Nash Equilibrium Seeking for Augmentation of Urban Traffic Light Control

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Submitted in total fulfilment of the requirements of the degree of Doctor of Philosophy

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June 2015

Produced on archival quality paper.
Abstract

The congestion problem in urban transportation has been becoming increasingly severe in recent years due to the growth of population and vehicle ownership. It was estimated that the cost of congestion is in the order of billions of dollar and is expected to keep raising. Besides increasing the average travel time which leads to loss of productivity, congestion also leads to public health, environmental and safety issues. As such, urban traffic congestion has become a major problem that needs to be urgently addressed.

Many urban traffic controllers have been developed with this goal in mind. The task of an urban traffic controller is to decide the traffic light setting, including the duration and the timing of the green lights. However, this decision is typically dependent upon some parameters of the controller. A set of poorly-tuned parameters will lead to poor performance of the controller. The problem is compounded when these parameters are determined by trial-and-error, where finding their optimal values is an arduous task, or even impossible. Thus, a systematic and practical approach to find the optimal parameters is required. This thesis provides a solution to this problem by developing a feedback controller, based on extremum-seeking theory, to continually fine-tune these parameters while the traffic network is running. The main advantage of using extremum-seeking is that no knowledge of the plant dynamics is required a priori. Since urban traffic is highly nonlinear and subject to a variety of unknowns, extremum-seeking is useful to avoid the need to model the dynamics of traffic.

This thesis outlines several extensions to extremum-seeking theory to address new areas of application and demonstrate its benefit in a real-world problem. In particular, these extensions accommodate the application of extremum-seeking for urban traffic control.
There are several issues posed by the considered extensions that are addressed in this thesis. Firstly, the number of parameters that needs to be selected increases proportionally with the size of the traffic network. Thus, a decentralisation of the extremum-seeking scheme is developed (more commonly referred to as Nash equilibrium seeking), which relaxes the requirement of the classical extremum-seeking result. This is done by exploiting the fact that a change in the setting of a traffic light will have a diminishing effect with distance. By doing this, the implementation of the Nash equilibrium seeker is simplified, in particular when dealing with large-scale systems. This concept is generalised, and a non-local stability result is developed for the Nash equilibrium seeking scheme acting on a family of systems with continuous nonlinear dynamics. The stability proof relies upon singular perturbation and averaging theories.

Secondly, an urban traffic network is naturally a hybrid system and there has been no extremum-seeking result for hybrid systems. Thus, a novel stability result for a class of singularly perturbed hybrid systems is presented. This is an extension to previous results, which instead consider a semi-globally practically asymptotically stable and continuous reduced system. Furthermore, differential inclusions are used such that systems with state disturbances are accommodated. The class of systems considered covers many engineering problems, such as switching control, controllers with internal timers, and urban traffic control. In addition, the singular perturbation result is used to produce the first stability result for extremum-seeking acting on hybrid systems.

Finally, the performance of the proposed scheme is investigated in simulation. Three existing strategies from three different categories of urban traffic light control are augmented with the proposed Nash equilibrium seeking scheme. The Nash equilibrium seeker is shown to improve the performance of the traffic controllers by fine-tuning their parameters, despite being subjected to various traffic conditions and settings. Therefore, it is demonstrated that the proposed scheme is able to increase the average travel speed, which consequently reduce congestion and assist in alleviating the associated problems.
Declaration

This is to certify that

1. the thesis comprises only my original work towards the PhD,

2. due acknowledgement has been made in the text to all other material used,

3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

__________________________________________
Ronny J. Kutadinata, June 2015
Acknowledgements

I would like to thank Professor Chris Manzie and Dr. Will Moase for their guidance and support over the last 4 years. I am grateful to have such an enjoyable learning experience under supervisors who are not only professional but also willing to go the extra mile in assisting me. In particular, I acknowledge the amount of patience Chris and Will have shown when dealing with all my questions and drafting all the related publications (along with this thesis).

I realise that at times I could be an inconvenience by working from home and I acknowledge their willingness to respond to all my emails in detail. And I acknowledge their extraordinary support during my third year when I struggled to focus on my research. I am especially grateful for all the deadlines Chris has given me, motivating me to keep going forward.

Throughout my PhD I have benefited from having discussions and collaborating with other researchers. In particular, the results of Chapter 5 were obtained by collaborating with Joyce (Lele) Zhang and Tim Garoni from Monash University. I acknowledge the use of their traffic simulation codes and I am especially grateful for Joyce’s constant support when I was having troubles with the codes. I would also like to thank Wei Wang from the Electrical Engineering department of the University of Melbourne, whose help greatly improved the quality of the results in Chapter 4. I also acknowledge the feedback I received from other researchers, including Majid Sarvi, Andrew Teel, Nicholas Gant and Eli Kwon. I am also grateful to those who introduced me to research: Doreen Thomas, Peter Grossman, and Marcus Brazil.

I would like to thank all my friends, especially fellow PhD students in the research group. In particular, special thanks to: Terence Kim and Chih Feng Lee for easing my
transition into research study; Denise Lam, whose thesis and reports have always been
the template of my own documents, and Alan Chang who both have aspired me to al-
ways work hard; Brett Bishop for our interesting conversations; Jalil Sharafi, my “best
buddy”, for all the useful discussions; Changfu Zou for the parking permit that solves
my parking dilemma; Vincent Bachtiar who was also my ex-housemate; Sai Hin Tse for
all the tennis sessions and the unforgettable “Hin & Ronny Tennis Open” cup that now
stands on a cabinet in my house (which I won with a score of 6–2, 4–6, 7–5, 6–3); and my
church cell group members for their support and encouragement. I would also like to
thank John Papandriopoulous for the use of his excellent thesis template.

Last, but certainly not least, I am grateful for the encouragement of my wife and our
family members. This PhD would not happen without their continuous support, both
financially and emotionally. In particular, I would like to express my appreciation for the
effort put forth by my wife to make sure that I could fully focus on my research and finish
the PhD in time.
To the love of my life, Paramitha.
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Chapter 1
Introduction

1.1 Motivation

Urban transportation is at the core of modern society but faces growing problems associated with increasing population densities and vehicle ownership. As a result, traffic congestion occurs in many major cities, which may have an associated cost in the order of billions of dollars. The cost originates from the decrease of productivity, increased fuel use, environmental and health impacts.

Several studies have identified the cost of congestion in various countries. An estimated cost of AU$9.4 billion was associated with congestion in Australia in 2005, and is expected to increase up to AU$20.4 billion in 2020 [1]. New Zealand suffers a loss of up to 1% of the country’s gross domestic product, which is contributed by the loss of income, time and and pollution due to congestion [115]. It was also found that in many metropolitan areas in the United States in 1994, traffic congestion has already cost each driver, on average, US$640 annually due to driving delay [17]. Hence, urban traffic congestion has imposed heavy financial burdens to many nations and is an issue worthy of special attention.

Urban traffic congestion also causes increased pollution from vehicle emissions. Up to 33% of carbon dioxide emissions in the United States comes from the transportation sector, 80% of which is produced by cars and trucks on motorways [19]. In addition, vehicle exhaust emissions have been pointed out as a dominant source of particulate matter both with diameter of 10 µm or less (PM$\text{_{10}}$) and fine particles with diameter of 2.5 µm or less (PM$\text{_{2.5}}$) [63]. Not only does this impact the environment, but vehicle emissions
also bring adverse health effect. Several studies have found that exposure to particulate matter leads to an increase in daily mortality associated with cardiopulmonary and lung cancer. It is stated in [83] that a 10 $\mu g/m^3$ increase in PM$_{2.5}$ from mobile combustion sources accounted for a 3.4% increase in daily mortality. Furthermore, [14] found that each 10 $\mu g/m^3$ elevation in fine particulate air pollution increases the risk of lung cancer mortality by up to 8%.

All of the aforementioned issues can potentially be lessened by reducing urban traffic congestion. Tackling urban traffic congestion involves many approaches on several different time-scales.

At a slower time-scale, urban traffic congestion can be mitigated by infrastructure expansion through which the network capacity is increased. Typically, infrastructure expansion requires years to complete and involves a large capital cost. For example, the widening of the CityLink–Tullamarine freeway in Melbourne has a budget of AU$850 million [33], and the upgrade of the M80 Ring Road in Melbourne is a AU$2.25 billion project [131]. Furthermore, the Federal Highway Administration in the United States has estimated in 2002 that the cost to add lanes in urban freeways and arterials was US$5.7 million and US$4.2 million per lane-mile respectively. Similarly, public transport improvement through infrastructure expansion has a slow time-scale and a large capital
1.2 Urban Traffic Light Control

Urban traffic can be represented by a collection of intersections (nodes) and roads (directed links) as depicted in Figure 1.2. The main focus of urban traffic control is the organisation of traffic flows at these intersections to avoid collisions of conflicting flows. Although it is possible to control an intersection using roundabouts, “Stop” or “Give Way/Yield” signs, when the traffic state is more congested, higher average traffic flow cost. For instance, Regional Rail Link in Victoria, Australia expanded its rail network taking 6 years to finish with a fund of AU$3.2 billion [111]. Furthermore, a AU$1 billion Gold Coast tram project, which started in 2006, took 8 years to complete [57].

On the other hand, public transport improvement through policy modification is of a medium time-scale, and can typically be carried out within months. An example policy is to allow free tram rides in the Melbourne CBD and cheaper fares effective January 2015 [32]. This encourages the use of public transport and reduces the number of private vehicles in the traffic network. Several less common examples of policy imposed to reduce the number of vehicles circulating in the traffic network are: road space rationing [62] or road pricing schemes [72].

This is then complemented by the improvement of traffic management, which has the fastest time-scale. In contrast to infrastructure expansion, more efficient utilisation of the existing infrastructure through improved traffic management has the potential to decrease congestion (or increase network capacity) at little cost. For instance, it is reported that, compared to a pre-timed/fixed traffic signal plan, the initial cost of the implementation of an adaptive traffic light control is only US$20,000 higher, with an average operating cost difference of only US$9,376 per mile per year [41].

Obviously, each of these alternatives needs to be explored in order to reduce traffic congestion. However, it can be argued that urban traffic flow is especially dependent on the coordination of conflicting traffic streams at intersections. Hence, this work considers tackling urban traffic congestion by improving traffic management through intersection control.
can be achieved by using traffic lights [42]. Hence, the majority of approaches of urban traffic control is by the use of traffic lights [22, 36, 67, 84, 113]. However, the formation of vehicle queues is unavoidable during red lights and there is potential upstream propagation of these queues [107]. Therefore, it is important that traffic lights at intersections are controlled efficiently to avoid congestion.

Figure 1.2: An example of traffic network as nodes/intersections (I) and links(L)

The application of traffic light control is not limited to urban traffic; traffic light control can also be applied to freeway traffic. For example, a traffic light control scheme has been applied to the Monash Freeway, Melbourne [110], resulting in an improvement of the average flow by up to 8.4% and the average speed by up to 58.6%. The traffic light controller applied to the Monash Freeway is feedback-based integral regulator on-ramp metering, which attempts to regulate the occupancy rate (equivalent to vehicle density) of the freeway to an optimum that provides maximum throughput [60]. The red light duration of the traffic signal on the on-ramp is the control variable/input, which is adjusted accordingly based on the occupancy measurement of the freeway downstream of the on-ramp. This strategy has been modified and extended to address the issues associated with the original version: the on-ramp queues management and the required knowledge of the critical occupancy. Firstly, there are a few on-ramp queue control methods proposed [116, 122] in the literature, which show that on-ramp queue spillback can be prevented
while still maximising the on-ramp storage space exploitation. Secondly, several real-time (online) critical occupancy estimation schemes are developed when its value cannot be estimated beforehand and/or time-varying [117]. The schemes use feedback traffic measurement to correct its estimation at each iteration, and have been shown to allow the use of the on-ramp metering without prior knowledge of the critical occupancy. These control approaches are not easily extendable to urban traffic networks, which (unlike freeways) consist of complex, interconnected, multi-directional traffic flows (as shown in Figure 1.3). Nevertheless, the success of on-ramp metering presents a strong case for the application of closed-loop traffic light control in urban traffic networks.

Figure 1.3: The comparison of complexity between freeway and urban traffic. Freeway traffic consists of only one main traffic flow with on-ramps merging from the sides whereas urban traffic is a complex network with many conflicting flows.

The difficulty of urban traffic control arises from the facts that [108]:

- Controlling the whole network can be a very large size problem which includes discrete variables.

- The system is always subject to disturbances.

- Measurements are limited and local with high noise.

Since a traffic light might have a widespread effect on traffic conditions well beyond its adjacent intersections, urban traffic light control involves more than just determining the
green light duration. In fact, there are four main key signal parameters when deciding the traffic lights plan of an intersection, namely phase, cycle, offset, and green split. At each intersection, there are several phases, each of which consists of a group of traffic flows. The flows within each group do not intersect with each other and are given green lights simultaneously. All of the phases at an intersection are typically run in a cycle. Within a cycle, each traffic flow typically receives a green light and a red light once. The proportion of the green light duration of a phase within a cycle is called the green split. In addition, the start time (with respect to some reference time) of a given phase at two adjacent intersections can also be adjusted to allow a particular traffic direction to flow continuously. The difference between the start time of the phase at these two intersections is referred to as the offset. These definitions (except offset) are illustrated in Figure 1.4.

It is clear that the control of multi-phase urban traffic lights at an intersection is not a straightforward problem. This can be further complicated by introducing multi-modal (multiple types of vehicles) traffic which might include buses, trucks, or trams. However, this work will only consider traffic with homogeneous fleets of vehicles, i.e. uni-modal traffic.

![Figure 1.4: An example of a typical traffic light cycle at an intersection.](attachment:image.png)
1.3 Thesis Layout

This research seeks to develop the underpinning theory to optimise urban traffic flow online and implement it on a realistic simulation. In particular, this project will focus on developing an optimisation scheme for multi-phase traffic lights in a uni-modal, urban environment.

The remainder of the thesis is laid out as follows. In Chapter 2, an assessment of existing urban traffic controllers is presented. It is found that existing approaches have some limitations that prevent them from performing optimally, due to the complexity of traffic dynamics, disturbances, time-varying traffic conditions and the size of the traffic network being large. These existing controllers can potentially be augmented by employing a model-free optimisation scheme such as extremum-seeking (ES). Thus, a comprehensive ES literature review is presented and the extensions required for applications on urban traffic light control are identified.

A decentralisation of a multi-input multi-output (MIMO) ES scheme via dither frequency re-use is developed in Chapter 3. Specifically, the MIMO ES scheme considered is a Nash equilibrium seeking (NES) scheme. Each dither typically requires a unique frequency such that the sensitivity rate of each output with respect to each input can be estimated. In a system where some of these sensitivities are small, a dither frequency can be re-used without introducing a significant estimate bias of the sensitivity. A stability proof is outlined that demonstrates the convergence of the proposed scheme under a sufficiently small amount of dither re-use. Due to the allowance of dither re-use, the tuning of the NES scheme is simplified during implementation, which alleviates the dimensionality issue associated with the size of the traffic network.

Following on from the result in Chapter 3, the ES scheme is extended to address a class of singularly perturbed hybrid systems in Chapter 4. The considered hybrid systems are represented by differential inclusions, which handles the issue of dealing with disturbances. The contributions are separated into two parts. Firstly, the stability of singularly perturbed hybrid systems with a hybrid fast subsystem, a continuous slow subsystem, and a continuous semi-globally practically asymptotically (SPA) stable reduced system is presented. In fact, the structure of the considered singularly perturbed hybrid sys-
tems encompasses the application of a vast array of ES schemes, including the proposed NES scheme in Chapter 3, onto a fairly general nonlinear hybrid system. Secondly, this singular perturbation result is used to demonstrate the stability of an ES scheme with second-order linear filters.

The application of ES for adaptation of urban traffic light controllers is presented in Chapter 5. The application is carried out in a simulation based on a cellular automata model. Three traffic light controllers are considered: a perimeter control scheme [2,12,59], SCATS [93,136], and self-organising traffic lights (SOTL) [31,136]. The simulation results demonstrate the existence of steady-state average behaviours of the traffic network under these considered traffic controllers. It is found that the steady-state average performance experiences an optimum with respect to some tuning parameters, yet the optimum depends upon the traffic condition. Then, it is shown that the use of ES as an augmentation is able to improve the performance of these traffic controllers by adapting their tuning parameters.

Finally, the summary of the contributions of this thesis and the highlight of potential areas for further research are laid out in Chapter 6.
Chapter 2
Literature Review

This chapter presents the literature review in five sections. Firstly, the different measures of traffic congestion are discussed to provide an insight into the specific objectives involved in minimising congestion. Secondly, existing urban traffic light controllers available in the literature are reviewed. From the review, potential shortcomings of existing strategies are discerned. In particular, the optimality of their performances depend on the selection of their tuning parameters, yet the optimal value depends on the traffic condition. Thus, an online calibration technique can be used to address these shortcomings, such as extremum-seeking. Section 2.3 provides the description of existing theories on extremum seeking (ES), which is the control technique that is proposed for the augmentation of existing urban traffic light controllers. In addition, it is discussed how existing ES schemes are not fully suited for urban traffic control. Thus, the required extensions for ES theory are identified. In the last two sections, the conclusions and the research aims are outlined.

2.1 Measure of Congestion

Broadly speaking, the aim of urban traffic control is to minimise congestion. However, the specific cost function/performance measure to be optimised for different traffic light control strategies can be different. Most of these various cost functions are different kinds of congestion indicator in urban traffic.

The most straightforward congestion indicator is the total travel time of the users while in the traffic network [86,88]. As the network gets more congested, the total travel time
increases proportionally. However, this performance measure is potentially misleading since some vehicles with the same origin and destination might travel through the network via different routes. The biggest problem with this cost function is that it is very difficult to measure in practice, as it is very impractical to track each individual vehicle and measure its total travel time.

An alternative is to measure the delay experienced by the commuters in the traffic network [5, 107]. As depicted in Figure 2.1, delay can be measured as the extra travel time required due to stopping (horizontal portions of the trajectories). Arguably, the delay is a better performance measure than total travel time since it is independent of the routes chosen by the drivers.

An approach to minimise delay is to maximise the amount of vehicles that experience a series of green traffic lights in an arterial road, which is referred to as the green wave (or the green band). By doing this, the delay caused by the stop-start behaviour is minimised [90]. The maximisation of the size of the green band is heavily related to having the offset as the decision variable (this will be explained in more detail in Section 2.2.1).

Alternatively, one of the most common objectives in urban traffic control is queue length minimisation [23, 30, 113]. By observing Figure 2.1, minimising the queue length is
2.1 Measure of Congestion

actually similar to minimising the delay of the traffic network. However, queue length minimisation is commonly preferable over minimising the delay due to the practicality of its measurement. For instance, the number of queuing vehicles can be estimated by measuring the arrival and exit flow rate of vehicles into a road using simple sensors such as inductance loop detectors [36]. The drawback of queue length minimisation is that it does not consider the possibility of one road being clear of vehicle queue while another being heavily congested, even if the total queue length is low. Alternatively, balancing the queues in the network [36] can prevent such a scenario and avoid queue spillback during heavy traffic conditions.

When the concern is emphasised on the congestion in the whole network rather than looking at each individual street, the concept of the Macroscopic Fundamental Diagram (MFD) is often relied upon. Originally, a fundamental diagram is a concave curve that describes the relationship between vehicle density and vehicle flow on uninterrupted and one-directional traffic flows (such as those on freeways) [85,112]. The existence of a similar relationship between density and flow in other road structures are investigated further [13,28,29,51,55,66,94,134]. In particular, analytical analysis reveals that the approximate shape of the curve of flow against density in an urban traffic network is concave [29] (as shown in Figure 2.2), which has been verified by experimental results [51]. Further studies for different scenarios of urban traffic networks also show that the concave nature of the MFD always presents [52, 136]. Therefore, the optimal throughput/flow of the network is typically sought (or equivalently, finding the corresponding network density) in network congestion control strategies, such as perimeter control schemes [2] (described later in Section 2.2.2).

Therefore, there are several ways to measure the level of congestion in a traffic network. The measure used is determined by the objective of a given strategy. The next section discusses the various urban traffic control strategies available in the literature.
2.2 Traffic Light Control Strategies

There have been major developments in intelligent control of urban traffic lights over the last few decades. These advanced control strategies have been implemented either in a traffic simulator or in practice with varying degrees of success reported under different operating conditions. An ideal urban traffic light control strategy should be able to cope with all traffic conditions, namely undersaturated (light traffic with small number of queues), saturated (heavy traffic with reasonable queues) and oversaturated (very heavy traffic with congestion leading to queue spillbacks). Furthermore, with the development of sensing technology and data processing, it should be adaptive to real-time traffic conditions and robust against disturbances, such as traffic demand fluctuations, accidents, road works or special events.

Urban traffic light control can be broadly classified into offline (open-loop) or online (closed-loop); and model based or non-model based. Table 2.1 shows some existing urban traffic light controllers in the literature based on these classifications.

2.2.1 Open-loop traffic control

Earlier work on traffic light control is typically characterised as open-loop. Due to the absence of any feedback measurements, open-loop traffic controllers require the use of traffic models\(^1\) to calculate the traffic light setting by using an approximated traffic de-

\(^{1}\)The reader is referred to [27] for further reading on traffic modelling.
2.2 Traffic Light Control Strategies

Table 2.1: Urban traffic light control classification

<table>
<thead>
<tr>
<th>Model based</th>
<th>Non-model based</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Open-loop</strong></td>
<td><strong>Closed-loop</strong></td>
</tr>
<tr>
<td>TRANSYT [113]</td>
<td>SCOOT [67]</td>
</tr>
<tr>
<td>MAXBAND [90]</td>
<td>Self-organising traffic light (SOTL) [31, 53, 84, 136]</td>
</tr>
<tr>
<td>MULTIBAND [49, 124]</td>
<td></td>
</tr>
<tr>
<td>Alvarez et al. [10]</td>
<td></td>
</tr>
<tr>
<td><strong>Closed-loop</strong></td>
<td></td>
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<tr>
<td>OPAC [47]</td>
<td></td>
</tr>
<tr>
<td>PRODYN [65]</td>
<td></td>
</tr>
<tr>
<td>CRONOS [22]</td>
<td></td>
</tr>
<tr>
<td>TUC [36]</td>
<td></td>
</tr>
<tr>
<td>RHODES [97]</td>
<td></td>
</tr>
<tr>
<td>Tettamanti &amp; Varga [130]</td>
<td></td>
</tr>
<tr>
<td>Lin et al. [89]</td>
<td></td>
</tr>
<tr>
<td>Perimeter control [2, 59, 61, 68]</td>
<td></td>
</tr>
</tbody>
</table>

mand profile obtained from historical data. The differences between these open-loop strategies are mainly the traffic model used and the optimised cost function. Open-loop approaches are still commonly used in many cities and are still important in the design phase of traffic networks. For instance, the TRANSYT strategy can be used as a benchmark to be compared with new traffic signal control strategies [38].

However, these controllers often suffer from a lack of robustness if faced with plant-model mismatch, disturbances or different traffic conditions. Firstly, their developments typically assume known and constant traffic demands and other conditions, such as turning probability and saturation flow. In addition, the model used to calculate the traffic light inputs has to be suitably accurate. The following various strategies are discussed to illustrate these points.

TRANSYT [113] is one of the earliest and best known traffic light controls, which is often used as a reference method to test new strategies. It is noteworthy that the average travel time of a network was reduced by 16% during its first field implementations [108]. Using platoon dispersion dynamics (first-order time-delay system) to model the traffic flow along a lane, an offset is calculated in addition to the splits and cycle time. Some assumptions used in this approach are constant traffic demand and constant (and known) turning rate. However, this strategy uses a hill-climbing optimisation algorithm which
only finds a local optimum. Also, the strategy requires the use of a common cycle time.

Another early approach in offline urban traffic control is the phase-based approach. This strategy analytically solves the optimal green splits at an intersection given that the phases and the traffic demand are known. This work approximates the delay of a particular link as a nonlinear function of the cycle time, total effective green split, traffic demand and the saturation flow rate. This approach has two different optimisation criteria, each implemented in two different programs, SIGSET and SIGCAP. The objective of SIGSET [5] is to minimise the sum of the weighted delay of all links. The control variables are the green split of each phase and the cycle time, and the cost function is the total weighted delay. On the other hand, SIGCAP [6] attempts to accommodate the fact that the actual demand might differ to the predicted demand. SIGCAP assumes that the actual demand is some percentage higher than the predicted demand. This strategy then maximises the amount of extra demand the network can accommodate without developing queues build up. Therefore, this will increase the robustness of the traffic signal plan.

An alternative to the phase-based approach is discussed in [9–11], which is an isolated intersection control. The approach is to consider the problem as a non-cooperative game where each link is a player that attempts to minimise its queue. Under the assumption that the average arrival flow rate is known, this strategy solves for the optimal green split, which is characterised by the Nash equilibrium, by using linear programming.

