Surface-alternating
knots and links

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Abstract

In this thesis we study several classes of knots and links which have alternating projections onto closed orientable surfaces. We are particularly interested in surface-alternating projections which have a pair of checkerboard surfaces that are topologically essential to the link exterior.

We study the structure of generalised alternating links and give a procedure for enumerating their projections. Through the study of their boundary slopes, we can prove some existence results.

We introduce a new class of surface-alternating links which we call weakly generalised alternating links. We show that both checkerboard surfaces associated to a weakly generalised alternating diagram are essential in the link exterior. For these links, there are no essential bigons between the two checkerboard surfaces, and we show this is equivalent to the relative 1-line property. This allows us to give a topological characterisation of weakly generalised alternating link exteriors.

We are also able to give a non-diagrammatic characterisation of planar alternating links, which answers a long-standing question of Fox. This means that alternating is a topological property of the link exterior, and not just a diagrammatic property.

Finally we use normal surface theory to produce algorithms which can decide if a knot is alternating or weakly generalised alternating, given either a link exterior or a non-alternating planar projection as input.
Declaration

This is to certify that:

1. the thesis comprises only my original work towards the PhD,

2. due acknowledgement has been made in the text to all other material used,

3. the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Joshua Andrew Howie
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Introduction

A link $L$ in $S^3$ is surface-alternating if there exists a projection onto a closed orientable embedded surface $F$,

$$\pi : F \times I \to F,$$

where $L \subset F \times I \subset S^3$, and $\pi(L)$ is alternating on $F$.

In this thesis we will consider several subclasses of surface-alternating links by adding extra conditions to this definition. For example we may be interested in whether $F$ is a Heegaard surface for $S^3$, whether $F \setminus \pi(L)$ consists of disks, or whether $\pi(L)$ is sufficiently complex in $F$. Sufficient complexity will be measured by the minimal number of intersections of $\pi(L)$ with essential curves on $F$, or the minimal number of intersections of $\pi(L)$ with boundaries of compressing disks for $F$.

We will be particularly interested in the classes of surface-alternating links whose projections separate $F$ giving rise to checkerboard surfaces which are essential in the link exterior. This means that the checkerboard surfaces are significant to the topology of the ambient 3-manifold.

The main theme of this thesis is the attempt to relate classes of surface-alternating diagrams to the topology of the link exterior. The link exterior $X$ is a compact orientable 3-manifold where each boundary component is a torus, and our greatest success in this thesis is to give topological characterisations of $X$ for certain classes of surface-alternating links.

Alternating links are one of the most studied and well loved classes of links. There are many important results which hold for them that do not hold for links in general, such as the Tait Conjectures. There have been many
proofs that the checkerboard surfaces associated to a reduced prime alternating planar diagram are both essential in $X$, the first due to Aumann [7].

The first higher genus examples of surface-alternating links were the toroidally alternating links introduced by Adams [3]. These were then extended to alternating projections onto higher genus surfaces by Adams, and also Hayashi [35], which we call an $F$-alternating link. In this definition, $F$ is Heegaard and all the regions of $F \setminus \pi(L)$ are disks. Given any non-split planar link diagram $\pi'(L)$, Adams has a method to construct an $F$-alternating projection by pushing any crossings which obstruct $\pi'(L)$ from being alternating onto the back of a handle in $F$.

This has been generalised to the concept of a Turaev surface [18], where an $F$-alternating projection is constructed from $\pi'(L)$ by pushing entire tangles onto handles. From our perspective, the main drawback of $F$-alternating link projections is that the associated checkerboard surfaces often fail to be essential in the link exterior.

Chapter 1 tells a fuller background story of alternating projections, essential surfaces, $F$-alternating projections, Turaev surfaces, and embeddings of links into closed orientable surfaces.

In Chapter 2, we discuss generalised alternating links which are another class of surface-alternating link. These were introduced by Ozawa [73], although Hayashi had previously studied a similar class of links. Here we no longer require $F$ to be Heegaard, but now need every essential curve on $F$ to meet $\pi(L)$ at least four times. Ozawa was able to show that both the checkerboard surfaces are $\pi_1$-essential in $X$.

Conway [13] described a method for enumerating planar link diagrams which involves inserting rational tangles into 4-regular graphs known as basic polyhedra. A method for enumerating basic polyhedra has been described by Bridgeman [10].

We extend their methods to the enumeration of generalised alternating link projections. This involves finding an initial finite list of generalised basic polyhedra from which all generalised basic polyhedra can be constructed with two simple operations on graphs. Fortunately for us, both these tasks have been studied by Nakamoto [67], and we are left to show that generalised basic
polyhedra are dual to irreducible bipartite quadrangulations of surfaces. This leads to the following theorem:

**Theorem 2.21.** There is an algorithm to enumerate generalised alternating link projections.

The study of the structure of generalised alternating projections allows us to place lower bounds on the number of crossings in a generalised alternating projection in terms of the genus of the projection surface. We then study boundary slopes of spanning surfaces for knots, and the diameter of the set of spanning slopes gives an upper bound on the number of crossings in a generalised alternating projection of a given knot. Together these two bounds allow us to show that there exist hyperbolic knots which are not generalised alternating onto any closed projection surface.

In Chapter 3, we introduce a new class of surface-alternating links, which we call weakly generalised alternating links. This is an extension of generalised alternating links, where we replace Ozawa’s condition that every essential curve on \( F \) intersects \( \pi(L) \) at least four times, with \( \pi(L) \) being weakly prime and the boundary of every compressing disk for \( F \) intersecting \( \pi(L) \) at least four times. We also allow projections onto generalised projection surfaces which are non-split collections of closed orientable surfaces embedded in \( S^3 \).

We study the checkerboard surfaces associated to a weakly generalised alternating projection and show that there are no essential bigons between them. This fact allows us to prove that both checkerboard surfaces are \( \pi_1 \)-essential in \( X \). This also provides a new proof that generalised alternating projections and reduced prime planar alternating projections have \( \pi_1 \)-essential checkerboard surfaces. We can then prove that a weakly generalised alternating link is non-trivial, non-split and prime.

The 1-line property was introduced by Freedman, Hass and Scott [25] and describes how the lift of an immersed \( \pi_1 \)-essential surface intersects itself in the universal cover. We discuss a modification called the relative 1-line property which is a condition on how the lifts of two embedded \( \pi_1 \)-essential surfaces intersect each other in the universal cover. Through several lemmas
we show that two $\pi_1$-essential spanning surfaces for a link have the relative 1-line property if and only if there are no bigons between the two spanning surfaces.

This allows us to give the following topological characterisation of weakly
generalised alternating link exteriors. Note that all the conditions are topological in nature and independent of any diagrammatic description of the link. Here $\sigma_j$ denotes the boundary component of $\Sigma$ embedded in the $j^{th}$ boundary component of $X$ and $i(\sigma_j, \sigma'_j)$ denotes the algebraic intersection number of $\sigma_j$ and $\sigma'_j$.

**Theorem 3.33.** An $m$-component link $L$ is weakly generalised alternating if and only if there exist a pair of non-split $\pi_1$-essential spanning surfaces $\Sigma$ and $\Sigma'$ for $L$ such that:

1. $i(\sigma_j, \sigma'_j) \in 2\mathbb{N}$ for each $j = 1, \ldots, m$.

2. There are isotopic representatives of $\Sigma$ and $\Sigma'$ which intersect only in standard arcs.

3. $\Sigma$ and $\Sigma'$ have the relative 1-line property.

An equation relating the Euler characteristics of the checkerboard surfaces to the Euler characteristic of the projection surface is proved in Theorem 3.10. Together with Theorem 3.33, this gives a non-diagrammatic characterisation of prime alternating links.

This answers the famous question of Ralph Fox, "What is an alternating knot?" This question has been interpreted as asking for a topological characterisation of alternating knots without mention of diagrams [57]. However we can do better than the characterisation described above, and instead we give a direct proof of the following theorem:

**Theorem 4.2.** Let $K$ be a non-trivial knot in $S^3$ with exterior $X$. $K$ has an alternating projection onto $S^2$ if and only if there exist a pair of spanning surfaces $\Sigma$, $\Sigma'$ for $K$ which satisfy

\[
\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma') = 2,
\]

where $i(\partial\Sigma, \partial\Sigma') \neq 0$.

Note that the conditions are topological since they only involve Euler characteristics and intersection numbers. Hence we have discovered a non-diagrammatic characterisation of alternating knots. We can also extend this
Theorem to alternating links by including an extra condition on the intersection numbers.

In Chapter 4, we also introduce a new class of surface-alternating links called essentially alternating links. They are defined to include all separating surface-alternating projections where both checkerboard surfaces are $\pi_1$-essential in $X$. This generalises both weakly generalised alternating links and adequate links and we give several other examples. However we are not able to establish a diagrammatic characterisation of such links.

We conclude the chapter with some results on when it is possible to see the geometry of a knot in an essentially alternating diagram. By geometry, we mean whether the knot is satellite, torus or hyperbolic in the sense of Thurston [90]. This generalises some of Menasco’s results [60] on when the geometry is visible in a planar alternating projection.

Chapter 5 begins with a review of normal surface theory, and we pay special attention to the theory of Jaco and Sedgwick [51], and their description of the boundary solution space and algorithm to list the boundary slopes of a knot.

We then show that there is a finite time algorithm to decide if a knot is prime and alternating. The input for this algorithm is assumed to be a 3-manifold with connected torus boundary. It is also possible to assume that the input is a planar non-alternating knot projection.

**Theorem 5.13.** Let $X$ be the exterior of a knot $K \subset S^3$. Given $X$, there is a normal surface algorithm to decide if $K$ is prime and alternating.

Furthermore, if $K$ is alternating, we have enough information to construct a planar alternating diagram of $K$ up to mirror image.

In the final section we give several algorithms to decide whether a knot is weakly generalised alternating. While we cannot decide this in general, we do have a finite time algorithm to decide if a knot has a weakly generalised alternating projection $\pi(K)$ where all the regions of $F \setminus \pi(K)$ are disks, given a knot exterior as input. We also have an algorithm which, given a hyperbolic knot exterior $X$ and a positive integer $g$, can decide if $X$ is the exterior of a knot $K$ which has a weakly generalised alternating projection onto a closed
orientable surface of genus at most $g$.

These last two algorithms rely on an algorithm we have developed to decide if some pair of surfaces, out of a given finite collection, has the relative 1-line property.

Throughout this thesis we use the Rolfsen names for links with up to 10 crossings [80], making the necessary modification to account for the Perko pair. For knots with 11 or more crossings, we use the Dowker-Thistlethwaite naming conventions [20].

The reader is directed to Adams [4], Lickorish [57], or Rolfsen [80] for any undefined concepts in knot theory, and to Hempel [37], Jaco [46], or Schultens [82] for background in 3-manifold topology.
Chapter 1

Alternating Projections

In this chapter, we provide a survey of known results about alternating link projections onto orientable surfaces. In Section 1.1, we provide an introduction to knot theory and planar alternating projections. In Section 1.2, we introduce incompressible and $\pi_1$-injective surfaces, look at the differences between the two concepts, paying special attention to one-sided surfaces. We consider the history of the problem of showing that both checkerboard surfaces are $\pi_1$-essential.

In Section 1.3, we introduce alternating projections onto orientable surfaces, including toroidally alternating links and $F$-alternating links. We also give the formal definition of a surface-alternating link. Section 1.4 provides a generalisation of a construction in the previous section for producing $F$-alternating projections from a planar diagram.

Section 1.5 deals with embeddings of links in orientable surfaces, and while these are not considered to be surface-alternating, they will be useful in constructing later examples.
1.1 Alternating Knots and Links

A knot \( K \) is a piecewise linear embedding of a circle in the 3-sphere,

\[
K : S^1 \hookrightarrow S^3.
\]

We will refer to the image of \( S^1 \) under this map as \( K \) also.

An \( m \)-component link \( L \) is a piecewise linear embedding of \( m \) circles into the 3-sphere, 

\[
L : \bigsqcup_{i=1}^{m} S^1 \hookrightarrow S^3.
\]

A link of one component is a knot.

The complement \( S^3 \setminus L \) of a link \( L \) is a connected, open 3-manifold, without boundary.

\( L \) has a regular neighbourhood \( N(L) \) that consists of \( m \) solid tori. The exterior \( X \) of a link \( L \) is the closure of \( S^3 \setminus N(L) \). The exterior is a connected, compact 3-manifold with boundary. Each component of \( \partial X \) is a torus.

A link projection or diagram is the image of a link \( L \) under a projection 

\[
\pi : S^3 \to S^2
\]

that also retains information about the crossings. More precisely, let \( P \) be a set of two distinct points in \( S^3 \setminus L \). Then \( S^3 \setminus P \cong S^2 \times I \), and the projection is actually the natural projection 

\[
\pi : S^2 \times I \to S^2.
\]

The link \( L \) is isotoped in \( S^2 \times I \) so that it projects to a 4-regular planar graph \( \Gamma \), called the projection graph. The projection \( \pi(L) \) is \( \Gamma \) together with crossing information. Crossing information tells us which of the two strands passing through a vertex of \( \Gamma \) is the over-strand or under-strand.

Later, we will study projections onto higher genus surfaces. We use the phrase planar diagram to specify a projection onto \( S^2 \).

We say that a link diagram is reduced if it does not contain any nugatory
crossings. A crossing is *nugatory* if there exists a simple closed curve $\ell \subset S^2$ which meets $\pi(L)$ in exactly this crossing and nowhere else.

We say that a link diagram $\pi(L)$ is *alternating* if when we trace along the knot, the crossings alternate between over and under crossings. Implicit in this definition is the fact that an alternating diagram contains at least one
crossing. We say that a link is alternating if it has a reduced alternating diagram. Throughout this thesis the convention is that the unknot is not considered to be an alternating knot.

We say that a link is prime if every embedded 2-sphere $S \subset S^3$ which meets the knot transversely in exactly two points, bounds a 3-ball which intersects $L$ in a trivial arc. An arc is trivial if it can be isotoped into $S$. We say that a knot diagram is prime if every simple closed curve in the projection surface which meets $\pi(L)$ exactly twice away from crossings bounds a disk $D$ which contains only a trivial arc. Here, an arc of the diagram is trivial if $\pi(L) \cap D$ is connected and there are no crossings inside $D$.

A link $L$ is split if there is an embedded sphere $S \subset S^3$ which separates components of $L$. A diagram $\pi(L)$ is split if there is an embedded curve $\ell \subset S^2$ which separates components of $\pi(L)$.

**Theorem 1.1** (Menasco [60]). Let $\pi(L)$ be a reduced alternating diagram of a link $L$.

1. $L$ is prime if and only if $\pi(L)$ is prime.

2. $L$ is split if and only if $\pi(L)$ is split.
1.1. ALTERNATING KNOTS AND LINKS

Alternating links are one of the most important and well-understood classes of links. This is because they have a special diagram from which it is possible to see many properties of the knot. The following three theorems are collectively known as the Tait Conjectures:

**Theorem 1.2 (Tait Conjectures).** Let $\pi(K)$ and $\pi'(K)$ be reduced alternating diagrams of the knot $K$. Then:

1. $\pi(K)$ and $\pi'(K)$ have the same writhe,
2. $\pi(K)$ and $\pi'(K)$ are related by a sequence of flypes,
3. $\pi(K)$ minimises crossing number over all projections of $K$.

The first and third conjectures were proved independently by Kauffman [54], Murasugi [63][64] and Thistlethwaite [86][87], using properties of the Jones polynomial. The second conjecture was proved by Menasco and Thistlethwaite [61]. The second and third conjectures also hold for reduced non-split alternating diagrams of the same link.

All prime knots up to 7 crossings are alternating, however as crossing number increases, the proportion of alternating links decreases to zero.

There have been several previous attempts at generalising properties of alternating knots. In this thesis we will consider alternating projections onto surfaces of higher genus than the sphere. We will do this in such a way that associated to the projection there will be two surfaces whose topology reflects the topology of the knot exterior.

Another generalisation is adequate and $\sigma$-adequate knots which we will describe in Section 1.4. There are also such notions as semi-alternating, semi-adequate, algebraic, and homogeneous which extend the concept of alternating links in various ways.
1.2 Essential Surfaces

A spanning surface for a link $L$ is an embedded surface $\Sigma \subset S^3$ such that $\partial \Sigma = L$. We will also call the restriction $\Sigma = \Sigma \cap X$ of $\Sigma$ to the knot exterior $X$ a spanning surface.

Every link has a constructible orientable spanning surface known as a Seifert surface, and there is a well-known algorithm to construct such a surface. If we start with a reduced alternating planar diagram $\pi(L)$, then Gabai [27] has shown that Seifert’s algorithm will produce an orientable spanning surface of minimal genus. This is another special property of alternating diagrams.

If we ignore crossing information, then a link projection $\pi(L)$ is a 4-regular planar graph $\Gamma$. Its dual graph is bipartite. Let us 2-colour the respective regions of $S^2 \setminus \pi(L)$ black and white. If $\pi(L)$ is non-split, then these regions are disks. At each crossing we can glue the two opposing black regions by adding a half-twisted strip so that the sides of the strip agree with $L$. We can also glue the two opposing white regions by adding a half-twisted strip. If we do this at every crossing, then we obtain a pair of spanning surfaces for $L$, known as the checkerboard surfaces. The checkerboard surfaces intersect in double arcs, one for each crossing in the projection.

In the following definitions, let $S$ be a surface that is neither a 2-sphere, a projective plane, nor a disk. A surface $S$ is properly embedded in a compact 3-manifold $M$ if $\partial M \cap S = \partial S$ and a collar of $\partial S$ in $S$ is transverse to $\partial M$. If $M$ is without boundary, then $S$ must be without boundary and the definition coincides with the definition of embedded.

A properly embedded surface $S$ is incompressible in a 3-manifold $M$ if for every embedded disk $D \subset M$ such that $D \cap S = \partial D$, we have that $\partial D$ bounds a disk in $S$. If $\partial D$ does not bound a disk in $S$, then we call $D$ a compressing disk for $S$.

A properly embedded surface $S$ is boundary-incompressible in a 3-manifold $M$ if for every embedded disk $D \subset M$ with $\partial D = D \cap (S \cup \partial M)$, with $\partial D \cap \partial M$ connected and $\partial D \cap \bar{S} \neq \emptyset$, we have that $\partial D$ bounds a disk in $S \cup \partial M$. If $\partial D$ does not bound a disk in $S \cup \partial M$, then we call $D$ a boundary-compressing
Figure 1.4: Black and white checkerboard surfaces for the knot $5_2$.

Figure 1.5: A compressing disk for the white checkerboard surface which sits above the under-strands and below the over-strands.

*disk* for $S$. Often we will write $\partial D = \alpha \cup \alpha'$ for the boundary of a boundary-compressing disk, where $\alpha$ is an arc in $S$ and $\alpha'$ is an arc in $\partial M$. 
A properly embedded surface $S$ is essential in a 3-manifold $M$ if it is incompressible, boundary-incompressible and not boundary-parallel. Incompressible, boundary-incompressible and essential are geometric definitions. We now give some analogous yet stronger algebraic definitions.

A properly embedded surface $S$ is $\pi_1$-injective in a 3-manifold $M$ if the homomorphism

$$\pi_1(S) \to \pi_1(M)$$

is injective. If $S$ is not $\pi_1$-injective, then there exists a map $f : D \to M$ such that $f(\partial D) \subset S$ and $f(\partial D)$ is not nullhomotopic in $S$. We call such a disk $D$ a singular compressing disk for $S$. If $f$ is an embedding and $f(\hat{D}) \cap S = \emptyset$, then $f(D)$ is a compressing disk for $S$.

A properly embedded surface $S$ is $\pi_1$-boundary-injective in a 3-manifold $M$ if the map

$$\pi_1(S, \partial S) \to \pi_1(M, \partial M)$$

is injective. This is not a homomorphism since $\pi_1(M, \partial M)$ is not a group. It is possible to give $\pi_1(M, \partial M)$ the structure of a groupoid, where the objects are
points in $\partial M$ and the morphisms are homotopy classes of paths in $M$ starting and ending in $\partial M$. We call $\pi_1(M, \partial M)$ the fundamental groupoid of $M$. The fundamental groupoid $\pi_1(S, \partial S)$ is defined similarly. Low-dimensional topologists usually think of $\pi_1(S, \partial S)$ as the set of paths in $S$ up to homotopy which start and end in $\partial S$.

A properly embedded surface $S$ is $\pi_1$-essential in a 3-manifold $M$ if it is $\pi_1$-injective, $\pi_1$-boundary-injective and not boundary-parallel.

Some authors prefer to use the terms geometrically incompressible and algebraically incompressible in place of incompressible and $\pi_1$-injective respectively.

A 3-manifold is irreducible if every embedded 2-sphere bounds an embedded 3-ball. An orientable 3-manifold $M$ is Haken if it is compact, irreducible and either $\partial M$ is incompressible and non-empty, or $M$ contains a closed incompressible surface that is not $S^2$. We are only considering orientable 3-manifolds since all knot and link complements in $S^3$ are orientable.

The boundary of a knot exterior $\partial X$ is $\pi_1$-injective in $X$ provided $K$ is not the unknot. A 3-manifold-with-boundary $M$ is boundary-irreducible, if every properly embedded disk $D$ in $M$ is parallel into $\partial M$. Hence $X$ is boundary-irreducible unless $K$ is the unknot.

A properly embedded surface $S$ is two-sided in a 3-manifold $M$ if it has a trivial normal bundle. Otherwise $S$ is one-sided. If $M$ is orientable, then $S$ is two-sided in $M$ if and only if $S$ is orientable.

When a surface is two-sided, the concepts of essential and $\pi_1$-essential agree. This makes use of the Loop Theorem which was originally proved by Papakyriakopoulos [78]. In this thesis, we will make use of the following form of the Loop Theorem:

**Theorem 1.3** (Loop Theorem). Let $D$ be a disk, $M$ a 3-manifold and $S$ a two-sided properly embedded surface in $M$. If there exists a map $f : D \to M$ such that $f(\partial D) \subset S$, and $f(\partial D)$ is not nullhomotopic in $S$, then there exists an embedding $g : D \hookrightarrow M$ such that $g(D)$ is a compressing disk for $S$.

Dehn’s Lemma, proved earlier by Papakyriakopoulos [77], is the special case of the Loop Theorem where $f|_{\partial D}$ is an embedding in $S$, so that all the
singularities of \( f(D) \) occur on its interior.

In practice, we use the loop theorem to replace a singular compressing disk \( f : D \to M \) for a surface \( S \) with an embedded compressing disk \( D' \) for \( S \), but we cannot be sure the \( f(\partial D) \) represents the same homotopy class in \( S \) as \( \partial D' \). This is particularly important for boundary-compressing disks since we need \( \partial D' \) to decompose into two arcs, one in \( S \) and one in \( \partial M \). However, \( f|_{\partial D} \) can be homotoped into a general position immersion, and in particular \( f(\partial D) \) consists of a curve in \( S \) where the only singularities are finitely many isolated double points. Let \( A \) be the collection of arcs formed by cutting \( f(\partial D) \) at each of its double points. From the proof of the Loop Theorem, it follows that we can always choose an embedded \( D' \) so that \( \partial D' \) consists of the union of the closures of a sub collection of \( A \).

**Lemma 1.4.** Let \( S \) be a properly embedded surface in a 3-manifold \( M \).

1. If \( S \) is \( \pi_1 \)-injective in \( M \), then \( S \) is incompressible in \( M \).

2. If \( S \) is incompressible and two-sided in \( M \), then \( S \) is \( \pi_1 \)-injective in \( M \).

**Proof.** Let \( D \) be a compressing disk for \( S \). Then \( \partial D \) is not nullhomotopic in \( S \), but is nullhomotopic is \( M \), showing that \( S \) is not \( \pi_1 \)-injective.

For the second part, suppose \( D \) is a singular compressing disk for the two-sided surface \( S \). Then, by the Loop Theorem, there exists an embedded compressing disk \( D' \) for \( S \).

Consider the diagram of the unknot with 2 crossings shown in Figure 1.7. The black checkerboard surface \( \Sigma \) is a punctured Klein bottle. Then \( \Sigma \) is incompressible in \( X \) since surgering along a compressing disk would either produce a closed embedded non-orientable surface in \( X \subset S^3 \) or produce a spanning disk at a non-zero slope. However \( \Sigma \) is not \( \pi_1 \)-injective in \( X \) since there is a singular compressing disk \( D \) for \( \Sigma \). While \( D \) is embedded, \( \partial D \) self-intersects. Note also that \( \Sigma \) is boundary-compressible in \( X \), since \( \pi(K) \) contains a nugatory crossing. A Dehn filling on \( X \) such that the meridian disk of a solid torus is glued to the boundary of the punctured Klein bottle, produces a closed Klein bottle embedded in the Lens space \( L(4,1) \). This is the standard example of an incompressible surface that is not \( \pi_1 \)-injective.
Lemma 1.5. Let $S$ be a properly embedded surface with boundary in an orientable 3-manifold $M$. If $S$ is $\pi_1$-boundary-injective in $M$, then $S$ is boundary-incompressible in $M$.

Proof. Let $D$ be a boundary-compressing disk for $S$ where $\partial D = \alpha \cup \alpha'$ with $\alpha$ and $\alpha'$ connected arcs in $S$ and $\partial M$ respectively. Then $\alpha$ is not homotopic into $\partial S$, but is homotopic into $\partial M$, showing that $S$ is not $\pi_1$-boundary-injective.

Lemma 1.6. Let $M$ be an orientable 3-manifold that is not a twisted $I$-bundle over a surface. A properly embedded surface $S$ with $\chi(S) \leq 0$ is $\pi_1$-essential in $M$ if and only if its orientable double cover $\hat{S}$ is $\pi_1$-essential in $M$.

Proof. Let $p : \hat{S} \rightarrow S$ be the covering map. Then the induced homomorphism $p_* : \pi_1(\hat{S}) \rightarrow \pi_1(S)$ is injective since $\pi_1(\hat{S})$ is an index two subgroup of $\pi_1(S)$. Let $f : S \hookrightarrow M$ be a proper embedding. Suppose that the induced homomorphism $f_* : \pi_1(S) \rightarrow \pi_1(M)$ is injective. Then $\hat{f}_* : \pi_1(\hat{S}) \rightarrow \pi_1(M)$ is injective since it is the composition of two monomorphisms, where $\hat{f} = f \circ p$. 

Figure 1.7: An incompressible spanning surface which is not $\pi_1$-injective. The immersed red curve bounds a singular compressing disk in the knot exterior.
Conversely, suppose that $x$ is a non-trivial element of $\ker(f_*)$. By assumption $\chi(S) \leq 0$ so $\pi_1(S)$ is torsion-free and $x$ must have infinite order in $\pi_1(S)$. The covering map $p$ has degree 2 so $p_*^{-1}(x)$ must have infinite order in $\pi_1(\hat{S})$. Hence $\ker(\hat{f}_*)$ is non-trivial.

Thus $S$ is $\pi_1$-injective if and only if $\hat{S}$ is $\pi_1$-injective. Similar statements about the induced functors on the fundamental groupoids of $S$ and $\hat{S}$ show that $S$ is $\pi_1$-boundary-injective if and only if $\hat{S}$ is $\pi_1$-boundary-injective.

