

## ON $f$ -DOMINATION NUMBER OF A GRAPH

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### 1. INTRODUCTION

The domination number  $\gamma(G)$  of a graph  $G$  is the smallest cardinality of a set  $D$  of vertices such that every vertex outside  $D$  has at least one neighbor in  $D$ . Extensive studies on domination number and domination-related topics have been done in the past thirty years. Recently, some new domination models have been proposed. For example, [4, 5] studied the  $k$ -domination number. For a positive integer  $k$ , a subset  $D$  of  $V(G)$  is a  $k$ -dominating set of  $G$  if each vertex of  $V(G) \setminus D$  is adjacent to at least  $k$  distinct vertices of  $D$ . A  $k$ -independent set  $T$  is a subset of  $V(G)$  such that the maximum degree of the induced subgraph  $G[T]$  of  $G$  is less than  $k$ . The  $k$ -domination number of  $G$ , denoted by  $\gamma_k(G)$ , is the cardinality of the smallest  $k$ -dominating set of  $G$  ([4, 5]). The  $k$ -independence number of  $G$  ([4, 5]),  $\beta_k(G)$ , is the cardinality of the largest  $k$ -independent set of  $G$ . Evidently,  $\gamma_1(G)$  and  $\beta_1(G)$  are, respectively, the ordinary domination number  $\gamma(G)$  and the ordinary independence number  $\beta(G)$ .

The following result was conjectured by J.F. Fink and M.S. Jacobson ([4, 5]) and proved in [3].

**Theorem 1.** *For any simple graph  $G$  and positive integer  $k$ , we have  $\gamma_k(G) \leq \beta_k(G)$ .*

This theorem generalizes the inequality  $\gamma \leq \beta$ . Another upper bound for  $\gamma_k$  is the following

**Theorem 2** ([1]). *Let  $n$  and  $k$  be positive integers, and  $G$  a graph with minimum degree  $\delta(G) \geq \frac{n+1}{n}k - 1$ . Then  $\gamma_k(G) \leq \frac{np}{n+1}$ , where  $p = |V(G)|$ .*

More upper bounds for  $\gamma_k$  can be found in [9]. In the same paper a general domination concept was introduced. For any integer-valued function  $f$  defined on

$V(G)$ , a subset  $D$  of  $V(G)$  is called an  $f$ -dominating set of  $G$  if  $|N_G(x) \cap D| \geq f(x)$  for each  $x \in V(G) \setminus D$ , where  $N_G(x)$  is the set of neighbors of  $x$  in  $G$ . Then the  $f$ -domination number of  $G$ , denoted by  $\gamma_f(G)$ , is defined to be the smallest cardinality of an  $f$ -dominating set of  $G$ . Obviously, if  $f$  is such that  $f(x) = k$  for all  $x \in V(G)$ , then  $\gamma_f(G)$  is exactly the  $k$ -domination number. If  $T \subseteq V(G)$  satisfies  $d_{G[T]}(x) < f(x)$  for all  $x \in T$ , then we call  $T$  an  $f$ -independent set of  $G$ . The maximum cardinality of  $f$ -independent sets of  $G$  is then defined to be the  $f$ -independence number, denoted by  $\beta_f(G)$ .

In this paper we initiate the study on  $\gamma_f$  and  $\beta_f$ . Basic results for these two invariants are discussed in the next section. Some upper bounds for  $\gamma_f$  are given in Section 3. In particular, Theorems 1–2 are generalized. In the last section some open problems are proposed. Throughout the paper  $G$  is a finite, undirected graph with no loops and multiedges, and  $f, V(G) \rightarrow Z$  is an integer-valued function. For  $D \subseteq V(G)$  and  $x \in V(G)$ , let  $N_D(x) = N_G(x) \cap D$  and  $d_D(x) = |N_D(x)|$ . Let  $p$  and  $\varepsilon(G)$  represent the number of vertices and the number of edges of  $G$ , respectively.

