectively the interval \([q_x^{\text{ABS} 05-07}, q_x^{\text{ABS} 05-07}]\) relocated such that the forecast of \(q_{x,2006}\) (\(\bar{q}_{x,2006}\)) is equal to \(q_x^{\text{ABS} 05-07}\) (for \(x = 60, 61, \ldots, 100\)) or \(q_x^{\text{ABS} 05-07}\) (for \(x = 101, 102, \ldots, 109\)).

Note that alternatively, the denominator in (2.31) could be replaced by \(\bar{q}_{x,2006+s}\) (which is calculated from the smoothed average of simulated values of \(\hat{m}_{x,2006+s}\)). However, this will result in a similar value of \(q_{x,2006+s}^{\text{Fin}}(\gamma) B\) as (2.31), since the average of simulated values of \(\hat{m}_{x,2006+s}\) is similar to \(\bar{m}_{x,2006+s}\) (Section 2.7.3).

Figure 2.25 presents \(\{q_{x,2006+s}^{\text{Fin}}(0.025) B\}\) in a selection of forecast years for females in fitting period 1921–2004.

### 2.7.5 Further comparison of the estimated 95% prediction interval (PI) between under a semi-parametric bootstrap and under the traditional method

In this section, we present a comparison of the estimated 95% PI of future \(\{q_x\}\) between under a semi-parametric bootstrap and under the traditional method (i.e. only by considering the uncertainty of future \(k_t\)).

Denote:

\[ q_{x,2006+s}^{(\gamma) T}: \text{the } q_{x,2006+s} \text{ value calculated from } \hat{m}_{x,2006+s}^{(\gamma) T}. \]

\[ q_{x,2006+s}^{\text{Fin}}(\gamma) T: \text{the final smoothed estimate of the } (100 \times \gamma)\text{th percentile point of the distribution of } q_{x,2006+s} \text{ under the traditional method.} \]
Figure 2.25: \( \left\{ \frac{q^{\text{Fin}}_{x:2006+s} (0.025)}{q^{(\gamma)}_{x:2006+s}} \right\} \) in a selection of forecast years.

![Figure 2.25: Females (1921-2004)](image)

We calculate \( q^{\text{Fin}}_{x:2006+s} (\gamma) \) under an equation similar to (2.31) with the term \( q^{(\gamma)}_{x:2006+s} \) replaced with \( q^{(\gamma)}_{x:2006+s} \).

Figure 2.26 presents \( \bar{e}_{60} \) and \( \bar{e}_{90} \) calculated from \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.025) \right\} \) (denoted “T (0.025)” in the figure), \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.975) \right\} \) (“T (0.975)”), \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.025) \right\} \) (“B (0.025)”) and \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.975) \right\} \) (“B (0.975)”) for females in fitting periods 1921–2004 and 1975–2004. An interval defined by a particular life expectancy calculated from each set of \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.025) \right\} \) and \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.975) \right\} \) is referred to as the estimated 95% PI of the corresponding life expectancy under a semi-parametric bootstrap (and similarly for the life expectancy calculated from each set of \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.025) \right\} \) and \( \left\{ q^{\text{Fin}}_{x:2006+s} (0.975) \right\} \)). Strictly, this estimate of the 95% PI of the life expectancy is not correct since it ignores the possible correlation of model parameters across ages (an accurate estimation of the PI of the life expectancy requires implementing the semi-parametric bootstrap with the life expectancy as the measure of interest). However, our
Figure 2.26: The $\hat{c}_{60}$ and $\hat{c}_{90}$ calculated from $\{q_{x,2006+s}^{Fin(\gamma)T}\}$ and $\{q_{x,2006+s}^{Fin(\gamma)B}\}$ for $\gamma = 0.025$ and $\gamma = 0.975$, females, fitting periods 1921–2004 and 1975–2004.

Purpose in illustrating the 95% PIs of the life expectancies is to present a comparison of the estimated 95% PIs of $q_{x,2006+s}$ across ages between under a semi-parametric bootstrap and under the traditional method and, therefore, the 95% PIs of these life expectancies are calculated from $\{q_{x,2006+s}^{Fin(\gamma)T}\}$ and $\{q_{x,2006+s}^{Fin(\gamma)B}\}$.

A semi-parametric bootstrap results in wider 95% PIs for $\hat{c}_{60}$ and $\hat{c}_{90}$ than the traditional method in both fitting periods (although for fitting period 1921–2004, the increase in width for $\hat{c}_{60}$ under a semi-parametric bootstrap is very small). This indicates that a semi-parametric bootstrap results in wider 95% PIs of future $\{q_{x}\}$ than the traditional method. The increase in width of the 95% PI under a semi-parametric bootstrap is less pronounced.
for \( \hat{e}_{00} \) (as compared to \( \hat{e}_{00} \)) and for fitting period 1921–2004 (as compared to fitting period 1975–2004). Lastly, the differences in width of the estimated 95% PIs between the semi-parametric bootstrap and the traditional method are generally increasing with forecast horizons. Note that these observations are in line with observations from Figures 2.21 and 2.23.

2.8 Further forecast assessment

In this section, we compare our forecasts of \( \{q_x\} \) with the forecasts calculated from the future mortality improvement factors presented in the Australian Life Tables (ALT) published by the Australian Government Actuary (AGA) (AGA, 1999, 2004).

2.8.1 Further assessment of the out of sample forecasts

In this subsection, we compare the performance of the out of sample forecasts of \( \{q_{x,1996+s}\} \) and their estimated 95% PIs under the traditional method with the forecasts of \( \{q_{x,1996+s}\} \) calculated from the future mortality improvement factors presented in ALT 1995-97 published by the AGA (AGA, 1999).

Let \( \{q_{x}^{AGA\ 95-97}\} \) denotes the \( \{q_x\} \) values published in ALT 1995-97 (AGA, 1999). The out of sample forecasts of mortality rates under model PLC(1) (as calculated in Section 2.6.4) are smoothed (by smoothing the \( \{\hat{b}_x\} \)) and the out of sample final forecasts of \( \{q_x\} \) are calculated by basing the out of sample forecasts of \( \{q_x\} \) (which are calculated from the smoothed out of sample forecasts of mortality rates) on \( \{q_{x}^{AGA\ 95-97}\} \) (under a similar method as in Section 2.6.6). The \( \{q_{x}^{AGA\ 95-97}\} \) are assumed to be applicable in 1996 and therefore, the out of sample (final) forecasts of \( \{q_x\} \) begin from 1997. In addition, we also estimate the out of sample 95% PIs of \( \{q_{x,1996+s}\} \) under the traditional method (with the estimates of the \((100 \times \gamma)\)th percentile point of the distribution of \( \{q_{x,1996+s}\} \) are smoothed and adjusted by basing it on \( \{q_{x}^{AGA\ 95-97}\} \)).

In ALT 1995-97, there are three types of future mortality improvement factors presented for the purpose of forecasting \( \{q_x\} \) (AGA, 1999):
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- “25 year” improvement factors: the improvement factors derived by extrapolating the historical trends in mortality improvement over the 25-year period ending in 1996,

- “100 year” improvement factors: the improvement factors derived by extrapolating the historical trends in mortality improvement over the 100-year period ending in 1996, and


In the comparison in this subsection, the “ABS” improvement factors are not considered since these improvement factors are not presented in ALT 2000-02 (AGA, 2004).

Denote:

- \( \hat{q}_{95-97}^{25} \) \( q_{x,1996+s} \): the forecast of \( q_{x,1996+s} \) calculated by applying the “25 year” improvement factors presented in ALT 1995-97 to \( q_{x}^{AGA} \) 95-97.

- \( \hat{q}_{95-97}^{100} \) \( q_{x,1996+s} \): the forecast of \( q_{x,1996+s} \) calculated by applying the “100 year” improvement factors presented in ALT 1995-97 to \( q_{x}^{AGA} \) 95-97.

- \( \hat{q}_{Fin}^{x,1996+s} \): the out of sample (final) forecast of \( q_{x,1996+s} \).

- \( q_{Fin}^{\gamma} T \) \( q_{x,1996+s} \): the out of sample (final) smoothed estimate of the \((100 \times \gamma)\)th percentile point of the distribution of \( q_{x,1996+s} \) under the traditional method.

Figure 2.27 presents the empirical values and forecast \( \hat{e}_{60} \) in 1997 to 2004 calculated from \( \{ \hat{q}_{x,1996+s}^{Fin} \} \) (denoted “Forecast” in the figure), \( \{ \hat{q}_{x,1996+s}^{(0.025)} T \} \) (“T (0.025)”), \( \{ \hat{q}_{x,1996+s}^{(0.975)} T \} \) (“T (0.975)”), \( \{ \hat{q}_{95-97}^{25} \} \) (“95-97 (25 year)”), and \( \{ \hat{q}_{95-97}^{100} \} \) (“95-97 (100 year)”) for males and females in fitting periods 1931–1994 and 1975–1994. The empirical values of \( e_{60} \) are obtained from ABS (2008a). In the following we present the observations from this figure.

The forecasts of \( e_{60} \) from fitting period 1931–1994 are similar to the forecasts from “100 year” improvement factors, while the forecasts from fitting
Figure 2.27: Comparison of the out of sample forecasts of $\hat{e}_{60}$ (with the 95% PIs estimated under the traditional method) with the forecasts from the improvement factors of ALT 1995-97, fitting periods 1931–1994 and 1975–1994.

Data analysis:

Period 1975–1994 are similar to the forecasts from “25 year” improvement factors. In addition, in all cases, all of the forecasts are lower than the empirical values with the forecast errors increasing with forecast horizons. This indicates that the forecasts from both fitting periods and both types of improvement factors under-predict the rate of mortality improvement. However, the forecasts from fitting period 1975–1994 (or “25 year” improvement factors) are more accurate than the forecasts from fitting period 1931–1994 (or “100 year” improvement factors). Lastly, in all cases other than females in fitting period 1975–1994, most of the empirical values of $\hat{e}_{60}$ are outside the 95% PIs which suggests that the estimated 95% PIs of future mortality rates
(especially at ages 60 to 90) under the traditional method are too narrow (note that for females in fitting period 1975–1994, the empirical values of \(\tilde{\epsilon}_{60}\) are close to the upper bound of its estimated 95% PI). Since at ages 60 to 90, the traditional method and semi-parametric bootstrap result in similar 95% PIs of future mortality rates (Section 2.7.3), therefore, a semi-parametric bootstrap is also likely to underestimate the width of the 95% PIs of future mortality rates at these ages.

2.8.2 Further assessment of the main forecasts

In this subsection, we compare our main forecasts of \(\{q_{x,2006+s}\}\) (i.e. \(\{\tilde{q}_{x,2006+s}^{\text{Fin}}\}\) (Section 2.6.6)) and their estimated 95% PIs (i.e. \(\left[ q_{x,2006+s}^{\text{Fin} (0.025) B}, q_{x,2006+s}^{\text{Fin} (0.975) B} \right] \) (Section 2.7.4)) with the forecasts of \(\{q_{x,2006+s}\}\) calculated from the future mortality improvement factors presented in ALT 2000-02 published by the AGA (AGA, 2004).

In ALT 2000-02, there are two types of future mortality improvement factors presented for the purpose of forecasting \(\{q_x\}\) (AGA, 2004):

- “25 year” improvement factors: the improvement factors derived by extrapolating the historical trends in mortality improvement over the 25-year period ending in 2001, and
- “105 year” improvement factors: the improvement factors derived by extrapolating the historical trends in mortality improvement over the 105-year period ending in 2001.

Denote:

- \(q_{x,2006+s}^{100-02 \ "25"}\) : the forecast of \(q_{x,2006+s}\) calculated by applying the “25 year” improvement factors presented in ALT 2000-02 to \(q_x^{ABS ~05-07}\) (for \(x = 60, 61, \ldots, 100\)) and \(q_x^{ABS ~05-07}\) (for \(x = 101, 102, \ldots, 109\)).
- \(q_{x,2006+s}^{100-02 \ "105"}\) : the forecast of \(q_{x,2006+s}\) calculated by applying the “105 year” improvement factors presented in ALT 2000-02 to \(q_x^{ABS ~05-07}\) (for \(x = 60, 61, \ldots, 100\)) and \(q_x^{ABS ~05-07}\) (for \(x = 101, 102, \ldots, 109\)).
Figure 2.28 presents the forecasts of $\delta_{60}$ from 2007 to 2080 calculated from $\{\hat{q}_{x:2006+s}^{\text{Fin}}\}$ (denoted “Forecast” in the figure), $\{\hat{q}_{x:2006+s}^{\text{Fin} (0.025)} B\}$ (“B (0.025)”), $\{\hat{q}_{x:2006+s}^{\text{Fin} (0.975)} B\}$ (“B (0.975)”), $\{\hat{q}_{x:2006+s}^{\text{00-02 “25”}}\}$ (“00-02 (25 year)”) and $\{\hat{q}_{x:2006+s}^{\text{00-02 “105”}}\}$ (“00-02 (105 year)”) for males and females in fitting periods 1921–2004 and 1975–2004. In the following we present the observations from this figure.

Firstly, the forecasts of $\delta_{60}$ from fitting period 1921–2004 are similar to the forecasts from the “105 year” improvement factors. Although the forecasts from fitting period 1975–2004 are lower than the forecasts from the “25 year” improvement factors, the differences between these forecasts in the next 40 years were

Figure 2.28: Comparison of the forecasts of $\delta_{60}$ (with the 95% PIs estimated under a semi-parametric bootstrap) with the forecasts from the improvement factors of ALT 2000-02, fitting periods 1921–2004 and 1975–2004.
years (which are important for the pricing of a reverse mortgage contract) are quite low. Secondly, the forecasts of $\hat{e}_{60}$ from the “25 year” improvement factors are mostly outside the 95% PIs of future $\hat{e}_{60}$ estimated from fitting period 1921–2004 and the forecasts from the “105 year” improvement factors are mostly outside the 95% PIs estimated from fitting period 1975–2004. This further indicates that the estimated 95% PIs of future $\{q_x\}$ under a semi-parametric bootstrap are too narrow and do not capture the possibility that the trend of mortality in the future is different than the trend of mortality in the fitting period.

Discussion on the suitability of the forecasts

Fitting period 1975–2004 results in higher forecasts of mortality improvement than fitting period 1921–2004. From the assessment of the out of sample forecasts, fitting period 1975–2004 is still likely to result in an under-prediction of future mortality improvement (as indicated by the fact that fitting period 1975–1994 results in an under prediction of mortality improvement in 1997 to 2004). However, the out of sample forecasts are assessed in a short period in which the rapid improvement of mortality is still observed. The duration and degree of the continuation of the mortality improvement in the future is hard to determine. As discussed in Section 2.3, by analysing the main causes of death, we are of the opinion that at ages 60 to 84, the rapid mortality improvement is unlikely to continue, while at ages 85+, the mortality improvement is likely to continue in the future. Better forecasts of mortality rates might be constructed by applying the forecasts of mortality improvement from fitting period 1975–2004 up to a certain time in the future when the improvement in mortality is expected to slow down (transition time); and the forecasts at years after the transition time are then calculated from the application of the forecasts of mortality improvement from fitting period 1921–2004. However, in such a forecasting method, it is difficult to determine the future transition time and this will result in a certain degree of forecast error. It is difficult to forecast mortality rates accurately and, therefore, for risk management of a reverse mortgage contract, we should instead consider
a variety of scenarios of future mortality improvement. We believe that, for this purpose, the \( \{ q_{x,2006+s}^{\text{Fin}}(0.025) \} \) estimated from fitting period 1975–2004 is conservative enough to capture the risk of high future longevity, while the \( \{ q_{x,2006+s}^{\text{Fin}}(0.975) \} \) estimated from fitting period 1921–2004 is conservative enough to capture the risk of low future longevity. Note that for the pricing of a reverse mortgage contract, in assessing the risk of high future longevity, the mortality improvements in (around) the next 40 years are important (since the youngest eligible age to borrow a reverse mortgage loan is 60). Although the \( \{ q_{x,2006+s}^{\text{Fin}}(0.025) \} \) estimated from fitting period 1975–2004 might underestimate the future mortality improvement in the short-term future, overall, it is likely to overestimate the future mortality improvement in the next 40 years.

### 2.9 Conclusion

In this chapter, we have determined the most suitable type of the LC models for forecasting mortality rates in our application. We restrict our analysis on the most common types of the LC models. In addition, we also estimated the 95% PIs of future mortality rates under a semi-parametric bootstrap. Two fitting periods are considered: 1921–2004 and 1975–2004, for the purpose of capturing the long-term trend and the recent short-term trend of mortality. The resulting forecasts of mortality rates from the chosen LC model are similar with the forecasts calculated from the related future mortality improvement factors presented in ALT 2000-02 published by the AGA (AGA, 2004), while the estimated 95% PIs of future mortality rates under a semi-parametric bootstrap are found to be too narrow. Nevertheless, the lower bounds of the estimated 95% PIs of future mortality rates under a semi-parametric bootstrap from fitting period 1975–2004 and the upper bounds of the estimated 95% PIs from fitting period 1921–2004 constitute PIs which are wide enough to cover a range of possibilities of future mortality rates for risk management of a reverse mortgage contract.
Chapter 3

A multi-state model to estimate disability transition probabilities

3.1 Introduction

The purpose of Chapters 3 and 4 is to present a new approach to estimating the probability of admission into an aged care home for Australian elderly using national data which are publicly available. The estimated probability of admission is intended to be an input for pricing analysis of a reverse mortgage contract. The approach adopted here is to first estimate the probability of disablement from a national disability survey. The probability of admission into an aged care home is estimated from the probability of disablement by assuming a relationship between the level of disability and incidence rates of entry into an aged care home. The motivation for this approach is the fact that disability is the major driver of demand for formal care in institutions (Jacobzone et al., 1999).

The disability definition which is relevant to our analysis should be measured in terms of the level of supervision required to conduct basic daily tasks. In Australia, there is a survey conducted by the Australian Bureau of Statistics (ABS) which measures this disability at the national level. In this
survey, namely the Survey of Disability, Ageing and Carers (ABS, 2004a), there is a measure of disability which is defined in terms of restriction in conducting so called core activities which are similar to basic daily tasks. There are four levels of restriction measured: mild, moderate, severe and profound. Although we believe that the severe and profound restrictions are more relevant in estimating the likelihood of admission into a nursing home, a multi-state model which also includes able (no restriction), mild and moderate restriction will better capture the dynamics of the disability process and result in a more accurate estimation of the probability of becoming severely or profoundly disabled.

Estimation of transition probabilities across disability states in a multi-state model ideally requires the total number of transitions between (disability) states, timing of the transitions, as well as total exposure for each state. If such data were available (and their amount is sufficiently large), we would be able to estimate transition intensities directly from the data and in turn the transition probabilities could be estimated. However, in practice, at best only longitudinal data are available (for example, National Long Term Care Survey (NLTCS) in the U.S.). Longitudinal data, which record the disability status of an individual at two points in time, only allow a direct estimation of transition probabilities for a discrete time interval where the interval is dictated by the timing of each of the two consecutive surveys. As for Australia, the available data at a national scale are even less ideal as only cross sectional data are available. Cross sectional data, which are less informative than longitudinal data, measure the disability status of an individual only at one point in time. Therefore, the methodology for the estimation of transition probabilities in Australia is limited by the available data.

One of the multi-state models for a disability process which uses cross sectional data is proposed by Rickayzen and Walsh (2002). This model assumes a functional form of the transition probabilities and employs a stationary population assumption in the fitting process. The parameters of the functional form are chosen to best replicate the observed disability prevalence rates data. This model has been fitted using disability data in Australia by Leung (2004) in the context of the projection of the cost of long term
3.2 Literature review

3.2.1 Probability of admission into an aged care home

There are a variety of methods which have been adopted to estimate the probability of admission into an aged care home. Conventionally, the risk of aged care home use is perceived as the prevalence rate of the aged care home population at a particular point in time (ratio of the population who are in aged care homes to total population who are in aged care homes and in the community) (Liu, 2000). Several commentators suggest that this measurement can be misleading (Manton, 1988; Liu, 2000). For example, Liu (2000), by applying a life table model to Australian data from a national Nursing Home Payment System (NHPS), found that the likelihood of entering a nursing home after turning age 65 is significantly higher than the prevalence rates of nursing homes’ population at ages 65 and over.

Considering the shortfall of the conventional approach described above, several more sophisticated methods have been adopted. The Equity Release Working Party (ERWP) of the Institute and Faculty of Actuaries in the U.K. estimates the incidence rates of residential long term care (LTC) using the prevalence rates of population residential LTC by employing two assumptions: a stationary population and additional mortality experienced by those who live in LTC communal establishments (ERWP, 2005). In Australia, a more recent analysis of the risk of admission into an aged care home has been performed by Mason et al. (2001) by employing a life table method similar to Liu (2000) with some additional refinements. They used a more recent dataset from the System for the Payment of Aged Residential Care (SPARC). Mason et al. (2001) found that the probability of admission over a lifetime is significantly higher than that commonly perceived based on a prevalence rate at a point in time; and over a life time, women have a much higher chance of using an aged care home than men. The analysis of Mason et al.
(2001) has been used by Standard & Poors to derive the initial assumptions relating to probabilities of reverse mortgage borrowers moving out of their homes (Standard & Poors, 2005). Liu’s (2000) life table method has also been adopted by Cullen (2006) to estimate an individual’s expected lifetime cost of permanent residential aged care. An alternative life table method is presented by Pollard (1995). This method infers the relationship between the survivorship function of those who are alive and not in nursing homes (who are subject to two decrements: death and entering a nursing home) with the survivorship function of those who are alive regardless of whether they are residing in nursing homes (who are only subject to the death decrement) from prevalence rates for the nursing home population.

Analysis of risk factors for admission into a nursing home has been done mainly in the U.S.. Reviews of these studies are provided by Gaugler et al. (2007) and Miller and Weissert (2000). In Australia, similar analysis from regional longitudinal studies (i.e. restricted to particular areas) has been done by Giles et al. (2007), McCallum et al. (2005) and Wang et al. (2001). The following risk factors are found to be significant: increasing age, lower income, lack of home-ownership, difficulty of hearing, incontinence, impaired respiratory flows, physical disability, depression, male gender, lower alcohol intake, fair or poor self rated health, walking difficulty and current smoking (Wang et al., 2001; McCallum et al., 2005; Giles et al., 2007). Note that the finding of females having a lower risk of admission than males from a regional longitudinal study (McCallum et al., 2005) might appear to contradict the finding of a study using a national dataset by Mason et al. (2001). However, this might be because of lower mortality rates of females, although it is also possible that regional experience is different to the national experience.

These longitudinal studies highlight that although disability is the major driver of admission, there are many other factors which need to be considered in assessing the risk of admission. In assessing the likely magnitude of future demand for aged care services in Australia, beside future level of disability, several other factors are often considered which include: access to informal support, changing preferences (towards at home or residential care) and expectations, trends in income and wealth, aged care prices and the level of
government subsidy towards aged care service, trends in mortality rates and medical advances (Madge, 2000; Aged Care Price Review Taskforce, 2004; Productivity Commission, 2008). These factors undoubtedly also have an impact on the likelihood of admission into an aged care home.

Estimation of the probability of admission into an aged care home under a regression model with covariates (to take into account other factors which might affect the likelihood of admission beside disability) has also been done mainly in the U.S.. Examples include Waidmann and Liu (2000) and Dick et al. (1994). Dick et al. (1994) employ a multi-state model with three states: living in the community, living in nursing homes and dead. Transition probabilities are estimated under a logistic model with covariates: age, sex, race, history of nursing home use and the duration of stay in the current state. Waidmann and Liu (2000) modelled the probability of an individual attaining a particular disability status under a multinomial logit specification. Being disabled and residing in a long term care facility is regarded as the highest disability status. The covariates included are: age, sex, race, years of education and marital status. The model also includes a time trend which allows the estimated probabilities to vary with time.

In Chapter 4 we present a new approach to estimate the probability of admission into an aged care home using a publicly available national dataset. As stated before, the probability of admission is estimated from the likelihood of disablement. Although there are many factors which affect the likelihood of admission beside disability, we opt for this approach for several reasons. Firstly, most empirical research shows that disability is the major driver of demand for formal care (Jacobzone et al., 1999). An overwhelming number of analyses of longitudinal data in the U.S. found that worse scores for activities of daily living (related to daily self-care activities, for example: self feeding, bathing, dressing, bowel management, etc) are associated with higher risk of admission (Miller and Weissert, 2000) with 3 or more activities of daily living (ADLs) dependencies amongst the strongest predictors of admission (Gaugler et al., 2007). Secondly, the publicly available data (at a national scale) of residential aged care statistics in Australia do not provide enough detail to carry out a regression analysis which takes into account other factors.
which might impact the likelihood of admission. Lastly, given the nature of the publicly available data (at a national scale), we feel that there is a stronger theoretical basis in the available methods to estimate the likelihood of disablement as compared to likelihood of admission into an aged care home. Note that the analysis of Mason et al. (2001) relied on the data from SPARC which are not publicly available. In the next section we review the available methods to estimate the probability of disablement.

3.2.2 Probability of disablement

Methods to estimate the probability of disablement are largely dictated by the type of available data. As stated before, in Australia, we only have cross sectional disability data (at a national scale). Papers which present a method to estimate the probability of disablement from cross sectional data include Nuttall et al. (1994), Brelivet et al. (2001), Davis et al. (2002), Rickayzen and Walsh (2002), Alegre et al. (2004), Leung (2004), Albarran et al. (2005) and Leung (2006). In the following we provide a brief description of the modelling methodology adopted in each paper and the justification for the methodology adopted in this chapter.

Nuttall et al. (1994) estimated disability incidence rates by employing a multi-state model with three states: healthy, disabled and dead. They showed that by assuming the absence of recovery from the disabled state, the disability incidence rates in the model can be estimated from the disability prevalence rates and mortality rates of the disabled. Albarran et al. (2005) further showed that under the same multi-state model and assumptions as Nuttall et al. (1994), the transition probability to the disabled state can be estimated from the disability prevalence rates and population mortality rates. However, an assumption of a relationship between mortality rates in the healthy state and mortality rates in the disabled state is required.

Alegre et al. (2004) analysed a multi-state model with four states of disability (including healthy). They also assumed the absence of recovery from the disabled states. They further assumed that the death probabilities of each state can be obtained by applying a loading (or discount) to the popu-
lation death probabilities, and the transition probabilities from each disabled state can also be obtained by applying a loading to transition probabilities from the healthy state. They showed that under these assumptions, the transition probabilities in the model can be estimated from the prevalence rates of each disability state and the assumed values of the loadings.

Brelivet et al. (2001) estimated the transition probability to the disabled state by following a cohort in two consecutive cross sectional surveys. This method requires estimates of mortality rates of those who are independent, dependent and living in the community, and those who are living in residential care (who are assumed to be dependent). This method also assumes the absence of recovery from dependence and exits from residential care. The mortality rates of the sub-populations are derived such that the estimated transition probabilities closely replicate the regression based probabilities which are estimated from regional longitudinal data.

Davis et al. (2002) estimated a so called marginal disability transition probability defined as the probability of becoming disabled at a later age conditional on being alive at the base age. The probability is estimated from the ABS disability survey data in 1981, 1988, 1993 and 1998. The estimation employed a multinomial modelling distribution. From the estimated probability, current and cohort health expectancies are calculated.

Rickayzen and Walsh (2002) built functional forms for transition probabilities to a lower disability state in a multi-state model. The functional forms are based on Perks (1932) formula for graduation of mortality rates. The recovery rates are included as an assumption. Under a stationary population hypothesis, parameters of the functional forms are chosen to closely replicate the observed disability prevalence rates. Leung (2004) applied Rickayzen and Walsh’s modelling framework to project the cost of long term care (LTC) in Australia, with a slight modification to the functional forms of the transition probabilities to adapt the model to Australian data. Leung (2006) presented a methodology to graduate transition intensities and calculate transition probabilities of the multi-state model to price and reserve a LTC insurance contract using Thiele’s differential equations, as described in Hoem (1969).
In this chapter we adopt a similar modelling framework to Rickayzen and Walsh (2002) and Leung (2004). This modelling framework is preferred as it offers a greater flexibility and realism: the model can be easily altered to include any number of disability states and a variety of relevant information can be easily incorporated into the various aspects of the modelling. There are several differences in our implementation of the model as compared with Leung (2004) as we have a different orientation. Firstly, we present a minor improvement in the fitting process to enhance the realism of the modelling framework. Secondly, we only cover persons aged 25 years and over as opposed to Leung (2004) who considered persons at all ages. The reason for this is because, while the implementation of the model requires us to start from a young age where the proportion of the disabled population is small, our focus is on the probability of becoming disabled for the elderly. Lastly, we update several aspects of the assumptions to reflect the relevant information available from the more recent related studies.

3.3 Data sources

There are a number of datasets which can be used to analyse the probability of admission into an aged care home in Australia. These include the statistical overview of residential aged care (Australian Institute of Health and Welfare (AIHW), 2009), a national disability survey (Survey of Disability, Ageing and Carers (SDAC)) and various local longitudinal surveys which measure both disability and aged care home use. In addition, one might also want to consider large scale overseas longitudinal data (for example National Long Term Care Survey (NLTCS) in U.S.). Given the methodology adopted here, we focus our attention on disability data.

The chosen set of disability data is SDAC. This national cross sectional disability survey is preferred over regional longitudinal data as national experience might be different to regional (as discussed in Section 3.2.1). Overseas longitudinal data (for example NLTCS), although representative of a national experience, might be of limited use as overseas data might not be comparable to Australian data (which might be due to different aged care policy,
disability experience, different surveying methods or disability definition). However, data from overseas and a local longitudinal survey will be used to set certain assumptions in the model which will be described in Section 3.4.

Data from the last two SDAC surveys will be used in this and subsequent chapters. In the following we provide a brief description of this survey.

SDAC is a national disability survey conducted by the ABS at five yearly intervals (generally) starting from 1981 (this survey was called the Survey of Handicapped Persons in 1981 and Survey of Disabled and Aged Persons in 1988). The latest SDAC (at the time of writing of this chapter) was conducted from June to November 2003. SDAC collects information from people with disability, from older people, and from their carers (ABS, 2004a). The survey covered people in both households and care accommodation. The latest survey comprised of 36,241 people for the household component and 5,145 for the cared-accommodation component. As in Leung (2004), the data that we use from SDAC are those that relate to limitations in conducting so-called core activities which are measured in three areas as described in Table 3.1.

<table>
<thead>
<tr>
<th>Core activity</th>
<th>Tasks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Communication</td>
<td>Understanding and being understood by family, friends and strangers.</td>
</tr>
<tr>
<td>Mobility</td>
<td>Getting into or out of a bed or chair; moving about at usual place of residence or at a place away from usual residence; walking 200 metres; walking up and down stairs without a handrail; bending and picking up an object from the floor; using public transport.</td>
</tr>
<tr>
<td>Self care</td>
<td>Showering or bathing; dressing; eating; toileting; bladder or bowel control.</td>
</tr>
</tbody>
</table>

*Source:* Australian Bureau of Statistics (ABS) (2004a)

Note that the failure or difficulty to perform core activities (core activity limitation (CAL)) will drive the demand for care, whether formal or informal. Therefore, under certain assumptions, the likelihood of moving into aged residential care can be determined from the likelihood of having some CAL.

There are four levels of CAL measured in the survey which are determined
based on whether a person needed help, had difficulty with, or used an aid or equipment with any of the core activities (ABS, 2004a). These are defined in Table 3.2 from the most severe to the relatively moderate category.

Table 3.2: Levels of CAL.

<table>
<thead>
<tr>
<th>Level of CAL</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profound</td>
<td>The person is unable to do, or always needs help with, a core activity task.</td>
</tr>
<tr>
<td>Severe</td>
<td>The person sometimes needs help with a core activity task; or, has difficulty understanding or being understood by family or friends; or, can communicate more easily using sign language or other non-spoken form of communication.</td>
</tr>
<tr>
<td>Moderate</td>
<td>The person needs no help but has difficulty with a core activity task.</td>
</tr>
<tr>
<td>Mild</td>
<td>The person needs no help and has no difficulty with any of the core activity tasks, but uses aids or equipments; or, cannot easily walk 200 meters; or, cannot walk up and down stairs without a handrail; or, cannot easily bend and pick up an object from the floor; or, cannot use public transport; or, can use public transport, but needs help or supervision; or, needs no help or supervision but has difficulty using public transport.</td>
</tr>
</tbody>
</table>

*Source: Australian Bureau of Statistics (ABS) (2004a)*

Note that analysis on profound and severe categories is more relevant in assessing the need to move into an aged care home. However, mild and moderate categories will also be included in our model to capture the dynamic of the disability process more accurately. In addition, since it is possible for people in the mild or moderate categories to have a need to move into an aged care home (as the need for formal care might be driven by reasons other than disability, for example an illness), ignoring these disability categories will underestimate the likelihood of admission.

Tables 3.3 and 3.4 show the estimated numbers of males and females in each age group for each disability category in Australia as reported in the latest SDAC survey. It is useful to present this information as the prevalence rates per 1,000 of population. These are shown in Tables 3.5 and 3.6. Several observations from these tables follow:
• The CAL prevalence rates increase with age (generally) for both males and females.

• The main exception to the above point is for the moderate CAL category where the prevalence rates fluctuate over age 75 for males and over age 65 for females.

• From age 75, the prevalence rates of profound CAL increase very rapidly for both males and females. At the last two age groups, this disability category has the highest prevalence rates for both genders.

• At ages 90 and above, most people (more than 90%) have some limitation on core activities.

• Overall, the prevalence rates of severe and profound CAL are higher for females. This is apparent from age 70 for severe CAL and from age 80 for profound CAL (however, the prevalence rate for severe CAL is higher for males at the last age group).

• The overall shapes of the prevalence rates (across ages) are similar for males and females for all CAL categories.
Table 3.3: ABS estimates of the number of disabled males with CAL (thousands).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>25–34</td>
<td>1,397.9</td>
<td>34.0</td>
<td>19.0</td>
<td>23.0</td>
<td>11.2</td>
<td>1,485.1</td>
</tr>
<tr>
<td>35–44</td>
<td>1,352.9</td>
<td>41.9</td>
<td>35.3</td>
<td>31.5</td>
<td>15.4</td>
<td>1,477.0</td>
</tr>
<tr>
<td>45–54</td>
<td>1,149.4</td>
<td>82.2</td>
<td>60.7</td>
<td>41.7</td>
<td>15.3</td>
<td>1,349.3</td>
</tr>
<tr>
<td>55–59</td>
<td>451.9</td>
<td>50.3</td>
<td>41.7</td>
<td>29.2</td>
<td>7.9</td>
<td>581.0</td>
</tr>
<tr>
<td>60–64</td>
<td>297.5</td>
<td>59.0</td>
<td>39.4</td>
<td>23.5</td>
<td>9.0</td>
<td>428.4</td>
</tr>
<tr>
<td>65–69</td>
<td>230.0</td>
<td>53.6</td>
<td>29.0</td>
<td>20.9</td>
<td>11.9</td>
<td>345.4</td>
</tr>
<tr>
<td>70–74</td>
<td>170.8</td>
<td>56.7</td>
<td>33.2</td>
<td>19.3</td>
<td>15.0</td>
<td>295.0</td>
</tr>
<tr>
<td>75–79</td>
<td>105.8</td>
<td>60.5</td>
<td>22.7</td>
<td>14.1</td>
<td>29.4</td>
<td>232.5</td>
</tr>
<tr>
<td>80–84</td>
<td>47.7</td>
<td>35.8</td>
<td>23.6</td>
<td>15.8</td>
<td>24.4</td>
<td>147.3</td>
</tr>
<tr>
<td>85–89</td>
<td>18.4</td>
<td>17.1</td>
<td>5.5</td>
<td>5.4</td>
<td>20.6</td>
<td>67.0</td>
</tr>
<tr>
<td>90 and over</td>
<td>2.2</td>
<td>4.0</td>
<td>3.2</td>
<td>3.7</td>
<td>9.8</td>
<td>22.9</td>
</tr>
</tbody>
</table>

Table 3.4: ABS estimates of the number of disabled females with CAL (thousands).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>25–34</td>
<td>1,395.2</td>
<td>30.1</td>
<td>10.9</td>
<td>23.9</td>
<td>9.4</td>
<td>1,469.5</td>
</tr>
<tr>
<td>35–44</td>
<td>1,342.2</td>
<td>50.2</td>
<td>35.7</td>
<td>43.6</td>
<td>8.3</td>
<td>1,480.0</td>
</tr>
<tr>
<td>45–54</td>
<td>1,121.4</td>
<td>78.7</td>
<td>73.4</td>
<td>50.1</td>
<td>24.3</td>
<td>1,347.9</td>
</tr>
<tr>
<td>55–59</td>
<td>414.0</td>
<td>56.8</td>
<td>45.9</td>
<td>29.7</td>
<td>16.6</td>
<td>563.0</td>
</tr>
<tr>
<td>60–64</td>
<td>290.8</td>
<td>48.5</td>
<td>42.1</td>
<td>28.1</td>
<td>13.6</td>
<td>423.1</td>
</tr>
<tr>
<td>65–69</td>
<td>243.2</td>
<td>41.7</td>
<td>34.6</td>
<td>17.8</td>
<td>18.9</td>
<td>356.2</td>
</tr>
<tr>
<td>70–74</td>
<td>181.3</td>
<td>47.1</td>
<td>41.9</td>
<td>31.4</td>
<td>25.4</td>
<td>327.1</td>
</tr>
<tr>
<td>75–79</td>
<td>138.2</td>
<td>58.1</td>
<td>33.5</td>
<td>23.3</td>
<td>39.6</td>
<td>292.7</td>
</tr>
<tr>
<td>80–84</td>
<td>73.9</td>
<td>36.1</td>
<td>20.2</td>
<td>28.3</td>
<td>60.4</td>
<td>218.9</td>
</tr>
<tr>
<td>85–89</td>
<td>27.6</td>
<td>12.3</td>
<td>13.3</td>
<td>14.4</td>
<td>56.9</td>
<td>124.5</td>
</tr>
<tr>
<td>90 and over</td>
<td>5.4</td>
<td>7.3</td>
<td>1.9</td>
<td>7.4</td>
<td>47.3</td>
<td>69.3</td>
</tr>
</tbody>
</table>

Source: Australian Bureau of Statistics (ABS) (2004a)
3.4 The multi-state model

In this section, we provide a detailed description of the multi-state model used to estimate the probability of disablement using the cross sectional data discussed in Section 3.3. Disability is defined as having any limitation
with core activities. We begin by describing the outline of the model and the fitting methodology, before presenting the fitting result and the estimated probabilities of disablement. The results presented here are related to the implementation of the model to the data from the 2003 SDAC survey.

3.4.1 Outline

The modelling methodology adopted here is similar to Rickayzen and Walsh (2002) and Leung (2004). Nevertheless, we present an improvement in the fitting process which results in a less restrictive assumption being required and more data being used. The formulae for the transition probabilities are similar to Leung (2004).

The model adopted is a discrete time multi-state model with six states as described below:

State 0: No CAL (i.e. able)  
State 1: Mild CAL  
State 2: Moderate CAL  
State 3: Severe CAL  
State 4: Profound CAL  
State 5: Dead.

Note that our modelling approach where we use of all five categories is different from some analysts', where the states mild CAL and moderate CAL, and severe CAL and profound CAL are usually grouped (for example Davis et al. (2001) and Australian Institute of Health and Welfare (AIHW) (2008)). We have taken our approach to fully utilize available information and to capture the dynamics of the disablement process more accurately.

The transition probabilities are annual. Deterioration to any worse disability state is possible over the course of a year, while improvement is only allowed by one category. The transition probabilities depend only on age, gender, year and disability category; no account is taken of how or when someone arrived in that category (i.e. we ignore duration and past disability experience). The possible transitions over a one-year period are presented in Figure 3.1 (adapted from Leung (2004)). Note that we assume only one transition is possible over a one-year period.

There are three types of transition probabilities: death probabilities, deter-
Figure 3.1: Multi-state model for disability.

3.4. THE MULTI-STATE MODEL

There are three types of transition probabilities: death probabilities, deterioration probabilities (probabilities of deteriorating to any lower disability state) and improvement probabilities (probabilities of recovering to a less disabled state). Mathematical functions are used to formulate deterioration probabilities and the additional mortality due to disablement, while the likelihood of improvement is included as an assumption.

3.4.2 Fitting methodology

The objective of the fitting process is to estimate the parameters of the formulae for the probabilities of deterioration. The probabilities of death and of improvement in disability are determined separately and included as an input in the fitting process. The parameters are estimated such that the disability prevalence rates reported in the SDAC survey (Tables 3.5 and 3.6) are closely replicated by the multi-state model. This is done as follows.

1. Under a set of assumptions (discussed next), the disability prevalence rates at any age will remain constant over time. The multi-state model will produce these prevalence rates at a single age.
2. We estimate the total population at each single age at the survey date (i.e. single age estimates of the last column of Tables 3.3 and 3.4).

3. We multiply prevalence rates and population from steps 1 and 2 to obtain the model estimate of the number of people at single ages for each disability category at the survey date.

4. From step 3, we calculate similar quantities as in Tables 3.5 and 3.6. These are regarded as the model prevalence rates.