MAXBAND is a control strategy which aims to maximise the size of the green band in the traffic [90]. The problem is formulated as a mixed-integer linear programming problem with the objective of creating an optimal traffic light pattern such that the width of the green band is maximised. However, the green splits have to be pre-determined prior to optimising the offsets. An extension of this work is presented in [48], which is known as MULTIBAND. In this case, the width of the band can vary between links. Some assumptions used in both approaches (MAXBAND and MULTIBAND) are pre-specified splits, constant and known queue clearance time and average speed. The drawback of this approach is the fact that this strategy does not include split and cycle time optimisation. Optimisation of opposing traffic flow might also be difficult to carry out.

The traffic model used in [3, 30] utilises the vehicle conservation equation (analogous
to fluid flow) for each link at each intersection, where the dynamics of the queue length is governed by the vehicle arrival flow and outflow rate. At a signalled intersection, the outflow rate when there is a queue can be assumed to be constant, which is referred to as the saturation flow rate. When there is no queue, it must mean that the outflow rate is equal to the arrival rate. Therefore, depending on the existence of a queue and the traffic light state, the outflow rate has a value that fluctuates in time. Via discretisation, this strategy reformulates the problem as a linear programming [30] or quadratic programming [3] depending on the objective/cost function of the optimisation. The outflow from a link is typically averaged over a cycle and is assumed to be time-invariant (i.e. only depends on the green split) [30], although an extension to include fluctuating outflow has been done [3]. The drawbacks of this strategy are the assumption of constant traffic demand and the use of averaged dynamics in modelling the plant, which potentially limit its application in practice.

The aforementioned open-loop controllers cannot always handle plant-model mismatches, changes in the traffic condition, or disturbances well. As a result, these strategies might not perform as well as expected during application. To address this shortcoming of the open-loop approach, more recent research has focused on closed-loop control.

2.2.2 Closed-loop traffic control

A closed-loop traffic controller utilises real-time measurements to correct for measured changes in the traffic. Therefore, this overcomes a major limitation of open-loop traffic controllers. The variety of the approaches used in closed-loop traffic control is large. However, closed-loop traffic control can be classified into two main groups: model based or non-model based strategies.

**Model based**

Closed-loop model based control has been increasingly popular as a control technique in recent years. A model based controller uses a prediction model of the traffic system to find an appropriate traffic light input. Two common examples of model based strate-
gies are: rolling horizon model predictive control (MPC) [47, 65, 87, 97, 130] and linear quadratic regulator/integrator (LQR/I) [2, 35]. A less common example is a type of self-organising traffic light (SOTL) control [84].

Despite the recent popularity of model based control strategies, they possess several drawbacks. Firstly, the computational requirement of a model based approach using a dynamical and realistic model is often high and, as a result, impractical. Subsequently, attempts to use more simplified models have been carried out, which typically use “averaged” parameters that might reduce the performance of these strategies. Besides these model parameters, the controllers also have tuning gains that are often manually selected. Therefore, their performances are dependant on how well these parameters and gains are approximated and selected. The following examples illustrate these points.

MPC is a control strategy that uses a prediction model to find the optimal control input over the future optimisation horizon. However, the optimal control input is only applied over a period much shorter than the optimisation horizon, after which new measurements are used to repeat the whole optimisation process. Furthermore, the control variables/inputs are not necessarily the split, offset and cycle length. Instead, some strategies [47, 65, 97] decide the next phase switching time. However, due to the rolling horizon approach of MPC that requires the optimisation to be solved online, the resulting computational difficulty is high such that the application of MPC when using a high-fidelity model is limited to controlling one intersection or a few at best [22, 47, 64, 65, 97]. In addition, some strategies might find only a local optimum (such as the Box’s algorithm [77] used in [22]). To address this computational issue, linear models are used [4, 21, 34–38, 86, 88, 89, 109], which might affect the performance of the MPC as previously discussed.

It is also possible to model the whole network as a group of smaller regions, where each region is modelled by an aggregate dynamical equation. This leads to the introduction of perimeter and boundary control [2, 59, 68], which essentially is a gating strategy that limits the incoming traffic flow such that the density in each region stays close to its critical value. The vehicle conservation concept is used, where the number of vehicle inside a region (equivalent to density) is changing based on the number of vehicles enter-
2.2 Traffic Light Control Strategies

ing the region (either from other regions or outside the network) and finishing their trips. The number of vehicles that complete their trips (throughput) is predicted by using the MFD. However, the number of vehicles that enter the network is actually just a proportion of the whole demand. This proportion is determined by the perimeter control such that the density of the region is regulated to the specified set-point. Therefore, the performance of perimeter control largely depends on the density set-point, which is manually selected to be close to its optimal value. However, the MFD of the network might vary for different traffic conditions [136], whereas these works assume it to be time-invariant.

Another approach of closed-loop model based control is Self-Orginising Traffic Light (SOTL) control. Essentially, this approach is priority-based, which means a green light is given to the phase that is most congested. However, the detail of each SOTL strategy may differ greatly. For instance, the objective of the strategy in [84] is to minimise the waiting time by allocating green time to the flow with the highest priority. The priority is calculated based on the anticipated waiting time caused by the control decision. The controller has two options: to clear the queue on the currently served flow; or to switch to serve another flow. The calculation of the anticipated waiting time is based on a very simple equation which uses the measurement of vehicle queue length as feedback.

Although model simplifications have been carried out to reduce the computational requirements of model based strategies, the model parameters that are present (including those introduced by the simplification process) still need to be properly selected, which potentially affect their performances. Due to the complexity of urban traffic, the selection of these parameters is not a straightforward task. Typically, they are chosen by trial-and-error or through a series of extensive experiments. Therefore, these strategies might benefit from the use of an online parameter calibration technique.

Non-model based

An online non-model based approach establishes “rules” to be followed. Generally, these rules introduce some tuning parameters, which have largely been manually calibrated. Similar to other categories of traffic control, non-model based strategies also use constant parameters that do not necessarily handle changes in traffic conditions. Thus, issues
similar to those discussed for other categories are also expected to occur. Some examples of non-model based strategies are: SCATS [93,136], SCOOT [67], and another type of self organising traffic lights (SOTL) [31,53].

A class of online non-model based traffic control strategy is a mixture between open-loop and closed-loop control. Two strategies that fall in this category are SCATS [93] and SCOOT [67]. These two strategies require a pre-specified signal plan which then is slightly modified based on the measurement of the real traffic condition.

SCOOT utilises a pre-specified traffic signal plan, which then is modified slightly based on the measurement of the degree of saturation. The degree of saturation is a measure of the green time utilisation, which indicates what proportion of the green time is used for vehicles to go through. Real-time measurements are used to investigate the effects of small changes applied to green splits, offsets and cycle length. A small percentage of this modification is made permanent if it is beneficial, which will result in a slowly evolving traffic signal plan. Due to the slow adaptation, the responsiveness of this strategy to a change in the traffic condition is still limited. In addition, the performance of the strategy largely depends on the optimality of the pre-specified signal plan.

SCATS has been used in many countries and cities, and has been used in Australia for decades. SCATS is a semi-decentralized control strategy where the traffic network is divided into subsystems, each consisting of several intersections. The controlled variables in this strategy are cycle time, green split, and offsets. Similar to SCOOT, SCATS operates based on a set of pre-specified traffic light setting plans, with a small amount of modification based on the measurement of the current traffic condition. The pre-specified plan is chosen in real-time by a voting algorithm which utilises the measurement of the degree of saturation.

SOTL control strategies are more commonly model-free methods [31,53,54,107,136], although they are still priority-based as their model-based counterpart. For the majority of the thesis, the author refers to the non-model based version when mentioning SOTL. A type of model-free SOTL is explained in detail in [31] and [136]. At each intersection, this controller measures the value of the threshold function of a phase, which represents the “busyness” of a phase. Both the traffic demand and how long the phase has been
idle influence the value of the threshold function. Then, when the threshold function
of a phase exceeds a threshold value, it becomes a candidate for the next active phase.
The selection of the threshold value changes how quickly the phase switches, and hence
affects the performance of SOTL [31].

In conclusion, similar to the other traffic light controls, the performance of a closed-
loop non-model based strategy depends on its tuning parameters, such as offset, maxi-
mum cycle time, green splits [67, 93, 136], and threshold values [31, 84, 136].

From the aforementioned classifications of urban traffic light controllers, it is clear that
parameter selection is an important consideration in the development of the closed-loop
system. These parameters are often approximated and assumed constant, although they
are sometimes time-varying. Furthermore, approximating these parameters might not be
a simple task. For instance, finding the optimal set point in a perimeter control strategy
[2] requires a rigorous investigation through simulations or experiments. Therefore, the
use of an adaptation technique to fine-tune these parameters is potentially beneficial for
the performance of these controllers One such technique is extremum-seeking, which in
fact has been used for exactly this purpose [24, 73–75, 120, 121]. Thus, extremum-seeking
forms the basis of the next section.

2.3 Extremum Seeking

Extremum-seeking (ES) is a non-model based steady-state optimisation scheme for dy-
amical plants. An ES controller regulates the input of a dynamical plant to the value
that optimises the steady-state output of the plant, without requiring knowledge of the
underlying dynamics. In order to achieve this, ES requires several components, namely
a gradient estimator that includes a dither signal, and an optimiser (Figure 2.3) operating
in progressively slower time-scales. Essentially, the input to the plant is perturbed by a
dither signal and its effect on the output is used to estimate the gradient of the steady-
state input-output (SSIO) mapping. The estimated gradient is then used by the optimiser
to drive the input-output pair to its extremum. The controller can be designed either in continuous time (e.g. [15]) or in discrete time (e.g. [25]).

![Figure 2.3: The components of a typical ES scheme](image)

Typically, there are three different time scales associated with the overall closed-loop system, with the plant being the fastest, the gradient estimator having a medium time scale, and optimiser as the slowest. This time scale separation is important for a typical ES scheme to work properly. To begin with, the plant’s transient response has to settle before the local gradient of the SSIO map can be estimated. Then, the optimiser needs to be slower than the gradient estimator so that the gradient can be properly estimated. Moreover, this time scale separation is exploited by using singular perturbation and averaging analysis in the stability proof [16,76,127]. In order to achieve time-scale separation (and therefore stabilise the scheme), certain controller parameters have to be chosen to be sufficiently small. Since the stability of the scheme depends on these parameters, the following stability concept is useful to capture such a situation.

One of the most common types of stability guaranteed for an ES scheme is weaker than a global yet stronger than a local stability, and is referred to as semi-global practical asymptotic (SPA) stability. Consider a parameterised family of systems

\[ \dot{x} = f(t, x, \epsilon), \]  

(2.1)

where \( x \in \mathbb{R}^{N_x} \) is the system’s state and \( f : \mathbb{R} \times \mathbb{R}^{N_x} \times \mathbb{R}^{N_x} \to \mathbb{R}^{N_x} \), \( \epsilon \in \mathbb{R}^{N_x} \) is a small parameter vector; reducing \( \epsilon \) tends to result in slower convergence of the closed-loop. Many ES schemes fit the structure of (2.1) when implemented on a general continuous nonlinear system. The system (2.1) is said to be SPA stable if for any \((\Delta, \nu) > 0\), there exists a sufficiently small \( \epsilon > 0 \) such that a closed ball of radius \( \nu \) around the origin is an
attractor given any initial conditions within a closed ball of radius $\Delta$ around the origin. This means that a sufficiently small closed “box” can be fixed around the origin in the parameter space $\epsilon$ and the scheme is stable for any parameters within this box.

The result is semi-global in a sense that the basin of attraction ($\Delta$) can be made arbitrarily large and it is practical because the attractor ($\nu$) can be made arbitrarily small. However, it is worthwhile to note that increasing $\Delta$ and reducing $\nu$ typically requires smaller $\epsilon$, thereby requiring slower convergence [100, 127].

### 2.3.1 Variations of ES schemes

In recent years, there have been significant developments in ES. Although many ES schemes operate on similar principles and require the aforementioned three different time-scales, there are many methods to realise the components described in Figure 2.3. There are also several other approaches and techniques that have been used to achieve various results, such as fast convergence and global stability. Hence, it is useful to discuss these various schemes available in the literature.

The simplest ES scheme is outlined in [127]. This scheme, shown in Figure 2.4, only consists of an integrator and a dither signal, which is used to perturb the input to the system and to demodulate the output signal. After the output undergoes demodulation, the resulting signal, on average, approximates the gradient of the SSIO map (which leads to the use of averaging analysis). Thus, the dither in combination with the demodulator is essentially the gradient estimator, whereas the integrator is a gradient descent (indicated by the negative gain, ascent if the gain is positive) optimiser. In this case, the small parameters are: the dither amplitude $a$, the dither frequency $\omega$, and the adaptation gain $k$. As previously mentioned, the small parameters $(a, k, \omega)$ have to be sufficiently small to achieve SPA stability of the full closed-loop system in Figure 2.4.

Alternatively, higher-order gradient estimators can be utilised by the ES scheme. An example of a higher-order ES scheme is shown in Figure 2.5 [16, 76]. In this scheme, there are a high-pass and a low-pass filters before and after the demodulation signal, respectively. Besides $(a, k, \omega)$, the design parameters of the ES scheme now include the cut-off frequency of the high-pass filter $\omega_H$ and of the low-pass filter $\omega_L$. Although it
involves more design parameter compared to the simple ES scheme, the filters, which are parts of the gradient estimator, assist in the extraction of the components of the output signal which lie within a range of frequencies corresponding to that of the dither signal. Similarly, the optimiser is simply a gradient descent, realised by the integrator and the negative gain. Another example of gradient estimators is a Luenberger observer [101]. In fact, the observer is able to extract the gradient and the higher derivative terms of the SSIO map such that a more complex optimiser can be used, such as a Newton-like adaptation scheme.

Although a sinusoidal perturbation is still the most commonly used type of dither [16, 43, 125–127], other periodic perturbations can also be used as long as they are persistently exciting (PE) [58, 100, 106]. Furthermore, the stability of ES under stochastic perturbations has been demonstrated in [92, 95, 119]. Similarly, these stability results (acting on a static plant) have been shown via various averaging techniques and, when dealing with plants with dynamics, a time-scale separation is also required [24, 120, 121].
2.3 Extremum Seeking

A method to achieve an arbitrarily fast convergence has been proposed in [98, 99, 101, 103] when dealing with Hammerstein plants. An extra linear output filter can be inserted before the gradient estimator to achieve an arbitrarily fast convergence rate [101] by allowing an arbitrarily large\(^2\) dither frequency \(\omega\). As a result, one of the time-scale separations between the plant and the ES scheme is not required.

Although some approaches are only able to seek local minima [16, 76, 102] (which implies semi-global stability with stronger assumptions [100, 127]), it is possible to achieve convergence to the global extremum in the presence of local extrema. The first approach is to use a decreasing dither amplitude [128]. By initially using a large amplitude, the dither “scans” a larger region and is able to “overshoot” the local extrema before eventually focusing on seeking the global extremum in its vicinity. Alternatively, the DIRECT algorithm can be used in an ES scheme operating in discrete time to achieve global stability [70, 71].

There have been significant developments of ES scheme which lead to various techniques being used to achieve different results as described above. However, the majority of these results address only single-input single-output (SISO) plants. Since urban traffic networks are systems with multiple inputs, these results for SISO plants are not widely applicable to urban traffic control. Thus, the next subsection focuses on multi-input ES schemes.

2.3.2 Multi-input extremum-seeking

The general concept of a multi-input ES scheme is largely the same as a single-input one. The difficulties that potentially arise when dealing with multiple inputs are: the gradient estimation bias that is caused by multiple perturbations (particularly for perturbation-based ES schemes), and the increasingly difficult optimisation problem due to high dimensionality. Furthermore, non-scalar ES schemes can have a single output or multiple outputs, depending on the nature of the plant.

\(^2\)Although there is a practical upper bound at which the effect of the dither is dissipated by the plant dynamics such that the effect of noise dominates the output.
Multi-input single-output (MISO) ES

One of the earliest perturbation based ES schemes for multi-input systems was presented in [15] for quadratic plants with linear dynamics. This result is extended to provide SPA stability with respect to the tuning parameters of the ES scheme [100], shown in Figure 2.6. Similar to its SISO counterpart in [127], this result requires global asymptotic stability (GAS) of the resulting gradient system (following the singular perturbation and averaging analysis) with robustness to small disturbances. In addition, the dither signal has to satisfy the aforementioned PE-like condition, and the most trivial choice of dither is a vector of sinusoids of different frequencies (each frequency can be used twice in an orthogonal fashion). Furthermore, the closed-loop system is required to have the same three time scales as in [127].

\[
\dot{x} = f(x, u) \\
y = g(x, u)
\]

\[
1 \\
s(K - 1) r(\omega t) \\
y \\
\bar{u} - k\omega a s(\omega t)
\]

Figure 2.6: The MISO ES analysed in [100], where \( y \in \mathbb{R}; \ u \in \mathbb{R}^{N_u} \) with \( N_u > 1; \ s(\cdot) \) and \( r(\cdot) \) are bounded, zero-mean, periodic signals with a unit period, such that \( K := \int_0^1 r(\cdot)s(\sigma)^T d\sigma \) is invertible.

Alternatively, it is possible to use multiple identical units of the system to get the estimated gradient using finite differences between the outputs of the units [71,123]. As an example, for a system with \( N_u \)-dimensional input \( u \in \mathbb{R}^{N_u} \), the number of units required are \( N_u + 1 \), where the first unit receives an unperturbed input while other units’ inputs are perturbed by \( \Delta \) in different directions [123]. However, the application of this approach to urban traffic light control is not practical as it is impossible to provide multiple units of the plant.

Another approach to optimise a system with multiple inputs is simultaneous perturbation stochastic approximation (SPSA), which is a discrete optimisation algorithm
In fact, this technique [24,120,121] and its modification (which is referred to as the AFT technique) [73–75] have been used as parameter adaptation schemes of urban traffic light control. At each iteration, the SPSA algorithm chooses a random perturbation vector and perturbs the inputs twice, by adding and subtracting the perturbation vector. As a result, each iteration requires two output measurements to be taken. The gradient of the output function at the current input value is then simply estimated by using finite differences of the two measurements for each element of the input vector.

The SPSA technique is modified in AFT algorithm by incorporating a different gradient estimator [73–75]. At each iteration of this technique, the inputs are perturbed for a number of times with different randomly generated dithers. The number of perturbation per iteration is determined by the user and is not limited to two as in the SPSA technique. Furthermore, the gradient estimator is allowed to use the output measurements of previous iterations. In addition, the measurements are not used to directly estimate the gradient. Instead, an approximator is used to locally estimate the output function. The approximator has weights that needs to be adjusted such that it fits the local shape of the output function, which is inferred by using the obtained measurements. Finally, the inputs are adjusted based on the gradient of the approximator function. By doing these extra steps, the behaviour of the AFT scheme can be made to approximate a standard gradient descent/ascent optimiser arbitrarily closely. Therefore, this algorithm potentially has a faster convergence by allowing a more complex gradient estimator scheme. However, the increasing difficulty of the optimisation problem due to high-dimensionality might still cause increasingly slow convergence. In addition, there has not been any rigorous stability proof of these techniques, both SPSA and AFT, applied to systems with dynamics; and extending the stability result to include dynamical systems might be a challenge due to the discrete nature of these techniques.

Therefore, a perturbation based ES approach is the most sensible approach to be used for urban traffic light control. However, as the number of inputs to a given system is increased, it is reasonable to expect that, in order to achieve acceptable convergence properties, the assignment of the frequencies to the dithers becomes an increasingly arduous task. Additionally, slow adaptation may be caused by the dithers and gradient estima-
tor occupying an increasingly wide range of time scales and, similar to SPSA and AFT, the increasing difficulty of the optimisation problem as its dimension grows. Therefore, the use of MISO ES schemes, which solve the optimisation problem centrally, is not fully suited for applications to large-scale systems such as urban traffic networks. One potential solution to these problems is to use decentralised ES techniques, which is the main focus of the next subsection.

Multi-input multi-output (MIMO) ES

A MIMO ES scheme can have two different meanings: finding a Pareto front [58]; or seeking a Nash equilibrium [40, 43–46, 92, 126]. This thesis focuses only on the latter. In this case, there are a number of ES controllers/agents, each of which control a group of inputs to the system (although typically each agent controls only one input) and has an associated output. Then, the control objective of each ES agent is to selfishly optimise its own output by changing its inputs, which might introduce adverse effects on the outputs of other agents. As a result, this creates a non-cooperative game being played among these ES agents, the equilibrium of which is a Nash equilibrium (NE). At the NE point, each agent cannot improve its own output by solely changing its own inputs.

A continuous Nash equilibrium seeking (NES) scheme for systems with nonlinear dynamics is developed in [44, 45], where each agent controls a single input (the case where the dynamics are absent is investigated in [43]). Each agent employs a simple SISO ES scheme. The Nash equilibrium \( u^* \in \mathbb{R}^{N_u} \) is defined as a point which satisfies (maximisation is considered in this case),

\[
\frac{\partial J_i}{\partial u_i}(u^*) = 0, \quad \frac{\partial^2 J_i}{\partial u_i^2}(u^*) < 0, \quad \forall i \in \{1, 2, \ldots, N_u\},
\]

where \( J_i \) is the SSIO map of agent \( i \). Besides the typical assumptions required for ES, it is also assumed that that the input \( u_i \) has a dominant effect on \( J_i \). Specifically, the assumption states that the Hessian matrix \( \frac{\partial^2 J_i}{\partial u^2} \) is diagonally dominant. However, the dither frequency of each agent still has to be unique such that the NES scheme can be expected to suffer similar limitations to traditional MISO ES approaches when applied to
large-scale systems.

One NES approach that addresses this issue is discussed in [126], which allows dithers to be re-used between agents who do not affect each other’s output/cost. However, such a result cannot be exploited in systems where each agent’s output/cost is potentially dependent upon all of the inputs. In addition, the analysis is limited to a specific class of systems with LTI dynamics, which is not representative of an urban traffic network.

It can be said that an urban traffic network falls into a class of systems that, in some sense, lies between those considered in [44] and [126]. When the signal plan of the set of traffic lights at a particular intersection is changed, it is expected that the effect of this change is observed only locally. However, “local” here does not only mean the traffic on the roads heading towards the intersection, but the effect can also extend over several intersections albeit with decaying intensity. Therefore, it is desired to develop a new decentralised ES scheme that exploits this particular “decaying effect” characteristic. In addition, it is important to note that a traffic network is naturally a hybrid system due to the existence of discrete events in the system (such as the traffic lights). However, none of the existing ES theories on the literature address the stability of the scheme acting on hybrid systems. Therefore, it is clear that extensions to ES theory are required to address
2.4 Summary of Findings

It can be concluded that the trend in urban traffic control development is to use closed-loop control strategies, which are more robust than open-loop strategies. However, these existing strategies lack any means to appropriately select their parameters, such that their performances are optimised. Since traffic demands generally fluctuate and undergo evolutions, the use of constant parameters might not be the best solution. In the long term, this could potentially lead to performance deterioration of these strategies. Therefore, the use of an online calibration technique to fine-tune these parameters presents an opportunity for performance improvement.

From the literature review, it can be concluded that despite the many recent developments in ES, some extensions are required to improve the applicability of an ES scheme for online calibration of existing urban traffic light controllers. Existing ES calibration schemes used for traffic control still utilise centralised approaches that, when faced with a large number of parameters, might suffer from slow convergence due to the high dimensionality of the optimisation problem. In addition, centralised approach most likely requires extensive communications among traffic lights. Therefore, urban traffic light control might benefit from the development of a decentralised ES approach.

Existing results on decentralised ES approaches are still impractical for use in urban traffic light control. Firstly, using different dither frequencies for each input in the system is an arduous task and might cause slow convergence. Although dither re-use has been shown to be possible, the analysis only applies to a class of systems which is not representative of an urban traffic network. Secondly, there has not been any stability result for ES schemes acting on hybrid systems. Therefore, there are still gaps in the existing literature on decentralised ES control that need to be addressed.
2.5 Research Aims

From the literature review, it seems reasonable to augment existing urban traffic control strategies with a real-time parameter calibration technique. As previously discussed, an extremum-seeking scheme that decentralises the optimisation problem is a promising approach. However, it is apparent that such extremum-seeking tools do not currently exist. In general, the goal of this project is to develop specialised ES results for parameter adaptation of existing urban traffic light controllers. The full closed-loop system as a result of this augmentation is illustrated in Figure 2.8.

![Figure 2.8: The overview of the adaptation of the parameters of existing urban traffic light controllers using extremum-seeking.](image)

This thesis mainly concerns on the design of the extremum-seeker that addresses the aforementioned issues, namely: high-dimensionality and the hybrid nature of the traffic network. Specifically, this work aims to:

1. develop a decentralised ES scheme for a class of nonlinear, continuous, MIMO systems that exploits the fact that the effect of an input on an output diminishes with some measure of proximity;

2. show stability of the developed ES scheme for implementation on a family of hybrid systems that admit a steady-state average behaviour;

3. demonstrate on a traffic simulator that improvements can be achieved by using the developed ES scheme as a parameter adaptation to optimise the performance of existing traffic light controllers, as shown in Figure 2.8.