## 2. BASIC RESULTS

A subset  $S$  of  $V(G)$  is called an  $f$ -transversal of  $G$  if it intersects all non- $f$ -independent sets of  $G$ . The minimum cardinality of  $f$ -transversals of  $G$  is then defined to be the  $f$ -transversal number of  $G$ , denoted by  $\alpha_f(G)$ . The following Gallai-type equality is in fact a consequence of a more general result of [6].

**Theorem 3.**  $\alpha_f(G) + \beta_f(G) = p$ .

*Proof.* It can be shown that  $S \subseteq V(G)$  is an  $f$ -transversal iff  $V(G) \setminus S$  is an  $f$ -independent set. Then the theorem follows.  $\square$

**Proposition 1.** (1) If  $H$  is a spanning subgraph of  $G$ , then  $\gamma_f(G) \leq \gamma_f(H)$ ;

(2) If  $f': V(G) \rightarrow Z$  is another function satisfying  $f(x) \leq f'(x)$  for all  $x \in V(G)$ , then  $\gamma_f(G) \leq \gamma_{f'}(G)$  and  $\beta_f(G) \leq \beta_{f'}(G)$ .

**Proposition 2.** (1) If  $f(x) > d(x)$  for some  $x \in V(G)$ , then  $x$  must belong to any  $f$ -dominating set of  $G$ ;

(2) If  $f(x) < 1$  for a vertex  $x$ , then  $x$  can not be in any minimal  $f$ -dominating set of  $G$ .

**Proposition 3.** Let  $M = \max_{x \in V(G)} f(x)$ , then

$$\gamma_f(G) \geq \frac{1}{M} \left( \sum_{x \in V(G)} f(x) - \varepsilon(G) \right).$$

**Proof.** Let  $D$  be an  $f$ -dominating set of  $G$  with the smallest cardinality. Then

$$\sum_{x \in V(G)} f(x) - \sum_{x \in D} f(x) = \sum_{x \in V(G) \setminus D} f(x) \leq \varepsilon(G).$$

So  $M \cdot \gamma_f(G) = M \cdot |D| \geq \sum_{x \in D} f(x) \geq \sum_{x \in V(G)} f(x) - \varepsilon(G)$ . This completes the proof.  $\square$

For any function  $f: V(G) \rightarrow Z$ , let  $f^*: V(G) \rightarrow Z$  be a companion function defined by  $f^*(x) = d(x) - f(x) + 1$ ,  $x \in V(G)$ . Then we have

**Proposition 4.**

- (1)  $\gamma_f(G) + \beta_{f^*}(G) \leq p$ ;
- (2)  $\gamma_{f^*}(G) + \beta_f(G) \leq p$ .

**Proof.** Let  $T$  be a maximum  $f^*$ -independent set of  $G$ . Then  $d_{G[T]}(x) \leq f^*(x) - 1$  for each  $x \in T$ . So  $d_{V(G) \setminus T}(x) \geq d(x) - f^*(x) + 1 = f(x)$  for each  $x \in T$ . Thus  $V(G) \setminus T$  is an  $f$ -dominating set of  $G$ , and (1) is true. Since  $(f^*)^* = f$ , (2) follows from (1) immediately.  $\square$

**Corollary 1.** If  $f(x) \leq \frac{d(x)+1}{2}$  for all  $x \in V(G)$ , then  $\gamma_f(G) + \beta_f(G) \leq p$ .

**Proof.** The given condition implies that  $f(x) \leq f^*(x)$  for each  $x \in V(G)$ . Hence  $\beta_f(G) \leq \beta_{f^*}(G)$  by Proposition 1(2). The corollary then follows from Proposition 4(1).

Corollary 1 generalizes a known result ([8]) that  $\gamma(G) + \beta(G) \leq p$  if  $p \geq 2$  and  $G$  contains no isolated vertices.  $\square$

### 3. SOME UPPER BOUNDS FOR $\gamma_f$

As shown in Theorem 1,  $\gamma_k(G) \leq \beta_k(G)$  for any positive integer  $k$ . Then we may naturally ask if  $\gamma_f(G) \leq \beta_f(G)$  for any function  $f$ . The answer is affirmative. In fact we have the following more general result.

