Anisotropic and Long-Range Vortex Interactions in Two-Dimensional Dipolar Bose Gases

B. C. Mulkerin,1 R. M. W. van Bijnen,2 D. H. J. O’Dell,3 A. M. Martin,1 and N. G. Parker4

1School of Physics, University of Melbourne, Victoria 3010, Australia
2Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, Netherlands
3Department of Physics and Astronomy, McMaster University, Hamilton, Ontario L8S 4M1, Canada
4Joint Quantum Centre Durham-Newcastle, School of Mathematics and Statistics, Newcastle University, Newcastle upon Tyne NE1 7RU, United Kingdom

(Received 13 July 2013; published 22 October 2013)

We perform a theoretical study into how dipole-dipole interactions modify the properties of superfluid vortices within the context of a two-dimensional atomic Bose gas of co-oriented dipoles. The reduced density at a vortex acts like a giant antidipole, changing the density profile and generating an effective dipolar potential centred at the vortex core whose most slowly decaying terms go as $1/\rho^2$ and $\ln(\rho)/\rho^2$. These effects modify the vortex-vortex interaction which, in particular, becomes anisotropic for dipole pairs polarized in the plane. Striking modifications to vortex-vortex dynamics are demonstrated, i.e., anisotropic corotation dynamics and the suppression of vortex annihilation.

Ferrohydrodynamics (FHD) describes the motion of fluids made of particles with magnetic (or electric) dipoles [1,2]. The interparticle dipole-dipole interactions (DDIs) are long range and anisotropic, giving rise to behavior such as magnetostriction and electrodstriction, geometric pattern formation and surface ripple instabilities [3]. The distinctive properties of ferrofluids are exploited in applications from tribology to information display and medicine [1]. Recently, quantum ferrofluids have been realized through Bose-Einstein condensates (BECs) of atoms with large magnetic dipole moments [4] such as $^{52}\text{Cr}$ [5,6], $^{164}\text{Dy}$ [7], and $^{168}\text{Er}$ [8]. Signatures of FHD have been observed in these gases, including magnetostriction [9] and $d$-wave collapse [10]. Pattern formation [11–13], linked to roton-like excitations [13–15], has been predicted but not yet seen. Superfluidity of semiconductor microcavity polaritons, which are inherently dipolar, has also recently been demonstrated [16].

In this Letter we consider the interplay between DDIs and vortices in a BEC. Vortices form the “sinews and muscles” of fluids [17,18] and drive phenomena such as mixing processes, sunspots, tornadoes, and synoptic scale weather phenomena [19]. In a superfluid BEC the vortices have a core of well-defined size and quantized vorticity. Single vortices, vortex rings, pairs, lattices, and turbulent states can be controllably generated [20] and imaged in real time [21]. In quasi-2D geometries, the paradigm of point vortices [22,23] can be realized. 2D Bose gases also provide a route to the Berezinsky-Kosterlitz-Thouless (BKT) transition [24], the thermal unbinding of vortex-antivortex pairs (VA).

Previous theoretical studies of a vortex in a trapped 3D dipolar BEC found density ripples about the core [25–27] and an elliptical core [25]. Here we consider the quasi-2D case and focus on the effective long-range and anisotropic potentials that are generated between vortices by DDIs. We demonstrate the striking implications of these potentials on the motion of pairs of vortices. Our results provide insight into the role of DDIs in large-scale superfluid phenomena, such as the vortex lattice phases [28], the BKT transition, and quantum turbulence.