5. Values of the parameters for the deterioration probabilities are estimated such that model prevalence rates closely replicate the reported prevalence rates in SDAC (i.e. closely replicate Tables 3.5 and 3.6).

For step 1 above, the following assumptions regarding the (actual) population process in the past are adopted:

(a) Constant prevalence rates of disability at birth (proportion of babies born into each disability category remains constant over time).

(b) Transition intensities between disability states and to the dead state are time invariant (but age variant).

(c) Disability prevalence rates for overseas migrants (immigrants and emigrants) at any age are the same as the disability prevalence rates of the population at that age.

(d) Immigrants experience the same disability transition probabilities (and death probabilities) as the population considered.

Note that due to active selection amongst migrants and current immigration policy, it is unlikely that assumption (c) holds. This assumption is adopted due to the limited amount of available data on the disability level of overseas migrants. Note that the inaccuracy caused by this assumption is unlikely to materially impact the resulting estimates of transition probabilities at ages 60+ since, at these ages, the size of overseas migration is very small (see Figure 3.6 of ABS (2005)).
3.4. THE MULTI-STATE MODEL

Under assumptions (a) to (d), the prevalence rates of each disability category at any age will remain constant over time. The multi-state model will produce these prevalence rates at a single age by adopting a stationary population model. That is, suppose we have another population where the number of babies born (alive) in each disability category is the same every year and the disability prevalence rates of these newborns is the same as the actual population. This alternative population is closed from overseas migration and experiences the same constant transition probabilities (and death probabilities) as the actual population. Note that this stationary population has the same disability prevalence rates as the actual population at any age (assuming (a) to (d) above applied to the actual population). Therefore, disability prevalence rates at a single year of age for the actual population (under simplifying assumptions of the past process of the actual population) can be calculated from this alternative stationary population. This is straightforward as only disability prevalence rates of the newborns are needed. In addition, as the proportion of disabled newborns is generally very small over time (as indicated from the prevalence rates at ages 0 to 4 years from the last five SDAC surveys), the constant disability prevalence rates assumption of these newborns is not too unrealistic.

Our actual fitting starts from age 25 rather than from newborns. Basically, we assume that the disability prevalence rates at age 25 years (which are assumed to be constant) are known. These are taken to be equal to the reported disability prevalence rates at ages 15 to 24 years. Note that the proportion of disabled people at age 25 is very small. The reason we start from age 25 is to improve the fit of the model and hence the accuracy of the estimated probability of disablement. Note that unlike in Leung (2004), as our objective is to estimate the probabilities of disablement for ages 60 and above, we are able to begin at an older age in our fitting process. However, the starting age should be chosen as the age where the proportion of disabled population can still be perceived to be small over time (so that the constant disability prevalence rates assumption for this starting population of the fitting process is not too unrealistic). The data suggest that age 25 provides the right balance.
To estimate the total population at a single age at the survey date, we rely on the observation that the age structure of the population does not change significantly from one year to the next. Figure 3.2 presents the proportion of the estimated resident population (the estimate of the number of people who live in Australia; for more details see ABS (2009)) at a single year of age (in the figure, the proportion is calculated as total population at a particular age divided by total population across all ages) for males and females at the middle of 2003 and 2004. The data are obtained from ABS (2009). Note that the proportion of the population at each age is similar for the populations at the middle of 2003 and 2004. This similarity is also evident for the populations at the middle of 1997 and 1998. Therefore, the estimate of the population at a single age interval at the survey date can be obtained by multiplying similar proportions to the total population at the survey date reported in SDAC (last column of Tables 3.3 and 3.4). For this purpose, the proportion is calculated relative to the population at each age group. We use the estimated resident population (ERP) at the closest date from the SDAC survey period (for example, we use ERP at the middle of 2003 for the implementation of the model to the data from the 2003 SDAC). Note that it is possible to apply linear interpolation to estimate the resident population at the middle of the survey period; however this is not considered necessary. Estimates for ages 100 and above use functions from the life tables of Australia in 2003 published by the Human Mortality Database (www.mortality.org).

Application of the above procedures will give an estimate of the total population at a single age interval at the survey date. Note that the age definition used by the ABS is age last birthday. We assume that the effective age for a life aged \( x \) last birthday (where \( x \) is a non-negative integer) is \( x + 0.5 \). Hence, in our fitting, we need to project the population on a yearly basis from exact age \( x + 0.5 \). Transition probabilities in the multi-state model need to reflect this.

Note that our fitting methodology is slightly different to that of Rickayzen and Walsh (2002). In our fitting process, we assume constant disability prevalence rates. However, the age structure of the population is allowed to vary.
with time. Under this approach, the age structure of the population at the survey date is estimated separately from the fitting process. In Rickayzen and Walsh’s methodology, the disability prevalence rates and the age structure of the population are assumed to be constant (and are estimated in the fitting process). Therefore, under the original methodology, it is possible that the estimated age structure of the population at the survey date (by the multi-state model) might be different to the actual age structure. In their fitting, Rickayzen and Walsh (2002) found that although their multi-state model produced the exact prevalence rates for the healthy category for the population aged 80 and over, it seemed to overestimate the prevalence rates of the healthy category for people both aged 80 to 84, and 85 and over. This could happen if the age structure produced by the multi-state model is different from the actual age structure.

In the following section, we provide detailed discussion of the transition probabilities estimated in the multi-state model. As a convention, the states (or categories) are denoted by \( \{n : n = 0, 1, \ldots, 5\} \) where \( n = 0 \) corresponds to no CAL (healthy), \( n = 1 \) corresponds to mild CAL, \( n = 2 \) corresponds to moderate CAL, \( n = 3 \) corresponds to severe CAL, \( n = 4 \) corresponds to profound CAL and \( n = 5 \) corresponds to the dead state. The estimated parameters of the formulae for transition probabilities are presented from the application of the model to data from the 2003 SDAC.
3.4.3 Mortality

We model mortality rates as being higher for people in a more severe disability category and, to reflect this, we split the mortality rate which applies to an individual in state $n$ ($n = 0, 1, \ldots, 4$) into two parts. The first part applies equally to people in any disability state, while the second is higher for more severely disabled people. Specifically:

$$Mortality(x + 0.5, n) = \text{Overall Mort}(x + 0.5) + \text{Additional Mort}(x + 0.5, n) \quad (3.1)$$

where

$Mortality(x + 0.5, n)$ is the one year death probability which applies to an individual aged exact $x + 0.5$ and in state $n$,

$\text{Overall Mort}(x + 0.5)$ is the component of the one year death probability which applies to any individual aged exact $x + 0.5$ regardless of the severity of his/her disability (this component is also called healthy mortality),

$\text{Additional Mort}(x + 0.5, n)$ is additional annual mortality due to disability for an individual aged exact $x + 0.5$ and in state $n$.

Similar to Leung (2004), the formula for the additional annual mortality due to disability is expressed as (for $n = 0, 1, \ldots, 4$):

$$\text{Additional Mort}(x + 0.5, n) = \frac{M}{1 + 1.1^{90-(x+0.5)}} \times \frac{\max(n - 2, 0)}{2} \quad (3.2)$$

where $M$ is the maximum additional annual mortality due to disability. The function in (3.2) has the following features:

- Weak age dependence (for ages 65 and above) in the disability related addition to healthy mortality.
- Additional mortality is low at younger ages.
• No additional mortality for people with mild or moderate CAL (these disability categories are not life threatening conditions).

• Age 50 is the pivotal age and 1.1 is the steepness factor.

There is limited information in Australia regarding the dependency of mortality on disability. One such information source is presented by Giles et al. (2004) from their study of longitudinal data from The Australian Longitudinal Study of Ageing (ALSA). In their study, there is a measurement of mortality rates from people with so called mobility and Nagi disability. However, mobility and Nagi disability measure disabling conditions which are less severe than mild CAL (discussed in Section 3.4.5). Note that to set the value of $M$, we need the estimate of mortality from the most severe CAL category (profound).

Considering the limitations of Australian data, we look at data from overseas. The most recent report by the Long Term Care (LTC) Experience Committee of the Society of Actuaries (SOA) in the U.S. presents useful information to set the value of $M$. This report examines the experience of private long term care (LTC) insurance plans in the U.S. over two decades (from 1/1/1984 to 31/12/2004) (Corliss et al., 2007). In this report, there are data on the discrepancy of mortality rates between active (“non-claim”) and disabled (“on-claim”) lives. Tables 3.7 and 3.8 present this information for the last few age groups. The figures in these tables are based on our own calculation from the data in Appendix H-8 of the report. Note that people with profound CAL can be associated with disabled (“on-claim”) lives defined in this report, as they always need assistance with core activities. (Leung (2006) stated that a profound CAL is equivalent to the failure of 5 or 6 ADLs).

Tables 3.7 and 3.8 suggest the values of $M$ of 0.28 for males and 0.19 for females. However, there is evidence that the “on-claim” lives in private LTC insurance plans in the U.S. are less healthy than the Australian population with profound CAL. Table 3.9 shows the prevalence rates of the severe diseases for the Australian population with profound CAL (as reported in the 1998 and 2003 SDAC) and for the “on-claim” lives in private LTC insurance
Table 3.7: Males, excess mortality due to disability, U.S. private LTC insurance experience.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Mortality rate</th>
<th>Difference disabled – active</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Active lives</td>
<td>Disabled lives</td>
</tr>
<tr>
<td>60–69</td>
<td>0.00490023</td>
<td>0.24062671</td>
</tr>
<tr>
<td>70–79</td>
<td>0.01377689</td>
<td>0.27649010</td>
</tr>
<tr>
<td>80–89</td>
<td>0.03735554</td>
<td>0.33345156</td>
</tr>
<tr>
<td>90 and above</td>
<td>0.11499330</td>
<td>0.43284773</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>0.28</td>
</tr>
</tbody>
</table>

Table 3.8: Females, excess mortality due to disability, U.S. private LTC insurance experience.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Mortality rate</th>
<th>Difference disabled – active</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Active lives</td>
<td>Disabled lives</td>
</tr>
<tr>
<td>60–69</td>
<td>0.00321742</td>
<td>0.21442357</td>
</tr>
<tr>
<td>70–79</td>
<td>0.00872466</td>
<td>0.18786510</td>
</tr>
<tr>
<td>80–89</td>
<td>0.02382519</td>
<td>0.20078075</td>
</tr>
<tr>
<td>90 and above</td>
<td>0.07781787</td>
<td>0.28530153</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>0.19</td>
</tr>
</tbody>
</table>

Data source: Corliss et al. (2007)

plans in the U.S. (as reported in Corliss et al. (2007)). These diseases are the top four leading causes of death in the high-income countries and contribute up to 46% of total deaths in 2008 for these countries (World Health Organization (WHO), 2011). The ranking in Table 3.9 is based on our own calculation from the data reported in WHO (2011).

Overall, the prevalence rate of the four leading diseases for the Australian population with profound CAL is around half of the prevalence rate for the “on-claim” lives. However, for cancer (which is the second leading cause of death), the prevalence rate for the Australian population (with profound CAL) is much lower than the prevalence rate for the “on-claim” lives. Therefore, we decided to adopt only 40% of the values of $M$ suggested by Tables 3.7 and 3.8. Note that Leung (2004) adopted the same level of discount to the maximum annual additional mortality (due to disability) suggested by LTC data from the U.S.. Table 3.10 presents the adopted values of $M$ for
### Table 3.9: Prevalence rates of the leading causes of death.

<table>
<thead>
<tr>
<th>Rank</th>
<th>Disease</th>
<th>Australia Profound CAL</th>
<th>Australia “On-claim” lives</th>
<th>U.S. “On-claim” lives</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Circulatory (incl. stroke)</td>
<td>14%</td>
<td>12%</td>
<td>22%</td>
</tr>
<tr>
<td>2</td>
<td>Cancer</td>
<td>2%</td>
<td>2%</td>
<td>10%</td>
</tr>
<tr>
<td>3</td>
<td>Alzheimer</td>
<td>11%</td>
<td>11%</td>
<td>24%</td>
</tr>
<tr>
<td>4</td>
<td>Respiratory</td>
<td>5%</td>
<td>6%</td>
<td>5%</td>
</tr>
<tr>
<td></td>
<td><strong>Total</strong></td>
<td><strong>32%</strong></td>
<td><strong>31%</strong></td>
<td><strong>61%</strong></td>
</tr>
</tbody>
</table>

Each gender and Figure 3.3 shows the additional mortality in the severe and profound CAL categories.

#### Table 3.10: Adopted values of $M$.

<table>
<thead>
<tr>
<th></th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>0.11</td>
<td>0.08</td>
</tr>
</tbody>
</table>

#### Figure 3.3: Additional mortality for severe CAL and profound CAL.

Once the additional mortality has been set, the healthy mortality is determined by the requirement that the total number of deaths from each disability category should match the total number of deaths if the whole population (total population across all disability categories) experience mortality rates as specified in Australian life tables (ALT) for the period 2002–2004 published by ABS (2008a). Note that this requires an iterative process in the
fitting of the multi-state model. We employ Gompertz’ formula to extrapolate the mortality rates of the ALT to age 109. The one-year life table death probability from exact age $x + 0.5$ ($q_{x+0.5}$) is estimated using

$$q_{x+0.5} = 1 - \exp \left\{ -0.5 \times (\mu_{x+0.5} + \mu_{x+1.5}) \right\}$$

(3.3)

where $\mu_x$ is the force of mortality at exact age $x$ which is estimated from the ALT for the period 2002–2004.

### 3.4.4 Deterioration probabilities

Formulæ for deterioration probabilities are mainly concerned with the probability of a healthy person making a transition into a particular CAL state over the year given that (s)he survives. This probability is split into two parts: the probability of a healthy person becoming disabled (attaining any CAL state) over the year and the probability of a healthy person attaining a particular CAL state given that (s)he became disabled over the year. The probability of a disabled person (a person who is already in a CAL state) making a transition into a more severe CAL state is obtained by applying a multiplier (deterioration factor) to the probability of a healthy person making a transition into that particular more severe CAL state. The formulæ are the same for males and females and similar to those in Leung (2004).

The formula for the probability that a healthy person aged $x$ makes a transition into any CAL state over one year is:

$$\text{New}_{.}\text{CAL}(x) = \alpha \left( \left( A + \frac{D - A}{1 + B^{c-x}} \right) \times \left( 1 - \frac{1}{3} \exp \left\{ - \left( \frac{x - E}{4} \right)^2 \right\} \right) \right)$$

(3.4)

where the six parameters are $A$, $B$, $C$, $D$, $E$ and $\alpha$.

The formula above is logistic in form and is based on Perks (1932) formula for the graduation of mortality rates. Note that the original formulation by Rickayzen and Walsh (2002) does not include the scale parameter $\alpha$. In this case, $A$ is the limit of the probability of becoming disabled (attaining
any CAL state) at young ages, while \( D \) is the limit at extremely high ages. Parameters \( B \) and \( C \) determine how rapidly the probability changes from \( A \) to \( D \) as age increases. Parameter \( E \) determines the age at which there is a “kink” in the function of the disablement probability. Table 3.11 presents the estimated values of the parameters for (3.4) for males and females. The parameter values presented in Tables 3.11, 3.12 and 3.13 are obtained by fitting the model to the 2003 SDAC data.

Table 3.11: Parameter values for \( \text{New\_CAL}(x) \).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>7.94291</td>
<td>7.72074</td>
</tr>
<tr>
<td>( A )</td>
<td>0.00005</td>
<td>0.00000</td>
</tr>
<tr>
<td>( B )</td>
<td>1.07730</td>
<td>1.06808</td>
</tr>
<tr>
<td>( C )</td>
<td>98.09020</td>
<td>97.38444</td>
</tr>
<tr>
<td>( D )</td>
<td>0.10171</td>
<td>0.06888</td>
</tr>
<tr>
<td>( E )</td>
<td>66.97864</td>
<td>66.46714</td>
</tr>
</tbody>
</table>

The probability of becoming disabled (attaining any CAL state) is mainly constrained by the observed sum of prevalence rates of disabled (CAL) categories. Figure 3.4 presents this sum as reported in the 2003 SDAC. For both genders, the following pattern is evident: disabled prevalence rates increase from age group 25–34 with the increase slowing down between age group 55–59 and 65–69 before it rapidly increases from age group 65–69. Roughly, this suggests a turning point of the trend of the prevalence rates at around age 67. This feature of the data (the “kink” feature) is captured by parameter \( E \) in (3.4). Note that its value is close to 67 for both genders.

Figure 3.5 presents the estimate of the probability of transition from a healthy state to any CAL state over a one year period using the parameter values presented in Table 3.11.

The formula for the probability that a healthy person aged \( x \) makes a transition into disability state \( n \) (1 \( \leq n \leq 4 \)) given that (s)he becomes disabled over the year is:

\[
\text{Severity}(x,n) = \frac{W(n) \times f(x)^{n-1}}{\text{Scale}(x)},
\]  

(3.5)
Figure 3.4: Sum of prevalence rates of CAL categories, 2003 SDAC (per 1000).

![Graph of prevalence rates for males and females across age groups.](image)

Figure 3.5: New CAL(x).

![Graph of probability across exact age for males and females.](image)

where

\[ f(x) = F + \frac{1 - F}{1 + G^{H-x}} \]  \hspace{1cm} (3.6)

and

\[ \text{Scale}(x) = \sum_{n=1}^{4} W(n) \times f(x)^{n-1}, \]  \hspace{1cm} (3.7)

where \( F, G, H, W(1), W(2), W(3) \) and \( W(4) \) are parameters.

The relative probability of attaining each disability state changes with age, with a higher likelihood of entering a more severe disability state for older ages. This age dependence feature is reflected in the parameters \( F, G \) and \( H \). There is a category width \( W(n) \) for each disability state which
allows for some categories to have more people than others. Lastly, the term $Scale(x)$ ensures that these probabilities sum up to one. The estimated values of parameters for $Severity(x, n)$ are presented in Table 3.12 for males and females.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>0.65189</td>
<td>0.40858</td>
</tr>
<tr>
<td>$G$</td>
<td>1.23455</td>
<td>1.32060</td>
</tr>
<tr>
<td>$H$</td>
<td>92.17871</td>
<td>84.54104</td>
</tr>
<tr>
<td>$W(1)$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$W(2)$</td>
<td>0.51221</td>
<td>1.90451</td>
</tr>
<tr>
<td>$W(3)$</td>
<td>0.31547</td>
<td>1.46407</td>
</tr>
<tr>
<td>$W(4)$</td>
<td>0.90409</td>
<td>5.42737</td>
</tr>
</tbody>
</table>

Figure 3.6 presents the comparison of $Severity(x, 4)$ (the values are derived using the parameter values presented in Table 3.12) and the reported prevalence rates of profound CAL (2003 SDAC). The shape of $Severity(x, 4)$ shows the following pattern: largely constant at “young ages” (i.e. from age 25 to around age 80 for males (age 70 for females)), rapidly increasing at old ages and stabilizing at very high ages. For females, the rapid increase starts from an earlier age (around age 75 as compared to age 85 for males) and has a steeper pattern. At all ages, the values of $Severity(x, 4)$ are higher for females, however, the difference is only significant from age 75 onwards (from this age, the difference is increasing with age). Note that these features of $Severity(x, 4)$ can be traced directly to the reported prevalence rates of profound CAL.

Finally, the probability of a healthy individual aged $x$ making a transition into disability category $n$ ($1 \leq n \leq 4$) over one year is:

$$Deteriorate(x, 0, n) = New_{CAL}(x) \times Severity(x, n). \quad (3.8)$$

The probability of an individual in disability category $m$ ($1 \leq m \leq 3$) de-
Figure 3.6: Severity($x, 4$) and reported prevalence rates (2003 SDAC) of profound CAL (per 1000).

Deteriorating to disability category $n$ ($m < n \leq 4$) is:

$$ Deteriorate(x, m, n) = Deteriorate(x, 0, n) \times I^m. \quad (3.9) $$

The parameter $I$ (deterioration factor) is required to be higher than one. This is because a disabled person is more likely to make a transition into a more severe disability state than a healthy person. The estimated values of $I$ for males and females are given in Table 3.13.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>1.524242</td>
<td>1.685492</td>
</tr>
</tbody>
</table>

Note that in fitting the multi-state model, we need to project the population from age $x + 0.5$. The one year transition probability from age $x + 0.5$ is estimated by:

$$ Deteriorate(x + 0.5, m, n) = 0.5 \times [Deteriorate(x, m, n) + Deteriorate(x + 1, m, n)]. \quad (3.10) $$
3.4.5 Improvements

We assume that people in any CAL state can only improve by one category over a one year interval if and only if they survive the year and do not deteriorate to a more severe disability state. We also assume that the recovery rates (conditional on survival and non-deterioration of disability) are age and gender invariant. These assumptions are the same as those adopted by Rickayzen and Walsh (2002) and Leung (2004). In reality, the recovery rates are likely to be different between genders and might be decreasing with age. In particular, given the strong policy position to get young people out of aged care homes (for example, Younger People in Residential Aged Care policy), younger people may be perceived to experience higher recovery rates. However, given the available data, we believe that any attempt to differentiate recovery rates by gender and (or) age is speculative. It is unlikely that such an attempt (given the possible error) will result in more accurate estimates of the disablement probabilities. In addition, given the lower disability prevalence rates of younger people, the inaccuracy of the estimated improvement rates at young ages is unlikely to materially impact the accuracy of the estimated disability transition probabilities at ages 60+ (the relevant ages for pricing analysis of a reverse mortgage contract). As discussed later, the data that we used to estimate the improvement rates are mainly related to old ages. Note that a simplified assumption regarding the recovery from disability is commonly adopted in disability analysis (discussed in Section 3.5.2). Our (unconditional) recovery rates are different between genders and decreasing with age due to the feature of the estimated mortality and disablement probabilities (which vary between genders and increase with age).

There is limited information in Australia regarding recovery from disability. The AIHW (2009) reports the number of discharges from aged care homes (ACH) due to death, return to community, transfer to hospital, transfer to continuing residential care and “other”. We refrain from using this data to estimate the disability improvement rates for two reasons. Firstly, the disability improvement rates of ACH residents might not be representative for the whole population (which might be due to the higher severity
of the disablement and the medical conditions of ACH residents). Secondly, in this survey, some of the reported discharges due to returns to community might not be due to improvement in disability, and similarly, some disability improvements might be captured in discharges due to transfer to continuing residential care and “other”.

Another useful information source is presented in a study by Giles et al. (2004). We use the information presented in this study to set our assumption regarding the recovery from mild and moderate CAL. The recovery from profound CAL is set using the data from the report by the LTC committee of the SOA as in Section 3.4.3.

Recovery rates from mobility and Nagi disability (defined below) from the first six waves of ALSA are presented by Giles et al. (2004). Mobility disability is defined as a failure of either (or both) of: walking up and down a flight of stairs; and walking half a mile without help. Nagi disability is defined as having more than a little difficulty in performing at least one of the following five tasks: pushing or pulling large objects, stooping or crouching or kneeling, lifting or carrying 10 pounds, reaching or extending arms and writing or handling small objects. Therefore, mobility and Nagi disability measure disabling conditions which are less severe than mild CAL. Nevertheless, this information is useful in setting the recovery assumption from mild and moderate CAL. The recovery rates from the first five waves of the ALSA study are presented in Table 3.14 (the interval between waves five and six is more than one year and therefore it is excluded). These rates are unconditional on survival or non-deterioration of disability.

Table 3.14: Recovery rates between consecutive waves of ALSA study.

<table>
<thead>
<tr>
<th>Disability definition</th>
<th>Waves 1 to 2</th>
<th>Waves 2 to 3</th>
<th>Waves 3 to 4</th>
<th>Waves 4 to 5</th>
<th>Average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mobility</td>
<td>24.3%</td>
<td>18.3%</td>
<td>15.1%</td>
<td>13.4%</td>
<td>17.78%</td>
</tr>
<tr>
<td>Nagi</td>
<td>20.9%</td>
<td>11.8%</td>
<td>15.4%</td>
<td>12.5%</td>
<td>15.15%</td>
</tr>
</tbody>
</table>

Source: Giles et al. (2004)

Therefore, the annual recovery rates assumption (conditional on survival and non-deterioration of disability) from mild and moderate CAL are set at
15%. We assume lower recovery rates than suggested by Table 3.14 since mobility and Nagi disability measure disabilities which are less severe than mild or moderate CAL.

To set the annual recovery rate assumption from profound CAL, we use data from the report by the LTC committee of the SOA as in Section 3.4.3, as people with profound CAL can be associated with “on-claim” lives in the report. However, only a rough approximation to the one year recovery rate can be made from this report. In addition, the approximation is further complicated by the fact that the claims which terminate during the elimination (waiting) period are excluded from the analysis and the persistency on claim (the duration of the claim) is measured from the end of the elimination period.

Around 50% of the claims analysed in the report have duration of less than a year. This figure is broadly consistent across ages (higher than 55) and gender. Table 3.15 presents the proportion of claims which persisted for at least 365 days by gender.

Table 3.15: Proportion of claims persisting at least 365 days, U.S. private LTC insurance experience.

<table>
<thead>
<tr>
<th></th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Males</td>
<td>50.37%</td>
</tr>
<tr>
<td>Females</td>
<td>52.24%</td>
</tr>
</tbody>
</table>

Source: Corliss et al. (2007)

A claim might terminate because of death, benefit expiry, recovery or transfer to a non-covered level of care. From all claims which have been closed, 27% are terminated because of recovery. Therefore, by assuming that 27% of claims which terminated in the first year were caused by recovery, the approximate one year recovery rate of the “on-claim” lives is around 13.5%. However, as the experience of people with LTC insurance coverage might be different to the experience of the general population (for example, they might be exposed to the “encouragement” to recover from insurers) and considering the approximation method adopted, the annual recovery rate (conditional on survival and non-deterioration of disability) that we adopt is 10%. Note that
these recovery rates are the same as the rates adopted by Rickayzen and Walsh (2002). Rickayzen and Walsh (2002) also refrained from using the high recovery rate suggested by U.S. data as it might represent the recovery from temporary disability (like breaking a bone) rather than characterising the recovery from long term disability.

The one-year recovery rate from severe CAL is assumed to be in between the recovery rate from moderate and profound CAL and is therefore set at 12.5%.

To summarise, the annual recovery rates assumption (conditional on survival and non-deterioration of disability) that we adopt for each CAL category are presented in Table 3.16 below.

<table>
<thead>
<tr>
<th>CAL Level</th>
<th>Mild</th>
<th>Moderate</th>
<th>Severe</th>
<th>Profound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recovery rate</td>
<td>15%</td>
<td>15%</td>
<td>12.5%</td>
<td>10%</td>
</tr>
</tbody>
</table>

3.4.6 Projection method for estimation of transition probabilities

We employ a similar projection method as described by Rickayzen and Walsh (2002) and Leung (2004). Projections are needed both in the fitting of the multi-state model and in the method to improve the transition probabilities estimated from the multi-state model (described in Chapter 4). For convenience, we describe the projection method applicable in Chapter 4: mortality probabilities vary with time, and overseas migration is included. In the fitting of the multi-state model, mortality probabilities are assumed to be time invariant, and overseas migration is ignored. In the following we describe the projection method. Note that we need to project the population from exact age $x + 0.5$, where $x$ is a non-negative integer.

Let $N^n_{x+0.5}(t)$ denote the size of the population aged $x + 0.5$ exact and in state $n$ (for $n = 0, 1, \ldots, 4$) at the middle of year $t$, and $NOM^n_{x+0.5}(t)$ denote the number of net overseas migrants from the middle of year $t$ to the middle
3.4. THE MULTI-STATE MODEL

of year $t + 1$ where the migrants are aged $x + 0.5$ exact and in state $n$ (for $n = 0, 1, \ldots, 4$) when they migrate. The estimation of $NOM_{x+0.5}^n(t)$ will be described in Chapter 4. We assume that half of overseas migrations between the middle of year $t$ and year $t + 1$ occur at the middle of year $t$ and half at the middle of year $t + 1$. The following equation determines $N_{x+0.5}^n(t)$:

$$N_{x+0.5}^n(t) = \left[ N_{x-0.5}^n(t - 1) + \frac{NOM_{x-0.5}(t - 1)}{2} \right] \times [1 - \text{Mortality}(x - 0.5, t - 1, n)] \times [1 - \text{Deteriorate\_From}(x - 0.5, n)] \times [1 - \text{Improve\_From}(x - 0.5, n)] + \text{Deteriorate\_To}(x + 0.5, t, n) + \frac{\text{Improve\_To}(x + 0.5, t, n) + NOM_{x+0.5}(t - 1)}{2}. \quad (3.11)$$

$\text{Mortality}(x + 0.5, t, n)$ is the probability that a person aged $x + 0.5$ exact and in state $n$ at the middle of year $t$ dies within one year. This probability is similar to (3.1) however it includes time dependence in the overall mortality component (more details in Section 4.1).

$\text{Deteriorate\_From}(x + 0.5, m)$ represents the probability that a person aged $x + 0.5$ exact and in state $m$ makes a transition to any more severe disability state within one year given that (s)he survives during the year. Therefore

$$\text{Deteriorate\_From}(x + 0.5, m) = \sum_{n=m+1}^{4} \text{Deteriorate}(x + 0.5, m, n). \quad (3.12)$$

$\text{Improve\_From}(x + 0.5, n)$ is the probability that a person aged $x + 0.5$ exact and in disability state $n$ (for $n = 1, 2, \ldots, 4$) who survives and does not deteriorate over one year, improves by one state during the year. As
described in Section 3.4.5 we set:

\[
    \text{Improve\,From} (x + 0.5, n) = \begin{cases} 
        0.1 & : \ n = 4, \\
        0.125 & : \ n = 3, \\
        0.15 & : \ n = 1, 2.
    \end{cases} \tag{3.13}
\]

\textit{Deteriorate\,To} (x + 0.5, t, n) represents the number of people aged \(x + 0.5\) exact at the middle of year \(t\) who made a transition to disability state \(n\) from any less severe disability state from the middle of year \(t - 1\) to the middle of year \(t\). The value is obtained using:

\[
    \text{Deteriorate\,To} (x + 0.5, t, n) = \left\{ \text{Exposed\,To\,Det} (x - 0.5, t - 1, m) \times \text{Deteriorate} (x - 0.5, m, n) \right\} \tag{3.14}
\]

where

\[
    \text{Exposed\,To\,Det} (x + 0.5, t, n) = \left[ N^n_{x+0.5}(t) + \frac{NOM^n_{x+0.5}(t)}{2} \right] \times [1 - \text{Mortality} (x + 0.5, t, n)]. \tag{3.15}
\]

\textit{Improve\,To} (x + 0.5, t, n) represents the number of people aged \(x + 0.5\) exact at the middle of year \(t\) who made a transition to state \(n\) from disability state \(n + 1\) from the middle of year \(t - 1\) to the middle of year \(t\). The value is obtained using:

\[
    \text{Improve\,To} (x + 0.5, t, n) = \text{Exposed\,To\,Imp} (x - 0.5, t - 1, n + 1) \times \text{Improve\,From} (x - 0.5, n + 1) \tag{3.16}
\]

where

\[
    \text{Exposed\,To\,Imp} (x + 0.5, t, n) =
\]
3.4.7 Fitting procedure and results

In subsection 3.4.2 we described the overall methodology for the fitting process. The exact steps in the fitting procedures are described below.

1. We start from a population aged 25 last birthday (which is assumed to be aged 25.5 exact). We assume that disability prevalence rates at age group 15 to 24 reported in the 2003 SDAC apply to this 25 year old population.

2. This starting population is projected forward at one year intervals up to the assumed limiting age (109.5 exact) under the transition probabilities described in Sections 3.4.3 to 3.4.5 and under the simplified version of the projection method described in Section 3.4.6; overseas migrations are ignored and mortality probabilities are assumed to be time invariant.

3. Every year, the same number of 25 year olds enters the population (also assumed to be aged 25.5 exact) with disability prevalence rates as described in point 1. They experience time invariant transition probabilities and overseas migrations are ignored as described in point 2. We assume that the actual numbers of transitions are the same as their expected values. Therefore, we have a stationary population.

4. We calculate disability prevalence rates of this stationary population at single year age intervals and multiply these rates with the single year age estimates of the total population at the survey date. Therefore, we obtain a model estimate of the population at single year age intervals for each disability category at the survey date. We regard this as the model population.
5. From the model population, we calculate similar quantities as in Tables 3.5 and 3.6. These are regarded as model prevalence rates.

6. We calculate the absolute value of the difference between model prevalence rates and the prevalence rates reported in the 2003 SDAC (Tables 3.5 and 3.6) for each age group and disability category and sum them up (in the summation we exclude the healthy category as the prevalence rate for this category is 1,000 minus the sum of the prevalence rates of all CAL states). Parameters of the deterioration probabilities are estimated such that this sum is minimised. Note that this approach follows Rickayzen and Walsh (2002).

7. We recalculate the overall mortality component of the one year mortality probability (see (3.1)) such that the total number of deaths from each disability category (including healthy) in the model population matches the total number of deaths if the whole model population experienced mortality rates as specified in Australian life tables for the period 2002–2004 published by ABS (2008a).

8. Iteratively repeat points 6 and 7 until we obtain convergence of deterioration probabilities and mortality probabilities.

A program was written in R to fit the multi-state model. Note that in the fitting process we need to set non-linear constraints (for example, we require all probabilities to be between 0 and 1). Optimisation of the multi-state program with non-linear constraints is performed using the function “constrOptim.nl” (Varadhan, 2011).

Given the large number of parameters to be estimated, the optimal fit is difficult to find. For example, there are many local minima encountered in the fitting process. However, we feel that, overall, a reasonably sufficient fit has been obtained (discussed below). Tables 3.17 and 3.18 present the differences between the prevalence rates reported in the 2003 SDAC survey (Tables 3.5 and 3.6) and those produced by the multi-state model, and Figure 3.7 presents a comparison between them.
Note that for profound CAL, the most important category for the estimation of probability of admission into an aged care home, the model replicates the reported prevalence rates well for both genders.

Admittedly, the relative errors (proportion of the difference between reported and model prevalence rate to the reported rate) are high for some disability categories and age groups. Note that the values of reported prevalence rates range from around 90% to 0.6%. In such a wide interval, it is difficult to obtain small relative errors for all reported prevalence rates. However, the relative errors are small for the prevalence rates which are significant which indicate that, in general, our model sufficiently captures the main dynamics of the disability process.

There seems to be a systematic error across age groups for some disability categories. For example, model prevalence rates for females with severe CAL are consistently below the reported prevalence rates from age groups 25–34 to 60–64 (see Figure 3.7). Note that the structure of the reported disability data is complex: prevalence rates do not vary smoothly across ages and disability categories, and the dependence of the prevalence rate on age is very different across categories. In addition, the formula for transition probabilities might not fully capture the dynamic of the disability process and the assumptions employed might not exactly reflect the actual disability experience of the population (for example, it might be possible for someone to improve by more than one disability category over one year). The complex structure of the disability data and the possible inadequacy of the modelling methodology might cause the systematic error in the model prevalence rates. However, in general, the pattern of the model prevalence rates sufficiently replicates the pattern of the reported prevalence rates.

The errors of model prevalence rates are small for prevalence rates which are significant and its pattern sufficiently replicates the pattern of the reported prevalence rates. Therefore, we conclude that overall, a reasonably sufficient fit has been obtained.
### Table 3.17: Difference in prevalence rates for males (as percentages), data - model.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>25–34</td>
<td>-0.63%</td>
<td>0.00%</td>
<td>0.27%</td>
<td>0.44%</td>
<td>-0.08%</td>
</tr>
<tr>
<td>35–44</td>
<td>-0.57%</td>
<td>-1.15%</td>
<td>0.65%</td>
<td>1.03%</td>
<td>0.04%</td>
</tr>
<tr>
<td>45–54</td>
<td>-1.56%</td>
<td>-0.97%</td>
<td>1.63%</td>
<td>1.46%</td>
<td>-0.57%</td>
</tr>
<tr>
<td>55–59</td>
<td>-1.99%</td>
<td>-2.05%</td>
<td>2.78%</td>
<td>2.60%</td>
<td>-1.34%</td>
</tr>
<tr>
<td>60–64</td>
<td>-4.00%</td>
<td>0.02%</td>
<td>3.35%</td>
<td>2.25%</td>
<td>-1.61%</td>
</tr>
<tr>
<td>65–69</td>
<td>-2.15%</td>
<td>-0.34%</td>
<td>1.37%</td>
<td>2.11%</td>
<td>-0.99%</td>
</tr>
<tr>
<td>70–74</td>
<td>-2.41%</td>
<td>0.00%</td>
<td>2.23%</td>
<td>1.45%</td>
<td>-1.27%</td>
</tr>
<tr>
<td>75–79</td>
<td>-1.04%</td>
<td>2.82%</td>
<td>-2.45%</td>
<td>-1.37%</td>
<td>2.04%</td>
</tr>
<tr>
<td>80–84</td>
<td>-1.34%</td>
<td>0.00%</td>
<td>1.01%</td>
<td>0.34%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>85–89</td>
<td>5.08%</td>
<td>3.64%</td>
<td>-8.13%</td>
<td>-5.52%</td>
<td>4.91%</td>
</tr>
<tr>
<td>90 and over</td>
<td>-2.47%</td>
<td>2.47%</td>
<td>0.00%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

### Table 3.18: Difference in prevalence rates for females (as percentages), data - model.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>25–34</td>
<td>0.88%</td>
<td>-0.26%</td>
<td>-0.83%</td>
<td>0.39%</td>
<td>-0.18%</td>
</tr>
<tr>
<td>35–44</td>
<td>-0.07%</td>
<td>-0.60%</td>
<td>-0.25%</td>
<td>1.56%</td>
<td>-0.64%</td>
</tr>
<tr>
<td>45–54</td>
<td>-1.47%</td>
<td>-0.78%</td>
<td>1.02%</td>
<td>1.55%</td>
<td>-0.31%</td>
</tr>
<tr>
<td>55–59</td>
<td>-3.91%</td>
<td>0.65%</td>
<td>1.60%</td>
<td>2.05%</td>
<td>-0.39%</td>
</tr>
<tr>
<td>60–64</td>
<td>-2.66%</td>
<td>-0.12%</td>
<td>1.62%</td>
<td>2.44%</td>
<td>-1.28%</td>
</tr>
<tr>
<td>65–69</td>
<td>1.14%</td>
<td>-1.32%</td>
<td>0.18%</td>
<td>0.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>70–74</td>
<td>-3.75%</td>
<td>-0.45%</td>
<td>0.99%</td>
<td>3.21%</td>
<td>0.00%</td>
</tr>
<tr>
<td>75–79</td>
<td>0.46%</td>
<td>3.83%</td>
<td>-3.07%</td>
<td>-1.08%</td>
<td>-0.14%</td>
</tr>
<tr>
<td>80–84</td>
<td>-0.73%</td>
<td>2.60%</td>
<td>-5.19%</td>
<td>1.01%</td>
<td>2.31%</td>
</tr>
<tr>
<td>85–89</td>
<td>-0.32%</td>
<td>1.35%</td>
<td>0.72%</td>
<td>-1.74%</td>
<td>0.00%</td>
</tr>
<tr>
<td>90 and over</td>
<td>-3.33%</td>
<td>7.17%</td>
<td>-1.80%</td>
<td>-2.03%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>
3.4. THE MULTI-STATE MODEL

Figure 3.7: Reported (2003 SDAC) and model prevalence rates for each CAL category (per 1000).
Tables 3.19 and 3.20 present the estimated one year (unconditional on survival and non-deterioration of disability) transition probabilities at 10-yearly age intervals and Figure 3.8 illustrates these probabilities (from the able state) for males and females respectively. These probabilities are assumed to be effective at the middle of 2003.

The following observations are true in general:

1. Deterioration probabilities are increasing with age while recovery probabilities are decreasing.

2. Deterioration probabilities are increasing with the severity of the disability while recovery probabilities are decreasing.

3. Deterioration is more likely into a less severe disability state than into a more severe one. There are several exceptions to this. In particular, at very high ages, the likelihood of deteriorating into the profound CAL state exceeds the likelihood of deteriorating into any other less disabled state. This can be attributable to the effect of the ageing process on frailty.

4. Likelihood of recovery is higher for males than females. However, for the profound CAL state the likelihood of recovery is higher for females (although the difference is not significant).

5. Likelihood of transition into severe or profound CAL states is higher for females.

6. Mortality rates are higher for males than females in all disability states with the difference between genders being highest for the profound CAL state.

Points 4, 5 and 6 indicate that females have a higher likelihood of using an aged care home over their lifetime than males, as they are more likely to spend more of their lifetime in severe or profound CAL states. However, this is true only if the relationship between disability and likelihood of admission is similar between genders.
### Table 3.19: Males, one-year disability transition probabilities.