Although the achievement of each research aim is an independent contribution on its own, together they contribute to the overall goal of the project in three steps, as illus-
trated in Figure 2.9. The first step is to address the first research issue, which allows the decentralisation of the ES scheme acting on a class of nonlinear, continuous systems that exhibit the “decaying effect” phenomena (Chapter 3). The second step involves proving stability of an arbitrary ES scheme (which includes the developed decentralised ES scheme) acting on a class of hybrid systems (Chapter 4). Thus, this guarantees the stability of the proposed ES scheme for adaptation of existing urban traffic controls, which is demonstrated in the third step (Chapter 5). Therefore, this research plan provides a cohesive approach to achieve the overall goal of the project.

![Figure 2.9: The three research contributions.](image-url)
Chapter 3

Dither Re-use in Nash Equilibrium Seeking

Many contemporary engineering challenges, such as urban traffic networks, irrigation networks and groups of UAVs, involve the optimisation of large-scale systems. The difficulty of addressing these challenges is compounded when important system parameters are unknown or even the system model itself is unknown. Such scenarios would benefit from the use of some form of online optimisation. In particular, the decentralisation of ES technique might be beneficial to simplify its implementation and the optimisation problem.

This chapter considers a simple NES scheme acting on a family of MIMO systems where the effect of an agent’s input on any agent’s output is bounded and dissipates as the “distance” between them (based on some measure of proximity) is increased. Then, this is used to guarantee that some of the interconnections among the ES agents are weak. In a sense, the family of systems examined in this chapter lies between those considered in [44], which looks at systems where all inputs affect all outputs, and [126], which deals with systems where each output is only influenced by a small subset of the inputs. In this chapter, it is shown that two agents whose effects on each other are sufficiently weak may use the same dither without adversely impacting stability. By re-using dithers, practical benefit can be gained by simplifying the task of assigning frequencies.

This is the first semi-global stability result for a NES scheme acting on a system with fairly general nonlinear dynamics. The convergence result is semi-global in the sense that the domain of attraction can be made arbitrarily large, assuming global asymptotic stabil-

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*A substantial proportion of this chapter has been published in [79].*
ity of the origin of the gradient system with robustness to small disturbances. Hence, the main contribution of this chapter is to show a SPA stability result of the proposed NES scheme when the agents who share a dither are affecting each other’s output sufficiently weakly.

Moreover, this chapter studies the relationship between the strength of the agent’s effect on the outputs and the extent of allowable dither re-use, which leads to the introduction of a specific assumption. The satisfaction of this assumption implies that some of the interconnections are indeed sufficiently weak, such that a certain amount of dither re-use is allowed without compromising SPA stability. This is followed by the application of the NES scheme to plants with quadratic costs, which results in more practical formulae that quantify these conditions. Lastly, simulation examples are presented to validate the main results and investigate the effect of dither re-use on the transient response of the NES scheme.

3.1 Preliminaries

Stability analysis in control theory often utilises comparison functions, namely class $\mathcal{K}$ function, class $\mathcal{K}_\infty$ function, and class $\mathcal{KL}$ function. A class $\mathcal{K}$ function, $\alpha \in \mathcal{K}$, is a function that satisfies: $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$, where $a \in \mathbb{R}_{\geq 0}$; $\alpha$ is continuous; $\alpha(0) = 0$; and $\alpha$ is strictly increasing. A class $\mathcal{K}_\infty$ function, $\alpha \in \mathcal{K}_\infty$ satisfies all the aforementioned properties of a $\mathcal{K}$ function, with $a = \infty$, and also satisfies $\lim_{s \to \infty} \alpha(s) = \infty$. A class $\mathcal{KL}$ function, $\beta \in \mathcal{KL}$, satisfies: $\beta : [0, a) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, where $a \in \mathbb{R}_{\geq 0}$; $\beta$ is continuous in both arguments; for each $s \in [0, a)$, $\beta(s, \cdot)$ is decreasing to zero; for each $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t) \in \mathcal{K}$.

The comparison functions are useful to state various stability results. Consider the following system

$$\dot{x} = f(t, x), \quad (3.1)$$

where $x \in \mathbb{R}^{N_x}$ and the origin is an equilibrium. The origin is uniformly (locally) asymptot-
ically stable if there exists $\Delta > 0$ and $\beta \in \mathcal{K}\mathcal{L}$ such that for any $\|x(t_0)\| \leq \Delta$,
\[
\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0.
\] (3.2)

The origin is uniformly globally asymptotically stable (UGAS) if for any initial conditions $x(t_0) \in \mathbb{R}^{N_x}$, (3.2) holds\(^1\).

In addition, consider a general continuous nonlinear dynamical system with small positive parameters $\varepsilon$:
\[
\dot{x} = f(t, x, \varepsilon),
\] (3.3)
where $x \in \mathbb{R}^{N_x}$, $\varepsilon \in \mathbb{R}^{N_\varepsilon}$, and $f : \mathbb{R} \times \mathbb{R}^{N_x} \times \mathbb{R}^{N_\varepsilon}$. The following is the definition of a semi-global practical asymptotic (SPA) stability of system (3.3).

**Definition 3.1.** System (3.3) is semi-globally practically asymptotically (SPA) stable for small $\varepsilon$ uniformly in small $(\varepsilon_1, \ldots, \varepsilon_l)$, $l \in \{1, \ldots, N_\varepsilon\}$ if there exists $\beta \in \mathcal{K}\mathcal{L}$ such that for any $(\Delta, \nu) \in \mathbb{R}^2_{>0}$, the following holds. There exists $(\varepsilon^*_1, \ldots, \varepsilon^*_l) \in \mathbb{R}^l_{>0}$ such that for any $(\varepsilon_1, \ldots, \varepsilon_l) \in (0, \varepsilon^*_1] \times \cdots \times (0, \varepsilon^*_l)$, there exists $(\varepsilon^*_{l+1}, \ldots, \varepsilon^*_{N_\varepsilon}) \in \mathbb{R}^{N_\varepsilon-l}_{>0}$ such that for any $(\varepsilon_{l+1}, \ldots, \varepsilon_{N_\varepsilon}) \in (0, \varepsilon^*_{l+1}] \times \cdots \times (0, \varepsilon^*_{N_\varepsilon})$ and any $\|x(0)\| \leq \Delta$, the solutions of (3.3) satisfy
\[
\|x(t)\| \leq \beta(\|x(0)\|, (\varepsilon_1 \cdot \varepsilon_2 \cdot \ldots \cdot \varepsilon_{N_\varepsilon})t) + \nu, \quad \forall t \geq 0.
\] (3.4)

## 3.2 System Description

### 3.2.1 Plant

The plant is described by the dynamical system:
\[
\begin{align*}
\dot{x} &= f(x, u), \quad \text{(3.5a)} \\
y &= g(x, u), \quad \text{(3.5b)}
\end{align*}
\]

\(^1\)Occasionally, the author abuses the terminology by referring to system (3.1) as being globally asymptotically stable instead of the origin.
where \( x \in \mathbb{R}^{N_x} \) and \( u, y \in \mathbb{R}^{N_u} \). For algebraic simplicity, define \( S = \{1, 2, \ldots, N_u\} \). For each \( i \in S \), there is an agent (or player) that is allowed to choose the \( i^{th} \) input, \( u_i \), based only on its measurement of the \( i^{th} \) output, \( y_i \).

**Remark 3.1.** The number of inputs and outputs is equal without loss of generality. It is, nonetheless, possible to have multiple inputs per agent. If, for example, there were \( N_y \) agents with the first one controlling two inputs, \( u_1 \) and \( u_2 \), then it is possible to treat this agent as two different SISO agents whose measurements are equal, so that

\[
y = \begin{bmatrix}
g_1(x, u_1) \\
g_1(x, u_2) \\
g_2(x, u_1) \\
\vdots \\
g_{N_y}(x, u_1)
\end{bmatrix}.
\]  

(3.6)

**Assumption 3.1.** There exists a function \( h : \mathbb{R}^{N_x} \to \mathbb{R}^{N_x} \) such that \( x = h(u) \) is a globally asymptotically stable equilibrium of (3.5a), uniformly in \( u \).

Let \( J(u) := g(h(u), u) \) be the input-to-output steady-state mapping of the system. The goal of each agent is to minimise its steady-state measurement.

**Assumption 3.2.** There exists a unique Nash equilibrium point \( u^* \) which, for all \( i \in S \), satisfies

\[
\frac{\partial J_i}{\partial u_i}(u^*) = 0, \quad \frac{\partial^2 J_i}{\partial u_i^2}(u^*) > 0.
\]  

(3.7)

The fact that an agent’s output is affected by other agents’ inputs at differing strength is going to be exploited to create a “neighbourhood”. The neighbourhood of agent \( i \) consists of all other agents whose input has a sufficiently strong influence on agent \( i \)'s output. Let \( N_i(R) \) denote the neighbours set of agent \( i \). The number of the neighbouring agents can be increased through the parameter \( R \). Thus, \( N_i(R_1) \subseteq N_i(R_2) \) for all \( 0 \leq R_1 \leq R_2 \), and \( N_i(R) = S \) for sufficiently large \( R \). For many physical systems, the influence of an input decays spatially. In such a case, it is sensible to choose \( R \) to be related to a physical distance.
3.2 System Description

**Assumption 3.3.** There exists $\alpha \in \mathcal{K}_\infty$ such that for any $R \in \mathbb{R}_{>0}$, there exists $\gamma_1(R) \in \mathbb{R}_{>0}$ such that for all $i \in S$,

$$\sum_{j \notin \mathcal{N}(i)(R)} \left| \frac{\partial J_i}{\partial u_j}(u) - \frac{\partial J_i}{\partial u_j}(u^*) \right| \leq \gamma_1(R) \alpha(\|u - u^*\|). \quad (3.8)$$

**Assumption 3.4.** For any $R$, there exists $\gamma_2(R) \in \mathbb{R}_{>0}$ such that for all $i \in S$,

$$\sum_{j \notin \mathcal{N}(i)(R)} \left| \frac{\partial J_i}{\partial u_j}(u^*) \right| \leq \gamma_2(R). \quad (3.9)$$

$\gamma_1(R)$ and $\gamma_2(R)$ will decrease to zero as $R$ is increased. This decrease will be particularly strong for systems where many of the interconnections between the agents are weak.

**Remark 3.2.** Assumption 3.3 is not restrictive since it only requires continuous differentiability of $J(u)$. Also, since $u^*$ is a constant, the LHS of (3.9) can always be upper bounded, which implies Assumption 3.4 can always be satisfied. Note that throughout this chapter, the arguments of $\gamma_1$ and $\gamma_2$ are dropped for notational compactness.

**Remark 3.3.** Assumptions 3.3 and 3.4 do not require specific knowledge of the bounding functions $\gamma_1$, $\gamma_2$ and $\alpha$. They only require the bounding functions to exist.

The convergence result for the proposed NES scheme is proven by showing that the NES scheme can be designed such that the behaviour of the full closed-loop system approximates

$$\frac{d\tilde{u}_i}{d\tau} = -\frac{\partial J_i}{\partial u_i}(\tilde{u} + u^*), \quad i \in S, \quad (3.10)$$

with arbitrary accuracy. Note that $\tilde{u}$ and $\tau$ are the auxiliary system’s states and time scale respectively. The following assumption provides global asymptotic stability of the origin of (3.10), which underpins the NES stability result.

**Assumption 3.5.** There exist $(c_1, c_2, c_3, c_4) \in \mathbb{R}^4_{>0}$ and a radially unbounded $V(\cdot) : \mathbb{R}^{N_u} \to \mathbb{R}$, such that for all $\tilde{u} \in \mathbb{R}^{N_u}$, the following hold

$$c_1 \alpha(\|\tilde{u}\|)^2 \leq V(\tilde{u}) \leq c_2 \alpha(\|\tilde{u}\|)^2, \quad (3.11a)$$
\[
\sum_{i=1}^{N_u} \frac{\partial V(\tilde{u})}{\partial \tilde{u}_i} \frac{\partial J_i(\tilde{u} + u^*)}{\partial \tilde{u}_i} \geq c_3 \alpha (\|\tilde{u}\|)^2, \tag{3.11b}
\]
\[
\|\nabla V(\tilde{u})\| \leq c_4 \alpha (\|\tilde{u}\|). \tag{3.11c}
\]

Note that \(\alpha(\cdot)\) is from Assumption 3.3. Assumption 3.5 also ensures (3.10) has an asymptotically stable solution close to the origin when subjected to sufficiently small perturbations.

### 3.2.2 NES scheme with dither re-use

The proposed Nash equilibrium seeking scheme is given by:

\[
\dot{x} = f(x, \tilde{u} + a \mathcal{s}(\omega t)), \tag{3.12a}
\]
\[
\dot{\tilde{u}} = -k \omega \text{diag}(s(\omega t)) \ g(x, \tilde{u} + a \mathcal{s}(\omega t)), \tag{3.12b}
\]

where \(s(\omega t)\) is both the dither and demodulation signal

\[
s(\omega t) = \begin{bmatrix}
\sin(\omega [\omega_1 t + \phi_1]) \\
\sin(\omega [\omega_2 t + \phi_2]) \\
\vdots \\
\sin(\omega [\omega_{N_u} t + \phi_{N_u}])
\end{bmatrix}, \tag{3.13}
\]

diag(\cdot) is a diagonal matrix where the diagonal elements are given by its arguments, and \(\omega_i \in \mathbb{Q}_{\geq \omega_{\text{min}}}\) for all \(i \in S\) and some \(\omega_{\text{min}} > 0\). The dither frequencies inside the neighbourhood of each agent must be distinct, that is for all \(i \in S\) and \(j \in N_i(R), \omega_j \neq \omega_i\). This property is a relaxation of the requirement in [44] of unique frequencies for all \(i \in S\). The stability of the NES scheme subject to this relaxation is provided in the next section.
3.3 Stability Results

3.3.1 Reduced system

First, a new time scale is defined \( \tau := \omega t \) and the closed loop system (3.12) can be expressed in this new time scale

\[
\begin{align*}
\omega \frac{dx}{d\tau} &= f(x, \bar{u} + as(\tau)) \quad (3.14a) \\
\frac{d\bar{u}}{d\tau} &= -k \text{diag}(s(\tau)) g(x, \bar{u} + as(\tau)) \quad (3.14b)
\end{align*}
\]

which is in the standard form for singular perturbation analysis. For small \( \omega \), the plant states \( x \) will quickly settle to their equilibrium \( x = h(\bar{x} + as(\tau)) \) and the reduced system is

\[
\frac{d\bar{u}_r}{d\tau} = -k \text{diag}(s(\tau)) J(\bar{u}_r + as(\tau)). \quad (3.15)
\]

3.3.2 Periodic average of reduced system

For small \( k \), a conclusion regarding the stability of the reduced system can be made by investigating the periodic average of the reduced system. Let \( T \) be the least common multiple of \( \left( \frac{2\pi}{\omega_1}, \frac{2\pi}{\omega_2}, \ldots, \frac{2\pi}{\omega_N} \right) \) which always exists since the dither frequencies are rational multiples of \( \omega \). Specifically, \( T \) is defined as follows:

\[
T = \min \left\{ t \left| \frac{t}{2\pi/\omega_i} \in \mathbb{N}, \forall i \right. \right\}. \quad (3.16)
\]

The average system is

\[
\frac{d\bar{u}_{av}}{d\tau} = -k T \int_0^T \text{diag}(s(\sigma)) J(\bar{u}_{av} + as(\sigma)) d\sigma. \quad (3.17)
\]

Consider the Taylor series expansion of \( J(\bar{u}_{av} + as(\tau)) \):

\[
J(\bar{u}_{av} + as(\tau)) = J(\bar{u}_{av}) + a \nabla J(\bar{u}_{av}) s(\tau) + \mathcal{O}(a^2) \quad (3.18)
\]
\[
\frac{d \bar{u}_{av,i}}{d \tau} = \frac{-k}{T} \int_0^T \left[ \sin(\omega_i \tau) J_i(\bar{u}_{av}) + a \sin(\omega_j \tau) \sum_j \frac{\partial J_i}{\partial u_j}(\bar{u}_{av}) \sin(\omega_j \tau) + O(a^2) \right] d\tau.
\]

(3.19)

The average of the first term is zero. Exploiting orthogonality properties of sinusoidal functions for the second term,

\[
\frac{1}{T} \int_0^T \sin(\omega_i \tau) \sum_j \frac{\partial J_i}{\partial u_j}(\bar{u}_{av}) \sin(\omega_j \tau) d\tau = \sum_j \frac{1}{T} \int_0^T \sin(\omega_i \tau) \sin(\omega_j \tau) \frac{\partial J_i}{\partial u_j}(\bar{u}_{av}) d\tau
\]

\[
= \frac{1}{2} \sum_{j: \omega_j = \omega_i, j \neq i} \frac{\partial J_i}{\partial u_j}(\bar{u}_{av})
\]

\[
= \frac{1}{2} \left( \frac{\partial J_i}{\partial u_i}(\bar{u}_{av}) + \sum_{j: \omega_j = \omega_i, j \neq i} \frac{\partial J_i}{\partial u_j}(\bar{u}_{av}) \right).
\]

(3.20)

For notational compactness, define

\[
\Gamma_i(\bar{u}_{av}) = \sum_{j: \omega_j = \omega_i, j \neq i} \left( \frac{\partial J_i}{\partial u_j}(\bar{u}_{av}) - \frac{\partial J_i}{\partial u_j}(u^*) \right),
\]

(3.21)

\[
\Lambda_i = \sum_{j: \omega_j = \omega_i, j \neq i} \frac{\partial J_i}{\partial u_j}(u^*).
\]

(3.22)

Now, substitution of (3.20)–(3.22) into (3.19) leads to the result:

\[
\frac{d \bar{u}_{av,i}}{d \tau} = -\frac{ka}{2} \left( \frac{\partial J_i}{\partial u_i}(\bar{u}_{av}) + \Gamma_i(\bar{u}_{av}) + \Lambda_i + O(a) \right).
\]

(3.23)

From Assumptions 3.3 and 3.4, \(|\Gamma_i(\bar{u}_{av})|\) and \(|\Lambda_i|\) are upper bounded by \(\gamma_1\) and \(\gamma_2\) respectively, and can be decreased by increasing \(R\). Thus, for sufficiently small \(a\), and sufficiently large \(R\), the averaged system can be expected to approximate (3.10).
3.3 Stability Results

3.3.3 Stability analysis of averaged and reduced systems

Before introducing the main result, it is useful to show the stability analysis of the averaged \((3.17)\) and the reduced \((3.15)\) systems. However, these stability results are only intended as stepping stones for proving the main result, and are of limited use practically. Since stability of the reduced system requires stability of the averaged system, the result for the reduced system is presented after the result for the averaged system. Lemma 3.1 and Lemma 3.2 state the stability of the averaged system \((3.17)\) and the reduced system \((3.15)\) respectively.

**Lemma 3.1.** Under Assumptions 3.2–3.5, there exists \(\beta_{av} \in \mathcal{K} \mathcal{L}\) such that for any \((\Delta, \nu) \in \mathbb{R}^2_{>0}\) there exist \((a^*, \gamma_2^*) \in \mathbb{R}^2_{>0}\) such that for all \((a, k, \gamma_1, \gamma_2) \in (0, a^*] \times \mathbb{R}_{>0} \times (0, c_3/(c_4\|1\|)) \times (0, \gamma_2^*],\) the solutions of \((3.17)\) with the initial condition \(\|\bar{u}_{av}(0) - u^*\| \leq \Delta,\) will satisfy

\[
\|\bar{u}_{av}(\tau) - u^*\| \leq \beta_{av} \left(\|\bar{u}_{av}(0) - u^*\|, ka\tau\right) + \nu, \quad \forall \tau \geq 0.
\]

**Proof.** The Lyapunov function from Assumption 3.5 can be used to prove stability of the average system. First define

\[
F_{av}(\bar{u}_{av}) = \frac{1}{T} \int_0^T \text{diag}(s(\tau)) J(\bar{u}_{av} + a s(\tau)) \, d\tau, \quad (3.24)
\]

\[
G_i(\bar{u}_{av}) = \frac{a}{2} \left( \frac{\partial J_i}{\partial \bar{u}_i} (\bar{u}_{av}) + \Gamma_i (\bar{u}_{av}) + \Lambda_i \right). \quad (3.25)
\]

Now \((3.17)\) can be expressed as:

\[
\frac{d\bar{u}_{av}}{d\tau} = -kG(\bar{u}_{av}) - k \left( F_{av}(\bar{u}_{av}) - G(\bar{u}_{av}) \right) \quad (3.26)
\]

Let \(\tilde{u} = \bar{u}_{av} - u^*\) and consider the Lyapunov function \(V(\tilde{u})\) satisfying Assumption 3.5.

\[
\frac{1}{k} \frac{dV}{d\tau} = -\nabla V(\tilde{u})^T G(\tilde{u} + u^*)
\]

\[
-\nabla V(\tilde{u})^T \left( F_{av}(\tilde{u} + u^*) - G(\tilde{u} + u^*) \right)
\]

\[
= -\frac{a}{2} \sum_{i=1}^{N_a} \frac{\partial V}{\partial \tilde{u}_i}(\tilde{u}) \frac{\partial J_i}{\partial \tilde{u}_i}(\tilde{u} + u^*)
\]
Dither Re-use in Nash Equilibrium Seeking

\[- \frac{\alpha}{2} \sum_{i=1}^{N_u} \frac{\partial V}{\partial \tilde{u}_i}(\tilde{u}_i (\bar{u}_{av} + \Lambda_i)) \]

\[-(\nabla V(\tilde{u}))^T (F_{av}(\tilde{u} + u^*) - G(\tilde{u} + u^*)) \]  

(3.27)

Note that from (3.23), \(\|F_{av}(u) - G(u)\|\) is of \(O(a^2)\). Consider the second term in (3.27), using Assumptions 3.3 and 3.4 and (3.11c) from Assumption 3.5

\[- \sum_{i=1}^{N_u} \frac{\partial V}{\partial \tilde{u}_i}(\tilde{u}_i (\bar{u}_{av} + \Lambda_i)) \leq \sum_{i=1}^{N_u} \left| \frac{\partial V}{\partial \tilde{u}_i}(\tilde{u}_i) \right| \left| \sum_{j \notin N_i(R)} \frac{\partial f_i}{\partial \tilde{u}_j}(\tilde{u} + u^*) - \frac{\partial f_i}{\partial \tilde{u}_j}(u^*) \right| \]

\[\leq (\gamma_1 \alpha(\|\tilde{u}\|) + \gamma_2) \|1\| c_4 \alpha(\|\tilde{u}\|), \]  

(3.28)

where \(\|1\|\) is the norm of a vector of ones with \(N_u\) elements. Similarly, after using (3.11b) from Assumption 3.5 for the first term and (3.11c) for the third term of (3.26), it follows that

\[\frac{1}{k} \frac{dV}{dt} \leq a(-c_3 + \gamma_1 c_4 \|1\|) \frac{\alpha(\|\tilde{u}\|)^2}{2} \]

\[+ (a \gamma_2 c_4 \|1\| + c_4 \|F_{av}(\tilde{u} + u^*) - G(\tilde{u} + u^*)\|) \frac{\alpha(\|\tilde{u}\|)}{2}. \]  

(3.29)

It is desired for the right hand side of (3.29) be negative.

\[0 \geq a(-c_3 + \gamma_1 c_4 \|1\|) \frac{\alpha(\|\tilde{u}\|)^2}{2} \]

\[+ (a \gamma_2 c_4 \|1\| + c_4 \|F_{av}(\tilde{u} + u^*) - G(\tilde{u} + u^*)\|) \frac{\alpha(\|\tilde{u}\|)}{2} \]  

(3.30)

Rearranging (3.30) to isolate \(\|\tilde{u}\|\) leads to the inequality,

\[\|\tilde{u}\| \geq \alpha^{-1} \left( \frac{\gamma_2 c_4 \|1\| + c_4 \|F_{av}(\tilde{u} + u^*) - G(\tilde{u} + u^*)\|}{c_3 - \gamma_1 c_4 \|1\|} \right) \]

\[=: \varepsilon. \]  

(3.31)
Therefore, (3.31) indicates the region in which \( dV/d\tau \) is negative, given any \((a, \gamma_2)\) and \(\gamma_1\) satisfying

\[
\gamma_1 < \frac{c_3}{c_4\|\mathbf{I}\|}. \tag{3.32}
\]

Equation (3.32) can always be satisfied by selecting sufficiently small \(R\). Also note that the terms in the numerator of (3.31) are of \(O(\gamma_2)\) and \(O(a)\) respectively.