Finally, if $S$ is boundary parallel, then $S$ is orientable, so $S$ is parallel to each component of $\hat{S}$. If $\hat{S}$ is boundary-parallel, then either $S$ is two-sided and also boundary-parallel, or $M$ is a twisted $I$-bundle over the one-sided surface $S$.

Let $\Sigma$ be a non-orientable spanning surface for a link $L$. Then $\Sigma$ has an orientable double cover $\hat{\Sigma}$ which is the boundary of a neighbourhood of $\Sigma$ in $X$. Hence by Lemma 1.6, $\Sigma$ is $\pi_1$-essential in $X$ if and only if $\hat{\Sigma}$ is essential in $X$.

In certain cases, as the following two lemmas show, it is possible to know that a surface is essential or $\pi_1$-essential, without having to check that it is boundary-incompressible or $\pi_1$-boundary-injective.

**Lemma 1.7.** Let $M$ be an irreducible 3-manifold with torus boundary. Every properly embedded two-sided incompressible surface-with-boundary in $M$ is either boundary-incompressible or a boundary-parallel annulus.

A proof of this Lemma is given by Ozawa [73] for the case when $M$ is a knot exterior. There is also an analogue for one-sided surfaces:

**Lemma 1.8 (Ozawa-Tsutsumi [76]).** Let $\Sigma$ be a $\pi_1$-injective non-orientable surface properly embedded in a knot exterior $X$. If $\Sigma$ is not $\pi_1$-boundary-injective in $X$, then $\Sigma$ is an unknotted, half-twisted Mobius band, and $X$ is a solid torus.

The advantage of this lemma is that once we have shown a spanning surface for a non-trivial knot is $\pi_1$-injective, it follows immediately the spanning surface is $\pi_1$-essential. Note the Klein bottle in Figure 1.7 is boundary-compressible.
The property of alternating knots that will interest us in this thesis is the following theorem. It shows that an alternating projection is special, because its checkerboard surfaces have topologies that capture some of the topology of the knot exterior $X$. The corresponding theorem for alternating links follows from Theorem 1.15.

**Theorem 1.9** (Aumann [7]). Let $\pi(K)$ be a reduced alternating diagram of a knot $K$, with associated checkerboard surfaces $\Sigma$ and $\Sigma'$. Both $\Sigma$ and $\Sigma'$ are $\pi_1$-essential in $X$.

Aumann does not state this Theorem in his paper, however he does prove that both surfaces are $\pi_1$-injective into certain handlebodies which give a Heegaard splitting of $S^3$ with $K$ embedded on the Heegaard surface. He uses this to show that $X$ is aspherical. A manifold $M$ is *aspherical* if all its higher homotopy groups vanish, ie $\pi_j(M) = 0$ for all $j \geq 2$.

In effect, he embeds the knot $K$ into a closed orientable surface $F$, where $F$ is the double cover of a checkerboard surface $\Sigma$ glued to an annulus sitting inside $N(K)$. He then shows that $\pi_1(F \setminus K)$ injects into $\pi_1(S^3 \setminus K)$. This shows that $\Sigma$ is $\pi_1$-essential since its orientable double cover is $F \setminus N(K)$.

A generalisation of Theorem 1.9 is proved by Ozawa [74] which we state as Theorem 1.15. In particular, Ozawa’s result extends Aumann’s Theorem
to alternating link diagrams. There are several other proofs of Aumann’s result in the literature and we will give another proof in Theorem 3.15.

A surface $S$ properly embedded in a link complement $S^3 \setminus L$ is meridionally incompressible if for every disk $D \subset S^3$ which meets $L$ tranversely in one point, with $D \cap S = \partial D$, we have that $\partial D$ is contractible in $S \cup L$. In the literature meridionally incompressible is sometimes called pairwise incompressible.

**Theorem 1.10** (Menasco [60]). Suppose that $K$ is a prime alternating knot in $S^3$. Then there are no incompressible meridionally incompressible surfaces in $S^3 \setminus K$.

Using work of Thurston, this gives us information about the geometry of alternating knots.

**Theorem 1.11** (Thurston [90]). Non-trivial knots in $S^3$ can be partitioned into three classes: torus knots, satellite knots, and hyperbolic knots.

This partition depends on the geometry of the knot exterior. A link $L$ is hyperbolic if $S^3 \setminus L$ admits a hyperbolic geometric structure. If $L$ is non-split and non-trivial and embeds in the Heegaard torus, then $L$ is a torus link. A torus knot exterior has the geometry of a Seifert fibred space.

Let $K'$ be a knot properly embedded in a solid torus $T$ with core $\tau$, such that $K'$ cannot be isotoped to be disjoint from some meridional disk of $T$, and $K'$ is not isotopic to $\tau$. Let $f : T \to S^3$ be a non-trivial embedding, meaning that $f(\tau)$ is not the unknot. Then $K = f(K')$ is a satellite knot in $S^3$. We call $K'$ the pattern of $K$, and $f(\tau)$ the companion of $K$. Note that $f(\partial T)$ is an incompressible torus in the knot exterior $X$, and $f(\partial T)$ is essential in $X$ provided $K'$ is not isotopic to $\tau$ in $T$. Satellite knots have non-trivial JSJ decompositions into geometric pieces. If the knot $K$ is not prime, then $K$ is a satellite knot.

These three classes also have a topological characterisation. A knot $K$ is hyperbolic if its exterior $X$ is atoroidal and anannular. A 3-manifold $M$ with non-empty boundary is atoroidal if there are no essential tori in $M$. This is equivalent to saying that the only incompressible tori are boundary
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parallel. A 3-manifold $M$ with non-empty boundary is an annular if there are no essential annuli properly embedded in $M$.

A knot $K$ is a satellite knot if its exterior $X$ contains an essential torus. A knot $K$ is a torus knot if its exterior $X$ is atoroidal but there is a properly embedded essential annulus in $X$.

This topological characterisation does not carry over to links, since there are torus links whose exteriors contain essential tori. Let $L$ be the $(6, 4)$-torus link embedded in the Heegaard torus $F$, and let $A$ be one of the two annular components of $F \setminus L$. Then $\partial N(A)$ is an essential torus in $X$, where $N(A)$ is a neighbourhood of $A$ in $S^3$.

Thurston’s result gives the following corollary to Theorem 1.10, which means we can tell from a prime alternating diagram whether a knot is hyperbolic. It is known that a torus link is alternating if and only if it is a $(p, 2)$-torus link.

**Theorem 1.12** (Menasco [60]). Suppose that $K$ is a prime alternating knot in $S^3$. Then $K$ is either a torus knot or a hyperbolic knot.

1.3 Toroidally Alternating and F-Alternating Links

The first to consider links which have alternating projections onto surfaces other than the 2-sphere was Adams [3]. Let $T$ be a Heegaard torus embedded in $S^3$. A link $L$ is toroidally alternating if there is a projection

$$\pi : T \times I \to T,$$

where $\pi(L)$ is alternating on $T$, $L \subset T \times I \subset S^3$, and each region of $T \setminus \pi(L)$ is homeomorphic to a disk.

An orientable surface $F$ embedded in an orientable 3-manifold $M$ is Heegaard if both components of $M \setminus N(F)$ are handlebodies. Adams actually defined toroidally alternating links more generally, by considering links in lens spaces, which also have genus one Heegaard splittings.
Almost alternating links were introduced by Adams et al [5]. A link $L$ is *almost alternating*, if it has a projection $\pi(L)$ onto $S^2$ such that changing one crossing from over to under would produce an alternating projection $\pi'(L')$ of some link $L'$. The crossing that we change is called the *dealternator*.

Adams showed that every almost alternating link is toroidally alternating. He also showed that every alternating link is toroidally alternating by first modifying its planar alternating diagram by a type II Reidemeister move to produce an almost alternating diagram. There is also a toroidally alternating projection of the unknot with three crossings.

From an almost alternating projection $\pi(L)$, if we add a handle to $S^2$ near the dealternator, and push the dealternator through to the other side of the handle, then we obtain a toroidally alternating projection $\pi'(L)$. See Figures 1.9 and 1.10 for a demonstration of this procedure. We make the convention in this thesis that all figures of surface-alternating projections will have all crossing drawn on the front of $F$. So in Figure 1.10, the crossing on the back of the handle has been rotated onto the front of $F$ by an isotopy of...
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Figure 1.10: A toroidally alternating projection of the knot $8_{19}$.

the diagram.

Hayashi [35] has shown that all Montesinos links are toroidally alternating. For knots up to 11 crossings, Adams et al [5] have shown that all but 3 are almost alternating and hence toroidally alternating. Stoimenow [83] subsequently showed that $11n_{183}$ has an almost alternating projection. It is not known if the exceptions $11n_{95}$ and $11n_{118}$ are almost alternating. There are also two links with at most 10 crossings for which this property is undecided, one with two components, the other with three. See [5] for details.

Adams has extended Theorem 1.10 of Menasco to toroidally alternating knots, which means we know where toroidally alternating knots sit in the Thurston trichotomy.

**Theorem 1.13** (Adams [3]). Suppose that $K$ is a toroidally alternating knot in $S^3$. Then there are no incompressible meridionally incompressible surfaces in $S^3 \setminus K$.

**Theorem 1.14** (Adams [3]). A prime non-trivial toroidally alternating knot is either a torus knot or a hyperbolic knot.
Stošić [84] has shown that the only almost alternating torus knots are the (4, 3) and (5, 3) torus knots. It is unknown if they are the only toroidally alternating torus knots.

Let $F$ be a Heegaard surface embedded in $S^3$. A link $L$ is $F$-alternating if there is a projection

$$
\pi : F \times I \to F,
$$

where $\pi(L)$ is alternating on $F$, $L \subset F \times I \subset S^3$, and each region of $F \setminus \pi(L)$ is homeomorphic to a disk.

A link $L$ is $m$-almost alternating, if it has a non-split projection $\pi(L)$ onto $S^2$ such that changing $m$ crossings from over to under or under to over would produce an alternating projection $\pi'(L')$ of some link $L'$. It follows that every $m$-almost alternating link has an $F$ alternating projection, where the genus of $F$ is $m$. Clearly every link is $m$-almost alternating for some $m \geq 0$, and thus $F$-alternating on some closed orientable surface $F$.

Hayashi [35] has also studied $F$-alternating links. He showed that if an $F$-alternating projection $\pi(L)$ is sufficiently complex then it is non-split and prime. We will state his results in Section 3.6 once we have introduced the terminology to define sufficiently complex.

While both Adams and Hayashi have made interesting generalisations of alternating knots, their main objective was to describe alternating projections onto surfaces in $S^3$ or the lens spaces, and subsequently describe the closed incompressible surfaces in the knot complement. We are interested in finding pairs of essential checkerboard surfaces, an issue that they have not addressed, however since their definitions are related to ours, we can use some of their results.

We point out in particular, that in the construction of an $F$-alternating projection $\pi'(L)$ from an $m$-almost alternating projection $\pi(L)$, both checkerboard surfaces are always boundary-compressible. This is because near a dealternator on $F$, there is a meridian disk for $F$ whose boundary meets $\pi'(L)$ in just this crossing. See Figure 1.11.

We will now make our formal definition of surface-alternating links, which is more general than that of Adams or Hayashi. A link $L$ is surface-alternating
if it has a projection onto an orientable embedded surface $F$,

$$\pi : F \times I \to F,$$

such that $\pi(L)$ is alternating on $F$ and $L \subset F \times I \subset S^3$.

This is more general since we do not require that the regions of $F\backslash\pi(L)$ are homeomorphic to disks, nor do we require that $F$ is Heegaard or connected. One extra condition that we do impose throughout this thesis is that $\pi(L)$ is separating on $F$. A diagram $\pi(L)$ is *separating* if every loop $\ell \subset F$ meets $\pi(L)$ an even number of times away from crossings. It is also possible to consider non-separating diagrams on $F$, however then it is not possible to define a pair of checkerboard spanning surfaces, and they are what we will be studying in this thesis. A non-separating surface-alternating projection is shown in Figure 1.12.
1.4 Turaev Surfaces

Let \( \pi(L) \) be a connected diagram on \( S^2 \) of a link \( L \), and let \( \mathcal{C} = \{c_1, \ldots, c_n\} \) be the set of crossings of \( \pi(L) \). A map

\[
\sigma : \mathcal{C} \to \{+, -\}
\]

is called a state for \( \pi(L) \). For each crossing \( c_i \) we perform a positive or negative smoothing at \( c_i \) depending on whether \( \sigma(c_i) \) is positive or negative. The conventions for a positive or negative smoothing are defined in Figures 1.13 and 1.14. Some authors refer to \( + \)-smoothing and \( - \)-smoothing as A-splice and B-splice respectively. For full details see [74] or [26].

Performing the smoothing at each crossing as per the state, resolves \( \pi(L) \) into a set of non-intersecting circles \( \sigma(\pi) \) in \( S^2 \). Each circle is called a state loop \( l_j \) and we write \( |\sigma(\pi)| \) for the number of circles. The state graph \( G_\sigma \) is defined to have a vertex for each state loop \( l_j \) and an edge between two vertices for each crossing \( c_i \) which is labelled by its state \( \sigma(c_i) \).

\( S^3 \setminus S^2 \) is a pair of open 3-balls \( B_+ \) and \( B_- \). A state surface \( \Sigma_\sigma \) is formed
1.4. TURAEV SURFACES

from a state $\sigma$ be attaching a disk $D_j$ to each state loop $l_j$, where the interior of each $D_j$ can be chosen to lie in either $B_+$ or $B_-$. The surface $\Sigma_\sigma$ is then completed by attached a half twisted band $b_i$ at each crossing $c_i$ that matches up with the crossing information from $\pi(L)$.

A state is called $\sigma$-adequate if $G_\sigma$ contains no loops. In a graph, a loop is an edge that connects a vertex to itself. This means that there are no crossings of $\pi(L)$ running from one state loop $l_j$ to itself. If $G_\sigma$ did contain such a loop, then it is likely, depending on choices of whether $D_j$ lies in $B_+$ or $B_-$, that $\Sigma_\sigma$ is boundary-compressible. A state is called $\sigma$-homogeneous if $G_\sigma$ has a decomposition into blocks where all the edges in each block have the same label.

Two particular states we are interested in are the positive state $\sigma_+$, where $\sigma_+(c_i) = +$ for each $i$ and the negative state $\sigma_-$, where $\sigma_-(c_i) = -$ for each $i$. Both the positive and negative states are automatically $\sigma$-homogeneous. For an alternating diagram, the positive and negative states correspond to the checkerboard surfaces.

One other state of interest is the Seifert state $\vec{\sigma}$ which is obtained by choosing from $\{+,-\}$ to smooth with an orientation of $L$. The Seifert state
Figure 1.15: An adequate diagram of the knot 10_{153} and its positive and negative smoothings along with their associated state graphs.
is automatically $\sigma$-adequate, and $\Sigma_\sigma$ is a Seifert surface.

**Theorem 1.15** (Ozawa [74]). Let $\Sigma_\sigma$ be a state surface coming from a planar link projection $\pi(L)$. If $\Sigma$ is $\sigma$-adequate and $\sigma$-homogeneous, then $\Sigma_\sigma$ is $\pi_1$-essential in $X$.

In particular, this shows that the checkerboard surfaces associated to a reduced alternating planar link diagram are both $\pi_1$-essential in $X$, extending the result of Aumann in Theorem 1.9.

A diagram $\pi(L)$ is **adequate** if it is $\sigma$-adequate for both the positive and negative states. A diagram $\pi(L)$ is **semi-adequate** if it is $\sigma$-adequate for either the positive or the negative state. A knot is adequate or semi-adequate if it admits an adequate or semi-adequate diagram respectively. All alternating knots are adequate and Thistlethwaite [88] has shown that an adequate diagram of a knot realises the crossing number of that knot.

Turaev [91] gave a simplified proof of the first and third Tait Conjectures (see Theorem 1.2). In the process, he gave a construction for producing an alternating projection onto an orientable surface, which we now describe.

Forgetting crossing information from a planar diagram $\pi(L)$ corresponding to a link $L$, $\pi(L)$ is a planar 4-regular graph $\Gamma$. Let $\Gamma \subset S^2 \times \{0\}$, $\sigma_+(\pi) \subset S^2 \times \{1\}$ and $\sigma_-(\pi) \subset S^2 \times \{-1\}$.

$\sigma_+(\pi) \subset S^2 \times \{1\}$ is naturally cobordant to $\sigma_-(\pi) \subset S^2 \times \{-1\}$ through a cobordism $W$ of genus $1 + \frac{1}{2}(|C| - |\sigma_+(\pi)| - |\sigma_-(\pi)|)$, since $W$ is a surface homotopic to $\Gamma$ and $\chi(\Gamma) = -|C|$. The surface $W$ can be thought of as gluing an annulus from each $l_j$ and $l_k$ to the corresponding edges of $\Gamma$ in $S^2 \times \{0\}$.

$S^3 \setminus (S^2 \times I)$ is a pair of 3-balls $B_+$ and $B_-$ such that $\partial B_+ = S^2 \times \{1\}$ and $\partial B_- = S^2 \times \{-1\}$. To each loop $l_j$ of $\sigma_+(\pi)$, attach a disk $D_j \subset B_+$, and to each loop $l_k$ of $\sigma_-(\pi)$, attach a disk $D_k \subset B_-$, so that all the disks are disjoint from each other.

The **Turaev surface** associated to $\pi(L)$ is the closed orientable surface $F$ obtained by gluing all the disks $D_j$ and $D_k$ to $W$. It follows that $\Gamma$ is embedded in $F$ such that $F \setminus \Gamma$ consists of disks. By remembering the crossing information, we obtain from $\Gamma$ an alternating projection of $L$ onto $F$. Note that $\chi(F) = |\sigma_+(\pi)| + |\sigma_-(\pi)| - |C|$. 


Theorem 1.16 (Dasbach-Futer-Kalfagianni-Lin-Stoltzfus [18]). Let $\pi(L)$ be a planar diagram for a link $L$. Then the Turaev surface $F$ corresponding to $\pi(L)$ is Heegaard in $S^3$ and $L$ has an alternating projection $\pi'(L)$ onto $F$, where the regions $F \setminus \pi'(L)$ are disks.

As for $F$-alternating link projections, the checkerboard surfaces associated to a Turaev projection are not necessarily essential in $X$. However, in the case of adequate links we can show that the checkerboard surfaces are $\pi_1$-essential. The author would like to thank Ana Dow [19] for the following observation:

Theorem 1.17. Let $\pi(L)$ be an adequate diagram for a link $L$, and let $F$ be the Turaev surface associated to $\pi(L)$. Then $L$ has an $F$-alternating projection $\pi'(L)$ onto $F$, such that both the checkerboard surfaces associated to $\pi'(L)$ are $\pi_1$-essential in $X$.

Proof. It follows from Theorem 1.16 that $L$ has an alternating projection $\pi'(L)$ onto $F$, and that the checkerboard surfaces exist. All the regions of $F \setminus \pi'(L)$ are disks by construction. The checkerboard surfaces are isotopic to the two state surfaces coming from the positive and negative smoothings, which by Theorem 1.15 are $\pi_1$-essential in $X$. 

The Turaev genus $g_T(L)$ of a link $L$ is defined to be the minimal genus of a Turaev surface over all planar diagrams of $L$. The Turaev projection is a generalisation of Adam’s construction of producing an $F$-alternating projection from an $m$-almost alternating projection. In the Turaev construction, we can push a whole subtangle of $\pi(L)$ through to the back of a handle, whereas Adam’s’ construction requires each subtangle to consist of a single crossing. Hence, the dealternating number is an upper bound for Turaev genus.

Theorem 1.18 (Dasbach-Futer-Kalfagianni-Lin-Stoltzfus [18]). A link $L$ is alternating on $S^2$ if and only if $g_T(L) = 0$.

It remains difficult to calculate the Turaev genus of a given link $L$ if $g_T(L)$ is suspected of being at least 2. It is known exactly for $m$-semi-alternating
links which have Turaev genus $m$ [1], and for $(p, 3)$-torus knots $K$, where $g_T(K) = k$ when $p$ is $3k + 1$ or $3k + 2$ [2].

1.5 Embedded Surface Links

In this section, we present links which have embeddings into closed orientable surfaces. In the case where the genus of the surface is at most two, these links have been well-studied and are known as torus links and double torus links. While we do not consider them to be alternating projections, these embeddings will be useful for constructing interesting examples in Section 3.10.

The unknot is the only non-split link which embeds in the 2-sphere, since the unknot is characterised as being the only knot which bounds an essential disk in $X$. 

Figure 1.16: A $F$-alternating diagram of the knot $10_{153}$ onto the Turaev surface associated to the diagram $\pi(K)$ from Figure 1.15. The checkerboard surfaces are $\pi_1$-essential since they are isotopic to the state surfaces $\Sigma_{\sigma_+}$ and $\Sigma_{\sigma_-}$. 

A torus link $L$ is a non-split link which embeds into the Heegaard torus $T$ of $S^3$. $L$ is isotopic to a $(p, q)$-curve on such a torus, which means that $L$ represents the element $p[\mu] + q[\lambda]$ of $\pi_1(T)$, where $\mu$ and $\lambda$ are the standard meridian and longitude on $T$. If $p$ and $q$ are coprime, then this is a knot $K$, and $T \setminus K$ is an annulus. Otherwise the number of components of $L$ is the greatest common divisor of $p$ and $q$, and $T \setminus L$ is a collection of annuli.

The $(p, q)$-torus knot is isotopic to the $(q, p)$-torus knot so we assume that $p \geq q$. If $q = 1$, then $K$ is the unknot. If $q > 3$, then the only $\pi_1$-essential surfaces in $X$ are the Seifert surface and the winding annulus $T \setminus K$. If $q = 2$, then $K$ is alternating, and there is also a $\pi_1$-essential Mobius band which is double covered by the winding annulus.

A cable knot is a satellite knot where the pattern is a torus knot. Hill [38] considered double torus links. These are links $L$ which embed into the Heegaard genus two surface $F$. Let $W$ and $W'$ be the genus two handlebodies bounded by $F$. Let $\alpha$ be any separating essential curve on $F$ which bounds disks in both $W$ and $W'$. In order to be classed as a double torus link Hill’s definition includes a clause which states that $L$ cannot be isotoped in $F$ to be disjoint from $\alpha$. Otherwise $L$ is a disjoint union of torus links and unlinks.

Hill shows that if we cut along $L$, then the pieces of $F \setminus L$ can be classified as falling into five topological categories. In particular, if $L$ is a knot $K$, then there are only two possibilities. Either $F \setminus K$ is a twice punctured genus two surface, or $F \setminus K$ consists of a pair of once-punctured tori. The former case we will describe as a non-separating double torus knot and the latter as a separating double torus knot. Non-separating double torus knots were further investigated by Hill and Murasugi [39].

Separating double torus knots include all genus one alternating knots and length three pretzel knots where each tangle contains an odd number of crossings. Non-separating double torus knots include genus one bridge one knots which in turn include all rational knots.

For links, $F \setminus L$ can also decompose into a punctured torus and thrice-punctured sphere, two thrice-punctured spheres, or a four-punctured sphere. For links, all five cases can also contain any number of annular components,
Figure 1.17: An embedding of $4_1$ as a separating double torus knot.

Figure 1.18: An embedding of $4_1$ as a non-separating double torus knot.
and consequently all five types can be either separating or non-separating.

Hill [38] showed that if \( K \) is a separating double torus knot embedded in the Heegaard genus 2 surface \( F \) with certain conditions on the way \( K \) intersects \( F \setminus \alpha \), then both components of \( F \setminus K \) are incompressible in \( X \), and \( K \) is a non-trivial knot.

Norwood [71] studied curves embedded in surfaces of arbitrary genus. He decomposes the genus \( g \) orientable surface \( F \) into \( 3(g - 1) \) cylinders and \( 2(g - 1) \) pairs of pants, where \( g \geq 2 \). An embedded curve on \( F \) is then described by a twist/intersection number on each cylinder.

Ozawa defined the representativity \( r(K, F) \) of a curve \( K \) embedded in a surface \( F \) to be the minimal number of intersections between \( K \) and \( \gamma \), where \( \gamma \) ranges over the boundaries of all compressing disks for \( F \). This gives a criterion to decide when the complement to a curve in a surface is incompressible or essential.

**Theorem 1.19** (Ozawa [73]). Let \( F \) be a closed orientable surface embedded in \( S^3 \) and let \( K \) be a knot embedded in \( F \).

1. If \( r(K, F) \geq 1 \), then \( F \setminus K \) is incompressible in \( X \).

2. If \( r(K, F) \geq 2 \), then \( F \setminus K \) is incompressible and boundary-incompressible in \( X \).

We define an embedded surface link to be a link \( L \) embedded in a closed orientable surface \( F \) such that \( r(L, F) \geq 1 \), and no component of \( L \) bounds a disk in \( F \). We do not require that \( F \) is Heegaard.

Embedding knots in surfaces such that the complement of the surface is essential is an important open problem. The original Neuwirth Conjecture [70] was an algebraic statement about how the knot group splits as a non-trivial free product with amalgamation. It was solved by Culler and Shalen [15], who showed that for any non-trivial knot \( K \), there exists a separating, orientable essential surface in the knot exterior \( X \). Neuwirth also made the following stronger geometric conjecture.
Conjecture 1.20 (Neuwirth Conjecture [70]). Suppose that $K$ is a non-trivial knot. Then there exists a closed orientable surface $F$ in $S^3$ such that $K$ can be embedded in $F$, $F \cap X$ is essential in $X$, and $F \setminus K$ is connected.

It is known that all knots up to 11 crossings satisfy the Neuwirth conjecture except for possibly $11_{118}$ and $11_{126}$. See the paper of Ozawa and Rubinstein [75] for details and several related conjectures, of which the following has particular relevance for us.

Conjecture 1.21 (Strong Neuwirth Conjecture). Suppose that $K$ is prime, non-trivial, and not a torus knot. Then $K$ bounds a non-orientable spanning surface which is $\pi_1$-essential in $X$.

The Strong Neuwirth Conjecture implies the Neuwirth Conjecture. Suppose $K$ bounds a $\pi_1$-essential non-orientable spanning surface $\Sigma$. Then its double cover $\hat{\Sigma} = \partial N(\Sigma)$ is orientable and has two parallel boundary components which have integral slope on $\partial X$. $\partial \hat{\Sigma}$ bounds an annulus in $N(K) = S^3 \setminus X$ which contains $K$. Gluing this annulus to $\hat{\Sigma}$ produces the desired closed surface $F$.

We will introduce a new class of knots in Chapter 3 and show that they satisfy the Strong Neuwirth Conjecture in Theorem 3.16.

As this thesis was about to be submitted, Dunfield [23] posted a preprint containing a counterexample to Conjecture 1.21. He produced a hyperbolic knot $K$ which does not bound a non-orientable $\pi_1$-injective spanning surface, however $K$ does satisfy Conjecture 1.20.
Chapter 2

Generalised Alternating Links

Generalised alternating knots were introduced by Ozawa. We give an overview of his results in the first section of this chapter.

The focus of this chapter is to study the structure of generalised alternating link projections, and describe an algorithm which can enumerate all generalised alternating projections onto the torus. This will be based on the work of Nakamoto on quadrangulations of surfaces, and is similar in spirit to the enumeration of planar link diagrams by Conway. We also produce examples of generalised alternating projections onto higher genus surfaces.