<table>
<thead>
<tr>
<th>Age</th>
<th>Able</th>
<th>Mild</th>
<th>Moderate</th>
<th>Severe</th>
<th>Profound</th>
<th>Dead</th>
</tr>
</thead>
<tbody>
<tr>
<td>60.5</td>
<td>0.950225</td>
<td>0.026370</td>
<td>0.008811</td>
<td>0.003540</td>
<td>0.006618</td>
<td>0.004436</td>
</tr>
<tr>
<td>70.5</td>
<td>0.907154</td>
<td>0.044473</td>
<td>0.014932</td>
<td>0.006029</td>
<td>0.011325</td>
<td>0.016087</td>
</tr>
<tr>
<td>80.5</td>
<td>0.789689</td>
<td>0.092344</td>
<td>0.032138</td>
<td>0.013449</td>
<td>0.026189</td>
<td>0.046190</td>
</tr>
<tr>
<td>90.5</td>
<td>0.611371</td>
<td>0.122838</td>
<td>0.050063</td>
<td>0.024536</td>
<td>0.055963</td>
<td>0.135230</td>
</tr>
<tr>
<td>100.5</td>
<td>0.428443</td>
<td>0.132600</td>
<td>0.064421</td>
<td>0.037633</td>
<td>0.102300</td>
<td>0.234603</td>
</tr>
<tr>
<td>109.5</td>
<td>0.270323</td>
<td>0.127502</td>
<td>0.064670</td>
<td>0.039441</td>
<td>0.111926</td>
<td>0.386138</td>
</tr>
<tr>
<td>Mild</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>60.5</td>
<td>0.144998</td>
<td>0.821653</td>
<td>0.013430</td>
<td>0.005396</td>
<td>0.010087</td>
<td>0.004436</td>
</tr>
<tr>
<td>70.5</td>
<td>0.140205</td>
<td>0.794496</td>
<td>0.022760</td>
<td>0.009189</td>
<td>0.017262</td>
<td>0.016087</td>
</tr>
<tr>
<td>80.5</td>
<td>0.126661</td>
<td>0.717744</td>
<td>0.048987</td>
<td>0.020500</td>
<td>0.039919</td>
<td>0.046190</td>
</tr>
<tr>
<td>90.5</td>
<td>0.099864</td>
<td>0.565899</td>
<td>0.076308</td>
<td>0.037399</td>
<td>0.085301</td>
<td>0.135230</td>
</tr>
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Table 3.20: Females, one-year disability transition probabilities.

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Figure 3.8: One-year transition probabilities from able (no CAL) state.
3.5 Estimation uncertainties

In this section we discuss uncertainties in the estimation in Section 3.4 which mainly arise because of data limitations and various issues in the adopted modelling methodology.

3.5.1 Data

The accuracy of the estimated probability of disablement depends on the quality of the data used. Below we discuss several aspects of the data which might potentially cause a problem.

Firstly, the grouping of all people aged 90 and above into one single age category might affect the accuracy of the estimated probability of disablement for these ages. It is likely that CAL prevalence rates at ages above 100 are significantly higher than the prevalence rates at age group 90 and over (especially for severe and profound CAL). Hence, the model might miss the dynamic of the disability process at these very high ages. However, as the probability of surviving to these ages is small, this problem is unlikely to be significant.

Secondly, the relative standard errors of the estimates of the number of people in certain disability categories at a few age groups are quite large as reported by ABS (2004a). However, as this is true for most disability categories only at the highest age group, this problem is unlikely to be significant. A possible way to address this problem is by conducting a sensitivity analysis.

Lastly, the SDAC survey covered both people who live in private dwellings and aged care homes. However, in pricing a reverse mortgage contract, we are only concerned with the disability experience of people who live at home. Therefore, for this pricing purpose, there are two types of discount which need to be applied to the estimated probability of admission into an aged care home (discussed in Chapter 4). The first discount reflects the lower likelihood of disablement of people who live in households (as compared to people who live in aged care homes), while the second reflects the assumed adverse selection of the borrowers of reverse mortgage loans.
3.5.2 Modelling methodology

In fitting the multi-state model, we employ various assumptions regarding the past population process such that disability prevalence rates at any age remain constant over time (Section 3.4.2). Among these, particular attention should be given regarding the assumption of constant mortality rates and transition rates between disability states. We think that the magnitude of overseas migration is insufficient to significantly affect the prevalence rates of the population. This assumption is employed to simplify the fitting procedure and, in particular, because of the difficulty in determining the past trend of transition rates between disability states. In reality, mortality rates vary with time and it is possible for other transition rates to also vary with time (for example, given the improvement in medical science and health awareness, it is likely that deterioration probabilities at a given age will decrease with time). As explained in Rickayzen and Walsh (2002), by employing the constant transition rates assumption, the estimated transition rates are effectively averages of past transition rates. Therefore, we might overestimate the transition rates which have been declining in the past and underestimate those which have been increasing. However, the possible magnitude of the error of the estimation and its net effect on the estimated probability of admission into an aged care home (discussed in Chapter 4) is difficult to quantify.

Another modelling issue is regarding the assumption that over a one year interval, disabled people can only improve by one category. However, this assumption is not too unrealistic for the majority of categories (given the severity of disablement, it is unlikely that people with severe or profound CAL will improve by more than one category over a year). Note that a simplified assumption regarding recovery from disability (such as the one that we adopt) is commonly adopted in disability analysis (Nuttall et al., 1994; Brelivet et al., 2001; Rickayzen and Walsh, 2002; Alegre et al., 2004; Leung, 2004; Albarran et al., 2005; Leung, 2006). Depending upon the availability of better data in the future, it might be possible to extend the model to allow a fuller set of recovery processes.
Possible inaccuracy in the adopted values of additional mortality due to
disability and improvement probabilities will also adversely impact the ac-
curacy of the estimation. However, given the availability of data, reasonable
estimated values have been adopted.

Lastly, there might be scope to improve the parametric functions of the
transition probabilities. However, we choose to improve the estimation under
an alternative modelling method which is described in Chapter 4.

3.6 Conclusion

In this chapter we have described a multi-state model to estimate disability
transition probabilities from cross sectional data. The model has been ap-
pliced to data from the 2003 SDAC. In Chapter 4 we describe an alternative
modelling method with the aim of improving the accuracy of the estimated
transition probabilities. This alternative modelling method requires the tran-
sition probabilities estimated in this chapter as an input of the model. The
probabilities of admission into an aged care home will be estimated from the
disability transition probabilities estimated under this alternative method.
Chapter 4

Estimation of disability transition probabilities under the IPF procedure

In this chapter we improve the accuracy of the disability transition probabilities estimated in Chapter 3. The method adopted in this chapter utilizes the disability data in the 1998 and 2003 SDAC and employs a so called Iterative Proportional Fitting (IPF) procedure. Fundamental to our estimation is the apparent stability of disability prevalence rates over two decades. Given this stability, we linearly interpolate the disability prevalence rates reported in the 1998 and 2003 surveys to estimate the prevalence rates in years 1999 to 2002. Applying these prevalence rates to the estimated resident population (ERP) in the relevant years, we obtain our estimate of the disabled population in years 1998 to 2003. We apply the IPF procedure to the estimated disabled population at each pair of subsequent years (i.e. 1998 and 1999, etc) to estimate one year disability longitudinal data. The idea of the procedure is to model the disability state of an individual in two subsequent years under a log-linear model. The row category represents the disability state at the base year while the column category represents the state at the following year. The interaction between row and column categories is estimated from the disability transition probabilities in Chapter 3 (initial transition probabil-
Practically, the procedure involves projecting the disabled population in the middle of a given year to the middle of the next year using the initial transition probabilities and adjusting the projection such that it satisfies the age structure of the disabled population at the middle of the next year. The results of the procedure are the estimates of one year disability longitudinal data. From these longitudinal data, the refined estimates of one year disability transition probabilities are estimated. Following that, we present the graduation of the estimated (refined) transition probabilities to obtain the final estimate of disability transition probabilities. From the graduated disability transition probabilities, the probabilities of admission into an aged care home (ACH) are estimated. Figure 4.1 presents the complete procedure for the estimation of ACH admission probabilities. Chapter 3 presents the analysis of step 1, while this chapter presents the analysis of steps 2 to 4.

Figure 4.1: Estimation steps of ACH admission probabilities.

Briefly, the steps are as follows.

1. Estimation of disability transition probabilities from a single set of cross sectional data under the multi-state model.

2. Estimation of one year disability longitudinal data (under the IPF procedure) and hence refined estimates of disability transition probabilities.

3. Graduation of the refined estimates of disability transition probabilities.

4. Estimation of probabilities of admission into an ACH.
4.1 Outline

As described in Section 3.3, in Australia we only have cross sectional disability data at a national scale which are given in the Survey of Disability, Ageing and Carers (SDAC). The method presented here, in a large part, consists of estimating one year disability longitudinal data using the data from the 1998 and 2003 SDAC surveys. These two surveys are the latest at the time of writing this chapter. We decided not to employ older SDAC survey data since surveys before 1998 were conducted under a method which is not comparable to the 1998 survey. There was a significant change of survey design in 1998 which resulted in a larger than expected increase in disabled people with severe or profound CAL (in SDAC (or the corresponding disability surveys) before 2003, CAL is referred to as core activity restriction (CAR)) identified in the survey (Davis et al., 2001). The survey in 2003, to a large extent, maintained the 1998 survey method structure with only some minor changes (Australian Institute of Health and Welfare (AIHW), 2008). While re-derived disability data for the 1988, 1993 and 1998 surveys under common disability criteria are presented in Davis et al. (2001), we opt not to use these adjusted disability data for two important reasons.

- The adjusted data are presented for “disability” (disability here is an umbrella category which includes any CAL, schooling/employment restrictions and disability without any specific limitations) and “severe” (combined category of severe and profound) categories, while in our study we consider each CAL category separately.

- There are several methodological effects which could not be adjusted for (Davis et al., 2001) which resulted in a significant increase in the identification of the disabled population in some age groups. For example, a significant proportion of the increase of the population with severe or profound limitation in age group 45 to 64 between the 1993 and 1998 surveys is because of better data capture due to survey developments which could not be adjusted for (Davis et al., 2001).
We reviewed relevant demographic literature and found two methods which are applicable for our purpose: iterative proportional fitting (Bishop et al., 2007) and relative state attraction (Schoen and Jonsson, 2003). Iterative proportional fitting (IPF) is a relatively classic method which has been applied as early as Kruithof (1937). This method has been widely applied in migration analysis (Nair, 1985; Willekens, 1999; Schoen and Jonsson, 2003). The statistical development and properties of this method are described in Bishop et al. (2007) and Willekens (1999). Relative state attraction (RSA) is a newer method which was proposed by Schoen and Jonsson (2003). This method typically results in similar estimates as IPF (Schoen and Jonsson, 2003). We choose the IPF method because we feel that it has a better theoretical foundation. For example, there is no general proof that the RSA method will always yield a unique and demographically realistic solution (Schoen and Jonsson, 2003). In the following, we describe the methodology to refine the transition probabilities estimated in Chapter 3 under the IPF method.

Consider the six states as described in Section 3.4.1. Suppose that at the middle of years $t$ and $t+1$ we have the population at a single year of age (last birthday) for each of the five living states. For simplification, we assume that the population and overseas migrants who are aged $x$ last birthday (where $x$ is a non-negative integer) are aged exactly $x+0.5$. Given our purpose of pricing a reverse mortgage contract, we only consider ages 60.5 and above. The possible transitions over a one-year period are as described in Section 3.4.1 and the notation for the states is as described in Section 3.4.2. Denote:

$N_{x+0.5, x+1.5}^{m,n}(t, t+1)$ : population who are aged $x+0.5$ and in state $m$ ($m = 0, 1, \ldots, 4$) at the middle of year $t$, and aged $x+1.5$ and in state $n$ ($n = 0, 1, \ldots, 5$) at the middle of year $t+1$.

$I_{x+0.5}^n(t)$ : number of net overseas migrants between the middle of years $t-1$ and $t$ who attain age $x+0.5$ and are in state $n$ ($n = 0, 1, \ldots, 5$) at the middle of year $t$.

$D_{x+0.5}(t)$ : number of deaths between the middle of years $t-1$ and $t$ from the population and net overseas migrants (those who migrate between
the middle of year \( t - 1 \) and \( t \) who are aged \( x - 0.5 \) at the middle of year \( t - 1 \).

Note that we have:

\[
N_{x+1.5}^n(t + 1) = \sum_{m=0}^{4} N_{x+0.5,x+1.5}^{m,n}(t, t + 1) + I_{x+1.5}^n(t + 1), \quad n = 0, 1, \ldots, 4
\]
\[
D_{x+1.5}(t + 1) = \sum_{m=0}^{4} N_{x+0.5,x+1.5}^{m,5}(t, t + 1) + I_5(t + 1) \tag{4.1}
\]

where \( N_{x+0.5}^{m}(t) \) is as described in Section 3.4.6.

Adjusting \( N_{x+1.5}^n(t + 1) \) and \( D_{x+1.5}(t + 1) \) to exclude net overseas migrants who migrate between the middle of years \( t \) and \( t + 1 \) we have:

\[
\hat{N}_{x+1.5}^n(t + 1) = \sum_{m=0}^{4} N_{x+0.5,x+1.5}^{m,n}(t, t + 1), \quad n = 0, 1, \ldots, 4
\]
\[
\hat{D}_{x+1.5}(t + 1) = \sum_{m=0}^{4} N_{x+0.5,x+1.5}^{m,5}(t, t + 1) \tag{4.2}
\]

where

\( \hat{N}_{x+0.5}^{n}(t) \) is the size of the population aged \( x + 0.5 \) and in state \( n \) (\( n = 0, 1, \ldots, 4 \)) at the middle of year \( t \) assuming the absence of overseas migration between the middle of years \( t - 1 \) and \( t \), and

\( \hat{D}_{x+0.5}(t) \) is the number of deaths between the middle of years \( t - 1 \) and \( t \) from the population aged \( x - 0.5 \) at the middle of year \( t - 1 \).

Equation (4.2) can be represented by Table 4.1 (for convenience, we denote \( N_{x+0.5,x+1.5}^{m,n} = N_{x+0.5,x+1.5}^{m,n}(t, t + 1) \)).
Table 4.1: Annual population transition from the middle of year \((t)\) to the middle of year \((t+1)\).

<table>
<thead>
<tr>
<th>Middle year ((t))</th>
<th>Age ((x + 0.5))</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(N_{x+0.5,x+1.5}^{0,0})</td>
<td>(N_{x+0.5,x+1.5}^{0,1})</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(N_{x+0.5,x+1.5}^{0,5})</td>
<td>(R_0)</td>
</tr>
<tr>
<td>1</td>
<td>(N_{x+0.5,x+1.5}^{1,0})</td>
<td>(N_{x+0.5,x+1.5}^{1,1})</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(N_{x+0.5,x+1.5}^{1,5})</td>
<td>(R_1)</td>
</tr>
<tr>
<td>2</td>
<td>(N_{x+0.5,x+1.5}^{2,0})</td>
<td>(N_{x+0.5,x+1.5}^{2,1})</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(N_{x+0.5,x+1.5}^{2,5})</td>
<td>(R_2)</td>
</tr>
<tr>
<td>3</td>
<td>(N_{x+0.5,x+1.5}^{3,0})</td>
<td>(N_{x+0.5,x+1.5}^{3,1})</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(N_{x+0.5,x+1.5}^{3,5})</td>
<td>(R_3)</td>
</tr>
<tr>
<td>4</td>
<td>(N_{x+0.5,x+1.5}^{4,0})</td>
<td>(N_{x+0.5,x+1.5}^{4,1})</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(\ldots)</td>
<td>(N_{x+0.5,x+1.5}^{4,5})</td>
<td>(R_4)</td>
</tr>
<tr>
<td>Total</td>
<td>(C_0)</td>
<td>(C_1)</td>
<td>(C_2)</td>
<td>(C_3)</td>
<td>(C_4)</td>
<td>(C_5)</td>
<td>(T)</td>
</tr>
</tbody>
</table>

where

\[
R_m = N_{x+0.5}^m(t) \quad m = 0, 1, \ldots, 4
\]

\[
C_n = N_{x+1.5}^n(t+1) \quad n = 0, 1, \ldots, 4
\]

\[
C_5 = \hat{D}_{x+1.5}(t+1)
\]

\[
T = \sum_{m=0}^{4} R_m = \sum_{n=0}^{5} C_n.
\]

From our cross sectional data, we do not know the values of \(N_{x+0.5,x+1.5}^{m,n}(t, t+1)\). However, if we have an idea of the likely transition probabilities between the middle of years \(t\) and \(t+1\), we are able to make an initial estimate of these one-year longitudinal data. Note that the transition probabilities estimated in Chapter 3 can serve for this purpose. We shall refer to these probabilities as the initial transition probabilities. Hence we have (for \(m = 0, 1, \ldots, 4\)): 
\[ \hat{N}_{x+0.5,x+1.5}(t, t+1) \]

\[
\left\{
\begin{array}{ll}
N_{x+0.5}(t) \times (1 - \text{Mortality}(x + 0.5, t, m)) \times \\
\text{Deteriorate}(x + 0.5, m, n) & : m < n \leq 4,
\end{array}
\right.
\]

\[
= \left\{
\begin{array}{ll}
N_{x+0.5}(t) \times (1 - \text{Mortality}(x + 0.5, t, m)) \times \\
(1 - \text{Deteriorate}_\text{From}(x + 0.5, m)) \times \\
(1 - \text{Improve}_\text{From}(x + 0.5, m)) & : n = m,
\end{array}
\right.
\]

\[
= \left\{
\begin{array}{ll}
N_{x+0.5}(t) \times (1 - \text{Mortality}(x + 0.5, t, m)) \times \\
(1 - \text{Deteriorate}_\text{From}(x + 0.5, m)) & : 0 \leq n = m - 1,
\end{array}
\right.
\]

\[
= \left\{
\begin{array}{ll}
N_{x+0.5}(t) \times \text{Mortality}(x + 0.5, t, m) & : n = 5,
\end{array}
\right.
\]

\[
= \left\{
\begin{array}{ll}
0 & : n < m - 1,
\end{array}
\right.
\]

where

\[ \hat{N}_{x+0.5,x+1.5}(t, t+1) \] is the estimate of \(N_{x+0.5,x+1.5}(t, t+1)\) from the transition probabilities in Chapter 3, and

\[ \text{Deteriorate}(x + 0.5, m, n), \text{ Deteriorate}_\text{From}(x + 0.5, m) \] and

\[ \text{Improve}_\text{From}(x + 0.5, m) \] are as estimated in Chapter 3.

Note that the one-year death probability varies with time. The time trend of this probability is incorporated through the healthy mortality component while the additional mortality due to the disability component is assumed to be time invariant. Specifically (for \(n = 0, 1, \ldots, 4\)):

\[ \text{Mortality}(x + 0.5, t, n) = \]

\[ \text{Overall}_\text{Mort}(x + 0.5, t) + \text{Additional}_\text{Mort}(x + 0.5, n) \quad (4.4) \]

where \(\text{Additional}_\text{Mort}(x + 0.5, n)\) and its values are as described in Section 3.4.3.
Table 4.2 is similar to Table 4.1 with the elements of Table 4.1 being replaced with $\hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1)$.

Table 4.2: Estimate of annual population transition from the middle of year $(t)$ to the middle of year $(t+1)$.

<table>
<thead>
<tr>
<th>Middle year $(t)$</th>
<th>Age $(x + 0.5)$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$N_{x+0.5,x+1.5}^{0,0}$</td>
<td>$N_{x+0.5,x+1.5}^{0,1}$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$N_{x+0.5,x+1.5}^{0,5}$</td>
<td>$R_0$</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>$N_{x+0.5,x+1.5}^{1,0}$</td>
<td>$N_{x+0.5,x+1.5}^{1,1}$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$N_{x+0.5,x+1.5}^{1,5}$</td>
<td>$R_1$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$N_{x+0.5,x+1.5}^{2,0}$</td>
<td>$N_{x+0.5,x+1.5}^{2,1}$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$N_{x+0.5,x+1.5}^{2,5}$</td>
<td>$R_2$</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$N_{x+0.5,x+1.5}^{3,0}$</td>
<td>$N_{x+0.5,x+1.5}^{3,1}$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$N_{x+0.5,x+1.5}^{3,5}$</td>
<td>$R_3$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$N_{x+0.5,x+1.5}^{4,0}$</td>
<td>$N_{x+0.5,x+1.5}^{4,1}$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$\cdots$</td>
<td>$N_{x+0.5,x+1.5}^{4,5}$</td>
<td>$R_4$</td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>$C_0^*$</td>
<td>$C_1^*$</td>
<td>$C_2^*$</td>
<td>$C_3^*$</td>
<td>$C_4^*$</td>
<td>$C_5^*$</td>
<td>$T$</td>
<td></td>
</tr>
</tbody>
</table>

Since transition probabilities sum to one, we have

$$R_m = N_{x+0.5}^m(t) = \sum_{n=0}^5 \hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1), \quad m = 0, 1, \ldots, 4. \quad (4.5)$$

However the following are unlikely to hold:

$$C_n = \hat{N}_{x+1.5}^n(t+1) = \sum_{m=0}^4 \hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1), \quad n = 0, 1, \ldots, 4 \quad (4.6)$$

$$C_5^* = \hat{D}_{x+1.5}(t+1) = \sum_{m=0}^4 \hat{N}_{x+0.5,x+1.5}^{m,5}(t, t+1). \quad (4.7)$$

Possible reasons why (4.6) and (4.7) do not hold are statistical variation and inaccurate estimation of the transition probabilities. However, if the exposures ($N_{x+0.5}^n(t)$ for $n = 0, 1, \ldots, 4$) are large, the error due to statistical variation is likely to be small and, hence, the remaining error is largely due to inaccurate estimation of the transition probabilities. Therefore, if we have accurate estimates of $N_{x+0.5}^n(t)$, $\hat{N}_{x+1.5}^n(t+1)$ (for $n = 0, 1, \ldots, 4$) and $\hat{D}_{x+1.5}(t+1)$ for each single year of age, we are able to employ the IPF algorithm to refine our initial estimates of one-year longitudinal data ($\hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1)$) such that (4.6) and (4.7) hold (by keeping (4.5) holding).
In doing so, we assume that the cross product ratios of \( \hat{N}_{m,n}^{x+0.5,x+1.5}(t, t+1) \) are similar to the cross product ratios of \( N_{m,n}^{x+0.5,x+1.5}(t, t+1) \) (which is hopefully the case; in the implementation of the multi-state model as part of the estimation of initial transition probabilities, we have tried to be as realistic as the data allow). The refined estimates of one-year disability longitudinal data are maximum likelihood estimates (MLEs) under a saturated log-linear model with the state at the middle of year \( t \) as variable 1 and the state at the middle of year \( t+1 \) as variable 2. Specifically (for \( m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5 \))

- the main effect of variable 1 is estimated from \( R_m \),
- the main effect of variable 2 is estimated from \( C_n \), and
- the interaction effect between variables 1 and 2 is estimated from the cross product ratios of \( \hat{N}_{m,n}^{x+0.5,x+1.5}(t, t+1) \).

We have described a method to estimate one-year disability longitudinal data assuming that we have the disabled population for each single year of age at the middle of years \( t \) and \( t+1 \) (one year interval). However, SDAC is conducted at five yearly intervals (1998 and 2003) and the disabled population is presented at five or ten year age groups with the very oldest placed in a single group. While it is possible to estimate five-year transition probabilities directly from SDAC data, due to the number of possible transitions over five year period, it is not however possible to estimate one-year transition probabilities from five-year transition probabilities. Therefore, our first step in finding the refined estimates of annual transition probabilities is to estimate the disabled population at single years of age (for ages 60 and above) for each disability category (i.e. able, mild, moderate, severe and profound CAL) at the middle of 1998, 1999, \ldots, 2003. We apply the method described above to the estimated disabled population at each pair of subsequent years (i.e. 1998 and 1999, 1999 and 2000, etc) to estimate one-year disability longitudinal data from 1998 to 2002. We take into account net overseas migration (NOM) in the implementation of the method. The refined estimate of the
one-year transition probability is calculated as:

\[ \hat{P}_{x+0.5,x+1.5}^{m,n} = \frac{\sum_{t=1998}^{2002} \tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t+1)}{\sum_{t=1998}^{2002} N_{x+0.5}^m(t)}, \quad m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5 \]  

(4.8)

where

\[ \hat{P}_{x+0.5,x+1.5}^{m,n} \] is the refined estimate of the probability that an individual aged \( x + 0.5 \) and in live state \( m \) makes a transition to state \( n \) before reaching age \( x + 1.5 \), and

\[ \tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t+1) \] is the refined estimate of \( \hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1) \).

Note that \( \hat{P}_{x+0.5,x+1.5}^{m,n} \) can only be regarded as a crude probability and hence it should be graduated.

In the following we describe the log-linear model and the IPF algorithm, which are the main tools in our procedure, followed by the estimation of the disabled population at each middle year from 1998 to 2003. We then calculate the refined estimates of one-year disability longitudinal data and (one-year) disability transition probabilities. Lastly, we present the graduation of the refined estimates of disability transition probabilities and followed by the estimation of probabilities of admission into an ACH.

4.2 Main tools (models)

4.2.1 Log-linear model

In this section we follow the methods described by Bishop et al. (2007). Suppose we have a single sample of size \( N \) which forms a rectangular array with \( I \) rows and \( J \) columns, corresponding to the \( I \) categories of variable 1 and \( J \) categories of variable 2. We use the following notation:

\( X_{ij} \) : random variable for the count in cell \((i, j)\).

\( x_{ij} \) : observed value of \( X_{ij} \).

\( p_{ij} \) : probability of an observation falling into cell \((i, j)\).
4.2. MAIN TOOLS (MODELS)

\[ m_{ij} = N p_{ij} = E(X_{ij}) : \text{expected value of } X_{ij}. \]

\[ \hat{m}_{ij}^{MLE} : \text{maximum likelihood estimate (MLE) of } m_{ij}. \]

Now since \( \sum_{i,j} \hat{m}_{ij}^{MLE} = 1 \) we have \( \sum_{i,j} \hat{m}_{ij}^{MLE} = N. \) As a convention, the sum over a subscript is denoted by replacing the subscript by “+”, so for example \( x_{i+} = \sum_{j=1}^J x_{ij} \) for \( i = 1, 2, \ldots, I. \)

The saturated log-linear model is defined as:

\[
\log(m_{ij}) = l_{ij} = u + \sum_{x=1}^I i_x u_{1(x)} + \sum_{y=1}^J j_y u_{2(y)} + \sum_{x=1}^I \sum_{y=1}^J i_x j_y u_{12(xy)}, \quad i = 1, 2, \ldots, I; j = 1, 2, \ldots, J \quad (4.9)
\]

where \( u, u_{1(i)}, u_{2(j)} \) and \( u_{12(ij)} \) are parameters, and

\[
i_x = \begin{cases} 1 & \text{for } x = i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad j_y = \begin{cases} 1 & \text{for } y = j, \\ 0 & \text{otherwise}. \end{cases}
\]

The following constraints are applied to the parameters:

\[
\sum_i u_{1(i)} = \sum_j u_{2(j)} = \sum_i u_{12(ij)} = \sum_j u_{12(ij)} = 0. \quad (4.10)
\]

The parameters are defined as:

\[ u = \frac{l_{++}}{I+J} : \text{overall mean.} \]

\[ u_{1(i)} = \frac{l_{i+}}{I} - \frac{l_{++}}{I+J} : \text{main effect of variable 1.} \]

\[ u_{2(j)} = \frac{l_{+j}}{J} - \frac{l_{++}}{I+J} : \text{main effect of variable 2.} \]

\[ u_{12(ij)} = l_{ij} - \frac{l_{i+}}{I} - \frac{l_{+j}}{J} + \frac{l_{++}}{I+J} : \text{two factor interaction effect between the two variables.} \]

Due to constraint (4.10), the number of independent parameters \((I \times J)\) is equal to the number of cells.

Each \( u \)-term can be expressed as a function of cross product ratios under different arrangements of the contingency table (see Bishop et al. (2007));
in particular, the terms $u_{12(ij)}$ are direct functions of cross product ratios under a standard arrangement of the table. Hence, the terms $u_{12(ij)}$ capture the interaction pattern between variable 1 (row categories) and 2 (column categories). Note that independence between variables 1 and 2 implies that the terms $u_{12(ij)}$ are zero (since $p_{ij} = p_{i+}p_{+j}$ under an independent model).

Under the saturated log-linear model the main effect of variable 1 is captured by the terms $u_{1(i)}$ while the main effect of variable 2 is captured by the terms $u_{2(j)}$. Further, the interaction between variables 1 and 2 (as measured by the cross product ratios under a standard arrangement of the contingency table) is captured by the terms $u_{12(ij)}$.

In our application, we fit the saturated log-linear model with the purpose of estimating all values in an $I \times J$ contingency table when only the marginal totals of the data are available. This is done by fitting a saturated log-linear model combining information from two different datasets: the incomplete dataset, containing only marginal totals and a second complete dataset containing observations for all $I \times J$ cells in the contingency table. The method assumes that the cross product ratios observed in the complete dataset are the same as the cross product ratios of the dataset that we try to estimate. By making this assumption, we estimate the values in the dataset for which only marginal totals are known by imposing the observed two-way interactions from our complete dataset to this incomplete dataset. In the following we describe the procedure along with the required algorithm (iterative proportional fitting) to fit the model.

### 4.2.2 Iterative proportional fitting (IPF) algorithm

We use the following notation.

- \{x_{ij}\} be the elements in dataset 1. Only row and column totals are observed here.

- \{w_{ij}\} be the elements in dataset 2. All elements are observed.

- $X_{ij}$ is the random variable associated with $x_{ij}$ as the observed value (as defined in Section 4.2.1); and similarly the $p_{ij}$, $m_{ij}$ and $\hat{m}_{ij}^{MLE}$ are
related to $x_{ij}$ as defined in Section 4.2.1.

- $\hat{m}_{ij}$ is an estimate of $m_{ij}$.

Note that by fitting a saturated log-linear model to $\{x_{ij}\}$, we have $\hat{m}_{ij}^{MLE} = x_{ij}$.

Now suppose that we only have the values of $x_{i+}$ and $x_{+j}$, and we want to estimate $x_{ij}$ for $i = 1,2,\ldots,I$ and $j = 1,2,\ldots,J$. In addition, suppose that we have $w_{ij}$ for all $i$ and $j$; and the cross product ratios of $\{w_{ij}\}$ are assumed to be the same as the cross product ratios of $\{x_{ij}\}$.

Firstly, we present the results by Birch (1963) which state that for a given log-linear model the following apply

- the $\{\hat{m}_{ij}^{MLE}\}$ are unique and replicate the values of the sufficient statistics, and

- the $\{\hat{m}_{ij}\}$ which replicate the values of the sufficient statistics are unique.

To fit a saturated log-linear model to the $\{x_{ij}\}$, the sufficient statistics are $x_{i+}$, $x_{+j}$ and the cross product ratios of $\{x_{ij}\}$. Specifically

- $u$ is estimated from $x_{++}$,

- $u_{1(i)}$ are estimated from $x_{i+}$,

- $u_{2(j)}$ are estimated from $x_{+j}$, and

- $u_{12(ij)}$ are estimated from the cross product ratios of $\{x_{ij}\}$.

Note that since the cross product ratios of the $\{w_{ij}\}$ are assumed to be the same as the cross product ratios of $\{x_{ij}\}$, we have the sufficient statistics to fit the saturated log-linear model to $\{x_{ij}\}$.

We now describe the IPF procedure (algorithm) which is used to estimate values in the incomplete dataset.

Let $\{\hat{f}^{(t)}_{ij}\}$ denote our estimate of $\{m_{ij}\}$ at step $t$ under the IPF algorithm, so that $\{\hat{f}^{(0)}_{ij}\}$ is the set of initial values. We set $\hat{f}^{(0)}_{ij} = w_{ij}$ for $i = 1,2,\ldots,I$
and \( j = 1, 2, \ldots, J \). Then we adjust the \( \{ \hat{f}^{(0)}_{ij} \} \) to fit successively to \( x_{i+} \) and \( x_{+j} \). Fitting to \( x_{i+} \) gives

\[
\hat{f}^{(1)}_{ij} = \frac{\hat{f}^{(0)}_{ij} x_{i+}}{\hat{f}^{(0)}_{i+}}
\]

and subsequent fitting to \( x_{+j} \) gives

\[
\hat{f}^{(2)}_{ij} = \frac{\hat{f}^{(1)}_{ij} x_{+j}}{\hat{f}^{(1)}_{+j}}.
\]

(4.11)

We repeat this two step cycle until convergence to the desired accuracy is obtained. At convergence we have

- \( \hat{f}_{i+} \approx x_{i+} \),
- \( \hat{f}_{+j} \approx x_{+j} \), and
- The cross product ratios of \( \{ \hat{f}_{ij} \} \) are the same (to any degree of accuracy specified for convergence of the IPF algorithm) as the cross product ratios of \( \{ w_{ij} \} \) (the final estimates retain the cross product ratios of the initial estimates). For the proof, see Bishop et al. (2007).

Since the cross product ratios of \( \{ w_{ij} \} \) are the same as the cross product ratios of \( \{ x_{ij} \} \), we know from above that \( \hat{f}_{ij} \approx \hat{m}_{ij}^{MLE} = x_{ij} \).

In practice, there are likely to be differences between the cross product ratios of \( \{ w_{ij} \} \) and \( \{ x_{ij} \} \). In addition, the \( x_{i+} \) and \( x_{+j} \) might not be exactly known. This results in a certain degree of estimation error.

In our application of the IPF algorithm to obtain the refined estimates of one-year disability longitudinal data, we set:

- \( x_{i+} = R_i \) for \( i = 0, 1, \ldots, 4 \)
- \( x_{+j} = C_j \) for \( j = 0, 1, \ldots, 5 \)
- \( \hat{f}^{(0)}_{ij} = \hat{N}_{x+0.5,x+1.5}^{ij}(t, t+1) \) for \( i = 0, 1, \ldots, 4; j = 0, 1, \ldots, 5 \).

Provided that the estimates of \( N_{x+0.5}^{n}(t), \hat{N}_{x+1.5}^{n}(t+1) \) and \( \hat{D}_{x+1.5}(t+1) \) are accurate, and the cross product ratios of \( \{ \hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1) \} \) are similar to the cross product ratios of \( \{ \hat{N}_{x+0.5,x+1.5}^{m,n}(t, t+1) \} \), at convergence we
have $\hat{f}_{mn} \approx N_{x+0.5,x+1.5}^{m,n}(t, t + 1)$ (for $m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5$). The convergence values under the above IPF algorithm are our refined estimates of one-year disability longitudinal data (i.e. $\tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t + 1) = \hat{f}_{mn}$).

### 4.3 Estimation of the disabled population at each mid-year

The ERP at a single age interval is provided by the ABS at the middle of each year (ABS (2009)). Therefore, if we have an idea of disability prevalence rates which apply at the middle of each year from 1998 to 2003, we are able to estimate the disabled population at these mid-years. We assume that disability prevalence rates reported in the 1998 SDAC (which was conducted from March to May 1998) are applicable at the middle of 1998 and the rates reported in the 2003 SDAC (conducted from June to November 2003) are applicable at the middle of 2003. Since the central dates of these surveys are close to the middle of the year, we believe this assumption is reasonable. The prevalence rates at years 1999 to 2002 (at the middle of each year) are interpolated from the prevalence rates presented in these two surveys. Note that we need to estimate the disabled population for single ages. For this purpose, we have two choices, both based on interpolation.

a. Estimate the prevalence rates at single ages at years 1998 and 2003, and then interpolate these single age prevalence rates from 1999 to 2002.

b. Interpolate the age group prevalence rates from 1999 to 2002. The disabled population at a single age will be estimated from the age group total.

We opt for approach (b) as the prevalence rate at a single age is likely to have a much higher fluctuation annually than the age group prevalence rate. Before considering a suitable interpolation method, we address the problem related to the reported prevalence rates in the surveys which is significant for the analysis in this chapter.
4.3.1 Interpolation of disability prevalence rates for the very high age group

In SDAC, disability prevalence rates of the very old are reported for a very wide age interval (85 and over for the 1998 survey and 90 and over for the 2003 survey). This could mask the dynamics of the disability process at high ages where the progress of disablement is the most significant. In our application, this could result in inaccurate estimates of disability longitudinal data for the very old. To remedy this problem, we estimate disability prevalence rates of the very old at narrower age bands (i.e., age groups 85–89, 90–94, . . . , 105–109). These estimates are obtained by implementing the multi-state model described in Section 3.4 separately for 1998 and 2003.

In implementing the multi-state model to the 1998 survey data, we adopt similar recovery rates and additional mortality (due to disability) assumptions as described in Section 3.4. We use the ERP (ABS (2009)) at the middle of 1998 to estimate the population at single ages at the survey date (see Section 3.4.2) and Australian life tables for the period 1997–1999 (ABS (2008a)) to determine the total number of deaths across disability categories in the fitting process (see Section 3.4.3).

For the purpose of estimation of disability longitudinal data, the estimated prevalence rates (at age groups 85–89, 90–94, . . . , 105–109) for some disability categories (from the multi-state model) do not sufficiently replicate the reported overall prevalence rates of the very old. This is due to model limitations and possibly inaccurate assumptions employed in the fitting process. For disability categories for which this is the case, we alternatively estimate the prevalence rates under a parametric formula. The estimated prevalence rates under a parametric formula need to satisfy the following criteria.

(a) Closely replicate the reported overall disability prevalence rates of the very old.

(b) The progression across ages is similar to the progression of the prevalence rates estimated from the multi-state model.

(c) The value is between zero and one.
(d) The sum of the estimated prevalence rates across all disability categories (including healthy) in a given age group is one.

To determine a suitable parametric formula, we experiment with polynomials of order 2 to 6. Each disability category is considered separately (i.e. a different polynomial might be used to estimate different categories). We consider fitting the polynomial functions to all age groups and only to age groups 60–64 and above. Reported prevalence rates for age groups 80–84 and below are assumed to relate to the ages at the middle of the age groups. For (reported) prevalence rates of the very old, we experiment with the ages to which they relate. Specifically, we consider ages around (and including) the weighted average age of the very old. The polynomials are fitted using ordinary least squares (OLS). From a given polynomial function, prevalence rates at age groups 85–89, 90–94, . . . , 105–109 are calculated by assuming that the prevalence rates at these age groups are related to the weighted average age of the corresponding age group.

We only estimate the prevalence rate for each CAL state; the prevalence rate for the healthy state is set to be one minus the sum of the prevalence rates of all CAL states. This ensures that criterion (d) is satisfied.

For some polynomials, criterion (c) can be enforced by adding an assumed prevalence rate at age 110 in the fitting. The value of this assumed prevalence rate is either 0 or 1 depending on whether the initial estimated prevalence rate is above 1 or below 0. If this procedure does not work, we alternatively adjust the initial estimates using the method illustrated in the next paragraph.

Table 4.3: Estimated prevalence rates for the 2003 survey, males.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>90–94</td>
<td>10.62%</td>
<td>19.19%</td>
<td>14.88%</td>
<td>16.04%</td>
<td>39.28%</td>
</tr>
<tr>
<td>95–99</td>
<td>4.84%</td>
<td>11.12%</td>
<td>10.52%</td>
<td>16.83%</td>
<td>56.68%</td>
</tr>
<tr>
<td>100–104</td>
<td>3.62%</td>
<td>2.94%</td>
<td>5.82%</td>
<td>15.86%</td>
<td>71.76%</td>
</tr>
<tr>
<td>105–109</td>
<td>3.90%</td>
<td>−1.84%</td>
<td>2.80%</td>
<td>14.27%</td>
<td>80.86%</td>
</tr>
<tr>
<td>Fitted (90+)</td>
<td>9.55%</td>
<td>17.53%</td>
<td>13.97%</td>
<td>16.16%</td>
<td>42.79%</td>
</tr>
<tr>
<td>Observed (90+)</td>
<td>9.61%</td>
<td>17.47%</td>
<td>13.97%</td>
<td>16.16%</td>
<td>42.79%</td>
</tr>
</tbody>
</table>
Table 4.3 gives the estimated prevalence rates for males in 2003. Note that the prevalence rate for the mild disability category at age group 105–109 is negative. To adjust this negative prevalence rate, we calculate the ratio of the prevalence rates for each of healthy and mild categories to the total prevalence rates of both healthy and mild categories at age group 100–104. We apply this ratio to the total prevalence rates of these categories (healthy and mild) at age group 105–109 to derive the adjusted prevalence rates for healthy and mild categories at this age group. The adjusted estimates are presented in Table 4.4.

Table 4.4: Estimated prevalence rates for the 2003 survey, males (adjusted for negative prevalence rates).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>90–94</td>
<td>10.62%</td>
<td>19.19%</td>
<td>14.88%</td>
<td>16.04%</td>
<td>39.28%</td>
</tr>
<tr>
<td>95–99</td>
<td>4.84%</td>
<td>11.12%</td>
<td>10.52%</td>
<td>16.83%</td>
<td>56.68%</td>
</tr>
<tr>
<td>100–104</td>
<td>3.62%</td>
<td>2.94%</td>
<td>5.82%</td>
<td>15.86%</td>
<td>71.76%</td>
</tr>
<tr>
<td>105–109</td>
<td>1.14%</td>
<td>0.93%</td>
<td>2.80%</td>
<td>14.27%</td>
<td>80.86%</td>
</tr>
<tr>
<td>Fitted (90+)</td>
<td>9.54%</td>
<td>17.53%</td>
<td>13.97%</td>
<td>16.16%</td>
<td>42.79%</td>
</tr>
<tr>
<td>Observed (90+)</td>
<td>9.61%</td>
<td>17.47%</td>
<td>13.97%</td>
<td>16.16%</td>
<td>42.79%</td>
</tr>
</tbody>
</table>

The chosen polynomial formula is the one which best satisfies criteria (a) and (b). Note that due to the adopted adjustment procedure, criteria (c) and (d) are satisfied by every polynomial formula.