Now let \(B(c) := \{\tilde{u} \in \mathbb{R}^N : \|\tilde{u}\| \leq c\}\) denote a ball of radius \(c\) and let \(D := \{\tilde{u} \in \mathbb{R}^N : V(\tilde{u}) \leq c_2a(\Delta)^2\}\). Using (3.11a) and realising that \(B(\Delta) \subseteq D\), \(V(\tilde{u})\) must increase if \(\tilde{u}\) is to leave \(D\) after being initialised in \(B(\Delta)\). Thus, for a sufficiently small \((a, \gamma_1, \gamma_2)\), \(V\) is guaranteed to decrease with time until \(\|\tilde{u}\| \leq \varepsilon\), in which case

\[
c_2a(\varepsilon)^2 \geq c_2a(\|\tilde{u}\|)^2 \geq V(\tilde{u}) \geq c_1a(\|\tilde{u}\|)^2 \tag{3.33}
\]

and therefore

\[
\|\tilde{u}\| \leq a^{-1} \left( a(\varepsilon) \sqrt{\frac{c_2}{c_1}} \right). \tag{3.34}
\]

Lemma 3.1 follows after noting that decreasing \((a, \gamma_1, \gamma_2)\) makes the region to which \(\tilde{u}\) converges arbitrarily small.

\[\square\]

**Lemma 3.2.** Under Assumptions 3.2–3.5, there exists \(\beta_r \in \mathcal{K}\mathcal{L}\) such that for any \((\Delta, v) \in \mathbb{R}^2_+\) there exist \((a^*, k^*, \gamma_2^* ) \in \mathbb{R}^3_+\) such that for all \((a, k, \gamma_1, \gamma_2) \in (0, a^*] \times (0, k^*] \times (0, c_3/(c_4\|\mathbf{I}\|)) \times (0, \gamma_2^*\] the solutions of (3.15) with the initial condition \(|\bar{u}_r(0) - u^*| \leq \Delta\), will satisfy

\[
|\bar{u}_r(\tau) - u^*| \leq \beta_r (|\bar{u}_r(0) - u^*|, ka\tau) + v, \quad \forall \tau \geq 0.
\]

**Proof.** Let \(p(q, \tau, a, k)\) represent the \(T\)-periodic (in \(\tau\)) solution of

\[
\frac{\partial p}{\partial \tau} = kF_{av}(q, a) - k \text{ diag } (s(\tau)) \cdot J(q + p + as(\tau)). \tag{3.35}
\]

For any bounded domain \(D_0 \in \mathbb{R}^N_u\) containing \(u^*\), the solution of (3.35) will be \(p(q, \tau, a, k) = O(k)\) uniformly in \((q, \tau) \in D_0 \times \mathbb{R}\) and small \((a, k)\). Specifically, for sufficiently small \((a, k)\), there exists some positive constant \(\sigma > 0\) such that \(\|p(q, \tau, a, k)\| \leq \sigma k\) uniformly
in \((q, \tau)\). Furthermore, there is a \(w\) such that \(p(w, \tau, a, k)\) exists and satisfies

\[ w + p(w, \tau, a, k) = \bar{u}_r \]  

(3.36)

for sufficiently small \((a, k)\) and any \(\bar{u}_r\) satisfying \(\bar{u}_r - u^* \in D_0\). Differentiating (3.36) with respect to \(\tau\) gives

\[ \frac{dw}{d\tau} = -k \left( I + \frac{\partial p}{\partial q} \bigg|_{q=w} \right)^{-1} F_{av}(w, a). \]  

(3.37)

Note that \(\partial p / \partial q\) can be made arbitrarily small by decreasing \(k\). Thus, the solution of (3.37) approaches the behaviour of the averaged system. Hence, using similar arguments to Lemma 3.1, \(V(w - u^*)\) can be used to prove convergence of \(\|w - u^*\|\) to a solution that can be made arbitrarily small by decreasing \((a, k, \gamma_1, \gamma_2)\). Lemma 3.2 follows after noting that the same conclusion can be taken for \(\bar{u}_r\) from (3.36).

**Remark 3.4.** Lemma 3.1 and 3.2 are stated in terms of \(\gamma_1\) and \(\gamma_2\). As previously mentioned, they are not intended to be used in practice since that implies one requires the knowledge of how \(\gamma_1\) and \(\gamma_2\) change with \(R\). The main theorem will address this so that such knowledge is unnecessary.

**Remark 3.5.** General averaging analysis can also be used instead of periodic averaging analysis. By following the steps in [69, Theorem 10.5], it can be shown that stability can be guaranteed without the condition \(\omega_i \in Q\). However, the result of the analysis will be SPA stability of the reduced system (3.15) in \((a, k, \gamma_1, \gamma_2)\), uniformly in \((a, \gamma_1, \gamma_2)\) instead of uniformly in \((a, k, \gamma_1, \gamma_2)\).

### 3.3.4 Main result

Before introducing the main stability result, define

\[ z = \begin{bmatrix} x - h(u) \\ \bar{u} - u^* \end{bmatrix}. \]  

(3.38)

**Theorem 3.1.** Under Assumptions 3.1–3.5, there exists \(\beta(\cdot, \cdot) \in KL\) such that for any \((\Delta, \nu) \in \mathbb{R}^2\) there exists \((a^*, k^*, R^*) \in \mathbb{R}^3\) such that for all \((a, k, R) \in (0, a^*] \times (0, k^*] \times \mathbb{R}^3\) there is
an $\omega^* \in \mathbb{R}_{>0}$ such that for all $\omega \in (0, \omega^*]$, the solutions of (3.12) with any $\|z(t_0)\| \leq \Delta$ satisfy

$$\|z(t)\| \leq \beta (\|z(t_0)\|, k\omega (t - t_0)) + \nu, \quad \forall t \geq t_0.$$ 

Proof. The proof follows three steps. Firstly, the average system (3.17) is shown to be SPA stable uniformly in small $(a, \gamma_1, \gamma_2)$ (see Lemma 3.1). Secondly, the reduced system (3.15) is proven to be SPA stable uniformly in $(k, a, \gamma_1, \gamma_2)$ (see Lemma 3.2). In addition, the stability of the boundary layer system is provided by Assumption 3.1. Finally, using a similar singular perturbation argument as in [127, Lemma 1], the full closed-loop system can be proven to be SPA stable with respect to $(k, a, \gamma_1, \gamma_2, \omega)$ uniformly in $(k, a, \gamma_1, \gamma_2)$. Theorem 3.1 follows after noting that the conclusion can be restated to incorporate $R$ instead of $\gamma_1$ and $\gamma_2$ (because $R$ is the actual design parameter) by using Assumption 3.3 and 3.4. \hfill \Box

Remark 3.6. Theorem 3.1 provides for a systematic procedure for selecting appropriate ES parameters. As is consistent with previous ES results [16, 127], the issue of gradient estimate bias is addressed by choosing sufficiently small $a$, $\omega$ and $k$. Similarly $R$ must be chosen to be sufficiently large to achieve an acceptable level of gradient estimate accuracy. For example, one can start with an arbitrary $R$. Then, if the scheme is unstable or has an unsatisfactory convergence properties, $R$ can be increased until a desired result is achieved. Alternatively, $R$ can potentially be decreased while the system closed loop behaviour remains satisfactory.

Remark 3.7. In some applications, game theoretical approaches are used to decentralise a large-scale optimisation problem, where the goal is to optimise a global cost function. Although in general the Nash equilibrium does not necessarily represent the global optimum of the system, several methods are available in the literature [7, 8, 18] which enable the Nash equilibrium to coincide with the global optimum. One condition that is common among these works is the fact that the global cost is the sum of each agent’s (local) cost.

3.3.5 Existence of $R^* < R_{\text{max}}$

Although Theorem 3.1 has provided stability and tuning guidelines for the scheme, it does not guarantee the existence of $R^* < R_{\text{max}}$, where $R_{\text{max}}$ is the smallest value of $R$ that
yields $N_i(R_{\text{max}}) = S$ for at least one of the agents. In other words, the result also means that one can always choose $R = R_{\text{max}}$ to guarantee convergence. Therefore, it is desirable to establish conditions for which one can guarantee the existence of $R^* < R_{\text{max}}$.

However, this can only be done if one has more knowledge of the plant. By exploiting inequality (3.32), the following assumption is produced.

**Assumption 3.6.** There exists $R_1 \in \mathbb{R}_{\geq 0}$ such that $\gamma_1 < c_3/(c_4 \|1\|)$ for all $R \in [R_1, R_{\text{max}})$ and $\gamma_1 \geq c_3/(c_4 \|1\|)$ for all $R \in (0, R_1)$.

If one has the luxury of possessing the knowledge to validate Assumption 3.6, Lemma 3.1 can be restated as follows.

**Corollary 3.1.** Under Assumptions 3.2–3.6, for any $(\Delta, \nu) \in \mathbb{R}^2_{\geq 0}$, there exists $a_1 \in \mathbb{R}_{\geq 0}$ such that for any $(a, k, R) \in (0, a_1) \times \mathbb{R}_{\geq 0} \times [R_1, R_{\text{max}}]$, the solutions of the average system (3.17) with the initial condition $\|\bar{u}_{av}(0) - u^*\| \leq \Delta$, will satisfy

$$\|\bar{u}_{av}(\tau) - u^*\| \leq \beta_{av}(\|\bar{u}_{av}(0) - u^*\|, ka\tau) + \nu, \quad \forall \tau \geq 0.$$ 

**Remark 3.8.** Indeed, to be able to calculate $R^*$, more specific knowledge of $\gamma_1$ is required, which might be difficult to obtain in practice. Yet the lack of this knowledge does not prevent the use of the main theorem. In addition, it is not always a simple task to verify that Assumption 3.6 is satisfied. The next section provides details on how this may be achieved for plants with quadratic costs, including outlining the knowledge required to validate Assumption 3.6 and the formula to be used in practice.

### 3.4 Plants with Quadratic Costs

It is of value to discuss the results developed in Section 3.3 with respect to plants with quadratic cost functions since these are very common in optimisation problems and many cost functions are well-approximated by quadratic functions in a region close to their minimum. Specifically, it is useful to investigate Assumption 3.5 and Corollary 3.1 when the cost function is quadratic. As a result, conditions which are more useful to the practitioner are determined, which can be used as a guide when choosing $R$. Specifically,
an upper bound for $\gamma_1$ is going to be determined. In addition, this section shows that it is relatively easy to satisfy the conditions required by the main result. Dynamics are ignored since they do not significantly affect the analysis.

The first step of analysing the plant is to investigate Assumption 3.5 for quadratic plants, which includes determining the gradient system, finding the Lyapunov function, $\alpha(.)$, $c_1$, $c_2$, $c_3$ and $c_4$. It is then possible to check whether the plant satisfies Assumption 3.6.

The cost functions in this instance can be stated as:

$$J_i = \frac{1}{2}u^T D^i u + H^T_i u + l_i$$

(3.39)

where $D^i = [d_{i,j,k}]$ is a symmetric matrix and $d_{i,j}^i$ is the $D^i$ matrix entry at row $j$ and column $k$; $d^i_j = [d^i_{j,1} \ d^i_{j,2} \ldots \ d^i_{j,Nu}]$; $H_i = [h^i_1 \ h^i_2 \ldots \ h^i_{Nu}]^T$; and $d^i_{j,k}, h^i_j, l_i \in \mathbb{R}$. For the quadratic cost function (3.39), the overall gradient system can be represented as a linear system,

$$\frac{du}{dt} = - \begin{bmatrix} d^1_1 \\ d^2_2 \\ \vdots \\ d^Nu_Nu \end{bmatrix} u - \begin{bmatrix} h^1_1 \\ h^2_2 \\ \vdots \\ h^Nu_Nu \end{bmatrix} := Au + B.$$  

(3.40)

The gradient system is stable if and only if $A$ is Hurwitz, however, it is uncertain whether this is true in general. To ensure this, some additional conditions that need to be satisfied by the system can be introduced. For example, [45] states that if $A$ is diagonally dominant with negative diagonal entries

$$d^i_{j,j} > 0, \quad \sum_{j \neq i} |d^i_{j,j}| \leq |d^i_{j,j}| \quad \forall i$$

(3.41)

then $A$ is Hurwitz based on Gershgorin’s discs theorem [104, Theorem 4.5].

As in previous sections, let $\tilde{u} = u - u^*$. Thus,

$$\frac{d\tilde{u}}{dt} = A(\tilde{u} + u^*) + B = A\tilde{u}.$$  

(3.42)
since $Au^* + B = 0$. Assuming $A$ is Hurwitz, the Lyapunov function to show stability of the gradient system can be chosen as $V(\bar{u}) = \bar{u}^TP\bar{u}$ where $P = P^T > 0$ satisfies the Lyapunov equation $A^TP + PA = -Q$ with any $Q = Q^T > 0$. As a result, the $\alpha$ function and all $c_i$’s in Assumption 3.5 can now be determined also by using linear Lyapunov theory. By selecting $\alpha(.)$ as an identity function, the following $c_i$’s are selected

$$c_1 = \lambda_{\text{min}}(P) \quad c_2 = \lambda_{\text{max}}(P) \quad (3.43a)$$
$$c_3 = \lambda_{\text{min}}(Q) \quad c_4 = 2\lambda_{\text{max}}(P) \quad (3.43b)$$

where $\lambda_{\text{min}}(.)$ and $\lambda_{\text{max}}(.)$ are the minimum and maximum eigenvalues respectively.

Now consider (3.8) in Assumption 3.3. Define $\gamma_1$ as

$$\gamma_1 = \max_i \left\{ \sum_{j \notin N_i(R)} |d_j| \right\}. \quad (3.44)$$

Thus, Assumption 3.3 is satisfied since

$$\sum_{j \notin N_i(R)} \left| \frac{\partial J_i}{\partial u_j}(u) - \frac{\partial J_i}{\partial u_j}(u^*) \right| = \sum_{j \notin N_i(R)} |d_j\tilde{u}| \leq \gamma_1 \|\tilde{u}\|. \quad (3.45)$$

Now it is possible to check whether Assumption 3.6 is satisfied. To remove the dependency of the result on $P$, the following inequality [135] is useful

$$\lambda_{\text{max}}(P) \leq -\frac{1}{\lambda_{\text{max}}\{(A + A^T)Q^{-1}\}}. \quad (3.46)$$

In addition, $Q = I$ is selected to simplify the analysis. Note that since $P > 0$ and $Q = I$, (3.46) also indicates that $\lambda_{\text{max}}(A + A^T) \in \mathbb{R}_{\leq 0}$.

$$\gamma_1 \leq \frac{c_3}{c_4\|\mathbf{1}\|} = \frac{\lambda_{\text{min}}(Q)}{2\lambda_{\text{max}}(P)\|\mathbf{1}\|} \leq \frac{-\lambda_{\text{max}}(A + A^T)}{2\|\mathbf{1}\|} \quad (3.47)$$

Hence, the upper bound on $\gamma_1$ largely depends on $A$. Furthermore, it is obvious that the fulfilment of (3.47) largely depends on how $\gamma_1$ changes with $R$. The following assumptions are going to be useful in that respect.
Assumption 3.7. There exists a known function $\rho_\gamma(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is decreasing to zero with respect to its argument, such that

$$\max_i \left\{ \sum_{j \notin N_i(R)} |d^i_j| \right\} \leq \rho_\gamma(R).$$

(3.48)

Assumption 3.8. Let $d_{\text{min}}, d_{\text{max}} \in \mathbb{R}_{\geq 0}$ such that for all $i \in S$, $d^i_{ij} \in [d_{\text{min}}, d_{\text{max}}]$. Then, there exist a known function $\rho(\cdot) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $i, j \in S$

$$|d^i_{ij}| \leq |d^i_{ij}| \rho(|i - j|),$$

and

$$\hat{\rho} := \max_i \left\{ \sum_{j \neq i} \rho(|i - j|) \right\} \leq \frac{2d_{\text{min}}}{d_{\text{min}} + d_{\text{max}}}.\quad (3.50)$$

Remark 3.9. It is important to note that Assumptions 3.7 and 3.8 only require bounds on quantities within $D^i$ to be known. Precise knowledge of $J$ is not required. In addition, Assumption 3.8 implies that $A$ is diagonally dominant and, consequently, Hurwitz. Also, it is worth noting that Assumption 3.8 is a property of the entire plant and is independent of the selection of the neighbourhood $N_i(R)$.

In order to eliminate the need to know $\lambda_{\text{max}}(A + A^T)$, Assumption 3.8 and Gershgorin’s disc theorem are used. Firstly,

$$\lambda_{\text{max}}(A + A^T) \leq \max_i \left\{ -2d^i_{ij} + \sum_{j \neq i} |d^i_{ij} + d^j_{ij}| \right\}$$

$$\leq \max_i \left\{ -2|d^i_{ij}| + \sum_{j \neq i} |d^i_{ij}| \rho(|i - j|) + |d^j_{ij}| \rho(|i - j|) \right\}$$

$$\leq \max_i \left\{ (\hat{\rho} - 2)|d^i_{ij}| + d_{\text{max}} \sum_{j \neq i} \rho(|i - j|) \right\}$$

$$\leq (\hat{\rho} - 2)d_{\text{min}} + \hat{\rho}d_{\text{max}}.\quad (3.51)$$
Taking the absolute value of both sides,

$$|\lambda_{\text{max}}(A + A^T)| \geq (2 - \hat{\rho})d_{\text{min}} - \hat{\rho}d_{\text{max}}.$$  (3.52)

From (3.44), (3.48) and (3.52), we have

$$\gamma_1 \leq \rho \gamma(R) \leq \rho \gamma \left( \frac{(2 - \hat{\rho})d_{\text{min}} - \hat{\rho}d_{\text{max}}}{2\|1\|} \right) \leq \left| \lambda_{\text{max}}(A + A^T) \right| \frac{2\|1\|}{2\|1\|}. \quad (3.54)$$

Therefore, (3.47) is satisfied if the following is satisfied

$$\rho \gamma(R) \leq \frac{(2 - \hat{\rho})d_{\text{min}} - \hat{\rho}d_{\text{max}}}{2\|1\|}. \quad (3.55)$$

In conclusion, by using (3.55), one can determine whether Assumption 3.6 is satisfied. Thus, for plants with quadratic costs, (3.55) is an important equation in determining whether dither re-use is possible.

### 3.5 Numerical Example

#### 3.5.1 Heating of a thin iron rod

The aim of the first example is to validate Theorem 3.1 by illustrating the NES scheme in a practical scenario. Consider a thin iron rod with length $l = 54$ cm, diameter $d = 2.5$ cm, specific heat $c_p = 447$ J/kg.K, and thermal conductivity $k_c = 80.2$ W/m.K. Several heat sources are placed along its length with equal spacing. Consider the problem of regulating the temperature of the rod at the heating points while minimising the heating effort. Thus, the control inputs are the heat flows from the heat sources, where each agent (i.e. each heat source) is trying to minimise the following cost function:

$$y_i = (T_i - T_r)^2 + w_u u_i^2.$$  (3.56)
where $T_i$ is the temperature of agent $i$, $T_r$ is the reference temperature, $u_i$ is the heat flow of agent $i$, and $w_u$ is a weighting coefficient. Thus, the Nash equilibrium $u^*$ is easily calculable by using heat transfer analysis.

The rod is modelled using MATLAB SIMSCAPE Library by segmenting the rod as a collection of thermal masses connected in series. In this case, each segment is 1 cm long, with conductive heat transfer at its end if it is connected to another segment, convective heat transfer via its surfaces, and direct heat flow into the element if it is being heated.

The following controller parameters are used: $T_r = 350 \text{ K}, w_u = 0.3 \frac{\text{K}^2}{\text{W}^2}, a = 1, k = 0.001$, and $\omega = 1/\tau$ where $\tau = 677.523$ seconds is the plant time constant. Furthermore, $\omega_i$ is determined by the following equation

$$
\omega_i = \omega_{\text{max}} - (i - 1)\Delta\omega \quad \text{if } i \in \{2Z_{\geq 0} + 1 \} \cup 0 \quad (\text{mod } 2R + 1),
$$

$$
\omega_i = \omega_{\text{max}} - (2R + 1 - i)\Delta\omega \quad \text{otherwise,}
$$

with $\omega_{\text{max}} = 1 \text{ rad/s}$ and $\Delta\omega = 0.1 \text{ rad/s}$. According to [26], such an arrangement maximises $\sum_{i=1}^{N_u} |\omega_i - \omega_{i-1}|^\sigma$ where $\sigma \geq 1$ and $\omega_0 \equiv \omega_{N_u}$. This arrangement ensures that the dither frequencies of two adjacent agents (which have the largest effect upon each other’s cost) have the largest difference.

Figure 3.1 shows the evolution of $\|\bar{u} - u^*\|$ and $\|y\|$ in time for two values of $R$, $R = 2$ and $R = 5$ (without dither re-use). It can be seen that the closed-loop system converges to the vicinity of the NE for multiple neighbourhood sizes, thus validating the steady-state result of Theorem 3.1. Note that in this particular case ($w_u = 0.3$) although the steady state values for $\|\bar{u} - u^*\|$ with $R = 2$ and $R = 5$ are almost identical, this may not be the case in general.

By using this simulation example, it has been demonstrated that convergence to the vicinity of the Nash equilibrium is achieved in the presence of dither re-use. As a result, it can be justified that dither re-use has practical benefits when tuning NES scheme acting on large-scale systems. Firstly, during the deployment of the controller into a large-scale system, the tuning process can be decentralised by dividing the system into smaller “neighbourhoods”, where the tuning of each can be carried out independently without having to concern oneself with the other neighbourhoods. Secondly, in applica-
tions where the system may increase in size in the future, the tuning of the dither frequencies of the additional agents does not necessarily require re-tuning of the entire scheme.

### 3.5.2 Plant with quadratic costs

While not specifically addressed in the theoretical results, there is some interest in investigating the implication of dither re-use on the transient behaviour of the NES scheme. In addition, an example application of the analysis in Section 3.4 is presented in order to show its use when dealing with a plant with quadratic cost functions.

Consider a nonlinear map with first-order lag output dynamics: \(0.2 \dot{y} + y = J(u)\), where \(u, y \in \mathbb{R}^{N_u}\) are the input and output respectively, and \(N_u = 21\). The cost function is

\[
J_i(u) = \frac{1}{2} u^T D_i u,
\]

where \(D_i \in \mathbb{R}^{N_u \times N_u}\) and for all \((i, j, k) \in S^3\),

\[
10 \geq d_{ij} \geq 8, \tag{3.59a}
\]

\[
0 \geq d_{jk} \geq -5e^{-(j-k)^2} (e^{-(j-j)^2} + e^{-(i-k)^2}), \quad \forall (j, k) \neq (i, i). \tag{3.59b}
\]

The exact values of the entries of \(D_i\)'s are unknown. Due to the exponential decay in (3.59), the neighbours set can be defined as \(N_i(R) = \{j \in S - \{i\} | |i - j| \leq \lceil R \rceil\}\) where \(\lceil R \rceil\) is the smallest integer larger than \(R\). Hence, \(R_{\text{max}} = 10\). Also, the Nash equilibrium
of this system is located at the origin.

**Application of the $R^*$ analysis**

Firstly, it is important to find $\rho$ and $\hat{\rho}$. From (3.59) and Assumption 3.8, it is found that:

$$\rho(|i - j|) = 0.625e^{-(i-j)^2} \left(1 + e^{-(i-j)^2}\right),$$  \hspace{1cm} (3.60)

and $\hat{\rho} = 0.3145$. In addition, since $8 \leq d_{ij}^\ell \leq 10$ for all $i$, then $d_{\text{min}} = 8$ and $d_{\text{max}} = 10$. Finally, using (3.48) and (3.59), $\rho_\gamma(R)$ can be determined for all $R \in [0, R_{\text{max}}]$.

<table>
<thead>
<tr>
<th>$R$</th>
<th>(0, 1)</th>
<th>(1, 2)</th>
<th>(2, 3)</th>
<th>(3, 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_\gamma$</td>
<td>1.5797</td>
<td>0.0991</td>
<td>0.0034</td>
<td>$3.2 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Then, the upper bound on $\rho_\gamma$ can be determined using (3.55).

$$\frac{(2 - \hat{\rho})d_{\text{min}} - \hat{\rho}d_{\text{max}}}{2\|1\|} = 1.1281.$$  

Comparing $\rho_\gamma(R)$ given in Table 3.1 with the maximum allowable value of $\rho_\gamma$ of 1.1281, it can be concluded that dither re-use is possible for any $R > 1$.

**Simulation result**

A proper tuning is essential to ensure that the convergence speed (i.e. the transient performance) of the scheme is suitably fast, or even to ensure that the scheme is stable. In order to focus on the effect of dither re-use, $a = \omega = 1$ for all simulations. Note that although $\omega$ is kept constant, the time-scale separation between the NES scheme and the plant can be achieved by choosing sufficiently small $\omega_i$. The other design parameters are the adaptation gain $k$, the neighbourhood parameter $R$ and the agents’ dither frequencies $\omega_i$. The main concern in this simulation is the tuning of $\omega_i$ and $k$ at $R = 10$ (maximum), $R = 5$ and $R = 2$. 
For each $R, \omega_i$ and $k$ are optimally determined using the MATLAB Global Optimisation Toolbox with the transient performance numerically improved through many iterations of convergent plant behaviour. As larger $R$ has more degrees of freedom, it is to be expected that the optimised transient performance should degrade as $R$ is decreased. The purpose of the optimisation is to quantify this degradation. However, this is not indicative of the typical or expected deployments, where the opportunity to optimise transient performance in this fashion does not exist.

The performance index used in the optimisation is the 10% settling time, averaged over 10 different initial conditions (such that the dependence of the result on the choice of the initial condition is reduced). Also, the 2-norm of all the initial conditions is set to be 50 (i.e. $\Delta = 50$). Furthermore, $R$ is decided before an optimisation and is not changed during an optimisation. Thus, the $R = 5$ and the $R = 2$ cases are an optimisation of 12 variables (11 dither frequencies) and 6 variables (5 dither frequencies) respectively, in contrast to 22 variables when $R = 10$ (no dither re-use).