Lastly, we introduce boundary slopes, and discuss the connection with generalised alternating knots. We show that there exist knots which do not admit any generalised alternating projections, and we show that there exist alternating knots whose spanning slope diameter is greater than twice the crossing number.
2.1 Definition of Generalised Alternating

A link \( L \) is \textit{generalised alternating} if it has a projection onto a closed orientable embedded surface \( F \),

\[
\pi : F \times I \to F,
\]

where \( L \subset F \times I \subset S^3 \) such that:

1. \( \pi(L) \) is alternating on \( F \), and

2. \( \pi(L) \) is prime.

A link projection \( \pi(L) \) onto a surface \( F \) is \textit{prime} if given any loop \( \ell \subset F \), such that \( \ell \) intersects \( \pi(L) \) transversely exactly twice, then \( \ell \) bounds a disk \( D \subset F \), such that \( D \) contains only a single unknotted arc of \( \pi(L) \). This definition is equivalent to the usual definition of prime for a diagram on \( S^2 \), however in the case of a higher genus projection surface \( F \), there exist loops \( \ell \subset F \) that do not bound disks in \( F \).

Define \( e(\pi(L), F) \) to be the minimum number of intersections between \( \ell \) and \( \pi(L) \) where \( \ell \) ranges over all essential loops in \( F \), such that \( \ell \) misses the crossings of \( \pi(L) \). We call \( e(\pi(L), F) \) the \textit{edge-representativity} of \( \pi(L) \) in \( F \). This is similar to the definition of edge-representativity from topological graph theory.

The \textit{representativity} of \( \pi(L) \) in \( F \), written \( r(\pi(L), F) \), is defined to be the minimum number of transverse intersections between \( \pi(L) \) and \( \ell \), where \( \ell \) ranges over the set of boundaries of all compressing disks for \( F \). If \( L \) is embedded in a surface \( F \) we may write \( r(L, F) \), which is how we defined representativity in Section 1.5.

Clearly \( r(\pi(L), F) \geq e(\pi(L), F) \). Let \( \Gamma \) be the 4-regular projection graph associated to \( \pi(L) \). Note that edge-representativity depends only on the embedding of \( \Gamma \) in \( F \), whereas the representativity also depends on the embedding of \( F \) in \( S^3 \).
2.1. DEFINITION OF GENERALISED ALTERNATING

In Figure 2.1, we present an example of a generalised alternating diagram on the torus. This is the non-alternating knot $10_{165}$. Ozawa proved the following results about generalised alternating links:

**Theorem 2.1** (Ozawa [73]). Let $\pi(L)$ be a generalised alternating projection of the link $L$ onto the closed orientable surface $F \subset S^3$. Then we have the following:

1. $F \setminus \pi(L)$ consists of open disks.
2. $F \setminus \pi(L)$ admits a checkerboard colouring.
3. Both checkerboard surfaces are connected. Let one of them be $\Sigma$.

4. $L$ can be isotoped into $\partial N(\Sigma)$, where $N(\Sigma)$ is the regular neighbourhood of $\Sigma$ in $S^3$.

5. $r(L, \partial N(\Sigma)) \geq 2$.

6. $L$ is non-split.

Proof. We will only provide a proof for the first three parts. Suppose some region $R$ of $F \setminus \pi(L)$ is not simply-connected. Then $R$ contains an essential loop $\ell$ which does not meet $\pi(L)$. If $\ell$ is essential in $F$, then part of $\ell$ can be isotoped in $F$ so it meets $\pi(L)$ exactly twice. But $\ell$ does not bound a disk in $F$, a contradiction to the primality of $\pi(L)$.

If $\ell$ is trivial in $F$, then $\pi(L)$ is disconnected and $\ell$ bounds a disk $D \subset F$. Then $\ell$ can be isotoped to a loop $\ell' \subset F$ so that $\ell'$ meets $\pi(L)$ exactly twice, and $\ell'$ bounds a disk $D' \subset F$ such that $D \subset D'$. But now $D'$ contains more than just a single embedded arc of $\pi(L)$, which cannot happen since $\pi(L)$ is prime.

Fix an orientation on $F$. Let $R$ be a disk region of $F \setminus \pi(K)$ and induce an orientation on $R$ from the orientation on $F$. This induces an orientation onto each segment of $\pi(L)$ adjacent to $R$. Since $\pi(L)$ is alternating on $F$, either every segment is oriented from an over crossing to an under crossing, or vice versa. These two cases give rise to a 2-colouring of the regions of $F \setminus \pi(L)$, where all the adjacent regions have different colours.

Suppose one of the checkerboard surfaces $\Sigma$ is not connected. Then there exists an embedded set of $k \geq 1$ curves $\ell_1, \ldots, \ell_k \subset F$ which are disjoint from $\Sigma$ and each component of $F \setminus \bigcup_{i=1}^k \ell_i$ contains part of $\pi(L)$. But then $\ell_1$ lies in a non-disk region of $F \setminus \pi(L)$, a contradiction to part 1.

He also has some extra results in the case of knots:

**Theorem 2.2** (Ozawa [73]). Let $\pi(K)$ be a generalised alternating diagram of the knot $K$ on the closed orientable surface $F \subset S^3$. Then we have the following:
2.1. DEFINITION OF GENERALISED ALTERNATING

1. At least one of the checkerboard surfaces relative to $F$ is non-orientable. Call it $\Sigma$.

2. $K$ can be isotoped into $\partial N(\Sigma)$ so that $\partial N(\Sigma) \setminus K$ is connected.

3. $r(K, \partial N(\Sigma)) \geq 2$.

4. $K$ is non-trivial.

Proof. We prove the first statement. Since an alternating projection has at least one crossing, the checkerboard surfaces must have different boundary slopes. Hence at least one of the checkerboard surfaces has a non-zero boundary slope and must be non-orientable since its boundary represents a non-trivial element of $H_1(X; \mathbb{Z})$. This thesis presents boundary slopes in Section 2.7.

Theorem 2.3 (Ozawa [73]). Let $\pi(L)$ be a generalised alternating projection of a link $L$ onto a closed orientable surface $F$. Then both of the checkerboard surfaces for $L$ relative to $F$ are $\pi_1$-essential in $X$.

This theorem follows from Part 5 of Theorem 2.1 and Theorem 1.19. In Chapter 3, we will extend this theorem to a larger class of links. The proof of Theorem 3.15 will provide an alternative proof to Theorem 2.3.

We now outline some other properties of generalised alternating projections which are implicit in Ozawa’s theorems, but we spell out in detail, since they will directly generalise in Chapter 3.

Lemma 2.4. Let $\pi(L)$ be a generalised alternating projection of a link $L$ onto a closed embedded surface $F \subset S^3$. Then:

1. $\pi(L)$ is separating in $F$.

2. $e(\pi(L), F) \geq 4$.

Proof. If some essential loop $\ell \subset F$ meets $\pi(L)$ an odd number of times, it contradicts the fact that $F \setminus \pi(L)$ admits a checkerboard colouring. Hence every essential loop $\ell \subset F$ meets $\pi(L)$ an even number of times, which is equivalent to $\pi(L)$ separating $F$. Therefore $e(\pi(L), F)$ is even.
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Suppose that \( e(\pi(L), F) = 2 \). Then there is an essential loop \( \ell \) which meets \( \pi(L) \) twice. However \( \ell \) does not bound a disk in \( F \) and hence \( \pi(L) \) cannot be prime, a contradiction.

Suppose that \( e(\pi(L), F) = 0 \). Let \( \ell \) be an essential loop in \( F \) which does not meet \( \pi(L) \). Then \( \ell \) lies within a region of \( F \setminus \pi(L) \) that is not a disk, a contradiction. Together this establishes that \( e(\pi(L), F) \geq 4 \).

Given a alternating projection \( \pi(L) \) onto a closed orientable surface \( F \), the easiest way to test whether it is generalised alternating is to apply the checkerboard colouring to the regions of \( F \setminus \pi(L) \). Then \( \pi(L) \) is generalised alternating if every region is a disk, and for each black region, each of its adjacent white regions are distinct. Otherwise there would be a loop \( \ell \subset F \), which only met \( \pi(L) \) twice but is either essential or bounds a disk \( D \subset F \) containing a crossing.

We note that Hayashi had previously studied generalised alternating projections onto Heegaard surfaces, however he used somewhat different terminology to Ozawa. Stating his results in our language gives the following theorems:

**Theorem 2.5** (Hayashi [35]). If a link \( L \) is generalised alternating, then \( L \) is non-split.

**Theorem 2.6** (Hayashi [35]). Let \( \pi(L) \) be a generalised alternating projection onto a Heegaard surface \( F \). If \( e(\pi(L), F) \geq 6 \), then \( X \) is atoroidal.

2.2 Arborescent Tangles and Basic Polyhedra

An \( n \)-tangle is a 3-ball \( B \) containing \( n \) disjoint properly embedded arcs. An \( n \)-tangle is trivial if the arcs are isotopic into the boundary sphere \( \partial B \). A trivial 2-tangle is also known as a rational tangle.

We often think of a rational tangle in terms of its diagrams. A diagram of a rational tangle, is a projection of the 3-ball \( B \) containing two properly embedded arcs, onto a disk \( D \subset S^2 \) such that the ends of the arcs project
2.2. ARBORESCENT TANGLES AND BASIC POLYHEDRA

to the NW, NE, SE, and SW points of $\partial D$. A rational link is formed by adding embedded arcs in $S^2 \setminus D$ between the NW end and the NE end, and between the SW end and the SE end.

Rational tangles get their name since each tangle can be associated to an element of $\mathbb{Q} \cup \{\infty\}$. There are two trivial 2-tangle diagrams which are denoted by 0 and $\infty$. A range of rational tangles and their sign conventions are depicted in Figure 2.2. We write $K(p/q)$ for the rational link that is the above described closure of the tangle associated to $p/q$.

![Figure 2.2: Some rational tangles.](image)

A rational link can also be thought of as gluing two trivial 2-tangles by a homeomorphism of their boundaries. The 4-punctured sphere $\partial B \setminus L$ of a rational tangle is compressible, since there is a properly embedded disk in $B \setminus L$ which separates the two embedded arcs.

All rational links are alternating. Rational links are also known as two-bridge links. Hodgson and Rubinstein gave a topological characterisation of rational links. This theorem is in the same spirit as the non-diagrammatic characterisations of surface-alternating links that we will prove in subsequent chapters.
Theorem 2.7 (Hodgson-Rubinstein [40]). A link \( L \) is rational if and only if its double branched cover is a Lens space.

Figure 2.3: The knot 6\(_3\) as the rational knot \( K(\frac{13}{5}) \).

Let \( \{ R_i \mid i = 1, \ldots, n \} \) be a collection of non-trivial rational tangles. Align the tangles along the equator of \( S^2 \) from west to east. We form a Montesinos tangle \( M \), by connecting the \( NE \) end of \( R_i \) to the \( NW \) end of \( R_{i+1} \), and connecting the \( SE \) end of \( R_i \) to the \( SW \) end of \( R_{i+1} \), for \( i = 1, \ldots, n - 1 \).

A Montesinos link is formed by adding embedded arcs in \( S^2 \setminus M \) between the \( NW \) end of \( R_1 \) and the \( NE \) end of \( R_n \), and between the \( SW \) end of \( R_1 \) and the \( SE \) end of \( R_n \). The length of a Montesinos link is \( n \), provided none of the \( R_i \) are integral tangles. A Montesinos link can be written as \( K(r_1, \ldots, r_n) \), where \( r_i \) is the fraction associated to the rational tangle \( R_i \).

Figure 2.4: The knot 9\(_{44}\) as the Montesinos knot \( K(\frac{2}{5}; \frac{2}{3}, -\frac{1}{2}) \).
A rational link is a Montesinos link of length 1 or 2. A pretzel link is a special case of a Montesinos link where each rational tangle $R_i$ has the form $\frac{1}{k_i}$ for some $k_i \in \mathbb{Z} \setminus \{0\}$. The double branched cover of a Montesinos link is a Seifert fibred space.

Let $\{M_i|i=1,\ldots,n\}$ be a collection of Montesinos tangles. We form an arborescent tangle $A$ inductively. Project $M_1$ onto $S^2$ and let $A_1 = M_1$. Project $M_i$ onto $S^2 \setminus A_{i-1}$. Then let $A_i$ be formed from $A_{i-1}$ and $M_i$ by connecting two adjacent ends of $A_{i-1}$ to two adjacent ends of $M_i$. The desired arborescent tangle $A$ is given by $A_n$. More generally, it is possible to start with a collection of arborescent tangles, and use this process to construct a larger arborescent tangle.

An arborescent link is either of the two choices of the closure of an arborescent tangle. Arborescent links are also known as algebraic links.

Conway [13] developed a method for enumerating all planar projections of knots, by replacing the vertices of a 4-regular planar graph with arborescent tangles. He defined a polyhedron to be an edge-connected, 4-regular planar graph. A basic polyhedron is a polyhedron, in which every face has degree at least 3.

A graph consisting of a single vertex and $n$ loops is known as the bouquet graph $B_n$. We will be particularly concerned with the embedding of $B_2$ in $S^2$, which we also denote by $B_2$. 
Let $\pi(L)$ be a reduced diagram on $S^2$ of a non-split non-trivial link $L$, and let $\Gamma$ be the corresponding projection graph obtained by forgetting crossing information. $\Gamma$ has at least two vertices. If $\Gamma$ contains a digon face $\phi$, then we obtain a new 4-regular planar graph by identifying the two vertices bounding $\phi$ and deleting both its edges. If $\Gamma$ contains a monogon face $\phi'$ with vertex $v$, then we obtain a new 4-regular graph by deleting $v$ and all its four edges, and adding a edge between the vertices adjacent to $v$. Not that the two neighbours of $v$ may in fact be the same vertex $v'$. These reduction operations are shown in Figures 2.7 and 2.8, and called digon collapse and monogon collapse respectively.

![Figure 2.7: A digon collapse.](image)

![Figure 2.8: A monogon collapse.](image)

Continue reducing $\Gamma$ until there are no digon faces or there is only one vertex. If the resulting graph has more than one vertex, it is a basic polyhedron $P$ and $\pi(L)$ can be reconstructed from $P$ by replacing the vertices with appropriate arborescent tangles. If the resulting graph has only one vertex,
2.3. OPERATIONS ON 4-REGULAR GRAPHS

it is $B_2$, and $\pi(L)$ can be reconstructed by replacing the vertex with a single arborescent tangle.

All non-split planar link projections can be obtained by replacing the vertices of either a basic polyhedron or $B_2$ with arborescent tangles. A rational, Montesinos or arborescent link projection can be thought of as replacing the single vertex in $B_2$ with a rational, Montesinos or arborescent tangle respectively.

Conway [13] used his notation for rational tangles, to enumerate all knots up to 11 crossings. As will be described in the following section, Bridge- man [10] discovered a method for enumerating all basic polyhedra.

Bonahon and Siebenmann [8] have also made a detailed, albeit unpublished study of arborescent links, where they study link exteriors by cutting them up along essential Conway spheres. Another reference is the thesis of Caudron [11].

A Conway sphere for a link $L$ is a sphere $S \subset S^3$ which transversely intersects $L$ exactly four times. A Conway sphere bounds a 2-tangle and possibly some closed components of $L$ on each side. A Conway sphere is essential if $S \cap X$ is essential in $X$. This implies that $S$ does not bound a rational tangle on either side, since the two arcs of a rational tangle can be separated by a compressing disk for $S \cap X$. All Montesinos links of length at least four, contain an essential Conway sphere.

2.3 Operations on 4-Regular Graphs

An embedding of a graph $G$ in a closed surface $F$ is an injective map

$$f : G \to F.$$ 

Let $\Gamma$ denote the image of $G$ under $f$.

We refer to a component of $F \setminus \Gamma$ as a face of $\Gamma$. An embedding is a 2-cell embedding if all the faces are homeomorphic to disks.

Let $h : F \to F$ be a homeomorphism of the closed orientable surface $F$, such that $h(\Gamma) = \Gamma'$ induces a graph isomorphism between $\Gamma$ and $\Gamma'$. Then
we say that $h$ is a graph homeomorphism between $\Gamma$ and $\Gamma'$.

Lickorish [56] has shown that the homeotopy group of a closed orientable surface is generated by a finite number of Dehn twists. Hence a graph homeomorphism of $\Gamma$ corresponds to a series of Dehn twists along essential loops in $F$, possibly composed with an orientation-reversing involution of $F$.

See the two graphs in the top row of Figure 2.20 for an example to two embedded graphs which are isomorphic as abstract graphs, but are not graph homeomorphic. Both are these graphs are isomorphic to the complete bipartite graph $K_{4,4}$, however this isomorphism is not induced by a homeomorphism of the torus.

A graph $\Gamma$ is $k$-edge-connected if it is still connected after the removal of any $k - 1$ edges. A graph $\Gamma$ is separating on a closed surface $F$, if every essential loop $\ell$ on $F$ that misses vertices, meets $\Gamma$ an even number of times. Note that a separating 4-regular graph is connected if and only if it is 2-edge-connected. Similarly, a separating 4-regular graph is 3-edge-connected if and only if it is 4-edge-connected.

A graph $\Gamma$ is defined to be $k$-edge-representative in a closed surface $F$, if every essential loop $\ell$ on $F$ meets $\Gamma$ at least $k$ times, where $\ell$ is isotoped to miss vertices of $\Gamma$. This is where our definition of edge-representativity for link projections onto surfaces comes from, and we will also write $e(\Gamma, F)$ for the edge-representativity of $\Gamma$ in $F$. We make the convention that a planar graph automatically satisfies any edge-representativity requirement placed on it, since there are no essential loops on the 2-sphere. In the graph theory literature, edge-representativity is also referred to as face-width.

We define a generalised basic polyhedron to be a separating, simple, 4-edge-connected, 4-edge-representative, 4-regular graph, 2-cell-embedded in a closed orientable surface $F$.

**Theorem 2.8.** Let $\pi(L)$ be a generalised alternating projection onto a closed orientable surface $F$. Forgetting crossing information, $\pi(L)$ is graph homeomorphic to a separating, 4-edge-connected, 4-edge-representative, 4-regular graph $\Gamma'$ 2-cell embedded in $F$. Furthermore, if $\Gamma$ is the graph obtained by collapsing all digons on $\Gamma'$ until none remain, then $\Gamma$ is a generalised basic polyhedron.
2.3. OPERATIONS ON 4-REGULAR GRAPHS

Proof. By Theorem 2.1, the regions of $F \setminus \pi(L)$ are homeomorphic to disks and are 2-colourable. Hence the faces of $F \setminus \Gamma'$ are disks, so $\Gamma'$ is separating and 2-cell embedded in $F$.

By definition $e(\Gamma', F) = e(\pi(L), F) \geq 4$. If $\ell$ is a loop in $F$ which meets the edges of $\Gamma$ exactly $k < 4$ times, then $\ell$ meets the edges of $\Gamma'$ exactly $k$ times, since the process of adding the digons back into $\Gamma$ to obtain $\Gamma'$ happens away from $\ell$. Thus $e(\Gamma, F) \geq 4$.

Suppose $\Gamma'$ is not 4-edge-connected. $\Gamma'$ is connected since $\pi(L)$ is non-split by Theorem 2.1. Then there exists a separating curve $\alpha \subset F$ which misses the vertices of $\Gamma'$ and intersects at most three edges of $\Gamma'$ such that deleting those three edges disconnects $\Gamma'$. But $\Gamma'$ is separating so $\alpha$ must intersect $\Gamma'$ exactly twice in different edges. It follows that $\alpha$ is trivial in $F$ by 4-edge-representativity. Let $D$ be the subdisk of $F$ bounded by $\alpha$. Then $D$ contains at least one vertex of $\Gamma'$, and therefore at least one crossing of $\pi(L)$ contradicting primality.

There cannot be a pair of non-separating curves on $F$ which together separate $\Gamma'$ and meet $\Gamma'$ less than four times combined, since applying 4-edge-representativity, one of them would not meet $\Gamma'$ at all.

It remains to show that $\Gamma$ is simple. There are no digon faces of $\Gamma$ since they have all been collapsed. If there is a monogon face in $\Gamma$, then $\pi(L)$ was not prime.

Suppose $\Gamma$ contains a loop $e$ which connects a vertex $v$ to itself. Then $e \cup v$ is equivalent to an essential closed curve $\ell \subset F$. Let $\beta$ be a component of $\partial N(\ell)$ where $N(\ell)$ is an annulus. But $\Gamma$ is separating so $\beta$ meets the edges of $\Gamma$ either twice or not at all, both of which contradict $\Gamma$ being 4-edge-representative.

Suppose $\Gamma$ has multiple edges $e$ and $e'$ between vertices $v$ and $v'$. Then $e \cup v \cup e' \cup v'$ is equivalent to a closed curve $\ell \subset F$. Let $\beta$ be one of the components of $\partial N(\ell)$. If $\beta$ intersects $\Gamma$ exactly twice or not at all, then $\Gamma$ is not 4-edge-representative. Otherwise $\beta$ intersects $\Gamma$ four times, in which case the other component of $\partial N(\ell)$ does not intersect $\Gamma$ at all, again a contradiction.

Thus $\Gamma$ has neither loops nor multiple edges and is therefore simple. □
Our goal is to enumerate all generalised basic polyhedra from which it will be possible to enumerate all generalised alternating link projections. We will first review the work of Bridgeman who has a method to enumerate all basic polyhedra on $S^2$.

Let $\Gamma$ be an embedding of a 4-regular graph in a closed surface $F$. We define an operation on $\Gamma$ called surgery. Choose a face of $\Gamma$ that is bounded by an $n$-cycle, $(v_1, \ldots, v_n), n \geq 4$. Choose any pair of non-adjacent edges $v_iv_{i+1}, v_jv_{j+1}$. We perform surgery on $\Gamma$ by adding a vertex $v$, removing the edges $v_iv_{i+1}$ and $v_jv_{j+1}$, and adding the edges $vv_i, vv_{i+1}, vv_j, vv_{j+1}$, and call the resulting graph $\Gamma^+$, which has one more vertex than $\Gamma$.

![Figure 2.9: Surgery on a 4-regular graph.](image)

We can undo surgery by cutting open at a vertex $v$. At each vertex, there are two ways to do this, however we do not proceed if this will result in the creation of a digon. A digon is a face bounded by a 2-cycle.

![Figure 2.10: Cutting open a 4-regular graph at $v$. Note that the other way of cutting open would produce digons.](image)
Let $\Gamma$ and $\Gamma'$ be 4-regular graphs embedded in closed orientable surfaces $F$ and $F'$. We form the vertex sum of $\Gamma$ and $\Gamma'$ by removing a neighbourhood of a vertex from each graph, and gluing the resulting half-edges together. Choose vertices $u \in \Gamma$ and $v \in \Gamma'$. Label the vertices adjacent to $u$ in $\Gamma$ as $u_1, u_2, u_3, u_4$ in a cyclic anti-clockwise order, and label the vertices adjacent to $v$ in $\Gamma'$ as $v_1, v_2, v_3, v_4$ in a cyclic clockwise order. Then the vertex sum $\Gamma \oplus \Gamma'$ is formed by adding the edges $u_iv_i$ for $i = 1, 2, 3, 4$ to $(\Gamma \setminus u) \sqcup (\Gamma' \setminus v)$. If $F$ has genus $g$ and $F'$ has genus $g'$, then $\Gamma \oplus \Gamma'$ embeds in a surface $F \# F'$ of genus $g + g'$.

![Figure 2.11: A vertex sum of two 4-regular planar graphs.](image)

The octahedral graph $O$ is the planar 4-regular graph with 6 vertices, 12 edges, and 8 triangular faces pictured in Figure 2.12. We call a vertex sum with the octahedral graph, an octahedral sum. The reverse of an octahedral sum, which involves the removal of a 4-cycle, which bounds a disk containing
a single vertex of degree 4, is called *octahedral removal*.

![Figure 2.12: The octahedral graph O.](image)

![Figure 2.13: The octahedral removal operation. The reverse operation is an octahedral sum.](image)

We call the non-simple 4-regular graph shown in Figure 2.14 the *trefoil graph*, and vertex sum with the trefoil graph is called *trefoil sum*. Note that trefoil sum is the reverse operation to a digon collapse.

We say that a generalised basic polyhedron $\Gamma$ embedded in a closed orientable surface is *initial* if it is not possible to cut open at any vertex or remove an octahedron without destroying simpleness, 4-edge-connectivity, or 4-edge-representativity.
On the 2-sphere, this means that a 4-regular graph $\Gamma$ is initial if it is not possible to cut open at any vertex or remove an octahedron without destroying simpleness or 4-edge-connectivity.

**Theorem 2.9** (Bridgeman [10]). The initial graphs for the 2-sphere are the anti-prisms $T_n$ for $n \geq 3$.

The anti-prism $T_n$ is constructed by taking two disjoint $n$-cycles in $S^2$ and triangulating the annular region between them by adding $2n$ edges to obtain a 4-regular planar graph. $T_3$ is graph homeomorphic to the octahedral graph $O$. 
CHAPTER 2. GENERALISED ALTERNATING LINKS

While Bridgeman did not use the same definition of initial as us, it follows that our result is equivalent since as we shall see in the following section, Theorem 2.9 is the dual statement to Theorem 2.12. Bridgeman has shown that it is possible to enumerate all basic polyhedra by performing sequences of surgeries and octahedral sums on the anti-prisms. It then follows from Conway [13] that we can enumerate all prime planar projections by either replacing each vertex of a basic polyhedron with an arborescent tangle, or closing up an arborescent tangle to get an arborescent link projection. Equivalently, we can enumerate all prime planar projection graphs by starting with $B_2$ or a basic polyhedron and taking a series of trefoil sums.

Note that Manca [58] had previously shown that all simple planar 4-regular graphs are obtained from $T_3$ by a sequence of surgeries, octahedral sum, and two other operations.

The first knot in the tables which is not arborescent is $8_{18}$. It’s minimal diagram is formed by replacing the vertices of $T_4$ with single crossing tangles in an alternating way. The knots $8_{16}, 8_{17}, 8_{19}, 8_{20}, 8_{21}$ have minimal crossing diagrams which are not arborescent, however they do admit arborescent diagrams, and the latter three admit Montesinos diagrams. All other prime knots with crossing number at most 8 are Montesinos knots.

We intend to construct a list of initial generalised basic polyhedra for each closed orientable surface. Then we will be able to construct all generalised basic polyhedra by a sequence of surgeries and octahedral sums.

2.4 Irreducible Quadrangulations

A quadrangulation $Q$ is a 2-cell embedding of a graph $G$ into a closed surface $F$ such that every face has degree 4.

A graph $G$ is bipartite if its vertex set can be decomposed into two disjoint sets, $X, Y$, such that any edge in $G$ has one end in $X$ and the other in $Y$. This is equivalent to saying that the vertices of $G$ are 2-colourable, or that $G$ contains no odd cycles.