**Theorem 4.** For any function  $f: V(G) \rightarrow Z$ , every  $f$ -independent set  $D$  of  $G$  such that  $\sum_{x \in D} f(x) - \varepsilon(D)$  is maximum is an  $f$ -dominating set of  $G$ , where  $\varepsilon(D)$  is the number of edges of  $G[D]$ .

**Proof.** The proof is similar to that used in [3].

Suppose otherwise; then there must exist  $v \in V(G) \setminus D$  such that  $d_D(v) < f(v)$ . Let  $B = N_D(v)$ , then  $0 \leq |B| < f(v)$ . Let

$$A = \{x \in B: d_D(x) = f(x) - 1\}$$

and let  $S$  be a maximal independent set of  $G[A]$ . Then  $\Phi \subseteq S \subseteq A \subseteq B \subseteq D$ . Let  $C = (D \setminus S) \cup \{v\}$ . Then  $C$  must be an  $f$ -independent set of  $G$ .

In fact,

$$\begin{aligned} d_C(v) &\leq |B| < f(v), \\ d_C(x) &\leq d_D(x) < f(x), \quad \forall x \in D \setminus B, \\ d_C(x) &\leq d_D(x) + 1 \leq (f(x) - 2) + 1 < f(x), \quad \forall x \in B \setminus A. \end{aligned}$$

Noting that  $S$  is a maximal independent set of  $G[A]$ , each  $x \in A \setminus S$  is adjacent to at least one vertex in  $S$ . Hence

$$d_C(x) \leq (d_D(x) - 1) + 1 < f(x), \quad \forall x \in A \setminus S.$$

Thus  $C$  is indeed an  $f$ -independent set of  $G$ . We have

$$\varepsilon(C) = \varepsilon(D) - \sum_{x \in S} (f(x) - 1) + |B| - |S| = \varepsilon(D) - \sum_{x \in S} f(x) + |B|.$$

Hence,

$$\begin{aligned} \sum_{x \in C} f(x) - \varepsilon(C) &= \left( \sum_{x \in D} f(x) - \sum_{x \in S} f(x) + f(v) \right) - \left( \varepsilon(D) - \sum_{x \in S} f(x) + |B| \right) \\ &= \sum_{x \in D} f(x) - \varepsilon(D) + f(v) - |B| > \sum_{x \in D} f(x) - \varepsilon(D), \end{aligned}$$

contradicting the choice of  $D$ . This completes the proof.  $\square$

**Corollary 2.** For any graph  $G$  and any function  $f: V(G) \rightarrow Z$ , we have  $\gamma_f(G) \leq \beta_f(G)$ .

*Proof.* By Theorem 4 there exists an  $f$ -dominating set  $D$  which is also an  $f$ -independent set. So  $\gamma_f(G) \leq |D| \leq \beta_f(G)$ .  $\square$

Let  $f^*$  be defined as in Section 2, then we have

**Corollary 3.**

- (1)  $\gamma_f(G) + \gamma_{f^*}(G) \leq p$ ;
- (2)  $\gamma_f(G) \cdot \gamma_{f^*}(G) \leq \left(\frac{p}{2}\right)^2$ .

Proof. By Theorem 4 we can choose an  $f$ -dominating set of  $G$  which is also an  $f$ -independent set. Thus for any  $x \in D$ ,

$$|N_G(x) \cap (V \setminus D)| \geq d_G(x) - (f(x) - 1) = f^*(x).$$

So  $V \setminus D$  is an  $f^*$ -dominating set of  $G$ . This implies (1). (2) is a direct consequence of (1).  $\square$

Combining Corollaries 1-2, we have

**Corollary 4.** Let  $f: V(G) \rightarrow Z$  be such that  $f(x) \leq \frac{1}{2}(d(x) + 1)$ ,  $\forall x \in V(G)$ , then  $\gamma_f(G) \leq \frac{1}{2}p$ .