When the dipoles are aligned by an external field, the DDIs are described by

$$U_{dd}(\mathbf{r} - \mathbf{r}') = \frac{C_{dd}}{4\pi} \frac{1 - 3\cos^2 \theta}{|\mathbf{r} - \mathbf{r}'|^3},$$

where $\theta$ is the angle between the polarization direction and the interparticle vector $\mathbf{r} - \mathbf{r}'$. For magnetic dipoles with moment $d$, $C_{dd} = \mu_0 d^2$, where $\mu_0$ is the permeability of free space. The strongly dipolar $^{164}\text{Dy}$ BEC [7] has $d = 10\mu_B$ (Bohr magnetons). The same interaction arises between polar molecules, which can possess huge electric dipole moments and have been cooled close to degeneracy [29]. A BEC with DDIs is described by the dipolar Gross-Pitaevskii equation (DGPE) [4], in which the DDIs are incorporated via Eq. (1), and the isotropic van der Waals interactions (vdWIs) via a local pseudopotential $U_{vdW}(\mathbf{r} - \mathbf{r}') = g_{3D} \delta(\mathbf{r} - \mathbf{r}')$ [30]. The relative strength of the DDIs is parametrized via the ratio $\varepsilon_{dd} = C_{dd}/3g_{3D}$ [4]. This parameter has a natural value $\varepsilon_{dd} \lesssim 1$. However, the ability to tune $g_{3D}$ between $-\infty$ and $+\infty$ via Feshbach resonance has enabled the realization of a purely dipolar gas ($\varepsilon_{dd} = \infty$) [31]. Furthermore, it is predicted that $C_{dd}$ can be reduced below its natural, positive value, including to negative values, by external field rotation [32]. As such, a large parameter space $-\infty < \varepsilon_{dd} < \infty$, with positive or negative $g_{3D}$ and $C_{dd}$, is possible.

We consider bosonic dipoles of mass $m$, free in the transverse ($\mathbf{p}$) plane and with harmonic trapping in the axial ($z$) direction. The axial trap frequency $\omega_z$ is sufficiently
modes. A collapse ensues via density ripples aligned either 
energy, the roton instability (RI) arises in finite momentum 
gas with dipoles polarized along 
the attractive part of the DDI. An exception arises for the 2D 
work in Fourier space and use the convolution theorem 
for the vdWIs and DDIs, respectively, where \( n = |\psi|^2 \) is the 
2D particle density, \( U_{\text{DDI}}^{(2)} \) is the effective 2D DDI 
potential, and \( g = g_{\text{3D}}/\sqrt{2\pi l_\perp} \). While an explicit 
expression for \( U_{\text{DDI}}^{(2)}(\rho) \) exists [35], it is convenient to 
work in Fourier space and use the convolution theorem 
\( \Phi(\rho, t) = \hat{F}^{-1} \left[ U_{\text{DDI}}^{(2)}(k) \hat{F}(k, t) \right] \) [34,36]. The 
Fourier transform of \( U_{\text{DDI}}^{(2)}(\rho) \) is \( U_{\text{DDI}}^{(2)}(q) = (4\pi g_{\text{dd}}/3)(F_\parallel(q)\sin^2 \alpha + F_\perp(q)\cos^2 \alpha) \), with 
\( q = k l_\perp/\sqrt{2} \), \( F_\parallel(q) = -1 + 3\sqrt{3}/\pi q^2 \) \( \text{erfc}(q) \), \( F_\perp(q) = 2 - 3\sqrt{3}/\pi q^2 \) \( \text{erfc}(q) \), and 
g_{\text{dd}} = C_{\text{dd}}/3\sqrt{2}\pi l_\perp. It follows that \( e_{\text{dd}} = g_{\text{dd}}/g \).

A homogeneous 2D dipolar gas of density \( n_0 \) has uniform 
dipolar potential \( \Phi_0 = n_0 g_{\text{dd}}(3\cos^2 \alpha - 1) \) and chemical 
potential \( \mu_0 = n_0 (g + g_{\text{dd}}[3\cos^2 \alpha - 1]) \) [33–35]. At the 
“magic angle” \( \alpha_0 = \arccos(1/\sqrt{3}) \approx 54.7^\circ \), \( \Phi_0 \) is zero. For 
\( \alpha < \alpha_0 \), \( \Phi_0 \) is net repulsive (attractive) for \( g_{\text{dd}} > 0 \) 
(\( < 0 \)), while for \( \alpha > \alpha_0 \) it is net attractive (repulsive) for 
\( g_{\text{dd}} > 0 \) (\( < 0 \)). The system suffers two key instabilities. The 
phonon instability (PI), familiar from conventional 
BECs [30], is associated with unsteady growth of zero 
momentum modes. It arises when the net local interactions 
become attractive, i.e., when \( \mu_0 < 0 \). In terms of \( e_{\text{dd}} \), it follows that the PI 
arises for \( e_{\text{dd}} < [1 - 3\cos^2 \alpha]^{-1} \) when 
\( g > 0 \) or \( e_{\text{dd}} > [1 - 3\cos^2 \alpha]^{-1} \) when \( g < 0 \) [4,15,36–38].