From a graph $\Gamma$ embedded in a closed surface $F$, we can form its dual graph $\Gamma^*$. $\Gamma^*$ has one vertex for each face of $\Gamma$, and two vertices of $\Gamma^*$ are
adjacent if and only if the corresponding faces are adjacent in $\Gamma$. The faces of $\Gamma^*$ correspond to vertices of $\Gamma$. If $\Gamma$ is a 2-cell embedding, then $\Gamma^{**}$ is graph homeomorphic to $\Gamma$.

**Lemma 2.10.** Let $\Gamma$ be 2-cell embedded in a closed orientable surface $F$. Then $\Gamma$ is separating on $F$ if and only if $\Gamma^*$ is bipartite.

**Proof.** Suppose $\Gamma^*$ is not bipartite. Then $\Gamma^*$ contains an odd cycle. It is always possible to choose an odd cycle which passes through each vertex of $\Gamma^*$ at most once. This odd cycle is isotopic to an embedded loop $\ell \subset F$ that meets the edges of $\Gamma$ an odd number of times, so $\Gamma$ is not separating.

Conversely, suppose that $\Gamma$ is not separating on $F$ so that there exists a loop $\ell \subset F$ which meets the edges of $\Gamma$ an odd number of times. $\ell$ can be isotoped to lie entirely on the edges and vertices of $\Gamma^*$, in which case it is equivalent to a closed walk on the edges of $\Gamma^*$ of odd length. Thus $\Gamma^*$ is not bipartite. \hfill $\square$

Let $\phi$ be a face of a quadrangulation $Q$ bounded by the 4-cycle $uvu'v'$. A **face contraction** at $v, v'$ is the identification of the vertices $v$ and $v'$, and the identification of the edges $uv$ and $uv'$ and the edges $u'v$ and $u'v'$.

![Figure 2.16: A vertex splitting. The reverse operation is a face contraction.](image)

The opposite operation is called a **vertex splitting**. Let $w$ be a vertex of a quadrangulation $Q$ and choose two vertices $u$ and $u'$ adjacent to $w$. Replace $w$ with two vertices $v$ and $v'$ and insert a new quad face $\phi$ which is bounded by the 4-cycle $uvu'v'$.
If every vertex in a graph $G$ has degree at least $k$, then we write $\delta(G) \geq k$. We will be particularly interested in quadrangulations $Q$ where $\delta(Q) \geq 3$. To ensure that a vertex splitting preserves minimum degree 3, it is necessary to choose $u$ and $u'$ so that they are not consecutive in the cyclic ordering of vertices adjacent to $w$.

Another operation that can be performed on a quadrangulation $Q$ is a 4-cycle addition. Choose a 4-cycle $v_1v_2v_3v_4$ that bounds a face $\phi$ of $Q$. Add a 4-cycle $u_1u_2u_3u_4$ to the interior of $\phi$ and connect $u_i$ to $v_i$ with an edge for each $i = 1, 2, 3, 4$. The opposite operation is called 4-cycle removal.

The four operations just described result in a new quadrangulation of $F$ and preserve bipartiteness.

A quadrangulation $Q$ is irreducible if it is simple and any face contraction or 4-cycle removal would destroy simplesness. A quadrangulation $Q$ is 3-irreducible if $Q$ is irreducible with $\delta(Q) \geq 3$ and any face contraction or 4-cycle removal introduces a vertex of degree 2.

**Theorem 2.11** (Nakamoto [67]). Let $Q$ be a quadrangulation of a closed orientable surface $F$ such that $\chi(F) \leq 0$. Then $Q$ is irreducible if and only if $Q$ is 3-irreducible.

If $\chi(F) \leq 0$, then there are no vertices of degree 2 in an irreducible quadrangulation of $F$, since it is always possible to contract one of the adjacent
faces. In the planar case, the only irreducible quadrangulation is the 4-cycle $C_4$ \[69\].

**Theorem 2.12** (Nakamoto \[67\]). The 3-irreducible quadrangulations of $S^2$ are the double wheels $W_{2n}$, $n \geq 3$.

The double wheel $W_{2n}$ is constructed by starting with a $2n$-cycle $u_1v_1 \ldots u_nv_n$. Add two vertices $u$ and $v$, one on each side of the $2n$-cycle, and add the edges $uu_i$ and $vv_i$ for $i = 1, \ldots, n$. The double wheel $W_{2n}$ is dual to the anti-prism $T_n$. In particular, $W_6$ is graph homeomorphic to the cube and dual to the octahedral graph.

Since the operations of face contraction and 4-cycle removal are respectively dual to the operations of cutting open and octahedral removal, it can be seen that Theorem 2.12 is equivalent to Theorem 2.9 of Bridgeman.

**Theorem 2.13** (Nakamoto \[66\]). There are exactly five irreducible bipartite quadrangulations of the torus up to graph homeomorphism.

The five irreducible bipartite quadrangulations are depicted in Figure 2.19. There are also three irreducible non-bipartite quadrangulations of the torus, but they do not concern us here since generalised alternating projections are separating.
We aim to extend this equivalence of Theorem 2.12 and Theorem 2.9 to quadrangulations and 4-regular graphs of arbitrary closed orientable surfaces. Nakamoto has the following result.

**Theorem 2.14** (Nakamoto [67]). An embedding $Q$ is a simple quadrangulation of a closed surface $F$ with $\delta(Q) \geq 3$ if and only if the dual $Q^*$ is a 3-edge-connected, 3-edge-representative, 4-regular, simple graph on $F$.

**Theorem 2.15.** Let $F$ be a closed orientable surface of positive genus. Then $\Gamma$ is a generalised basic polyhedron in $F$ if and only if $\Gamma^*$ is a simple bipartite quadrangulation of $F$ with $\delta(\Gamma^*) \geq 3$.

*Proof.* Suppose that $\Gamma$ is a generalised basic polyhedron on $F$. Then $\Gamma$ is a separating 4-edge-representative 4-regular simple graph 2-cell embedded...
in a closed orientable surface $F$. It follows from Theorem 2.14 that $\Gamma^*$ is a simple quadrangulation of $F$ with $\delta(\Gamma^*) \geq 3$. Lemma 2.10 shows that $\Gamma^*$ is bipartite.

Conversely, suppose that $\Gamma^*$ is a simple bipartite quadrangulation of $F$ with $\delta(\Gamma^*) \geq 3$. Then Theorem 2.14 tells us that $\Gamma$ is a 3-edge-connected, 3-edge-representative, 4-regular, simple graph on $F$. Lemma 2.10 shows that $\Gamma$ is separating, so every loop $\ell \subset F$ meets $\Gamma$ away from vertices an even number of times and it follows that $\Gamma$ is 4-edge-representative and 4-edge-connected.

There are no digon or monogon faces of $\Gamma$ since $\delta(\Gamma^*) \geq 3$. There can be no loops or multiple edges in $\Gamma$, since otherwise they would be isotopic to an essential closed curve $\ell \subset F$. Then at least one component of $\partial N(\ell)$ would meet $\Gamma$ less than four times, contradicting 4-edge-representativity. Therefore $\Gamma$ is simple.

**Theorem 2.16.** Let $F$ be a closed orientable surface of positive genus. Then $\Gamma$ is an initial generalised basic polyhedron embedded in $F$ if and only if $\Gamma^*$ is an irreducible, bipartite quadrangulation of $F$.

**Proof.** Initial and irreducible are dual concepts, since it is possible to perform a face contraction or 4-cycle removal on $\Gamma^*$ without destroying simpleness if and only if it is possible to cut open a vertex or perform octahedral removal on $\Gamma$ without destroying 4-edge-representativity, 4-edge-connectivity or simpleness. Hence this theorem follows immediately from Theorem 2.15.

Theorem 2.16 gives us the following corollary to Theorem 2.13. These five initial generalised basic polyhedra are depicted in Figure 2.20.

**Corollary 2.17.** There are exactly five initial generalised basic polyhedra on the torus up to graph homeomorphism.

**Theorem 2.18** (Nakamoto [67]). Let $F$ be a closed surface with $\chi(F) \leq 0$. Then any quadrangulation $Q$ of $F$ with $\delta(Q) \geq 3$ can be obtained from an irreducible quadrangulation of $F$ by a sequence of vertex splittings and 4-cycle additions preserving minimum degree at least 3.
Corollary 2.19. Let $F$ be a closed orientable surface with $\chi(F) \leq 0$. Then any generalised basic polyhedron embedded in $F$ can be obtained from an initial generalised basic polyhedron for $F$ by a sequence of surgeries and octahedral sums.

We finish this section by presenting all non-initial generalised basic polyhedra for the torus up to graph homeomorphism with at most 10 vertices in Figure 2.21.
Figure 2.21: The non-initial generalised basic polyhedra of the torus with at most 10 vertices up to graph homeomorphism.
2.5 Enumeration of Generalised Alternating Projections

The final ingredient that we need to enumerate generalised alternating link projections is a way to enumerate embeddings of projection surfaces into the 3-sphere.

Up to ambient isotopy, there is only one embedding of $S^2$ into $S^3$. If $F$ is a torus embedded in $S^3$, then $F$ bounds a solid torus on at least one side, so embeddings of tori correspond to ambient isotopy classes of knots $K \subset S^3$, where $F \cong \partial N(K)$. Knots in $S^3$ are well-known to be enumerable.

When the genus of $F$ is at least two, the enumeration of surface embeddings is more complicated since $F$ does not necessarily bound a handlebody. If $F$ does bound a handlebody on at least one side, then an embedding of $F$ corresponds to a genus $g$ handlebody-knot. A genus $g$ handlebody-knot is by definition an embedding of the genus $g$ handlebody into the 3-sphere, where two handlebody-knots are equivalent if there is an ambient isotopy of $S^3$ which maps one homeomorphically onto the other. Ishii, Kishimoto, Moriuchi, and Suzuki [45] have enumerated the genus 2 handlebody-knots up to 6 crossings. This problem is closely related to the enumeration of spatial graphs in $S^3$, however there are non-isomorphic graphs whose embeddings correspond to equivalent handlebody-knots.

The following algorithm is impractical, but there does not appear to be a systematic study of surface embeddings in $S^3$ where the surface bounds a handlebody on neither side.

**Lemma 2.20.** There is an algorithm to enumerate embeddings of closed orientable surfaces into $S^3$.

**Proof.** Assume that it is possible to enumerate embeddings of closed orientable surfaces of genus $g$ into $S^3$. There is only one embedding of $S^2$ into $S^3$.

Let $F$ be a closed orientable surface of genus $g$ embedded in $S^3$. Let $\mathcal{T}$ be a simplicial triangulation of $S^3$ such that $F$ is contained within the 2-skeleton.
of $\mathcal{T}$. Let $Y$ and $Y'$ be the 3-manifolds that result by cutting $S^3$ along $F$ and let $\mathcal{T}_F$ and $\mathcal{T}_Y$ be the induced triangulations on $F$ and $Y$ respectively.

Let $\alpha$ be an embedded arc in the 1-skeleton of $\mathcal{T}_Y$ such that $\partial \alpha = \alpha \cap F$, and the link of $\alpha$ in $\mathcal{T}_Y$ is an annulus $A$ such that $\partial A = A \cap F$. Let $E$ be the star of $\partial \alpha$ in $\mathcal{T}_F$, so that topologically $E$ is a pair of embedded disks.

Let $F' = (F \setminus E) \cap A$. Then $F'$ is a closed orientable surface of genus $g + 1$ embedded in $S^3$. Let $\mathcal{T}^i$ be the $i^{th}$ barycentric subdivision of $\mathcal{T}$, a process which is clearly enumerable. For each $\mathcal{T}^i$, there are only finitely many embedded arcs $\alpha$ which satisfy the conditions in the previous paragraph, so there are only finitely many surfaces of genus $g + 1$ constructed from each barycentric subdivision of $F$.

Therefore, by induction it is possible to enumerate all embeddings of closed orientable surfaces into $S^3$. 

Now we are ready to give a procedure for enumerating generalised alternating link projections.

**Theorem 2.21.** There is an algorithm to enumerate all generalised alternating link projections:

1. Choose an initial generalised basic polyhedron $\Gamma_0$ for a closed orientable surface $F$.

2. Perform a sequence of surgeries and octahedral sums to obtain a generalised basic polyhedron $\Gamma$ for $F$.

3. Embed $F$ into $S^3$.

4. Perform a graph homeomorphism of $\Gamma$.

5. Replace the vertices of the graph with alternating arborescent tangles to form a link, in such a way that the alternating property extends globally.

**Proof.** Step 1 relies on being able to enumerate all initial generalised basic polyhedra for a given surface $F$. If $F$ is the 2-sphere or torus, then this list is known by Theorem 2.9 and Corollary 2.17. Methods for generating generalised basic polyhedra for higher genus surfaces will be discussed in
Section 2.6. The number of initial graphs for the 2-sphere is countable, while the number of initial graphs is finite for all other surfaces, since Nakamoto and Ota [68] have shown that there are only a finite number of irreducible quadrangulations for each surface.

Step 2 corresponds to Corollary 2.19. Note that there are only finitely many choices of surgery or octahedral sum that can be performed on a given generalised basic polyhedron.

The third step is taken care of by Lemma 2.20. Graph homeomorphisms of $\Gamma$ correspond to elements of the homeotopy group of $F$, which are enumerable.
since the homeotopy group is generated by Dehn twists along a finite number of essential loops \( \ell \subset F \) [56].

In Step 5, we choose an alternating arborescent tangle for each vertex. Arborescent tangles can be enumerated and, depending on the symmetry of the tangle, there are up to eight ways to replace the first vertex of \( \Gamma \) with a given arborescent tangle. Then for each subsequent vertex, there are up to four ways to insert a given arborescent tangle since we want the alternating property to extend globally to the link projection.

As an example of this process, consider the generalised alternating diagram of the knot 10\(_{159}\) depicted in Figure 2.22. This has been obtained by starting with the initial graph \( \Gamma \) at the top right of Figure 2.20. We have performed two surgeries on \( \Gamma \). The surface \( F \) has been embedded as a Heegaard torus for \( S^3 \) so that the red edges in Figure 2.22 are mapped to the canonical meridian and longitude of \( F \). One vertex has been replaced by a tangle with two crossings, the other nine vertices have been replaced by single crossing tangles.

Note that if reflect each arborescent tangle in the projection surface, we can obtain a generalised alternating projection of a different knot. In the example above, we get 11\(_a_{125}\).

### 2.6 Higher Genus Quadrangulations

We now consider quadrangulations of closed orientable surfaces with negative Euler characteristic. We show that all such surfaces admit generalised alternating projections and place a lower bound on the number of crossings which depends on the genus of the surface.

The examples of irreducible or 3-irreducible bipartite quadrangulations of the 2-sphere or the torus with the minimal number of crossings are \( r \)-regular for \( r = 2, 3, 4 \). We show that this is not necessarily the case for higher genus surfaces.

Suppose that \( Q \) is an \( r \)-regular quadrangulation of a closed orientable surface \( F \) of genus \( g 
eq 1 \). Then \( 2e = rv \), \( 2e = 4f \) and \( v - e + f = 2 - 2g \).
This implies that

\[ v = \frac{8(g - 1)}{r - 4}, e = \frac{4r(g - 1)}{r - 4}, f = \frac{2r(g - 1)}{r - 4}, \]

where \( r \neq 4 \) and \( v, e, f \in \mathbb{N} \).

**Lemma 2.22.** There are no \( r \)-regular simple bipartite quadrangulations of the double torus. If the triple torus has an \( r \)-regular bipartite quadrangulation, then it must be 5-regular.

**Proof.** Suppose that \( Q \) is a bipartite quadrangulation but not necessarily regular. Then \( Q \) must be a subgraph of the complete bipartite graph \( K_{m,n} \), with \( v = m + n \). \( Q \) is simple so it cannot have more edges than \( K_{m,n} \). Hence \( e \leq mn \leq \left( \frac{v}{2} \right)^2 \). If \( Q \) is also \( r \)-regular, then \( e \leq \left( \frac{v}{2} \right)^2 \) implies that

\[(r - 2)^2 \leq 4g,\]

for \( g > 1 \), which has no solutions for \( g = 2, r > 4 \), and the only solution for \( g = 3 \) is \( r = 5 \). \( \square \)

**Theorem 2.23.** Suppose that \( Q \) is a simple bipartite quadrangulation of the closed orientable surface \( F \) of genus \( g > 0 \). Then \( Q \) has at least \( 4(1 + \sqrt{g}) \) vertices and at least \( 2(1 + \sqrt{g})^2 \) faces.

**Proof.** \( Q \) is a quadrangulation so \( f - v = 2g - 2 \) and \( Q \) is simple and bipartite so the equation \( e \leq \left( \frac{v}{2} \right)^2 \) from the previous lemma implies that \( 8f \leq v^2 \). It follows that

\[ v^2 \geq 8v + 16g - 16 \]
\[ (v - 4)^2 \geq 16g \]
\[ v \geq 4(1 + \sqrt{g}) \]

and

\[ f = v + 2g - 2 \geq 2 + 2g + 4\sqrt{g} = 2(1 + \sqrt{g})^2. \]

\( \square \)
Theorem 2.24. Suppose that $K$ has a generalised alternating projection $\pi(K)$ onto the closed orientable surface $F$ of genus $g$. Then $\pi(K)$ has at least $2(1 + \sqrt{g})^2$ crossings.

Proof. If $g = 0$, the minimal crossing diagram of the Hopf link has 2 crossings. Any other planar diagram with one or two crossings is not generalised alternating.

If $g > 0$, $\pi(L)$ is obtained by replacing the vertices of a generalised basic polyhedron $\Gamma$ with non-trivial alternating arborescent tangles. $\Gamma$ is dual to a simple bipartite quadrangulation $Q$ which must have at least $2(1 + \sqrt{g})^2$ faces by Theorem 2.23. Each tangle contains at least one crossing, so $\pi(L)$ contains at least $2(1 + \sqrt{g})^2$ crossings.

It is shown in [9] that the complete bipartite graph $K_{m,n}$ has a 2-cell embedding in the closed orientable surface of genus $\lceil \frac{(m-2)(n-2)}{4} \rceil$, where $\lceil x \rceil$ is the smallest integer greater than $x$. In particular the graphs $K_{2g+2,4}$ for $g \geq 1$ and $K_{2+g,6}$ for $g \geq 2$ quadrangulate the surface of genus $g$, showing that there are generalised alternating projections for orientable surfaces of every genus.

Applying Euler’s formula, we can see that the embeddings of $K_{6,4}$, $K_{6,5}$, $K_{6,6}$, and $K_{7,6}$ in the surfaces of genus 2, 3, 4, and 5 respectively are quadrangulations which realise the minimums of Theorem 2.23. For the surfaces of genus 3, 4, and 5, these are the only abstract graphs which realise the minimum, however they may have many embeddings which are not graph homeomorphic. For the surface of genus 2, there is also a quadrangulation by the graph isomorphic to $K_{5,5}$ with one edge removed, and one such quadrangulation is depicted in Figure 2.23.

The full list of irreducible bipartite quadrangulations is unknown for surfaces of genus greater than one, however these lists are known to be finite [68]. Based on the corresponding work for irreducible triangulations [85], we would expect the number of irreducible quadrangulations to increase rapidly with genus.

We produce a concrete example of a generalised alternating projection onto the Heegaard double-torus in Figure 2.24. The underlying generalised
basic polyhedron is dual to $K_{6,4}$.

We finish this section with one other way of producing generalised basic polyhedra for higher genus surfaces. Let $Q$ and $Q'$ be simple bipartite quadrangulations of closed orientable surfaces $F$ and $F'$ respectively. We form the quad sum of $Q$ and $Q'$ by identifying one face of each quadrangulation. We denote the quad sum by $Q \Box Q'$, and the resulting quadrangulation embeds in $F \# F'$. Note that 4-cycle addition is equivalent to quad sum with the double wheel $W_6$. 

Figure 2.23: A simple bipartite quadrangulation of the genus two surface. The graph is isomorphic to the complete bipartite graph $K_{5,5}$ minus one edge.
Figure 2.24: A generalised alternating projection of the knot 12a_{603} on the double torus.

Lemma 2.25. Let \( Q \) and \( Q' \) be simple bipartite quadrangulations of closed orientable surfaces \( F \) and \( F' \) respectively. Then \( Q \square Q' \) is a simple bipartite quadrangulation of \( F \# F' \). Furthermore, if \( Q \) and \( Q' \) are both irreducible and \( F \) and \( F' \) have positive genus, then \( Q \square Q' \) is irreducible.

Proof. Let \( \phi \) and \( \phi' \) be the identified faces in the quad sum. Let \( C \) be the 4-cycle in \( Q \square Q' \) that is the image of \( \partial \phi \) and \( \partial \phi' \) under the identification.

\( Q \square Q' \) is clearly bipartite and there are no loops in \( Q \square Q' \) since there were no loops in either \( Q \) or \( Q' \).

Suppose there are multiple edges \( e \) and \( e' \) between vertices \( v \) and \( v' \) in \( Q \square Q' \). Then \( v \) and \( v' \) must lie on \( C \) since there were no multiple loops in either \( Q \) or \( Q' \). But the only edges between \( v \) and \( v' \) also lie on \( C \) since \( Q \) and \( Q' \) are simple. But \( C \) does not contain any multiple edges. Therefore
Suppose that $Q$ and $Q'$ are irreducible and $F$ and $F'$ have positive genus. Then $Q$ and $Q'$ are both 3-irreducible by Theorem 2.11, so $\delta(Q) \geq 3$ and $\delta(Q') \geq 3$. This means that $\delta(Q \Box Q') \geq 3$ and in particular that each vertex of $C$ has degree at least four. If it is possible to perform a 4-cycle removal on $Q \Box Q'$, then there is a face $\phi$ of $Q \Box Q'$ where each vertex of $\partial \phi$ has degree exactly three. So $\phi$ must occur away from $C$, and hence it was possible to perform a 4-cycle removal on either $Q$ or $Q'$, a contradiction.

Suppose it is possible to perform a face contraction on the face $\psi$ of $Q \Box Q'$ contracting the vertices $v$ and $v'$. Assume without loss of generality that $\psi$ is a face of $Q$. Then there is no vertex $w$ in $Q \Box Q'$ which is adjacent to both $v$ and $v'$. But then there is no vertex $w$ in $Q$ or $Q'$ which is adjacent to both $v$ and $v'$ so it is possible to perform a face contraction on $\psi$ in $Q$, a contradiction to the irreducibility of $Q$. 

Lemma 2.25 has the following corollary for generalised basic polyhedra, which allows us to construct many generalised basic polyhedra for higher genus surfaces. Not all generalised basic polyhedra can be constructed this way, since we have examples satisfying the lower bounds of Theorem 2.23.

**Corollary 2.26.** Let $\Gamma$ and $\Gamma'$ be generalised basic polyhedra embedded in closed orientable surfaces $F$ and $F'$ respectively. Then $\Gamma \oplus \Gamma'$ is a generalised basic polyhedron embedded in $F \# F'$. Furthermore, if $\Gamma$ and $\Gamma'$ are initial and $F$ and $F'$ have positive genus, then $\Gamma \oplus \Gamma'$ is initial.

Nakamoto [65] has also shown that if we have a simple bipartite quadrangulation $Q$ on a closed orientable surface $F$ of positive genus $g$ with $k \geq \eta(g)$ vertices for some function of genus $\eta(g) \in \mathbb{N}$, then any simple bipartite quadrangulation on $F$ with $k$ vertices can be obtained from $Q$ using two operations called diagonal rotation and diagonal slide. Not all quadrangulations obtained this way satisfy $\delta(Q) \geq 3$. Nevertheless, this gives another method of generating generalised basic polyhedra of higher genus surfaces.
2.7 Boundary Slopes

Let \( \Sigma \) be a properly embedded surface with boundary in a knot exterior \( X \). \( \partial \Sigma \) is a family of parallel simple closed curves on the torus \( \partial X \). If \( \alpha \) is a component of \( \partial \Sigma \), then \( [\alpha] = p[\mu] + q[\lambda] \) with \( p, q \in \mathbb{Z} \) and where \((\mu, \lambda)\) is a preferred meridian-longitude basis of \( \partial X \).

A simple closed curve \( \mu \) on \( \partial X \) is a meridian if \( \mu \) bounds a disk \( D \) in \( S^3 \) such that \( D \cap X = \partial D \), and \( D \) intersects \( K \) transversely exactly once. The meridian \( \mu \) is a generator for \( H_1(X; \mathbb{Z}) \cong \mathbb{Z} \cong <\mu> \).

A simple closed curve \( \lambda' \) on \( \partial X \) is a longitude if \( \lambda' \) intersects the meridian \( \mu \) transversely exactly once. The preferred longitude \( \lambda \) is a longitude which bounds a properly embedded orientable surface in \( X \). This is a Seifert surface for the knot \( K \) and \( [\lambda] \) represents the trivial element of \( H_1(X; \mathbb{Z}) \).

If \( \Sigma \) is \( \pi_1 \)-essential in \( X \), then we say that \( s = \frac{p}{q} \) is a boundary slope for \( K \). Let \( \mathcal{B}(K) \) be the set of boundary slopes for \( K \). Then

\[
\mathcal{B}(K) \subset \mathbb{Q} \cup \{\infty\},
\]

where the \( \infty \) slope is represented by a meridian. An example of an infinite boundary slope is an essential Conway sphere.

A general 3-manifold \( M \) with a single torus boundary component is called a knot manifold.

**Theorem 2.27** (Culler-Shalen [15]). Let \( M \) be a knot manifold. Then \( M \) has at least two boundary slopes.

In the context of knot exteriors, this bound is realised by torus knots, which have exactly two boundary slopes, 0 and \( pq \), which represent a Seifert surface and the winding annulus respectively.

**Theorem 2.28** (Hatcher [32], Jaco-Sedgwick [51]). Let \( M \) be a knot manifold. Then there only exist a finite number of boundary slopes which bound \( \pi_1 \)-essential surfaces in \( M \).

Theorem 2.28 is not true if \( M \) has more than one toroidal boundary component. For example, the exterior \( X \) of the Hopf Link contains essen-
tial annuli at every slope on one of the boundary components, since \(X\) is homeomorphic to \(T^2 \times I\).

Algorithms to calculate all the boundary slopes for a rational knot and more generally a Montesinos knot were developed by Hatcher and Thurston [34] and Hatcher and Oertel [33] respectively. Dunfield [21] has produced a table of all the boundary slopes of Montesinos knots up to 10 crossings, which corrected some errors in the list in [33]. Dunfield [22] has a further unpublished update to [21].

The *boundary slope diameter* of a knot \(K\) is

\[
d(K) = \max\{|s - s'| : s, s' \in B(K) \setminus \{\infty\}\}.
\]

Let \(c(K)\) denote the crossing number of \(K\).

**Theorem 2.29** (Ichihara-Mizushima [43]). If \(K\) is a Montesinos knot, then

\[
d(K) \leq 2c(K),
\]

with equality if \(K\) is alternating and Montesinos.

This theorem was proved for the case of two-bridge knots in [59]. Ichihara and Mizushima [44] have also found a lower bound of

\[
d(K) \geq 2c(K) - 6
\]

on the diameter of Montesinos knots. The pretzel knot \(K(\frac{1}{3}, \frac{1}{3}, \frac{1}{n})\) has crossing number \(n + 6\), and a diameter that approaches \(2n + 6\) as \(n \to \infty\), however no knot is currently known to strictly obtain the lower bound.