Table 3.2: The relative settling time increase and runtime reduction when $R = 10$, $R = 5$ and $R = 2$

<table>
<thead>
<tr>
<th>$R$</th>
<th>Number of distinct frequencies</th>
<th>Relative settling time increase</th>
<th>Wall time reduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>21</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>5.17%</td>
<td>58.1%</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>27.3%</td>
<td>84.8%</td>
</tr>
</tbody>
</table>

For all tested cases, convergence to the intended region is achieved. Figure 3.2 shows examples of convergence of the NES scheme for each $R$. Furthermore, the settling time relative to the scheme with completely independent dithers is compared in Table 3.2, along with the wall time taken to optimally tune the scheme for different dithers. The latter is indicative of the calibration effort required for each scheme. It can be seen that the average transient performance degrades as $R$ is decreased. However, there is significant reduction in the time taken to produce an optimal calibration of the NES parameters, implying this task is now significantly easier. Therefore, a proper amount of dither re-use can allow for decentralised tuning without significantly impacting upon convergence.
3.6 Conclusion

Semi-global stability of the Nash equilibrium with respect to the design parameters is demonstrated for the proposed control scheme applied to a nonlinear MIMO system satisfying certain conditions. These conditions are consistent with existing ES results, with one extra condition that requires the effect of an agent’s input on another agent’s cost function to decay to a sufficiently small value as the “distance” between them is increased. This “decaying effect” characteristic allows any given dither frequency to be used by multiple agents if those agents have a sufficiently weak effect on each other’s steady-state cost function. In addition, another extra condition is provided, which guarantees that a non-trivial amount of dither re-use is possible. This condition is then quantified in an example application to a plant with general quadratic cost functions. Finally, the benefits of re-using dither frequencies, which are simplification and decentralisation of the tuning process, were demonstrated in simulations.

This chapter has focused on the decentralisation process of a NES scheme by introducing dither re-use to exploit the “decaying effect” characteristic of the system. This result is useful not only for application of ES to urban traffic control, but also to the opti-
misation of other large-scale systems that exhibit this characteristic.
Chapter 4

Semi-Global Practical Stability of a Class of Hybrid Systems with Application to Extremum-Seeking

Urban traffic networks are hybrid systems due to the existence of discrete events such as those introduced by traffic lights. Therefore, this chapter considers the necessary extension to accommodate the application of ES schemes to hybrid systems.

Many engineering problems involve hybrid systems, such as urban traffic control, switching control, systems subjected to impulses, and internet congestion. Many of these systems have tunable parameters which affect their steady-state performance. The selection of these parameters is important, and can be posed as an optimisation problem. When the influence of these parameters on performance is not known a priori, real time optimisation can be achieved through the use of extremum-seeking.

As typical ES controllers act on a slower time scale than that of the plant dynamics, it is usual to rely on singular perturbation analysis to prove stability of an ES scheme. However, the classical singular perturbation result [69] only applies to systems with continuous dynamics. It is only recently that singular perturbation results for a class of hybrid systems have been developed [114, 132, 133]. However, these results for singularly perturbed hybrid systems only consider systems with a globally-asymptotically stable (GAS) reduced system, whereas most ES schemes are semi-globally practically asymptotically (SPA) stable [16, 79, 91, 100, 127]. In addition, some studies [114, 132] only deal with continuous boundary layer systems, i.e. the jump is caused by the slow state. On the

The content of this chapter has been submitted as a journal paper [80].
other hand, applying ES to a hybrid system results in a singularly perturbed hybrid system with a continuous slow subsystem and a hybrid fast subsystem. Hence, the results by [114,132,133] are not directly applicable and must be extended.

This chapter considers an extension of [133] by dealing with a class of singularly perturbed system with a continuous slow state and a SPA stable reduced system. This is then used to prove SPA stability of an extremum-seeker (ES) acting on a hybrid plant. The convergence of the ES is then numerically demonstrated for a simple problem involving bioprocess optimisation.

4.1 Preliminaries

A set-valued mapping $F : \mathbb{R}^{N_x} \Rightarrow \mathbb{R}^{N_x}$ is said to be outer semi-continuous at $x$ if for all sequences $x_i \rightarrow x$ and $y_i \in F(x_i)$ such that $y_i \rightarrow y$, then $y \in F(x)$. If this applies for all $x \in \mathbb{R}^{N_x}$, then it is said that $F$ is outer semi-continuous. Furthermore, $F$ is locally bounded if for any compact set $K \subset \mathbb{R}^{N_x}$, there exists $r$ such that $F(K) := \cup_{x \in K} F(x) \subset rB$, where $rB$ is a closed ball of radius $r$.

The time domain of a solution to a hybrid system consists of two elements $(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. A compact hybrid time domain is a set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ which is defined as $S = \cup_{j=0}^{j-1}([t_j,t_{j+1}],j)$ with a finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_j$. A set $S$ is said to be a hybrid time domain if $S \cap ([0,t],[0,1,\ldots,j])$ is a compact hybrid time domain for all $(t,j) \in S$. In addition, a solution to a hybrid system $x$ is maximal if there is no other solution $x'$ such that dom $x$ is a proper subset of dom $x'$ and $x'$ is equal to $x$ on dom $x$.

A maximal solution to a hybrid system that is forward complete is characterized by its time domain being unbounded. It is said to be forward pre-complete if the time domain is compact or unbounded. A hybrid system is forward pre-complete from a compact set $K$ if all solutions starting from $K$ that are maximal are forward pre-complete.
4.2 Motivation

Before presenting the results of the chapter, the motivation of this work is going to be discussed in more detail. A typical approach to prove stability of an ES scheme is to use a singular perturbation analysis followed by an averaging analysis. In particular, consider the following general continuous nonlinear dynamical plant:

\[ \dot{x} = f(x, u), \quad (4.1a) \]
\[ y = g(x, u), \quad (4.1b) \]

where \( x \in \mathbb{R}^N_x \), \( u, y \in \mathbb{R} \), \( f : \mathbb{R}^N_x \times \mathbb{R} \to \mathbb{R}^N_x \), and \( g : \mathbb{R}^N_x \times \mathbb{R} \to \mathbb{R} \). The control objective is to regulate the input \( u \) to a point that optimises the steady-state value of the output \( y \). Specifically, (4.1) has to satisfy the following two assumptions.

**Assumption 4.1.** There exists a map \( h : \mathbb{R} \to \mathbb{R}^N_x \) such that \( x = h(u) \) is an asymptotically stable equilibrium of (4.1a) uniformly in \( u \in \mathbb{R} \).

**Assumption 4.2.** Defining \( J(\cdot) := g(h(\cdot), \cdot) \), there exists a unique extremum \( u^* \) minimising \( J(\cdot) \), and the following hold: \( J'(u^*) = 0 \); and \( \exists \zeta > 0 \) such that \( J'(u^* + \delta)/\delta > \zeta \) for all \( \delta \neq 0 \).

Furthermore, consider an ES scheme with the following structure:

\[ \dot{\kappa} = \varepsilon_f G_\kappa(\kappa, y, d(\varepsilon_f t, \varepsilon_s), \varepsilon_s), \quad (4.2a) \]
\[ u = G_u(\kappa, d(\varepsilon_f t, \varepsilon_s), \varepsilon_s), \quad (4.2b) \]

where \( \kappa \in \mathbb{R}^N_\kappa ; \varepsilon_f \in \mathbb{R}^{N_\kappa} \) and \( \varepsilon_s \in \mathbb{R}^{N_\varepsilon} \) are small positive parameters; \( d(\varepsilon_f t, \varepsilon_s) \in \mathbb{R} \) is a “dither” signal; \( G_\kappa : \mathbb{R}^{N_\kappa} \times \mathbb{R}^2 \times \mathbb{R}^{N_\varepsilon} \to \mathbb{R}^{N_\kappa} \); and \( G_u : \mathbb{R}^{N_\kappa} \times \mathbb{R} \times \mathbb{R}^{N_\varepsilon} \to \mathbb{R} \). The dither is used to perturb the input to the plant such that the gradient of \( J(\cdot) \) can be estimated. Furthermore, let \( N_{\varepsilon_f} = 1 \) for notational compactness and \( \varepsilon_f \) corresponds to \( \omega \) for typical ES schemes, including that in Chapter 3. Following singular perturbation arguments [69], by defining a new time-scale \( \tau := \varepsilon_f t \) and setting \( \varepsilon_f \) small such that the dynamics of the ES are slow compared to those of the plant, a reduced system can be derived from the full closed-loop system (4.1)–(4.2) by replacing \( y \) in (4.2) with the steady-state input-output.
map \( J(u) \) as follows,

\[
\frac{d\kappa}{d\tau} = G_\kappa(\kappa_r, J(G_u(\kappa_r, d(\tau, \epsilon_s), \epsilon_s)), d(\tau, \epsilon_s), \epsilon_s).
\]  

(4.3)

Typically, it is shown that (4.3), i.e. the ES scheme acting on the map \( J(u) \), is SPA stable for small \( \epsilon_s \). Then, the SPA stability of the full closed-loop system follows from singular perturbation theory [69], given sufficiently small \( \epsilon_f \). Many ES schemes fit the structure of (4.2) and satisfy the SPA stability property for sufficiently small \( \epsilon_s \). Two examples of such ES schemes are presented below.

**Example 4.1.** One of the simplest forms of ES scheme [127] is as follows

\[
\begin{align*}
\dot{\kappa} &= -k\omega \sin(\omega t) y, \\
u &= \kappa + a \sin(\omega t),
\end{align*}
\]  

(4.4a)

(4.4b)

with \( \epsilon_f \equiv \omega, \epsilon_s \equiv (a, k), \) and \( d(\cdot, \epsilon_s) \equiv a \sin(\cdot) \). Thus, the reduced system derived from the system (4.1) and (4.4) is

\[
\frac{d\kappa}{d\tau} = -k \sin(\tau) J(\kappa + a \sin(\tau)).
\]  

(4.5)

By making \( k \) small and using periodic averaging theory, it can be shown that the reduced system (4.5) approximates the following averaged system

\[
\frac{d\kappa_{av}}{d\tau} = -\frac{k}{2\pi} \int_0^{2\pi} \sin(s) J(\kappa_{av} + a \sin(s)) \, ds.
\]  

(4.6)

Then, it can be shown that as \( a \to 0 \), the averaged system (4.6) approaches the following gradient system

\[
\frac{d\kappa_{gr}}{d\tau} = -\frac{ka}{2} J'(\kappa_{gr}).
\]  

(4.7)

Under Assumption 4.2, (4.7) is GAS and it can be shown that (4.5) is SPA stable for small \( \epsilon_s = (a, k) \).

**Example 4.2.** Alternatively, a linear filter can be inserted before the demodulation signal as fol-
4.2 Motivation

\[
\dot{\xi} = \omega A \xi + \omega B y, \quad (4.8a)
\]
\[
\dot{\bar{u}} = -k \omega \sin(\omega t - \phi) C \xi, \quad (4.8b)
\]
\[
u = \bar{u} + a \sin(\omega t), \quad (4.8c)
\]

where \( A \in \mathbb{R}^{N_x \times N_x}, B \in \mathbb{R}^{N_x}, C^T \in \mathbb{R}^{N_y}, \kappa \equiv (\xi, \bar{u}), \epsilon_f \equiv \omega, \epsilon_s \equiv (a, k), \text{ and } d(\cdot, \epsilon_s) \equiv a \sin(\cdot) \). System (4.8) with \( y \equiv f(u) \) is SPA stable for small \((a, k)\). A sketch of proof for this result is given in Section 4.4.1. The role of \( \phi \) is to compensate for any phase shift (at the dither frequency, \( \omega \)) caused by the filter (4.8a). For example, let \( H(s/\omega) = C(sI/\omega - A)^{-1}B \) be the transfer function for the filter, where \( I \) is the identity matrix. Then, the dither experiences a phase shift, \( H(i) \), due to the filter. This can be compensated for by letting \( \phi = H(i) \), thereby bringing the demodulation signal in-phase with the oscillatory component of the plant output attributed to the dither. \( \phi \) can be chosen to also compensate for the plant dynamics, if they are known [99].

Now consider the stability of an arbitrary ES scheme acting on a general nonlinear hybrid system of the form

\[
x^+ \in M(x, u), \quad x \in \Psi(u), \quad (4.9a)
\]
\[
x \in F(x, u), \quad x \in \psi(u), \quad (4.9b)
\]
\[
y = g(x), \quad (4.9c)
\]

where \( x \in X \subset \mathbb{R}^{N_x}; u, y \in \mathbb{R}; M, F : \mathbb{R}^{N_x} \times \mathbb{R} \Rightarrow \mathbb{R}^{N_x}; \Psi, \psi : \mathbb{R} \Rightarrow \mathbb{R}^{N_x}; \text{ and } g : \mathbb{R}^{N_x} \rightarrow \mathbb{R} \) is continuous in \( x \). The hybrid system (4.9) can represent many engineering systems, such as: engine combustion dynamics, systems with sampling, PWM control, and urban traffic light control. Since continuous systems are special cases of hybrid systems, it is desired to replace Assumption 4.1 when dealing with a more general system such as (4.9). Although singular perturbation results for hybrid systems are available in the literature [114, 132, 133], they do not encompass systems that result from the application of ES schemes to hybrid systems, which have a SPA stable reduced system. Thus, it is clear that an extension is required to serve this purpose.
4.3 Singularly Perturbed Hybrid Systems

In this section, the required extension to the singular perturbation results is discussed. Consider the system (4.9) with the ES scheme (4.2), resulting in the following closed-loop system stated in the $\tau$ time-scale,

\begin{align}
    x^+ &\in M(x,u), \quad x \in \Psi(u), \quad (4.10a) \\
    \varepsilon f \frac{dx}{d\tau} &\in F(x,u), \quad x \in \psi(u), \quad (4.10b) \\
    \frac{dx}{d\tau} & = G_x(\kappa, g(x), d(\tau, \varepsilon_s), \varepsilon_s), \quad (4.10c) \\
    u & = G_u(\kappa, d(\tau, \varepsilon_s), \varepsilon_s) \quad (4.10d)
\end{align}

In order to use results on robustness of hybrid systems [56, 132, 133], the set-valued mappings have to satisfy the following properties obtained from the so-called basic assumptions of hybrid systems [56, Assumption 6.5].

**Assumption 4.3.** The following hold:

- The set-valued mappings $\Psi$ and $\psi$ are outer semi-continuous and locally bounded. For each $u \in \mathbb{R}$, $\Psi(u) \subset X$ and $\psi(u) \subset X$ are nonempty.

- The set-valued mapping $F$ is outer semi-continuous and locally bounded. For each $(x,u) \in \psi(u) \times \mathbb{R}$, the set-valued mapping $F(x,u)$ is nonempty and convex.

- The set-valued mapping $M$ is outer semi-continuous and locally bounded. For each $(x,u) \in \Psi(u) \times \mathbb{R}$, the set-valued mapping $M(x,u)$ is nonempty and $M(x,u) \subseteq \Psi(u) \cup \psi(u)$.

4.3.1 Time-scale properties of the system

The stability of (4.10) relies upon two key assumptions: the system (4.10) admits an averaged system (i.e. the reduced system); and the averaged system admitted is SPA stable. In addition, some properties need to be imposed on the boundary layer system, as when dealing with continuous systems.

Firstly, the boundary layer system is obtained by setting $\varepsilon_f = 0$ as follows (stated in $t$...
4.3 Singularly Perturbed Hybrid Systems

(4.11) \begin{align*}
    &\dot{x}_{\text{bl}} \in M(x_{\text{bl}}, u_{\text{bl}}), \quad x_{\text{bl}} \in \Psi(u_{\text{bl}}), \quad (4.11a) \\
    &\dot{x}_{\text{bl}} \in F(x_{\text{bl}}, u_{\text{bl}}), \quad x_{\text{bl}} \in \psi(u_{\text{bl}}), \quad (4.11b) \\
    &\dot{\kappa}_{\text{bl}} = 0, \quad u_{\text{bl}} = G(\kappa_{\text{bl}}, d(0, \varepsilon_s), \varepsilon_s). \quad (4.11c)
\end{align*}

Assumption 4.4. No complete solution $x_{\text{bl}}$ to the boundary layer system (4.11) has the time domain $\{0\} \times \mathbb{Z}_{\geq 0}$.

Remark 4.1. Assumption 4.4 is required such that the boundary layer system does not have any instantaneous Zeno solution, i.e. solutions that jump infinitely often at $t = 0$.

Then, it is assumed that (4.10) admits an averaged system.

Assumption 4.5. There exists a map $J : \mathbb{R} \to \mathbb{R}$ such that for each compact set $K \subset \mathbb{R}^N_{\kappa}$, there exists $\sigma_K \in \mathcal{L}$, such that for:

- each $L > 0$,
- each $\kappa \in K$,
- each solution $x_{\text{bl}}$ to the boundary layer system (4.11) with $\text{dom} \ x_{\text{bl}} \cap ([L, \infty) \times \mathbb{Z}_{\geq 0}) \neq \emptyset$,
- each measurable function $g_{\text{bl}} : [0, L] \to \mathbb{R}$ that satisfies $g_{\text{bl}}(s) = g(x_{\text{bl}}(s, j))$ for each $(s, j) \in \text{dom} \ x_{\text{bl}}$,
- each $\varepsilon_s > 0$ and each $\tau \geq 0$,

the following holds with $u = G_u(\kappa, d(\tau, \varepsilon_s), \varepsilon_s)$,

$$
\frac{1}{L} \left\| \int_0^L G_\kappa(\kappa, g_{\text{bl}}(s), d(\tau, \varepsilon_s), \varepsilon_s) - G_\kappa(\kappa, J(u), d(\tau, \varepsilon_s), \varepsilon_s) \, ds \right\| \leq \sigma_K(L). \quad (4.12)
$$

Example 4.3. When considering the ES schemes in Examples 4.1 and 4.2, (4.12) can be simplified due to the affinity of $G_\kappa$ with respect to $y$. For instance in Example 4.2, by defining $\sigma'_K(\cdot) := \sigma_K(\cdot)/\|B\|$, if the following holds

$$
\frac{1}{L} \left\| \int_0^L (g_{\text{bl}}(s) - J(u)) \, ds \right\| \leq \sigma'_K(L), \quad (4.13)
$$
then (4.12) also holds since

\[
\sigma_K(L) = \sigma_K'(L)\|B\|
\geq \frac{\|B\|}{L} \left\| \int_0^L (g_{bl}(s) - J(u)) \, ds \right\|
\geq \frac{1}{L} \left\| \int_0^L B(g_{bl}(s) - J(u)) \, ds \right\|
= \frac{1}{L} \left\| \int_0^L \left[ A\zeta + Bg_{bl}(s) - A\zeta - Bf(u) \right] \, ds \right\|.
\]  

(4.14)

Thus, (4.12) can be replaced by (4.13).

**Remark 4.2.** Assumption 4.5 states that given a constant \( u \), the output of the plant can be represented by an average steady-state map \( J(u) \) when evaluating the vector field \( G_\kappa \). Hence, the averaged/reduced system is simply the ES scheme acting on a static plant \( J(u) \). In essence, Assumptions 4.3–4.5 apply to a more broader range of systems compared to Assumption 4.1 when dealing with hybrid systems.

**Remark 4.3.** Although not stated explicitly, Assumption 4.5 implies that the fast state is also well-behaved such that the output \( y \) is able to converge, on average, to some mapping \( J(\cdot) \) when evaluating \( G_\kappa \).

Then, as a consequence of Assumption 4.5, the resulting reduced/averaged system is (as for the continuous plant) given by (4.3).

**Assumption 4.6.** The system (4.3) is SPA stable uniformly in small \( \varepsilon_s \).

**Remark 4.4.** Under Assumption 4.6, the stability of an arbitrary ES scheme acting on a static map \( J(u) \) is guaranteed. Given a specific ES scheme, this assumption can be replaced with the stability proof of the considered ES scheme, as demonstrated in Section 4.4. In addition, if the equilibrium of (4.3) is not at the origin, it can always be translated such that the origin becomes the equilibrium.
4.3.2 SPA stability of the closed-loop hybrid system

This section essentially provides a guarantee that the slow state of (4.10) has a similar behaviour to the solution of (4.3). However, the time domains of the solutions of (4.10) and (4.3) are different since the former is a hybrid system while the latter is a continuous system. Therefore, define a mapping $\kappa_c$ that satisfies

$$\text{graph}(\kappa_c) = \bigcup_{(\tau, j) \in \text{dom}(x, \kappa)} (\tau, \kappa_c(\tau, j)).$$

(4.15)

Hence, $\kappa_c$ is now a continuous signal, with values identical to those of $\kappa$, and will be used in the result statement.

The stability proof of the system uses the following results on the closeness of $\kappa_c$ with the solution of the reduced system $\kappa_r$.

**Lemma 4.1** (Theorem 1 of [133]). Consider system (4.10) under Assumptions 4.3–4.5. If the reduced system (4.3) is forward pre-complete from a compact set $K \subset \mathbb{R}^{N_k}$, then for any $(T, \rho) \in \mathbb{R}^2_{>0}$, there exists $\epsilon_f^r$ such that for any $\epsilon_f \in (0, \epsilon_f^r]$, and each solution of (4.10c) with $\kappa(0, 0) \in K$, there exists some solution $\kappa_r$ to the reduced system (4.3) with $\kappa_r(0) \in K$ such that for each $\tau \leq T$, there exists $s$ such that $|\tau - s| \leq \rho$ and $\|\kappa_c(\tau) - \kappa_r(s)\| \leq \rho$.

Thus, the stability of the full system can be stated.

**Theorem 4.1.** Consider system (4.10) under Assumptions 4.3–4.6. There exists $\beta \in \mathcal{KL}$ such that for any $(\Delta, \nu)$ that satisfy $\Delta \geq \nu > 0$, there exists $\epsilon_s^r > 0$ such that for any $\epsilon_s \in (0, \epsilon_s^r]$, there exists $\epsilon_f^r > 0$ such that for any $\epsilon_f \in (0, \epsilon_f^r]$, and any solutions of (4.10c) with $\|\kappa(0, 0)\| \leq \Delta$, the following holds

$$\|\kappa_c(\tau)\| \leq \beta(\|\kappa_c(0)\|, \epsilon_s \tau) + \nu, \quad \forall \tau \geq 0.$$

**Proof.** The proof follows a similar idea to the proof of [129, Theorem 1] using the following steps.

1. Choose $(\Delta, \nu)$ that satisfy $\Delta \geq \nu > 0$.

2. From Assumption 4.6, use $(\Delta, \nu/3)$ to generate $\beta$ and $\epsilon_s^r$ from the SPA stability definition of (4.3).
3. Let $T$ be large enough that $\beta(\Delta, \varepsilon, \tau) \leq v/3$ for all $\tau \geq T$.

4. Let $\eta > 0$ be sufficiently small such that there exists $\rho^* > 0$ such that $\forall \rho \in (0, \rho^*]$, the following hold.

   (a) $\forall \|\kappa_r(0)\| \leq \Delta$ and $\forall \tau \leq \rho$, $\|\kappa_r(0) - \kappa_r(\tau)\| \leq \eta$.

   (b) The following holds

   $$\sup_{\delta \in (0, \Delta]} \left[ \beta \left( \delta + \eta + \rho, \varepsilon s \max \left\{ \frac{\tau - \rho}{0} \right\} \right) - \beta(\delta, \varepsilon, \tau) \right] + \rho \leq v/3. \quad (4.16)$$

   The existence of $\rho^*$ for sufficiently small $\eta$ is guaranteed by the continuity of $\kappa_r$ and the definition of a $KCL$ function.

5. Use $(2T, \rho)$ to generate $\varepsilon^*_T$ from Lemma 4.1, such that:

   (a) For all $\tau \leq 2T$, $\exists s_1 \geq 0$ such that $|\tau - s_1| \leq \rho$, $\|\kappa_c(\tau) - \kappa_r(s_1)\| \leq \rho$. Also from Lemma 4.1, $\exists s_2 \geq 0$ such that $s_2 \leq \rho$, $\|\kappa_c(0) - \kappa_r(s_2)\| \leq \rho$. It follows that,

   $$\|\kappa_c(\tau)\| \leq \|\kappa_r(s_1)\| + \rho$$

   $$\leq \beta(\|\kappa_r(0)\|, \varepsilon, s_1) + \rho + \frac{v}{3} \quad \text{(by Assumption 6)}$$

   $$\leq \beta(\|\kappa_r(s_2)\| + \eta, \varepsilon, s_1) + \rho + \frac{v}{3} \quad \text{(from Step 4a)}$$

   $$\leq \beta \left( \|\kappa_r(0)\| + \eta + \rho, \varepsilon \max \left\{ \frac{\tau - \rho}{0} \right\} \right) + \rho + \frac{v}{3} \quad \text{(by Lemma 4.1)}$$

   $$\leq \beta(\|\kappa_r(0)\|, \varepsilon \tau) + \frac{2v}{3} \quad \text{(from Step 4b).}$$

   (b) More specifically, by using Step 3, it follows that for all $\tau \in [T, 2T]$,

   $$\|\kappa_c(\tau)\| \leq \beta(\Delta, \varepsilon \tau) + \frac{2v}{3} \leq v.$$

6. Noting that $\Delta > v$, re-apply Lemma 4.1 for $\tau \in [T, 3T]$. It follows that for all $\tau \in [2T, 3T]$,

   $$\|\kappa_c(\tau)\| \leq \beta(\Delta, \varepsilon \tau) + \frac{2v}{3} \leq v. \quad (4.17)$$
7. Theorem 4.1 follows from repeating Step 6 to obtain:

\[ \| \kappa_c(\tau) \| \leq \nu, \quad \forall \tau \in [T, \infty). \]

\[ \square \]

**Remark 4.5.** The low convergence speed that is caused by \( \epsilon_f \) and \( \epsilon_s \) being small is typical of an ES stability result. Since ES is a steady-state optimisation that is not aimed to handle the transient response of the plant, the slow convergence is generally acceptable.