DDIs can induce a roton dip at finite momentum in the 
excitation spectrum [14,15,27]. When this softens to zero 
energy, the roton instability (RI) arises in finite momentum 
modes. A collapse ensues via density ripples aligned either 
along the polarization axis (for \( C_{\text{dd}} > 0 \)) or perpendicular 
to it (for \( C_{\text{dd}} < 0 \)) due to the preferred alignment of dipoles 
end to end or side by side, respectively [11]. Close to the RI 
stable density ripples arise when the roton mode mixes into 
the ground state [25,27]. Typically, the RI is induced by the 
attractive part of the DDI. An exception arises for the 2D 
with dipoles polarized along \( z \); the attractive part of the 
DDI lies out of the plane where the particles cannot access it 
[36] but a roton can be induced via attractive local 
interactions [39]. For dipoles polarized in the plane, the 
particles can access the attractive part of the DDI, and the 
“conventional” dipolar roton is supported.

As a precursor to understanding the vortex-vortex (VV) 
interaction we first explore single vortex solutions in 
our 2D homogeneous system (in contrast to the previous 
analyses in 3D [25,27]). We obtain vortex solutions and 
dynamics by numerically solving Eq. (2) [40]. Density, 
energy, and length are scaled in units of \( n_0, \mu_0 \) and the 
corresponding healing length \( \xi_0 = h/\sqrt{m\mu_0} \), respectively.

The vortex core size is of the order of \( \xi_0 \), which diverges 
as \( \mu_0 \to 0 \). The 2D approximation is valid for \( \sigma = 1/\xi_0 \leq 1 \); we choose \( \sigma = 0.5 \). We consider the 
representative cases of \( \alpha = 0 \) (dipoles parallel to the \( z \) axis) and 
\( \alpha = \pi/4 \) (dipoles tipped partly along \( x \) axis).

\( \alpha = 0 \).—Figure 1(a) shows the vortex density along \( x \) 
and \( y \) as a function of \( e_{\text{dd}} \). The dipolar potential, and hence the 
density profile, are axisymmetric. For \( e_{\text{dd}} = 0 \) (left inset) the 

vortex has the standard axisymmetric core of 
vanishing density of width \( \xi_0 \) [30]. For \( g > 0 \) the system is 
stable from \( e_{\text{dd}} = -0.5 \), the PI threshold, upwards. No RI 
is observed for \( g > 0 \), as expected [36]. Meanwhile, for 
\( g < 0 \) we only find solutions for \( e_{\text{dd}} \approx -1.16 \), the RI 
threshold (red dashed line).

The vortex structure for \( e_{\text{dd}} \neq 0 \) is almost identical to the 
\( e_{\text{dd}} = 0 \) case [41], apart from in two regimes. As one 
approaches \( e_{\text{dd}} = -0.5 \) from above, the vortex core 
becomes increasingly narrow with respect to \( \xi_0 \) (middle 
 inset), due to the cancellation of explicit contact interactions. Meanwhile, as the RI is approached from below, 
axisymmetric density ripples appear around the vortex 
(like in the 3D case [25–27]), decaying with distance and 
with an amplitude up to \( \approx 20\% n_0 \) (third inset). The 
ripple wavelength is \( \approx 4\xi_0 \) [42], implying that our treatment 
is self-consistently 2D because \( \sigma = l_\perp/\xi_0 = 0.5 \).