More generally, Culler and Shalen [16] have shown that for a non-trivial knot \(K\), the diameter \(d(K)\) is at least 2. However, in general, no upper bound on diameter in terms of crossing number is known.

Dunfield and Garoufalidis [24] have shown that alternating knots with non-integral boundary slopes are quite common. However, this cannot happen for an alternating Montesinos knot [33].
2.8 Spanning Slopes

\(\Sigma\) is a **spanning surface** for a knot \(K\) if \(\Sigma\) is embedded in \(S^3\) such that \(\partial \Sigma = K\). \(\Sigma = \Sigma \cap X\) is a properly embedded surface in the knot exterior \(X\) which we also refer to as a spanning surface. The boundary slope of an essential spanning surface \(\Sigma\) is defined to be the boundary slope of \(\Sigma\) in \(X\).

If \(\Sigma\) is a spanning surface, then \([\partial \Sigma] = p[\mu] + [\lambda]\) since \(\partial \Sigma\) winds only once around the longitude. If \(\Sigma\) is orientable then \(s = p = 0\), and if \(\Sigma\) is non-orientable, then \(p\) must be even since \([\partial \Sigma]\) must represent a trivial element in \(H_1(X; \mathbb{Z}_2)\) in order to bound. Therefore if \(s\) is the boundary slope of a spanning surface, then \(s \in 2\mathbb{Z}\).

Let \(\mathcal{S}(K)\) be the subset of \(\mathcal{B}(K)\) that consists of boundary slopes that are realised by spanning surfaces. We refer to an element of \(\mathcal{S}(K)\) as a **spanning slope** of \(X\). Note that \(\mathcal{S}(K) \subset 2\mathbb{Z}\), whereas \(\mathcal{B}(K) \subset \mathbb{Q} \cup \{\infty\}\).

Boundary slope diameters can be used to show that many non-alternating knots with low crossing number do not have any generalised alternating projections. Let \(d_{S}(K)\) be the diameter of \(\mathcal{S}(K)\), and we call \(d_{S}(K)\) the **spanning slope diameter**. Note that \(d_{S}(K) \subset 2\mathbb{Z}\), whereas \(d(K) \subset \mathbb{Q}\).

**Theorem 2.30.** A knot \(K\) has a generalised alternating projection onto the orientable surface of genus \(g\) only if \(d_{S}(K) > 0\) and

\[
g \leq \frac{1}{4} \left( \sqrt{d_{S}(K)} - 2 \right)^2.
\]

**Proof.** By Theorem 2.28, \(K\) has only finitely many boundary slopes, and therefore the slope diameter is finite. If \(K\) has a generalised alternating projection \(\pi(K)\) onto a surface \(F\) of genus \(g\) with \(n\) crossings, then both the checkerboard surfaces are \(\pi_1\)-essential and the distance between their slopes is \(2n\). Hence \(2n \leq d_{S}(K) \leq d(K)\). Theorem 2.24 shows that \(n \geq 2(1 + \sqrt{g})^2\). Therefore

\[
g \leq \left( \sqrt{\frac{n}{2}} - 1 \right)^2 \leq \left( \sqrt{\frac{d_{S}(K)}{4}} - 1 \right)^2,
\]

which gives the result. \(\square\)
This shows that a knot can have generalised alternating projections onto orientable surfaces of genus \( g \) for only finitely many \( g \). If we know the spanning slope diameter of a knot, then we can use this result to show that some knots are not generalised alternating.

Note that there exist knots where \( d_S(K) = 0 \), however by Theorem 2.3, these knots cannot have generalised alternating projections onto any closed orientable surface. One family of knots known to satisfy \( d_S(K) = 0 \) are the \((p,q)\)-torus knots where \( p > q \geq 3 \), so no non-alternating torus knot has a generalised alternating projection onto any closed orientable surface. We will state this fact later as Theorem 4.10.

Dunfield’s counterexample [23] to Conjecture 1.21 also satisfies \( d_S(K) = 0 \), which gives an example of a knot which is neither generalised alternating nor a torus knot.

**Lemma 2.31.** Suppose that \( K \) is a non-alternating, Montesinos knot with crossing number 8 or 9. Then \( K \) does not have a generalised alternating projection on to any orientable surface \( F \).

**Proof.** From Dunfield’s tables of boundary slopes for Montesinos knots [22], it can be seen that all non-alternating, Montesinos knots with 8 or 9 crossings have diameter at most 15. Theorem 2.30 implies that a non-alternating, Montesinos knot with 8 or 9 crossings can only have a generalised alternating projection onto \( S^2 \) but \( K \) is non-alternating. \( \square \)

This argument also works for all non-alternating, Montesinos knots with ten crossings except for \( 10_{139} \) and \( 10_{145} \). These two exceptions have a diameter of 20, so potentially could have generalised alternating projections with 9 or 10 crossings onto the torus, whereas all other non-alternating, Montesinos knots with ten crossings have diameter strictly less than 18. Lemma 2.23 shows that a knot with diameter strictly less than 18 but at least 16 could only have a non-planar generalised alternating projection onto the torus with 8 crossings. However the only generalised alternating projections onto the torus with 8 crossings, come from inserting single crossing tangles into the two toroidal embeddings of the graph \( K_{4,4} \), which give rise to links with 2 or 4 components respectively.
Using Theorem 1.17, the final statement in Theorem 2.29 can be slightly strengthened.

**Theorem 2.32.** If $K$ is an adequate Montesinos knot, then

$$d_S(K) = 2c(K).$$

**Proof.** If $K$ has an adequate planar projection $\pi(K)$, then it has an alternating projection $\pi'(K)$ onto the Turaev surface $F$ obtained from $\pi(K)$ and the positive and negative states. The two checkerboard surfaces relative to $\pi'(K)$ are $\pi_1$-essential by Theorem 1.17. The checkerboard surfaces realise the diameter of $2c(K)$, since the crossing number of an adequate knot is realised by its adequate diagram [88].

If $K$ is non-Montesinos, such an upper bound as demonstrated in Theorem 2.29 does not apply. A partial result was claimed by Curtis and Taylor, however we will show that there exist counterexamples to Theorem 2.33.

**Theorem 2.33 (Curtis-Taylor [17]).** If $K$ is an alternating knot, then

$$d_S(K) = 2c(K).$$

The proof given in [17, page 1350] contains a mistake, where a result of Adams and Kindred [6] has been incorrectly stated. Adams and Kindred showed that given an essential spanning surface $\Sigma$ for an alternating knot $K$, then a spanning surface $\Sigma'$ can be obtained from a basic layered spanning surface $S$ for $K$, by adding some number of handles or crosscaps to $S$, such that $\Sigma'$ has the same orientability, slope and genus as $\Sigma$. Basic layered surfaces are also known as $\sigma$-adequate state surfaces, which were introduced in Section 1.4.

Curtis and Taylor have quoted this result as: given an essential spanning surface $\Sigma$ for an alternating knot $K$, then there exists a basic layered surface $S$ which has the same orientability, slope and genus as $\Sigma$. This is not true because adding a crosscap changes the slope by $\pm 2$, so it is possible to obtain spanning surfaces for a knot $K$ whose slopes lie outside the range of slopes.
of basic layered surfaces. In particular, there may exist essential spanning surfaces for an alternating knot which has a slope bigger or smaller than both the checkerboard surfaces associated to a reduced, alternating diagram. We will show that such surfaces can exist for non-Montesinos knots.

Let $K = 8_{17}$. In [28], it was stated that by work of Kabaya [53],

$$B(K) = \{ -14, -8, -6, -4, -2, 0, 2, 4, 6, 8, 14, \infty \},$$

so clearly $d(K) > 2c(K) = 16$. We will show that in fact $d_S(K) = 28 > \ldots$
2.8. **SPANNING SLOPES**

Figure 2.25 shows a generalised alternating projection of $K$. In it the black surface has slope $+14$ and the white surface has slope $-8$. A method for calculating the slopes of spanning surfaces is outlined in [6]. The black surface is non-orientable and has Euler characterstic $-8$.

There is a unique reduced alternating diagram of $8_{17}$ on $S^2$. The two checkerboard surfaces have slopes $+8$ and $-8$. Each surface has Euler characteristic $-3$ and is non-orientable. If we take the planar checkerboard surface with slope $+8$ as our basic layered surface, add three crosscaps in the appropriate way, and add one handle, then we obtain a spanning surface which is non-orientable, has slope $+14$ and Euler charcteristic $-8$, demonstrating that this example does not contradict [6].

If we reflect the given toroidal projection of $8_{17}$ in the plane, we obtain a different projection of $8_{17}$. This projection is also generalised alternating on a torus. It has checkerboard surfaces with slopes $-14$ and $+8$. Note that $8_{17}$ is amphichiral. Therefore $d_S(K) = 28$ since both the slopes $-14$ and $+14$ are realised by essential spanning surfaces.

We give another example, this time of a non-alternating, non-Montesinos knot. Figure 2.26 shows the knot $K = 10_{161}$. We will show that $d_S(K) \geq 22 > 2c(K)$. Every knot $K$ has a Newton polygon $N_K$ coming from the $A$-polynomial. This gives a list of visible boundary slopes for $K$. For $K = 10_{161}$, see Culler [14], $N_K$ has 16 sides, which give five slopes,

$$B(K) \supseteq \{-20, -18, -9, -8, 2\}.$$ 

Of course 0 is also a slope for $K$.

The black surface has slope $+2$ and the white surface has slope $-20$. Therefore the diameter of $S(K)$ is at least 22, whereas its crossing number is only 10.

Together these two examples establish the following theorem. See also Figures 0.1 and 2.24 for further examples of this phenomenon.
Theorem 2.34. There exist non-Montesinos knots, both alternating and non-alternating, such that

\[ d_S(K) > 2c(K). \]

Kabaya [53] had already shown that there exist alternating knots with
We have been able to show that all prime alternating non-Montesinos knots with at most 9 crossings have generalised alternating projections onto the torus. Of the remaining non-alternating knots up to 9 crossings, only 9₄₇ and 9₄₉ are not Montesinos. The former has a generalised alternating projection onto the torus shown in Figure 0.1.

It is unknown if $K = 9_{49}$ is generalised alternating. Consideration of its boundary slopes, calculated using the A-polynomial shows that

$$\mathcal{B}(K) \supseteq \{0, 4, 6, 12, 15\}.$$ 

Similar reasoning to Lemma 2.31 would suggest that $K$ does not have a generalised alternating projection, however it is known that the character variety does not detect all boundary slopes [12].
Chapter 3

Weakly Generalised
Alternating Links

In this chapter, we define weakly generalised alternating knots. In the first section we show that this new class includes all prime alternating knots and all generalised alternating knots.

The second section gives several lemmas which describe how spanning surfaces intersect in a link exterior.

We show in Section 3.3 that there are no essential bigons between checkerboard surfaces in standard position, which is then used to prove the related theorem that there are no essential trigons between the boundary of a link exterior and the checkerboard surfaces in standard position. These two results are the key properties of checkerboard surfaces arising from a weakly generalised alternating projection.

Throughout the next three sections, we use these key properties to show that both checkerboard surfaces are $\pi_1$-essential, and that a weakly generalised alternating link is prime, non-split and non-trivial.

In Section 3.7, we construct a closed 3-manifold from the link exterior and show that all the above properties carry over to embedded closed surfaces. By taking an appropriate double cover, we are then able to lift this whole setup to a pair of closed orientable surfaces embedded in a 3-manifold.

These two-sided closed embedded surfaces lift to families of planes in the
universal cover. We then define the relative 1-line property in terms of the intersections of planes in the universal cover. The relative 1-line property allows us to give a topological characterisation of the class of weakly generalised alternating links, the main result of this chapter.

Finally, in Section 3.10 we give several constructions for producing examples of weakly generalised alternating diagrams.

### 3.1 Weakly Generalised Alternating

We begin by defining a new class of knots which have surface-alternating projections. We will extend our concept of projection surface to include a collection of closed orientable surfaces embedded in $S^3$.

Let $F_i$ be a closed orientable surface for $i = 1, \ldots, p$. A *generalised projection surface* $F$ is a piecewise linear embedding,

$$F : \bigsqcup_{i=1}^{p} F_i \hookrightarrow S^3,$$

such that $F$ is non-split in $S^3$. Since $F$ is an embedding we will also denote the image of $F$ in $S^3$ by $F$. A collection of surfaces $F$ is *non-split* if every embedded 2-sphere in $S^3 \setminus F$ bounds a 3-ball in $S^3 \setminus F$.

A link $L$ is *weakly generalised alternating* if it has a projection onto a generalised projection surface $F$,

$$\pi : F \times I \to F,$$

where $L \subset F \times I \subset S^3$, which satisfies all of the following:

1. $\pi(L)$ is alternating on $F$.
2. $\pi(L)$ is separating on $F$.
3. $r(\pi(L), F) \geq 4$.
4. $\pi(L)$ is weakly prime.
3.1. WEAKLY GENERALISED ALTERNATING

5. \( \pi(L) \cap F_i \neq \emptyset \) for each \( i = 1, \ldots, p \).

6. each component of \( L \) projects to at least one crossing in \( \pi(L) \).

7. if \( F \cong S^2 \), then \( \pi(L) \) contains at least two crossings.

A generalised diagram is weakly prime whenever a disk \( D \subset F \) where \( \partial D \) intersects \( \pi(L) \) transversely exactly twice, we have that \( \pi(L) \cap D \) is a single embedded arc. Clearly prime implies weakly prime but the converse is not true. For example, we may have a weakly generalised alternating projection where \( e(\pi(L), F) = 2 \), as in Figure 3.1 which is weakly prime but not prime since there exists a loop \( \ell \subset F \) which intersects \( \pi(K) \) twice but \( \ell \) does not bound a disk in \( F \).

We must also insist that \( \pi(L) \) is separating on \( F \). This is because we wish to study the two surfaces which arise from a checkerboard colouring of the regions of \( F \setminus \pi(L) \). Recall that alternating implies there is at least one crossing in \( \pi(L) \).

We make the assertion that planar projections contain at least two crossings in order to rule out the one crossing planar diagrams of the unknot (Figure 1.8 and its mirror), since we do not wish to consider the unknot as weakly generalised alternating. We also assume that the condition \( r(\pi(L), F) \geq 4 \) is vacuously satisfied when \( F \cong S^2 \), since there are no compressing disks for \( S^2 \subset S^3 \). We will show that our definition corresponds to the usual definition of an alternating diagram on \( S^2 \) being reduced, prime and non-split.

The requirement that each component of \( L \) is part of at least one crossing in \( \pi(L) \) is to avoid the situation where the checkerboard surfaces have the same slope on some component of \( L \).

If one component of \( F \) is a 2-sphere \( S \), then \( F \cong S \), since otherwise, one of the components of \( \partial(S \times I) \) would be a splitting sphere for \( F \). If \( L \) is a knot, then \( F \) is connected.

**Lemma 3.1.** Let \( \pi(L) \) be an alternating diagram on \( S^2 \). Then \( \pi(L) \) is weakly generalised alternating if and only if \( \pi(L) \) is reduced, non-split and prime.
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Figure 3.1: A weakly generalised alternating projection of 11_{n,99}. This projection is not generalised alternating since it is weakly prime but not prime.

Proof. $\pi(L)$ is alternating so contains at least one crossing. Forgetting crossing information, $\pi(L)$ is a 4-regular graph $\Gamma$. The dual graph $\Gamma^*$ is a quadrangulation of $S^2$, and every quadrangulation of $S^2$ is bipartite. Hence $\Gamma$ and $\pi(L)$ are separating on $S^2$. On $S^2$, the definitions of prime and weakly prime agree.

If $\pi(L)$ has only a single crossing, then it is neither reduced nor weakly generalised alternating, so assume that $\pi(L)$ contains at least two crossings.

If $\pi(L)$ is split, then there is an embedded loop $\ell$ on $S^2$ which bounds disks $D$ and $D'$ on each side, each of which contains part of $\pi(L)$. This loop can be isotoped to meet $\pi(L)$ exactly twice, such that neither complementary region
3.1. WEAKLY GENERALISED ALTERNATING

contains only a single embedded arc, since at least one of $D$ or $D'$ contains a crossing. Hence $\pi(L)$ is not weakly prime.

Assume that $\pi(L)$ is non-split. If $\pi(L)$ is not reduced, then it is not prime, since there is a disk $D \subset S^2$ such that $\partial D$ meets $\pi(L)$ exactly twice, and $D$ contains at least one crossing. Since $\pi(L)$ contains at least two crossings, $D$ can be chosen so that $S^2 \setminus D$ also contains a crossing, so $\pi(L)$ is not weakly prime.

Lemma 3.2. Let $\pi(L)$ be a generalised alternating projection onto a closed orientable surface $F$. Then $\pi(L)$ is a weakly generalised alternating projection onto a closed orientable surface $F$.

Proof. $\pi(L)$ is alternating and separating on $F$ by Lemma 2.4. $r(\pi(L), F) \geq 4$ since $e(\pi(L), F) \geq 4$. $\pi(L)$ is weakly prime since it is prime.

The regions $F \setminus \pi(L)$ of a weakly generalised alternating projection are not necessarily disks. Examples can be found in Figures 3.18 and 4.10. Weakly generalised alternating projections with exactly one crossing do exist on any closed orientable surface of positive genus.

Analogous to the results of Ozawa on generalised alternating knots (see Theorems 2.1 and 2.2), we have the following lemma. Note that now, the regions of $F \setminus \pi(L)$ are not necessarily disks and when $L$ is a link with multiple components, then neither checkerboard surface is necessarily connected.

Lemma 3.3. Let $\pi(L)$ be a weakly generalised alternating diagram of the link $L$ onto the closed surface $F \subset S^3$. Then $F \setminus \pi(L)$ admits a checkerboard colouring. If $L$ is a knot, then at least one of the surfaces arising from the checkerboard colouring is non-orientable.

Proof. The first statement is true by the definition, since $\pi(L)$ separates $F$. Let $K$ be a knot. $\pi(K)$ is alternating on $F$ so has at least 1 crossing. This means that the boundaries of the two checkerboard surfaces must have different slopes on $\partial X$, and so at most one of them can be bounded by the preferred longitude, which is necessary to be orientable. Thus at least one of the checkerboard surfaces is non-orientable.
We now mention a result of Hayashi on toroidally-alternating links. We will generalise this result to all weakly generalised alternating links in Section 3.6.

**Theorem 3.4** (Hayashi [35]). Let \( \pi(L) \) be a toroidally alternating projection onto a Heegaard torus \( F \). If \( r(\pi(L), F) \geq 4 \) and \( \pi(L) \) is weakly prime, then \( L \) is prime and non-split.

### 3.2 Intersections of Spanning Surfaces

In this section, we will prove several lemmas that describe how spanning surfaces intersect in a knot exterior. We prove some of these results for a setting more general than weakly generalised alternating, since we will need them in Chapter 4.

A pair of spanning surfaces in general position intersect in a set of properly embedded arcs and embedded loops since each spanning surface is properly embedded. In particular, there are no triple points or branch points.

Recall that we use the notation \( \Sigma \) for a compact surface embedded in \( S^3 \) such that \( \partial \Sigma = L \), and \( \Sigma \) for a properly embedded surface in the link exterior \( X \) where \( \Sigma = X \cap \Sigma \). Both \( \Sigma \) and \( \Sigma ' \) are referred to as spanning surfaces.

**Lemma 3.5.** Let \( \Sigma \) and \( \Sigma ' \) be \( \pi_1 \)-essential spanning surfaces in a link exterior \( X \). Then \( \Sigma \) and \( \Sigma ' \) can be isotoped so that any loop of intersection is essential in both \( \Sigma \) and \( \Sigma ' \).

**Proof.** Let \( \ell \) be a loop of intersection. If \( \ell \) bounds a disk in \( X \), then \( \ell \) must bound a disk in each of \( \Sigma \) and \( \Sigma ' \) since they are incompressible in \( X \). Choose an innermost such loop \( \ell ' \). The two disks in \( \Sigma \) and \( \Sigma ' \) bounded by \( \ell ' \) form a sphere which bounds a 3-ball since \( X \) is irreducible, so \( \Sigma \) and \( \Sigma ' \) can be isotoped so that \( \ell ' \) disappears.

If \( \ell \) is essential in \( \Sigma \), then \( \ell \) must be essential in \( \Sigma ' \) since otherwise \( \ell \) would bound a disk in \( \Sigma ' \) which would be a compressing disk for \( \Sigma \). \( \square \)

**Lemma 3.6.** Let \( \Sigma \) and \( \Sigma ' \) be \( \pi_1 \)-essential spanning surfaces for a knot \( K \) which have the same boundary slope. If at least one of \( \Sigma \) or \( \Sigma ' \) is non-
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orientable and $\partial \Sigma \cap \partial \Sigma' = \emptyset$, then $\Sigma$ and $\Sigma'$ must intersect in an essential loop.

Proof. Suppose $\Sigma$ and $\Sigma'$ do not meet in any loops. Connecting the boundaries of $\Sigma$ and $\Sigma'$ by an annulus in $\partial X$ would produce a closed non-orientable surface embedded in $S^3$, a contradiction.

Given a separating projection $\pi(L)$ onto a closed orientable surface $F$, there is a standard position for the associated checkerboard surfaces. Away from a crossing the checkerboard surfaces are embedded in $F$, but in a small regular neighbourhood of a crossing, we think of the link lying on the surface of a ball $U$. $U$ intersects $F$ in an equatorial disk, and the over strand runs over the upper hemisphere, while the under strand runs under the lower hemisphere. Each checkerboard surface intersects $U$ in a half-twisted band. The ball $U$ is called a bubble and this viewpoint of checkerboard surfaces was introduced by Menasco [60] for planar alternating diagrams, and extended to diagrams on the Heegaard torus by Adams [3]. It extends to any separating projection onto a generalised projection surface $F$.

Figure 3.2: A checkerboard surface in standard position near a bubble $U$. 
In standard position, the checkerboard surfaces $\Sigma$ and $\Sigma'$ do not intersect in any loops, and intersect only in arcs corresponding to the north-south axis of each bubble. The intersection of the corresponding spanning surfaces $\Sigma$ and $\Sigma'$ is a union of trivalent graphs and a collection of loops, consisting of the link $L$ and the collection of axes of the bubbles. Since we are assuming that every component of $L$ is involved in a crossing of $\pi(L)$, then $\Sigma \cap \Sigma'$ forms a graph $\Gamma'$, where each connected component is 3-regular.

Let $\Sigma$ be a spanning surface for a link $L$ with $m$ components. Let $\{L_j \mid j = 1, \ldots, m\}$ denote the components of $L$. Let $\partial X = \{C_j \mid j = 1, \ldots, m\}$ where each $C_j$ is a torus and $C_j = \partial N(L_j)$

A curve $\mu_j \subset C_j$ is a meridian of $X$ if $\mu_j$ bounds an embedded disk in $N(L_j)$ which intersects $L_j$ transversely exactly once. We define a preferred longitude $\lambda_j$ of $C_j$ to be the unique non-trivial curve in $C_j$ which meets $\mu_j$ exactly once and bounds an orientable surface $S_j$ in $S^3 \setminus N(L_j)$. Note that $S_j$ is not necessarily embedded in $X$ since it may intersect other components of $L$. Let $\{\sigma_j \mid j = 1, \ldots, m\}$ be the components of $\partial \Sigma$. Then $[\sigma_j] = p_j[\mu_j] + [\lambda_j]$ where $p_j \in \mathbb{Z}$.

Let $\Sigma$ and $\Sigma'$ be checkerboard surfaces in standard position with respect to a separating alternating projection $\pi(L)$ onto a generalised projection surface $F$. If for some $j \in \{1, \ldots, m\}$, the projection of $L_j$ is embedded in $F$, then $\sigma_j$ does not intersect $\sigma'_j$ on $C_j$. In this case $\sigma_j$ and $\sigma'_j$ divide $C_j$ into a pair of annuli. For a weakly generalised alternating projection, we assume that this case does not happen.

For components $L_j$ whose projection has at least one crossing in $\pi(L)$, it follows that $\sigma_j$ and $\sigma'_j$ must intersect. Fix an orientation on $L_j$ and let $\sigma_j$ and $\sigma'_j$ inherit that orientation via $\lambda_j$. For an alternating projection in standard position it can be seen that as we traverse $L_j$, $\sigma_j$ rotates in a positive manner say, with respect to $F$, while $\sigma'_j$ rotates in a negative manner.

For a weakly generalised alternating projection $\pi(L)$, $\sigma_j$ and $\sigma'_j$ form a quadrangulation $Q_j$ of each boundary component $C_j$. We state this for a more general class of alternating projections as the following lemma. We call the set of quadrangulations of $\partial X$, the boundary quadrangulation of $X$.

If $\phi$ is a quadrilateral face of $Q_j$, then one pair of non-adjacent vertices
of $\partial \phi$ are identified in $Q_j$. Since $\Sigma$ and $\Sigma'$ are spanning surfaces, they each intersect the meridian $\mu_j$ exactly once. Hence one diagonal of $\phi$ is isotopic to $\mu_j$, so the vertices at the end of this diagonal are identified in $Q_j$.

Define $n_j$ to be the number of crossings which are encountered as we traverse once along the projection of the link component $L_j$. The count $n_j$ may count some crossings twice. In particular, if $n$ is the number of crossings in $\pi(L)$, then

$$2n = \sum_{j=1}^{m} n_j.$$ 

Note that $n_j$ is the intersection number $|i(\sigma_j, \sigma'_j)|$ on $C_j$, and $Q_j$ contains $n_j$ quadrangular faces. Define the intersection number of $\partial \Sigma$ and $\partial \Sigma'$ to be

$$i(\partial \Sigma, \partial \Sigma') = \sum_{j=1}^{m} |i(\sigma_j, \sigma'_j)|.$$ 

We collect the above information in the following two lemmas.

**Lemma 3.7.** Let $\Sigma$ and $\Sigma'$ be checkerboard surfaces associated to a separating and alternating link projection $\pi(L)$ onto the generalised projection surface $F$, where the projection of each component of $L$ contains a crossing in $\pi(L)$. Let $C$ be a component of $\partial X$. Then $C \cap \Sigma$ and $C \cap \Sigma'$ form a quadrangulation of $C$.

**Lemma 3.8.** Let $L$ be a link with $m$ components. Suppose that $\pi(L)$ is a separating and alternating projection of $L$ onto a generalised projection
surface $F$, which has $n$ crossings. Then

$$n = \frac{1}{2} l(\partial \Sigma, \partial \Sigma') = \frac{1}{2} \sum_{j=1}^{m} |i(\sigma_j, \sigma'_j)|.$$

The Euler characteristics of the checkerboard surfaces coming from a separating projection are related to the Euler characteristic of the projection surface.

**Lemma 3.9.** Suppose that $L$ has a separating projection $\pi(L)$ onto a closed orientable surface $F$, which has $n$ crossings. Then

$$\chi(\Sigma) + \chi(\Sigma') + n = \chi(F).$$

**Proof.** Let $\Gamma$ be the graph embedded in $F$ formed by ignoring the crossing information of $\pi(L)$. Each connected component of $\Gamma$ is either a 4-regular graph, or an embedded loop consisting of a single vertex and a single edge. Let $l$ be the number of embedded loops. Then $\Gamma$ has $n + l$ vertices and $2n + l$ edges.