Corollary 4 generalizes an early result of Ore which states that  $\gamma(G) \leq \frac{1}{2}p$  if  $G$  has no isolated vertices.

The idea used in [1] can be applied to prove the following result, which generalizes Theorem 2.

**Theorem 5.** Let  $n$  be a positive integer and let  $f: V(G) \rightarrow Z$  be such that  $f(x) \leq \frac{n}{n+1}(d_G(x) + 1)$ ,  $\forall x \in V(G)$ . Then  $\gamma_f(G) \leq \frac{np}{n+1}$ .

Proof. Let  $V_1, V_2, \dots, V_{n+1}$  be a partition of  $V(G)$  such that  $E' = E(G) \setminus \bigcup_{i=1}^{n+1} E(G[V_i])$  contains as many edges as possible. Then by a theorem of Erdős ([2])  $d_H(x) \geq \left\lceil \frac{n}{n+1} d_G(x) \right\rceil$ ,  $\forall x \in V(G)$ , where  $H = (V(G), E')$  and  $\lceil a \rceil$  is the smallest integer not less than  $a$ . The condition  $f(x) \leq \frac{n}{n+1}(d_G(x) + 1)$  implies  $d_G(x) \geq \frac{n+1}{n}f(x) - 1$ . This gives

$$d_H(x) \geq \left\lceil \frac{n}{n+1} \left( \frac{n+1}{n} f(x) - 1 \right) \right\rceil = \left\lceil f(x) - \frac{n}{n+1} \right\rceil = f(x), \quad \forall x \in V(G).$$

Without loss of generality we may suppose  $|V_1| = \max_{1 \leq i \leq n+1} |V_i|$ . By the above discussion,  $\bigcup_{i=2}^{n+1} V_i$  is an  $f$ -dominating set of  $G$ . Thus

$$\gamma_f(G) \leq p - |V_1| \leq p - \frac{p}{n+1} = \frac{np}{n+1}.$$

$\square$

**Corollary 5.** Let  $n_0 = \max_{x \in V(G)} \left\lceil \frac{f(x)}{f^*(x)} \right\rceil$  ( $f(x) \neq d(x) + 1$  for all  $x \in V(G)$ ). Then  $\gamma_f(G) \leq \frac{n_0 p}{n_0 + 1}$ .

Note that this corollary generalizes Corollary 4.

For any  $f: V(G) \rightarrow Z$  with  $1 \leq f(x) \leq d_G(x)$ ,  $x \in V(G)$ , define a function  $f-1: V(G) \rightarrow Z$  such that

$$(f-1)(x) = \max\{1, f(x) - 1\}, \quad x \in V(G).$$

Inductively define the function  $f - (i + 1) = (f - i) - 1$  for any positive integer  $i$ . Then it is not difficult to see that  $\gamma_{f-m} = \gamma(G)$ , where  $m = \max_{x \in V(G)} f(x) - 1$ . To investigate the relation between  $\gamma_f$  and  $\gamma$ , we prove the following

**Theorem 6.** For any function  $f: V(G) \rightarrow Z$  satisfying  $1 \leq f(x) \leq d(x)$ ,  $x \in V(G)$ , we have

$$\gamma_f(G) \leq \frac{1}{2}(p + \gamma_{f-1}(G)).$$

*Proof.* Let  $D_1$  be an  $(f-1)$ -dominating set of  $G$  with the cardinality  $\gamma_{f-1}(G)$ , and

$$S = \{x \in V(G) \setminus D_1 : f(x) = 1\}.$$

Then

$$(f-1)(x) = \begin{cases} 1, & x \in S, \\ f(x) - 1, & x \in V(G) \setminus (D_1 \cup S). \end{cases}$$

Let  $A, B$  be, respectively, the set of non-isolated vertices and the set of isolated vertices of  $G[V(G) \setminus (D_1 \cup S)]$ . Let  $T$  be a minimum dominating set of  $G[A]$ . Then by Ore's theorem (mentioned earlier),  $|T| \leq \frac{1}{2}|A|$ .