The estimated prevalence rates can be further adjusted using the IPF procedure. This is done by taking disability category as the column variable and age group as the row variable. Note that in a given age group (say 90–94), we can reliably estimate the total disabled population across disability categories (including healthy). This can be estimated from the ERP at the closest central date of the survey (see Section 3.4.2). In addition, in a given disability category, the total population of the very old is given by the survey. We multiply the estimated prevalence rates with the total population to obtain the starting values for the IPF procedure. From the solution of the IPF procedure, we calculate our final estimate of disability prevalence rates. These are shown in Tables 4.5 to 4.8.
### Table 4.5: Estimated prevalence rates for the 2003 survey, males (IPF adjusted).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>90–94</td>
<td>10.69%</td>
<td>19.12%</td>
<td>14.88%</td>
<td>16.04%</td>
<td>39.28%</td>
</tr>
<tr>
<td>95–99</td>
<td>4.88%</td>
<td>11.08%</td>
<td>10.52%</td>
<td>16.84%</td>
<td>56.68%</td>
</tr>
<tr>
<td>100–104</td>
<td>3.64%</td>
<td>2.93%</td>
<td>5.82%</td>
<td>15.86%</td>
<td>71.75%</td>
</tr>
<tr>
<td>105–109</td>
<td>1.15%</td>
<td>0.92%</td>
<td>2.80%</td>
<td>14.27%</td>
<td>80.86%</td>
</tr>
</tbody>
</table>

### Table 4.6: Estimated prevalence rates for the 2003 survey, females (IPF adjusted).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>90–94</td>
<td>9.18%</td>
<td>11.21%</td>
<td>3.32%</td>
<td>11.02%</td>
<td>65.27%</td>
</tr>
<tr>
<td>95–99</td>
<td>3.00%</td>
<td>8.65%</td>
<td>0.72%</td>
<td>9.58%</td>
<td>78.06%</td>
</tr>
<tr>
<td>100–104</td>
<td>1.42%</td>
<td>4.08%</td>
<td>0.34%</td>
<td>8.98%</td>
<td>85.18%</td>
</tr>
<tr>
<td>105–109</td>
<td>1.89%</td>
<td>5.43%</td>
<td>0.45%</td>
<td>3.52%</td>
<td>88.71%</td>
</tr>
</tbody>
</table>

### Table 4.7: Estimated prevalence rates for the 1998 survey, males (IPF adjusted).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>85–89</td>
<td>21.26%</td>
<td>18.34%</td>
<td>12.45%</td>
<td>12.86%</td>
<td>35.09%</td>
</tr>
<tr>
<td>90–94</td>
<td>4.96%</td>
<td>13.71%</td>
<td>5.20%</td>
<td>12.52%</td>
<td>63.61%</td>
</tr>
<tr>
<td>95–99</td>
<td>0.11%</td>
<td>7.56%</td>
<td>2.52%</td>
<td>12.23%</td>
<td>77.58%</td>
</tr>
<tr>
<td>100–104</td>
<td>1.23%</td>
<td>1.62%</td>
<td>1.22%</td>
<td>11.42%</td>
<td>84.50%</td>
</tr>
<tr>
<td>105–109</td>
<td>0.37%</td>
<td>0.49%</td>
<td>0.57%</td>
<td>10.61%</td>
<td>87.96%</td>
</tr>
</tbody>
</table>
Table 4.8: Estimated prevalence rates for the 1998 survey, females (IPF adjusted).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>85–89</td>
<td>20.23%</td>
<td>9.54%</td>
<td>8.61%</td>
<td>13.89%</td>
<td>47.72%</td>
</tr>
<tr>
<td>90–94</td>
<td>11.02%</td>
<td>3.96%</td>
<td>3.85%</td>
<td>12.62%</td>
<td>68.55%</td>
</tr>
<tr>
<td>95–99</td>
<td>4.97%</td>
<td>1.35%</td>
<td>1.51%</td>
<td>11.42%</td>
<td>80.74%</td>
</tr>
<tr>
<td>100–104</td>
<td>1.88%</td>
<td>0.45%</td>
<td>0.60%</td>
<td>10.54%</td>
<td>86.54%</td>
</tr>
<tr>
<td>105–109</td>
<td>0.62%</td>
<td>0.14%</td>
<td>0.20%</td>
<td>9.85%</td>
<td>89.18%</td>
</tr>
</tbody>
</table>

4.3.2 Trend of disability prevalence rates

Underlying prevalence rates of disability are affected by many factors such as incidence rates of disability, recovery rates, age at onset and survival rates of people with disability (AIHW, 2003). In addition to factors which affect underlying prevalence rates, there are factors which might affect reported prevalence rates even if underlying prevalence rates remain unchanged. Examples of these factors include changes in survey design, changes in the community’s perception and awareness of disability and medical advances which result in a better diagnosis. These factors are more likely to have an impact on the reported prevalence rates of mild disability and less on more severe disability.

Recent reviews of the trends of disability prevalence rates are provided by Davis et al. (2001) and AIHW (2003 and 2008). These reviews suggest that for ages 65 and above, overall “disability” (disability here is an umbrella category as described in Section 4.1) prevalence rates have been roughly constant from 1988 to 2003 (see Table 8.1 of AIHW (2003) or Table 1 of AIHW (2008)). While there was a substantial increase in overall “disability” prevalence rates (for ages 65 and above) between the 1981 survey and the four later surveys, this was largely because of the increasing emphasis in the later surveys on ageing rather than reflecting an actual increase of the underlying prevalence rates (AIHW, 2003). The overall “severe” (combined category of severe and profound CAL) prevalence rates for ages 65 and above are relatively stable over the two decades to 2003 (Table 1 of AIHW (2008)).
Previous studies also suggest the stability of “severe” prevalence rates over time (Wen et al., 1995; Walsh and De Ravin, 1995; Madden and Wen, 2001; Leung, 2004). For example, Wen et al. (1995), by performing a decomposition analysis, found that the overall age-standardised “severe” prevalence rate in Australia was relatively stable during the 1980s and early 1990s.

Adjusted age-specific “disability” and “severe” prevalence rates from the 1988, 1993 and 1998 surveys (adjusted according to common disability criteria) are presented by Davis et al. (2001). Figures 4.2 and 4.3 below show the ratio of adjusted age-specific “disability” and “severe” prevalence rates in 1993 and 1998 to the rates in 1988.

In general, age-specific “disability” prevalence rates reported in the 1988, 1993 and 1998 surveys are roughly at a similar level for both genders. For males, the prevalence rates at age group 70–74 in 1993 and age group 75–79 in 1998 are relatively high and this seems due to a cohort effect which will be described later.

For “severe” prevalence rates, there is a significant increase for males at age group 60–64 and 75–79 in the 1998 survey, and at age group 85 and over in both the 1993 and 1998 surveys. For females, there is a significant increase only at age group 60–64 in the 1998 survey. The increase at age group 60–64 for both genders in the 1998 survey seems due mostly to the more accurate capture due to survey improvements: out of 0.45% of the increase in prevalence rates from the 1993 to 1998 surveys contributed by 45–64 population, around 0.35% is due to better capture because of survey developments (Davis et al., 2001). The increase at age group 75–79 in 1998 for males seems due to a cohort effect: because of the trend in healthy cooking and exercise, economic improvement and medical advances; males who were aged 70–79 in 1998 live longer than the older cohorts, however, their higher survival rates result in a higher proportion of disablement (Davis et al., 2001). This cohort effect, however, is not present in the 1993 survey for the “severe” category. The increase in prevalence rates at age group 85 and over for males is due to improvement of survival rates which results in a higher proportion of males with disability. Note that the prevalence rates at this age group for males are progressively higher from 1988 to 1998. There is no increase of
prevalence rates for females at this age group (85 and over) which might be due to lower improvement of survival rates for females over this period.

**Figure 4.2:** Ratio of the age-specific “disability” prevalence rates to 1988 survey*.

**Figure 4.3:** Ratio of the age-specific “severe” prevalence rates to 1988 survey*.

* Data adjusted according to common disability criteria.

Figure 4.4 shows the ratio of “disability” and “severe” prevalence rates in 2003 to the rates in 1998. Note that the prevalence rates are relatively more stable for both genders in these two surveys. The major fluctuation occurs for males in the “severe” category at age groups 65–69 and 85 and over for which the reason is not clear. It is possible that some changes in survey
4.3. ESTIMATION OF THE DISABLED POPULATION

Figure 4.4: Ratio of the age-specific “disability” and “severe” prevalence rates between 2003 and 1998 surveys.

To summarise, “disability” and “severe” prevalence rates are relatively stable from the 1988 to the 2003 surveys for both genders. There was a significant increase of “severe” prevalence rates in the 1998 survey. However, only the increase for males at age groups 75–79 and 85 and over, which is due to cohort effect and increased longevity respectively, might reflect the actual increase of underlying prevalence rates. Note that both cohort and increased longevity effects seem to reduce greatly in the 2003 survey. There are some insignificant fluctuations of prevalence rates from the 1998 to the 2003 survey which changes in survey design might partly explain.

Prevalence rates for each CAL category in the 1998 and 2003 surveys are presented in Figures 4.5. While there are some fluctuations, in general, the prevalence rates across ages for each CAL category are somewhat similar between the two surveys.
Figure 4.5: Prevalence rates for each CAL category.
4.3.3 Estimation of age group prevalence rates at each mid-year

A possible method to estimate disability prevalence rates at the middle of each year from 1999 to 2002 is by modelling the disability status using a multinomial logit specification (see Waidmann and Liu (2000)). By including time (in years) as one of the covariates, disability prevalence rates at these years can be estimated from the model. However, since we only have two observations (i.e. prevalence rates in 1998 and 2003) to estimate the time trend term, this method is not feasible.

Due to the limited amount of available data and given the apparent stability of disability prevalence rates from 1988 to 2003 (Section 4.3.2), we decided to linearly interpolate the prevalence rates in 1998 and 2003 to estimate the prevalence rates in years 1999 to 2002.

Multiplying the estimated prevalence rates by the ERP, we obtain our estimate of the disabled population at the middle of each year from 1998 to 2003. The estimate is obtained for all age groups and disability categories. ERP at single ages for ages 100 and above is estimated by using functions from the Australian life tables in the corresponding year published by The Human Mortality Database (www.mortality.org). Given our purpose of estimating transition probabilities, rounding of the estimates (to the closest integer) is not necessary. Tables 4.9 and 4.10 below present the estimates at year 2000.

4.3.4 Estimation of disabled population at single ages (ages 60 and above)

There are a variety of methods to estimate the population at single ages from an age group total. These include estimation by employing a rectangular assumption, the prorating method, and graphic and parametric interpolation (Siegel and Swanson, 2004). Under a rectangular assumption, the population in a given age group is assumed to be evenly distributed at each single age, while under the prorating method, the age structure of the population
Table 4.9: Estimate of disabled population at the middle of year 2000, males.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–4</td>
<td>634,083.30</td>
<td>0.00</td>
<td>1,178.66</td>
<td>9,452.15</td>
<td>11,155.89</td>
</tr>
<tr>
<td>5–14</td>
<td>1,254,285.82</td>
<td>29,448.00</td>
<td>10,513.76</td>
<td>43,144.39</td>
<td>39,909.03</td>
</tr>
<tr>
<td>15–24</td>
<td>1,262,438.15</td>
<td>30,606.57</td>
<td>11,337.90</td>
<td>18,182.38</td>
<td>10,445.99</td>
</tr>
<tr>
<td>25–34</td>
<td>1,333,539.74</td>
<td>42,881.55</td>
<td>19,854.16</td>
<td>20,500.37</td>
<td>13,924.18</td>
</tr>
<tr>
<td>35–44</td>
<td>1,328,209.72</td>
<td>53,060.86</td>
<td>41,489.60</td>
<td>31,644.46</td>
<td>13,783.36</td>
</tr>
<tr>
<td>45–54</td>
<td>1,093,211.93</td>
<td>76,237.02</td>
<td>66,992.68</td>
<td>51,883.16</td>
<td>13,239.21</td>
</tr>
<tr>
<td>55–59</td>
<td>371,536.61</td>
<td>45,975.13</td>
<td>34,426.09</td>
<td>28,611.20</td>
<td>9,649.98</td>
</tr>
<tr>
<td>60–64</td>
<td>274,567.16</td>
<td>56,790.57</td>
<td>37,359.41</td>
<td>21,942.51</td>
<td>10,139.35</td>
</tr>
<tr>
<td>65–69</td>
<td>218,723.01</td>
<td>52,478.58</td>
<td>32,615.83</td>
<td>17,005.05</td>
<td>11,212.53</td>
</tr>
<tr>
<td>70–74</td>
<td>170,560.20</td>
<td>61,908.26</td>
<td>31,974.57</td>
<td>16,260.98</td>
<td>18,882.99</td>
</tr>
<tr>
<td>75–79</td>
<td>99,821.97</td>
<td>49,619.82</td>
<td>28,748.36</td>
<td>16,311.88</td>
<td>25,087.97</td>
</tr>
<tr>
<td>80–84</td>
<td>46,133.00</td>
<td>29,280.37</td>
<td>13,245.97</td>
<td>10,856.79</td>
<td>19,452.87</td>
</tr>
<tr>
<td>85–89</td>
<td>13,482.78</td>
<td>12,047.85</td>
<td>6,106.42</td>
<td>6,213.47</td>
<td>18,941.49</td>
</tr>
<tr>
<td>90–94</td>
<td>1,213.79</td>
<td>2,655.22</td>
<td>1,517.93</td>
<td>2,329.84</td>
<td>9,014.23</td>
</tr>
<tr>
<td>95–99</td>
<td>70.59</td>
<td>313.56</td>
<td>199.90</td>
<td>491.98</td>
<td>2,419.97</td>
</tr>
<tr>
<td>100–104</td>
<td>10.39</td>
<td>10.15</td>
<td>14.48</td>
<td>62.43</td>
<td>375.60</td>
</tr>
<tr>
<td>105–109</td>
<td>0.17</td>
<td>0.17</td>
<td>0.37</td>
<td>3.04</td>
<td>21.43</td>
</tr>
</tbody>
</table>

and hence the number at each single age is estimated from another information source. For graphic interpolation, the estimates at a single age are obtained from graphical analysis. Lastly, under parametric interpolation, the estimates are obtained from interpolation with a mathematical formula.

We opt to adopt parametric interpolation. The rectangular assumption is likely to result in inaccurate estimates at very high ages where probabilities of disablement are significant, while the prorating method is difficult to implement since there is no suitable information source to accurately estimate the age structure of the disabled population across the whole age range considered (60 and above). Note that it is unlikely that the age structure of the disabled population is similar to the age structure of the general population (the total population across disability categories) at this age range. Lastly, graphical analysis is also difficult to implement in our situation.

To implement parametric interpolation, there are two possible methods:
4.3. ESTIMATION OF THE DISABLED POPULATION

Table 4.10: Estimate of disabled population at the middle of year 2000, females.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–4</td>
<td>610,410.32</td>
<td>246.81</td>
<td>1,607.52</td>
<td>4,258.86</td>
<td>6,576.49</td>
</tr>
<tr>
<td>5–14</td>
<td>1,251,480.21</td>
<td>13,089.07</td>
<td>3,893.81</td>
<td>20,288.30</td>
<td>21,044.62</td>
</tr>
<tr>
<td>15–24</td>
<td>1,224,056.41</td>
<td>27,311.18</td>
<td>9,494.17</td>
<td>13,690.56</td>
<td>10,730.68</td>
</tr>
<tr>
<td>25–34</td>
<td>1,359,026.80</td>
<td>35,045.24</td>
<td>16,763.81</td>
<td>25,795.72</td>
<td>8,700.42</td>
</tr>
<tr>
<td>35–44</td>
<td>1,332,776.21</td>
<td>54,677.81</td>
<td>41,520.27</td>
<td>44,629.98</td>
<td>11,716.73</td>
</tr>
<tr>
<td>45–54</td>
<td>1,074,766.43</td>
<td>75,461.53</td>
<td>66,334.07</td>
<td>58,974.91</td>
<td>21,725.05</td>
</tr>
<tr>
<td>55–59</td>
<td>348,885.39</td>
<td>47,649.25</td>
<td>38,805.15</td>
<td>27,779.88</td>
<td>10,363.32</td>
</tr>
<tr>
<td>60–64</td>
<td>276,604.32</td>
<td>45,314.99</td>
<td>37,218.27</td>
<td>25,105.15</td>
<td>12,610.27</td>
</tr>
<tr>
<td>65–69</td>
<td>233,364.33</td>
<td>46,679.25</td>
<td>31,874.31</td>
<td>18,238.81</td>
<td>14,924.30</td>
</tr>
<tr>
<td>70–74</td>
<td>190,248.76</td>
<td>52,134.95</td>
<td>37,965.69</td>
<td>24,980.59</td>
<td>28,313.02</td>
</tr>
<tr>
<td>75–79</td>
<td>134,955.83</td>
<td>54,373.23</td>
<td>30,731.48</td>
<td>25,154.02</td>
<td>42,529.44</td>
</tr>
<tr>
<td>80–84</td>
<td>65,547.82</td>
<td>38,213.85</td>
<td>14,851.15</td>
<td>19,129.01</td>
<td>52,258.18</td>
</tr>
<tr>
<td>85–89</td>
<td>24,512.90</td>
<td>11,287.42</td>
<td>11,017.83</td>
<td>15,125.92</td>
<td>54,745.93</td>
</tr>
<tr>
<td>90–94</td>
<td>4,677.57</td>
<td>3,120.01</td>
<td>1,654.08</td>
<td>5,449.76</td>
<td>30,581.60</td>
</tr>
<tr>
<td>95–99</td>
<td>475.39</td>
<td>485.20</td>
<td>135.47</td>
<td>1,213.91</td>
<td>9,051.02</td>
</tr>
<tr>
<td>100–104</td>
<td>25.69</td>
<td>28.90</td>
<td>7.48</td>
<td>150.49</td>
<td>1,305.15</td>
</tr>
<tr>
<td>105–109</td>
<td>4.67</td>
<td>2.18</td>
<td>0.29</td>
<td>7.07</td>
<td>85.96</td>
</tr>
</tbody>
</table>

midpoint and cumulation-differencing. Under the midpoint method, the average of the population in a given age group (i.e. the population in a given age group divided by the length of the age group) is assumed to relate to the age at the middle of the age group. By interpolating these values, the population at a single age is estimated. The shortcoming of this method (theoretically) is that the average of the population (in a given age group) seldom relates to the age at the middle of the age group. It is possible to instead consider the weighted average age; however, in our application, this is difficult to estimate since we do not know the age structure of the disabled population. Considering this, we opt to adopt the cumulation-differencing method which is described below.

Denote:

\[ f(x) : \text{a parametric function in } x. \]
\( l(x) \): observed number of the population aged \( x \) last birthday.

\[ L(x) = \sum_{n=0}^{x-1} l(n) \]: observed number of population in age interval \([0, x)\).

Hence

\[ l(x) = L(x + 1) - L(x). \tag{4.13} \]

For each disability category, the values of \( L(x) \) for \( x = 5, 15, \ldots, 55, 60, \ldots, 110 \) can be obtained from the estimated age group total (see Tables 4.9 and 4.10). Note that these values relate to the precise age at which they apply. We fit a parametric function \( f(x) \) to these values to estimate \( L(x) \) for \( x = 60, 61, \ldots, 110 \). From these estimates, we use (4.13) to estimate \( l(x) \) for \( x = 60, 61, \ldots, 109 \). In implementing this method, we assume that the pattern of accumulation of the population at a five or ten year age interval is a good indicator of the pattern of accumulation at a single age interval.

In applying the cumulation-differencing method, we consider a variety of parametric functions: natural cubic spline, Karup-King, Sprague, Beers “Ordinary” and Beers “Modified”. These functions are commonly employed (together with the cumulation-differencing method) in the estimation of a single year age distribution from five yearly or other regularly grouped demographic data (Bijak and Kupiszewska, 2008; Siegel and Swanson, 2004; Smith et al., 2004). For example, cumulation-differencing by employing either Karup-King, Sprague or Beers formula is among the methods proposed by Bijak and Kupiszewska (2008) to disaggregate five-year age groups of population into single years in the estimation of population size for 31 European countries. In the following, we provide a description for each of the functions.

**Natural cubic spline**

Under a cubic spline, we fit a cubic polynomial to each interval of data. The cubic polynomials are chosen in a way such that they are continuous up to the second order derivative at the joining points or knots. To uniquely define the parameters of the cubic polynomials, additional constraints are required. Usually, this is done by requiring the spline to be linear for the interval before the first knot and after the last knot. If such constraints
are employed, the spline is called a natural cubic spline. For a description of cubic splines, see Benjamin and Pollard (1993) or Siegel and Swanson (2004).

In interpolating $L(x)$ with a natural cubic spline, we place the knots at each age $x$ at which the values of $L(x)$ can be obtained from the estimated age group total (i.e., we place the knots at ages 0, 5, 15, \ldots, 55, 60, \ldots 110; with $L(0)$ assumed to be 0). Therefore, our estimates of $l(x)$ at single ages under a natural cubic spline satisfy the estimated age group totals (for example, the estimates of $l(x)$ for year 2000 satisfy the values of age group totals presented in Tables 4.9 and 4.10).

**Karup-King’s third-difference and Sprague’s fifth-difference equations**

Karup-King and Sprague are among the widely used formulae in osculatory interpolation. Osculatory formulae are constructed by joining two successive interpolation curves in such a way that for a certain abscissa they have a common ordinate, tangent or radius of curvature. A general derivation of osculatory formulae is given in Glover (1910). In the following we provide a brief description for Karup-King and Sprague formulae.

The Karup-King formula is a third-difference equation based on two overlapping polynomials of the second degree. Four given points, $y_n, y_{n+1}, y_{n+2}$ and $y_{n+3}$, are required in the interpolation, and must be equally spaced along the abscissa. The equation is designed to interpolate between the abscissae $n + 1$ and $n + 2$ (limited to middle interval interpolation), where at these abscissae, the two overlapping polynomials are forced to have common ordinates and tangents.

Sprague’s formula is a fifth-difference equation based on two overlapping polynomials of the fourth degree (Sprague, 1881). Six given points, $y_n, y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}$ and $y_{n+5}$, are required in the interpolation, and must be equally spaced along the abscissa. The equation is designed to interpolate between abscissae $n + 2$ and $n + 3$ (limited to middle interval interpolation), where at these abscissae, the two overlapping polynomials are forced to have common ordinate, tangent and radius of curvature.

For fuller details of the Karup-King and Sprague formulae and their ap-
application, see Siegel and Swanson (2004).

Our estimates of $l(x)$ at single ages under the Karup-King and Sprague formulae also satisfy the estimated age group totals.

**Beers’ six-term “ordinary” and “modified” formulae**

The interpolation coefficients specified in the Beers’ “Ordinary” formula (six-term formula correct to fourth differences and replicates the given values) are designed to minimise, on average, the sum of the squares of the fifth differences of the interpolated values (Beers, 1944). Under this method, the interpolated points are not considered as lying on any particular curve, but rather, the interpolation coefficients are constructed under the assumption that any two consecutive fifth differences of given data are independently distributed random variables with zero mean and equal variances (Greville, 1948).

Beers’ “Modified” formula combines interpolation with some smoothing or graduation of the given values (Beers, 1945). By relaxing the requirement to reproduce the given values, this formula is able to produce smoother interpolated results (compared to Beers’ “Ordinary” formula) (Siegel and Swanson, 2004). This formula minimises, on average, the fourth differences of the interpolated values.

For details of application of Beers’ “Ordinary” and “Modified” formulae, see Siegel and Swanson (2004).

The cumulation-differencing method has been applied to estimate the disabled population at a single year of age ($l(x)$) from the age group totals estimated in Section 4.3.3 by employing each of the above interpolation formulae. For convenience, the natural cubic spline, Beers’ “Ordinary” and Beers’ “Modified” formulae are also referred to as osculatory formulae. For each formula, the estimates are obtained for each disability category and each middle year from 1998 to 2003. These formulae, except Beers’ “Modified”, result in similar estimates. The reason for Beers’ “Modified” formula resulting in slightly different estimates (compared to other formulae) is because this formula combines interpolation with some smoothing (which results in
4.3. ESTIMATION OF THE DISABLED POPULATION

smoother interpolated values but does not replicate the given values). Figure 4.6 below presents the estimate of \( l(x) \) for males in the profound category at the middle of year 2000.

Figure 4.6: Estimates of \( l(x) \) for profound CAL at the middle of year 2000, males.

For some disability categories, the estimates of \( l(x) \) are negative for certain ages above 90. This problem occurs for all interpolation formulae considered. The reason for this is that as the number in the disabled population (in a given disability category) decreases rapidly at high ages (see Table 4.9 and 4.10), the slope of the curve of \( L(x) \) changes rapidly at these ages and this causes the interpolation curve to be non-monotonic (therefore the differences of \( L(x) \) will be negative). This problem is more prevalent for less severe disability categories (healthy, mild and moderate CAL) than for more severe categories (severe and profound CAL) since, at high ages, the population in the less severe categories decreases more rapidly with age. Table 4.11 below presents the estimates of \( l(x) \) for females at the middle of year 2000 under...
Table 4.11: Estimate of $l(x)$ at the middle of year 2000, females.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>105</td>
<td>−3.11</td>
<td>4.50</td>
<td>−7.43</td>
<td>0.61</td>
<td>34.27</td>
</tr>
<tr>
<td>106</td>
<td>−0.89</td>
<td>1.79</td>
<td>−2.44</td>
<td>1.14</td>
<td>22.88</td>
</tr>
<tr>
<td>107</td>
<td>0.77</td>
<td>−0.24</td>
<td>1.31</td>
<td>1.55</td>
<td>14.35</td>
</tr>
<tr>
<td>108</td>
<td>1.88</td>
<td>−1.59</td>
<td>3.80</td>
<td>1.82</td>
<td>8.65</td>
</tr>
<tr>
<td>109</td>
<td>2.43</td>
<td>−2.27</td>
<td>5.05</td>
<td>1.95</td>
<td>5.81</td>
</tr>
</tbody>
</table>

cubic splines. The estimates are presented for each disability category.

A similar problem has been encountered by Wilmoth et al. (2007) in the estimation of the number of deaths at a single age using cubic splines. In Wilmoth’s case, the flattening of the mortality curve in the second year of life causes the interpolation curve to be non-monotonic and this results in the negative estimated number of deaths. Wilmoth et al. (2007) dealt with this problem by imposing a constraint on the slope of the spline function. Smith et al. (2004) proposed an alternative method to ensure monotonicity of an interpolating cubic spline by applying a so-called ‘Hyman filer’. This method has been applied by Smith et al. (2004) to estimate the number of deaths at a single age with a satisfactory result.

To deal with negative estimates of $l(x)$, we instead consider several alternative interpolation methods. In the following we describe the methods considered.

(a) Cumulation-differencing by employing an appropriate polynomial with a restriction such that the estimates of $l(x)$ are not negative.

Estimation of the parameters of the polynomials with a restriction described above was done in R using the function “constrOptim.nl”; see Varadhan (2011).

(b) Method (a) with further refinement by the prorating method.

Under Method (b), the estimates under Method (a) are only used to estimate the age structure of the disabled population in a given age group. For
example, suppose that under Method (a), we have 20% of the disabled population (in a given disability category) in age group 60–64 at age 60. Then, the estimate of the disabled population aged 60 (in the corresponding category) under Method (b) is 20% of the observed disabled population in the age group 60–64.

(c) Method (b) with further refinement under the IPF procedure.

This is done by taking the disability category as the column variable and age as the row variable. Note that in a given disability category, the size of the disabled population for each age group is known. In addition, at a given age, the sum of the disabled population across disability categories (including healthy) is also known (which is the general population (ERP) at that age). Under Method (c), the estimates under Method (b) are further refined under the IPF procedure such that they satisfy the known age group totals for each disability category and the known ERP for each age.

To compare the performance of the methods considered, we experimented with the ERP data. We summed the reported single-age distributions of ERP into five year age groups and estimated the ERP at a single age from these age groups under Methods (a), (b) and (c). We compared the estimates under each method with the reported single-age distributions of the ERP. We experimented with data from both genders and from each year from 1998 to 2003. In this experiment, Method (c) was implemented by taking year as the column category and age as the row category. We consider polynomials of orders 2 to 6, and OLS and WLS fitting methods. Several conclusions from this experiment are as follows.

- Method (c) results in the best estimate, while Method (b) performs better than Method (a). Therefore, implementation of prorating and the IPF procedure improves the accuracy of the estimates.

- Polynomials which result in accurate estimates after being refined with prorating and IPF procedures (Method (c)), typically, before being refined with these procedures, also result in an appropriate age structure.
On the other hand, if the polynomial initially results in an inappropriate age structure (for example, increasing with age), the resulting estimates under Method (c) will also not be accurate.

Figure 4.7 illustrates the accuracy of Method (c) in estimating the single-age distributions of the ERP. The relative error is defined as the absolute value of the difference between the observed and the estimated values divided by the observed value. Note that the estimation of single-age distributions of disabled population (for each disability category) is a more difficult problem than the similar estimation of ERP especially for estimation at very high ages (disabled population, especially those in less severe categories, decreases more rapidly with age). Therefore, a similar level of accuracy of Method (c) in the estimation of the disabled population cannot be expected.

Figure 4.7: Absolute value of the relative error of the estimation.

Arguably, in the estimation of the single-age distributions of ERP, an accurate result under Method (c) requires a suitable estimate of age and time
(in year) interaction, while in the estimation of the disabled population, it instead requires a suitable estimate of age and disability interaction. However, a suitable estimate of age and disability interaction can be obtained from appropriate estimates of the disabled population at a single age for each disability category. Therefore, if Method (b) results in such estimates, the further refinement under the IPF procedure is also likely to increase the accuracy of these estimates.

Therefore, we adopt Method (c) to deal with the negative values of the estimated disabled population. We consider polynomials of orders 2 to 6, and OLS and WLS fitting methods.

Choosing a polynomial which is likely to result in accurate estimates is difficult. However, we are able to determine polynomials which are likely to result in poor estimates. These are polynomials which result in an inappropriate age structure (for example, an increasing age structure). After discarding such polynomials, the chosen polynomial is the one which best replicates the observed accumulated values.

In applying Method (c), we opted to keep the estimates under osculatory formulae up to age 89. The alternative method (Method (c)) is employed only to estimate the disabled population at ages 90 and above. This is because in another experiment with ERP data, we found that osculatory formulae consistently result in better estimates than ordinary polynomials (in the experiment, the estimates from the polynomials have been refined with the prorating method). The cutting age 89 is chosen for convenience. There is little impact on the accuracy of the estimation associated with the chosen cutting age as osculatory formulae do not perform very well at very high ages.

Estimates for ages 60 to 89 under osculatory formulae are also refined under the IPF procedure such that it satisfies both the known age group total for each disability category and the known ERP for each age (similar to Method (c)).

A further refinement in the estimation is possible by maintaining the continuity of the given osculatory curve with the chosen polynomial curve. The parameters of the chosen polynomial can be restricted such that its
derivatives (up to a certain desired degree) at exact age 90 are equal with the derivatives of the osculatory curve at this age. We have experimented with this method with ERP data and decided not to employ this refinement. Under an appropriate polynomial, there is no additional gain in the accuracy of the estimation to compensate for the additional complexity of the fitting process under this refinement.

Estimates of the single-age distributions of the disabled population could also be obtained by implementing the multi-state model described in Section 3.4. We implemented this model to the disabled population at the middle of each year from 1998 to 2003 estimated in Section 4.3.3. The recovery rates and the additional mortality (due to disability) assumptions are as described in Section 3.4. ERP and Australian life tables at the relevant year are used in the estimation. The estimates were also refined under the IPF procedure similar to Method (c). Due to the methodology of the estimation, the problem of the negative estimated values is not encountered for the estimation under the multi-state model.

To summarise, we have two sets of estimates of the single-age distributions of the disabled population:

a. Estimates under the cumulation-differencing method. For ages 60 to 89, the estimates are obtained under each of the osculatory formulae, while for ages 90 and above, the estimates are obtained under Method (c).

b. Estimates under the multi-state model. The estimates are obtained for all ages considered.

Figure 4.8 presents the estimates at the middle of year 2000. Note that the estimates for ages 90 and above under the osculatory formulae are obtained under Method (c).
Figure 4.8: Estimates of the single-age distributions \( l(x) \) of the disabled population at the middle of year 2000.

- Males, mild CAL
- Females, mild CAL
- Males, moderate CAL
- Females, moderate CAL
- Males, severe CAL
- Females, severe CAL
- Males, profound CAL
- Females, profound CAL
4.4 Estimation of net overseas migration (NOM)

In this section we describe the estimation of \(NOM_{x+0.5}(t)\) (defined in Section 3.4.6) for \(n = 0, 1, \ldots, 4\) and \(t = 1998, 1999, \ldots, 2002\). The data for the estimation are from the overseas migration statistics published by the ABS (ABS, 2000, 2001, 2003, 2004b). Overseas migration consists of four components:

- Permanent movement.
- Long-term movement (arrivals and departures involving a period of 12 months or more).
- Short-term movement (arrivals and departures involving a period of less than 12 months).
- Category jumping (the change between actual and intended duration of stay of travelers to and from Australia).

In adjusting the estimated disabled population for overseas migration, we only consider permanent and long-term movement (in line with the measurement of ERP). The estimates of permanent and long-term movement from one mid-year to the next from 1998 (i.e. from the middle of 1998 to the middle of 1999) to 2002 are provided by the ABS. The estimates from 1998 to 2000 are not adjusted for category jumping (ABS, 2005). The NOM is calculated as total permanent and long-term arrivals minus total permanent and long-term departures. Table 4.12 below presents the estimate of NOM in 1999 for males and females.

For ages 60 and over, the NOM is very small in comparison with the size of the disabled population (see Tables 4.9 and 4.10). Therefore, error in the estimation of NOM (including the error which arises from category jumping) is likely to have an insignificant effect to the estimated transition probabilities.
Table 4.12: NOM for the year ending 30 June 2000.

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>Males</th>
<th>Females</th>
</tr>
</thead>
<tbody>
<tr>
<td>60 – 64</td>
<td>300</td>
<td>198</td>
</tr>
<tr>
<td>65 – 69</td>
<td>159</td>
<td>−6</td>
</tr>
<tr>
<td>70 – 74</td>
<td>30</td>
<td>−15</td>
</tr>
<tr>
<td>75 and over</td>
<td>−44</td>
<td>−195</td>
</tr>
</tbody>
</table>

Source: Australian Bureau of Statistics (2001)

The NOM at single ages is estimated under the cumulation-differencing method using Sprague’s formula. In the estimation of one-year disability longitudinal data (Section 2.5), we assume the following.

1. Zero overseas migration for ages 95 and above.

2. Overseas migrants who migrate between the middle of years $t$ and $t + 1$ have the same disability prevalence rates as the population at the same age at the middle of year $t + 1$.

3. Half of overseas migration between the middle of years $t$ and $t + 1$ occur at the middle of year $t$ and half at the middle of year $t + 1$.

4. The one-year death probability (4.4) and the conditional disability transition probabilities (i.e. probabilities of deteriorating, improving or staying in the same disability state conditional on survival) estimated in Chapter 3 are applicable to the overseas migrants.

4.5 Estimation of one-year disability longitudinal data

In this section we describe the steps to estimate one-year disability longitudinal data between the middle of years $t$ and $t + 1$ assuming we have the disabled population (at single ages) at the middle of years $t$ and $t + 1$ and the NOM at the corresponding period. The projection method used is as
described in Section 3.4.6 (with some adjustments where necessary). The steps are as follows.

(a) Estimate the healthy mortality component of the one-year death probability (4.4) which is applicable at the middle of year \( t \) (\( \text{Overall}_t \text{Mort}(x + 0.5, t) \)).

The \( \text{Overall}_t \text{Mort}(x + 0.5, t) \) is estimated by matching the total of estimated deaths between the middle of years \( t \) and \( t + 1 \) from each disability category (where the one-year death probability of disability category \( n \) (for \( n = 0, 1, \ldots, 4 \)) calculated according to (4.4)) with the total deaths implied from the Australian life tables (ALT) in the relevant year (i.e. total deaths if the whole population (total population across disability categories) experiences mortality rates as specified in ALT in the relevant year; the ALTs are obtained from the ABS (2008a)). In the estimation, deaths from overseas migrants are considered. The one-year life table death probability from exact age \( x + 0.5 \) which is applicable at the middle of year \( t \) (\( q_{x+0.5}^t \)) is calculated as:

\[
q_{x+0.5}^t = 1 - \exp \left( -0.5 \times \left( \mu_{x+0.5}^t + \mu_{x+1.5}^{t+1} \right) \right)
\]  

(4.14)

where \( \mu_{x}^t \) is the force of mortality at exact age \( x \) which is estimated from the ALT applicable in year \( t \). The forces of mortality at ages above 100 are extrapolated using Gompertz formula.

(b) Adjust the disabled population at the middle of year \( t + 1 \) for net overseas migration which occurs between the middle of years \( t \) and \( t + 1 \) (adjust \( N_{x+1.5}^n(t + 1) \) from \( I_{x+1.5}^n(t + 1) \) for \( n = 0, 1, \ldots, 4 \)).

Using the one-year death probability (4.4) (with the \( \text{Overall}_t \text{Mort}(x + 0.5, t) \) estimated in step (a)) and the conditional disability transition probabilities (i.e. probabilities of deteriorating, improving or staying in the same disability state conditional on survival) estimated in Chapter 3, the net overseas migrants who migrate at the middle of year \( t \) (note assumption (3) of Section 4.4) are projected into the middle of year \( t + 1 \). The disabled population at the middle of year \( t + 1 \) is adjusted from these projected migrants and from
the net overseas migrants who migrate at the middle of year $t + 1$. In this step, we calculate the first five column totals of Table 4.1 in Section 4.1 (i.e. $C_0, C_1, \ldots, C_4$).

(c) Estimate the actual number of deaths between the middle of years $t$ and $t + 1$ from the population excluding the deaths from overseas migrants who migrate between the middle of years $t$ and $t + 1$.

The required quantity is estimated as (using the notation in Section 3.4.6 and 4.1):

$$
\hat{D}_{x+1.5}(t + 1) = N_{x+0.5}(t) - N_{x+1.5}(t + 1) + I_{x+1.5}(t + 1) \tag{4.15}
$$

where

$$
N_{x+0.5}(t) = \sum_{n=0}^{4} N^n_{x+0.5}(t),
$$

$$
I_{x+0.5}(t) = \sum_{n=0}^{4} I^n_{x+0.5}(t).
$$

In this step, we calculate the sixth column total of Table 4.1 ($C_5$).

(d) Project the disabled population at the middle of year $t$ to the middle of year $t + 1$. The projection is carried out separately for each initial disability category (i.e. disability category at the middle of year $t$).

The one-year death probability (4.4) (with the $\text{Overall}_\text{Mort}(x + 0.5, t)$ estimated in step (a)) and the conditional disability transition probabilities estimated in Chapter 3 are used in the projection. In this step, we calculate the initial element of Table 4.2 (i.e. $\hat{N}^{i,j}_{x+0.5,x+1.5}(t, t + 1)$ for $i = 0, 1, \ldots, 4$ and $j = 0, 1, \ldots, 5$).

(e) For each age, apply the IPF procedure described in Sections 4.1 and 4.2.
From step (e), we obtain the one-year disability longitudinal data between the middle of years $t$ and $t + 1$.

We apply the above procedures to the disabled population estimated in Section 4.3. The procedures are implemented to the disabled population estimated at each pair of subsequent years (1998 and 1999, etc). Therefore, we obtain the refined estimates of one-year disability longitudinal data from 1998 (i.e. from the middle of 1998 to the middle of 1999) to 2002. Specifically, we obtain $\tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t+1)$ for $x = 60, 61, \ldots, 108; m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5$ and $t = 1998, 1999, \ldots, 2002$ (defined in Section 4.1). The $\{\tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t+1)\}$ are obtained from the disabled population estimated under each method considered in Section 4.3.

### 4.6 Estimation of crude disability transition probabilities

From the one year disability longitudinal data estimated in Section 4.5, we use (4.8) to calculate the refined estimates of one year disability transition probabilities. As described in Section 4.1, these probabilities are regarded as crude probabilities. The crude probabilities are obtained from the disabled population estimated under each method considered in Section 4.3. Two observations about the estimated crude probabilities are as follows.

- For crude transition probabilities from exact ages 60.5 to 89.5, different osculatory formulae (to estimate the disabled population) result in similar estimates.

- The disabled population estimated from the multi-state model results in different estimates of crude transition probabilities than the disabled population estimated under the cumulation-differencing method (especially for ages above 90).

We choose the crude transition probabilities obtained from the disabled population estimated under the cumulation-differencing method (we choose
estimates from cubic splines at ages 60.5 to 89.5, however, other osculatory formulae are also appropriate at these ages). This is because the cumulation-differencing method is more likely to result in more accurate estimates of the single age distribution of the disabled population than the multi-state model due to the following reasons.