Let the checkerboard surfaces $\Sigma$ and $\Sigma'$ be in standard position relative to $\pi(L)$. Consider the graph $\Gamma'$ which is formed by the intersections of $\Sigma$ and $\Sigma'$. Each connected component of $\Gamma'$ is either a trivalent graph whose vertices are the north and south poles of each bubble, or an embedded loop consisting of a single vertex and a single edge. Then $\Gamma'$ has $2n + l$ vertices and $3n + l$ edges. By collapsing the axis of each bubble to a point, $\Gamma'$ collapses to $\Gamma$, and $\chi(\Gamma') = \chi(\Gamma) = -n$.

Let $F' = \Sigma \cup \Sigma'$. Collapsing the axis of each bubble to a point, collapses $F'$ onto $F$. This induces a homeomorphism between $F' \setminus \Gamma'$ and $F \setminus \Gamma$, so $\chi(F') = \chi(F)$.

Since $\Sigma \cap \Sigma' = \Gamma'$,

$$\chi(F') = \chi(\Sigma) + \chi(\Sigma') - \chi(\Gamma').$$

Clearly $\chi(\Sigma) = \chi(\Sigma)$ and $\chi(\Sigma') = \chi(\Sigma')$, so the result follows. \qed
When the projection is alternating, this relationship is purely topological since it is in terms of the intersection number of the boundaries of the checkerboard surfaces. Lemmas 3.8 and 3.9 combine to give us the following theorem:

**Theorem 3.10.** Let \( \pi(L) \) be a separating and alternating projection of \( L \) onto a closed orientable surface \( F \), which has \( n \) crossings. Then

\[
\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial \Sigma, \partial \Sigma') = \chi(F).
\]

Let \( \Sigma \) and \( \Sigma' \) be two spanning surfaces for a link \( L \). Let \( \alpha \) be an arc of intersection between the interiors of \( \Sigma \) and \( \Sigma' \). There are two types of intersection arc.

![Figure 3.4: A parallel arc of intersection between two spanning surfaces.](image)

Let \( U \) be a regular neighbourhood of \( \alpha \) in \( S^3 \). Let \( \beta \) and \( \beta' \) be the two components of \( U \cap L \). Let \( V = U \cap X \). We can choose \( U \) so that \( \partial (\Sigma \cap V) \)
and \( \partial(\Sigma' \cap V) \) are both disks. Then \( V \) is a compact handlebody of genus two.

Fix an orientation on \( \beta \). This induces orientations on the disks \( \Sigma \cap V \) and \( \Sigma' \cap V \). If these both induce the same orientation on \( \beta' \), then \( \alpha \) is a parallel arc. If they induce opposite orientations on \( \beta' \), then \( \alpha \) is a standard arc.

This is equivalent to saying \( \partial(\Sigma \cap V) \) and \( \partial(\Sigma' \cap V) \) have algebraic intersection number zero if \( \alpha \) is a parallel arc, and algebraic intersection number two if \( \alpha \) is a standard arc.

If we collapse a standard arc \( \alpha \) to a point, then \( (\Sigma \cup \Sigma') \cap U \) collapses to a disk. An arc of intersection between two checkerboard surfaces in standard position relative to some separating projection onto a closed orientable surface is a standard arc. If we try to collapse a parallel arc to a point, then \( (\Sigma \cup \Sigma') \cap U \) collapses to an object homeomorphic to a compact annulus with its core collapsed to a point.

**Lemma 3.11.** Let \( \Sigma \) and \( \Sigma' \) be spanning surfaces for a link \( L \), isotoped so that their boundaries realise the intersection number \( i(\partial \Sigma, \partial \Sigma') \). If

\[
i(\partial \Sigma, \partial \Sigma') = |\sum_{j=1}^{m} i(\sigma_j, \sigma'_j)|,
\]

then every arc of intersection between \( \Sigma \) and \( \Sigma' \) is standard.

**Proof.** We need to define the intersection number \( i(\sigma_j, \sigma'_j) \) more carefully than we have up to now. Fix an orientation on each longitude \( \lambda_j \) and define the orientation of each meridian \( \mu_j \) so that \( (\lambda_j, \mu_j) \) form a right-handed basis for each torus boundary component \( C_j \). Then if \( [\sigma_j] = [\lambda_j] + p_j[\mu_j] \) and \( [\sigma'_j] = [\lambda'_j] + p'_j[\mu'_j] \), we define the intersection number as \( i(\sigma_j, \sigma'_j) = p_j - p'_j \).

By definition \( i(\partial \Sigma, \partial \Sigma') = \sum_{j=1}^{m} |i(\sigma_j, \sigma'_j)| \). If

\[
\sum_{j=1}^{m} |i(\sigma_j, \sigma'_j)| = \sum_{j=1}^{m} |i(\sigma_j, \sigma'_j)|
\]

for each \( j \), then every intersection between \( \sigma_j \) and \( \sigma'_j \) must be positive, or every intersection between \( \sigma_j \) and \( \sigma'_j \) must be negative. A parallel arc only
occurs when an arc of intersection connects a positive intersection to a negative intersection.

Note that the previous Lemma implies that parallel arcs of intersections only occur between different boundary components of $X$. Hence if $K$ is a knot, every arc of intersection between two spanning surfaces realising minimal intersection number, must be standard.

### 3.3 Essential Bigons and Trigons

Let $S$ and $S'$ be properly embedded surfaces in general position in an orientable irreducible 3-manifold $M$. A bigon is a disk $B$ embedded in $M$, such that $\partial B = \beta \cup \beta'$, where $\beta \subset S$ and $\beta' \subset S'$ are connected, $\beta \cap \beta'$ consists of two distinct points of $S \cap S'$, and $B \cap (S \cup S') = \partial B$.

![Figure 3.5: A bigon between two surfaces.](image)

A component of the intersection $S \cap S'$ is known as a double arc or double loop.
A bigon is \textit{inessential} if it can be homotoped into a double arc or double loop. Otherwise, it is \textit{essential}. Here homotopy means that a homotopy restricted to the boundary of $B$ satisfies the property that $\beta$ and $\beta'$ must remain in $S$ and $S'$ respectively, throughout the homotopy.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{essential_bigon.png}
\caption{An essential bigon between two checkerboard surfaces associated to an $F$-alternating projection of $10_{153}$.}
\end{figure}

\textbf{Theorem 3.12.} If $L$ has a weakly generalised alternating projection $\pi(L)$ on $F$, then there are no essential bigons between the checkerboard surfaces arising from $\pi(L)$.

\textit{Proof.} Suppose that $L$ has a weakly generalised alternating projection $\pi(L)$, with checkerboard surfaces $\Sigma$ and $\Sigma'$ in standard position. Suppose $B$ is a bigon between $\Sigma$ and $\Sigma$. $B$ is a disk whose boundary lies on and whose interior is disjoint from $\Sigma \cup \Sigma' \cup L$. By collapsing the double arcs between $\Sigma$ and $\Sigma'$, it can be seen that $\Sigma \cup \Sigma' \cup L$ collapses to a surface isotopic to $F$. 


It follows that either $B$ is a compressing disk for $F$ or $B$ is parallel to a trivial disk in $F$. The bigon $B$ can be isotoped so that $\partial B$ intersects $\pi(L)$ transversely exactly twice, just push the points where $\partial B$ meets a double arc away from the crossings and onto $\pi(L)$. However, if $B$ is a compressing disk for $F$ then $\partial B$ must meet $\pi(L)$ at least four times since $r(\pi(L), F) \geq 4$.

Suppose then that $\partial B$ is trivial in $F$. Let $D$ be the subdisk of $F$ bounded by $\partial B$. $\pi(L)$ is weakly prime, so $D$ must contain only a single embedded arc $\alpha$ of $\pi(L)$. The projection is alternating, so following along $L$ from $\alpha$ must lead to an undercrossing in one direction and an overcrossing in the other direction. Assume that the interior of $B$ sits above the projection surface $F$. Then push $\partial B$ off $L$ and into $\Sigma \cup \Sigma'$. To do this, $B$ must be isotoped so that both points of intersection $\partial \alpha$ are pushed towards the undercrossing, and hence into the same double arc. Therefore $B$ is an inessential bigon. 

Let $S_1$, $S_2$ and $S_3$ be properly embedded surfaces in general position in an orientable irreducible 3-manifold $M$. A trigon is a disk $\Delta$ embedded in $M$ with $\partial \Delta = \alpha_1 \cup \alpha_2 \cup \alpha_3$ where $\alpha_i \subset S_i$ is connected, $\alpha_i = \Delta \cap S_i$, and $\alpha_i \cap \alpha_j$ is a single point of $S_i \cap S_j$ for $i \neq j$.

A component of the intersection $S_1 \cap S_2 \cap S_3$ is known as a triple point. A trigon is inessential if it can be homotoped into a triple point. Otherwise, it is essential. Again, when this homotopy is restricted to the boundary of the trigon, each subarc $\alpha_i$ of $\partial \Delta$ must remain on $S_i$ throughout the homotopy.

**Theorem 3.13.** Let $\pi(L)$ be a weakly generalised alternating projection onto a generalised projection surface $F$. Then there are no essential trigons between the two checkerboard surfaces and $\partial X$.

**Proof.** Let $\Sigma$ and $\Sigma'$ be the checkerboard surfaces in standard position with respect to the weakly generalised alternating projection $\pi(L)$.

Suppose that $\Delta$ is a trigon, with $\partial \Delta = \alpha \cup \alpha' \cup \beta$, where $\alpha \subset \Sigma$, $\alpha' \subset \Sigma'$, and $\beta \subset \partial X$ are connected, and the interiors of $\alpha$, $\alpha'$, and $\beta$ are pairwise disjoint.

Let $C_i$ be a component of $\partial X$. By Lemma 3.7, the boundaries of the checkerboard surfaces on $C_i$ form a quadrangulation $Q_i$. Each quadrilateral face of $Q_i$ has two non-adjacent edges belonging to $\sigma_i \subset \Sigma$ and the other
two non-adjacent edges belonging to $\sigma'_i \subset \Sigma'$.  $\beta$ is contained within one quadrilateral face $\phi$ and must run from one edge to an adjacent edge. Thus $\beta$ bounds a triangular subdisk $D \subset \phi$. Let $q$ be the vertex of $D$ opposite the edge $\beta$.

Let $B = D \cup \Delta$. $B$ is an embedded bigon between $\Sigma$ and $\Sigma'$ where one endpoint is $q$. Since $L$ is weakly generalised alternating, Theorem 3.12 implies that $B$ is an inessential bigon, so it can be homotoped into a double arc keeping $q$ fixed, and hence can be homotoped into the triple point $q$. Therefore $\Delta$ is inessential.
3.4 Essential Checkerboard Surfaces

We will now use the results of the previous section to prove that both checkerboard surfaces associated to a weakly generalised alternating projection are essential in the link exterior.

**Theorem 3.14.** Let \( \pi(L) \) be a weakly generalised alternating projection onto a generalised projection surface \( F \). Then both the checkerboard surfaces associated to \( \pi(L) \) are essential in the link exterior \( X \).

**Proof.** Let \( \Sigma \) and \( \Sigma' \) be the checkerboard surfaces in standard position relative to \( \pi(L) \). Suppose that \( D \) is a compressing disk for \( \Sigma \). Consider the intersection of \( D \) with \( \Sigma' \). \( D \) can be isotoped to be in general position relative to \( \Sigma' \) so that the intersection consists of simple closed loops and properly embedded arcs in \( D \) (Figure 3.8).

Let \( \ell \) be an innermost loop of intersection. This means that \( \ell \) bounds a subdisk \( D' \subset D \), such that the interior of \( D' \) is disjoint from \( \Sigma' \). Then either \( \ell \) bounds a disk \( E \) in \( \Sigma' \) disjoint from \( \partial X \), or \( \ell \) bounds a compressing disk for \( \Sigma' \). In the former case, we can make a disk exchange, to isotope \( D' \) across \( E \) and remove \( \ell \). In the latter case, \( D' \) must be either a compressing disk for \( F \) whose boundary does not meet \( \pi(L) \), or \( D' \) is parallel to a subdisk \( E' \subset F \). However \( r(\pi(L),F) \geq 4 \), so \( \ell \) does not bound a compressing disk \( D' \) for \( F \), and \( \pi(L) \) is weakly prime so \( E' \) cannot contain any components of \( \pi(L) \).

Therefore we can remove all closed loops of intersection between \( D \) and \( \Sigma' \), proceeding each time from an innermost such loop.

Let \( \alpha \) be an outermost arc of intersection between \( D \) and \( \Sigma' \), so that \( \partial \alpha \subset \partial D \). Let \( \gamma \subset \partial D \) such that \( \alpha \cup \gamma \) bounds a subdisk \( B \subset D \), such that \( \hat{B} \cap \Sigma' = \emptyset \). Then \( B \) is a bigon between \( \Sigma \) and \( \Sigma' \). By Theorem 3.12, \( B \) is an inessential bigon, and can be homotoped into a double arc. Therefore we can remove all arcs of intersection between \( D \) and \( \Sigma' \).

This means that \( D \cap \Sigma' = \emptyset \), and therefore \( \partial D \) lies entirely within one region \( R \) of \( \Sigma \setminus \Sigma' \). We have assumed that \( D \) is a compressing disk for \( \Sigma \) so \( \partial D \) cannot bound a disk in \( R \). If \( \partial D \) is essential in \( F \), then \( D \) is a compressing disk for \( F \) which contradicts \( r(\pi(L),F) \geq 4 \). If \( \partial D \) is trivial in \( F \), then we
have a contradiction to the weak primality of $\pi(L)$.

Therefore $\Sigma$ is incompressible in $X$ and the same argument with the roles of $\Sigma$ and $\Sigma'$ reversed shows that $\Sigma'$ is also incompressible in $X$.

Suppose that $D$ is a boundary compressing disk for $\Sigma$. Consider the intersection of $D$ with $\Sigma'$. $D$ can be isotoped into general position such that the intersection with $\Sigma'$ consists of simple closed loops and properly embedded arcs in $D$ (Figure 3.9).

By the same argument as for the compressing disk, any innermost closed loop of intersection can be removed, and any innermost arc of intersection $\alpha$, where the ends of $\alpha$ both lie on $\Sigma$, can be removed.

Let $\beta$ be an outermost arc of intersection between $D$ and $\Sigma'$ such that both ends of $\beta$ lie on $\partial X$. Since both checkerboard surfaces are spanning surfaces, whose boundaries wrap exactly once along the longitude, Lemma 3.7 shows that $\partial \Sigma$ and $\partial \Sigma'$ divide $\partial X$ into an even number of quadrilateral disks. The boundary of such a quadrilateral disk $Q$, consists of two edges from $\partial \Sigma$ and two edges from $\partial \Sigma'$, where no two edges from the same set are
adjacent. Furthermore the boundary of such a disk consists of only three distinct vertices, since two of the four must be identified because $\partial \Sigma$ and $\partial \Sigma'$ are spanning surfaces. Call the identified vertex $q$.

Let $\gamma$ be the part of $\partial D$ that lies on $\partial X$ and connects the two ends of $\beta$ and let $E \subset D$ be the subdisk bounded by $\beta \cup \gamma$. Suppose that $\gamma$ connects one edge of $Q$ coming from $\partial \Sigma'$ with the same edge. $\gamma$ is isotopic into an arc $\gamma' \subset \partial \Sigma' \cap \partial Q$ through a subdisk of $\partial X$, which fixes its endpoints. $\gamma \cup \gamma'$ bounds a disk in $Q$, call it $E'$.

Then either $\beta \cup \gamma'$ bounds a disk in $\Sigma'$ or it doesn’t. If it does, then $\beta \cup \gamma$ bounds a disk in $\Sigma' \cup \partial X$. Since $X$ is irreducible, it is possible to isotope $\Sigma'$ through $E \cup E'$ and remove the arc of intersection $\beta$.

Suppose then that $\beta \cup \gamma'$ does not bound a disk in $\Sigma'$. We can push $E'$ into the interior of $X$, with $\gamma'$ being pushed into the interior of $\Sigma'$. Let $\gamma''$ be the pushoff of $\gamma'$ into $\Sigma'$. Then $\beta \cup \gamma''$ does not bound a disk in $\Sigma'$, but $\beta \cup \gamma''$ does not meet $\pi(L)$. Thus we have a contradiction to either $r(\pi(L), F)$ being at least 4 or $\pi(L)$ being weakly prime.
Suppose that $\gamma$ connects one edge of $Q$ coming from $\partial \Sigma'$ with the other edge coming from $\partial \Sigma'$. Both these edges have the vertex $q$ in common. $\partial X$ projects down to an annulus in $F$ and $\pi(L)$ is the core of this annulus. The ends of $\beta$ can both be slid towards $q$ so that $\gamma$ crosses this annulus once. Hence $\gamma$ can be isotoped to meet $\pi(L)$ once. $q$ also represents one end of a double arc in $X$ and hence a crossing of $\pi(L)$. Since there is a crossing, $\gamma$ also crosses the other strand of $\pi(L)$ which meets this crossing. This means that $\beta \cup \gamma$ transversely intersects $\pi(L)$ exactly twice.

If $\beta \cup \gamma$ bounds a disk in $F$, then this is a contradiction to $\pi(L)$ being weakly prime. If $\beta \cup \gamma$ does not bound a disk in $F$, then this is a contradiction to $r(\pi(L), F) \geq 4$.

Let $\delta$ be an outermost arc of intersection between $D$ and $\Sigma'$, such that one end of $\delta$ lies on $\partial X$, and the other end lies on $\Sigma$. $\delta$ bounds a triangular subdisk $\Delta \subset D$. By theorem 3.13, $\Delta$ is an inessential triangular disk, therefore $\Sigma'$ can be isotoped so that $\delta$ is isotoped off $D$.

Therefore all arcs and loops of intersection between $D$ and $\Sigma'$ can be isotoped off $D$. Thus $\partial D \cap \Sigma$ lies entirely within one region of $\Sigma$, and $\partial D \cap \partial X$ lies entirely within one quad $Q$ of $\partial X$. Similar arguments to the arc $\beta$ above, then show that $\partial D \cap (\Sigma \cup \partial X)$ is isotopic to an essential curve in $F$ that either
meets \( \pi(L) \) twice or not at all, depending on whether \( \partial D \cap \partial X \) runs between opposite arcs or the same arc of \( \partial \Sigma \cap \partial Q \). Both cases cause a contradiction with either \( r(\pi(L), F) \) being at least 4 or \( \pi(L) \) being weakly prime.

Therefore \( \Sigma \) is boundary-incompressible in \( X \). Similar arguments show that \( \Sigma' \) is boundary-incompressible in \( X \). Therefore both \( \Sigma \) and \( \Sigma' \) are essential in \( X \).

\[ \square \]

### 3.5 \( \pi_1 \)-Essential Checkerboard Surfaces

In this section, we will show that the checkerboard surfaces arising from a weakly generalised alternating projection are actually \( \pi_1 \)-essential. Recall that a surface \( S \) is essential in a 3-manifold \( M \) whenever \( S \) is \( \pi_1 \)-essential in \( M \), but the converse cannot be guaranteed unless \( S \) is two-sided in \( M \). Since at least one of the checkerboard surfaces is non-orientable and hence one-sided in \( X \), we must be very careful trying to strengthen Theorem 3.14 to the conclusion that the checkerboard surfaces are \( \pi_1 \)-essential.

The difficulty is working with one-sided surfaces stems from the fact that Dehn’s Lemma and the Loop Theorem do not necessarily apply. We will use the version of the Loop Theorem which we stated as Theorem 1.3. We will avoid the issue of one-sidedness by cutting any singular compressing disks into pieces to which the Loop Theorem can be applied.

We form the boundary pattern \((P, \Lambda)\) from \( X \) by cutting along both the checkerboard surfaces \( \Sigma \) and \( \Sigma' \). This cuts \( X \) into two 3-manifolds-with-boundary \( Y \) and \( Y' \), where \( \partial Y \cong \partial Y' \cong P \). Here \( P \) is an orientable surface homeomorphic to \( F \), and \( \Lambda \) is an embedded trivalent graph.

Recall that \( \Gamma \) is the 4-regular graph embedded in \( F \) that is obtained from \( \pi(L) \) by forgetting crossing information. The boundary pattern graph \( \Lambda \) is formed from \( \Gamma \) by blowing up every vertex into a quad. Thus if \( \Gamma \) has \( v \) vertices, \( e \) edges, and \( f \) faces, then \( \Lambda \) has \( 4v \) vertices, \( e + 4v \) edges, and \( f + v \) faces. \( \Lambda \) is cubic, bipartite, 3-edge-colourable, and 3-face-colourable. The face-colouring corresponds to \( \Sigma, \Sigma' \) and \( \partial X \), while the edge-colouring corresponds to \( \Sigma' \cap \partial X, \partial X \cap \Sigma \) and \( \Sigma \cap \Sigma' \). Note that the faces of \( \Lambda \) are not necessarily disks, however they are orientable subsurfaces of \( P \).
Figure 3.11: A planar projection graph and the associated boundary pattern graph \( \Lambda \) on \( P \cong S^2 \).

**Theorem 3.15.** Let \( \pi(L) \) be a weakly generalised alternating projection onto a generalised projection surface \( F \). Then both the associated checkerboard surfaces are \( \pi_1 \)-essential in the link exterior \( X \).

**Proof.** Let \( f : D \to X \) be a map from a disk \( D \) into the knot exterior, such that \( f(\partial D) \subset \Sigma \), where \( \Sigma \) is one of the checkerboard surfaces. Suppose that \( f(\partial D) \) is essential in \( \Sigma \). Of course \( f(\partial D) \) is nullhomotopic in \( X \).

If \( f \) is an embedding, then we are in the case of Theorem 3.14, so assume that \( f(D) \) is a singular disk. If \( \Sigma \) is two-sided in \( X \), then we can use the loop theorem to replace \( f(D) \) with an embedded disk \( D' \). This reduces to the case of Theorem 3.14, so we may assume that \( \Sigma \) is one-sided in \( X \). We can homotope \( f \) so that \( f(\partial D) \cap \Sigma = \emptyset \), since otherwise \( f \) can be replaced by a map \( g : D' \to X \), where \( D' \) is a subdisk of \( D \).

Consider the intersection of the other checkerboard surface \( \Sigma' \) with \( f(D) \). We can isotope \( \Sigma' \) into general position so that it misses the double points of \( f(\partial D) \) and the triple points and branch points of \( f(D) \). Then look at the pre-image \( f^{-1}(\Sigma') \) in \( D \). It consists of a collection of embedded loops and properly embedded arcs.

Let \( \alpha \) be an innermost loop of intersection between \( f^{-1}(\Sigma') \) and \( D \), which bounds a disk \( E \subset D \). Then \( f \) maps \( E \) to a singular disk \( f(E) \), but \( f(\alpha) \) lies
on the projection surface $F$ which is two-sided in $X$. If $f(\alpha)$ is nullhomotopic in $F$, then $f(E)$ can be homotoped off $F$. If $f(\alpha)$ is essential in $F$, then the Loop Theorem tells us that there is an embedded disk $E'$ such that $\partial E'$ is made up of subarcs of $f(\partial E)$ and $\partial E'$ is essential in $F$. But then $E'$ is an embedded compressing disk for $\Sigma'$ which contradicts Theorem 3.14. Hence $\alpha$ can be removed.

Let $\beta$ be an outermost arc of intersection between $f^{-1}(\Sigma')$ and $D$, which bounds a disk $E \subset D$, where $\partial E = \beta \cup \gamma$ with $\gamma \subset \partial D$. Now $f(E)$ is a singular bigon between $\Sigma$ and $\Sigma'$. If $f(E)$ is homotopic into a double arc, then $f$ can be homotoped to remove $\beta$. Otherwise $f(E)$ is an essential singular bigon. Consider the intersection of $f(\partial E)$ with the boundary pattern $P$. $f(\beta)$ lies in a face corresponding to $\Sigma'$ and $f(\gamma)$ lies in a face corresponding to $\Sigma$ while $f(E)$ lies entirely in either $Y$ or $Y'$ since $E$ is outermost. Without loss of generality, assume $f(E) \subset Y$. Thus since $P$ is two-sided in $Y$, we can apply the Loop Theorem to conclude that there is a properly embedded disk $E' \subset Y$ in such a way that $\partial E'$ is made up of subarcs of $f(\partial E)$. This contradicts either Theorem 3.12 if $E'$ is an embedded bigon, or Theorem 3.14 if $E'$ is a compressing disk for either $\Sigma$ or $\Sigma'$. Hence $\beta$ can be removed.

Thus the intersection between $D$ and $f^{-1}(\Sigma')$ is empty, and without loss of generality $f(D)$ lies in $Y$. Then $f(\partial D)$ is contained entirely within a single face of $\Lambda$ corresponding to $\Sigma$. Using that $P$ is two-sided in $Y$ an applying the Loop Theorem, there exists an embedded disk $D'$ such that $\partial D'$ comprises subarcs of $f(\partial D)$. But then $D'$ is an embedded compressing disk for $\Sigma$, a contradiction to Theorem 3.14. Therefore the homomorphism $\pi_1(\Sigma) \rightarrow \pi_1(X)$ is injective.

To prove $\pi_1$-boundary-injectivity, let $D$ be a disk such that $\partial D = \zeta \cup \zeta'$ where $\zeta$ and $\zeta'$ are connected arcs intersecting only in their endpoints. Let $f : D \rightarrow X$ be a map from the disk $D$ into the knot exterior, such that $f(\zeta) \subset \Sigma$, and $f(\zeta') \subset \partial X$. Suppose that $f(\partial D)$ is essential in $\Sigma \cup \partial X$.

Again, we may assume that $f$ is singular, that $\Sigma$ is one-sided in $X$, and that $\partial D \cap f^{-1}(\Sigma) = \emptyset$. Put $f(D)$ into general position with $\Sigma$, $\Sigma'$ and $\partial X$ and consider the intersection of $f^{-1}(\Sigma')$ with $D$.

If $\alpha$ is an innermost arc of intersection between $f^{-1}(\Sigma')$ and $D$, or $\beta$ is
an outermost arc of intersection between \( f^{-1}(\Sigma') \) and \( D \) with \( \partial \beta \subset \Sigma \), then \( \alpha \) and \( \beta \) can be homotoped off in the same manner as the first part of this proof. The rest of this proof uses variations on the same trick so we will brush over some of the details.

If \( \gamma \) is an outermost arc of intersection between \( f^{-1}(\Sigma') \) and \( D \) with \( \partial \gamma \subset \partial X \), then \( \gamma \) can be homotoped away, since otherwise the Loop Theorem shows the existence of an embedded boundary compressing disk for \( \Sigma' \) which contradicts Theorem 3.14.

If \( \delta \) is an outermost arc of intersection between \( f^{-1}(\Sigma') \) and \( D \) with one end of \( \delta \) on \( \Sigma \) and the other end on \( \partial X \), then \( \delta \) can be homotoped away. Otherwise the Loop Theorem shows the existence of an embedded trigon between \( \Sigma, \Sigma' \) and \( \partial X \), which contradicts Theorem 3.13.