**Remark 4.6.** Theorem 4.1, along with Assumptions 4.3 and 4.5, can be easily modified to address a slightly more general class of singularly perturbed hybrid systems:

\( x^+ \in M(x, \kappa, \epsilon_f), \quad x \in \Psi(\kappa), \quad (4.18a) \)

\( \epsilon_f \frac{dx}{d\tau} \in F(x, \kappa, \epsilon_f), \quad x \in \psi(\kappa), \quad (4.18b) \)

\( \frac{d\kappa}{d\tau} = G(x, \kappa, \epsilon_f, \epsilon_s), \quad (4.18c) \)

where \( x \in X \subset \mathbb{R}^{N_x}; \kappa \in \mathbb{K} \subset \mathbb{R}^{N_\kappa}; M, F : \mathbb{R}^{N_x} \times \mathbb{R}^{N_\kappa} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}^{N_x}; \Psi, \psi : \mathbb{R}^{N_\kappa} \Rightarrow \mathbb{R}^{N_x}; \) and \( G : \mathbb{R}^{N_x} \times \mathbb{R}^{N_\kappa} \times \mathbb{R}_{>0}^2 \rightarrow \mathbb{R}^{N_\kappa}. \) The reduced system obtained from (4.18) still has to be SPA stable uniformly in \( \epsilon_s. \)

The singular perturbation result of Theorem 4.1 is summarised in the second row of Table 4.1 and its continuous counterpart is provided in the first row for comparison.

### 4.4 Application of Theorem 4.1 to an Extremum-seeking Scheme with a Second-order Linear Filter

In this subsection, Theorem 4.1 is used to prove stability of an ES scheme acting on a hybrid system (4.9). Essentially, this subsection demonstrates that the proposed ES scheme (4.8) in Example 4.2 satisfies Assumption 4.6 when acting on (4.10).

The proposed ES scheme is equipped with a second-order linear filter, through which the output \( y \) is processed. An example of a second-order linear filter is a band-pass filter,
Table 4.1: Singular perturbation results of Theorems 4.1 and 4.2 compared to their continuous time counterparts.

<table>
<thead>
<tr>
<th>Boundary layer</th>
<th>Reduced system</th>
<th>Full system</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous UGAS (Assumption 4.1)</td>
<td>SPA stable uniform in small $\epsilon_s$ (Assumption 4.6)</td>
<td>SPA stable for small $(\epsilon_f, \epsilon_s)$ uniform in $\epsilon_s$</td>
</tr>
<tr>
<td>Hybrid (Assumptions 4.3–4.5)</td>
<td>SPA stable uniform in small $\epsilon_s$ (Assumption 4.6)</td>
<td>Slow dynamics are SPA stable for small $(\epsilon_f, \epsilon_s)$ uniform in $\epsilon_s$ (Theorem 4.1)</td>
</tr>
<tr>
<td>Continuous UGAS (Assumptions 4.1 &amp; 4.2)</td>
<td>SPA stable ES uniform in small $\epsilon_s$ (Assumption 4.6)</td>
<td>SPA stable for small $(\epsilon_f, \epsilon_s)$ uniform in $\epsilon_s$ (Theorem 4.1)</td>
</tr>
<tr>
<td>Hybrid (Assumptions 4.3–4.5 &amp; 4.7)</td>
<td>SPA stable ES uniform in small $\epsilon_s$ (Assumption 4.6)</td>
<td>ES dynamics are SPA stable for small $(\epsilon_f, \epsilon_s)$ uniform in $\epsilon_s$ (Theorem 4.2)</td>
</tr>
</tbody>
</table>

Figure 4.1: The proposed ES scheme.

which assists in extracting the sinusoidal component of the output signal corresponding to the effect of the dither.

**Assumption 4.7.** There exists a unique extremum $u^*$ such that the following hold for $J(u)$ from Assumption 4.5: $J'(u^*) = 0$; and $\exists \zeta > 0$ such that $J'(u^* + \delta)/\delta > \zeta$ for all $\delta \neq 0$.

**Remark 4.7.** Assumption 4.7 is the counterpart of Assumption 4.2 when dealing with hybrid systems (4.9).
4.4 Application to an Extremum-seeking Scheme

4.4.1 Reduced system

The reduced system is obtained, as in Section 4.3, by replacing $y$ in (4.8) with $J(u)$, which leads to

$$\frac{d\xi_r}{d\tau} = A\xi_r + BJ(\bar{u}_r + a\sin(\tau)), \quad (4.19a)$$
$$\frac{d\bar{u}_r}{d\tau} = -k\sin(\tau - \phi)C\xi_r. \quad (4.19b)$$

Note that $J(\bar{u}_r + a\sin(\tau))$ is $2\pi$-periodic in $\tau$ (and satisfies Dirichlet conditions by Assumption 4.7). Then, using Fourier series representation (while considering $\bar{u}_r$ as a constant)

$$J(\bar{u}_r + a\sin(\tau)) = \sum_{n \in \mathbb{Z}} \hat{J}_n(\bar{u}_r)e^{in\tau}, \quad (4.20)$$

where $i$ is the imaginary unit and

$$\hat{J}_n(\bar{u}_r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} J(\bar{u}_r + a\sin(\tau))e^{-in\tau} d\tau, \quad (4.21)$$

the following approximate equilibrium solution of $\xi_r$ can be obtained:

$$h(\bar{u}_r, \tau) = \sum_{n \in \mathbb{Z}} (in I - A)^{-1}B\hat{J}_n(\bar{u}_r)e^{in\tau}, \quad (4.22)$$

where $I$ is the identity matrix.

Define the error coordinates $\tilde{u}_r = \bar{u}_r - u^*, \tilde{\xi}_r = \xi_r - h(\bar{u}_r + u^*, \tau)$. Then, the reduced system in the error coordinate is as follows:

$$\frac{d\tilde{\xi}_r}{d\tau} = A\tilde{\xi}_r - \frac{\partial h(\bar{u}_r + u^*, \tau)}{\partial \bar{u}_r} \frac{d\bar{u}_r}{d\tau}, \quad (4.23a)$$
$$\frac{d\bar{u}_r}{d\tau} = -k\sin(\tau - \phi)C(\tilde{\xi}_r + h(\bar{u}_r + u^*, \tau)). \quad (4.23b)$$

Finally, recall the definition $H(s) = C(sI - A)^{-1}B$ and introduce the following assumption.

**Assumption 4.8.** $\text{Re}\{e^{i\phi}H(i)\} > 0$ and $A$ is Hurwitz.

**Remark 4.8.** Assumption 4.8 can be satisfied by appropriate design.
Lemma 4.2. Under Assumptions 4.7–4.8, (4.19) is SPA stable uniformly in small \((a,k)\).

Proof. Only a sketch of a proof is provided here due to space limitations. For the sake of review, further details are provided in the attachment. The proof consists of three steps. Firstly, following the approach by [129], the reduced system (4.23) is separated into: a boundary-layer system by setting \(k = 0\); and an averaged system by ignoring (4.23a) and averaging the dynamics of (4.23b) with \(\bar{\xi}_r = 0\). Secondly, the stability of the averaged system is proven by following a similar approach to Lemma 3.1, while the boundary layer system is stable since \(A\) is Hurwitz. Lastly, Lemma 4.2 follows from [129, Theorem 1].

4.4.2 Stability of the full closed-loop system

Now consider the full closed-loop system in the error coordinate \(\bar{u} = \bar{u} - u^*\), \(\bar{\xi} = \xi - h(\bar{u} + u^*, \omega t)\), and the \(\tau\) time-scale

\[
\begin{align*}
  x^+ &\in M(x, \bar{u} + a \sin(\tau)), \quad x \in \Psi(\bar{u} + a \sin(\tau)), \\
  \omega \frac{dx}{d\tau} &\in F(x, \bar{u} + a \sin(\tau)), \quad x \in \psi(\bar{u} + a \sin(\tau)), \\
  \frac{d\bar{\xi}}{d\tau} &\quad= A_2 \bar{\xi} + B(g(x) - J(\bar{u} + u^* + a \sin(\tau))), \\
  \frac{d\bar{u}}{d\tau} &\quad= -k \sin(\tau - \phi) C(\bar{\xi} + h(\bar{u} + u^*, \tau)).
\end{align*}
\]

As before, define a mapping \(\bar{\kappa}_c\) that satisfies:

\[
\text{graph}(\bar{\kappa}_c) = \bigcup_{(\tau,j) \in \text{dom}(x,\bar{\xi},\bar{u})} (\tau, \bar{\xi}(\tau,j), \bar{u}(\tau,j)).
\]  

The stability of the complete closed-loop system (4.24) follows from Theorem 4.1.

Theorem 4.2. Consider system (4.24) under Assumptions 4.3–4.5, 4.7 and 4.8, for any \((\Delta, \nu)\) that satisfy \(\Delta \geq \nu > 0\), there exist \((a^*, k^*) \in \mathbb{R}^2_{\geq 0}\) such that for all \((a,k) \in (0, a^*] \times (0, k^*],\) there exists \(\omega^* \in \mathbb{R}_{>0}\) such that for any \(\omega \in (0, \omega^*],\) any solutions of (4.24c)–(4.24d) with
\[\|\tilde{\xi}(0,0), \tilde{u}(0,0)\| \leq \Delta, \text{the following holds}\]

\[\|\tilde{\kappa}_c(\tau)\| \leq \beta (\|\tilde{\kappa}_c(0)\|, k\tau) + \nu, \quad \forall \tau \geq 0.\]

Table 4.1 summarises the difference between the ES stability result of Theorem 4.2 (fourth row) and its continuous counterpart (third row).

**Remark 4.9.** Theorem 4.1 accommodates a wide range of ES schemes as long as they satisfy Assumption 4.6, such as: the family of ES schemes considered in the unifying ES framework of [105]; the Nash equilibrium seeking (NES) scheme of Chapter 3; and the simple scheme in Example 4.1, leading to subsequent restatement of Theorem 4.2. Special attention is given to the NES scheme of Chapter 3 in order to illustrate this point. Under Assumptions 3.2–3.5, one can choose sufficiently small \((a,k)\) and sufficiently large \(R\) such that the NES scheme acting on a static map is SPA stable (based on Lemma 3.2). This shows that the NES scheme satisfies Assumption 4.6. Therefore, applying the NES scheme to system (4.9) and if Assumptions 4.3–4.5 also hold, one can choose sufficiently small \(\omega\) such that the resulting full closed-loop system is SPA stable.

### 4.5 Simulation Example

Consider the problem of maximising the steady-state trade-off between yield and productivity in a bioprocess reaction [20]. For simplicity, the dynamics of the enzymatic reaction is normalised by using the maximum reaction rate \(v_m\) and the liquid volume \(V\), which is kept constant. The resulting chemical model becomes

\[
\begin{align*}
\dot{x}_1 &= -\frac{x_1}{K_m + x_1} + q(c - x_1), \\
\dot{x}_2 &= \frac{x_1}{K_m + x_1} - qx_2,
\end{align*}
\]

(4.26a) (4.26b)

where \(x_1\) is the reactant’s concentration, \(x_2\) is the product’s concentration, \(K_m = 0.1\) is the half-saturation constant, \(c = 3\) is the reactant’s concentration in the feed stream, and \(q\) is the volumetric flow rate of both the reactant feed stream and the reaction medium withdrawal stream (feed and withdraw flow rate is equal so that \(V\) is constant). Also,
q and time are normalised by $v_m V$ and $v_m^{-1}$ respectively. The flow rates $q$ are controlled by solenoid valves, which only have fully-open or fully-closed states. PWM is used to achieve flow rates that are, in an average sense, in the range $[0, q_{\text{open}}]$, where $q_{\text{open}} = 1.6 v_m V$ is the flow rate when the valves are open. The output considers a linear combination of the productivity (the rate of harvesting the product $= q x_2$) and the yield (the amount of product made per unit of substrate fed $= x_2/c$) with a weighting factor $\lambda = 0.5$ as follows

$$y := \lambda q x_2 + (1 - \lambda) c^{-1} x_2.$$ (4.27)

It is assumed that $x_2$ is measurable such that $y$ can be computed. The control objective is to determine the duty cycle $u$ of the “PWM valve” that maximises $y$ in the “steady-state” (on average) given a cycle time $C = 0.1 v_m^{-1}$.

The ES scheme uses a band-pass filter as follows,

$$A = \begin{bmatrix} -\omega_H & 0 \\ -\omega_L & -\omega_L \end{bmatrix}, \quad B = \begin{bmatrix} \omega_H \\ \omega_L \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$ (4.28)

where $\omega_H = 0.2$ and $\omega_L = 5$ are (after being multiplied by $\omega$) the cut-off frequencies. The following ES parameters are used: $a = 0.05, k = 0.8, \omega = 0.1 \text{ rad}/v_m^{-1}$, and $\phi = 20^\circ$.

In order to validate the convergence of the scheme to the optimum, $u^*$ is estimated by converting the optimum value $q^* = 0.31$ of the continuous case in [20] as follows: $u^* = q^*/q_{\text{open}} = 0.19$. Similarly, the optimal value of the output is expected, on average, to be $y^* = 0.83$. For the sake of presentation, the output is averaged to smooth out the effects of the PWM valves. Figure 4.2 shows the trajectories of $\|\tilde{\kappa}_c\|$, $u$, and $y_{\text{av}}$, which is the output $y$ subjected to a moving average filter with $T_{\text{av}} = C v_m^{-1}$. As predicted by Theorem 4.2, $\tilde{\kappa}_c$ converges to the vicinity of the origin. In addition, it can be seen that the input of the closed-loop system converges to the vicinity of the optimum $u^*$, and consequently $y_{\text{av}}$ converges to a vicinity of $y^*$.
4.6 Conclusions

Conditions for semi-global practical asymptotic stability of a class of singularly perturbed hybrid systems with a hybrid fast subsystem, a continuous slow subsystem, and a SPA stable reduced system (rather than GA stable) have been presented. The stability result encompasses hybrid systems equipped with an arbitrary ES scheme. In particular, this result is used to prove the stability of an ES scheme with a second-order linear filter acting upon a hybrid system. This expands the classes of systems for which extremum-seeking has been shown to be stable. Therefore, following the result of this chapter, the stability of ES schemes acting on hybrid systems, such as a NES scheme acting on urban traffic networks, can be guaranteed.
Figure 4.2: The evolution of $\|\tilde{\kappa}\|$, $u$ and $y_{av}$ in time. The dashed lines indicate the expected optima $u^*$ and $y^*$. 
Chapter 5

Enhancing the Performance of Urban Traffic Light Control Through Extremum-seeking

This chapter presents the benefits of using a NES scheme for fine-tuning the parameters of several existing urban traffic light control strategies, as illustrated in Figure 5.1. In particular, the NES scheme used is the one proposed in Chapter 3 augmented with band-pass filters to improve its transient response. The stability of the NES scheme in this chapter relies on the theoretical results of Chapters 3 and 4. Firstly, the stability of the NES scheme acting on a static map can be easily validated by combining the approach used in Lemmas 3.2 and 4.2. Secondly, the singularly perturbed hybrid systems result of Theorem 4.1 provides the stability of the full closed-loop system.

This chapter investigates the calibration of the parameters of three existing urban traffic light controllers taken from the closed-loop model-free and model-based families of traffic controllers: SCATS’s internal and orthogonal offsets; SOTL’s threshold value; and the density set-point of a perimeter controller. It is then demonstrated that the NES scheme is able to seek the optimal parameters, with respect to a certain performance measure, for each of these traffic light controllers in an urban, uni-modal traffic environment.

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A substantial proportion of this chapter has been submitted as a journal paper [82].
5.1 Overview of the Nash Equilibrium Seeking Scheme

In this section, the proposed extremum-seeking scheme is outlined and some of its features are explained. Specifically, Figure 5.1 is to be explained in more detail to clarify the control objective and the process of the whole closed-loop system.

5.1.1 Control objective in a non-cooperative game setting

There are three parts of Figure 5.1 that are of main concern in this section: the plant’s input $u$, the plant’s output $y$, and the NES scheme (the bottom dashed box). From the perspective of the NES scheme, the combination of the traffic network and the traffic light controller is a single plant. Thus, the plant’s input $u$ is the parameter vector of the traffic light controller, and the output $y$ is the performance measure.

The parameter selection affects the performance of the traffic light controller, which is measured as the plant’s output $y$. The plant’s output $y$ can represent various performance measures, such as (but not limited to) average speed or flow. Given the dimension of $u$ for the full network, it is desirable to decentralise the NES scheme and as a result, the output is not a network-wide performance measure, but rather a group of “local” performance measures (i.e. $y$ is a vector), each only considers a smaller part of the network. Then,
5.1 Overview of the Nash Equilibrium Seeking Scheme

Each parameter is associated with a “local” performance measure/cost function/output (these three terms are used interchangeably throughout this chapter) and it is desired to find the value that optimises its associated local output in the steady-state (recall that ES is a steady-state optimisation).

By using only local performance measures, it is possible that optimising one local output results in the deterioration of other local outputs. For instance, increasing the flow out of an intersection might create congestion at downstream intersections, which in turn decreases the outflows of those downstream intersections. This eventually leads to a non-cooperative game being played and each parameter acts as a player. Since each “player” is only optimising (in this chapter, maximisation is considered) the local reward selfishly, this leads to the Nash equilibrium. Specifically, let \( u \in \mathbb{R}^{N_u} \) and \( u_i \) be the \( i^{th} \) input/parameter and \( J_i(u) \) be the associated steady-state local output of \( u_i \) (i.e. the output associated with the \( i^{th} \) parameter \( y_i \to J_i(u) \) as \( t \to \infty \)). Then, the Nash equilibrium \( u^* \) is defined as in Assumption 3.2, which is a point that satisfies (maximisation is considered in this chapter)

\[
\frac{\partial J_i}{\partial u_i}(u^*) = 0, \quad \frac{\partial^2 J_i}{\partial u_i^2}(u^*) < 0,
\]

for all \( i \in \{1, 2, \ldots, N_u\} := S \). Having defined the Nash equilibrium, the next subsection describes a NES scheme that is slightly modified from that proposed in Chapter 3 by adding output linear filters equivalent to those discussed in Section 4.4.

5.1.2 Nash equilibrium seeking with linear filters

The NES scheme consists of two parts. Firstly, the output associated with each input \( y_i \in \mathbb{R} \) is processed through a group of \( N_u \) linear filters. Each filter is as follows:

\[
\dot{\xi}_i = \omega A_i \xi_i + \omega B_i y_i, \quad (5.1a)
\]
\[
\hat{y}_i = C_i^T \xi_i \quad (5.1b)
\]

where \( \xi_i \in \mathbb{R}^{N_i} \) is the filter’s state, \( A_i \in \mathbb{R}^{N_i \times N_i} \) is chosen to be Hurwitz, \( B_i \in \mathbb{R}^{N_i} \), \( C_i^T \in \mathbb{R}^{N_n} \), \( \hat{y}_i \in \mathbb{R} \) is the filter’s output, and \( \omega > 0 \) is a parameter used to introduce time-
scale separation between the plant and the NES scheme, i.e. by making $\omega$ sufficiently small, the NES scheme is made to act slowly relative to the plant dynamics. The linear filters used in this work are a group of identical band-pass filters,

$$
A_i = \begin{bmatrix} -\omega_H & 0 \\ -\omega_L & -\omega_L \end{bmatrix}, \quad B_i = \begin{bmatrix} \omega_H \\ \omega_L \end{bmatrix}, \quad C_i = \begin{bmatrix} 0 & 1 \end{bmatrix},
$$

with cut-off frequencies at $\omega_H$ (high-pass filter) and $\omega_L$ (low-pass filter). These band-pass filters “extract” the components of the output signals which lie within a range of frequencies corresponding to those of the dither signals.

The second part of the NES scheme involves correlating the dither with the plant output,

$$
\dot{u}_i = k\omega \sin(\omega_i t - \phi_i) \hat{y}_i,
$$

where $\bar{u}_i$ is the nominal value of the $i^{th}$ input, $\omega_i \in Q_{\geq \omega_{\min}}$ for all $i \in S$ and some $\omega_{\min} > 0$, and $\phi_i \in [0, 2\pi)$. The resulting effect is to adapt $\bar{u}_i$ at a rate that is, on average, proportional to $\partial J_i / \partial u_i$. Hence, the complete NES scheme can be written:

$$
\dot{\xi} = \omega A \xi + \omega B y, \quad (5.4a)
$$

$$
\dot{\bar{u}} = -k\omega \text{diag}(s(\omega t, \phi)) C \xi, \quad (5.4b)
$$

$$
u = \bar{u} + a s(\omega t, 0), \quad (5.4c)
$$

where:

- $\xi \in \mathbb{R}^{N_\xi}$ and $N_\xi = \sum_{i=1}^{N_i} N_i$;
- $\bar{u} \in \mathbb{R}^{N_u}$;
- $\text{diag}(\cdot)$ is a diagonal matrix where the diagonal elements are given by its argument;
- $A := \bigoplus_{i=1}^{N_i} A_i, B := \bigoplus_{i=1}^{N_i} B_i, C := \bigoplus_{i=1}^{N_i} C_i$, where $\bigoplus$ is the direct sum;
- $s(\omega t, \phi)$ is a dither/perturbation vector whose $i^{th}$ element is given by $\sin(\omega_i t - \phi_i)$;
- $\omega \omega_i$ is the dither frequency of input $i$;
• $a \in \mathbb{R}_{>0}$ is the dither amplitude;

• and $k \in \mathbb{R}_{>0}$ is the adaptation gain.

The role of $\phi$ is to partially compensate for the phase lag introduced by the plant dynamics and filters.

5.1.3 Dither re-use

It is reasonable to expect that, when changing the setting of a given set of traffic lights at an intersection, the effect of this change decays with “distance”. Note that this decay is not solely determined by physical distance, as there are potentially other factors that contribute to the decay, such as network topography and origin-destination profile (which determines the direction of the majority of the traffic flows). Similarly, changing a particular parameter of a given traffic light controller is expected to have this “decaying effect” phenomena. This is due to the fact that each parameter affects only several local intersections at most. For instance, changing a threshold value in SOTL only affects one intersection and an offset in SCATS is used by only a few intersections in series. Therefore, the concept of dither re-use introduced in Chapter 3 is exploited to simplify the tuning process of the NES scheme. This provides some practical benefits by allowing the use of fewer unique dither frequencies within the NES scheme such that the task of assigning frequencies is simplified, as discussed in Chapter 3.

The implementation of the NES scheme requires the knowledge of neither the bounding functions in the assumptions of Chapter 3 nor the value of $R^*$, However, it is still required that the neighbourhood to be configured by the user. In this case, it will be intuitively clear that the effect of each parameter decays with physical distance. Thus, it is reasonable to configure the neighbourhood and the dither re-use based on physical proximity; and it is expected that the NES scheme to work properly.

5.1.4 Urban traffic as a hybrid system

In relation to the results of Chapter 4, it is clear that an urban traffic network is a hybrid system due to the involvement of discrete events, such as the traffic lights. In addition,
Enhancing the Performance of Urban Traffic Light Control Through Extremum-seeking

as previously mentioned, the network and the existing traffic light control are considered as one entity, which corresponds to the hybrid fast subsystem in Chapter 4. Intuitively, it is clear that the hybrid system considered is “well-behaved”, in a sense that: given any initial conditions, at least a solution exists; all maximal solutions are complete and non-Zeno; it is robust against a very small change of parameter values, i.e. the solutions when using two almost identical parameter values are close. Moreover, the NES scheme corresponds to the continuous slow subsystem, which can be shown to be SPA stable following the approaches in Lemmas 3.2 and 4.2. Although it is not yet clear whether the output measurement of each agent, on average, is converging to a steady-state map, it will be shown later that this is the case. Therefore, it is expected that the NES scheme is able to converge properly.