1. For the disabled population estimated under the multi-state model, the IPF adjustment results in estimates with an abrupt pattern (Figure 4.8), and a significant adjustment to the initial estimates (Figure 4.9).

2. For ages 90 and over, the cumulation-differencing method (Method (c)) has performed satisfactorily in the estimation of the single age distribution of ERP.

Figure 4.9: Estimates of the single age distribution \( l(x) \) of the population with severe CAL at the middle of year 2000, males.

Tables 4.13 and 4.14 present the estimates of the crude disability transition probabilities for males and females, while Figure 4.10 illustrates these crude probabilities for the no CAL category. In Figure 4.10, the transition data at ages above 85 are grouped into quinquennial age bands (discussed later).
Table 4.13: Males, refined estimates of (crude) one-year disability transition probabilities.

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Table 4.14: Females, refined estimates of (crude) one-year disability transition probabilities.

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Figure 4.10: One-year crude transition probabilities from able (no CAL) state.
There is a lack of randomness in the estimated crude probabilities as indicated by the systematic oscillation in the movement of the estimates from age to age. This is due to various estimation methods employed in estimating the annual disabled population at a single age and in deriving the initial estimates of one-year disability longitudinal data (i.e. $\hat{N}^{m,n}_{x+0.5,x+1.5}(t,t+1)$).

While there is a lack of randomness, we believe the levels of estimated crude probabilities are indicative of the true levels of crude probabilities.

The patterns of the crude improvement probabilities, with the exception of improvement from mild CAL, are peculiar. Figure 4.11 presents the crude improvement probabilities from moderate CAL and from profound CAL for males. This peculiarity is because while in reality, improvement by more than one disability category is possible over the year, the initial transition probabilities do not allow this. Note that the accuracy of the results of the IPF procedure also depends on the accuracy of the initial transition probabilities used in the estimation. The crude improvement probabilities from moderate CAL show a more peculiar pattern than the crude improvement probabilities from either severe CAL or profound CAL, as improvement by more than one category is more likely for the less disabled state.

**Figure 4.11:** Crude improvement probabilities from moderate CAL and from profound CAL, males.

For females, the following crude transition probabilities from mild CAL also have peculiar patterns.

(a) Mild CAL to mild CAL: non-decreasing at ages above 86.
(b) Mild CAL to moderate CAL: decreasing rapidly at ages above 86.

(c) Mild CAL to severe CAL: decreasing rapidly at ages above 86.

(d) Mild CAL to dead: lower than the crude death probabilities of no CAL at ages above 89.

Figure 4.12 presents the crude transition probabilities from mild CAL to mild CAL and from mild CAL to severe CAL for females. The peculiar patterns of the above crude transition probabilities for females might be caused by the inaccuracy of the initial deterioration probabilities from mild CAL to moderate CAL and to severe CAL. The crude probabilities suggest that these deterioration probabilities should decrease with age at the advanced ages. However, the initial deterioration probabilities are non-decreasing with age over this age range. If in fact these deterioration probabilities do decrease with age (which might be due to the rapid increase of deterioration to profound CAL and death probabilities), the inaccuracy of the initial transition probabilities will distort the results of the IPF procedure at the relevant ages. Therefore, the decreasing patterns of (b) and (c) implied from the estimated crude probabilities might be too strong. The possible distortion might not be limited to the related probabilities (i.e. (a) and (d) might also be the result of distortion).

Figure 4.12: Selected crude transition probabilities from mild CAL, females.
The crude probabilities also suggest a decreasing pattern of the deterioration probabilities from no CAL to moderate CAL for females at ages above 86. However, since the decreasing pattern happens to only one deterioration probability, the distortion is likely to be small. Other crude transition probabilities from no CAL for females generally behave as expected.

It is counterintuitive for deterioration probabilities to decrease with age. However, the disability longitudinal data from the U.S. show that this could be the case. In Table 6 of Rickayzen and Walsh (2002), the disability transition rates from the National Long-Term Care Surveys of 1982 and 1984 are presented. In that table, the transition probabilities from failing one ADL to failing two ADLs for males and from failing two ADLs to failing three or more ADLs for females are decreasing with age (although the stay probabilities (probabilities of staying at the same category) of these categories are high).

For some transition probabilities, the crude probabilities at ages above 85 or 90 are spurious. This is because of the difficulty in estimating the disabled population at these ages. In the graduation, we resolve this problem by grouping the data at high ages into quinquennial age bands and calculating crude probabilities at the weighted mean age.

In the following we describe the main observations from the estimated crude probabilities.

- Additional mortality due to disability is also experienced in the moderate CAL category.

- At very high ages, the observed mortality rates of the population are close to the crude mortality rates of profound CAL, while at younger ages, they are close to the crude mortality rates of no CAL. This is in line with the reported disability prevalence rates at these ages.

- Generally, crude mortality rates are higher for males than females across disability categories with the difference between genders being much less pronounced at very high ages. The difference between genders also seems to increase with the severity of disablement.
• For females, the crude mortality rates of mild CAL at ages above 89 are lower than the crude mortality rates of no CAL.

• The crude probabilities suggest lower mortality rates than the initial probabilities for no CAL and mild CAL at ages above 89. For females, the crude probabilities also suggest higher mortality rates for severe CAL and profound CAL at ages above 98.

We make some observations about crude deterioration probabilities.

• Generally, the crude deterioration probabilities are increasing with age with the increase slowing down at the very advanced ages. There are several exceptions to this. In particular, for females, the crude deterioration probabilities from no CAL to moderate CAL, and from mild CAL to moderate CAL and severe CAL are decreasing with age at ages above 86.

• Generally, deterioration is more likely into the less disabled states. However, crude deterioration probabilities into profound CAL are peculiar. Specifically, for males, deterioration into profound CAL is more likely than deterioration into severe CAL. For females, deterioration into profound CAL is more likely than deterioration into any less disabled states at ages above 80.

• Crude deterioration probabilities into a given disability state are higher for more disabled states than the less disabled ones.

• Crude deterioration probabilities are generally higher for females with the difference between genders seeming to increase with the severity of the disablement. The exceptions are deterioration into mild CAL (at ages 60 and above) and moderate CAL (at ages above 89).

• The crude deterioration probabilities suggest lower deterioration rates than the initial probabilities for deterioration into moderate CAL, severe CAL and profound CAL at ages above 89.

We now make some observations about crude improvement probabilities.
The crude improvement probabilities from moderate CAL are very spurious.

Generally, crude improvement probabilities are decreasing with age (excluding improvement from moderate CAL; in the discussion below we exclude improvement from moderate CAL).

For males, the crude improvement probabilities are generally decreasing with the severity of disablement.

For females, at ages between 60 and 89, the improvement from mild CAL is the highest, while the improvements from severe CAL and profound CAL are roughly at similar levels. However, at ages above 89, the improvement from profound CAL is slightly higher than the improvement from either mild CAL or severe CAL.

The improvements from mild CAL and severe CAL are roughly similar between genders at ages below 89 and higher for males at ages above 89. For profound CAL, the improvements are higher for males at ages between 60 and 70, slightly higher for females at ages between 70 and 90, and similar between genders at ages above 90.

For males, the crude probabilities suggest higher improvement rates than the initial probabilities for mild CAL at ages above 89 and for profound CAL at ages between 60 and 70. For females, the crude probabilities suggest lower improvement rates from mild CAL at ages above 70 and from profound CAL at ages above 90.

We make some observations about crude stay probabilities.

Generally, the crude stay probabilities are decreasing with age.

At ages below 83, the crude stay probabilities of no CAL are the highest amongst the disability categories. At ages above 83, excluding profound CAL, the crude stay probabilities are decreasing with the severity of disablement (the crude stay probabilities of profound CAL
are relatively high at these ages at about a similar level as the crude stay probabilities of no CAL).

- The crude stay probabilities of no CAL and mild CAL are roughly at similar levels between genders. The crude stay probabilities of moderate CAL and severe CAL are higher for males at ages above 89 (due to the higher likelihood of deteriorating into profound CAL for females). The crude stay probabilities of profound CAL are higher for females (due to the higher death probabilities for males).

- For males, the crude probabilities suggest higher stay probabilities than the initial probabilities for no CAL, mild CAL and moderate CAL at ages above 89. For females, the crude probabilities suggest higher stay probabilities for mild CAL and lower stay probabilities for moderate CAL at ages above 89.

4.7 Graduation of disability transition probabilities

In this section, we graduate the crude disability transition probabilities calculated in Section 4.6. We begin by discussing the graduation method, followed by the graduation of the transition probabilities and finally the assessment of the graduated probabilities.

4.7.1 Graduation method

We consider graduation under Generalised Linear and Non-Linear models (GLM/NLM) and the Whittaker-Henderson (W-H) method. Graduation under GLM/NLM is discussed in Renshaw (1991) while W-H method is discussed in Joseph (1952). In the following we describe the adopted GLM/NLM. Denote (for $m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5$):

$$N_{x+0.5}^m = \sum_{t=1998}^{2002} N_{x+0.5}^m(t).$$
4.7. GRADUATION OF TRANSITION PROBABILITIES

$T_{x+0.5,x+1.5}^{m,n}$ : random variable of the number of transitions from state $m$ to state $n$ from the exposure $N_{x+0.5}^{m}$ where the transitions occur over the age interval $[x + 0.5, x + 1.5]$.

$\hat{T}_{x+0.5,x+1.5}^{m,n}$ : sample value of $T_{x+0.5,x+1.5}^{m,n}$ obtained from the investigation from the middle of 1998 to the middle of 2003.

$\hat{P}_{x+0.5,x+1.5}^{m,n}$ : graduated value of $\hat{T}_{x+0.5,x+1.5}^{m,n}$.

Since we assume that only one transition is possible over a one year interval and the population who are aged $x$ last birthday are aged $x + 0.5$ exact, then the following is roughly correct (using the notation in Section 4.1):

$$\hat{T}_{x+0.5,x+1.5}^{m,n} = \sum_{t=1998}^{2002} \tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t + 1) \quad m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5.$$  

(4.16)

The reason (4.16) is not exactly correct is because by definition $\tilde{N}_{x+0.5,x+1.5}^{m,n}(t, t + 1)$ is affected by the transitions which occur at exact age $x + 1.5$. However, practically, this error is negligible.

We adopt the following binomial modelling distribution:

$$T_{x+0.5,x+1.5}^{m,n} \sim \text{Bin} \left( N_{x+0.5}^{m}, \hat{P}_{x+0.5,x+1.5}^{m,n} \right) \quad m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 5$$  

(4.17)

where

$$\hat{P}_{x+0.5,x+1.5}^{m,n} = GM_{\beta}^{r,s}(x + 0.5) \text{ or } LGM_{\beta}^{r,s}(x + 0.5),$$

$$GM_{\beta}^{r,s}(x + 0.5) = \sum_{i=1}^{r} \beta_{i}(x + 0.5)^{i-1} + \exp \left\{ \sum_{i=r+1}^{r+s} \beta_{i}(x + 0.5)^{i-r-1} \right\}, \quad r, s \geq 0$$  

(4.18)

$$LGM_{\beta}^{r,s}(x + 0.5) = \frac{GM_{\beta}^{r,s}(x + 0.5)}{1 + GM_{\beta}^{r,s}(x + 0.5)}, \quad r, s \geq 0$$  

(4.19)

with the convention that $r = 0$ implies the exponentiated polynomial term only, and $s = 0$ implies the polynomial term only.
Equations (4.18) and (4.19) are called Gompertz-Makeham (GM) and Logit Gompertz-Makeham (LGM) formulae respectively. These formulae are often used in the graduation of mortality, disability and health data (for example, Forfar et al. (1988); Renshaw (1991); Leung (2006); Pritchard (2006); Continuous Mortality Investigation (CMI) (2010)).

The parameters \( \{\beta_1, \beta_2, \ldots, \beta_{r+s}\} \) are estimated under the maximum likelihood (ML) method. Note that if both \( r \) and \( s \) are at least 1, we will have a non-linear predictor (GNLM). In this case, the parameters can be estimated by expanding the non-linear terms in the predictor using a Taylor series expansion (Renshaw, 1991). However, for convenience, we instead fit the GNLM by utilizing the “gnm” function in R; see Turner and Firth (2011). Starting values of the iteration can be obtained using the weighted least squares (WLS) method.

For each transition probability, to determine the suitable model, we experiment with (4.18) and (4.19) with a variety of values of \( r \) and \( s \).

As described in Section 4.6, some crude transition probabilities are spurious at ages above 85. For transitions for which this is the case, we group the data at these ages into quinquennial age bands and calculate crude transition probabilities at the weighted mean age. The graduated probabilities are calculated from these recalculated crude probabilities.

Graduation under the W-H method is done using the Graduation Tools Spreadsheet developed by the Department of Actuarial Studies at Macquarie University. The weights in the fitting are chosen to be the inverse of the variance of the crude probabilities. Under this method, in the case where the data are grouped, the graduated probabilities at a single age are obtained by fitting a polynomial to the graduated probabilities at the relevant ages. While this approach is not standard, for some transitions it produces satisfactory graduated values.

### 4.7.2 Graduation of death probabilities

For graduation of death probabilities (i.e. \( P_{x+0.5,x+1.5}^{m,5} \) for \( m = 0, 1, \ldots, 4 \)), in addition to the methods described in Section 4.7.1, we also consider GLMs
by targeting the force of mortality ($\mu_{x+0.5}$) and graduation by reference to a standard table (Benjamin and Pollard, 1993).

In implementing the GLM by targeting $\mu_{x+0.5}$, we limit our investigation to the following formula:

$$\mu_{x+0.5} = GM^\alpha_s(x + 0.5) \quad s \geq 0.$$ 

Under graduation by reference to a standard table, the chosen standard table is Australian Life Tables 2000-02 published by the Australian Government Actuary (AGA) (2004). We consider a variety of possible relationships between the mortality rates of the standard table and the graduated mortality rates. The parameters of the graduation formula are estimated under the WLS method.

For females, the graduated mortality rates of mild CAL are lower than the graduated mortality rates of no CAL at the advanced ages. Note that this is because of the peculiarity of the crude rates. As discussed in Section 4.6, for females, there might be a distortion in the results of the IPF procedure for transitions from mild CAL at the advanced ages. Comparing the crude mortality rates of no CAL with mild CAL, we find that at younger ages, the crude mortality rates of these categories are similar. Therefore, for females, we decide to ignore the crude mortality rates of mild CAL at the advanced ages and instead set the graduated mortality rates of mild CAL to be equal to the graduated mortality rates of no CAL.

### 4.7.3 Graduation of improvement, stay and deterioration probabilities

Improvement, stay and deterioration probabilities are graduated under the methods described in Section 4.7.1. Since for a given disability category, the transition probabilities sum to one, we only need to graduate two of the three transition probabilities as one can be set as residuals. Except for moderate CAL and profound CAL categories, the stay probabilities are set as residuals. For these categories, we instead set the improvement probabilities as resid-
uals. This is because, as discussed in Section 4.6, the crude improvement probabilities from moderate CAL are spurious. For profound CAL, we find that by setting the improvement probabilities as residuals, more satisfactory graduated values can be obtained.

For some deterioration probabilities (e.g. $P^{0,1}_{x+0.5,x+1.5}$, $P^{0,2}_{x+0.5,x+1.5}$, $P^{0,3}_{x+0.5,x+1.5}$, $P^{0,4}_{x+0.5,x+1.5}$, $P^{2,3}_{x+0.5,x+1.5}$ and $P^{2,4}_{x+0.5,x+1.5}$ for females), the methods described in Section 4.7.1 could not produce satisfactory graduated values over the entire considered age range. We resolve this problem using the following method. Firstly, we determine the optimal GM or LGM formula (the parameters are estimated under the ML method as described previously). For ages at which the chosen GM (or LGM) formula is unsuitable, we alternatively calculate the graduated values by fitting a suitable polynomial at these ages. The parameters of the polynomial are estimated under the ordinary least squares (OLS) method and chosen such that the polynomial curve is continuous from the GM (or LGM) curve at the transition age (the age at which the graduation curve changes from the GM (or LGM) curve to the polynomial curve). Since the ages at which an alternative method is required are typically very high, a simple alternative method is adopted.

For females, in the graduation of $P^{1,1}_{x+0.5,x+1.5}$, $P^{1,2}_{x+0.5,x+1.5}$, $P^{1,3}_{x+0.5,x+1.5}$ and $P^{1,4}_{x+0.5,x+1.5}$, we choose not to fully capture the pattern of the crude probabilities at ages above 86. For example, for $P^{1,2}_{x+0.5,x+1.5}$ and $P^{1,3}_{x+0.5,x+1.5}$, we choose not to fully capture the rapid decreasing pattern of the crude probabilities at these ages. This is because for females, there is a possible distortion in the results of the IPF procedure for transitions from mild CAL at these ages as discussed in Section 4.6.

For some improvement probabilities (e.g. $P^{1,0}_{x+0.5,x+1.5}$ and $P^{2,2}_{x+0.5,x+1.5}$ for females), the chosen graduation formula results in negative graduated values at the extremely high ages. In this case, we simply replace the negative graduated values with zero. We could instead choose an alternative graduation formula which results in non-negative graduated values over the entire considered age range. This method is not preferable since the negative graduated values only occur at the extremely high ages (there will be more cost for
switching into an alternative formula as it results in less optimal graduated values over the majority of ages).

Tables 4.15 and 4.16 present the graduated disability transition probabilities for males and females at a selection of ages.
Table 4.15: Males, graduated one-year disability transition probabilities.

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### 4.7. GRADUATION OF TRANSITION PROBABILITIES

Table 4.16: Females, graduated one-year disability transition probabilities.

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<td>0.416433</td>
</tr>
</tbody>
</table>
4.7.4 Assessment of graduated transition probabilities

To assess the graduated transition probabilities, we consider smoothness and goodness of fit tests. A smoothness test is required for graduation under the W-H method and in the case where we blend a polynomial with a GM (or LGM) formula in the graduation.

For smoothness, we adopt the criteria as described in Barnett (1985): the graduated probabilities should have third order smoothness (i.e. $\left| \Delta^3 \hat{P}_{x,x+0.5,x+1.5}^{m,n} \right| < \hat{P}_{x,x+0.5,x+1.5}^{m,n}$) over the majority of the age range. In all of our graduated probabilities, the proportion of ages which satisfy Barnett’s (1985) third order smoothness is higher than 50% (see Tables 4.17 and 4.18).

We find that passing the $\chi^2$ test is very difficult. Note that we do not have the data of the exposure (at a single age) and the number of transitions. These quantities are estimated and there are estimation uncertainties. The unmodified implementation of the $\chi^2$ test (i.e. by assuming that the estimated exposure at a single age is the real exposure and the estimated number of transitions is the real number of transitions) ignores these uncertainties and therefore penalises too much on the differences between crude and graduated probabilities. However, incorporation of these uncertainties into the $\chi^2$ test is not straightforward. Therefore, for goodness of fit criteria, as in Leung (2006), we instead use the Theil Inequality Coefficient (TIC), Theil (1958), which is expressed as:

$$TIC = \sqrt{\frac{\sum_{i=1}^{w} \left( \hat{P}_{x_i,x_{i+1}} \right)^2}{w}} \sqrt{\frac{\sum_{i=1}^{w} \left( \hat{P}_{x_i,x_{i+1}} - \hat{P}_{x,x_{i+1}}^{m,n} \right)^2}{w}} \quad (4.20)$$

where $x_1, x_2, \ldots, x_w$ are ages considered in the graduation. The TIC lies between 0 and 1, with 0 being a perfect fit. The statistical properties of the TIC are discussed in Theil (1958). As in Leung (2006), we accept the graduated probabilities with a TIC of 10% or less. While our assessment of the goodness of fit does not consider the estimated exposure, it is taken into
account in our graduation method (except in the case where the graduated probabilities are alternatively calculated under OLS method (where we blend a polynomial with the GM or LGM formula), however this alternative method is only required at the very high ages as discussed before).

In addition, due to the lack of randomness of the estimated crude probabilities, passing the individual standardised deviations, runs and serial correlations tests appears to be very difficult. However, most of the graduated probabilities pass both the cumulative deviations and sign tests (at the 5% significance level).

Tables 4.17 and 4.18 present the results of the chosen goodness of fit and smoothness assessments. The heading for these Tables are:

(a) : Cumulative Deviations Test.
(b) : Sign Test.
(c) : Theil Inequality Coefficient.
(d) : Proportion of Ages Which Satisfy Barnett’s (1985) Third Order Smoothness Criteria.

There are several cases where we accept graduated probabilities although they do not meet the goodness of fit criteria:

- \( \hat{P}_{x+0.5,x+1.5}^{1.1}, \hat{P}_{x+0.5,x+1.5}^{1.2}, \hat{P}_{x+0.5,x+1.5}^{1.3}, \hat{P}_{x+0.5,x+1.5}^{1.4} \) and \( \hat{P}_{x+0.5,x+1.5}^{1.5} \) for females. Note that for these transitions (for females), we choose not to fully capture the pattern of the crude probabilities at ages above 86 in the graduation (for reasons discussed before).

- \( \hat{P}_{x+0.5,x+1.5}^{2.1} \). Note that for this transition, the pattern of the crude probabilities is very spurious (discussed in Section 4.6) and hence, in the graduation, this transition is set as a residual. Therefore, the graduated probabilities of this transition are determined from other graduated transition probabilities from moderate CAL.

- \( \hat{P}_{x+0.5,x+1.5}^{4.3} \) for males and \( \hat{P}_{x+0.5,x+1.5}^{0.3}, \hat{P}_{x+0.5,x+1.5}^{2.2}, \hat{P}_{x+0.5,x+1.5}^{2.5} \) and \( \hat{P}_{x+0.5,x+1.5}^{2.5} \) for females. Due to the systematic oscillation, for some transitions, it is hard
to pass both the cumulative deviations and sign tests. However, note that the TIC of these transitions are low which indicate that, overall, the graduated probabilities are sufficiently close to the crude probabilities (the TIC of $\hat{P}^{0.3}_{x+0.5,x+1.5}$ for females is only marginally higher than 10%).

Overall, the results presented in Tables 4.17 and 4.18 suggest that satisfactory graduated values have been obtained. Figure 4.13 presents the crude and graduated transition probabilities from mild CAL state for males. Note that the graduated probabilities capture the pattern of the crude probabilities. Similar qualities of fit are observed for other transition probabilities in cases other than those discussed above. Lastly, generally, similar degrees of oscillation of the crude probabilities are observed in other transition probabilities.
4.7. GRADUATION OF TRANSITION PROBABILITIES

Table 4.17: Males, goodness of fit and smoothness assessment of graduated transition probabilities.

<table>
<thead>
<tr>
<th>P-value</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Death probabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>100.00%</td>
<td>71.11%</td>
<td>7.01%</td>
<td>100.00%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>100.00%</td>
<td>45.83%</td>
<td>8.80%</td>
<td>100.00%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>87.90%</td>
<td>26.49%</td>
<td>7.82%</td>
<td>100.00%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>48.95%</td>
<td>44.27%</td>
<td>5.41%</td>
<td>100.00%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>99.60%</td>
<td>47.09%</td>
<td>3.85%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Deterioration probabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>100.00%</td>
<td>100.00%</td>
<td>4.13%</td>
<td>82.98%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>47.09%</td>
<td>71.11%</td>
<td>6.87%</td>
<td>85.11%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
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<td>100.00%</td>
<td>4.84%</td>
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<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>92.20%</td>
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<td>5.66%</td>
<td>95.74%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
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<td>7.13%</td>
<td>84.62%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>93.41%</td>
<td>71.11%</td>
<td>7.12%</td>
<td>95.74%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>92.27%</td>
<td>100.00%</td>
<td>6.22%</td>
<td>89.36%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
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<td>71.11%</td>
<td>3.58%</td>
<td>100.00%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>98.24%</td>
<td>45.83%</td>
<td>9.60%</td>
<td>97.87%</td>
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<td>100.00%</td>
<td>100.00%</td>
<td>6.58%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Improvement probabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
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<td>71.11%</td>
<td>4.01%</td>
<td>100.00%</td>
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<td>$p_{x+0.5,x+1.5}$</td>
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<td>29.62%</td>
<td>8.23%</td>
<td>90.00%</td>
</tr>
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<td>$p_{x+0.5,x+1.5}$</td>
<td>69.65%</td>
<td>1.33%</td>
<td>7.22%</td>
<td>100.00%</td>
</tr>
<tr>
<td>Stay probabilities</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>65.30%</td>
<td>71.11%</td>
<td>0.71%</td>
<td>95.74%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>95.34%</td>
<td>71.11%</td>
<td>1.53%</td>
<td>95.74%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
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<td>71.11%</td>
<td>2.03%</td>
<td>97.87%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
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<td>66.55%</td>
<td>1.87%</td>
<td>97.87%</td>
</tr>
<tr>
<td>$p_{x+0.5,x+1.5}$</td>
<td>79.53%</td>
<td>47.09%</td>
<td>1.60%</td>
<td>100.00%</td>
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Table 4.18: Females, goodness of fit and smoothness assessment of graduated transition probabilities.

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<th>P-value</th>
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<tr>
<td></td>
<td>(a)</td>
<td>(b)</td>
<td>(c)</td>
<td>(d)</td>
</tr>
<tr>
<td><strong>Death probabilities</strong></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( P_{0,5}^{0.5} )</td>
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<td>3.06%</td>
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<td>( P_{1,5}^{0.5} )</td>
<td>0.00%</td>
<td>6.14%</td>
<td>24.69%</td>
<td>61.70%</td>
</tr>
<tr>
<td>( P_{2,5}^{0.5} )</td>
<td>0.00%</td>
<td>26.49%</td>
<td>6.34%</td>
<td>61.70%</td>
</tr>
<tr>
<td>( P_{3,5}^{0.5} )</td>
<td>99.95%</td>
<td>76.60%</td>
<td>3.43%</td>
<td>97.62%</td>
</tr>
<tr>
<td>( P_{4,5}^{0.5} )</td>
<td>100.00%</td>
<td>23.27%</td>
<td>2.98%</td>
<td>100.00%</td>
</tr>
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<td><strong>Deterioration probabilities</strong></td>
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<td></td>
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<tr>
<td>( P_{0,1}^{0.5} )</td>
<td>99.53%</td>
<td>100.00%</td>
<td>6.57%</td>
<td>61.70%</td>
</tr>
<tr>
<td>( P_{1,2}^{0.5} )</td>
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<td>45.83%</td>
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<td>95.74%</td>
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<tr>
<td>( P_{2,3}^{0.5} )</td>
<td>99.43%</td>
<td>100.00%</td>
<td>12.07%</td>
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<tr>
<td>( P_{3,4}^{0.5} )</td>
<td>99.87%</td>
<td>100.00%</td>
<td>8.23%</td>
<td>89.36%</td>
</tr>
<tr>
<td>( P_{4,5}^{0.5} )</td>
<td>99.52%</td>
<td>26.49%</td>
<td>15.76%</td>
<td>100.00%</td>
</tr>
<tr>
<td><strong>Improvement probabilities</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( P_{1,0}^{0.5} )</td>
<td>99.56%</td>
<td>26.49%</td>
<td>2.96%</td>
<td>89.36%</td>
</tr>
<tr>
<td>( P_{2,1}^{0.5} )</td>
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<td>26.49%</td>
<td>16.95%</td>
<td>91.49%</td>
</tr>
<tr>
<td>( P_{3,2}^{0.5} )</td>
<td>99.35%</td>
<td>100.00%</td>
<td>7.97%</td>
<td>85.11%</td>
</tr>
<tr>
<td>( P_{4,3}^{0.5} )</td>
<td>78.95%</td>
<td>100.00%</td>
<td>9.77%</td>
<td>100.00%</td>
</tr>
<tr>
<td><strong>Stay probabilities</strong></td>
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<td></td>
</tr>
<tr>
<td>( P_{0,0}^{0.5} )</td>
<td>13.38%</td>
<td>71.11%</td>
<td>0.78%</td>
<td>100.00%</td>
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<tr>
<td>( P_{1,1}^{0.5} )</td>
<td>0.00%</td>
<td>71.11%</td>
<td>3.61%</td>
<td>91.49%</td>
</tr>
<tr>
<td>( P_{2,2}^{0.5} )</td>
<td>0.00%</td>
<td>6.14%</td>
<td>3.64%</td>
<td>87.23%</td>
</tr>
<tr>
<td>( P_{3,3}^{0.5} )</td>
<td>95.36%</td>
<td>23.27%</td>
<td>2.07%</td>
<td>91.49%</td>
</tr>
<tr>
<td>( P_{4,4}^{0.5} )</td>
<td>99.52%</td>
<td>76.60%</td>
<td>1.11%</td>
<td>100.00%</td>
</tr>
</tbody>
</table>

Figure 4.13: Crude and graduated transition probabilities from the mild CAL state, males.
4.8 Estimation of probability of admission into an aged care home

In this section, we estimate the probabilities of admission into an aged care home (ACH). The probabilities of admission are estimated for each disability category. Due to the limited amount of data, only rough estimates of the probabilities of admission can be obtained. Nevertheless, these estimates are useful to set plausible scenarios of termination probabilities in the pricing analysis of a reverse mortgage contract.

One of the relevant sets of data for our estimation is the measurement of dependency levels of the residents of aged care homes reported in Residential Aged Care in Australia 2002-03 (RACA 02-03) (AIHW, 2004). In this report, the dependency levels are measured by the Resident Classification Scale (RCS). There are 8 levels of RCS reported with level 1 being the highest level of dependency and level 8 being the lowest. The RCS level is determined from the response to the RCS questionnaire which measures the degree of assistance required for personal care and health care activities (AIHW, 2007b). Different weights are allocated to each question and the sum of these weights determines the level of RCS of the residents.

We chose not to use the RCS data, mainly due to the difficulty in linking the RCS level with the measurement of CAL level under SDAC. This is due to several reasons. Firstly, the measurement of the level of assistance required under the RCS questionnaire might be different from the measurement of the severity of CAL under SDAC. Secondly, the RCS level is determined from the overall severity of the areas covered under RCS questionnaire, while under SDAC, the level of CAL can be heavily influenced by the severity of disablement in a particular area (for example, a person who requires continuous assistance with communication will be categorized as having a profound CAL). Lastly, unlike the CAL level measurement, the RCS questionnaire also covers some areas of health care.

Another relevant set of data are the living arrangements of the disabled population reported in Table 6 of the 2003 SDAC (ABS, 2004a). From these data, the prevalence rates of the ACH population for each disability category
can be calculated and, hence, the probability of admission can be estimated under the stationary population model. However, there are some limitations of these data. Firstly, they are not separated by gender. Secondly, the disabled population who live in ACH (which can be obtained from the data by summing the disabled population who live in nursing homes or aged care hostels and in non-private dwelling category (d) (includes hostels for people with disabilities and some cared components of retirement villages)) is only reported for age groups 60–79 and 80 and over. Nevertheless, these data are useful for our estimation.

Table 4.19 presents the CAL prevalence rates of ACH residents (population) calculated from 2003 SDAC data. As a comparison, Table 4.20 presents the CAL prevalence rates of the population (i.e. community and ACH population). Note that the prevalence rates of profound CAL for ACH residents are significantly higher than the prevalence rates (of profound CAL) for the population especially at younger ages. The prevalence rates of profound CAL for ACH residents are increasing with age although at a lower rate as compared with the prevalence rates of the population. This is because the prevalence rates (of profound CAL) for ACH residents are already very high at younger ages. The reduction of the prevalence rates of mild CAL and severe CAL for ACH residents from ages 60–79 to ages 80+ are due to the increase of the prevalence rates of profound CAL between these ages. Note that these prevalence rates are already very high at ages 60–79.

Table 4.19: Prevalence rates of ACH residents calculated from 2003 SDAC data (per 1000).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>60–79</td>
<td>45.8</td>
<td>18.9</td>
<td>13.5</td>
<td>110.5</td>
<td>811.3</td>
</tr>
<tr>
<td>80+</td>
<td>27.8</td>
<td>11.6</td>
<td>15.2</td>
<td>89.6</td>
<td>855.7</td>
</tr>
</tbody>
</table>

There are two types of residents of an ACH: permanent and respite. A permanent resident is admitted into an ACH for permanent or long-term care, while a respite resident is admitted for short-term care (with the primary purpose of giving the carer or a care recipient a short-term break from their
Table 4.20: Prevalence rates of population calculated from 2003 SDAC data (per 1000).

<table>
<thead>
<tr>
<th>Age (years)</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>60–79</td>
<td>613.8</td>
<td>157.5</td>
<td>102.4</td>
<td>66.1</td>
<td>60.3</td>
</tr>
<tr>
<td>80+</td>
<td>269.6</td>
<td>173.3</td>
<td>104.2</td>
<td>115.4</td>
<td>337.6</td>
</tr>
</tbody>
</table>

usual care arrangement) (AIHW, 2004). For pricing of a reverse mortgage contract, the probability of admission for permanent care is more relevant (the average length of stay of respite residents is only three weeks (AIHW, 2004)). Note that by employing the data from the 2003 SDAC, we estimate the sum of the probabilities of admission for permanent and respite care. However, since the proportion of permanent residents in ACHs is very high, the estimated probability of admission can be considered to be applicable for admission for permanent care (at the middle of 2003, the proportions of permanent residents are higher than 97% for all age groups and for both genders (AIHW, 2004)).

Intuitively, the ACH prevalence rates of the disabled population are different between genders. To estimate the ACH prevalence rates for each gender, we need to estimate the disabled population who live in ACHs (ACH disabled population) for each gender. This is done by applying the sex ratio of the ACH residents at the middle of 2003 (obtained from AIHW (2004)) to the ACH disabled population reported in the 2003 SDAC. We assume that the calculated sex ratio is applicable for all disability categories. The ACH prevalence rates for each gender can then be obtained by dividing the ACH disabled population by the total disabled population reported in the 2003 SDAC for each gender.

To estimate the probability of admission into an ACH, we need to estimate the ACH prevalence rates at single ages. The initial estimates of the ACH prevalence rates at single ages are obtained under interpolation using the following logistic function:

\[ IntPrev(x + 0.5) = \frac{1}{1 + BC^{x+0.5}} \]  

(4.21)
where $\text{IntPrev}(x + 0.5)$ is the initial estimate of the ACH prevalence rate at age $x + 0.5$, and $B$ and $C$ are parameters. The parameters $B$ and $C$ determine how rapidly the prevalence rate changes from 0 to 1 as age increases.

The motivation for (4.21) is because the ACH prevalence rates should be bounded between 0 and 1 and display the following features: initially increasing with age at an increasing rate, however, at very high ages, the rate of increase is decreasing. In fitting (4.21), the reported ACH prevalence rate in a given age group is assumed to relate to the weighted mean age of the corresponding age group. To calculate the weighted mean age, we assume that the age structure of the disabled population in the 2003 SDAC is similar to the age structure of the disabled population at the middle of 2003.

Multiplying the initial estimates of ACH prevalence rates with the disabled population in the 2003 SDAC, we obtain the initial estimates of the ACH disabled population at single ages in the 2003 SDAC (we assume that the prevalence rate at age $x + 0.5$ is applicable to the population aged $x$ last birthday). These initial estimates are adjusted under the prorating method using the age group total of the ACH disabled population reported in the 2003 SDAC (i.e. under a similar approach to method (b) in Section 4.3). The final estimates of ACH prevalence rates at single ages are obtained from the adjusted estimates of the ACH disabled population.

The probabilities of admission into an ACH are estimated from the ACH prevalence rates by adopting a stationary population model. Specifically, the probabilities of admission are estimated such that the ACH prevalence rates of the stationary population replicate the estimated ACH prevalence rates. We assume that the stationary population who are aged $x$ last birthday are aged $x + 0.5$ exact and over a one-year period, only one transition between the community and an ACH is possible. Denote (for $m = 0, 1, \ldots, 4$):

- $\text{POP}^{m}_{x+0.5}$ : the stationary population who are aged $x + 0.5$ and in state $m$.
- $\text{ACH}^{m}_{x+0.5}$ : the stationary population who are aged $x + 0.5$, in state $m$ and live in an ACH.
- $\text{COM}^{m}_{x+0.5}$ : the stationary population who are aged $x + 0.5$, in state $m$ and live in the community.
\( \text{Prev}(x + 0.5, m) \) : the estimate of ACH prevalence rate at age \( x + 0.5 \) for state \( m \).

\( P_{x+0.5,x+1.5}^{m,ACH} \) : the probability that an individual aged \( x + 0.5 \), in state \( m \) and living in the community moves to an ACH before reaching age \( x + 1.5 \) and survives to this age. This probability is referred to as the (unconditional) one-year ACH admission probability.

\( CP_{x+0.5,x+1.5}^{m,ACH} \) : the probability that an individual aged \( x + 0.5 \), in state \( m \) and living in the community moves to an ACH before reaching age \( x + 1.5 \) given that (s)he stays in state \( m \) during the year. This probability is referred to as the conditional one-year ACH admission probability.

\( P^{COM} \) : the probability that an individual who lives in an ACH returns to the community in one year’s time.

We assume that the transition from state \( m \) to \( n \) between ages \( x + 0.5 \) and \( x + 1.5 \) occurs at age \( x + 1 \) and hence, the probability of admission into an ACH for an individual who makes such a transition is estimated as \( \frac{1}{2} \left( CP_{x+0.5,x+1.5}^{m,ACH} + CP_{x+0.5,x+1.5}^{n,ACH} \right) \). In addition, we also assume that the graduated disability transition probabilities are applicable to the community and ACH populations. Therefore, \( P_{x+0.5,x+1.5}^{m,ACH} \) is calculated from \( CP_{x+0.5,x+1.5}^{m,ACH} \) according to (for \( m = 0, 1, \ldots, 4 \)):

\[
P_{x+0.5,x+1.5}^{m,ACH} = \frac{1}{2} \left( CP_{x+0.5,x+1.5}^{m,ACH} + CP_{x+0.5,x+1.5}^{n,ACH} \right)
\]

To estimate the disability transition probabilities of the ACH population, it might be desirable to adjust the graduated disability transition probabilities using the death rates of the ACH population estimated by Mason et al. (2001). This can be done by firstly adjusting the graduated death rates of each CAL category such that the overall graduated death rates across CAL categories match the death rates of the ACH population estimated by Mason et al. (2001). Then, for each CAL category, the graduated transition probabilities can then be adjusted using the ratio of the graduated death probabilities before and after adjustment. However, we choose not to apply
4.8. ESTIMATION OF ACH ADMISSION PROBABILITY

this adjustment. This is mainly due to the fact that we estimate the disabil-
ity prevalence rates of the ACH population and, therefore, there is likely
to be an estimation error. This adjustment, which relies on these estimated
prevalence rates, might compound the estimation error and, hence, reduce
the accuracy of the estimated ACH admission probabilities. As discussed
later, despite our simplistic assumption, the estimated ACH admission prob-
abilities appear to be reasonable and similar with other studies. Lastly, as
discussed in Chapter 5, admission into an ACH only represents a small pro-
portion of total terminations of reverse mortgage contracts and, therefore,
a certain degree of estimation error of ACH admission probabilities will not
materially impact the results of our pricing analysis.

In the following we describe the steps to estimate $CP_{x+0.5,x+1.5}^{m,ACH}$ (for
$m = 0, 1, \ldots, 4$).

(a) Estimate $POP_{x+0.5}^m$ for $x = 60, 61, \ldots, 109$ and $m = 0, 1, \ldots, 4$.
This is done by projecting the disabled population aged 60.5 in the 2003
SDAC to age 109.5 using the graduated disability transition probabilities
estimated in Section 4.7. This stationary population is separated into com-
munity and ACH populations using the estimated ACH prevalence rates.
Specifically (for $x = 60, 61, \ldots, 109; m = 0, 1, \ldots, 4$):

\[
ACH_{x+0.5}^m = POP_{x+0.5}^m \times \text{Prev}(x + 0.5, m),
\]

\[
COM_{x+0.5}^m = POP_{x+0.5}^m - ACH_{x+0.5}^m.
\]

(b) Estimate $CP_{x+0.5,x+1.5}^{n,ACH}$ for $x = 60, 61, \ldots, 108$ and $m = 0, 1, \ldots, 4$.
These probabilities are estimated such that the following equation holds (for
$x = 60, 61, \ldots, 108; n = 0, 1, \ldots, 4$):

\[
ACH_{x+1.5}^n = \sum_{m=0}^{4} (COM_{x+0.5}^m \times \hat{P}_{x+0.5,x+1.5}^{m,n} \times \frac{1}{2} (CP_{x+0.5,x+1.5}^{n,ACH} + CP_{x+0.5,x+1.5}^{n,ACH})
+ ACH_{x+0.5}^m \times \hat{P}_{x+0.5,x+1.5}^{m,n} \times (1 - P^{COM})).
\]

Note that we assume the $P^{COM}$ is age and disability invariant. This
assumption is adopted due to the limited amount of data. This probability is
estimated from AIHW (2004) by dividing the total returns to the community from ACHs from the middle of 2002 to the middle of 2003 by the total residents of ACHs at the middle of 2003. The estimated values are 31.23% for males and 20.05% for females.