Hence all intersections of \( D \) with \( f^{-1}(\Sigma') \) can be homotoped away. Thus \( f(D) \) is a singular boundary compressing disk for \( \Sigma \). Assume that \( f(D) \subset Y \). But \( f(\partial D) \) lies in \( P \) and meets \( \Lambda \) transversely. We apply the Loop Theorem again so show the existence of an embedded disk \( D' \) with \( \partial D' \) made up of subarcs of \( f(\partial D) \). Now \( D' \) is either a compressing disk or boundary-compressing disk for \( \Sigma \), both of which contradict Theorem 3.14. Note \( D' \) cannot be a compressing disk for \( \partial X \), since \( \partial D' \) would be contained in a single quadrilateral disk face of \( \Lambda \), but the Loop Theorem forces \( \partial D' \) to be essential in \( P \).

Hence the map \( \pi_1(\Sigma, \partial \Sigma) \to \pi_1(X, \partial X) \) is injective, and \( \Sigma \) is \( \pi_1 \)-essential in \( X \). Similarly, for \( \Sigma' \). Therefore both \( \Sigma \) and \( \Sigma' \) are \( \pi_1 \)-essential in \( X \). \( \square \)

In the process of proving the above theorem, we have given new proofs of Theorem 1.9 and Theorem 2.3.

We note that if \( K \) was a knot, to prove \( \pi_1 \)-boundary injectivity in Theorem 3.15, we could invoke Lemma 1.8 once we have established \( \pi_1 \)-injectivity. We would be able to apply this Lemma, since Theorem 3.14 guarantees that \( K \) bounds an essential non-orientable spanning surface, which implies that \( K \) is non-trivial.

Since one of the checkerboard surfaces is non-orientable, we have established that weakly generalised alternating knots satisfy the Strong Neuwirth
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Conjecture which we stated as Conjecture 1.21.

**Theorem 3.16.** Suppose that $K$ is a weakly generalised alternating knot. Then $K$ satisfies the Strong Neuwirth Conjecture.

We finish this section by giving a proof that there are no singular essential bigons between the two checkerboard surfaces associated to a weakly generalised alternating projection. A *singular bigon* is a map

$$f : B \to X$$

where $B$ is a disk and $\partial B$ consists of two connected arcs $\beta$ and $\beta'$ where $f(\beta) \subset \Sigma$ and $f(\beta') \subset \Sigma'$. A singular bigon is inessential if it can be homotoped into a double arc of $\Sigma \cap \Sigma'$ where $B$ remains a singular bigon throughout the homotopy. Otherwise, it is essential.

**Theorem 3.17.** Let $\pi(K)$ be a weakly generalised alternating projection onto a closed orientable surface $F$. Then there are no singular essential bigons between the checkerboard surfaces $\Sigma$ and $\Sigma'$.

**Proof.** Let $f : B \to X$ be a singular bigon between $\Sigma$ and $\Sigma'$. By transversality we can assume that $f^{-1}(\Sigma)$ and $f^{-1}(\Sigma')$ are properly embedded arcs and loops in $B$.

Let $B'$ be an innermost bigon between $f^{-1}(\Sigma)$ and $f^{-1}(\Sigma')$. Suppose there is a loop of intersection between $\hat{B}'$ and $f^{-1}(\Sigma)$. Let $\ell$ be an innermost such loop. Note that $\ell$ does not intersect $f^{-1}(\Sigma')$ since otherwise $B'$ would not have been innermost. $\ell$ bounds a subdisk $D$ of $B'$. Then $f(D)$ is either a (possibly singular) compressing disk for $\Sigma$ which contradicts the $\pi_1$-injectivity of $\Sigma$ shown in Theorem 3.15, or $\ell$ can be homotoped off $B'$. Thus all loops of intersection between $\hat{B}'$ and $f^{-1}(\Sigma)$ or $f^{-1}(\Sigma')$ can be homotoped away.

Therefore $f(B') \cap (\Sigma \cup \Sigma') = f(\partial B')$. Look at the intersection of $f(\partial B')$ with the boundary pattern $(P, \Lambda)$. Let $\partial B' = \beta \cup \beta'$ where $f(\beta) \subset \Sigma$ and $f(\beta') \subset \Sigma'$. Then $f(\beta)$ lies entirely within one face of $\Lambda$ corresponding to $\Sigma$ and $f(\beta')$ lies entirely within one face of $\Lambda$ corresponding to $\Sigma'$.

If $f(\partial B')$ is essential in $P$, then $f(B)$ is a singular compressing disk for $P$ and hence $F$. We can use the loop theorem to find an embedded compressing
disk $E$ for $P$ such that $E \cap \Lambda \subset f(\partial B') \cap \Lambda$. But then $E$ is either an embedded bigon between $\Sigma$ and $\Sigma'$ contradicting Theorem 3.12, or $E$ is an embedded compressing disk for one of $\Sigma$ or $\Sigma'$ contradicting Theorem 3.14. Hence $f$ can be homotoped to remove $B'$.

This reduces the number of double points between $f^{-1}(\Sigma)$ and $f^{-1}(\Sigma')$. By repeating these steps we can show that $f$ can be homotoped so that $f(B)$ is disjoint from $\Sigma$ and $\Sigma'$.

The innermost bigon case now applies to the whole singular bigon $f(B)$, and we get a contradiction. \hfill \Box

### 3.6 Primality and Splittability

In this section, we will show that a link with a weakly generalised alternating projection is prime, non-split, and non-trivial.

**Theorem 3.18.** If $L$ is a weakly generalised alternating link, then $L$ is non-split.

**Proof.** Let $\pi(L)$ be a weakly generalised alternating projection onto a generalised projection surface $F$, with checkerboard surfaces $\Sigma$ and $\Sigma'$ in standard position. Then $L$, $\Sigma$ and $\Sigma'$ are contained in a small neighbourhood $F \times I$ of $F$.

Suppose that $S$ is a 2-sphere properly embedded in the link exterior $X$. If $S$ is disjoint from $F \times I$, then $S$ must bound a 3-ball in $X$, since $F$ is non-split. Thus $S$ must intersect $F \times I$ and we assume that $S$ is in general position with respect to $\Sigma$ and $\Sigma'$.

$S$ intersects each of $\Sigma$ and $\Sigma'$ in a collection of loops (Figure 3.12). Let $\ell$ be an innermost loop of intersection between $S$ and $\Sigma$. If $\ell$ is disjoint from $\Sigma'$, then $\ell$ must bound a disk in $\Sigma$ since Theorem 3.14 shows that $\Sigma$ is essential in $X$. Thus $S$ can be isotoped to remove the intersection $\ell$.

Suppose $B \subset S$ is an innermost bigon where $\partial B = \beta \cup \beta'$ with $\beta \subset \Sigma$ and $\beta' \subset \Sigma'$. Theorem 3.12 shows that $B$ must be an inessential bigon, so both endpoints $\beta \cap \beta'$ must lie on the same double arc $\alpha$. Therefore we can isotope $S$ across $\alpha$ to remove $B$, and reduce the number of points of $S \cap \Sigma \cap \Sigma'$ by
two. Note that if the number of points of $S \cap \Sigma \cap \Sigma'$ is nonzero, then there must be a bigon.

We can continue in this fashion to remove all bigons and all isolated loops of intersection until $S$ is disjoint from both $\Sigma$ and $\Sigma'$. Hence $S$ is disjoint from $F \times I$ and must bound a 3-ball in $X$. So there are no splitting spheres for $L$.

Note that this gives a new proof of the classical result of Menasco that a reduced non-split prime alternating planar diagram represents a non-split link, which we stated in Theorem 1.1, and we have also generalised Theorems 2.5 and 3.4.

Menasco and Thistlethwaite [61] gave the first geometric proof that reduced prime alternating planar diagrams represent non-trivial knots, and this result was extended to generalised alternating knots by Ozawa, as we stated in Theorem 2.2. We will now extend this result to weakly generalised alternating knots.

The unknot does not bound any essential non-orientable spanning surfaces, so it follows from Theorem 3.15 and Lemma 3.3 that the unknot is not weakly generalised alternating. We provide another direct proof below, using the familiar techniques of this chapter.
Theorem 3.19. Let $\pi(K)$ be a weakly generalised alternating projection of a knot $K$ onto a closed orientable surface $F$. Then $K$ is non-trivial.

Proof. Suppose that $D$ is a disk properly embedded in the knot exterior $X$. Let $\Sigma$ and $\Sigma'$ be checkerboard surfaces in standard position to $\pi(K)$. Isotope $D$ to be in general position with respect to $\Sigma$ and $\Sigma'$.

Suppose that $\ell$ is an innermost arc of intersection between $D$ and $\Sigma$. By Theorem 3.14, $\Sigma$ is essential in $X$, so $\ell$ must bound a disk in $\Sigma$. Since $X$ is irreducible, we can isotope $D$ to remove $\ell$. Similarly, all innermost loops of intersection between $D$ and $\Sigma'$ can be removed.

Suppose $B \subset D$ is an innermost bigon where $\partial B = \beta \cup \beta'$ with $\beta \subset \Sigma$ and $\beta' \subset \Sigma'$. Theorem 3.12 shows that $B$ must be an inessential bigon, so we can isotope $D$ to remove $B$, and reduce the number of points of $D \cap \Sigma \cap \Sigma'$ by two.

Suppose $\alpha$ is an outermost arc of intersection between $D$ and $\Sigma$. Then $\alpha$ bounds a disk $D' \subset D$ between $\Sigma$ and $\partial X$. Theorem 3.14 shows that $D'$ cannot be a boundary-compressing disk, so we can isotope $D$ to remove $\alpha$. Similarly, all outermost arcs of intersection between $D$ and $\Sigma'$ can be removed.

Suppose $\Delta \subset D$ is an outermost trigon where $\partial \Delta = \gamma \cup \gamma' \cup \delta$ with $\gamma \subset \Sigma$, $\gamma' \subset \Sigma'$, and $\delta \subset \partial X$. Theorem 3.13 shows that $\Delta$ must be an inessential trigon, so we can isotope $D$ to remove $\Delta$, and reduce the number of points of $D \cap \Sigma \cap \Sigma'$ by one.

By choosing to perform the above isotopies in an appropriate order, $D$ can be isotoped to be disjoint from $\Sigma$ and $\Sigma'$. Thus $\partial D$ must be contained in some region of $\partial X \setminus (\partial \Sigma \cup \partial \Sigma')$. But $\partial \Sigma$ and $\partial \Sigma'$ cut $\partial X$ into a set of quadrilateral disks, so $D$ must be parallel into $\partial X$, proving that $K$ is non-trivial.

It follows from the previous two theorems that the exterior $X$ of a weakly generalised alternating link is boundary-irreducible.

Lemma 3.20. Let $X$ be the exterior of a weakly generalised alternating link $L$. Then $X$ is boundary-irreducible.
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Proof. If \( X \) is not boundary-irreducible, then there is a disk \( D \) properly embedded in \( X \) such that \( \partial D \) is essential in \( \partial X \). If \( L \) is a knot, this contradicts Theorem 3.19.

Suppose then that \( L \) has at least two components. Let \( C \) be the component of \( \partial X \) which contains \( \partial D \). Let \( V \) be a regular neighbourhood of \( D \cup C \) in \( X \). Then \( \partial V \) is a splitting sphere for \( L \), contradicting Theorem 3.18. \( \square \)

We complete this section, by showing that a weakly generalised alternating link is prime. In the process we give a new proof that a reduced prime planar alternating diagram represents a prime link (Theorem 1.1), and prove that a generalised alternating diagram represents a prime link.

By Theorem 3.4, it was already known that a weakly generalised alternating projection \( \pi(L) \) onto a Heegaard torus \( F \), where all the regions of \( F \setminus \pi(L) \) are disks, represents a prime link.

Theorem 3.21. If \( L \) is a weakly generalised alternating link, then \( L \) is prime.

Proof. Suppose \( L \) is not prime, and let \( A \) be an essential annulus at meridional slope. Cutting along \( A \) decomposes \( X \) into non-trivial summands. Let the checkerboard surfaces \( \Sigma \) and \( \Sigma' \) be in standard position with respect to a weakly generalised projection \( \pi(L) \) onto a generalised projection surface \( F \).

Recall that \( \partial \Sigma \) and \( \partial \Sigma' \) form a 4-regular quadrangulation of \( \partial X \).

Let the boundary components of \( A \) be \( \alpha \) and \( \alpha' \). \( A \) has meridional slope and \( \Sigma \) and \( \Sigma' \) are spanning surfaces, so \( A \) can be isotoped into general position with \( \Sigma \) and \( \Sigma' \) so that each of \( \alpha \) and \( \alpha' \) meets each of \( \partial \Sigma \) and \( \partial \Sigma' \) exactly once. Therefore there must be an arc of intersection \( \beta \) between \( A \) and \( \Sigma \) where one endpoint of \( \beta \) meets \( \alpha \) and the other endpoint meets \( \alpha' \). Any essential arc of intersection which had both endpoints on \( \alpha \) say, would contradict the \( \pi_1 \)-boundary-injectivity of \( \Sigma \). Similarly, there is an arc of intersection \( \beta' \) between \( A \) and \( \Sigma' \) where one endpoint of \( \beta' \) meets \( \alpha \) and the other endpoint meets \( \alpha' \).

Consider the intersection of \( A \) with \( \Sigma \cup \Sigma' \) (Figure 3.13). Any loop of intersection \( \ell \) between \( A \) and \( \Sigma \) must be trivial in \( A \) since otherwise \( \ell \) would intersect \( \beta \), which is impossible because \( \Sigma \) is properly embedded in \( X \).

Let \( \ell \) be an innermost loop of intersection between \( A \) and \( \Sigma \) so that \( \ell \) bounds a disk \( E' \subset A \) and \( E' \) is disjoint from \( \Sigma' \). Theorem 3.14 shows that
Σ is essential in X, so ℓ bounds a disk in Σ and we can isotope A to remove ℓ.

Let $B \subset A$ be a bigon between Σ and Σ'. Then A can be isotoped to remove $B$ and reduce the number of points of $A \cap \Sigma \cap \Sigma'$ by two, since every bigon between Σ and Σ' is inessential.

Now the only possible intersection between Σ and Σ' in A is a single intersection of the arc $\beta$ with the arc $\beta'$. In that case, there is a trigon $\Delta$ between $\beta$, $\beta'$, and $\partial A$. Theorem 3.13 shows that $\Delta$ is inessential, so it is possible to isotope $A$ to remove $\Delta$ and $\beta \cap \beta'$.

Therefore $\beta$ and $\beta'$ cut $A$ into two disks, whose interiors are disjoint from both of Σ and Σ'. Let $D$ be one of these disks, where $\partial D = \beta \cup \gamma \cup \beta' \cup \gamma'$, and $\gamma$ and $\gamma'$ are subarcs of $\alpha$ and $\alpha'$ respectively (Figure 3.14). Now $\partial D$ is a loop embedded in $F$ which meets $\pi(L)$ exactly twice, so $\partial D$ cannot be essential in $F$ because $r(\pi(L), F) \geq 4$. Thus $\partial D$ must be trivial in $F$ but $\pi(L)$ is weakly prime so $\partial D$ bounds a disk $E \subset F$ which intersects $\pi(L)$ is a
Let $B'$ be one component of $E \setminus \pi(L)$. Then $B'$ is a boundary-compressing disk for $A$, a contradiction to $A$ being essential. Alternatively, let $D' = A \setminus D$, where it can be seen by the previous paragraph that $D'$ also bounds $E \subset F$. Therefore $A$ is parallel into $\partial X$, a contradiction to $A$ being essential.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.14.png}
\caption{The subdisk $D$ of the annulus $A$.}
\end{figure}

3.7 Doubling along the Boundary

The checkerboard surfaces are compact surfaces properly embedded in the link exterior. However the knot exterior, while compact, is not closed. We would prefer to work with closed manifolds and closed embedded surfaces, so that we can more readily apply the theory of Freedman-Hass-Scott.

In order to do this, we will choose to double the knot exterior along its boundary. Let $X$ be a link exterior. Take another copy of $X$ with its orientation reversed and call it $\tilde{X}$. Then if we glue $\partial X$ to $\partial \tilde{X}$ by the identity map, we call $2X = X \cup \tilde{X}$ the double of the link exterior.
Note that $2X$ is a closed orientable 3-manifold which contains an essential torus arising from the boundary of each component of the link. Provided $L$ is non-split and not the unknot, then $2X$ is irreducible. Note that if $K$ is the unknot, then $X$ is a solid torus and $2X \cong S^2 \times S^1$, which is not irreducible.

If $\Sigma$ is a properly embedded surface in $X$, then $2\Sigma$ is a closed surface in $2X$ where $2\Sigma = \Sigma \cup \hat{\Sigma}$ and the identification is induced by the identity gluing of $\partial X$ to $\partial \hat{X}$. We call $2\Sigma$ the double of $\Sigma$. We must be careful not to confuse the double of a manifold $2M$, with the double cover of that manifold, which we denote as $\hat{M}$.

We will now prove three lemmas showing that the properties of $\pi_1$-essential surfaces and essential bigons transfer to the doubled spanning surfaces and doubled link exteriors.

**Lemma 3.22.** $\Sigma$ is a properly embedded essential surface in an irreducible and boundary-irreducible link exterior $X$ if and only if $2\Sigma$ is a closed essential surface embedded in $2X$.

*Proof.* Suppose $2\Sigma$ is essential in $2X$. Any compressing disk for $\Sigma$ in $X$ is also a compressing disk for $2\Sigma$ in $2X$. Suppose that $E$ is a boundary-compressing disk for $\Sigma$. Then $\hat{E}$ is a boundary compressing disk for $\hat{\Sigma}$. Thus $\partial(2E)$ is not contractible in $2\Sigma$, so $2E$ is a compressing disk for $2\Sigma$ in $2X$, which is a contradiction. Hence $\Sigma$ is essential in $X$.

For the converse, $2X$ has no boundary so it is only necessary to check that $2\Sigma$ is incompressible in $2X$. Suppose then to the contrary, that $D$ is a compressing disk for $2\Sigma$. If $D$ can be isotoped to lie entirely within either $X$ or $\hat{X}$, then $D$ would be a compressing disk for $\Sigma$ or $\hat{\Sigma}$ respectively, a contradiction. Thus $D$ must lie partially in $X$, with the remainder in $\hat{X}$.

Isotope $D$ to be in general position with each of $2\Sigma$ and $\partial X$ (Figure 3.15). Let $\ell$ be an innermost loop of intersection between $D$ and $\partial X$. Then $\ell$ must bound a disk in $\partial X$ since $\partial X$ is incompressible in $2X$. Thus we can isotope $D$ to remove $\ell$. By repeating this procedure on each innermost closed loop, we can remove all closed loops of intersection.

Let $\alpha$ be an outermost arc of intersection between $D$ and $\partial X$. Then $\alpha$ bounds a subdisk $D' \subset D$ where $\partial D' = \alpha \cup \gamma$ and $\gamma \subset \partial D$. Without
loss of generality, we may assume that \( D' \subset X \) rather than \( D' \subset \tilde{X} \). Since \( \Sigma \) is boundary-incompressible in \( X \), \( \gamma \) must bound a disk \( E' \subset \Sigma \) where \( \partial E' = \gamma \cup \gamma' \), where \( \gamma' \subset \partial \Sigma \). Thus we can isotope \( D \) to remove \( \alpha \).

Hence we can isotope \( D \) to lie completely in \( X \) or \( \tilde{X} \), in which case \( D \) is a compressing disk for \( \Sigma \) or \( \tilde{\Sigma} \). If \( D \) is a compressing disk for \( \tilde{\Sigma} \), then \( \tilde{D} \) is the reflection of \( D \) across \( \partial X \). Either way, we have a contradiction, which proves that \( 2\Sigma \) is incompressible in \( 2X \).

**Lemma 3.23.** \( \Sigma \) is a properly embedded \( \pi_1 \)-essential spanning surface in a knot exterior \( X \) if and only if \( 2\Sigma \) is a closed \( \pi_1 \)-essential surface embedded in \( 2X \).

**Proof.** Suppose \( 2\Sigma \) is \( \pi_1 \)-essential in \( 2X \). Then \( \Sigma \) is \( \pi_1 \)-injective in \( X \), since any essential loop in \( \Sigma \) which is nullhomotopic in \( X \) would also be nullhomotopic in \( 2X \). Let \( D \) be a singular boundary-compressing disk for \( \Sigma \). Then \( 2D \) is a non-trivial singular compressing disk for \( 2\Sigma \), since \( \partial(2D) \) is not contractible in \( 2X \). This contradicts the \( \pi_1 \)-injectivity of \( 2\Sigma \).

Conversely, suppose \( \Sigma \) is \( \pi_1 \)-essential in \( X \) and let \( f : D \to 2X \) be a singular compressing disk for \( 2\Sigma \). \( f(D) \) cannot lie entirely inside \( \Sigma \) or \( \tilde{\Sigma} \).
Without loss of generality, let $E$ be a subdisk of $D$ such that $\partial E = \alpha \cup \beta$ where $f(\alpha) \subset \Sigma$ and $f(\beta) \subset \partial X$. If $f(\alpha)$ is homotopic into $\partial X$ through $\Sigma$, then we can homotope $f(E)$ into $\bar{\Sigma}$. If $f(\alpha)$ is essential in $(\Sigma, \partial \Sigma)$, then $f(E)$ is a singular boundary-compressing disk for $\Sigma$.

By repeating this procedure, either $D$ contains a subdisk $E$ such that $f(\alpha)$ is non-trivial in $\pi_1(\Sigma, \partial \Sigma)$ but trivial in $\pi_1(X, \partial X)$ or $f(D)$ can be homotoped into either $\Sigma$ or $\bar{\Sigma}$, both of which are contradictions.

**Lemma 3.24.** Let $\Sigma$ and $\Sigma'$ be properly embedded $\pi_1$-essential spanning surfaces in the exterior $X$ of a non-trivial non-split link $L$. Then there are no essential bigons between $\Sigma$ and $\Sigma'$ in $X$, and no essential trigons between $\Sigma$, $\Sigma'$, and $\partial X$ if and only if there are no essential bigons between $2\Sigma$ and $2\Sigma'$ in $2X$.

**Proof.** If $B'$ is an essential bigon between $\Sigma$ and $\Sigma'$, then $B'$ is an essential bigon between $2\Sigma$ and $2\Sigma'$. If $\Delta'$ is an essential trigon between $\Sigma$, $\Sigma'$, and $\partial X$, then $2\Delta'$ is a bigon between $2\Sigma$ and $2\Sigma'$.

Let $p$ and $\bar{p}$ be the two points of $2\Delta' \cap (2\Sigma \cap 2\Sigma')$. If $2\Delta'$ is inessential, then $2\Delta'$ can be homotoped into an arc $\alpha'$ joining $p$ to $\bar{p}$. Let $q = \alpha' \cap \partial X$. Then $\Delta'$ can be homotoped into $q$ which contradicts that $\Delta'$ is essential. Hence $2\Delta'$ is an essential bigon between $2\Sigma$ and $2\Sigma'$.

Conversely, suppose that $B$ is an essential bigon between $2\Sigma$ and $2\Sigma'$. If $B$ lies entirely within $X$ or $\bar{X}$, then $B$ is an essential bigon either between $\Sigma$ and $\Sigma'$, or between $\bar{\Sigma}$ and $\bar{\Sigma}'$ respectively. If not, then $\partial X$ must meet $B$. Isotope $B$ to be in general position with respect to $2\Sigma$, $2\Sigma'$, and $\partial X$ (Figure 3.16).

Suppose there is an innermost loop of intersection $\ell$ between between $\partial X$ and $B$. $\ell$ bounds a subdisk of $B$, and since $\partial X$ is incompressible in $2X$, $\ell$ bounds a subdisk of $\partial X$. Therefore we can isotope $B$ to remove $\ell$, and all other innermost loops of intersection.

Suppose there is an outermost arc of intersection $\alpha$ between between $\partial X$ and $B$ where both ends of $\alpha$ lie on $2\Sigma$. Then there is a subdisk $D \subset B$ between $\partial X$ and $2\Sigma$, where $\partial D = \alpha \cup \gamma$ with $\gamma \subset \partial B$. Assume that $\gamma \subset \Sigma$, an analogous argument will work if $\gamma \subset \bar{\Sigma}$. Since $\Sigma$ is boundary-incompressible
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Figure 3.16: An essential bigon between $2\Sigma$ and $2\Sigma'$ and its intersections with $\partial X$.

in $X$, $\gamma$ must cobound with $\partial \Sigma$ a disk $D' \subset \Sigma$. Thus we can isotope $B$ to remove $\alpha$. Similarly we can isotope away all arcs of intersection between $\partial X$ and $B$ when both endpoints lie on $2\Sigma'$.

Suppose there is an arc of intersection $\delta$ between between $\partial X$ and $B$ where one end of $\delta$ lies on $2\Sigma$ and the other on $2\Sigma'$. Then there is a subdisk $\Delta \subset B$ which is an outermost trigon, and can be assumed to lie in $X$ rather than $\tilde{X}$. The components of $\partial \Delta$ lie in $\Sigma$, $\Sigma'$ and $\partial X$ respectively.

If $\Delta$ is an essential trigon in $X$, the proof is complete. If $\Delta$ is an essential trigon in $\tilde{X}$, then its reflection $\tilde{\Delta}$ is an essential trigon in $X$.

So assume that $\Delta$ is inessential. Then $\Delta$ can be homotoped into a triple point, which means that $B$ can be isotope in such a way that $\Delta$ is isotope into $\tilde{X}$.

Therefore we can reduce intersections between $B$ and $\partial X$ until $B$ lies entirely within either $X$ or $\tilde{X}$. Hence $B$ is an essential bigon in either $X$ or $\tilde{X}$. If $B$ ends up in $\tilde{X}$, then $\tilde{B}$, which is the reflection of $B$ across $\partial X$, is an essential bigon in $X$. \qed
3.8 Relative 1-Line Property

In this section, we introduce the relative 1-line property which restricts how the lifts of two $\pi_1$-injective immersed surfaces intersect in the universal cover. The work of Freedman-Hass-Scott [25] then details how this geometric definition is equivalent to an algebraic condition involving fundamental groups. We then consider the relative 1-line property in our simplified situation where, both surfaces are embedded downstairs and the ambient manifold is orientable. We will then show that the relative 1-line property is equivalent to there being no essential bigons between $\pi_1$-essential spanning surfaces.

Let $S$ be a closed surface that is neither $S^2$ nor $P^2$, and let $M$ be a closed $P^2$-irreducible 3-manifold. Let $f : S \rightarrow M$ be a two-sided $\pi_1$-injective immersion. This means that there is an induced injective homomorphism $f_* : \pi_1(S) \rightarrow \pi_1(M)$, and $\pi_1(S)$ is a subgroup of $\pi_1(M)$.

We assume that $f$ is PL least area in its homotopy class. It is a theorem of Jaco and Rubinstein [48] that every $\pi_1$-injective map has a PL least area representative in its homotopy class. It is implicit here that we are working in the piecewise-linear category and that $M$ has a fixed triangulation $T$. PL area is measured first by weight which is the number of intersections of $S$ with the 1-skeleton of $T$, and then by total arc length of intersections of $S$ with the 2-skeleton of $T$, where $T^{(2)}$ has an appropriate metric. For instance, each face of $T$ could be equipped with the hyperbolic metric of an ideal triangle. All the following results of Freedman-Hass-Scott follow in the PL category by [48]. This allows us to avoid the smooth category, where the much more difficult existence results of Meeks and Yau must be used. See also Hass and Scott [31] for further existence results.