\( \alpha = \pi/4 \).—The axisymmetry of the dipolar potential 
and density is lost [Fig. 1(b)]. For \( g_{\text{dd}} > 0 \) (\( < 0 \)) the dipoles 
prefectly along \( x \) (\( y \)). For \( g < 0 \) we find no stable 
solutions (down to \( e_{\text{dd}} = -20 \)) due to the RI. For \( g > 0 \) we 
observe the RI, unlike for \( \alpha = 0 \), since the atoms now feel 
the attractive part of \( U_{\text{dd}} \). The RI occurs for \( e_{\text{dd}} \approx 14.9 \)

![FIG. 1 (color online). Vortex density profiles in the presence of DDIs parametrized by \( e_{\text{dd}} = g_{\text{dd}}/g \). The left-hand (right-hand) side of each figure shows the profile along \( x \) (\( y \)). (a) Polarization along \( z \) (\( \alpha = 0 \)). Stable density ripples form close to the onset of the RI (dashed red line). (b) For off-axis dipoles (\( \alpha = \pi/4 \)) the vortex becomes highly anisotropic. Insets: Vortex density \( n(x, y) \) for examples of \( e_{\text{dd}} \) (area \( 40\xi_0^2 \) for each). Gray bands indicate unstable regimes where no steady state solutions exist, while the 
labels \( g > 0 \) and \( g < 0 \) on the individual solution sheets indicate that they are only stable for the specified value of \( g \).](Image:170402-2.png)
by assuming the vortex ansatz and vanishes for a homogeneous system. The decay of $\Phi_{\text{NL}}/\Phi_0$, compared with $1/\rho^2$ (gray line). (c) The decay of $\Phi_{\text{NL}}/\Phi_0$, compared with $(A \ln \rho' + B)/\rho^3$. Parameters are $\alpha = 0$, $\sigma = 0.5$.

We see the same qualitative behavior for all $0 < \alpha < \alpha_0$, albeit with shifted RI and PI thresholds. For cases with $\alpha > \alpha_0$, we see a reversal in the $e_{\text{dd}}$ dependence, with no solutions for $g > 0$ and ripples polarized along $y$.

From the FHFD perspective, the depleted density due to a vortex acts like a lump of “antidipoles” whose charges have been reversed [43]. Let us calculate the mean-field dipolar potential $\Phi(\rho)$ generated by such a defect located at $\rho = 0$; we shall see shortly how it modifies the vortex-vortex interaction. For simplicity we consider $\alpha = 0$ and the illustrative cases of $e_{\text{dd}} = -1.2$ (vortex with ripples, $g < 0$) and $5$ (vortex with no ripples, $g > 0$) [Fig. 2(a)]. As $\rho \to \infty$, $\Phi \to \Phi_0 = n_0 g_{\text{dd}}(3 \cos^2 \alpha - 1)$, the homogeneous result, while for $\rho \leq 5 \xi_0$, $\Phi$ is dominated by the core structure and ripples (where it is positive). It is insightful to consider $\Phi$ as the sum of a local term $\Phi_L(\rho) = n(\rho) g_{\text{dd}}(3 \cos^2 \alpha - 1)$ and a nonlocal term $\Phi_{\text{NL}}(\rho)$ [33,35,37]. The latter is generated by variations in the density [33,37] and vanishes for homogeneous systems. The generic functional form of $\Phi$ at long range is revealed by assuming the vortex ansatz $n(\rho) = n_0 (1 - 1/(1 + \rho^2))$, where $\rho' = \rho / \xi_0$ [44,45]. We employ a first-order expansion in $\sigma$ of $U_{\text{dd}}(k) = g_{\text{dd}} (2 - \sqrt{9 \pi/2} k \sigma^2 + O(\sigma^3))$ and the Hankel transform $h(k) = n_0 \delta(k)/k + n_0 K_0(k/\sqrt{2})/2$, where $K_0(k)$ is a modified Bessel function of the second kind [46]. Then, to first order in $\sigma$ and third order in $1/\rho'$,