In solving (4.25) for \( n = 0, 1, \ldots, 4 \), we have five linear equations with five unknowns and hence, unique estimates of \( CP_{x+0.5,x+1.5}^{0,ACH} \), \( CP_{x+0.5,x+1.5}^{1,ACH} \), \( \ldots \), \( CP_{x+0.5,x+1.5}^{4,ACH} \) can be obtained.

The estimated conditional one-year ACH admission probabilities are inappropriate (i.e. negative or higher than one) at some ages for some disability categories. This is because of the possible inaccuracies of the estimated ACH prevalence rates and the adopted return to the community assumption. Note that the data for the estimation are very limited. To fix the inappropriate estimates of conditional admission probabilities, we re-estimate these probabilities by scaling the conditional admission probabilities of other disability category. The scale is the ratio of the ACH prevalence rates of the corresponding disability categories. For example, to re-estimate \( CP_{x+0.5,x+1.5}^{m,ACH} \) by scaling \( CP_{x+0.5,x+1.5}^{n,ACH} (n \neq m) \), the following equation is employed:

\[
CP_{x+0.5,x+1.5}^{m,ACH} (\ast) = \frac{\text{Prev}(x + 0.5, m)}{\text{Prev}(x + 0.5, n)} CP_{x+0.5,x+1.5}^{n,ACH}
\]

where \( CP_{x+0.5,x+1.5}^{m,ACH} (\ast) \) is the re-estimated value of \( CP_{x+0.5,x+1.5}^{m,ACH} \).

The motivation for (4.26) is because the conditional admission probabilities are positively related to the ACH prevalence rates (given all other things constant, higher ACH prevalence rates imply higher conditional admission probabilities).

To re-estimate inappropriate values of \( CP_{x+0.5,x+1.5}^{1,ACH} \) and \( CP_{x+0.5,x+1.5}^{2,ACH} \) we choose to scale \( CP_{x+0.5,x+1.5}^{0,ACH} \), while to re-estimate \( CP_{x+0.5,x+1.5}^{3,ACH} \) we choose to scale \( CP_{x+0.5,x+1.5}^{4,ACH} \). This choice is made by considering the closeness of the severity of disablement between disability categories.

Tables 4.21 and 4.22 present the estimates of conditional admission probabilities for each disability category for males and females in a selection of ages. Several observations from these tables are as follows.
• The conditional admission probabilities are increasing with age and generally also increasing with the severity of disablement.

• The conditional admission probabilities are generally higher for females.

• The conditional admission probabilities of mild CAL are lower than the conditional admission probabilities of no CAL. This feature can be traced directly to the ACH prevalence rates reported in the 2003 SDAC.

Table 4.21: Males, conditional one-year ACH admission probabilities.

<table>
<thead>
<tr>
<th>Exact age</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>60.5</td>
<td>0.000085</td>
<td>0.000214</td>
<td>0.000183</td>
<td>0.006080</td>
<td>0.058565</td>
</tr>
<tr>
<td>65.5</td>
<td>0.000189</td>
<td>0.000325</td>
<td>0.000361</td>
<td>0.010247</td>
<td>0.086536</td>
</tr>
<tr>
<td>70.5</td>
<td>0.000424</td>
<td>0.000492</td>
<td>0.000716</td>
<td>0.016650</td>
<td>0.121901</td>
</tr>
<tr>
<td>75.5</td>
<td>0.000953</td>
<td>0.000684</td>
<td>0.001428</td>
<td>0.026730</td>
<td>0.167349</td>
</tr>
<tr>
<td>80.5</td>
<td>0.003108</td>
<td>0.001434</td>
<td>0.003872</td>
<td>0.046526</td>
<td>0.239706</td>
</tr>
<tr>
<td>85.5</td>
<td>0.006595</td>
<td>0.003020</td>
<td>0.007049</td>
<td>0.075471</td>
<td>0.318431</td>
</tr>
<tr>
<td>90.5</td>
<td>0.015253</td>
<td>0.005319</td>
<td>0.014486</td>
<td>0.122070</td>
<td>0.421164</td>
</tr>
<tr>
<td>95.5</td>
<td>0.034251</td>
<td>0.009234</td>
<td>0.029070</td>
<td>0.202804</td>
<td>0.568235</td>
</tr>
<tr>
<td>100.5</td>
<td>0.073417</td>
<td>0.015797</td>
<td>0.056461</td>
<td>0.337248</td>
<td>0.769773</td>
</tr>
<tr>
<td>105.5</td>
<td>0.147964</td>
<td>0.026988</td>
<td>0.106102</td>
<td>0.516741</td>
<td>0.975476</td>
</tr>
<tr>
<td>109.5</td>
<td>0.317921</td>
<td>0.054454</td>
<td>0.222869</td>
<td>0.590603</td>
<td>0.975476</td>
</tr>
</tbody>
</table>

Analysis of the risk of admission into an ACH in Australia has been done previously by Mason et al. (2001) and Cullen (2006). These studies use the comprehensive national dataset on the use of residential aged care at the individual level which is administered by the Department of Health and Ageing. The dataset collects the date of birth, sex, date of each admission and separation of each ACH resident in Australia. Mason et al. (2001) analysed the residential aged care data in 1999-00 while Cullen (2006) analysed the data in 2004-05. Mason et al. (2001) and Cullen (2006) reported the lifetime risk of receiving permanent residential aged care at different ages for men and women irrespective of whether they have received permanent residential aged care before that age ($LP(x)$). From the graduated disability
transition probabilities, estimated conditional ACH admission probabilities and estimated ACH prevalence rates, we are able to estimate $LP(x)$. Figure 4.14 presents the comparison of $LP(x)$ estimated from our model and as reported in Mason et al. (2001) and Cullen (2006) (Cullen (2006) presented the numerical results of $LP(x)$ only at selected ages).

In general, our estimates of $LP(x)$ are very close with the estimates of Mason et al. (2001). Since we use the ACH prevalence rates in the 2003 SDAC, we expect our estimates to be closer to the estimates of Cullen (2006). However, this is true only for females at ages 65.5 to 85.5. Note that we estimate the conditional ACH admission probabilities from a very limited amount of data. Nevertheless, over the majority of ages, the differences between our estimates (of $LP(x)$) and the estimates of Cullen (2006) are not significant.

Using the data from the Appendix tables A1 and A2 of Mason et al. (2001), we are able to calculate the one-year probabilities of ACH admission for permanent care for those who have not used an ACH before (first time ACH admission probabilities) (denoted $P(B)$). In Figure 4.15, we compare these probabilities with our estimates of (unconditional) one-year ACH admission probabilities (denoted $P(A)$ (Model)). Note that our estimates of

<table>
<thead>
<tr>
<th>Exact age</th>
<th>No CAL</th>
<th>Mild CAL</th>
<th>Moderate CAL</th>
<th>Severe CAL</th>
<th>Profound CAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>60.5</td>
<td>0.000056</td>
<td>0.000176</td>
<td>0.000067</td>
<td>0.003151</td>
<td>0.039721</td>
</tr>
<tr>
<td>65.5</td>
<td>0.000150</td>
<td>0.000338</td>
<td>0.000186</td>
<td>0.005957</td>
<td>0.058321</td>
</tr>
<tr>
<td>70.5</td>
<td>0.000397</td>
<td>0.000679</td>
<td>0.000512</td>
<td>0.012010</td>
<td>0.090199</td>
</tr>
<tr>
<td>75.5</td>
<td>0.001045</td>
<td>0.001291</td>
<td>0.001400</td>
<td>0.024287</td>
<td>0.137952</td>
</tr>
<tr>
<td>80.5</td>
<td>0.004865</td>
<td>0.002675</td>
<td>0.003026</td>
<td>0.054886</td>
<td>0.216560</td>
</tr>
<tr>
<td>85.5</td>
<td>0.010684</td>
<td>0.003051</td>
<td>0.009576</td>
<td>0.100178</td>
<td>0.282427</td>
</tr>
<tr>
<td>90.5</td>
<td>0.027205</td>
<td>0.004184</td>
<td>0.037857</td>
<td>0.173371</td>
<td>0.367603</td>
</tr>
<tr>
<td>95.5</td>
<td>0.065937</td>
<td>0.005549</td>
<td>0.121270</td>
<td>0.297168</td>
<td>0.491107</td>
</tr>
<tr>
<td>100.5</td>
<td>0.142969</td>
<td>0.012952</td>
<td>0.321554</td>
<td>0.488110</td>
<td>0.665798</td>
</tr>
<tr>
<td>105.5</td>
<td>0.250976</td>
<td>0.070799</td>
<td>0.645643</td>
<td>0.768379</td>
<td>0.922900</td>
</tr>
<tr>
<td>109.5</td>
<td>0.608980</td>
<td>0.363527</td>
<td>0.804413</td>
<td>0.876160</td>
<td>0.989940</td>
</tr>
</tbody>
</table>
ACH admission probabilities are applicable to those who live in the community regardless of their history of ACH use. Our estimates of ACH admission probabilities are higher than the first time ACH admission probabilities at most ages (the only exception is age 95.5, however, the estimate of first time ACH admission probability at this age seems to be peculiar). This is reasonable since those who have not used an ACH are likely to be healthier than those who have a history of ACH use. In addition, the differences between our estimates and the first time ACH admission probabilities are increasing with age. This is also reasonable since an older person who has not used an ACH belongs to a more select group of population than a younger person who has not used an ACH.

In comparison with the first time ACH admission probabilities, our estimate of (unconditional) ACH admission probabilities generally behave as expected. Therefore, we conclude that, notwithstanding the limited amount of data, our estimates of ACH admission probabilities appear to be reasonable, at least as a benchmark in setting plausible scenarios of termination probabilities in the pricing analysis of a reverse mortgage contract.
4.9 Estimation of termination probability of a reverse mortgage contract

Using the graduated disability transition probabilities and the estimated conditional ACH admission probabilities, we are able to estimate the termination probabilities of a reverse mortgage (RM) contract due to death and admission into an ACH. In the following, the termination probabilities are calculated by considering only termination due to death and ACH admission and without considering any selection effect (i.e. we ignore the possibility that the experience of RM borrowers is different from the general population). Denote:

\[ T(x, n) : \text{a random variable for the future term of a RM contract taken by a single person who is aged } x \text{ and in disability category (state) } n \text{ at the origination of the contract.} \]

\[ T(x) : \text{a random variable for the future term of a RM contract taken by a single person who is aged } x \text{ at the origination of the contract.} \]

\[ T(x, y, m, n) : \text{a random variable for the future term of a RM contract taken by a couple where the husband is aged } x \text{ and in disability category } m \text{ and the wife is aged } y \text{ and in disability category } n \text{ at the origination of the contract.} \]
4.9. ESTIMATION OF RM TERMINATION PROBABILITY

The quantities \( \Pr (T(x, n) > t) \), \( \Pr (T(x) > t) \) and \( \Pr (T(x, y, m, n) > t) \) are calculated under a life table method by assuming that a return to the community from an ACH is not possible. In the following we describe the calculation of \( \Pr (T(x, n) > t) \) followed by the calculation of \( \Pr (T(x) > t) \) and \( \Pr (T(x, y, m, n) > t) \).

**Calculation of \( \Pr (T(x, n) > t) \)**

Let \( l^p_x \) denote the life table population who are aged \( x \) (for \( x = 60.5, 61.5, \ldots, 109.5 \)) and in disability state \( p \) (for \( p = 0, 1, \ldots, 4 \)). Then \( l^p_{x+t} \) is calculated according to (for \( t = 1, 2, \ldots, 110.5 - x \) (we assume a limiting age of 110.5)):

\[
l^p_{x+t} = \sum_{m=0}^{4} l^m_{x+t-1} \times \hat{p}^{m,p}_{x+t-1,x+t} \times \left( 1 - \frac{1}{2} \left( CP^{m,ACH}_{x+t-1,x+t} + CP^{p,ACH}_{x+t-1,x+t} \right) \right). \tag{4.27}
\]

To calculate \( \Pr (T(x, n) > t) \) (for \( x = 60.5, 61.5, \ldots, 109.5 \) and \( n = 0, 1, \ldots, 4 \)), firstly, we do the following:

- set \( l^n_x \) to be a suitably large number (e.g. 100,000),
- set \( l^p_x = 0 \) for \( p \neq n \), and
- calculate \( l^p_{x+t} \) for \( p = 0, 1, \ldots, 4 \) and \( t = 1, 2, \ldots, 110.5 - x \) using (4.27).

Then \( \Pr (T(x, n) > t) \) is calculated according to:

\[
\Pr (T(x, n) > t) = \frac{1}{l^n_x} \sum_{p=0}^{4} l^p_{x+t}. \tag{4.28}
\]

**Calculation of \( \Pr (T(x) > t) \)**

We calculate \( \Pr (T(x) > t) \) according to (for \( x = 60.5, 61.5, \ldots, 109.5 \)):

\[
\Pr (T(x) > t) = \sum_{n=0}^{4} \Pr (\text{InState}(x) = n) \times \Pr (T(x, n) > t), \tag{4.29}
\]

where \( \Pr (\text{InState}(x) = n) \) is the probability that the disability state of the borrower at the origination of the contract (initial disability state), where the
borrower is aged \( x \), is \( n \) (for \( n = 0, 1, \ldots, 4 \)). This probability is estimated as the prevalence rate of disability state \( n \) at age \( x \) of the community population estimated from the 2003 SDAC data.

**Calculation of** \( \Pr(T(x, y, m, n) > t) \)

To calculate \( \Pr(T(x, y, m, n) > t) \) we assume that the mortality, disability and ACH admission of the husband are probabilistically independent of the mortality, disability and ACH admission of the wife. This survival probability is calculated according to (for \( x = 60.5, 61.5, \ldots, 109.5; y = 60.5, 61.5, \ldots, 109.5; m = 0, 1, \ldots, 4 \) and \( n = 0, 1, \ldots, 4 \)):

\[
\Pr(T(x, y, m, n) > t) = 1 - \Pr(T^m(x, m) < t) \times \Pr(T^f(y, n) < t), \tag{4.30}
\]

where \( \Pr(T^m(x, m) < t) \) is \( \Pr(T(x, m) < t) \) calculated using the males’ disability transition and conditional ACH admission probabilities, and similarly for \( \Pr(T^f(y, n) < t) \).

Figure 4.16 presents the survival probabilities for a RM contract over time for a single male, a single female and a couple. For convenience, we denote \( T \) as a random variable for the future term of the RM contract considered. The contract is assumed to be taken by a single person (or a couple) who is aged 60.5 and has no CAL (or for a couple, both are aged 60.5 and have no CAL) at the origination of the contract. The survival probabilities for a contract taken by a male are lower than for a contract taken by a female due to the higher mortality rates of males (which outweigh the higher ACH admission probabilities of females). The survival probabilities for a contract taken by a couple are the highest as expected.

Figure 4.17 presents the probabilities of termination in year \( t \) of a RM contract with the probabilities being presented separately for termination due to death (left hand side) and due to ACH admission (right hand side). The probabilities are presented for a single male and a single female. The probabilities of termination due to death are initially higher for males, however, in later time, the differences between genders are nullified by the lower
survival probability of the contract to time $t$. On the other hand, the probabilities of termination due to ACH admission are higher for females with the differences between genders initially increasing with time before being nullified by the lower survival probability of the contract to time $t$.

Figure 4.18 presents the survival probabilities of a RM contract over time for a single male by different disability state and age of the borrower at the origination of the contract. Note that in the case where the initial disability state is unknown, the survival probability is given by $\Pr(T(x) > t)$. The survival probabilities are decreasing with increasing severity of the initial disability state with the initial states of severe CAL and profound CAL having a significant impact on the survival probabilities. In addition, the survival probabilities are also decreasing with increasing initial ages.
4.10 Estimation uncertainties

In this section, we discuss the uncertainties in the estimation of disability transition probabilities and ACH admission probabilities. The uncertainties are due to the limited amount of data and the limitations of the modelling methodology.

4.10.1 Data

Most uncertainties of the estimation are due to the limited amount of data. The available data are limited in several aspects. In particular, the disability prevalence rates are observed only in 1998 and 2003. The actual prevalence rates in 1999 to 2002 are likely to be different to the estimated prevalence rates in these years. However, there are several reasons to believe that reasonable estimates of transition probabilities have been obtained. Firstly, the disability prevalence rates in 1999 to 2002 are unlikely to be significantly different to the prevalence rates in 1998 or 2003 (given the relatively stable prevalence rates of disability over two decades). Note that the prevalence rates in 1998 and 2003 are generally similar. Secondly, the IPF procedure is not overly sensitive to the input of the estimation (from our experiment, a small difference of input in the IPF procedure did not result in significantly
4.10. ESTIMATION UNCERTAINTIES

Another material limitation of the data is the grouping of the very old in the disabled population into a very wide age interval. This caused difficulty in the estimation of the disabled population at single ages for these ages. Although we have attempted to remedy this problem, it is likely that a certain degree of estimation error still remains. Note that the estimated crude transition probabilities are spurious at these ages for some disability categories (which is partly due to the inaccuracy of the estimated disabled population). By grouping the transition data at these ages in the graduation, reasonable estimates of the transition probabilities can be obtained. However, more accurate estimates can be obtained if the disabled population is reported at older age groups in the survey.

Lastly, given the data used for the estimation, the estimated ACH admission probabilities are applicable to the general community population. However, the experience of RM borrowers might be different to the experience of the general community population. This could be due to selection effects (for example, RM borrowers might be wealthier and hence healthier than the general community population) and moral hazard (for example, RM borrowers might have more incentive to stay at home for a longer period than the general community population). Therefore, an adjustment to the estimated ACH admission probabilities might be necessary in the pricing
4.10.2 Modelling methodology

The accuracy of the results of the IPF procedure depends, among other things, on the accuracy of the initial transition probabilities. Therefore, inaccuracy and limitations of the initial transition probabilities will adversely impact the results of the IPF procedure. For example, the limitation of possible annual recovery of the initial transition probabilities is also borne by the transition probabilities estimated under the IPF procedure. Nevertheless, we believe that the implementation of the IPF procedure improves the accuracy of the initial transition probabilities.

The ACH admission probabilities are estimated by assuming that the estimated disability transition probabilities (which are estimated from the data of the total population (i.e. the sum of community and ACH population)) are applicable to the community and to the ACH population. This assumption is adopted due to the difficulty, given the limited amount of data, in determining the appropriate adjustment to the estimated disability transition probabilities which is suitable for each of the community and ACH population. Assuming that the community population are healthier than the ACH population, we are likely to overestimate the deterioration probabilities and underestimate the improvement probabilities of the community population (and vice versa for the ACH population). Therefore, the exposure of risk of moving into an ACH are overestimated for severe disability categories and underestimated for mild disability categories, and these result in an underestimation of the conditional ACH admission probabilities for severe disability categories and overestimation for mild disability categories. This error is likely to be material only at very high ages and only for the most severe disability categories (profound) where the ACH prevalence rates are quite high. In addition, this error is, in a certain degree, being offset by the inaccuracy of the assumed disability transition probabilities of the ACH population. Overall, the resulting estimates of the (unconditional) ACH admission probabilities seem to be generally consistent with other stud-
ies and hence, the error of the assumed disability transition probabilities is likely to have only minimal impact (whether because the error is small or due to the offsetting impact from the ACH population) on the estimated ACH admission probabilities.

4.11 Conclusion

In this chapter, we propose a new estimation method for disability transition probabilities. The method utilizes data in the 1998 and 2003 SDAC. There are uncertainties in the estimation which are mainly due to the limited amount of data. However, reasonable estimates are obtained with several informative observations emerging. The method is flexible (easily adjusted if desirable (for example, to suit better data)) and will produce increasingly accurate results when better data become available.

There are several ways in which better data can be obtained. The obvious one is to increase the frequency of SDAC surveys. However, this is unlikely given the cost of conducting such a large scale national survey. An improvement to the survey that might materially improve the accuracy of the estimation is to present the estimate of disabled population at older age groups (say up to age 105 and over). However, this is difficult given the small numbers of people in each CAL category at oldest ages. Better data can also be obtained by measuring CAL with a similar scale as in SDAC in regional disability longitudinal studies. If such data were available, we would be able to obtain more accurate initial transition probabilities (in the application of the IPF procedure) and, hence, more accurate refined estimates of disability transition probabilities could be obtained. Note that the availability of such data would circumvent many of the modelling limitations associated with the estimation of initial transition probabilities (for example, it would allow a fuller set of recovery processes in the estimation). Furthermore, as a non-Markovian model can be constructed for the estimation of initial transition probabilities, this might open the possibility of constructing a non-Markovian (or proxy non-Markovian) model for the estimation of disability transition probabilities from national cross sectional datasets.
In this chapter, we also estimate the conditional probabilities of admission into an ACH. The conditional ACH admission probabilities are estimated from the estimates of disability transition probabilities and ACH prevalence rates. Despite the limited amount of data, the resulting estimates of ACH admission probabilities are generally consistent with other studies.

Using the estimated disability transition probabilities, the conditional ACH admission probabilities and the projected mortality improvement factors (Chapter 2), we are able to undertake a pricing analysis of a RM contract.
Chapter 5

Pricing reverse mortgages in Australia

5.1 Introduction

The current funding for superannuation in Australia will not be sufficient to provide for the needs during retirement of the majority of older Australians (Rice Warner Actuaries, 2012). This problem is exacerbated by the rising life expectancy and the ageing of the Australian population. For the majority of older Australians, home equity constitutes a substantial proportion of their wealth (see Table 22 of Australian Bureau of Statistics (ABS) (2011a)). Hence, a financial product which enables them to conveniently unlock the equity in their homes while providing a certain degree of financial protection presents an attractive solution to address the inadequacy of superannuation savings. Naturally, this type of product, which is called an equity release product, has experienced rapid growth in Australia recently (see the research reports by SEQUAL/Deloitte). Amongst this type of product, the reverse mortgage is the most popular (Bridge et al., 2010).

A reverse mortgage (RM) is a financial product that allows people to borrow against their house. Generally, the house must be owned outright. The minimum age to borrow a RM loan is generally around 60 and for a couple, the age requirement is determined by the age of the younger spouse. The
maximum which can be borrowed as a proportion of the value of the house, called the loan to value ratio (LVR), depends on the age of the borrower (or in the case of a couple, the age of the younger spouse). The older the borrower is, the higher the maximum allowable LVR is. The proceeds from the loan can be accessed as a lump sum, an income stream or a line of credit. The loan can be accumulated at a fixed or a variable interest rate. For a more detail overview of RM products in Australia, see Bridge et al. (2010).

There are two features of a RM which distinguish this product from a standard loan. Firstly, under a RM, no repayments of the principal or interest are required until termination of the loan which can only occur because the borrower or, in the case of a couple, the last surviving spouse, dies, moves out from his/her house permanently or voluntarily repays the loan early. During the life of the loan, the borrower is allowed to stay in his/her house and retains full ownership of it. Secondly, the RM incorporates a no negative equity guarantee (NNEG) which implies that the maximum the lender can recoup from the loan is the value of the house at the termination time. Therefore, if at the termination time, the accumulated loan is higher than the house value, the lender can only recoup up to the value of the house. Note that these two features, which are designed to protect the borrower, make the RM a risky product to issue for the lender.

There are two significant risks for a lender issuing RM products. The first is the risk that at the termination time, the value of the house is less than the accumulated loan and hence, due to the NNEG, the full amount of the accumulated loan cannot be recouped (crossover risk). The crossover risk is significant since the duration of a RM contract can be quite long (more than 30 years) and, hence, during an adverse price movement in the housing market, the lender might suffer a significant amount of loss. The second is the liquidity risk: since for a given RM contract, there will be no cash inflow for the lender until termination of the contract, in a given portfolio of RM loans, a significant amount of the lender’s capital might be tied up for a long period of time. This liquidity risk, if not properly managed, might cause several adverse consequences to the lender such as missing valuable investment opportunities, losses (due to the costs associated
with the mismatch between assets and liabilities) and even insolvency.

In this chapter, we seek to contribute to the existing literature on the measurement of crossover and liquidity risks for RMs in Australia. Note that the measurement of these risks requires a suitable modelling of house prices, interest rates and termination rates of RM contracts. The house prices and the interest rates are modelled and simulated under a developed house price inflation linked LIBOR market model. The termination probabilities of RMs in Australia are estimated under a multi-state modelling framework to take into account a variety of reasons for termination and by using data for termination experience from the U.S. and the U.K.. We restrict our analysis on the lump sum type of RM (i.e. the proceeds from the loan are accessed as a lump sum at the origination of the RM contract) which is the most popular type of RM in Australia (e.g. SEQUAL/Deloitte (2012)).

The work in this chapter is part of a wider project on RMs and can only be viewed as a subset of this project. In this particular chapter, we use the work of other people. Specifically, we use a program which was written in C++ (with an Excel spreadsheet as the interface) which has been developed to simulate future house price index and LIBOR rates under a developed house price inflation linked LIBOR market model. The development of the house price inflation linked LIBOR market model was a significant task and is beyond the scope of our thesis.

5.2 An overview of reverse mortgages

5.2.1 Risks for the lender of reverse mortgages

In the following we briefly describe a variety of underlying risks that the lender of RM faces.

- **Maturity risk** – The risk that the RM loan terminates sooner or later than expected. Sooner than expected termination results in a lower than expected profit for the lender since the interest is earned in a shorter than expected period of time. In addition, if the loan terminates too early, the origination costs (i.e. the costs associated with originating
a RM loan) might not be fully recouped. On the other hand, a longer duration of a loan results in higher crossover and liquidity risks. Note that the adverse consequences of a longer than expected duration of the loan are likely to be more severe than a shorter than expected duration.

- **House price and interest rate risks** – The risk of an adverse movement in the house prices and interest rates which could escalate the NNEG cost and amount of loss. The difference between these two risks is that the house price risk is partially diversifiable (by holding a large number of RM loans across areas) while the interest rate risk is not. In addition, based on U.S. experience, Davidoff and Welke (2007) found that RM termination rates are sensitive to house price appreciation rates. Therefore, very low house price appreciation rates could result in lower termination rates which could further escalate the NNEG cost and amount of loss.

- **Moral hazard risk** – The risk that the borrower does not maintain the house, reducing its appeal to future buyers. This risk is especially high if the accumulated loan is almost or already above the house value.

- **Reputation risk** – The risk that a dispute might arise at the settlement of the contract which might result in reputational damage to the lender. The borrowers of RMs are perceived as vulnerable members of society.

- **Expense risk** – The risk that the expenses associated with originating and managing a portfolio of RM are higher than expected.

Risk management of a portfolio of RMs is a complicated exercise due to the variety of risks involved. For comprehensive discussions on this risk management see Szymanoski (1994), Equity Release Working Party (ERWP) (2005) and Hosty et al. (2008). Since our aim is to measure the crossover and liquidity risks, we focus on maturity, house price and interest rate risks.
5.2. AN OVERVIEW OF RM

5.2.2 The no negative equity guarantee (NNEG)

Figure 5.1 presents sample trajectories of house value, accumulated RM loan, accumulated funding cost (i.e. the price of obtaining the loan capital) from providing the capital for the RM loan and net cash flow (i.e. receipts from the loan minus the accumulated funding cost given that the loan terminates in the corresponding policy year) with and without NNEG across policy years (i.e. time elapsed since the origination of the RM contract measured in year). In this illustration, the crossover point (the earliest time at which the accumulated loan exceeds the house value) occurs in policy year 36. If the RM terminates before the crossover point, the lender will be able to recoup the full amount of the accumulated loan and hence the cost of the NNEG is zero. However, if the RM terminates after the crossover point, the lender will only be able to recoup the value of the house and the difference between the house value and the accumulated loan at the termination time represents a NNEG cost to the lender. Note that after the crossover point, the net cash flow without NNEG is higher than the net cash flow with NNEG with the difference increasing very rapidly with policy year. Assuming the absence of any costs other than the funding cost, the lender will suffer a loss if at the termination time the accumulated funding cost is higher than the house value. In this illustration, the loss to the lender is unlikely to happen (bearing in mind that the homeowner would be at least 60 at origination).

5.2.3 Reasons for termination

A RM might terminate due to the following reasons.

- Death of the borrower or, in the case of a couple, the death of the last surviving spouse.

- The borrower or, in the case of a couple, the last surviving spouse, moves into an aged care home (ACH).

- Termination due to reasons other than above (voluntary prepayment) which include the following (ERWP, 2005).
Changes in personal or financial circumstances of the borrower which result in the borrower moving out permanently from his/her house (for example, to live with family) or voluntarily repaying the loan early.

- Remortgaging.

5.2.4 An overview of the reverse mortgage market in Australia

In Australia, RMs were introduced to the market as early as the 1980s (Beal, 2001). However, this product has been growing rapidly only recently. SEQUAL/Deloitte publishes twice yearly research reports which provide a comprehensive overview of the Australian RM market (with the first report pub-
lished in October 2006). In the following we present several observations from these research reports which are relevant for our analysis.

- At 31 December 2011, the total amount of outstanding loans was $3.32 billion which represents 119% growth from 31 December 2006.

- The new loans are consistently dominated by variable interest rate and lump sum type of loan. At 31 December 2009, variable interest rate type of loan represents 85% of outstanding loans (Hickey et al., 2009).

- At 31 December 2009, the average age of existing borrowers was 74 with the proportions of borrowers in age groups less than 65, 65–69, 70–74, 75–79, 80–84 and 80+ being 12%, 18%, 21%, 18% and 30% respectively (Hickey et al., 2009).

- The total termination rate (i.e. the total number of terminations divided by the total number of outstanding loans) was around 10% p.a. in which termination due to death and entry into ACH accounts for only around 1% to 1.5%.

- At 31 December 2009, couples represent the largest category of existing borrowers (45% of outstanding loans) followed by single females (37%) and single males (18%) (Hickey et al., 2009). However, the proportion of new borrowers who are single females is increasing over time.

5.2.5 An overview of reverse mortgage market in the U.S.

The RM products in the U.S. are dominated by the Home Equity Conversion Mortgage (HECM) product which represents over 90% of all RM loans in the U.S. (Bishop and Shan, 2008). A particular feature of a HECM is that it is insured by the Federal Housing Administration (FHA) of the U.S. Department of Housing and Urban Development (HUD). This insurance protects the lender from NNEG losses. In addition, the lenders of HECM loans have the option to assign an active HECM loan to the U.S. HUD in the event the
accumulated loan reaches the maximum claim covered by FHA insurance (Szymanoski et al., 2007). Comparing the HECM market in the U.S. with the RM market in Australia, we observe the following similarities.

- HECMs, as RMs in Australia, are dominated by variable interest rate type of loan (Bishop and Shan, 2008).

- The distribution by age of borrowers of HECM loans (which can be inferred from Figure 8 of Bishop and Shan (2008)) is similar to that of RM borrowers in Australia. In addition, the median age of HECM borrowers (Table 2 of Bishop and Shan (2008)) is similar to the average age of existing RM borrowers in Australia.

The above similarities between the market for HECMs in the U.S. and RMs in Australia gives us a basis to estimate the voluntary prepayment probabilities (i.e. termination probabilities due to reasons other than death and admission into an ACH) of RMs in Australia from the experience of HECM loans in the U.S. (described in Section 5.5.4).

5.3 Literature review

5.3.1 Valuation of NNEG

In general, there are two methods to value the NNEG: the no arbitrage option pricing method (Equity Release Working Party (ERWP), 2005; Hosty et al., 2008; Chen et al., 2010; Li et al., 2010; Ji et al., 2012) and the “real world” stochastic modelling method (Szymanoski, 1994; Chia and Tsui, 2005; Hosty et al., 2008; Feng, 2010; Sun and Sherris, 2010). In the following we briefly describe these methods.

Under the no arbitrage option pricing method, the cost of the NNEG is given by the value of the portfolio which (supposedly) replicates the payoff of the NNEG. Early attempts on this pricing method assume that the returns on the underlying house (i.e. the house against which the RM loan is borrowed) follow a geometric Brownian motion (GBM) process (ERWP,
5.3. LITERATURE REVIEW

Under this assumption, due to the similarity between the payoff of the NNEG and a European put option, the derived pricing formula for the NNEG is similar to the Black-Scholes pricing formula for a vanilla European put option (Equity Release Working Party (ERWP), 2005; Ji et al., 2012). Since there are no historical data on prices of the underlying house, the GBM model is typically fitted to a house price index. However, typically, the returns on house price index exhibit a strong autocorrelation, which contradicts the GBM assumption. While it is possible to identify an alternative stochastic process which is more suitable to model the returns on house price index, usually, the chosen stochastic process will imply market incompleteness (i.e. there exists more than one equivalent martingale measure) (Chen et al., 2010; Li et al., 2010). In such a situation, the conditional Esscher transform (Bühlmann et al., 1996) can be utilized to identify an equivalent martingale measure which gives an economically consistent and justifiable price for the NNEG (Chen et al., 2010; Li et al., 2010). Recently, Kogure et al. (2012) developed a pricing methodology for reverse mortgages under a multivariate Bayesian risk-neutral method. This methodology is an extension of the univariate Bayesian risk-neutral pricing method developed by Kogure and Kurachi (2010).

Under the “real world” stochastic modelling method, we fit a stochastic model to the house price index (as a proxy for the prices of the underlying house) and the interest rates. The house price index and the interest rates can be modelled separately (Szymanoski, 1994; Chia and Tsui, 2005; Hosty et al., 2008; Feng, 2010) or jointly to taken into account their interactions (Sun and Sherris, 2010). The cost of the NNEG is then determined, typically, from a simulation method.

In relation to the valuation of the NNEG, there are a variety of stochastic models which have been adopted to model house price index (or their returns) and the interest rates. On modelling house price index (or their returns), the adopted models include GBM (Szymanoski, 1994; Equity Release Working Party (ERWP), 2005; Hosty et al., 2008; Wang et al., 2008; Ji et al., 2012), ARMA-GARCH (Chen et al., 2010) and ARMA-EGARCH (Li et al., 2010). On modelling the interest rates, the adopted models include Vasicek (Wang
et al., 2008), AR1 (Feng, 2010) and the Cox-Ingersoll-Ross short rate model (Chia and Tsui, 2005). Recently, Sun and Sherris (2010) jointly modelled house price index and interest rates under a Vector Autoregressive (VAR) model to take into account their interactions.

In this chapter, we evaluate the cost of the NNEG under a “real world” stochastic modelling method. The house price index and the interest rates are jointly modelled under a house price inflation linked LIBOR market model. The cost of the NNEG (and other risk measures) is then determined under a simulation method. To the best of our knowledge, market models have not previously been used to price RM products. Market models, especially the LIBOR market model, are considered to be the current standard approach for pricing exotic interest rates derivatives. The RM is an exotic product which is not just an interest rate derivative, but also a property derivative since its price is derived from both interest rates and house prices. The cross-currency LIBOR market model has been developed to price exotic interest rate derivatives that derive their prices from the interest rates in more than one currency (Mikkelsen, 2002). Furthermore, the cross-currency LIBOR market model has been used to price inflation indexed products by interpreting the foreign currency as the inflation adjusted real forward rate and the exchange between domestic and foreign currency as the consumer price index (CPI) (assuming the products are indexed by the CPI) (Brigo and Mercurio, 2006). The idea of a house price inflation linked LIBOR market model is similar to the idea of the implementation of a cross-currency LIBOR market model to price inflation indexed products with a change of definition of CPI with house price index (HPI).

5.3.2 Estimation of termination probabilities

In Australia, estimation of termination probabilities of RMs is difficult due to the limited amount of publicly available termination data. From the research reports by SEQUAL/Deloitte, we only obtain the overall termination rates per year due to each of death, ACH admission and voluntary prepayment. Papers which deal with the pricing and risk management of RMs for the Aus-
In the following we describe the estimation of termination probabilities of RM contracts in these papers.

Wang et al. (2008) assumed that the termination probabilities of RM contracts due to reasons other than death are 30% of one-year death probabilities of RM borrowers. This assumption is based on the “move-out” rate of 85 year-olds in the general population in the U.S., estimated by Jacobs (1988). The death probabilities of RM borrowers are assumed to be the same as the death probabilities of the general population. Sun and Sherris (2010) considered a range of scenarios of termination probabilities of RM contracts which include: 100% of females (general population) death probabilities across all ages, 130% of females death probabilities (following the original assumption in the pricing of HECM loans in the U.S. as described in Szymanoski et al. (2007)) until age 93 (at which point it reverts to 100% of females death probabilities) and the termination probabilities of HECM loans in the U.S. estimated by Szymanoski et al. (2007). In their scenarios, they also considered constant longevity improvement rates of 1% and 1.5% p.a. across all age groups which are based on historical data from the Australian Life Tables 1995-97 (Australian Government Actuary (AGA), 1999).

In this chapter, we estimate the termination probabilities of RM contracts by estimating the death, ACH admission and voluntary prepayment probabilities of RM borrowers. The death and ACH admission probabilities of RM borrowers are estimated using the disability transition and conditional ACH admission probabilities estimated in Chapter 4. Since the disability transition probabilities are estimated from the experience of the general population, a range of scenarios of selection effect are considered to anticipate possible differences of disability and mortality experience between the RM borrowers and the general population. In addition, we also consider a range of scenarios of future improvement factors for death and deterioration (i.e. the probabilities of deteriorating to any worse disability state) probabilities and future conditional ACH admission probabilities. Lastly, two scenarios of voluntary prepayment probabilities are considered: the voluntary prepayment probabilities which are estimated from the experience of HECM loans in the
U.S. (as the estimates of current voluntary prepayment probabilities of RM in Australia) and the voluntary prepayment probabilities assumed by Hosty et al. (2008) in their analysis of RM in the U.K. market (as the estimates of these probabilities when the Australian RM market matures).

5.4 Measures of profit, liquidity, NNEG cost and loss

As described above, we restrict our analysis to the lump sum type of RM. In evaluating the cash flows of RM contracts, we assume the following.

(a) The only cost for the lender is the cost of providing capital to finance the loan.

(b) The LIBOR rate represents the funding (capital) cost rate and the investment return rate for the lender. The future net cash flows are discounted to the present under the future investment return rate (future LIBOR rate). This can be rationalised as follows. Suppose a loss occurs at future time $t$, the present value of this loss can be considered as the amount of money we need now to cover this loss at time $t$ by taking into account the investment return from the present to time $t$. Similarly, the present value of profit can be considered as the current amount of money which if being accumulated at the future investment return rate will accumulate to the amount of this profit at time $t$.

(c) The cash flows associated with RM contracts which terminate in policy year $t$ occur at the end of policy year $t$. In addition, given that the RM terminates in policy year $t$, the loan is accumulated until the end of policy year $t$.

(d) The spread (i.e. excess of the interest rate charged to the borrower over the funding cost rate) is constant over time.

(e) Borrowers who are aged $x$ last birthday are aged exactly $x + 0.5$. 
(f) All borrowers are assumed to die by age 110.5.

(g) The RM is initiated at 1 January 2011.

Denote:

\( t \): the policy year. Since the RM is assumed to be initiated at 1 January 2011, policy year 1 corresponds to calendar year 2011.

\( HPI_t \): the House Price Index (HPI) at the December quarter of calendar year \( 2010 + t \) with \( HPI_0 \) as the HPI at the December quarter of calendar year 2010.

\( LIB_t \): the LIBOR rate in calendar year \( 2010 + t \).

\( H_t \): the house price at the end of policy year \( t \).

\( L_t \): the accumulated loan (at the LIBOR rate plus spread (see below)) at the end of policy year \( t \).

\( F_t \): the accumulated funding cost (at the LIBOR rate) at the end of policy year \( t \) from providing a RM loan of amount \( L_0 \) at time 0 (the time of the origination of the RM contract). Note that \( F_0 = L_0 \).

\( s \): the excess of the interest rate charged to the borrower over the LIBOR rate (spread).

\( CI_t \): the cash inflow for the lender at the end of policy year \( t \) given that the RM terminates in policy year \( t \).

\( NNEG_t \): the NNEG cost for the lender at the end of policy year \( t \) given that the RM terminates in policy year \( t \).

\( Loss_t \): the amount of loss for the lender at the end of policy year \( t \) given that the RM terminates in policy year \( t \).

\( p(t) \): the probability that the RM terminates in policy year \( t \).

\( p(NNEG) \): the probability that the lender incurs a NNEG cost.
$p(Loss)$ : the probability that the lender incurs a loss.

The profit, liquidity, NNEG cost and loss are evaluated under a simulation method. Let $HPI_t^n$ denotes the value of $HPI_t$ in the $n$th simulation, and similarly for $LIB_t^n, H_t^n, L_t^n, F_t^n, CI_t^n, NNEG_t^n$ and $Loss_t^n$. In the $n$th simulation, we obtain $\{LIB_t^n, HPI_t^n\}$ for $t = 1, 2, \ldots, T$. Note that since all borrowers are assumed to die by age 110.5, we have $T = 110.5 - (x + 0.5)$, where $x + 0.5$ (for $x = 60, 61, \ldots, 109$) is the age of the borrower (or for a couple, the age of the younger spouse) at the origination of the RM contract. From $\{LIB_t^n, HPI_t^n\}$, the values of other variables of interest in the $n$th simulation are given by (for $t = 1, 2, \ldots, T$):

\begin{align}
H_t^n &= \frac{H_t HPI_t^n}{HPI_0}, \quad (5.1) \\
L_t^n &= L_0 \prod_{j=1}^{t} \left(1 + \left(LIB_j^n + s\right)\right), \quad (5.2) \\
F_t^n &= F_0 \prod_{j=1}^{t} \left(1 + LIB_j^n\right), \quad (5.3) \\
CI_t^n &= \min(H_t^n, L_t^n), \quad (5.4) \\
NNEG_t^n &= \max(L_t^n - H_t^n, 0), \quad (5.5) \\
Loss_t^n &= \max(F_t^n - H_t^n, 0). \quad (5.6)
\end{align}

In the following we assume that $N$ simulations are performed.