It is easy to see that  $D_1 \cup B \cup T$  is an  $f$ -dominating set of  $G$ , so

$$(1) \quad \gamma_f(G) \leq \gamma_{f-1}(G) + |B| + |T| \leq \gamma_{f-1}(G) + |B| + \frac{|A|}{2}.$$

On the other hand  $D_1 \cup S \cup T$  is also an  $f$ -dominating set of  $G$ . In fact for any  $x \in B$ ,  $d_{D_1}(x) \geq (f-1)(x) = f(x) - 1$ . If  $d_{D_1}(x) = f(x) - 1 \leq d_G(x) - 1$ , then  $x$  must be adjacent to a vertex of  $S$ . Thus  $d_{D_1 \cup S \cup T}(x) \geq f(x)$ . If  $d_{D_1}(x) = f(x)$ , then  $d_{D_1 \cup S \cup T}(x) \geq f(x)$  as well. It is obvious that  $d_{D_1 \cup S \cup T}(x) \geq f(x)$  for any  $x \in A \setminus T$ . So  $D_1 \cup S \cup T$  is indeed an  $f$ -dominating set. Thus

$$(2) \quad \gamma_f(G) \leq \gamma_{f-1}(G) + |S| + |T| \leq \gamma_{f-1}(G) + |S| + \frac{|A|}{2}.$$

Combining (1) and (2) we get

$$\gamma_f(G) \leq \gamma_{f-1}(G) + \frac{1}{2}(|A| + |B| + |S|) = \frac{1}{2}(p + \gamma_{f-1}(G)).$$

□

**Corollary 6.** For any  $i$  with  $1 \leq i \leq m = \max_{x \in V(G)} f(x) - 1$ ,

$$\gamma_f(G) \leq p - \frac{1}{2^i} (p - \gamma_{f-i}(G)).$$

In particular,

$$\gamma_f(G) \leq p - \frac{1}{2^m} (p - \gamma(G)).$$

For each  $k$ ,  $1 \leq k \leq \Delta(G)$ , let

$$\mathcal{X}_k = \{X \subseteq V(G) : |X| = k\}.$$

Denote  $\Gamma_G(X) = \bigcap_{x \in X} N_G(x)$  for each  $X \in \mathcal{X}_k$ , and  $\Delta_k(G) = \max_{X \in \mathcal{X}_k} |\Gamma_G(X)|$ . Then  $\Delta_k(G) \geq 1$ . Let  $X \in \mathcal{X}_k$  be such that  $|\Gamma_G(X)| = \Delta_k(G)$  and  $S = V(G) \setminus (X \cup \Gamma_G(X))$ . Then  $V(G) \setminus \Gamma_G(X)$  is a  $k$ -dominating set of  $G$ . Thus

$$\gamma_k(G) \leq p - \Delta_k(G)$$

or, equivalently,

$$(3) \quad |\Gamma_G(X)| = \Delta_k(G) \leq p - n,$$

where  $n = \gamma_k(G)$ . Suppose

$$(4) \quad |\Gamma_G(X)| = p - n - r, \quad 0 \leq r < p - n.$$

Then

$$(5) \quad |S| = p - |X| - |\Gamma_G(X)| = n + r - k.$$

Note that for any  $x \in X$  and  $y \in \Gamma_G(X)$ ,  $(S \setminus N_G(y)) \cup \{x, y\}$  is a dominating set of  $G$ . Hence

$$|S| - |S \cap N_G(y)| + 2 \geq \gamma(G),$$

i.e.