$$\frac{\Phi(\rho)}{\Phi_0} \sim \left(1 - \frac{1}{\rho'^2}\right) + \left(\frac{A \ln \rho' + B}{\rho'^3}\right) \sigma,$$

with constants $A = -\sqrt{9 \pi/8} = -1.88$ and $B = (\ln 2 - 1)A = 0.577$. The first term corresponds to the local contribution of the vortex dipolar potential [Fig. 2(b), gray line]. It physically arises from the $1/\rho^2$ decay of vortex density at long range, and is also present for nondipolar vortices (albeit controlled by $g$ and not $e_{\text{dd}}$) [47]. The second term describes the nonlocal contribution [Fig. 2(c), gray line]. This vanishes in the true 2D limit $\sigma = 0$ since the volume of antidipoles in the core vanishes. Equation (3) agrees with numerical calculations in the limit $\rho' \gg 1$ [Figs. 2(b) and 2(c)]. The dominant scaling of $\Phi_{\text{NL}}$ as $\ln \rho' / \rho^3$, and not $1/\rho^3$, shows us that the vortex does not strictly behave as a pointlike collection of dipoles at long range. This is due to the slow, power-law recovery of the vortex density to $n_0$. We have checked that exponentially decaying density profiles, e.g., $\exp(-\rho^2)$, do lead to a $1/\rho^3$ scaling of $\Phi_{\text{NL}}$ at long range. For $\alpha \neq 0$, $\Phi$ also varies anisotropically as $\cos^2 \theta$ at long range.

We now explore the vortex-vortex interaction through the interaction energy [40]. For a negative (positive) $\Phi(\rho'')$ for VA (VV) pairs due to the cancellation (reinforcement) of velocity fields at large distance. For $d \geq 4 \xi_0$, $E_{\text{int}}(d) = (2 \pi \eta_1 q^2 q_2 / m \ln(R/d))$, the hydrodynamic (coreless vortex) prediction [30] for system size $R$. For $d \leq 4 \xi_0$, the cores overlap causing $|E_{\text{int}}|$ to flatten off.

When $e_{\text{dd}} \neq 0$, the logarithmic, hydrodynamic contribution to $E_{\text{int}}(d)$ dominates the dipolar contribution for large $d$. Elsewhere, significant deviations to $E_{\text{int}}(d)$ arise, particularly at short range $d \leq 3 \xi_0$. This deviation is most striking for $-0.5 < e_{\text{dd}} < 0$ (g > 0), e.g., $e_{\text{dd}} = -0.45$ (red lines), for which $|E_{\text{int}}|$ increases dramatically as $d$ decreases. This is due to the narrowing of the vortex core (relative to $\xi_0$) as $e_{\text{dd}} \to -0.5$, reducing the core overlap as $d \to 0$. Outside of this range, for small $d$ ($d < 3 \xi_0$), DDIs reduce $|E_{\text{int}}|$. For values of $e_{\text{dd}}$ that support density ripples, $|E_{\text{int}}|$ features a small peak at $d \sim 4 \xi_0$ due to ripple overlap. Further out, $E_{\text{int}}$ decays as $1/\rho^2$ towards the $e_{\text{dd}} = 0$ result. Note that as $|e_{\text{dd}}|$ is increased, $E_{\text{int}}(d)$ tends towards a fixed behavior, independent of the sign of $e_{\text{dd}}$.
For $\alpha \neq 0$, $E_{\text{int}}$ depends on the orientation angle of the pair $\eta$, defined as the angle between the line joining the vortices and the in-plane polarization direction. We illustrate this in Fig. 4 for $\alpha = \pi/4$ and $\epsilon_{\text{dd}} = 5$. For the VA pair [4(a)], $E_{\text{int}}$ is maximal for $\eta = \pi/2$ and decreases monotonically to 0 as $\eta \to 0, \pi$. For moderate separations $d \approx 6\xi_0$ (dotted red line and solid black line), the vortex ripples dominate the vortex interaction (see inset). For $\eta = 0$, the ripples from each vortex (aligned in $x$) merge side by side. As $\eta$ is increased, this side-by-side formation is broken at energetic cost. For $\eta \sim \pi/2$ one might expect, for appropriate $d$, a scenario where the inner ripples from each vortex overlap at energetic benefit. However, the intervortex density is suppressed due to the high flow velocity there (Bernoulli’s principle). Beyond the ripples, $E_{\text{int}}$ approaches a sinusoidal dependence on $\eta$ (dashed green line).