**Measure of profit**

To estimate the profitability of a portfolio of homogenous RM loans, we calculate the expected present value (EPV) of profit ($EPV_{Profit}$) which is given by:

\[
EPV_{Profit} = \sum_{t=1}^{T} p(t) \times \frac{1}{N} \sum_{n=1}^{N} \left( CI_t^n \times \prod_{j=1}^{t} \left(1 + LIB_j^n\right)^{-1} \right) - F_0. \quad (5.7)
\]
5.4. MEASURES OF PROFIT AND RISKS

Measure of liquidity

To measure the liquidity of a portfolio of homogenous RM loans, we calculate the expected value of the discounted payback period \(E_{DPP}\) which is given by:

\[
E_{DPP} = \begin{cases} 
Z(t) : & EPV_{Profit} \geq 0, \\
\infty : & \text{otherwise}, 
\end{cases}
\]  
(5.8)

where

\[
Z(t) = \inf \left\{ t \in A : \sum_{k=1}^{t} p(k) \times \frac{1}{N} \sum_{n=1}^{N} \left( CI^n_k \times \prod_{j=1}^{k} \left( 1 + LIB^n_j \right)^{-1} \right) > F_0 \right\},
\]

and \(A = \{1, 2, \ldots, T\}\).

Measures of NNEG risk and cost

The probability that the lender incurs a NNEG cost \(p(NNEG)\) is given by:

\[
p(NNEG) = \sum_{t=1}^{T} p(t) \times \Pr (H_t < L_t) = \sum_{t=1}^{T} p(t) \times \frac{1}{N} \sum_{n=1}^{N} I (H^n_t, L^n_t) \]  
(5.9)

where

\[
\Pr (H_t < L_t) \text{ is the probability that } H_t < L_t,
\]

\[
I (A, B) = \begin{cases} 
1 : & A < B, \\
0 : & \text{otherwise}. 
\end{cases}
\]

To estimate the cost of the NNEG, we calculate the EPV of the NNEG cost \(EPV_{NNEG}\) which is given by:

\[
EPV_{NNEG} = \sum_{t=1}^{T} p(t) \times \frac{1}{N} \sum_{n=1}^{N} \left( NNEG^n_t \times \prod_{j=1}^{t} \left( 1 + LIB^n_j \right)^{-1} \right). \]  
(5.10)
Measures of loss risk and amount

The probability that the lender incurs a loss \( p(\text{Loss}) \) is given by:

\[
p(\text{Loss}) = \sum_{t=1}^{T} p(t) \times \Pr(H_t < F_t) = \sum_{t=1}^{T} p(t) \times \frac{1}{N} \sum_{n=1}^{N} I(H^n_t, F^n_t). \tag{5.11}
\]

To estimate the amount of loss, we calculate the EPV of loss \( \text{EPV}_{\text{Loss}} \) which is given by:

\[
\text{EPV}_{\text{Loss}} = \sum_{t=1}^{T} p(t) \times \frac{1}{N} \sum_{n=1}^{N} \left( \text{Loss}^n_t \times \prod_{j=1}^{t} \left(1 + \text{LIB}^n_j\right)^{-1} \right). \tag{5.12}
\]

As described in Section 5.1, we used a program which was written in C++ (with an Excel spreadsheet as the interface) to simulate the future house price index and LIBOR rates. Specifically, this program takes the initial house price \( (H_0) \), LVR, spread \( (s) \) and the number of simulations \( (N) \) as the input and returns the following simulated variables (for \( t = 1, 2, \ldots, T \)).

- \( \frac{1}{N} \sum_{n=1}^{N} \left( CI^n_t \times \prod_{j=1}^{t} \left(1 + \text{LIB}^n_j\right)^{-1} \right) \),
- \( \frac{1}{N} \sum_{n=1}^{N} I(H^n_t, L^n_t) \),
- \( \frac{1}{N} \sum_{n=1}^{N} \left( \text{NEG}^n_t \times \prod_{j=1}^{t} \left(1 + \text{LIB}^n_j\right)^{-1} \right) \),
- \( \frac{1}{N} \sum_{n=1}^{N} I(H^n_t, F^n_t) \), and
- \( \frac{1}{N} \sum_{n=1}^{N} \left( \text{Loss}^n_t \times \prod_{j=1}^{t} \left(1 + \text{LIB}^n_j\right)^{-1} \right) \).

In Section 5.5, we describe the construction of scenarios of death, disability transition, conditional ACH admission and voluntary prepayment probabilities for the pricing analysis of RM contracts in Australia.
5.5 Scenarios of death, disability transition, conditional ACH admission and voluntary prepayment probabilities

The termination probabilities of RM contracts depend on the death, ACH admission and voluntary prepayment probabilities of RM borrowers. In addition, in our modelling, the ACH admission probabilities depend on disability transition and conditional ACH admission probabilities (Section 4.8). Therefore, to construct scenarios of termination probabilities of RM contracts, we need to construct scenarios of death, disability transition, conditional ACH admission and voluntary prepayment probabilities of RM borrowers. In this section, we construct scenarios of these probabilities. We begin by adjusting the disability transition probabilities estimated in Chapter 4 to reflect the progression of these probabilities over time. Following that, we construct scenarios of selection effect (to take into account the possibility that the experience of RM borrowers is different from that of the general population) and future improvement factors for death and deterioration probabilities. Lastly, we construct scenarios of future conditional ACH admission probabilities and voluntary prepayment probabilities of RM borrowers.

In the following, the notation for the disability states is the same as in Section 3.4.1.

5.5.1 Adjusting the disability transition probabilities from 2001 to 2009

In Chapter 4, the disability transition probabilities are estimated from the 1998 and 2003 SDAC data and therefore are (approximately) effective in 2001. Therefore, these transition probabilities need to be adjusted to reflect the progression of these probabilities over time. Due to the absence of data, the progression of these transition probabilities over time is difficult to determine. Therefore, we simply assume that the transition probabilities to the living states conditional on survival are constant over time. Under this
assumption, if the progression of death probabilities over time for each disability category is known, then for all disability categories, the progression of transition probabilities to the living states can be determined. We further assume that the death probabilities of each disability category experience the same percentage improvement over time as the death probabilities of total population (i.e. total population across disability categories). Therefore, the progression of disability transition probabilities over time can be determined from the progression of the death probabilities of the total population.

We choose to adjust the disability transition probabilities to 2009 since at the time (when work on this chapter started), the most recent Australian life table (ALT) is the ALT for the period 2008–2010 (which is assumed to be applicable in 2009) published by ABS (2011b). The procedure for the adjustment is described in the following.

Let \( \hat{P}_{m,n}^{x+0.5}(t) \) (for \( m = 0, 1, \ldots, 4 \) and \( n = 0, 1, \ldots, 5 \)) denote the probability \( \hat{P}_{m,n}^{x+0.5,x+1.5} \) applicable in year \( t \) (\( \hat{P}_{m,n}^{x+0.5,x+1.5} \) is described in Section 4.7.1). Note that, due to our assumption, \( \hat{P}_{m,n}^{x+0.5}(2001) \) is \( \hat{P}_{m,n}^{x+0.5,x+1.5} \) estimated in Chapter 4. Firstly, we adjust the death probabilities of each disability category to 2009 according to (for \( x = 60, 61, \ldots, 108 \)):

\[
\hat{P}_{x+0.5}^{m,5}(2009) = \hat{P}_{x+0.5}^{m,5}(2001) \times \frac{q_{ABS}^{08-10}}{q_{ABS}^{00-02}} \quad m = 0, 1, \ldots, 4 \quad (5.13)
\]

where \( q_{ABS}^{00-02} \) and \( q_{ABS}^{08-10} \) are estimated from the ALT for the period 2000–2002 (ABS, 2008a) and 2008–2010 (ABS, 2011b) respectively.

The \( q_{ABS}^{00-02} \) and \( q_{ABS}^{08-10} \) at ages above 100 are extrapolated by inferring from the mortality rates presented in the Australian life tables in the relevant period published by the Human Mortality Database (HMD) under a method described in Section 2.6.6.

We adjust only up to age 108.5 since we assume \( \hat{P}_{109.5}^{m,5}(t) = 1 \) for \( m = 0, 1, \ldots, 4 \) and \( \forall t \).

Following that, the transition probabilities to the living states are ad-
justed to 2009 according to (for \( x = 60, 61, \ldots, 108 \)):

\[
P_{m,n}^{x+0.5}(2009) = P_{m,n}^{x+0.5}(2001) \times \frac{1 - P_{m,5}^{x+0.5}(2009)}{1 - P_{m,5}^{x+0.5}(2001)} \quad m = 0, 1, \ldots, 4; n = 0, 1, \ldots, 4.
\] (5.14)

Note that under adjustment (5.14), the sum of transition probabilities from a given disability category is retained at 1.

The conditional ACH admission probabilities (i.e. \( CP_{m,ACH}^{x+0.5,x+1.5} \) for \( x = 60, 61, \ldots, 108 \) and \( m = 0, 1, \ldots, 4 \)) which are described in Section 4.8) are assumed to be time invariant. Therefore, the estimates of these conditional probabilities in Section 4.8 are applicable in 2009.

5.5.2 Scenarios of selection effect

In this subsection, we construct scenarios of selection effect for death and deterioration probabilities of RM borrowers. We assume that there is no selection effect for other disability transition probabilities and conditional ACH admission probabilities. Therefore, the selection effect for ACH admission probabilities of RM borrowers is determined only from the selection effect of deterioration probabilities.

Death probabilities

The RM products are traditionally perceived to be attractive for people who are healthy. This is due to the perception that those who benefit more from this product are those who will stay in their homes for a longer period of time. However, Davidoff and Welke (2007) found that in the U.S., the death probabilities of HECM borrowers are marginally higher than for the general population. In Australia, due to the absence of data, we do not know whether the death probabilities of RM borrowers are higher or lower than the death probabilities of the general population. As discussed above, both shorter and longer than expected duration of RM loan could result in an adverse financial impact to the lender with the adverse consequences of longer than expected duration of the loan being likely to be more severe...
than shorter than expected duration. Therefore, in constructing scenarios of selection effect, we consider both cases of lower (adverse selection) and higher (advantageous selection) death probabilities of RM borrowers as compared to the death probabilities of general population. In the following, we begin by describing the construction of adverse selection scenarios, followed by the construction of an advantageous selection scenario.

The adverse selection scenarios are constructed by following the suggestion of ERWP (2005) to adopt immediate annuitant mortality tables as suitable mortality tables for RM borrowers. This is due to the fact that both immediate annuity and RM products are voluntarily purchased and expose the provider to the risk of people living too long (ERWP, 2005). In addition, the adverse selection scenarios are also constructed by following the approach of Hosty et al. (2008) who adopted a pensioners’ mortality table due to the good fit between the socio-economic profile of lives buying equity release contracts and those buying pension annuities. However, in Australia, there are no mortality tables for either immediate annuitants or pensioners. Therefore, the death probabilities of immediate annuitants and pensioners in Australia are estimated from the experience from the U.K. and the U.S. This is done by firstly calculating, for the data from each of the U.K. and the U.S., the ratios of death probabilities of each of immediate annuitants and pensioners to the death probabilities of the general population. These ratios are then applied to the death probabilities of the Australian general population to estimate the death probabilities of immediate annuitants and pensioners in Australia. In the following, these ratios are referred to as the “selection” factors. In the U.K., suitable mortality tables for immediate annuitants and pensioners are IM/FL00 (i.e. IML00 for males and IFL00 for females) and PNM/FA00 tables respectively (the details of these mortality tables are described in Continuous Mortality Investigation (CMI) (2009)), while in the U.S., suitable mortality tables are Annuity 2000 (Mortality) (Johansen, 1995) and RP-2000 (Combined Healthy) (Society of Actuaries (SOA), 2000) tables respectively. The death probabilities of the general population in the U.K. and the U.S. are obtained from the English Life Table (ELT) No. 16 (Office for National Statistics (ONS), 2009) (with a suitable
adjustment to reflect the differences of applicable year between ELT 16 and 00-series mortality tables (i.e. IM/FL00 and PNM/FA00 tables)) and the U.S. life tables for 2000 (Arias, 2002) respectively.

The advantageous selection scenario is constructed from Davidoff and Welke (2007) who found that, averaging across all ages, male and female borrowers of HECM loans are 2.9% and 1.4% likelier to die than males and females respectively in the general population in the U.S.. In our advantageous selection scenario, these ratios are assumed to be applicable to all ages.

It might be desirable to adjust the estimated “selection” factors (i.e. the multiplication factors for the death probabilities of the general population to take into account the selection effect) to 2009 to align them with our disability transition probabilities in 2009. However, due to the limited amount of data, this adjustment is a speculative exercise. Therefore, we chose not to adjust the estimated “selection” factors and simply assumed that they are also applicable in 2009.

Table 5.1 presents the scenarios of selection effect for death probabilities of RM borrowers, while Figure 5.2 presents the “selection” factors across ages in each selection effect scenario for males and females. For males, the “selection” factors estimated from IML00 and RP-2000 are higher than 1 at ages (around) 88 and above. This feature, which is counter-intuitive, might be a result of the way these tables were graduated given the possible lack of credible experience at these very high ages. Therefore, in these cases, we chose to cap the “selection” factors at 1.

In a given scenario, the “selection” factors are assumed to be applicable to the death probabilities of all disability categories.

**Deterioration of disability probabilities**

There are no data to estimate the selection effect for deterioration probabilities of RM borrowers. Therefore, the selection effect for these probabilities is assumed to be the same as the selection effect for death probabilities of RM borrowers (i.e. the scenarios of selection effect for deterioration probabilities
Table 5.1: Scenarios of selection effect for death probabilities of RM borrowers.

<table>
<thead>
<tr>
<th>Selection effect scenario</th>
<th>Estimated from</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mort.Select.1</td>
<td>IM/FL00 tables.</td>
</tr>
<tr>
<td>Mort.Select.2</td>
<td>PNM/FA00 tables.</td>
</tr>
<tr>
<td>Mort.Select.3</td>
<td>Annuity 2000 (Mortality) tables.</td>
</tr>
<tr>
<td>Mort.Select.4</td>
<td>RP-2000 (Combined Healthy) tables.</td>
</tr>
<tr>
<td>Mort.Select.5</td>
<td>Davidoff and Welke (2007) (i.e. constant “selection” factors of 102.9% and 101.4% across ages for males and females respectively).</td>
</tr>
</tbody>
</table>

are the same as the scenarios for death probabilities). Note that in the case where the RM borrowers are healthier (less healthy) than the general population, both their death and deterioration probabilities are lower (higher) than the general population. In a given scenario, the “selection” factors are assumed to be applicable to deterioration probabilities (to any worse disability categories) of all disability categories.
Figure 5.2: “Selection” factors in each selection effect scenario, males and females.
5.5.3 Scenarios of future death, deterioration and conditional ACH admission probabilities

In this subsection, we construct scenarios of future death, deterioration and conditional ACH admission probabilities of RM borrowers. The future stay (i.e. staying in the same disability category) and improvement of disability probabilities are determined only from the spill over effect of the progression of death and deterioration probabilities over time.

Death probabilities

In constructing scenarios of future death probabilities of RM borrowers, we assume that the percentage improvement of death probabilities of RM borrowers in any disability category is the same as the percentage improvement of death probabilities of the general population. This assumption is adopted due to the absence of mortality data for the disabled population and for the RM borrowers. Under this assumption, the scenarios of future improvement factors of death probabilities for RM borrowers can be constructed from the mortality experience of the general population. In the following we describe the construction of these scenarios.

The scenarios of future improvement factors of death probabilities of RM borrowers are constructed from the application of the Lee-Carter (LC) mortality rates forecasting model to the mortality data of the general population. In the Human Mortality Database (HMD), at the time when the work in this chapter started, the Australian mortality data were provided from 1921 to 2009. We find that there are minor differences between the current and “old” (i.e. the mortality data that we used for the analysis in Chapter 2) mortality data at ages 80 and above from 1990 onwards. This indicates that there might be a revision to the “old” mortality data. Therefore, for the analysis in this chapter, we fit the LC model to the current Australian mortality data. Two fitting period are considered: 1921–2009 and 1975–2009 to capture both the long-term trend and recent short-term trend of mortality. As in Chapter 2, we only consider ages 60 and above. The forecasts of mortality rates are calculated for period 2010 to 2085. In our current application, we do not
attempt to identify the most suitable type of LC model and simply choose model PLC(1) (which was chosen in Chapter 2) for forecasting. The forecasts of mortality rates are calculated under forecasting equation (2.14). Note that since there are only minor differences between the current and “old” mortality data, model PLC(1) is also likely to be the most suitable type of LC model in our current application. In our current application, the quality of fit of model PLC(1) is good and the resulting forecasts of mortality rates are reasonable. As in Chapter 2, the 95% prediction interval (PI) of future mortality rates (and hence one-year death probabilities) is estimated under a semi-parametric bootstrap. A variety of adjustments adopted in Chapter 2 are also adopted to obtain the final forecasts and 95% PI of future one-year death probabilities.

In addition to the future improvement factors obtained from the application of the LC model, we also consider the “25 year” and “100 year” future mortality improvement factors presented in the Australian Life Tables (ALT) 2005-07 (which are published by the Australian Government Actuary (AGA) (2009)) in our scenarios.

Table 5.2 presents the scenarios of future improvement factors for death probabilities of RM borrowers. The $\hat{q}_{x,t}^{\text{Fin}}$ and $\hat{q}_{x,t}^{\text{Fin} (\gamma) B}$ are as described in Chapter 2. In a given scenario, the future improvement factors are assumed to be applicable to the death probabilities of all disability categories.

In the following we compare the future improvement factors for death probabilities between the scenarios.

Denote:

$q_{x}^{\text{ABS} 08-10}$ : the mortality rate $q_{x}$ presented in Australian Life Tables for the period 2008–2010, published by the ABS (ABS, 2011b).

$q_{x,2009+s}^{05-07 “25”}$ : the forecast of $q_{x,2009+s}$ (for $s = 1, 2, \ldots$) calculated by applying the “25 year” improvement factors presented in ALT 2005-07 to $\{q_{x}^{\text{ABS} 08-10}\}$ (and their extrapolated values (for ages above 100)).

$q_{x,2009+s}^{05-07 “100”}$ : the forecast of $q_{x,2009+s}$ (for $s = 1, 2, \ldots$) calculated by applying the “100 year” improvement factors presented in ALT 2005-07 to $\{q_{x}^{\text{ABS} 08-10}\}$ (and their extrapolated values).
Table 5.2: Scenarios of future improvement factors for death probabilities of RM borrowers.

<table>
<thead>
<tr>
<th>Future improvement scenario</th>
<th>Estimated from</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mort Imp_1</td>
<td>{\hat{q}^{\text{Fin}}_{x, 2009+s}} (fitting period 1921–2009).</td>
</tr>
<tr>
<td>Mort Imp_2</td>
<td>{F^{\text{Fin}}<em>{x, 2009+s} (0.025) B {q^{\text{Fin}}</em>{x, 2009+s}}} (fitting period 1921–2009).</td>
</tr>
<tr>
<td>Mort Imp_3</td>
<td>{F^{\text{Fin}}<em>{x, 2009+s} (0.975) B {\hat{q}^{\text{Fin}}</em>{x, 2009+s}}} (fitting period 1921–2009).</td>
</tr>
<tr>
<td>Mort Imp_4</td>
<td>{\hat{q}^{\text{Fin}}_{x, 2009+s}} (fitting period 1975–2009).</td>
</tr>
<tr>
<td>Mort Imp_5</td>
<td>{F^{\text{Fin}}<em>{x, 2009+s} (0.025) B {q^{\text{Fin}}</em>{x, 2009+s}}} (fitting period 1975–2009).</td>
</tr>
<tr>
<td>Mort Imp_6</td>
<td>{F^{\text{Fin}}<em>{x, 2009+s} (0.975) B {q^{\text{Fin}}</em>{x, 2009+s}}} (fitting period 1975–2009).</td>
</tr>
<tr>
<td>Mort Imp_7</td>
<td>“100 year” improvement factors of ALT 2005-07.</td>
</tr>
</tbody>
</table>

Figure 5.3 presents the forecasts of $\tilde{e}_{60}$ in 2010 to 2085 calculated from $\{\hat{q}^{\text{Fin}}_{x, 2009+s}\}$ (denoted “Forecast” in the figure), $\{F^{\text{Fin}}_{x, 2009+s} (0.025) B \{q^{\text{Fin}}_{x, 2009+s}\}\}$ (“B (0.025)”), $\{F^{\text{Fin}}_{x, 2009+s} (0.975) B \{q^{\text{Fin}}_{x, 2009+s}\}\}$ (“B (0.975)”), $\{q^{\text{Fin}}_{x, 2009+s}\}$ (“05–07 “25” year”), $\{q^{\text{Fin}}_{x, 2009+s}\}$ (“05–07 “100” year”) for males and females in fitting periods 1921–2009 and 1975–2009. In this figure, $\{\hat{q}^{\text{Fin}}_{x, 2009+s}\}$, $\{F^{\text{Fin}}_{x, 2009+s} (0.025) B \{q^{\text{Fin}}_{x, 2009+s}\}\}$ and $\{F^{\text{Fin}}_{x, 2009+s} (0.975) B \{q^{\text{Fin}}_{x, 2009+s}\}\}$ are calculated by basing $\{q^{\text{ABS}}_{x, 2009+s}\}$, $\{q^{(0.025)}_{x, 2009+s}\}$ and $\{q^{(0.975)}_{x, 2009+s}\}$ on $\{q^{\text{ABS}}_{x, 08–10}\}$ (and their extrapolated values) under the approach described in Sections 2.6.6 and 2.7.4. Note that, in the next 40 years (which are important for the pricing of a RM contract), the forecasts from fitting period 1921–2009 are similar to the forecasts from the “100 year” improvement factors, while the forecasts from fitting period 1975–2009 are similar to the forecasts from the “25 year” improvement factors. In addition, $\{F^{\text{Fin}}_{x, 2009+s} (0.025) B \{q^{\text{Fin}}_{x, 2009+s}\}\}$ estimated from fitting period 1975–2009 result in the highest mortality improvement amongst the scenarios, while $\{F^{\text{Fin}}_{x, 2009+s} (0.975) B \{q^{\text{Fin}}_{x, 2009+s}\}\}$ estimated from fitting period 1921–2009 result in the lowest mortality improvement.
Deterioration of disability probabilities

We assume that the future improvement factors for deterioration probabilities are the same as the future improvement factors for death probabilities (i.e. the scenarios of future improvement factors for deterioration probabilities are the same as the scenarios for death probabilities). Note that in the case of an improvement (deterioration) of health, it is likely that both death and deterioration probabilities will decrease (increase). In a given scenario, the future improvement factors are assumed to be applicable to the deterioration probabilities (to any worse disability categories) of all disability categories.
Conditional ACH admission probabilities

The trend of ACH admission probabilities is determined by the trend of health factors (i.e. disability) and non-health factors (for example, access to informal support, changing preferences (towards at home or residential care) and expectations, level of government subsidy towards aged care service, etc). Note that the uncertainties of future health factors are captured by the scenarios of future disability transition probabilities. Given the definition of conditional ACH admission probabilities (Section 4.8), the trend of these probabilities depends on the trend of non-health factors. However, the future trend of these non-health factors is difficult to predict. Therefore, for sensitivity analysis, we choose to adopt simple scenarios of future 

\[
CP^{m,ACH}_{x+0.5,x+1.5}(t) \text{ (for } x = 60, 61, \ldots, 108 \text{ and } m = 0, 1, \ldots, 4) \text{ is } CP^{m,ACH}_{x+0.5,x+1.5} \text{ applicable in year } t \text{ with } CP^{m,ACH}_{x+0.5,x+1.5}(2009) \text{ being } CP^{m,ACH}_{x+0.5,x+1.5} \text{ estimated in Section 4.8.}
\]

Table 5.3: Scenarios of \(CP^{m,ACH}_{x+0.5,x+1.5}(t)\) for \(t = 2011, 2012, \ldots\)

<table>
<thead>
<tr>
<th>Future scenario</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACH_Fut_1</td>
<td>(CP^{m,ACH}<em>{x+0.5,x+1.5}(t) = CP^{m,ACH}</em>{x+0.5,x+1.5}(2009)).</td>
</tr>
<tr>
<td>ACH_Fut_2</td>
<td>(CP^{m,ACH}<em>{x+0.5,x+1.5}(t) = \frac{1}{2} \times CP^{m,ACH}</em>{x+0.5,x+1.5}(2009)).</td>
</tr>
<tr>
<td>ACH_Fut_3</td>
<td>(CP^{m,ACH}<em>{x+0.5,x+1.5}(t) = \min \left(1, 1.5 \times CP^{m,ACH}</em>{x+0.5,x+1.5}(2009)\right)).</td>
</tr>
</tbody>
</table>

5.5.4 Scenarios of voluntary prepayment probabilities

The publicly available data of voluntary prepayments of RMs in Australia are very limited. From the research reports by SEQUAL/Deloitte, we only know that the annual voluntary prepayment rate (i.e. total number of terminations due to voluntary prepayment over total number of outstanding loans) of RM is around 9% p.a. in Australia. Note that the estimation of voluntary prepayment probabilities is very important since in Australia, currently, most terminations of RMs are due to voluntary prepayment (Section 5.2.4). Therefore, voluntary prepayment probabilities are estimated from the experience of HECM loans in the U.S. where more detailed termination rate
5.5. SCENARIOS OF DECREMENTS OF TERMINATION PROBABILITIES

Data are available. In addition to the experience from the U.S., we also consider the voluntary prepayment rates assumed by Hosty et al. (2008) which are suitable for RMs in the U.K. market.

**Estimation from the U.S. experience**

Szymanoski et al. (2007) estimated one-year termination probabilities (due to death, ACH admission and voluntary prepayment of HECM borrowers) of HECM loans in the U.S. across policy years using the HECM data from 1989 to 2006. Two types of termination probabilities are estimated: termination probabilities including and excluding loan assignment to the U.S. HUD. In our estimation, we use the estimates of termination probabilities which exclude loan assignment to the U.S. HUD. From these estimates of (one-year) termination probabilities, we are able to estimate (one-year) voluntary prepayment probabilities of HECM loans across policy years. The voluntary prepayment probabilities of RMs in Australia are then assumed to be the same as the voluntary prepayment probabilities of HECM loans. However, firstly, we need to assess the reasonableness of this assumption.

Figure 5.4 presents the termination rates p.a. (due to any reason) of HECM loans in the U.S. across calendar years calculated from the data presented in McConaghy (2004). From these data, we are only able to calculate the termination rates (p.a.) until 1999. These termination rates of HECM loans are comparable with the termination rates of RM loans in Australia since both HECM in the U.S. and RM in Australia are dominated by variable interest rate type of loans and the age distribution of borrowers of the U.S. HECM and Australian RM seems to be similar (Section 5.2). Note that initially, the termination rates of HECM loans were very low. However, the termination rates were increasing over time. The termination rate in 1998 was close to 10% (which was close with the termination rate of RMs in Australia) although it fell to 5.68% in 1999.

Figure 5.5 presents the number of originations (p.a.) of HECM loans in the U.S. from 1989 to 2006 which is compiled from the data presented in Szymanoski et al. (2007). Note that in this period, the bulk of the loans
Figure 5.4: Termination rates (p.a.) of HECM loans in the U.S..

were originated from 2001 onwards. Therefore, the termination probabilities estimated in Szymanoski et al. (2007) are heavily influenced by the termination experience of the loans which were originated from 2001 onwards. From Figure 5.4, we note that the termination rates of HECM loans were increasing over time with the termination rates in years close to 2000 being close to the current termination rate of RMs in Australia. Note that due to the increasing number of loan originations over time and the decreasing proportion of surviving loans for the loans originated in earlier years (older loans), the older loans contribute less to the termination rate in a given year and, therefore, the termination rates reflect more of the experience of the loans which are more recently originated. However, this observation only indicates the possibility that the overall termination rate of HECM loans in 1989 to 2006 might be similar to the termination rate of RMs in Australia. In addition, it does not indicate the similarity of the termination rates at a sin-
gle age for each gender and marital status between the borrowers of HECM loans and RMs in Australia. Therefore, we are not able to conclude that the voluntary termination experience of HECM loans in 1989 to 2006 is similar to the termination experience of RMs in Australia. Nevertheless, given the limited amount of data, the approach to estimate the voluntary prepayment probabilities of RMs in Australia is very limited. In such a situation, we believe that our approach to estimate the voluntary prepayment probabilities of RMs in Australia from the experience of HECM loans in the U.S. is reasonable. An alternative approach is to simply make a subjective opinion on the level of voluntary prepayment probabilities in Australia (across policy years); however, we believe that this method will result in more inaccurate estimates than the estimation from the experience of HECM loans. In the following we describe the estimation of one-year voluntary termination probabilities of HECM loans.

Figure 5.5: Total number of HECM loan originations (p.a.) in the U.S.
Szymanoski et al. (2007) presented the estimated one-year termination probabilities of HECM loans for single males, single females and couples for age groups 64–66, 74–76 and 84–86 at origination of the loan. To estimate the voluntary prepayment probabilities of HECM loans, we firstly estimate the death and ACH admission probabilities of HECM borrowers. In estimating the death and ACH admission probabilities of HECM borrowers, we assume that the effective age for borrowers in a given age group is the middle age of the group. For example, for borrowers in age group 64–66 at origination of the loans, we assume they are effectively aged 65.5 exact at origination.

Szymanoski et al. (2007) estimated (one-year) termination probabilities from the experience of HECM loans from 1989 to 2006 and, therefore, these termination probabilities represent the overall experience of HECM loans in this period. We simply assume that the RP-2000 (Combined Healthy) (SOA, 2000) tables are suitable to estimate the death probabilities of HECM borrowers over this period. This is due to the difficulty in estimating the death probabilities of HECM borrowers over this period; we are unable to find any literature which estimates the death probabilities of HECM borrowers at a single age interval. Note that Hosty et al. (2008), based on the U.K. data, found a good fit between the socio-economic profiles of lives buying equity release contracts and those buying pension annuities. Since the RP-2000 tables are applicable in 2000, it might be desirable to adjust the death probabilities presented in these tables to reflect the differences in applicable year between the RP-2000 tables and the termination probabilities estimated by Szymanoski et al. (2007). However, it is difficult to determine the applicable year of termination probabilities estimated by Szymanoski et al. (2007). As discussed above, these termination probabilities are likely to be heavily influenced by the HECM loans which were originated from 2001 onwards. In addition, the sum of weighted years (weighted by the proportion of loan originations in a given year over total loan originations from 1989 to 2006) of HECM loan originations (from 1989 to 2006) is 2003. Although this information does not shed much light on the applicable year of these termination probabilities, it indicates that their applicable year appears to be close to 2000. In addition to the difficulty in determining the applicable year of these
termination probabilities, there are likely to be discrepancies between the death probabilities presented in the RP-2000 tables and the death probabilities of HECM borrowers over the period 1989 to 2006 with the direction of the discrepancies (over-estimate or under-estimate) being difficult to determine. Therefore, we decide not to adjust the death probabilities presented in the RP-2000 tables. In the face of these uncertainties, it is unclear whether any attempts to adjust the death probabilities presented in these tables will result in more accurate estimates of death probabilities of HECM borrowers.

The ACH admission probabilities of HECM borrowers have been investigated even less in the literature than their death probabilities. Therefore, to take into account termination due to ACH admission, we simply scale up the death probabilities presented in the RP-2000 tables with the scaling factors calculated from the disability transition probabilities and conditional ACH admission probabilities estimated in Chapter 4. The scaling factors are the ratios of termination probabilities due to death and ACH admission over termination probabilities due to death only. There might be differences in applicable year between the estimated disability transition and ACH admission probabilities (which are approximately applicable in 2001) and the termination probabilities estimated by Szymanoski et al. (2007). However, due to reasons similar to above, we choose not to adjust these probabilities (i.e. the disability transition and ACH admission probabilities). In addition, since these probabilities are estimated from Australian data, there is likely to be an additional estimation error.

For a couple, the termination probabilities due to death and ACH admission are estimated by assuming that the mortality, disability and ACH admission of the husband are probabilistically independent from the mortality, disability and ACH admission of the wife.

The voluntary prepayment probabilities of HECM loans are estimated by deducting the estimated death and ACH admission probabilities of HECM borrowers from the termination probabilities of HECM loans estimated by Szymanoski et al. (2007). The estimated voluntary prepayment probabilities are then smoothed to obtain the final estimates of voluntary prepayment probabilities. Figure 5.6 presents the original (i.e. the estimates obtained by
deducting the estimated death and ACH admission probabilities from the termination probabilities estimated by Szymanoski et al. (2007)) and smoothed estimates of voluntary prepayment probabilities for couples aged 74–76 at origination of the loans, while Figure 5.7 presents the smoothed estimates of these probabilities for male, female and couples aged 64–66, 74–76 and 84–86 at origination of the loans. The estimated voluntary prepayment probabilities generally behave as expected: initially increasing with policy year (this is because the borrowers who find that a RM is not suitable for them are likely to terminate during early durations of the loan) and then declining in later policy years (this is because the incentive to terminate the RM is diminishing with the duration of the loan in later policy years). In addition, the voluntary prepayment probabilities generally decrease with age especially in later policy years, which is reasonable due to the competing decrements of mortality and disability, and the lower likelihood of older borrowers to move out from their homes due to reasons other than death or ACH admission.

The estimated voluntary prepayment probabilities for females at age groups 74–76 and 84–86 are peculiar: the estimated probabilities are similar at these age groups. This is likely due to the inaccuracy of the estimates of death and ACH admission probabilities of female HECM borrowers (over the period 1989 to 2006) at these age groups. Note that the data for the estimation are very limited. We could arbitrarily adjust the estimated probabilities to obtain a desirable pattern; however, it is unclear whether this will result in more accurate estimates. Therefore, we choose not to adjust the estimated probabilities. Due to their possible inaccuracy, these estimates of voluntary prepayment probabilities should be used with caution.

In the following, these voluntary prepayment probabilities are referred to as the U.S. voluntary prepayment probabilities.
Figure 5.6: The one-year voluntary prepayment probabilities estimated from Szymanoski et al. (2007), original and smoothed, couple aged 74–76 at origination of the loans.
Figure 5.7: The smoothed one-year voluntary prepayment probabilities estimated from Szymanoski et al. (2007), aged 64–66, 74–76 and 84–86 at origination of the loans.

Estimates adopted by Hosty et al. (2008)

Hosty et al. (2008), in analyzing RMs in the U.K. market, assumed a set of voluntary prepayment rates p.a. which are presented in Table 5.4. These rates are assumed to be applicable to all RM borrowers.

Note that the voluntary prepayment rates assumed by Hosty et al. (2008) appear to be too low for the current RM market in Australia. These voluntary prepayment rates are considered as a scenario of one-year voluntary prepayment probabilities of RMs in Australia when the market matures. This scenario of voluntary prepayment probabilities is applicable to all RM borrowers. As discussed in Hosty et al. (2008), as the RM products become more flexible and more accepted by borrowers, the voluntary prepayment
Table 5.4: Voluntary prepayment rates (p.a.) assumed by Hosty et al. (2008).

<table>
<thead>
<tr>
<th>Policy year</th>
<th>Voluntary prepayment rates p.a.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1–2</td>
<td>1.00%</td>
</tr>
<tr>
<td>3</td>
<td>2.00%</td>
</tr>
<tr>
<td>4–5</td>
<td>2.50%</td>
</tr>
<tr>
<td>6–8</td>
<td>2.00%</td>
</tr>
<tr>
<td>9–10</td>
<td>1.00%</td>
</tr>
<tr>
<td>11–20</td>
<td>0.50%</td>
</tr>
<tr>
<td>20+</td>
<td>0.25%</td>
</tr>
</tbody>
</table>

rates are likely to decrease. Note that in assessing the risk of NNEG and loss, it is important to consider the possibility that the voluntary prepayment probabilities will decrease to a low level.

In the following, these voluntary prepayment probabilities are referred to as the U.K. voluntary prepayment probabilities.

5.6 Pricing

Table 5.5 presents the base case scenario for pricing analysis of RM contracts. In the base case scenario, we assume the absence of selection effect for death and deterioration probabilities, and the absence of improvement for deterioration probabilities of RM borrowers. For couple borrowers, we assume that both the husband and the wife are aged 60 at origination of the contract. For pricing under the U.S. voluntary prepayment probabilities, we adopt the voluntary prepayment probabilities applicable for borrowers aged 64–66 at origination of the contracts. For pricing analysis, we perform $10^6$ simulations under the house price inflation linked LIBOR market model to obtain $\{LIB^n_t, HPI^n_t\}$ for $t = 1, 2, \ldots, T$ and $n = 1, 2, \ldots, 10^6$. 
Table 5.5: Base case scenario for pricing analysis of RM contracts.

<table>
<thead>
<tr>
<th>Age at origination</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loan to value ratio (LVR)</td>
<td>0.15</td>
</tr>
<tr>
<td>House value</td>
<td>$500,000</td>
</tr>
<tr>
<td>Spread (s)</td>
<td>3%</td>
</tr>
<tr>
<td>Termination scenario</td>
<td>Mort_imp.8</td>
</tr>
<tr>
<td>Improvement of death probabilities</td>
<td>ACH_Fut_1</td>
</tr>
<tr>
<td>Conditional ACH admission probabilities</td>
<td></td>
</tr>
</tbody>
</table>

5.6.1 Simulation results from the house price inflation linked LIBOR market model

The sample trajectories of house values, accumulated loan and accumulated funding cost presented in Figure 5.1 are the average of simulated values of the corresponding variables under the house price inflation linked LIBOR market model under the base case scenario.

Figure 5.8 presents the EPV without taking into account the termination probabilities of \( CI_t, NNEG_t \) and \( Loss_t \) across policy years under the base case scenario. Note that without taking into account the termination probabilities, the EPV of \( CI_t \) is given by \( \frac{1}{10^6} \sum_{n=1}^{10^6} \left( CI_n^t \times \prod_{j=1}^t \left( 1 + LIB^t_j \right)^{-1} \right) \), and similarly for the EPV of \( NNEG_t \) and \( Loss_t \).

The EPV of \( CI_t \) is initially increasing with policy year. However, due to the NNEG, the rate of increase is decreasing from policy year 23 and it is decreasing from policy year 37. Note that on average, early termination of the RM loan results in a low present value of cash inflow. In addition, it is not optimal for a RM loan to terminate at a later time than policy year 36 since, on average, after policy year 36, a longer duration of the loan results in a lower present value of cash inflow. The EPV of \( NNEG_t \) and \( Loss_t \) are insignificant in early policy years; however, they are increasing very rapidly from (around) policy year 31. Note that the EPV of \( Loss_t \) is significantly lower than the EPV of \( NNEG_t \) in later policy years.
5.6. PRICING

Figure 5.8: EPV (without taking into account the termination probabilities) of $CI_t$, $NNEG_t$ and $Loss_t$, base case scenario.

5.6.2 Pricing results

The termination probabilities of RM contracts are estimated by firstly estimating the termination probabilities due to death and ACH admission using the estimated disability transition probabilities and conditional ACH admission probabilities as described in Section 4.9. We assume that the severity of disablement of the RM borrowers is unknown and, hence, the termination probabilities are estimated by assuming that the disability prevalence rates at origination age (i.e. age at origination of the RM contract) of RM borrowers are the same as the disability prevalence rates of the community population at the same age in the 2003 SDAC. The final termination probabilities are then estimated by adding the estimated voluntary prepayment probabilities to the estimated termination probabilities due to death and ACH admission.
Figure 5.9 presents the EPV of cash inflows (i.e. $p(t) \times \frac{1}{10^6} \sum_{n=1}^{10^6} \left( CI_i^n \times \prod_{j=1}^{n} \left( 1 + LIB_j^n \right)^{-1} \right)$) across policy years for couple borrowers under the U.S. and the U.K. voluntary prepayment probabilities under the base case scenario. As discussed above, the U.S. voluntary prepayment probabilities represent the estimates for the current Australian RM market, while the U.K. voluntary prepayment probabilities represent a scenario of these probabilities when the Australian RM market matures. Note that the pattern of EPV of cash inflows is very different under the U.S. and the U.K. voluntary prepayment probabilities. Under the U.S. voluntary prepayment probabilities, significant amounts of cash inflows are expected in early policy years, while under the U.K. voluntary prepayment probabilities, most cash inflows are expected in later policy years. As a result, these scenarios of voluntary prepayment probabilities result in significantly different pricing results. Therefore, we present the pricing results under the U.S. and the U.K. voluntary prepayment probabilities separately.

Table 5.6 presents the pricing results for single males, single females and couple borrowers under the U.S. and the U.K. voluntary prepayment probabilities under the base case scenario. The pricing results under the column headings U.S. and U.K. are the pricing results under the U.S. and the U.K. voluntary prepayment probabilities respectively. The numbers for EPV_P, EPV_NNEG and EPV_Loss are presented as a proportion of $L_0$. The ratio of EPV_P to $L_0$ is referred to as the expected return on capital. In the following we present observations from this table.