$$|S \cap N_G(y)| \leq n + r - k - \gamma(G) + 2.$$

Similarly, we have

$$|S \cap N_G(x)| \leq n + r - k - \gamma(G) + 2.$$

Let  $h_k$  be the maximum number of edges in a subgraph of  $G$  with  $|S| = p - \Delta_k(G) - k$  vertices. Then we have

$$\begin{aligned}
 (6) \quad 2\varepsilon(G) &\leq 2\varepsilon(G[S]) + |X|(|X| - 1) + |X| |\Gamma_G(X)| \\
 &\quad + \sum_{y \in N_G(X)} (|N_G(y) \cap S| + |N_G(y)|) + \sum_{x \in X} |N_G(x) \cap S| \\
 &\leq 2h_k + k(k-1) + k\Delta_k + \Delta_k[(n+r-k-\gamma(G)+2) + \Delta] \\
 &\quad + k(n+r-k-\gamma(G)+2) \quad (\Delta_k = \Delta_k(G), \Delta = \Delta(G)) \\
 &= 2h_k + k(k-1) + k\Delta_k + (\Delta_k + k)(p - \Delta_k - k - \gamma(G) + 2) + \Delta_k \Delta \\
 &= 2h_k + k(p - \gamma(G) + 1) - \Delta_k^2 + \Delta_k(p - k + \Delta(G) - \gamma(G) + 2) \\
 &\leq 2h_k + k(p - \gamma(G) + 1) - \Delta_k^2 + (p - n)(p - k + \Delta(G) - \gamma(G) + 2).
 \end{aligned}$$

This leads to the following

**Theorem 7.** For each  $k$ ,  $1 \leq k \leq \Delta(G)$ , let  $\Delta_k$  and  $h_k$  be as before. Then

$$\gamma_k(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - h_k) + \Delta_k^2 - k(p - \gamma(G) + 1)}{p - k + \Delta(G) - \gamma(G) + 2} \right\rceil.$$

Since  $\gamma(G) \geq 1$  and  $h_k \leq \frac{1}{2}(p - \Delta_k - k)(p - \Delta_k - k - 1)$ , we obtain the following two corollaries.

**Corollary 7.** For each  $k$  with  $1 \leq k \leq \Delta(G)$ ,

$$\gamma_k(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - h_k) + \Delta_k^2 - kp}{p - k + \Delta(G) + 1} \right\rceil.$$

**Corollary 8.**  $\gamma_k(G) \leq p - \left\lceil \frac{2\varepsilon(G) + 2(p-k)\Delta_k + k(p+\gamma(G)-k-2) - p(p-1) - \Delta_k}{p-k+\Delta(G)-\gamma(G)+2} \right\rceil$ .

Taking  $k = 1$  in (6) we get

$$2\varepsilon(G) \leq 2h_1 + (p - \gamma(G) + 1) - \Delta^2 + \Delta(p + \Delta + 1 - \gamma(G)).$$

This implies a new upper bound for  $\gamma(G)$ .

**Corollary 9.** Let  $h_1$  be the maximum number of edges in a subgraph of  $G$  having  $p - \Delta - 1$  vertices. Then

$$(7) \quad \gamma(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - h_1)}{\Delta(G) + 1} \right\rceil + 1.$$



**Example 1.** Let  $G$  be the graph obtained from the cycle of five edges by adding a chord. Then  $p = 5$ ,  $\varepsilon(G) = 6$ ,  $\Delta(G) = 3$ ,  $\gamma(G) = 2$ ,  $\Delta_2(G) = 2$  and  $h_2 = 0$ . Theorem 7 gives  $\gamma_2(G) \leq 3$ . But it is easy to see  $\gamma_2(G) \geq 3$ . So  $\gamma_2(G) = 3$ . This shows that the upper bound in Theorem 7 is attainable.

**Example 2.** If  $G$  is the cycle with four edges, then it is easy to see that both sides of (7) equal 2. So the upper bound in Corollary 9 is attainable.

For any subgraph  $H$  of  $G$ , the restriction of the function  $f: V(G) \rightarrow Z$  to  $V(H)$  is also denoted briefly by  $f$ . Thus  $\gamma_f(H)$  is well-defined. The technique used in the proof of Theorem 7 can be applied to prove the next result, which shows the connection of  $\gamma_f(G)$  and  $\gamma_f(H)$ .