For the VV pair [4(b)] no such suppression occurs. As the pair is rotated, and for suitably large $d$ ($d \approx 2\xi_0$), the inner ripples combine at significant energetic benefit such that $E_{\text{int}}$ is minimal for $\eta = \pi/2$. At smaller separations, this overlap cannot occur and $E_{\text{int}}(\eta)$ is dominated by the effect of the outer ripples (as for the VA pair), such that the $\eta = 0$ case is most energetically favored. For intermediate $d$ ($d \sim 3\xi_0$), there is energetic competition between the inner and outer ripples. At larger $d$, the modulation becomes approximately sinusoidal with $\eta$.

The effects of DDIs upon vortex pair dynamics are now considered. For $\epsilon_{\text{dd}} = 0$, VA pairs form moving solitary wave solutions for $d \approx 2\xi_0$ [48]. For $d \approx 2\xi_0$ the cores overlap and the pair is unstable to annihilation [Fig. 5(a)]. In Fig. 1(a), we saw significant core narrowing (relative to $\xi_0$) for $\epsilon_{\text{dd}} = -0.4$. These conditions can stabilize the VA pair against annihilation [Fig. 5(a), red lines] [49]. For $\epsilon_{\text{dd}} = 0$ (or $\alpha = 0$), VV pairs undergo circular corotation [50] [black curve, Fig. 5(b)]. Since in-plane ($\alpha \neq 0$) DDIs cause $E_{\text{int}}$ to vary as the VV pair rotates, the vortices corotate in an anisotropic path [blue and magenta curves, Fig. 5(b)]. Moreover, with strong anisotropic ripples for small separations, the vortices are unable to corotate, instead “wobbling” about their initial positions [green curves, Fig. 5(b)].

We have shown that the interaction of two vortices can be significantly different in quantum ferrofluids than in conventional superfluids. At short range the vortex-vortex interaction is strongly modified by the changed shape and peripheral density ripples of each vortex. At longer range, each vortex experiences the dipolar mean-field potential of the other, with $1/\rho^2$ and $\ln(\rho)/\rho^3$ contributions. The vortex-vortex interaction is most significantly modified up to mid-range separations ($d \approx 10\xi$), beyond which it reduces to the usual hydrodynamic behavior. When the dipoles have a component in the plane, the vortex-vortex interaction becomes anisotropic. The vortex-vortex interaction is a pivotal building block for understanding macroscopic superfluid phenomena. For example, it is the key input parameter for models of quantum turbulence [51], the BKT transition [24], and vortex crystals [52]. The striking effects of DDIs on the dynamics of vortex pairs point to interesting new regimes for macroscopic systems of superfluid vortices.

B. C. M. acknowledges support from the Overseas Research Experience Scholarship, University of Melbourne, and D. H. J. O. acknowledges support from NSERC (Canada).

[41] This is our motivation for scaling length in terms of $\xi_0$.
[42] More generally the ripple wavelength and location of the first peak varies with $q_{\text{det}}$, $\alpha$, and $\sigma$, but the latter remains within a few $\xi_0$ of the vortex origin.
[49] The initial motion of the vortices towards each other is a transient effect associated with starting the simulation with a vortex pair solution obtained in the static frame.
Minerva Access is the Institutional Repository of The University of Melbourne

Author/s:
Mulkerin, BC; van Bijnen, RMW; O'Dell, DHJ; Martin, AM; Parker, NG

Title:
Anisotropic and Long-Range Vortex Interactions in Two-Dimensional Dipolar Bose Gases

Date:
2013-10-22

Citation:

Persistent Link:
http://hdl.handle.net/11343/43018