Note that there is significant variability in $p(NNEG)$, EPV_NNEG, $p(Loss)$ and EPV_Loss amongst different gender and marital status of the borrowers, and different scenarios of voluntary prepayment probabilities. For measurement of risks, $p(Loss)$ and EPV_Loss are more relevant than the $p(NNEG)$ and EPV_NNEG since the NNEG cost only represents the profit foregone due to the NNEG. Note that in the case where the crossover point is reached (i.e. the accumulated loan exceeds the house value), the lender still gains a profit if, at termination of the RM contract, the cash inflow is greater than the accumulated funding cost (note that we assume the absence of any costs other than the funding cost). To measure the riskiness of
Figure 5.9: EPV of $C_I$ for couple borrowers under the U.S. and the U.K. voluntary prepayment probabilities, base case scenario.

RM contracts, the insufficiency of cash inflow at termination of the contract to cover the accumulated funding cost, which is measured by the amount of loss, is more relevant than the profit foregone due to the NNEG. Note that although there is significant variability in $p(\text{Loss})$ and $EPV_{\text{Loss}}$, in all cases, the values of these risk measures are small. Therefore, assuming that the funding cost is accumulated at the LIBOR rate, the risk and severity of loss are expected to be small. On the other hand, the liquidity risk (measured by $E_{\text{DPP}}$) is quite significant in all cases. Under the base case scenario, the lender of a RM will have their capital tied up, on average, for a period between 14 to 28 years, depending on the gender and marital status of the borrowers. Therefore, liquidity management is very important for the lender of a RM. Lastly, although in general, a higher liquidity risk results in higher expected profit, NNEG cost and loss amount, the relationship be-
Table 5.6: Pricing results for single male, single female and couple borrowers under the U.S. and the U.K. voluntary prepayment probabilities, base case scenario.

<table>
<thead>
<tr>
<th></th>
<th>Single males</th>
<th></th>
<th>Single females</th>
<th></th>
<th>Couples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$EPV_{Profit}/L_0$</td>
<td>33.76%</td>
<td>74.78%</td>
<td>37.67%</td>
<td>77.02%</td>
<td>45.50%</td>
</tr>
<tr>
<td>$E_{DPP}$</td>
<td>14</td>
<td>25</td>
<td>19</td>
<td>25</td>
<td>24</td>
</tr>
<tr>
<td>$p(NNEG)$</td>
<td>0.0232</td>
<td>0.0904</td>
<td>0.0256</td>
<td>0.0818</td>
<td>0.0521</td>
</tr>
<tr>
<td>$EPV_{NNEG}/L_0$</td>
<td>2.03%</td>
<td>7.82%</td>
<td>1.99%</td>
<td>6.32%</td>
<td>4.51%</td>
</tr>
<tr>
<td>$p(Loss)$</td>
<td>0.0035</td>
<td>0.0137</td>
<td>0.0036</td>
<td>0.0116</td>
<td>0.0079</td>
</tr>
<tr>
<td>$EPV_{Loss}/L_0$</td>
<td>0.14%</td>
<td>0.55%</td>
<td>0.14%</td>
<td>0.43%</td>
<td>0.32%</td>
</tr>
</tbody>
</table>

Between these measures is not straightforward. For example, under the U.K. voluntary prepayment probabilities, although the $E_{DPP}$ of single male borrowers and single female borrowers are the same, $EPV_{Profit}$ is higher for females, while $EPV_{NNEG}$ and $EPV_{Loss}$ are higher for males. This is because of differences in importance of $p(t)$ in different policy years between these measures. Specifically, $E_{DPP}$ significantly depends on $p(t)$ in early policy years (since it represents the expected time required to recoup the initial capital ($L_0$) and, therefore, it significantly depends on cash inflows in early policy years), $EPV_{NNEG}$ and $EPV_{Loss}$ significantly depend on $p(t)$ in later policy years (since without taking into account the termination probabilities, the EPV of $NNEG_t$ and $Loss_t$ are significant only in later policy years (Figure 5.8)) and $EPV_{Profit}$ depends on the overall pattern of $p(t)$ across policy years (since it depends on the total expected cash inflows across policy years). The dependence of $EPV_{Profit}$, $EPV_{NNEG}$ and $EPV_{Loss}$ on $p(t)$ is illustrated in Figures 5.10, 5.11 and 5.12. Figure 5.10 presents $p(t)$ for males and females under the U.S. and the U.K. voluntary prepayment probabilities, while Figures 5.11 and 5.12 present the EPV of $CI_t$, $NNEG_t$ and $Loss_t$ for males and females under these scenarios of voluntary prepayment probabilities respectively. Note that the pattern of $p(t)$ and EPV of $CI_t$ are similar which indicates that $EPV_{Profit}$ (which is the sum of EPV of $CI_t$ across policy years) depends on the pattern of $p(t)$ across policy years. The dependence of $EPV_{NNEG}$ and $EPV_{Loss}$ on $p(t)$ is
In the following we discuss the differences in pricing results under different scenarios of voluntary prepayment probabilities, gender and marital status of the borrowers.

The liquidity, NNEG and loss risks, and expected amount of NNEG cost and loss are significantly lower under the U.S. than under the U.K. voluntary prepayment probabilities. However, the expected profit is also significantly lower under the U.S. voluntary prepayment probabilities. Note that \( EPV_{NNEG} \) and \( EPV_{Loss} \) under the U.S. voluntary prepayment probabilities are around a third of those under the U.K voluntary prepayment probabilities, while \( EPV_{Profit} \) is only around a half. This is because the U.S. voluntary prepayment probabilities are significantly higher than the U.K voluntary prepayment probabilities. As discussed above, although shorter loan duration results in lower liquidity, NNEG and loss risks, and lower expected amount of NNEG cost and loss, it also results in a lower expected profit (since the interest is earned in a shorter period of time).

The fact that the pricing results depend significantly on the assumed voluntary prepayment probabilities is, although expected (since currently, a significant proportion of RM terminations in Australia are due to voluntary prepayment), is unfortunate. Due to the limited amount of data, it is
Figure 5.11: EPV of $CI_t$, $NNEG_t$ and $Loss_t$ for single male and single female borrowers under the U.S. voluntary prepayment probabilities, base case scenario.

difficult to estimate the voluntary prepayment probabilities accurately. In addition, there is likely to be a significant variability of voluntary prepayment experience amongst the lenders (due to differences in product design and distribution method) and in different economic circumstances (Hosty et al., 2008). Note that our conservative estimate of expected profit is given under the U.S. voluntary prepayment probabilities, while our conservative estimates of liquidity, NNEG and loss risks, and expected amount of NNEG cost and loss are given under the U.K. voluntary prepayment probabilities.

The liquidity, NNEG and loss risks, and expected amount of NNEG cost and loss are higher for couple borrowers than for single borrowers. However, the expected profit for couple borrowers is also higher. This is because the
termination probabilities of the RM loan are lower for couple borrowers than for single borrowers.

The expected amount of NNEG cost and loss are higher for males than females under both scenarios of voluntary prepayment probabilities (although under the U.S. voluntary prepayment probabilities, the differences between genders are very small). The reason for this is discussed next.

From Figure 5.10, we note that under the U.S. voluntary prepayment probabilities, \( p(t) \) values for males are slightly higher than for females in policy years above 35, while under the U.K. voluntary prepayment probabilities, \( p(t) \) values are higher for males in policy years above 32. The reason for the very small differences in \( p(t) \) between genders in policy years above
under the U.S. voluntary prepayment probabilities is because these voluntary prepayment probabilities are very high which results in low values of \( p(t) \) in these later policy years. From Figure 5.8, we note that, without taking into account the termination probabilities, the EPV of \( NNEG_t \) and \( Loss_t \) are significant only from policy years above 31. Since in these policy years, overall, \( p(t) \) values for males are higher than those for females under both scenarios of voluntary prepayment probabilities, the expected amount of NNEG cost and loss are higher for males (with the differences under the U.S. voluntary prepayment probabilities being very small, since the differences in \( p(t) \) between genders are very small (in these policy years) under these voluntary prepayment probabilities).

The reason that \( p(t) \) values in later policy years are higher for males (than females) is because, under the base case scenario, the one-year termination probabilities (i.e. the probabilities that the RM contract terminates within one year given that the contract is still active) due to death and ACH admission are higher for females in policy years above 20 (when the borrowers are aged above 80). This is due to several reasons. Firstly, the ACH admission probabilities, which are significant at old ages, are higher for females than males. Secondly, the future mortality improvement factors under scenario Mort_Imp_8 (which are derived by extrapolating the historical trends in mortality improvement over the 25-year period ending in 2006 (AGA, 2009)) result in higher mortality improvement for males than females. Lastly, the one-year termination probabilities (due to death and ACH admission) are heavily influenced by the one-year death probabilities of mild disability categories which are higher for females (than males) at young ages. This is because the assumed prevalence rates of mild disability categories for RM borrowers at age 60 (i.e. the origination age under the base case scenario) are quite high. Note that the disability prevalence rates of RM borrowers at origination age are assumed to be the same as the disability prevalence rates of the community population at the same age in the 2003 SDAC, and at age 60, the prevalence rates of mild disability categories of the community population in the 2003 SDAC are quite high.
5.6. Pricing

5.6.3 Sensitivity analysis

Sensitivity to spread, LVR and origination age

Tables 5.7 and 5.8 present the pricing results under a variety of spread ($s$), LVR and origination age for couple borrowers under the U.S. and the U.K. voluntary prepayment probabilities under the base case scenario. In these tables, the column and sub-column headings represent the variable which is being varied. For example, under the column heading ‘spread’ and sub-column heading ‘2%’, the pricing results are presented under a spread assumption of 2% while the other pricing variables are as in the base case scenario. For pricing under the U.S. voluntary prepayment probabilities, for borrowers aged 75 at origination of the RM contract, we adopt the U.S. voluntary prepayment probabilities applicable for borrowers aged 74–76 at origination, while for borrowers aged 85 at origination, the U.S. voluntary prepayment probabilities applicable for borrowers aged 84–86 at origination are adopted. In the following we present observations from these tables.

Table 5.7: Pricing results under a variety of spread, LVR and origination age, couple borrowers, U.S. voluntary prepayment probabilities, base case scenario.

<table>
<thead>
<tr>
<th>Spread</th>
<th>LVR</th>
<th>Origination age</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2%</td>
<td>4%</td>
</tr>
<tr>
<td>$EPV_{Profit}/L_0$</td>
<td>28.48%</td>
<td>63.17%</td>
</tr>
<tr>
<td>$E_{DPP}$</td>
<td>27</td>
<td>21</td>
</tr>
<tr>
<td>$p(NNEG)$</td>
<td>0.0269</td>
<td>0.0977</td>
</tr>
<tr>
<td>$EPV_{NNEG}/L_0$</td>
<td>1.79%</td>
<td>11.37%</td>
</tr>
<tr>
<td>$p(Loss)$</td>
<td>0.0079</td>
<td>0.0079</td>
</tr>
<tr>
<td>$EPV_{Loss}/L_0$</td>
<td>0.32%</td>
<td>0.32%</td>
</tr>
</tbody>
</table>

Note that the sensitivity of pricing results to spread, LVR and origination age is significantly higher under the U.K. voluntary prepayment probabilities than under the U.S. voluntary prepayment probabilities (with the exception of the sensitivity of $E_{DPP}$ to spread).

A higher spread results in higher expected profit and lower liquidity risk. In addition, the risk and expected amount of loss are not affected by the
variability of spread. Note that although a higher spread results in higher risk and expected amount of NNEG cost, as discussed above, NNEG cost only represents the profit foregone due to the NNEG and, therefore, for measurement of risk, the risk and expected amount of loss are more relevant. Therefore, a higher spread is beneficial for the lender. However, the ability of the lender to increase the spread is limited by the competition in the market.

A higher LVR results in higher risk and expected amount of NNEG cost and loss. Note that the liquidity risk (i.e. the $E_{DPP}$) is not sensitive to LVR (we find that even a significant increase in LVR results only in a small increase (1 or 2 years) in $E_{DPP}$). A higher LVR results in a higher expected profit, however, the expected profit as a proportion of initial capital ($L_0$) is decreasing with LVR. Therefore, it is more beneficial for the lender to issue two identical RM loans to two identical borrowers than to issue a single RM loan with an LVR twice of the original LVR (note that we assume the absence of any costs other than the funding cost; other costs, for example, origination cost, might result in a change to this finding). In addition, at very high LVR (for example, under the U.K. voluntary prepayment probabilities, an LVR of 25%), an increase in LVR instead results in a lower expected profit.

A higher origination age results in lower liquidity, NNEG and loss risks, and a lower expected amount of NNEG cost and loss. However, a higher origination age also results in a lower expected profit. This is because the termination probabilities of RM contracts for older borrowers are higher than
for younger borrowers.

**Sensitivity to future death, disability transition, conditional ACH admission probabilities and selection effect**

In this section we analyse the sensitivity of pricing results for couple borrowers under the base case scenario to future death, disability transition, conditional ACH admission probabilities and selection effect. For this purpose, we construct 5 sets of scenarios of termination probabilities and 2 special scenarios which are described in the following.

A. This set of scenarios is to analyse the sensitivity of pricing results (for couple borrowers under the base case scenario) to the future improvement factors of death probabilities. This set of scenarios consists of scenarios 1 to 8 where in scenario 1 we apply the future improvement factors in scenario Mort.Imp.1 to the death probabilities of each disability category, in scenario 2 we apply the future improvement factors in scenario Mort.Imp.2, and similarly for scenarios 3, 4, ..., 8.

B. This set of scenarios is to analyse the sensitivity of pricing results to the future improvement factors of death and deterioration probabilities. This set of scenarios consists of scenarios 9 to 16 where in scenario 9 we apply the future improvement factors in scenario Mort.Imp.1 to the death and deterioration probabilities of each disability category, in scenario 10 we apply the future improvement factors in scenario Mort.Imp.2, and similarly for scenarios 11, 12, ..., 16.

C. This set of scenarios is to analyse the sensitivity of pricing results to the selection effect of death probabilities. This set of scenarios consists of scenarios 17 to 21 where in scenario 17 we apply the “selection” factors in scenario Mort.Select.1 to the death probabilities of each disability category, in scenario 18 we apply the “selection” factors in scenario Mort.Select.2, and similarly for scenarios 19, 20 and 21.

D. This set of scenarios is to analyse the sensitivity of pricing results to the selection effect of death and deterioration probabilities. This set of
scenarios consists of scenarios 22 to 26 where in scenario 22 we apply the “selection” factors in scenario Mort.Select.1 to the death and deterioration probabilities of each disability category, in scenario 23 we apply the “selection” factors in scenario Mort.Select.2, and similarly for scenarios 24, 25 and 26.

E. This set of scenarios is to analyse the sensitivity of pricing results to the future conditional ACH admission probabilities. This set of scenarios consists of scenarios 27 to 29 where in scenario 27 the future conditional ACH admission probabilities are as in scenario ACH_Fut.1, in scenario 28 the future conditional ACH admission probabilities are as in scenario ACH_Fut.2, and similarly for scenario 29.

s1. This is a scenario of lowest termination probabilities. This scenario is constructed by applying the future improvement factors in scenario Mort.Imp.5 (the scenario of highest future improvement) and the selection factors in scenario Mort.Select.3 (the scenario of highest adverse selection effect) to death and deterioration probabilities of each disability category, and future conditional ACH admission probabilities in this scenario are as in scenario ACH_Fut.2. Note that this scenario is a stress testing scenario for liquidity, NNEG and loss risks, and expected amount of NNEG cost and loss.

s2. This is a scenario of highest termination probabilities. This scenario is constructed by applying the future improvement factors in scenario Mort.Imp.3 (the scenario of lowest future improvement) and the selection factors in scenario Mort.Select.5 (the scenario of an advantageous selection effect) to death and deterioration probabilities of each disability category, and future conditional ACH admission probabilities in this scenario are as in scenario ACH_Fut.3. The future improvement factors under scenario Mort.Imp.3 are higher than one (which indicates deterioration) in the majority of ages and, therefore, in this scenario (s2), these improvement factors are also applied to the deterioration probabilities. Note that this scenario is a stress testing scenario for
expected profit.

In a given scenario, the residual disability transition probabilities (i.e. the disability transition probabilities which are not subject to improvement or selection effect) in a given projection year are calculated by applying the ratio of the sum of these residual disability transition probabilities in the corresponding projection year to the sum of these residual transition probabilities in 2009 to the corresponding residual transition probabilities in 2009. For example, in scenario 9, where the death and deterioration probabilities are subject to an improvement, the stay and improvement probabilities in the projection years are calculated according to (for \( x = 60, 61, \ldots, 108 \) and \( t = 2011, 2012, \ldots \)):

\[
\hat{P}_{m,n}^{x+0.5}(t) = \frac{1 - \sum_{k=m+1}^{5} \hat{P}_{m,k}^{x+0.5}(t)}{1 - \sum_{k=m+1}^{5} \hat{P}_{m,k}^{x+0.5}(2009)} \quad 0 \leq n \leq m < 5.
\]

Note our approach in projecting the residual disability transition probabilities retains the sum of disability transition probabilities from a given disability category at 1.

Tables 5.9 and 5.10 present the minimum and maximum values of the pricing results under scenario sets A, B, \ldots, E, and the pricing results under scenarios s1 and s2 under the U.S. and the U.K. voluntary prepayment probabilities respectively. In these tables, the column heading represents the scenario set (or scenario) which is being considered. In the following we present observations from these tables.

Note that the sensitivity of pricing results to death, disability transition, conditional ACH admission probabilities and selection effect is higher under the U.K. voluntary prepayment probabilities than under the U.S. voluntary prepayment probabilities.

Under the U.S. voluntary prepayment probabilities, the expected profit of RM contracts is not sensitive to the variability of future death, deterioration, conditional ACH admission probabilities and selection effect of death and deterioration probabilities (these are referred to as the termination decrements). Note that under scenario s2 (scenario of highest termination probabilities),
the expected return on capital is only 2.4% lower than under the base case scenario (note that the pricing results under the base case scenario is given in Table 5.6). The risk and expected amount of NNEG cost are moderately sensitive to the variability of each of the termination decrements; however, as discussed above, for measurement of risk, the risk and expected amount of loss are more relevant. Although the risk and expected amount of loss are moderately sensitive to the variability of each of the termination decrements (and very sensitive to the combined variability of the termination decrements), the expected amount of loss is small relative to the expected amount of profit. Note that even under scenario s1 (scenario of lowest termination probabilities), \( EPV_{Loss} \) is small compared to \( EPV_{Profit} \) under the base case scenario. Of more concern is the fact that the combined adverse movement of the termination decrements results in a significant increase in

Table 5.9: Pricing results under different sets of scenarios of termination probabilities, couple borrowers, U.S. voluntary prepayment probabilities, base case scenario.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
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Table 5.10: Pricing results under different sets of scenarios of termination probabilities, couple borrowers, U.K. voluntary prepayment probabilities, base case scenario.

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<td>16.96%</td>
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<tr>
<td><em>p (Loss)</em></td>
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<tr>
<td>Minimum</td>
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<td>0.65%</td>
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<td>1.12%</td>
<td>1.96%</td>
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</tbody>
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liquidity risk. Note that the $E_DPP$ under scenario s1 is significantly higher than under the base case scenario.

Under the U.K. voluntary prepayment probabilities, the expected profit and liquidity risk are not sensitive to the variability of each the termination decrements. However, these measures are sensitive to the combined variability of the termination decrements. Note that the combined adverse movement of the termination decrements results in a significant increase in liquidity risk (the value of $E_DPP$ under scenario s1 is significantly higher than under the base case scenario). The risk and expected amount of NNEG cost and loss are sensitive to the variability of each of the termination decrements (and very sensitive to the combined variability of the termination decrements). However, even under scenario s1, the expected amount of loss is small relative to the expected amount of profit under the base case scenario.
5.7 Modelling limitations

House price model

The most significant modelling limitation is due to the fact that we model and simulate the house price index, not the price of the underlying house at which the RM loan is borrowed against. The future returns on the price of the underlying house are then assumed to be the same as the returns on house price index. However, there is likely to be discrepancy between the future returns on the price of the underlying house and on house price index (basis risk). Although the basis risk can be reduced by holding a large number of RM loans across different areas, a certain degree of basis risk still remains.

Estimation of termination probabilities

The main limitation in estimating the termination probabilities is due to the fact that, due to the limited amount of data, it is difficult to estimate the voluntary prepayment probabilities accurately. In addition, the voluntary prepayment experience is likely to vary significantly amongst the lenders and in different economic circumstances. As discussed above, currently, voluntary prepayment is the most significant termination decrement in the Australian RM market. In our analysis, we consider two scenarios of voluntary prepayment probabilities, where one scenario represents the estimates of these probabilities for the current Australian RM market, while the other scenario represents the estimates when the Australian RM market matures. However, as discussed above, there is a considerable uncertainty in our estimation.

Other limitation

In our modelling, we only consider the funding cost. In practice, there are many other costs that the lender needs to consider (e.g. administrative cost, marketing cost, distribution cost, legal cost, etc). Therefore, we are likely to overestimate the expected profit and underestimate the liquidity, NNEG (assuming that the cash inflow at the termination time is the minimum between the accumulated loan and house value net of selling cost) and loss risks, and
expected amount of NNEG cost and loss.

5.8 Conclusion

In this chapter we present a methodology for the measurement of profitability, liquidity, NNEG and loss risks for RMs in Australia. The future prices of the underlying house and the funding cost rates (which are assumed to be the same as the LIBOR rates) are simulated under the house price inflation linked LIBOR market model (by assuming that the returns on the price of the underlying house are the same as the returns on a house price index) while the termination probabilities are estimated under the multi-state model to take into account a variety of modes of termination. The differences in the pricing results between gender and marital status of the borrowers are analysed. The sensitivity of pricing results for couple borrowers to the spread, LVR, origination age and termination decrements are investigated.

There are uncertainties in our estimation which are, in particular, due the basis risk and the difficulty in estimating the voluntary prepayment probabilities. Nevertheless, several informative observations are obtained, in particular, the fact that the liquidity risk is significant and the loss risk is insignificant. However, in our analysis, we ignore costs other than the funding cost and, therefore, these risks are likely to be underestimated.

The adopted modelling methodology is easily extended to incorporate a variety of costs that the lender of RMs faces. In addition, it is straightforward to modify the methodology to make it suitable for pricing other types of RM products (for example, a RM where the loan is accessed as an income stream or as a line of credit) or other equity release products (i.e. share appreciation mortgage, home reversion scheme, etc).

We believe that there is scope for future research on the measurement and management of basis risk and on the estimation of voluntary prepayment probabilities. On the issue of basis risk, the effectiveness of the diversification of RM portfolios in reducing the basis risk might be analysed. In addition, a variety of mechanisms (for example, securitization) might be investigated for their feasibility in reducing or transferring the basis risk. Future research
on measurement of voluntary prepayment probabilities might begin with the collection of a finer voluntary termination data of RM in Australia. Using these data, more accurate voluntary termination probabilities might be estimated and the relationship between these probabilities and the borrowers’ characteristics (i.e. gender, marital status and origination age) or product features (i.e. LVR, spread, early prepayment penalty, etc) might be investigated. Although the voluntary prepayment experience of the lender might be different from the experience of the market, accurate estimates of the market’s voluntary prepayment experience are useful as a starting point for the lender in estimating the voluntary prepayment probabilities which are suitable for their particular circumstances.
Chapter 6

Conclusion

In Chapter 2, we investigated the suitability of the Gaussian and Poisson Lee-Carter (LC) models for forecasting mortality rates at ages 60 and above in Australia. We restricted our investigation to one-term and two-term variants of the LC model. We found that the two-term Poisson LC model (PLC(2)) was not suitable for our application, while the one-term Poisson LC model (PLC(1)) was preferable over the Gaussian LC models. This is because model PLC(1) resulted in more accurate weighting than the Gaussian LC models. In addition, the heteroscedasticity of the random variables for the number of deaths under model PLC(1) was more suitable for a semi-parametric bootstrap (to estimate the 95% prediction intervals (PIs) of future mortality rates) than the homoscedastic error structure of the Gaussian LC models. The forecasts of mortality rates were then calculated under model PLC(1) from two fitting periods, 1921–2004 and 1975–2004, for the purpose of capturing the long-term trend and the recent short-term trend of mortality. The resulting forecasts were found to be similar with the forecasts calculated from the future mortality improvement factors presented in the Australian life tables (ALT) 2000-02 published by the Australian Government Actuary (AGA) (AGA, 2004). In this chapter, we also estimated the 95% PIs of future mortality rates under a semi-parametric bootstrap. As compared to the 95% PIs of future mortality rates estimated under the traditional method, where only the uncertainty of future mortality indexes was considered, the 95% PIs esti-
mated under a semi-parametric bootstrap are wider only at ages (around) 90 and above. In addition, from the assessment of the out of sample forecasts, the 95% PIs estimated under a semi-parametric bootstrap were found to be too narrow. Nevertheless, the lower bounds of the estimated 95% PIs under a semi-parametric bootstrap from fitting period 1975–2004 and the upper bounds of the estimated 95% PIs from fitting period 1921–2004 constitute PIs which are wide enough to cover a range of possibilities of future mortality rates for risk management of a reverse mortgage (RM) contract.

Chapters 3 and 4 are concerned with the estimation of disability transition probabilities in Australia. In Chapter 3, we applied the multi-state model adopted by Rickayzen and Walsh (2002) and Leung (2004) to estimate the disability transition probabilities using the 2003 Survey of Disability, Ageing and Carers (SDAC) data. In Chapter 4, we presented a method to improve the accuracy of the disability transition probabilities estimated in Chapter 3. This method employs the Iterative Proportional Fitting (IPF) procedure and uses the disability transition probabilities estimated in Chapter 3 (initial transition probabilities) as an input of the estimation. Fundamental to the implementation of this method is the apparent stability of disability prevalence rates over the past two decades. Practically, this method involves projecting the disabled population from one year to the next year (using the initial transition probabilities) and adjusting the projection such that it satisfies the age structure of the disabled population at the following year. The results of the procedure are the estimates of one year disability longitudinal data. From these estimates of longitudinal data, the refined estimates of one year disability transition probabilities were calculated. Following that, we graduated the refined estimates of disability transition probabilities to obtain the final estimates of disability transition probabilities. There are uncertainties in our estimation which are mostly due to the limited amount of data and the possible inaccuracy of the initial transition probabilities. The available data are limited in several aspects. In particular, the disability prevalence rates are observed only in 1998 and 2003 and, therefore, there are likely to be differences between the actual and estimated disability prevalence rates in 1999 to 2002. In addition, the grouping of the very old into
a very wide age interval in the reported number of disabled population in
the SDAC results in the difficulty in the estimation of disabled population
at single ages for these ages. The obvious limitation of the initial transi-
tion probabilities is the restriction on the possible annual recovery from the
disabled states. In addition, due to the limited amount of data, there are
uncertainties in the setting of some of the assumptions in the implementa-
tion of the multi-state model to estimate these initial transition probabilities.
Despite these uncertainties, reasonable estimates were obtained with several
informative observations emerging. The method is flexible (easily adjusted
if desirable (for example, to suit better data)) and will produce increasingly
accurate results when better data become available. There are several ways
in which better data can be obtained. For example, a simple improvement to
the SDAC that might materially improve the accuracy of the estimation is
to present the estimate of disabled population at older age groups (say up to
age 105 and over). Better data can also be obtained by measuring CAL with
a similar scale as in the SDAC in regional disability longitudinal studies. If
such data were available, we would be able to obtain more accurate initial
transition probabilities (in the application of the IPF procedure) and, hence,
more accurate refined estimates of disability transition probabilities could be
obtained. Note that the availability of such data would circumvent many of
the modelling limitations associated with the estimation of initial transition
probabilities (for example, it would allow a fuller set of recovery processes in
the estimation). Furthermore, as a non-Markovian model can be constructed
for the estimation of initial transition probabilities, this might open the pos-
sibility of constructing a non-Markovian (or proxy non-Markovian) model in
the estimation of disability transition probabilities from national cross sec-
tional datasets. In Chapter 4, we also estimated the conditional probabilities
of admission into an ACH. The conditional ACH admission probabilities were
estimated from the final estimates of disability transition probabilities and
the ACH prevalence rates estimated from the 2003 SDAC data. Despite the
limited amount of data, the resulting estimates of ACH admission probabil-
ities are generally consistent with other studies.

In Chapter 5 we presented a methodology for measurement of crossover
and liquidity risks for a RM contract in Australia. We focused our analysis on the lump sum type of RM loan. The house prices and the interest rates were modelled and simulated under the house price inflation linked LIBOR market model, while the termination probabilities of RM contracts were estimated under a multi-state modelling framework to take into account a variety of modes of termination. The work in this chapter is part of a wider project on RMs and can only be viewed as a subset of this project. In this particular chapter, we used the work of other people. Specifically, we used a program (which was written in C++) which has been developed to simulate future house price index and LIBOR rates under the developed house price inflation linked LIBOR market model. The differences of the pricing results between gender and marital status of the borrowers were analysed. The sensitivity of pricing results for couple borrowers to the spread, loan to value ratio, origination age and termination decrements were also investigated. There are uncertainties in our estimation which are, in particular, due to the basis risk and the difficulty in estimating the termination probabilities. The basis risk arises because there are likely to be differences between future returns on prices of the underlying house (the house against which the RM loan is borrowed) and on the house price index. The main limitation in estimating the termination probabilities is due to the fact that, due to the limited amount of data, it is difficult to estimate the voluntary prepayment probabilities accurately. In addition, the voluntary prepayment experience is likely to vary significantly amongst the lenders and in different economic circumstances. This is unfortunate since, currently, voluntary prepayment is the most significant termination decrement in the Australian RM market. Despite these uncertainties, several informative observations were obtained, in particular, we found that the liquidity risk is significant and the loss risk is insignificant. However, in our analysis, we ignored costs other than the funding cost and, therefore, these risks are likely to be underestimated. The adopted modelling methodology is easily extended to incorporate a variety of costs that the lender of RM faces. In addition, it is straightforward to modify the methodology to make it suitable for pricing other types of RM products (for example, RM where the loan is accessed as an income stream or as a
line of credit) or other equity release products (i.e. share appreciation mortgage, home reversion scheme, etc). We believe that there is scope for future research on the measurement and management of basis risk and on the estimation of voluntary prepayment probabilities. On the issue of basis risk, the effectiveness of the diversification of RM portfolios in reducing the basis risk might be analysed. In addition, a variety of mechanisms (for example, securitization) might be investigated for their feasibility in reducing or transferring the basis risk. Future research on measurement of voluntary prepayment probabilities might begin with the collection of finer voluntary termination data of RM in Australia. Using this data, more accurate voluntary termination probabilities might be estimated and the relationship between these probabilities and the borrowers’ characteristics (i.e. gender, marital status and origination age) or product features (i.e. LVR, spread, early prepayment penalty, etc) might be investigated. Although the voluntary prepayment experience of the lender might be different from the experience of the market, accurate estimates of the market’s voluntary prepayment experience are useful as a starting point for the lender in estimating the voluntary prepayment probabilities which are suitable for their particular circumstances.
Bibliography


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BIBLIOGRAPHY


Appendices
APPENDIX A. RESULTS OF A SEMI-PARAMETRIC BOOTSTRAP

A Results of a semi-parametric bootstrap for females in fitting period 1921–2004

In this Appendix, we present the results of a semi-parametric bootstrap for females in fitting period 1921–2004. The results are presented for parameters $a_x$, $b_x$, $k_t$ and $k_s$ (for $s = 2005, 2010, \ldots, 2080$) in Tables A.1 to A.4 respectively. The original estimate of parameter $\theta$ obtained from the fitting of model PLC(1) to the observed data is $\hat{\theta}$, $\bar{\theta}$ is the average of the simulated values of $\theta$, $\hat{\sigma}_\theta$ is the standard deviation of the simulated values of $\theta$, $\hat{\theta}_p$ is the 100th quantile of the 10,000 simulated values of $\theta$.

Table A.1: Results of a semi-parametric bootstrap for $a_x$.

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APPENDIX A. RESULTS OF A SEMI-PARAMETRIC BOOTSTRAP

Table A.2: Results of a semi-parametric bootstrap for $b_x$.

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<tbody>
<tr>
<td>60</td>
<td>0.19956</td>
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<td>0.0028</td>
<td>0.19391</td>
<td>0.19489</td>
<td>0.19950</td>
<td>0.20412</td>
<td>0.20501</td>
</tr>
<tr>
<td>65</td>
<td>0.20452</td>
<td>0.20450</td>
<td>0.0024</td>
<td>0.19981</td>
<td>0.20061</td>
<td>0.20450</td>
<td>0.20837</td>
<td>0.20918</td>
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<tr>
<td>70</td>
<td>0.19721</td>
<td>0.19716</td>
<td>0.0020</td>
<td>0.19315</td>
<td>0.19377</td>
<td>0.19717</td>
<td>0.20047</td>
<td>0.20119</td>
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<tr>
<td>75</td>
<td>0.19574</td>
<td>0.19570</td>
<td>0.0018</td>
<td>0.19217</td>
<td>0.19275</td>
<td>0.19571</td>
<td>0.19866</td>
<td>0.19931</td>
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<tr>
<td>80</td>
<td>0.16423</td>
<td>0.16418</td>
<td>0.0017</td>
<td>0.16094</td>
<td>0.16144</td>
<td>0.16417</td>
<td>0.16690</td>
<td>0.16743</td>
</tr>
<tr>
<td>85</td>
<td>0.13507</td>
<td>0.13500</td>
<td>0.0018</td>
<td>0.13155</td>
<td>0.13207</td>
<td>0.13502</td>
<td>0.13785</td>
<td>0.13840</td>
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<tr>
<td>90</td>
<td>0.09654</td>
<td>0.09648</td>
<td>0.0022</td>
<td>0.09217</td>
<td>0.09284</td>
<td>0.09648</td>
<td>0.10007</td>
<td>0.10073</td>
</tr>
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<td>95</td>
<td>0.05624</td>
<td>0.05615</td>
<td>0.0037</td>
<td>0.04893</td>
<td>0.05010</td>
<td>0.05612</td>
<td>0.06225</td>
<td>0.06336</td>
</tr>
<tr>
<td>100</td>
<td>0.02746</td>
<td>0.02735</td>
<td>0.0087</td>
<td>0.01001</td>
<td>0.01302</td>
<td>0.02739</td>
<td>0.04175</td>
<td>0.04442</td>
</tr>
</tbody>
</table>

Table A.3: Results of a semi-parametric bootstrap for $k_t$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\hat{k}_t$</th>
<th>$\bar{k}_t$</th>
<th>$\hat{\sigma}_{k_t}$</th>
<th>$\hat{k}_t^{0.025}$</th>
<th>$\hat{k}_t^{0.05}$</th>
<th>$\hat{k}_t^{0.5}$</th>
<th>$\hat{k}_t^{0.95}$</th>
<th>$\hat{k}_t^{0.975}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1921</td>
<td>1.9159</td>
<td>1.9161</td>
<td>0.0580</td>
<td>1.8023</td>
<td>1.8198</td>
<td>1.9164</td>
<td>2.0104</td>
<td>2.0310</td>
</tr>
<tr>
<td>1930</td>
<td>1.4828</td>
<td>1.4822</td>
<td>0.0502</td>
<td>1.3839</td>
<td>1.4001</td>
<td>1.4829</td>
<td>1.5638</td>
<td>1.5804</td>
</tr>
<tr>
<td>1940</td>
<td>1.5680</td>
<td>1.5685</td>
<td>0.0407</td>
<td>1.4882</td>
<td>1.5027</td>
<td>1.5683</td>
<td>1.6364</td>
<td>1.6493</td>
</tr>
<tr>
<td>1950</td>
<td>1.2805</td>
<td>1.2814</td>
<td>0.0363</td>
<td>1.2098</td>
<td>1.2215</td>
<td>1.2816</td>
<td>1.3413</td>
<td>1.3523</td>
</tr>
<tr>
<td>1960</td>
<td>0.5132</td>
<td>0.5135</td>
<td>0.0333</td>
<td>0.4484</td>
<td>0.4589</td>
<td>0.5133</td>
<td>0.5681</td>
<td>0.5786</td>
</tr>
<tr>
<td>1970</td>
<td>0.5629</td>
<td>0.5633</td>
<td>0.0295</td>
<td>0.5055</td>
<td>0.5149</td>
<td>0.5631</td>
<td>0.6118</td>
<td>0.6215</td>
</tr>
<tr>
<td>1980</td>
<td>−1.2528</td>
<td>−1.2531</td>
<td>0.0304</td>
<td>−1.3120</td>
<td>−1.3021</td>
<td>−1.2531</td>
<td>−1.2030</td>
<td>−1.1935</td>
</tr>
<tr>
<td>1990</td>
<td>−2.1435</td>
<td>−2.1437</td>
<td>0.0283</td>
<td>−2.1986</td>
<td>−2.1895</td>
<td>−2.1443</td>
<td>−2.0970</td>
<td>−2.0867</td>
</tr>
<tr>
<td>2004</td>
<td>−4.1701</td>
<td>−4.1717</td>
<td>0.0288</td>
<td>−4.2282</td>
<td>−4.2196</td>
<td>−4.1716</td>
<td>−4.1242</td>
<td>−4.1158</td>
</tr>
</tbody>
</table>
Table A.4: Results of a semi-parametric bootstrap for $k_s$.  

<table>
<thead>
<tr>
<th>$s$</th>
<th>$\hat{k}_s$</th>
<th>$\bar{k}_s$</th>
<th>$\hat{k}_s \sigma$</th>
<th>$\hat{k}_s^{0.025}$</th>
<th>$\hat{k}_s^{0.05}$</th>
<th>$\hat{k}_s^{0.5}$</th>
<th>$\hat{k}_s^{0.95}$</th>
<th>$\hat{k}_s^{0.975}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>-4.5559</td>
<td>-4.5550</td>
<td>0.3469</td>
<td>-5.2295</td>
<td>-5.1172</td>
<td>-4.5570</td>
<td>-3.9734</td>
<td>-3.8706</td>
</tr>
<tr>
<td>2020</td>
<td>-5.2931</td>
<td>-5.2915</td>
<td>0.5286</td>
<td>-6.3346</td>
<td>-6.1678</td>
<td>-5.2859</td>
<td>-4.4260</td>
<td>-4.2853</td>
</tr>
<tr>
<td>2030</td>
<td>-6.0303</td>
<td>-6.0334</td>
<td>0.6652</td>
<td>-7.3554</td>
<td>-7.1292</td>
<td>-6.0233</td>
<td>-4.9363</td>
<td>-4.7298</td>
</tr>
<tr>
<td>2040</td>
<td>-6.7675</td>
<td>-6.7729</td>
<td>0.7817</td>
<td>-8.3064</td>
<td>-8.0593</td>
<td>-6.7742</td>
<td>-5.4883</td>
<td>-5.2537</td>
</tr>
</tbody>
</table>

In Table A.4, $\tilde{k}_s$ is the forecast of $k_s$ calculated under the standard time series method.
B Peculiarity of the ERP for females

Table B.1 below presents the estimate of resident population (ERP) for females from the middle of 1998 to the middle of 2001.

Table B.1: ERP for females.

<table>
<thead>
<tr>
<th>Age Last Birthday</th>
<th>Mid-Year 1998</th>
<th>Mid-Year 1999</th>
<th>Mid-Year 2000</th>
<th>Mid-Year 2001</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>79,609</td>
<td>81,724</td>
<td>83,455</td>
<td>86,322</td>
</tr>
<tr>
<td>61</td>
<td>77,082</td>
<td>79,500</td>
<td>81,882</td>
<td>83,488</td>
</tr>
<tr>
<td>62</td>
<td>74,775</td>
<td>77,075</td>
<td>79,340</td>
<td>82,077</td>
</tr>
<tr>
<td>63</td>
<td>71,289</td>
<td>75,052</td>
<td>76,980</td>
<td>79,204</td>
</tr>
<tr>
<td>64</td>
<td>69,705</td>
<td>70,891</td>
<td>75,196</td>
<td>76,951</td>
</tr>
<tr>
<td>65</td>
<td>69,260</td>
<td>69,158</td>
<td>70,380</td>
<td>75,465</td>
</tr>
</tbody>
</table>


Let $P(x,t)$ denote the size of the female population aged $x$ last birthday at the middle of year $t$. Note that $P(60, 1999) < P(61, 2000) < P(62, 2001)$ and $P(62, 1998) < \ldots < P(65, 2001)$. We find that the overseas migrations could not explain the observed increase of these female cohorts over these years. This problem occurs only in two cohorts and persists until 2001. This problem results in the negative estimated transitions in the estimation of longitudinal data under the iterative proportional fitting (IPF) procedure. To remedy this problem, we instead estimate the number of deaths of these cohorts over these years using the Australian Life Tables (ALT) in the relevant year (the ALTs are obtained from the ABS (2008)).
Author/s:
Hariyanto, Evan A.

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Mortality and disability modelling with an application to the pricing of a reverse mortgage contract

Date:
2014

Citation:

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File Description:
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