5.1.5 Dealing with day-to-day traffic fluctuations

In some situations, it is desirable to have multiple parameter set-points in a traffic controller during a day, reflecting the presence of varying load demands. For instance, a day can be divided into three periods each with its own traffic demand, namely evening, peak, and off-peak. In order to accommodate this situation, several parallel NES controllers are set-up, where each NES controller is activated for each period (as in [120]). In this thesis, the switching time of the NES scheme has been pre-defined. This configuration can also incorporates special events, such as holidays; but it is not discussed in this thesis. Therefore, the NES scheme remembers its state from the previous cycle and can continue the optimisation without having to start over. In particular, the NES resets the value of $\xi$, $\bar{u}$, and the dithers’ phases at the beginning of a period to the final values the last time that period was entered.
5.2 Simulation Description

5.2.1 Traffic network

In this chapter, the considered square network consists of 16 intersections (nodes), shown in Figure 5.2. Each road (link) consists of two lanes and one right-turning lane\(^1\), as shown in Figure 5.3a. The lengths of the road and the right-turning lane are 300\(m\) and 45\(m\) respectively. This is indicative of the length of roads in typical Australian city centres. In this paper, a cellular automata model, which is described in detail by [31], is used in the simulations (Figure 5.3b). Furthermore, each intersection uses these four phases: an east/west phase, a north/south turning phase, a north/south phase, an east/west turning phase (Figure 5.4). Note that in this work, the simulated traffic is uni-modal, i.e. all vehicles are identical. In addition, to simplify the terminology, each road is categorised as either a boundary road (i.e. a road that has an entry/exit point from/to the unsimulated region) or a bulk road (otherwise). For the remainder of this chapter, the terms intersection and node are used interchangeably, as are road and link.

Driver behaviour in the simulation is modelled via a series of stochastic processes. Firstly, at each time step, there is a probability of \(\alpha\) that a vehicle enters the network through each inbound boundary road. To simulate congestion outside the network, a vehicle exits the network with probability \(\beta\) when it reaches the end of an outbound boundary road. \(\alpha\) and \(\beta\) can be given a subscript that indicates its direction, as shown in Figure 5.2. In addition, there are internal traffic sources and sinks located near the middle of each lane of each bulk road, representing, for example, car-parks. The probabilities of a vehicle entering and exiting the network via the internal sources and sinks are \(\gamma\) and \(\delta\) respectively. Various traffic conditions can be simulated by varying \(\alpha\), \(\beta\), \(\gamma\), and \(\delta\). Furthermore, a predetermined turning probability, both left and right, enables each vehicle, at each node, to make a random decision regarding its direction. In this study, the turning probability is fixed at 10\% in each direction. Table 5.1 summarises the traffic condition parameters that are varied in this study.

\(^1\)Vehicles drive on the left in this model.
5.2.2 Traffic light controllers

In this chapter, three traffic light controllers are studied: SCATS, SOTL and perimeter control. SCATS [93,136] is chosen to demonstrate the usefulness of the proposed NES
5.2 Simulation Description

Figure 5.4: The phases at each intersection simulation: east/west phase (phase 1), north/south turning phase (phase 2), north/south phase (phase 3), east/west turning phase (phase 4). A dashed line indicates a traffic flow that has to give way to other flows, whereas a solid line indicates a flow that has the right of way.

<table>
<thead>
<tr>
<th>Entry probability</th>
<th>Exit probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary link</td>
<td>α</td>
</tr>
<tr>
<td>Bulk link</td>
<td>γ</td>
</tr>
</tbody>
</table>

Table 5.1: Varied traffic condition parameters

scheme when applied to one of the most widely used traffic light controllers. Meanwhile, the chosen version of SOTL [136] and perimeter controller [2, 12] are two of the latest urban traffic light controllers proposed in the non-model and model based categories, respectively. Thus, these two are chosen to emphasise the versatility of the proposed NES scheme to future implementations. Each of the three controllers has parameters that are selected adaptively using the NES scheme as summarised in Table 5.2.

**Perimeter control**

The first traffic light controller considered is perimeter control. The development of this strategy largely depends on the existence of a macroscopic fundamental diagram (MFD) within a certain homogeneous region/network. The MFD demonstrates traffic flow decreases away from a unique critical density. This controller is essentially a gating strategy aimed to regulate the density inside the network to a certain set point by restricting the incoming traffic flow into the network (refer to Appendix B for a more detailed description of the perimeter control). The NES scheme is then used to fine-tune the selection
of the density set-point such that the performance measure is optimised. In this case, the whole network is considered as a region that involves only one input and output, in which case the NES scheme degenerates and becomes a single-input, single-output ES scheme.

Although the aim of the perimeter control is to achieve maximum flow in the network, it is common for the desired density set-point to be just below the critical density \([2, 12, 59]\). In order to achieve this, it is proposed that a performance measure that is a linear combination of the average speed and flow be used. Specifically, let \(v_j(t)\) be the speed of vehicle \(j\) at time \(t\) and \(H_i(t)\) be the set of the vehicles that are on the roads that are considered by parameter \(i\), at time \(t\). Also, to ease presentation and to reduce the fluctuations in the performance measure, it is subjected to a moving average filter with averaging period of \(T_{av} = 5\) minutes. Then, the average speed associated with parameter \(i\) is defined as:

\[
y_{speed\text{-}i}(t) = \frac{1}{T_{av}} \int_{t-T_{av}}^{t} \left( \frac{1}{|H_i(s)|} \sum_{j \in H_i(s)} v_j(s) \right) ds. \tag{5.5}
\]

Furthermore, let \(f_l(t)\) be the impulse signal from the sensor on lane \(l\) (located 45 m downstream of the upstream intersection) at time \(t\), and \(L_i\) be the set of lanes that are considered by parameter \(i\). The average flow (per lane) associated with parameter \(i\), \(y_{flow\text{-}i}(t)\), is calculated in a similar manner to the average speed in (5.5). Then, the \textit{mixed network}
Performance measure is as follows

\[ y_{\text{mixed},i}(t) = y_{\text{flow},i}(t) + w_s y_{\text{speed},i}(t), \]  

(5.6)

where \( y_{\text{flow},i} \) and \( y_{\text{speed},i} \) are the aforementioned average speed and flow associated with parameter \( i \), and \( w_s \) has units veh/m. In addition, the performance measure at an intersection refers to the averaged performance measure of the links that are approaching the intersection.

**SCATS**

The version of SCATS in this chapter adaptively controls two key traffic light settings: cycle length and split time, at each intersection. A comprehensive description is provided in [136]. On top of that, the NES scheme adjusts the internal and orthogonal offsets in the network.

The internal offset determines the delay of the start of the east/west phase of a node relative to that of the node directly to its west. Since each link is identical, the same internal offset is used along a series of nodes (from east to west). This provides green waves for the east bound traffic and creates four one-dimensional subsystems in the network (Figure 5.5) and four internal offsets, one for each subsystem. Furthermore, the orthogonal offset represents the delay of the start of the east/west phase of a node that is located at the east edge of the network relative to that of the node directly at its south. Lastly, a master node indicates the “starting point” of the green wave within a subsystem and the super master node is the starting point of the “orthogonal green wave” along the master nodes (otherwise a node is referred to as a slave). Hence, there are four internal offsets and three orthogonal offsets. Since the main aim of using offsets is to create a green wave and reduce the chance that vehicles are stopped by red light, it is sensible to use the average speed as the performance measure of SCATS.
Figure 5.5: The set-up of the subsystems, the master and the super master nodes. Each shape/node represents an intersection, and the links represent the roads connecting the intersections.

Self-organising traffic light

The self-organising traffic light scheme used in this work is explained in detail by [31] and [136]. SOTL is a priority based control that measures in real-time the value of the threshold function of a phase, which represents the “busyness” of a phase, at each intersection. When the measure of the threshold function exceeds a threshold value, it becomes a candidate for the next active phase. The selection of the threshold value changes how quickly the phase switches, and hence affects the performance of SOTL [31]. Then, the NES scheme is used to regulate the threshold values to the Nash equilibrium. Lastly, since SOTL in its essence is an isolated intersection control, it is mainly concerned with reducing the delay of vehicles attempting to cross an intersection. Thus, the measure of average speed used for SCATS is also suitable for SOTL.
5.3 Simulation Results for a Model Based Strategy Augmented with NES

An example of a model based strategy that is investigated is the network perimeter control proposed in [2]. In this chapter, the perimeter controller is implemented by varying $\alpha$. Since perimeter control in practice is only applied when the traffic demand is sufficiently high, then it is assumed that at all times there are always vehicles that want to enter the network. Hence, varying $\alpha$ is equivalent to the process of restricting the proportion of the traffic flow demand that is allowed to enter the network. Since the perimeter controller adjusts the value of $\alpha$, the traffic condition is determined by the value of $\beta$, $\delta$ and $\gamma$. Moreover, SOTL is used as the intersection control inside the network with threshold value equal to 5. It was shown that SOTL performs well on homogenising the density on the links in a network [136], which is essential to have a well-defined and robust MFD.

This section provides the simplest implementation of NES among the three traffic light controllers considered as there is only one parameter updated (from Table 5.2). There are two aspects being investigated. The steady-state input-output (SSIO) maps for several traffic conditions are first obtained. The SSIO map shows the relationship between the calibrated parameter and the average steady-state value of the performance measure, under a given time-invariant traffic condition. Note that in practice, obtaining the SSIO map is often not possible and is unnecessary other than to justify NES as an appropriate augmentation algorithm. The benefit of the proposed NES scheme is then tested on several relevant time-invariant traffic scenarios. Finally, it is also shown that a time-varying traffic scenario can also be dealt with by implementing the NES scheme.

5.3.1 Steady-state input-output map

Note that the boundary roads are not considered to be parts of the network and simply seen as buffers to couple the bulk network to the unsimulated region. As a result, the network aggregate observables are calculated by considering only the bulk roads. The steady-state value of the mixed performance measure at two time-invariant traffic conditions is plotted against the density set-point of the perimeter control (Figure 5.6), which
is the SSIO map for the NES scheme. Note that the flatness of the maps above a certain density set-point is caused by the fact that, due to the traffic conditions (i.e. $\beta$, $\gamma$ and $\delta$) being constant, the system is unable to reach the specified density. The NES scheme cannot be guaranteed to converge to the optimum if initialised in this region. Nevertheless, the optimal density set-point is not located in that flat region and is able to be sought when the NES scheme is initialised properly (based on these cases, low initial condition is preferred and slope-seeking method can be used when the NES is stuck in the flat region). Moreover, it can be observed that the optimal density set-point can vary greatly for different traffic conditions (especially for different $\gamma$ and $\delta$). Therefore, the NES scheme can potentially provide benefits to the performance of the perimeter control by fine-tuning the set point.

![SSIO map](image)

(a) $\beta = 0.6$, $\gamma = 0.05$, and $\delta = 0.1$

(b) $\beta = 0.4$, $\gamma = 0.01$, and $\delta = 0.2$

Figure 5.6: (Perimeter control) The SSIO map of the network by using the mixed performance measure at various $w_s$.

### 5.3.2 Time-invariant traffic demand

The NES parameters are set up as follows: dither amplitude $a = 0.025$; $k = 4$; $\omega = 5 \times 10^{-4}$; $\omega_H = 9 \times 10^{-5}$ rad/s; $\omega_L = 2.5 \times 10^{-3}$ rad/s; and $\phi = 70^\circ$.

For this section, two time-invariant traffic demand profiles are considered (shown in Table 5.3), which approximate two different types of peak conditions. Case 1 reflects a peak period of a CBD region where most vehicles are entering car parking spaces inside the network (large sinks), whereas Case 2 is aimed to approximate a peak condition of a
suburban city center where only some of the vehicles finish their trips inside the network, while the other are passing through. Note that traffic conditions with large sources are not considered since in that case, perimeter control is unlikely to perform efficiently (the perimeter control would just restrict cars from entering the network at its boundaries).

Table 5.3: Perimeter control time-invariant traffic demand

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0.60</td>
<td>0.05</td>
<td>0.10</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.40</td>
<td>0.01</td>
<td>0.20</td>
</tr>
</tbody>
</table>

The NES scheme uses $w_s = 0.005$ such that it emphasises maximising flow rather than speed, and the mixed performance measure should just push the optimal set point slightly lower than the critical density. It can be observed from Figures 5.7–5.8 that the NES scheme is able to converge to two different equilibria for these two cases. In addition, the mixed performance measure for both cases is improved.

5.3.3 Time-varying traffic demand

Since the use of perimeter control is limited to high external demand scenarios, both Case 1 and 2 from Table 5.3 are going to be used again for the time-varying demand profile as shown in Table 5.4. Since both cases are approximating peak periods, their durations in the time-varying demand profile are set to 4.5 and 3.5 hours respectively, which are typical durations of peak periods. In between these two peak periods, two identical off-peak periods are inserted to simulate the early stage of a peak period, during which the network density is increasing after being emptied by the preceding off-peak period. As a result, the simulation should more closely approximate a real-life traffic situation.

Figure 5.9 shows the convergence of the ES scheme to the optima of both cases. Note that due to the dynamics introduced by the time-varying demand, the optima for both cases are slightly increased. Furthermore, from Figure 5.10, it can be seen that the mixed performance measure of each periods and their average are improved after every cycle. Hence, it has been demonstrated that the ES scheme is able to find the optimal density set-point even under a time-varying traffic condition.
Figure 5.7: (Perimeter control) The simulation result for Case 1. Note that density is calculated similar to average speed in (5.5).

Table 5.4: Perimeter control traffic demand cycle

<table>
<thead>
<tr>
<th>Traffic profile</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>α = 0.06</td>
<td>β = 0.01</td>
<td>α = 0.06</td>
</tr>
<tr>
<td>β = 0.01</td>
<td>γ = 0.01</td>
<td>β = 0.01</td>
</tr>
<tr>
<td>γ = 0.01</td>
<td>δ = 0.01</td>
<td>γ = 0.01</td>
</tr>
<tr>
<td>ρ = 0.01</td>
<td>θ = 0.01</td>
<td>ρ = 0.01</td>
</tr>
<tr>
<td>Duration (hours)</td>
<td>4.5</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
5.3 Simulation Results for a Model Based Strategy Augmented with NES

Figure 5.8: (Perimeter control) The simulation result for Case 2.

Figure 5.9: (Perimeter control) The trajectories of the density set-point during both Case 1 (black) and 2 (red) periods.
Figure 5.10: (Perimeter control) The simulation result for the time-varying traffic condition.
Therefore, it has been demonstrated that the NES scheme is capable of seeking the optimal density set-point, improving the performance of the perimeter control. Similar results can be achieved when augmenting other model-based traffic control strategies, tuning parameters and performance measures that satisfy the general assumptions of NES.

5.4 Simulation Results for SCATS Augmented with NES

In this section, NES is used to augment SCATS. Compared to perimeter control, this task involves real-time adaptation of more parameters. Similar to Section 5.3, the SSIO maps of SCATS are presented and the convergence of the NES scheme when calibrating the offsets is demonstrated for several time-invariant traffic conditions and a time-varying one. Note that the maximum cycle length used is 90 seconds (excluding four 3-second waiting periods between phases).

5.4.1 Steady-state input-output map

The aim of this section is to understand the effect of the offset (either orthogonal or internal) on the average speed of each intersection. In order to do that, there are several traffic conditions that are investigated, as shown in Table 5.5. Essentially, the cases considered are traffic conditions which are biased in one direction, with lower demand from all other directions. The intensity of the lower demand is varied among all five cases to cover the scenarios that might be encountered in practice. Since the phase-linking is most efficient when the vehicles are free-flowing, $\beta = 0.99$ and $\gamma = \delta = 0$ for all cases such that the vehicle platoons are not interrupted.

Since the main aim of the internal offset is to benefit the prioritised flow, i.e. the east bound flow, it is desired to find an internal offset at which the average speed of the east bound flow is maximum. The internal offset of subsystem 2 is varied, while the other internal offsets are set to 20 seconds (based on an approximated travel speed of 55 km/h) and the orthogonal offsets are set to 0 seconds. Firstly, Case 1 is considered. As shown in Figure 5.11a, the optimal internal offset is roughly 15 seconds indicating free-flowing
traffic through multiple intersections, i.e. a green wave exists. In addition, because Case 1 almost does not have any incoming traffic from the other directions, the effect of the internal offset on the average speed of the flow in the other directions is insignificant.

However, when the demand from the other directions is higher (Case 5 of the east bound biased), using the optimal internal offset found earlier would result in a slower (or even slowest) average speed for the west bound traffic (Figure 5.11b). In fact, the shapes of the plots of the average speed of the east and west bound traffic are opposite. Furthermore, the north and south bound traffic are more affected by the internal offset due to higher demand in those directions (Figure 5.12). The adjacent subsystems are also partly affected by the internal offset in this case. However, these plots do not exhibit optima that are similar across the nodes in the subsystem, as those of Figure 5.11a.

**Table 5.5: (SCATS) East bound biased traffic**

<table>
<thead>
<tr>
<th></th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 3</th>
<th>Case 4</th>
<th>Case 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_E$</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>$\alpha_W$</td>
<td>0.01</td>
<td>0.03</td>
<td>0.05</td>
<td>0.07</td>
<td>0.09</td>
</tr>
<tr>
<td>$\alpha_N$</td>
<td>0.01</td>
<td>0.03</td>
<td>0.05</td>
<td>0.07</td>
<td>0.09</td>
</tr>
<tr>
<td>$\alpha_S$</td>
<td>0.01</td>
<td>0.03</td>
<td>0.05</td>
<td>0.07</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Figure 5.11: (SCATS) The average speed of the east/west bound traffic within subsystem 2 for various values of the internal offset of subsystem 2 using Case 1 and 5. The average speed plots of the east bound traffic for Case 5 are of similar shape to those of Case 1. The average speed of all other directions and all other nodes are unaffected.
Figure 5.12: (SCATS) The average speed of the north and south bound traffic approaching the nodes within subsystems 1–3 for various values of the internal offset of subsystem 2. The traffic demand is East bound biased Case 5. The north/south bound traffic in subsystem 4 is unaffected.
Moreover, Figure 5.13 shows the plots of the average speed from all directions at the intersections of subsystem 2. It can be observed that the shapes of the plots for Case 1 follow closely the shapes of the east bound plots in Figure 5.11a. When the demand from the other directions is increased (changing from Case 1 to Case 5), the plots in Figure 5.13 become more similar to those of Node 5–8 in Figure 5.12 and, similarly, do not share a clear optimum among the nodes in the subsystem. This shows that the benefit of the internal offset is most obvious when the traffic is heavily biased in the east direction.

Figure 5.13: (SCATS) The average speed of the traffic approaching the nodes within subsystem 2 for various values of the internal offset of subsystem 2. The traffic demand profiles are East bound biased Case 1 (◦), Case 2 (×), Case 3 (□), Case 4 (△), and Case 5 (⋄).

The effect of the orthogonal offset on the north/south bound traffic is also similar to the effect of the internal offset on the east/west bound traffic. From Figure 5.14, it can
be observed that the orthogonal offset affects the north bound flow of the “downstream” (away from the super master node) subsystem and the south bound flow of the “upstream” subsystem. The shapes of the plots of the average speed of the north bound flow of the downstream subsystem are also the opposite of those of the south bound flow of the upstream subsystem.

![Subsystems](image)

**Figure 5.14:** (SCATS) The average speed of the north (○) and south bound (×) traffic approaching the nodes within subsystems 2 (“upstream”) and 3 (“downstream”) for various values of orthogonal offset 2. The traffic demand is East bound biased Case 5. The north/south bound traffic in subsystem 1 and 4 is unaffected.

In conclusion, with this choice of parameter combination and set-up, the offsets are beneficial only to the prioritised traffic flow and, for the link length simulated in this study, have an opposing effect on the opposite flow. As a result, when the demand of these opposing directions is approximately equal, the offset does not provide any clear overall benefit. Therefore, the usefulness of offsets is most obvious when the traffic demand is heavily biased in only non-opposing directions. For instance, the traffic condition during peak hours can most likely benefit from properly tuned offsets. However, note that in practice, it might be possible that adjusting both the internal offset and the cycle length benefits the traffic flow from both opposing directions. This thesis focuses only on varying the offsets, and incorporating cycle length to the NES scheme might be a future research work.
5.4.2 Time-invariant traffic demand

Since, in the current set-up the offsets are only clearly beneficial when the traffic demand is biased in non-opposing directions, the proposed NES scheme is tested only on a traffic demand scenario that is biased on the East and North directions. However, two different speed limits are applied to demonstrate that the NES scheme is able to locate different optima. The practice of using different speed limits during different periods of the day is common in some cities in Australia, such as in school zones where the speed limit switches between 40 and 60 km/h. However, the speed of a vehicle in the simulation is expressed in cells/s (each cell is 7.5 m long), ranging from 0 to $v_{\text{max}}$ cells/s. Due to the stochasticity in the simulation (particularly the random deceleration done by each vehicle; see [136]), the average free-flow speed that can be achieved is actually lower than $v_{\text{max}}$. By setting the maximum vehicle speed to 2 cells/s and 3 cells/s, the average free-flow speed achieved is approximately 45 km/h and 60 km/h respectively. This closely represents the speed limit decrease of school zones. The complete traffic conditions used are shown in Table 5.6.

Table 5.6: SCATS time-invariant speed limit conditions

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_E$</th>
<th>$\alpha_W$</th>
<th>$\alpha_N$</th>
<th>$\alpha_S$</th>
<th>$v_{\text{max}}$</th>
<th>$\gamma$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0.16</td>
<td>0.01</td>
<td>0.09</td>
<td>0.01</td>
<td>3 cells/s</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.16</td>
<td>0.01</td>
<td>0.09</td>
<td>0.01</td>
<td>2 cells/s</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Based on the observations in the previous subsection, the “cost/output” associated with each offset is set up as follows by using the subsystems shown previously in Figure 5.5. Orthogonal offset 1 considers the average speed of subsystem 1 and 2; orthogonal offset 2 considers the average speed of subsystem 2 and 3; orthogonal offset 3 considers the average speed of subsystem 3 and 4; and each internal offset considers the average speed of its own subsystem. Note that the average speed of a subsystem means the average speed of the vehicles on all of the links that are directed towards the intersections within the subsystem. As a result of this set-up, there is some degree of cooperativeness between each orthogonal offset and the corresponding internal offsets.

The NES parameters are set up as follows. Dither amplitude $a = 3$; $k = 2.0$ and
5.4 Simulation Results for SCATS Augmented with NES

$k = 1.0$ for the orthogonal and internal offsets respectively; $\omega = 10^{-3}$; $\omega_H = 1 \times 10^{-4}$ \text{rad/s}; $\omega_L = 5 \times 10^{-3}$ \text{rad/s}; $\omega_{ort1} = \omega_{ort3} = 0.88$, $\omega_{ort2} = 0.82$, $\omega_{int1} = \omega_{int3} = 1.0$, and $\omega_{int2} = \omega_{int4} = 0.94$, where the subscript ort and int indicate orthogonal and internal offset respectively. Some of the dithers are re-used since each offset only significantly affects the performance measure of the “adjacent offsets”. Lastly, $\phi = 35^\circ$ and $\phi = 23^\circ$ for the orthogonal and internal offset respectively, determined by observing the plant’s frequency response.

In the simulation, the offsets are initialised based on the approximated travel time between intersections. With an approximate average speed of 10 m/s (from Figure 5.13) and link length of 300 m, the travel time, and hence the initial offset, is approximated to be 30 s (for both internal and external offsets). The simulation result is shown in Figures 5.15–5.16. It can be observed that the optimal offsets for Case 1 of both internal and orthogonal offsets are approximately 15 seconds. In addition, it can be observed that the output corresponding to each offset is improved. On the other hand, the optimal offset for Case 2 is higher compared to Case 1 due to the slower speed. Also, the improvement in the output corresponding to each offset is predictably smaller because the initial offsets are closer to their optima.

5.4.3 Time-varying traffic demand

In order to demonstrate the ability of the proposed NES scheme to handle a varying speed limit, the traffic profile shown in Table 5.7 is used, which approximates the speed limit condition during a day including evening, morning peak, off-peak, and afternoon peak periods.

<table>
<thead>
<tr>
<th>Traffic profile</th>
<th>Case 1</th>
<th>Case 2</th>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration (hours)</td>
<td>10</td>
<td>4</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

Figure 5.17 shows the convergence of the orthogonal and internal offsets respectively. It can be observed that the offsets converge to their corresponding optima during each period. In addition, the speed during each period (Case 1 and 2) is averaged for each
Figure 5.15: (SCATS) The trajectories of the offsets when $v_{max} = 3$ cell/s (black) and $v_{max} = 2$ cell/s (red).
Figure 5.16: (SCATS) The output associated with each offset when $v_{max} = 3$ cell/s (black) and $v_{max} = 2$ cell/s (red). The subscript int and ort refers to internal and orthogonal offset respectively.
day, and is shown in Figure 5.18. Figure 5.18 also shows the speed of each subsystem averaged throughout the whole day/cycle. The average speed improvements of each subsystem are approximately up to 15%, 5%, and 13% during the Case 1 period, Case 2 period, and the whole cycle respectively.

Figure 5.17: (SCATS) The trajectories of the offsets during Case 1 (black) and Case 2 (red) period.
5.5 Simulation Results for a Non-model Based Strategy Augmented with NES

It has been demonstrated that augmenting the existing SCATS algorithm with a Nash extremum seeker operating on the offset parameters has enabled the optimal average network speed to be obtained in reasonable time. Other tuning parameters and alternative measures of network performance satisfying the general requirements of NES could be substituted without compromising the ability of the augmented system to achieve optimal performance. Furthermore, similar results will be obtained if SCATS were replaced by another non-model based alternative.