**Theorem 8.** Let  $t_q$  be the maximum number of edges in a subgraph of  $G$  with  $p - q$  vertices,  $1 \leq q \leq p - 1$ . Then for any subgraph  $H$  of  $G$  with  $q (> \gamma_f(H))$  vertices and without isolated vertices,

$$\gamma_f(G) \leq p - \left\lceil \frac{2(\varepsilon(G) - t_q) + (q - a)^2 - (p - \gamma(G))\gamma_f(H)}{p + \gamma_f(H) + \Delta(G) - \gamma(G) + 1} \right\rceil.$$

*Proof.* Suppose that  $X$  is an  $f$ -dominating set of  $H$  with  $\gamma_f(H) = a$  vertices, and that  $Y = V(H) \setminus X$ ,  $S = V(G) \setminus V(H)$ . Since  $X \cup S = V(G) \setminus Y$  is an  $f$ -dominating set of  $G$ , we have

$$\gamma_f(G) \leq p - |Y|,$$

or equivalently,  $|Y| \leq p - n$ , where  $n = \gamma_f(G)$ . Suppose

$$|Y| = p - n - r, \quad 0 \leq r < p - n.$$

Then  $|S| = p - a - |Y| = n + r - a$ .

For any  $y \in Y$ ,  $(S \setminus N_G(y)) \cup X \cup \{y\}$  is a dominating set of  $G$ , hence

$$|S| - |S \cap N_G(y)| + |X| + 1 \geq \gamma(G),$$

i.e.  $|S \cap N_G(y)| \leq n + r - \gamma(G) + 1$ . Similarly,  $|S \cap N_G(x)| \leq p - a - \gamma(G) + 1$  for each  $x \in X$ . We have

$$\begin{aligned}
 2\varepsilon(G) &\leq 2\varepsilon(G[S]) + |X|(|X| - 1) + |X||Y| \\
 &\quad + \sum_{y \in Y} (|S \cap N_G(y)| + |N_G(y)|) + \sum_{x \in X} |S \cap N_G(x)| \\
 &\leq 2t_q + a(a - 1) + a|Y| + |Y|(n + r - \gamma(G) + 1 + \Delta(G)) \\
 &\quad + a(p - a - \gamma(G) + 1) \\
 &= 2t_q + a(p - \gamma(G)) + a|Y| + |Y|(p - |Y| - \gamma(G) + \Delta(G) + 1) \\
 &= 2t_q + a(p - \gamma(G)) - |Y|^2 + |Y|(p + a + \Delta(G) - \gamma(G) + 1) \\
 &\leq 2t_q + a(p - \gamma(G)) - (q - a)^2 + (p - n)(p + a + \Delta(G) - \gamma(G) + 1).
 \end{aligned}$$

This gives

$$n \leq p - \frac{2(\varepsilon(G) - t_q) + (q - a)^2 - (p - \gamma(G))a}{p + a + \Delta(G) - \gamma(G) + 1}.$$

This completes the proof.  $\square$

#### 4. REMARKS

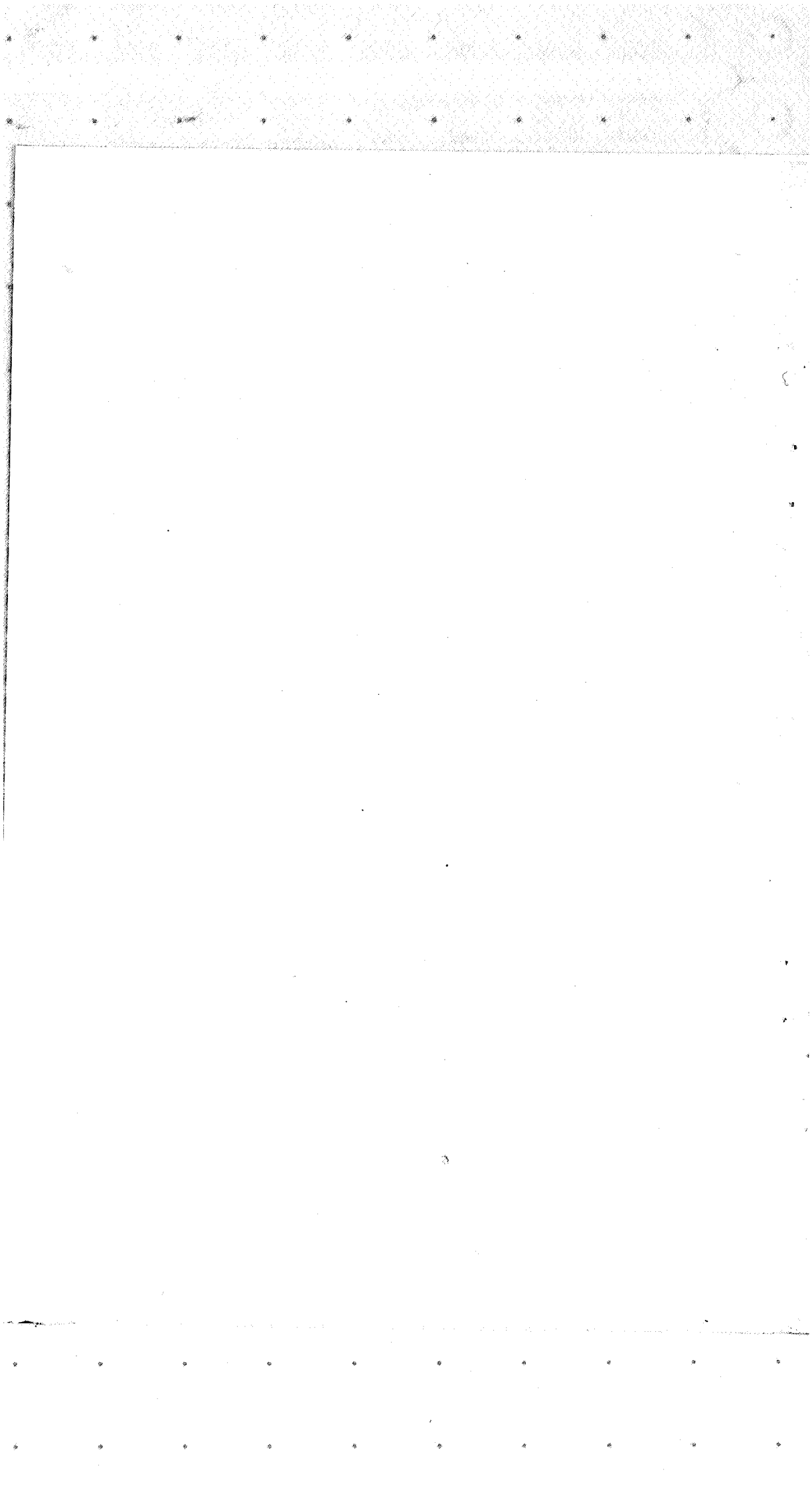
A lot of problems concerning the  $f$ -domination number and the  $f$ -independence number can be proposed. Perhaps the most attractive one is whether there exist the Nordhaus–Gaddum type inequalities for  $\gamma_f$ . Such inequalities for  $\gamma$  have been shown in [7]. Naturally we can define the upper  $f$ -domination number  $\Gamma_f(G)$  of  $G$  to be the maximum cardinality of a minimal  $f$ -dominating set of  $G$ . Also we can define the  $f$ -domatic number,  $d_f(G)$ , to be the maximum order of a partition of  $V(G)$  into  $f$ -dominating sets. Another interesting invariant is  $i_f(G)$ , which is defined to be the smallest non-negative integer  $i$  such that  $\gamma_{f-i}(G) = \gamma(G)$ . Studies on these invariants are necessary, as well as interesting. For example, relations among  $\gamma_f$ ,  $\Gamma_f$ ,  $\beta_f$ ,  $d_f$ ,  $i_f$  and other graphical invariants, e.g. the domination number, the independence number, are valuable research topics. The lower bounds and the upper bounds for  $\gamma_f$  and  $\beta_f$  deserve further study as well.

This work is on-going and results will be published later.

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