5.5 Simulation Results for a Non-model Based Strategy Augmented with NES

In this study, an example of a traffic light control strategy from the non-model based category that is investigated is SOTL [31, 136]. This version of SOTL is a highly adaptive signal system that has been shown in simulations to perform as well as, and often better than, SCATS [136]. In this section, it is shown that SSIO maps for SOTL exist and the NES scheme is able to seek the optimal parameters. In addition, it is also shown that NES can be implemented to deal with a time-varying traffic scenario, depicting periodic day-to-day traffic fluctuations.

5.5.1 Steady-state input-output map

In SOTL, each node has its own threshold value. Interestingly, when SOTL is used in the network studied, it appears that each node’s output is independent of other nodes’
inputs. When the threshold of each node is perturbed by sinusoid signals of different frequencies, Figure 5.19 shows that the spike in each node’s output (average speed) power spectrum curve occurs predominantly at the frequency corresponding to its own input (threshold). In conclusion, each node is affected most significantly by its own input. Thus, when SOTL is used, the NES scheme has 16 parameters and the problem can be treated as multiple SISO optimisations, which implies that unrestricted dither re-use is allowed. Although the independence between intersections is intuitively surprising and is an interesting aspect to investigate, this thesis does not pursue it further since it is tangential to the main research objectives.

![Figure 5.19: (SOTL) Power spectrum of the output (steady-state average speed) signal of each node with $\alpha = 0.15$, $\beta = 0.9$ and $u = 3$.](image)

The SSIO maps of Node 6 (which is typical of each node), for various values of $\alpha$ and $\beta$, when using SOTL is shown in Figure 5.20. Firstly, it is found that when $\alpha \approx 0.11$, the traffic in the network has an average speed of approximately 40 km/h (compared to the average free-flow speed of 60 km/h), whereas $\alpha \geq 0.18$ corresponds to a congested network where the speed drops to below 15 km/h. Secondly, it is found that when $\alpha$ is not very high, $\beta$ does not affect the speed in the network unless it is very low, as shown in Figure 5.20b. Most importantly, it can be observed that the optimal threshold value
varies for different values of $\alpha$. Note that in this case, it is sufficient to demonstrate the usefulness of the NES scheme when adapting SOTL without having to introduce internal sources and sinks.

![SSIO maps of Node 6](image)

Figure 5.20: SOTL SSIO maps of Node 6.

### 5.5.2 Time-invariant traffic demand

In this section, the performance of the NES scheme for adapting the threshold value of SOTL by using constant traffic demand profiles is to be investigated. As before, the purpose of this section is to demonstrate the convergence of the NES scheme to the various optima when faced with different traffic conditions. Furthermore, the NES parameters are set up as follows: dither amplitude $a = 1$; $k = 0.4$; $\omega = 10^{-3}$; $\omega_H = 9 \times 10^{-5}$ rad/s; $\omega_L = 3 \times 10^{-3}$ rad/s; and $\phi = 60^\circ$. Note that each node is using the same dither frequency.

The first case considered is a uniform traffic demand with $\alpha = 0.13$ and $\beta = 0.9$. As can be observed from Figure 5.21, the NES scheme is able to adapt the threshold values to an equilibrium value of approximately 4.5 and the average speed is improved, especially the case with initial conditions equal to 10. In addition, shown in Figure 5.22 is a typical convergence trajectory of the NES scheme with the second traffic demand case, which is uniform with $\alpha = 0.17$ and $\beta = 0.9$. It can be observed that the NES scheme again converges to the optimum, and the threshold is different to the first case.
Figure 5.21: (SOTL) The simulation result of Node 6, which is typical of all nodes, using two initial threshold values when $\alpha = 0.13$ and $\beta = 0.9$.

Figure 5.22: (SOTL) The simulation result of Node 6, which is typical of all nodes, using two initial threshold values when $\alpha = 0.17$ and $\beta = 0.9$. 
5.5.3 Time-varying traffic demand

To demonstrate the usefulness of the proposed NES scheme when using SOTL on a daily basis (as mentioned in Section 5.1.5), a time-varying demand profile, as shown in Table 5.8, is used in the simulation. Although in this case the demand is uniform from all directions, it is sufficient to show that the NES scheme is able to seek the optima for all three periods: night time, peak and off-peak.

Table 5.8: SOTL traffic demand cycle for each day

<table>
<thead>
<tr>
<th>Period</th>
<th>Night time</th>
<th>Morning peak</th>
<th>Off-peak</th>
<th>Afternoon peak</th>
<th>Evening off-peak</th>
</tr>
</thead>
<tbody>
<tr>
<td>Duration (h)</td>
<td>6.5</td>
<td>4.5</td>
<td>6</td>
<td>3.5</td>
<td>3.5</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.07</td>
<td>0.16</td>
<td>0.12</td>
<td>0.16</td>
<td>0.12</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td>0.70</td>
<td>0.90</td>
<td>0.70</td>
<td>0.90</td>
</tr>
</tbody>
</table>

Figure 5.23 shows the typical convergence of the SOTL threshold under the traffic demand profile in Table 5.8. It can be observed that for each period, the NES scheme converges to the vicinity of its corresponding optimum (approximately at 2, 4, and 6 for night time, off-peak and peak respectively). Furthermore, Figure 5.24 shows the improvement in the average speed of each node during each day. Except during peak period, the improvement in the average speed is approximately up to 6.9%, 6%, and 5.4% during the night time, off-peak, and whole-day period respectively. Note that for the peak period, the average speed does not improve significantly. The reasons for this are twofold. Firstly, the threshold has been initialised sufficiently close to the optimum. Secondly, as previously shown in Figure 5.20a, when $\alpha = 0.16$, the average speed is almost the same for threshold values from 6 to 8. Thus, the average speed during the peak period is already close to the maximum. This also explains why the threshold value chosen by the NES scheme varies quite a lot during peak when compared to other time periods.

In conclusion, the NES scheme is able to adapt the threshold of the SOTL controller to a value that optimises the average speed at each node, for each traffic condition. By using the NES scheme, it is also possible to calibrate other tuning parameters of SOTL that optimise a different performance measure, as long as they satisfy the general properties.
Figure 5.23: (SOTL) The convergence of the threshold of Node 6, which is typical of all nodes, during the night time (black), off-peak (red), and peak (green) period.

Figure 5.24: (SOTL) The average speed of each node for each day. The top group of lines represents the corner nodes (Node 1, 4, 13, and 16); the middle group of lines represent the edge nodes (Node 2, 3, 5, 8, 9, 12, 14, and 15); and the bottom group of lines represent the internal nodes (Node 6, 7, 10, and 11).
required by the NES scheme.

5.6 Conclusions

Using ES as an augmentation to existing urban traffic control strategies has been demonstrated to improve their performance in a uni-modal urban environment. This improvement is achieved by real-time fine tuning of their parameters and has been demonstrated for three different traffic light controllers, namely: perimeter control, SCATS, and SOTL. It has been shown in each case that the proposed NES scheme is able to seek the optimal parameters with respect to a range of different performance measures, irrespective of the type of the traffic light control strategy. It is also pointed out that the performance improvement is up to 29% for the perimeter control, 15% for SCATS, and 6.9% for SOTL, compared to using parameter values that are expected to produce a reasonable performance based on the investigation of the SSIO maps. This clearly demonstrates the versatility of the NES scheme for performance optimisation without requiring an overhaul of the existing traffic light controllers. Furthermore, it is expected that the same conclusion still holds when unsymmetrical traffic demand profile is used. It is then expected that, although the resulting optimal parameters might not be symmetrical, the NES scheme is still able to seek these optimal parameter values. In addition, one of the most important benefits of using the proposed NES scheme comes from the fact that the stability has been theoretically guaranteed in Chapters 3 and 4. Consequently, the NES scheme can always be fine-tuned, adhering to the guidelines provided by the main theorems of Chapters 3 and 4, such that it is able to seek the optimal parameters of the traffic controller.
Chapter 6
Contributions and Further Work

6.1 Summary of Contributions

This thesis has outlined the research carried out to achieve the overall goal of the project, which is to develop specialised ES results for parameter adaptation of existing urban traffic light controllers (as previously illustrated in Figure 2.8). In particular, the three research aims outlined in Section 2.5 have been achieved that leads to the following contributions:

1. Development of a decentralised ES scheme

   (a) Dither re-use in Nash equilibrium seeking

   As discussed in the literature review, each input typically requires a unique dither frequency to prevent gradient estimate bias. This can be proven to be an issue when dealing with large-scale systems as the frequency assignment can become arduous in practice; and the gradient estimator and the dithers occupying a wide range of time-scales causing slow convergence. Therefore, it is beneficial to relax this requirement for systems where the effect of an agent’s input on another agent’s output decays as the “distance” between them (based on some measure of proximity) is increased. In Chapter 3, the concept of dither frequency re-use is introduced for a Nash equilibrium seeking scheme. The dithers are re-used to exploit the “decaying effect” characteristic of a continuous, nonlinear MIMO systems. It is proposed that agents are allowed to share dither frequency as long as they are not inside each other’s “neighbourhood”,
which implies that they are not in the proximity of to each other. It is rigorously shown that if the “size” of the neighbourhood is sufficiently large, the gradient estimate bias introduced by dither re-use is sufficiently small such that it does not prevent SPA stability of the full-closed loop system. Consequently, this allows decentralisation during the implementation of the NES scheme, facilitating the assignment of the dither frequency especially for large-scale systems. Moreover, Theorem 3.1 identifies the stability region of the NES scheme in its parameter space, which serves as a tuning guideline when deploying the proposed NES scheme. This simplifies the design process of the NES scheme in practice.

(b) Condition for dither re-use guarantee and its quantification

The stability result of Theorem 3.1 relies on the robustness of the gradient system under sufficiently small perturbations. Following from this result, it is found that, when more knowledge of the plant is available, the NES scheme is guaranteed to be able to exploit a certain amount of dither re-use without compromising the SPA stability of the full closed-loop system. Specifically, if the relationship between the size of the agents’ neighbourhoods and the magnitude of the estimate bias introduced by the dither re-use is known, a lower bound on the size of neighbourhood can be determined. It is shown that if the neighbourhood has a size larger than the lower bound, SPA stability of the full closed-loop system is achieved. Moreover, an example application is presented that further quantifies this condition, along with the main theorem of Chapter 3, when dealing with plants with quadratic costs. Quadratic cost functions are commonly found in engineering problems such that this result provides more practical formulae for the specific case considered, which is useful for determining the amount of dither re-use in practice.

2. Stability of ES schemes on a family of hybrid systems

(a) Singularly-perturbed hybrid systems with SPA stable reduced system

In Chapter 4, the SPA stability of a class of singularly perturbed hybrid sys-
tems is proven. The investigated family of hybrid systems is of a special class which consist of a hybrid fast subsystem and a continuous slow subsystem. In particular, the condition imposed on the reduced system resulting from the time-scale separation is SPA stability, which is more relaxed than a GA stability assumption required by existing results in the literature. The considered slow subsystem encompasses various ES schemes available in the literature (including the NES scheme proposed in Chapter 3).

(b) *Stability of ES schemes on a class of hybrid systems*

By using the singular perturbation result described above, the stability of an ES scheme is demonstrated when acting on a hybrid system. In particular, the ES scheme discussed includes a general second-order linear output filter as a part of the gradient estimator. An example of such filter is a band-pass filter that assists in extracting the sinusoidal component of the output signal corresponding to the effect of the dither. When the slow subsystem of the singularly perturbed hybrid system is replaced with the considered ES scheme, the resulting reduced system is indeed SPA stable. Thus, the ES scheme acting on the hybrid plant is also SPA stable. Consequently, this is the first stability result of ES acting on a fairly general nonlinear hybrid plant. In addition, the approach used can similarly be applied to prove stability of a wide array of ES schemes acting on hybrid systems, as long as it has a SPA stability property when applied to a static plant.

3. **Implementation of a NES scheme for augmentation of traffic light control**

Simulations are used to demonstrate the improvement of the performance of existing traffic light controllers as a result of using the proposed NES scheme as an augmentation. The proposed NES scheme is used as an online calibration tools of the tuning parameters of these traffic light controllers, which attempts to improve a certain performance measure associated with each parameter, as previously illustrated in Figure 2.8. This improvement is shown for three existing strategies from three different categories of urban traffic light controller, namely perimeter control, SCATS, and SOTL, under both time-invariant and time-varying traffic conditions.
The performance improvement achieved is up to 29% for the perimeter control, 15% for SCATS, and 6.9% for SOTL, compared to using reasonable parameter values obtained from the investigation of the steady-state performance maps.

### 6.2 Publications

The research carried out for this thesis led to the following publications (including recent submissions).


### 6.3 Further Work

Opportunities for further research stemming from the contributions of this thesis have been identified as follows:
1. **Cost function design to achieve Pareto efficiency**
   As illustrated by the well-known “Prisoner’s Dilemma” example and rigorously discussed in [39], the Nash equilibrium is not necessarily the best solution when the optimality of all agents’ costs is considered, which is characterised by the Pareto front. In this case, previous works have shown that by appropriately introducing some penalty terms into each agent’s cost function, Pareto optimality can be achieved when using a NES scheme [7,8]. However, the penalty term used in these existing results require communication among all agents. Potential extensions can be considered by exploiting the “decaying effect” characteristics of the system such that the penalty term of each agent only considers other agents that strongly influence the cost function.

2. **Faster convergence**
   The results in this thesis only consider the steady-state of the NES scheme due to the time-scale separation requirement of the stability proof. The improvement of the transient performance of the NES scheme, such as a faster convergence, can also be investigated. As discussed in the literature review, arbitrarily fast convergence of an ES scheme can be achieved for some classes of plants [99–101]. It would be beneficial to extend these results to a larger class of systems and investigate the resulting trade-off between transient and steady-state performance that may arise from achieving faster convergence.

3. **Consideration of more classes of hybrid systems**
   The singular perturbation result in Chapter 4 can potentially be extended to cover a wider range of classes of hybrid systems. This might include the consideration of different types of fast and slow subsystems. For instance, considering a SPA stable, hybrid reduced system is potentially useful since many controllers are of hybrid nature, such as discrete controller (including the discrete ES schemes discussed in Chapter 2), switching control, and controller with internal timers.
4. **Implementation on more complex and realistic scenarios**

The application of ES schemes for urban traffic light control can be extended to include more complex and realistic scenarios. This includes: adaptation of more parameters; simulating more realistic networks, such as considering unsymmetrical traffic load and simulating real networks in Melbourne; consideration of multimodal traffic; accommodating flow prioritisation; estimation of performance metrics that cannot be directly measured; more thorough investigation about other adaptation strategies followed by a comparison study with the developed NES scheme; and replacing the time-based switching method with one that depends on a traffic load detection approach.

5. **Other approaches of implementing ES schemes for urban traffic control**

A way to extend the results in Chapter 5 is by considering other approaches of implementing ES schemes for urban traffic control. One approach to achieve this is by taking into account the capability of parameters adaptation provided by the ES scheme when designing the urban traffic light controller. For instance, the traffic controller is designed to handle incident to assist the convergence of the ES scheme. This should result in an improved performance of the traffic light control. Alternatively, if the ES is able to achieve faster convergence as discussed in point 2, the use of ES to directly optimise the `green splits`, `offsets`, and `cycle lengths` is also possible.
Appendix A
Proofs

A.1 Proof of Lemma 4.2

Firstly, using the approximate equilibrium solution of $\xi_r$, the error coordinate $(\hat{\xi}_r, \hat{u}_r)$, and defining (for algebraic simplicity) $\tilde{f}(\sigma) := f(\sigma + \mu^*), \tilde{h}(\sigma, t) := h(\sigma + \mu^*, t), Q_u(\sigma) := \tilde{f}_u(\sigma + \mu^*)$, and $\tilde{\sigma} := d\sigma/d\tau$ (instead of using the $t$ time-scale), the reduced system written in the error coordinate is as follows:

\begin{align*}
\dot{\hat{\xi}}_r &= A\hat{\xi}_r - \frac{\partial \tilde{h}(\hat{u}, \tau)}{\partial \hat{u}_r} \hat{u}_r, \\
\dot{\hat{u}}_r &= -k \sin(\tau - \phi) C\hat{\xi}_r - k \sin(\tau - \phi) \tilde{h}(\hat{u}, \tau),
\end{align*}

From (A.1), it can be observed that by making $k$ small, the dynamics of $\hat{u}_r$ is at a slower time-scale than those of $\hat{\xi}_r$. Hence, using the result in [129], the stability of (A.1) can be studied by analysing the stability of its boundary layer system and its averaged system.

A.1.1 Boundary layer system

The boundary layer system is obtained by setting $k = 0$, such that:

\begin{align*}
\dot{\xi}_{bl} &= A\xi_{bl}, \quad \dot{u}_{bl} = 0, \quad \hat{u}_{bl} = \hat{u}(0).
\end{align*}

It is easy to see that the origin of the boundary layer system is exponentially stable.
A.1.2 Averaged system

To analyse the averaged system, consider the average dynamics of (A.1b) when $\xi_r = 0$. Using the definition of $H(s) = C(sI - A)^{-1}B$, consider the second term in (A.1b):

$$\sin(\tau - \phi)C\tilde{h}(\tilde{u}; \tau) = \frac{1}{2i} \sum_{n \in \mathbb{Z}} H(in)Q_n(\tilde{u}) \left( e^{i(n+1)(\tau - \phi)} - e^{i(n-1)(\tau + \phi)} \right).$$  \hfill (A.3)

It follows that the average is non-zero only when $n = \pm 1$. In that case,

$$\frac{1}{2i} \left( e^{-i\phi}H(-i)Q_{-1}(\tilde{u}) - e^{i\phi}H(i)Q_1(\tilde{u}) \right)$$

$$= -\text{Im} \left\{ e^{i\phi}H(i)Q_1(\tilde{u}) \right\}$$

$$=: F_{\text{av}}(\tilde{u}).$$  \hfill (A.4)

Thus, the averaged system is simply

$$\dot{\tilde{u}}_{\text{av}} = -kF_{\text{av}}(\tilde{u}_{\text{av}}).$$  \hfill (A.5)

Expanding (A.4),

$$\text{Im} \left\{ e^{i\phi}H(i)Q_1(\tilde{u}_{\text{av}}) \right\} = \frac{\Theta}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(\tilde{u}_{\text{av}} + a \sin(\tau)) \cos(\tau) \, d\tau$$

$$- \frac{\Phi}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(\tilde{u}_{\text{av}} + a \sin(\tau)) \sin(\tau) \, d\tau,$$  \hfill (A.6)

where $\Theta := \text{Im} \left\{ e^{i\phi}H(i) \right\}$ and $\Phi := \text{Re} \left\{ e^{i\phi}H(i) \right\}$. Using Taylor series expansion:

$$\tilde{f}(\tilde{u}_{\text{av}} + a \sin(\tau)) \approx \tilde{f}(\tilde{u}_{\text{av}}) + a \sin(\tau) \tilde{f}'(\tilde{u}_{\text{av}}) + O(a^2),$$

we can observe that the average of the first term of (A.6) is $O(a^2)$, and the average of the second term is:

$$\frac{a}{2} \tilde{f}'(\tilde{u}_{\text{av}}) + O(a^2).$$

Then, using small $a$, the averaged system would approximate a simple gradient system. Next, the following stability result of the averaged system is required for the stability of

Proofs
Lemma A.1. Under Assumptions 4.7 and 4.8, there exist $\beta_{av} \in \mathcal{K} \mathcal{L}$ such that for any $(\Delta, \nu) \in \mathbb{R}^2_{>0}$, there exist $a^* \in \mathbb{R}_{>0}$ such that for all $(a, k) \in (0, a^*] \times \mathbb{R}_{>0}$, the solutions of (A.5) with the initial condition $\|\tilde{u}_{av}(0)\| \leq \Delta$, will satisfy
\[
\|\tilde{u}_{av}(\tau)\| \leq \beta_{av}(\|\tilde{u}_{av}(0)\|, k\Delta) + \nu, \quad \forall \tau \geq 0.
\]

Proof. Refer to Appendix A.2

Then, Lemma 4.2 follows from [129, Theorem 1].

A.2 Proof of Lemma A.1

The proof follows a similar approach to Lemma 3.1. Using the Lyapunov function $V = \frac{1}{2} \tilde{u}_{av}^2$, the stability of the averaged system can be proven. First define
\[
\Gamma(\tilde{u}_{av}) := \frac{1}{a} \text{Im} \left\{ e^{i\phi} H(i) Q_1(\tilde{u}) \right\} + \frac{\Phi}{2} f'(\tilde{u}_{av}). \tag{A.7}
\]
Then, (A.5) can be expressed as
\[
\dot{\tilde{u}}_{av} = -\frac{ka\Phi}{2} f'(\tilde{u}_{av}) + ka\Gamma(\tilde{u}_{av}). \tag{A.8}
\]

Therefore, by using Assumption 4.7,
\[
\frac{1}{ka} \dot{V} = -\frac{\Phi}{2} f'(\tilde{u}_{av}) \tilde{u}_{av} + \Gamma(\tilde{u}_{av}) \tilde{u}_{av}
= -\frac{\Phi}{2} \frac{f'(\tilde{u}_{av})}{\tilde{u}_{av}} \tilde{u}_{av}^2 + \Gamma(\tilde{u}_{av}) \tilde{u}_{av}
= -\frac{\Phi}{2} \frac{f'(\tilde{u}_{av})}{\tilde{u}_{av}} \|\tilde{u}_{av}\|^2 + \Gamma(\tilde{u}_{av}) \tilde{u}_{av}
\leq -\frac{\Phi}{2} \zeta \|\tilde{u}_{av}\|^2 + \|\Gamma(\tilde{u}_{av})\| \|\tilde{u}_{av}\|. \tag{A.9}
\]
It is desired that $\dot{V}$ be negative, which is true when the following holds:

$$-\frac{\Phi}{2} \zeta \| \tilde{u}_{av} \|^2 + \| \Gamma(\tilde{u}_{av}) \| \| \tilde{u}_{av} \| \leq 0. \quad (A.10)$$

The inequality (A.10) can be solved for $\| \tilde{u}_{av} \|$ which yields

$$\| \tilde{u}_{av} \| \geq \frac{2\| \Gamma \|}{\Phi \zeta}. \quad (A.11)$$

Lemma A.1 follows after noting that $\| \Gamma \|$ is of $O(a)$ and hence, decreasing $a$ makes the region to which $\tilde{u}_{av}$ converges arbitrarily small.
Appendix B

Details of the Perimeter Control

The perimeter control used in Chapter 5 is based on the work by [2, 12]. In this strategy, the number of vehicles inside the network, \( n \), is assumed to be governed by the following dynamics:

\[
\dot{n} = \theta Q_{in} - Q_{out}(n),
\]  

(B.1)

where \( Q_{in} \) and \( Q_{out} \) is the flow of the incoming and outgoing vehicles respectively, and \( \theta \in [0, 1] \) is the proportion of \( Q_{in} \) that is actually allowed to enter the network. Note that \( Q_{in} \) is assumed to be constant in (B.1), unlike its counterpart in the work by [2] where \( Q_{in} \) is proportional or analogous to \( Q_{out}(n) \). Furthermore, \( Q_{out}(n) \) represents the MFD of the network if input to output dynamics are not instantaneous and any delays are comparable with the average travel time across the network.

It is desired to regulate the system to a set point \((n^*, \theta^*)\), around which the system is to be linearised. The set point \( n^* \) should be selected as close as possible to the critical accumulation, whereas \( \theta^* \) needs to be determined by solving \( 0 = \theta^* Q_{in} - Q_{out}(n^*) \), which yields \( \theta^* = Q_{out}(n^*) / Q_{in} \). Note that \( \theta^* < 1 \) only when \( Q_{in} > Q_{out}(n^*) \). This implies that the perimeter control is only applicable when the traffic demand is sufficiently high. Furthermore, by defining the error coordinates \( \tilde{n} = n - n^* \), \( \tilde{\theta} = \theta - \theta^* \) and using first-order Taylor approximation, the linearised system can be written as follows:

\[
\dot{\tilde{n}} = -\frac{dQ(n^*)}{dn} \tilde{n} + Q_{in} \tilde{\theta}.
\]

Moreover, in order to remove the requirement of the knowledge of \( \theta^* \), an LQI regulator
is used. Thus, the system is augmented by the following error variable:

\[ \dot{e} = \tilde{n}. \]

Then, the system is discretized and a state-feedback control law of the following form can be derived:

\[ \theta(t_D) = \theta(t_D - 1) - k_p(n(t_D) - n(t_D - 1)) - k_i(n(t_D) - n^*), \]  \hspace{1cm} (B.2)

where \( t_D \) is the discrete time index, \( k_p \) and \( k_i \) are the controller gains. [2] outlines the derivation of (B.2) in detail.
Bibliography


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Title:
Nash Equilibrium Seeking for Augmentation of Urban Traffic Light Control

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2015

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