Let $S$ and $S'$ be closed surfaces embedded in a 3-manifold $M$. A product region between $S$ and $S'$ is a handlebody $V$ such that $\partial V = R \cup R'$, $R \cong R'$, $V \cong R \times I$, and $R \cap R' = \partial R = \partial R'$, where $R$ and $R'$ are connected subsurfaces of $S$ and $S'$ respectively.

**Lemma 3.25** (Freedman-Hass-Scott [25]). Let $f : S \rightarrow M$ and $f' : S' \rightarrow M$ be embeddings of closed orientable surfaces. If $f$ and $f'$ are least area, then
there are no product regions between \( f(S) \) and \( f'(S') \).

**Theorem 3.26** (Freedman-Hass-Scott [25]). Let \( f : S \to M \) be a least area map which is homotopic to an embedding \( f' : S \to M \). Then either \( f \) is an embedding or \( f(S) \) double covers a 1-sided surface in \( M \).

Let \( M_S \) be the cover of \( M \) such that \( \pi_1(M_S) = \pi_1(S) \). Then \( f \) lifts to an embedding \( f_S : S \to M_S \). \( f_S \) is a homotopy equivalence, and is also least area.

Consider the universal cover \( \tilde{M} \). Let \( \tilde{S} \) denote the pre-image in \( \tilde{M} \) of \( f_S \). \( \tilde{S} \) is a collection of embedded planes. Let \( P \) be one such plane, and let \( hP \) denote its translate by \( h \in \pi_1(M) \).

Let \( f' : S' \to M \) be an embedding of another closed orientable surface \( S' \) into \( M \), and let \( M_{S'} \) be the cover of \( M \) associated to \( \pi_1(S') \). Then \( f_{S'} \) is a least area map which is a homotopy equivalence between \( S' \) and \( M_{S'} \). Let \( \tilde{S}' \) denote the lift of \( f_{S'}(S') \) to \( \tilde{M} \), which is again a collection of embedded planes. Let \( P' \) be one such plane, and let \( gP' \) denote its translate by \( g \in \pi_1(M) \).

The stabiliser of a plane \( P \), written \( \text{stab}(P) \) is the subgroup of \( \pi_1(M) \) that fixes \( P \) set-wise. \( \pi_1(S) \) is a subgroup of \( \text{stab}(P) \) of index either one or two. The index of \( \text{stab}(P) \) in \( \pi_1(S) \) is 2 if and only if \( f(S) \) double covers a one-sided surface in \( M \). The stabiliser of the plane \( gP \) is conjugate to \( \text{stab}(P) \), and in particular \( g\pi_1(S)g^{-1} \) is a subgroup of index one or two in \( \text{stab}(gP) \).

Two embedded surfaces \( S \) and \( S' \) satisfy the relative 1-line property in \( M \) if the intersection in \( \tilde{M} \) of \( gP \) and \( hP \) is empty or a single line, for any planes \( gP \) in \( \tilde{S} \) and \( hP \) in \( \tilde{S}' \).

Let \( G = g\pi_1(S)g^{-1} \cap h\pi_1(S')h^{-1} \). Suppose that \( S \) and \( S' \) satisfy the relative 1-line property in \( M \). If \( gP \) and \( hP' \) are disjoint in \( \tilde{M} \), then \( \text{stab}(gP) \cap \text{stab}(hP') \) is trivial. If \( gP \) intersects \( hP' \) is a single line \( l \), then \( \text{stab}(gP) \cap \text{stab}(hP') = \text{stab}(l) = \mathbb{Z} \). But \( g\pi_1(S)g^{-1} \leq \text{stab}(gP) \) and \( h\pi_1(S')h^{-1} \leq \text{stab}(hP') \), so \( G \) is a subgroup of \( \mathbb{Z} \), and is therefore either trivial or infinite cyclic. The converse was proved by Freedman, Hass and Scott.
Theorem 3.27 (Freedman-Hass-Scott [25]). Let \( f : S \to M \) and \( f' : S' \to M \) be least area two-sided embeddings of closed surfaces \( S \) and \( S' \) in an orientable 3-manifold \( M \). Then \( S \) and \( S' \) have the relative 1-line property in \( M \) if and only if \( G = g\pi_1(S)g^{-1} \cap h\pi_1(S')h^{-1} \) is trivial or infinite-cyclic for all \( g, h \in \pi_1(M) \).

Theorem 3.27 was actually proved for the case where \( f \) and \( f' \) are \( \pi_1 \)-essential immersions in \( M \), but we will only be interested in the case where \( f \) and \( f' \) are embeddings of the doubles of \( \pi_1 \)-essential spanning surfaces. They also defined the 1-line property for a single \( \pi_1 \)-essential immersion, in which case it is necessary to consider the intersections between \( P \) and its translate \( gP \).

The spanning surfaces in a knot exterior are compact with boundary. In order to apply Theorem 3.27, we now define some closed orientable surfaces embedded in a closed manifold. Let \( \Sigma \) and \( \Sigma' \) be spanning surfaces for an \( m \)-component link \( L \), whose exterior is \( X \).

Let \( 2X \) be the double of \( X \). It is possible to calculate a presentation for \( \pi_1(2X) \) from a presentation of \( \pi_1(X) \) by taking a free product with amalgamation over the first boundary component, and HNN-extensions over the other components of \( \partial X \).

The abelianisation of \( \pi_1(X) \) is \( H_1(X) \cong \mathbb{Z}^m \), so any homomorphism \( \pi_1(X) \to \mathbb{Z}_2 \) must factor through \( \mathbb{Z}^m \),

\[
\pi_1(X) \to \mathbb{Z}^m \to \mathbb{Z}_2.
\]

Let \( h : \mathbb{Z}^m \to \mathbb{Z}_2 \) be the epimorphism that sends every meridian \( \mu_j \) of \( X \) to the non-trivial element of \( \mathbb{Z}_2 \).

\( H_1(2X) \cong \mathbb{Z}^m \), since a generator of \( H_1(X) \) is freely homotopic to a generator of \( H_1(\tilde{X}) \) in \( 2X \). Thus any homomorphism \( \pi_1(2X) \to \mathbb{Z}_2 \) must similarly factor through \( \mathbb{Z}^m \). Therefore \( \pi_1(2X) \) has a canonical index two subgroup \( H \) which corresponds to the kernel of \( h \). Let \( W \) be the double cover of \( 2X \) such that \( \pi_1(W) = H \).

Geometrically, \( W \) is formed by cutting \( 2X \) along \( 2\Sigma \), where \( \Sigma \) is a \( \pi_1 \)-essential spanning surface for \( L \), and gluing two copies of \( 2X \setminus 2\Sigma \) in a cyclic
fashion. Usually, one forms this double cover cutting by letting \( \Sigma \) be any Seifert surface and hence two-sided in \( X \), however it is also possible to cut along a one-sided spanning surface. In that case \( 2X \setminus 2\Sigma \) will only have one boundary component, but the resulting \( W \) will be homeomorphic.

Let \( \Omega \) be the lift of \( 2\Sigma \) to \( W \). Then \( \Omega \) is homeomorphic to the orientable double cover of \( 2\Sigma \) if \( \Sigma \) is non-orientable, or homeomorphic to two copies of \( 2\Sigma \) if \( \Sigma \) is orientable. Similarly, define \( \Omega' \) to be the lift of \( 2\Sigma' \) to \( W \).

We say that \( \Sigma \) and \( \Sigma' \) have the relative 1-line property if \( \Omega \) and \( \Omega' \) have the relative 1-line property in \( W \). This makes sense since \( \Omega \) and \( \Omega' \) are two-sided closed surfaces in \( W \).

**Lemma 3.28.** Let \( L \) be a non-trivial non-split link. Then \( W \) is aspherical.

**Proof.** Since \( L \) is non-trivial and non-split, \( X \) is irreducible. Lemma 3.20 shows that \( X \) is also boundary-irreducible. Therefore \( 2X \) is irreducible. Since \( \pi_1(2X) \) is infinite, it follows from the Sphere Theorem [77] that \( 2X \) is aspherical, and so must be \( W \). \( \square \)

**Theorem 3.29.** Let \( \Omega \) and \( \Omega' \) be closed \( \pi_1 \)-essential surfaces embedded in a closed aspherical 3-manifold \( W \). Then \( \Omega \) and \( \Omega' \) have the relative 1-line property if and only if there are no essential bigons between \( \Omega \) and \( \Omega' \).

**Proof.** Suppose \( \Omega \) and \( \Omega' \) have the relative 1-line property, where \( \Omega \) and \( \Omega' \) are least area in their homotopy classes. Let \( \hat{W} \) be the universal cover of \( W \), and let \( g \in \pi_1(W) \). By Theorem 3.27, this means that for some plane \( gP \) from \( \hat{\Omega} \), we can choose \( h \in \pi_1(W) \) such that the component \( hP' \) from \( \hat{\Omega}' \) meets \( gP \) in a single line \( l \). If \( B \) is a bigon between \( gP \) and \( hP' \), then \( \partial B = \alpha \cup \alpha' \) where \( \alpha \subset gP \) and \( \alpha' \subset hP' \). Both points of \( \alpha \cap \alpha' \) lie on \( l \), and since \( \hat{W} \) is contractible by Lemma 3.28, \( B \) must be homotopic into \( l \). Thus \( B \) is inessential, and the projection of \( B \) to \( W \) must be inessential as well.

For the converse, let \( \tilde{P} \) be a plane of \( \tilde{\Omega} \) and let \( \tilde{P}' \) be a plane of \( \tilde{\Omega}' \). Perturb \( \tilde{\Omega} \) and \( \tilde{\Omega}' \) so that all their intersections are transverse. Suppose that for some \( g, h \in \pi_1(W) \), \( gP \) intersects \( hP' \) in two lines \( l \) and \( l' \). Then there is some path \( \beta \) in \( gP \) connecting \( l \) with \( l' \), and there is some path \( \beta' \) in \( hP' \) connecting \( l \) to \( l' \). We can always choose such paths so that \( \partial \beta = \partial \beta' \), \( \beta \cap hP' = \partial \beta \), and
\[ \beta' \cap gP = \partial \beta'. \] Since \( \tilde{W} \) is simply-connected, \( \beta \) and \( \beta' \) bound an essential bigon \( B \) between \( gP \) and \( hP' \).

Let \( p : \tilde{W} \rightarrow W \) be the covering map. Then \( p(B) \) must be an essential bigon between \( \Omega \) and \( \Omega' \).

If it were not the case that \( p(B) \) was essential, then there would be a homotopy \( \eta \) which homotopes \( p(B) \) into some double loop \( \alpha \) which belongs to \( \Omega \cap \Omega' \). The homotopy \( \eta \) lifts to the universal cover \( \tilde{W} \). Both endpoints of \( p(B) \) lie on \( \alpha \), and using the lifted homotopy \( \tilde{\eta} \), \( \alpha \) lifts to a line of intersection which contains both endpoints of \( \beta \), contradicting that \( l \) and \( l' \) are distinct lines.

Therefore, \( gP \) and \( hP' \) meet in at most one line, so \( \Omega \) and \( \Omega' \) have the relative 1-line property.

\[ \square \]

**Lemma 3.30.** There are no essential bigons between \( 2\Sigma \) and \( 2\Sigma' \) in \( 2X \) if and only if there are no essential bigons between \( \Omega \) and \( \Omega' \) in \( W \).

**Proof.** Let \( p : W \rightarrow 2X \) be the double covering map. \( W \) is formed by gluing two copies of \( 2X \) cut along \( 2\Sigma \). Recall that \( 2Y \) and \( 2Y' \) are the components of \( 2X \) cut along \( 2\Sigma \) and \( 2\Sigma' \). Hence \( p^{-1}(2Y) \) is homeomorphic to two copies of \( 2Y \), whose interiors are disjoint.

Let \( B \) be a bigon between \( 2\Sigma \) and \( 2\Sigma' \) in \( 2X \). Then \( B \) lies entirely inside \( 2Y \) or \( 2Y' \). Assume the former. Thus \( p^{-1}(B) \) consists of two disjoint bigons, each lying inside a component of \( p^{-1}(2Y) \). It follows that \( B \) is essential if and only if either component of \( p^{-1}(B) \) is essential. \[ \square \]

Together Theorem 3.29 and Lemmas 3.24 and 3.30 combine to give the following theorem:

**Theorem 3.31.** Let \( \Sigma \) and \( \Sigma' \) be \( \pi_1 \)-essential surfaces properly embedded in an irreducible and boundary-irreducible link exterior \( X \). Then \( \Sigma \) and \( \Sigma' \) have the relative 1-line property if and only if there are no essential bigons or trigons between \( \Sigma \) and \( \Sigma' \).
3.9 Characterisation

We are now ready to give the main theorem of this thesis. This is a non-diagrammatic characterisation of weakly generalised alternating knots. Hence weakly generalised alternating is a topological property of the knot exterior. But first we have one more lemma. A spanning surface $\Sigma$ for a link $L$ is non-split if every 2-sphere properly embedded in $X \setminus \Sigma$ bounds a 3-ball in $X \setminus \Sigma$.

**Lemma 3.32.** Let $\Sigma$ be a $\pi_1$-essential spanning surface for a link $L$. Then $\Sigma$ is split if and only if $L$ is split.

**Proof.** If $\Sigma$ is split, then $L$ is split, since every component of $\Sigma$ meets $L$.

Suppose that $S$ is a splitting sphere for a link $L$. Then $S$ is properly embedded in $X$. If $\Sigma \cap S = \emptyset$, then we are done. Otherwise, suppose $\ell$ is an innermost loop of intersection between $\Sigma$ and $S$ so that $\ell$ bounds a disk $D' \subset S$. Since $\Sigma$ is $\pi_1$-essential, $\ell$ also bounds a disk $D \subset \Sigma$. Let $S'$ be the 2-sphere formed by gluing $D$ to $D'$. Then $S'$ can be isotoped to be disjoint from $\Sigma$ since $D$ is two-sided in $X$. If $S'$ does not bound a 3-ball in $X$, then $S'$ is a splitting sphere for $\Sigma$. If $S'$ does bound a 3-ball $B$ in $X$, then $S$ can be isotoped across $B$ and $D$ to remove $\ell$. Hence all loops of intersection between $\Sigma$ and $S$ can be removed, and $S$ can be isotoped to be a splitting sphere for $\Sigma$. \qed

Recall that we denote the boundary components of a spanning surface $\Sigma$ for an $m$-component link by $\sigma_1, \ldots, \sigma_m$.

**Theorem 3.33.** An $m$-component link $L$ is weakly generalised alternating if and only if there exist a pair of non-split $\pi_1$-essential spanning surfaces $\Sigma$ and $\Sigma'$ for $L$ such that:

1. $i(\sigma_j, \sigma'_j) \in 2\mathbb{N}$ for each $j = 1, \ldots, m$.

2. There are isotopic representatives of $\Sigma$ and $\Sigma'$ which intersect transversely only in standard arcs.

3. $\Sigma$ and $\Sigma'$ have the relative 1-line property.
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Proof. Let $\pi(L)$ be a weakly generalised alternating projection of an $m$-component link $L$ onto a generalised projection surface $F$. Let $\Sigma$ and $\Sigma'$ be the checkerboard surfaces in standard position with respect to $\pi(L)$. Hence $\Sigma$ and $\Sigma'$ intersect in neither loops nor parallel arcs, or equivalently, every intersection between $\Sigma$ and $\Sigma'$ is standard.

Since $\pi(L)$ is weakly generalised alternating, every component of $L$ is involved in some crossing of $\pi(L)$. Hence $\partial \Sigma$ and $\partial \Sigma'$ have different slopes on each component of $\partial X$, so $i(\sigma_j, \sigma'_j) \neq 0$. As $\pi(L)$ is alternating, $i(\sigma_j, \sigma'_j)$ is actually an even positive integer.

By Theorem 3.15, $\Sigma$ and $\Sigma'$ are both $\pi_1$-essential in $X$, and by Theorem 3.12, there are no essential bigons between them. It then follows by Theorem 3.31 that $\Sigma$ and $\Sigma'$ satisfy the relative 1-line property.

By Theorem 3.18, $L$ is non-split, so by Lemma 3.32, both $\Sigma$ and $\Sigma'$ are non-split.

Conversely, suppose that $\Sigma$ and $\Sigma'$ are non-split $\pi_1$-essential spanning surfaces for $L$ properly embedded in the link exterior $X$, satisfying the three listed properties.

Let $\overline{\Sigma}$ be the extension of $\Sigma$ to $S^3$, such that $\partial \overline{\Sigma} = L$, which is obtained by gluing $\Sigma$ to a set of $m$ annuli $A_j \subset N(L_j)$, where $\partial A_j = L_j \cup (\partial \Sigma \cap C_j)$. Similarly, let $\overline{\Sigma}'$ be the extension of $\Sigma'$ to $S^3$. If $\alpha$ is an arc of intersection between $\Sigma$ and $\Sigma'$, let $\overline{\alpha}$ be the extended arc of intersection between $\overline{\Sigma}$ and $\overline{\Sigma}'$, such that $\partial \overline{\alpha} \subset L$.

Isotope $\overline{\Sigma}$ and $\overline{\Sigma}'$ so that $i(\sigma_j, \sigma'_j)$ is realised on each boundary component $C_j$ for $j = 1, \ldots, m$. Then isotope the interiors of $\overline{\Sigma}$ and $\overline{\Sigma}'$ to remove any trivial loops of intersection. This can be done since it is assumed that there are isotopic representatives of $\Sigma$ and $\Sigma'$ which do not meet in loops. In particular, these can be chosen to be least area representatives of $\Sigma$ and $\Sigma'$, which by Lemma 3.25 do not cobound any product regions.

Let $F'$ be the pseudo 2-complex defined by $\overline{\Sigma} \cup \overline{\Sigma}'$ with 1-skeleton $\Gamma' = \overline{\Sigma} \cap \overline{\Sigma}'$. We call $F'$ a pseudo 2-complex, since neither $F'$ nor $\Gamma'$ is necessarily connected and that the 2-cells of $F'$ are not necessarily simply connected. It follows that the graph $\Gamma$ is 3-regular since $i(\sigma_j, \sigma'_j) \neq 0$ for each $j = 1, \ldots, m$. If on some component $C_j$ of $\partial X$, $\partial \Sigma \cap C_j$ and $\partial \Sigma' \cap C_j$ had the same slope,
then \( \Gamma' \) would also contain some loops corresponding to the component \( L_j \) of \( L \), and \( \Gamma' \) would not be 3-regular. Note that \( \Gamma' = L \cup \overline{A} \), where \( A \) is the collection of all standard arcs of intersection between \( \Sigma \) and \( \Sigma' \), and \( \overline{A} \) is the set of extended arcs in \( S^3 \).

The number of arcs of intersection between \( \Sigma \) and \( \Sigma' \) is \( \frac{1}{2} i(\partial \Sigma, \partial \Sigma') \). By hypothesis, all the arcs of \( A \) are standard. Let

\[ p : F' \rightarrow F \]

be the projection which collapses each arc of \( \overline{A} \) to a point. During this process, \( \Gamma' \) collapses to a 4-regular graph \( \Gamma \).

As discussed in Section 3.2, a neighbourhood of a standard arc collapses to a disk \( D' \) which contains a vertex of \( \Gamma \), and the regions of \( D' \setminus \Gamma \) are 2-coloured by \( \Sigma \) and \( \Sigma' \). A point on \( L \setminus \overline{A} \) has a regular neighbourhood in \( F' \) which contains part of \( \partial \Sigma \) joined to part of \( \partial \Sigma' \). This neighbourhood is homeomorphic to its image under \( p \). Each region of \( F' \setminus \Gamma' \) is homeomorphic to a surface. Hence each region of \( F \setminus \Gamma \) is homeomorphic to a surface. Therefore \( F \) is homeomorphic to a surface embedded in \( S^3 \) so \( F \) must be orientable. The graph \( \Gamma \) is embedded in \( F \).

Since both \( \Sigma \) and \( \Sigma' \) are non-split, it follows that \( F' \) and \( F \) are non-split.

Since \( F \) is two-sided in \( S^3 \), we also have crossing information coming from the projection \( p \). Let \( \pi(L) \) be \( \Gamma \) equipped with this crossing information.

Let \( \ell \) be a closed loop embedded in \( F \) which is transverse to \( \Gamma \). Then \( \ell \) passes through regions of \( F \setminus \pi(L) \) which come alternately from \( \Sigma \) and \( \Sigma' \). Hence \( \pi(L) \) separates \( F \).

Suppose that \( \pi(L) \) is not alternating. Traversing along the image of some link component in the projection, at some point there will be two undercrossings in a row. Consider a simple closed curve composed of one connected arc \( \beta \) in \( \Sigma \setminus \Sigma' \) and one connected arc \( \beta' \) in \( \Sigma' \setminus \Sigma \), where the arcs intersect at the two vertices of \( \Gamma \) corresponding to the two undercrossings. This loop bounds a disk \( B \) whose interior lies above the projection surface and \( \beta \cup \beta' \) is trivial in \( F \). Then the disk \( p^{-1}(B) \) is an essential bigon between \( \Sigma \) and \( \Sigma' \) in \( X \), which by Theorem 3.31 contradicts that \( \Sigma \) and \( \Sigma' \) have the
Thus we may assume that \( \pi(L) \) is alternating. Suppose that \( \pi(L) \) is not weakly prime. Then there exists a loop \( \gamma \) in \( F \) which meets \( \Gamma \) exactly twice and bounds a disk \( E \) in \( F \) such that \( E \) contains at least one vertex of \( \Gamma \). Also \( \gamma \) must meet different edges of \( \Gamma \). Push the interior of \( E \) off the projection surface. Near each point of \( \gamma \cap \Gamma \), isotope \( E \) so that \( \gamma \cap \Gamma \) slides along \( \pi(L) \) in the direction of an undercrossing. Exactly one direction leads to an undercrossing since \( \pi(L) \) is alternating. Then at each of the two crossings (possibly the same crossing if \( \pi(L) \) only has one crossing), slide the points of \( \partial E \cap \Gamma \) into the double arc. \( E \) is now an essential bigon between \( \Sigma \) and \( \Sigma' \) in \( X \), which by Theorem 3.31 contradicts that \( \Sigma \) and \( \Sigma' \) have the relative 1-line property.

Suppose that \( r(\pi(L), F) = 2 \). Then there exists a loop \( \delta \) in \( F \) which meets \( \pi(L) \) exactly twice and \( \delta \) bounds a disk \( D \) which is a compressing disk for \( F \). \( \delta \) must meet different edges of \( \Gamma \). The interior of \( D \) is disjoint from \( F \). Look at \( F \) from the side which contains the interior of \( D \). Near each point of \( \delta \cap \Gamma \), isotope \( D \) so that \( \delta \cap \Gamma \) slides along \( \pi(L) \) until it hits an undercrossing. Exactly one direction leads to an undercrossing since \( \pi(L) \) is alternating. Then at each of the two crossings (possibly the same crossing if \( \pi(L) \) only has one crossing), slide the point into the double arc. \( D \) is now an essential bigon between \( \Sigma \) and \( \Sigma' \) in \( X \), which by Theorem 3.31 contradicts that \( \Sigma \) and \( \Sigma' \) have the relative 1-line property.

Suppose that \( r(\pi(L), F) = 0 \). Then there is a compressing disk \( D \) for \( F \) such that \( \partial D \cap \pi(L) = \emptyset \). But then \( \partial D \) is an essential loop embedded in either \( \Sigma \) or \( \Sigma' \), which implies that \( \Sigma \) or \( \Sigma' \) is compressible, a contradiction. Therefore \( r(\pi(L), F) \geq 4 \).

Finally, suppose \( F \cong S^2 \), and \( \pi(L) \) has only one crossing. Then \( L \) is the unknot but one of \( \Sigma \) or \( \Sigma' \) is a boundary-compressible Mobius band. So if \( F \cong S^2 \), then \( \pi(L) \) contains at least two crossings.

Therefore \( \pi(L) \) satisfies all the conditions listed in Section 3.1 required to be a weakly generalised alternating projection. Hence \( L \) is weakly generalised alternating.
3.10 Constructions

In this section, we give two constructions for producing weakly generalised alternating projections that are not generalised alternating. Recall that in Chapter 2, we detailed a method for constructing all generalised alternating projections onto a closed orientable surface $F$.

Figure 3.17: An embedded separating 4-regular graph with representativity 4 but edge-representativity 2.

Let $\Gamma$ be a generalised basic polyhedron embedded in some closed orientable surface $F$, where $F$ is embedded in $S^3$. We can cut open some vertices of $\Gamma$, so long as the representativity $r(\Gamma', F)$ remains at least four, where $\Gamma'$ is the result of cutting open some vertices of $\Gamma$. Replace the vertices of $\Gamma'$ with alternating 2-tangles, in such a way that we obtain an alternating link projection $\pi(L)$. 
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Starting with an initial generalised basic polyhedron trivially embedded in a Heegaard torus, we can cut open a vertex to obtain the 4-regular graph $\Gamma'$ pictured in Figure 3.17. Choosing appropriate rational tangles to replace the vertices produces the weakly generalised alternating projection of $11_{n99}$ shown in Figure 3.1.

![Figure 3.18: A weakly generalised alternating projection of 10_{139} where one of the white regions is homeomorphic to an annulus.](image)

For our second method, let $F$ be a closed orientable surface embedded in $S^3$. Let $L'$ be a separating embedded surface link such that $r(L', F) \geq 4$. We defined embedded surface links in Section 1.5. Let $D$ be a subdisk of $F$ such that $D \cap L'$ is two properly embedded arcs. Replace $D$ by an alternating 2-tangle. We call this process tangle insertion. The resulting link projection $\pi(L)$ is weakly generalised alternating. We can iterate this process to produce more complicated links. See Figure 3.18, where we have constructed a weakly generalised alternating knot projection from the $(4, 4)$ torus link, after three tangle insertions, where each tangle has a single crossing.
Chapter 4

Essentially Alternating Links

In this chapter, we give a non-diagrammatic characterisation of alternating knots. We do not assume that such knots are prime, but in this case we get a characterisation with much simpler conditions than Theorem 3.33. We give two proofs of our result. We also extend our result to characterise alternating links.

In the second section, we define a new class of links called essentially alternating links. These are the most general class of separating surface-alternating links, where both checkerboard surfaces are \( \pi_1 \)-essential in the exterior. We give several examples which do not fit into any of the previous classes we have studied. However, we show that it is not obvious how to give a diagrammatic characterisation of this class.

In the third section, we introduce a new algebraic condition which is sufficient to ensure that spanning surfaces do not meet in essential loops. This allows us to give a topological characterisation of knots which have weakly generalised alternating projections onto the torus.

In the final section, we give criteria to decide whether a knot is hyperbolic, satellite, or torus, given a surface-alternating diagram.
4.1 Characterisation of Alternating Knots

Let $\Sigma$ and $\Sigma'$ be surfaces properly embedded in a link exterior $X$. If $\Sigma$ and $\Sigma'$ are in general position, then the intersection $\Sigma \cap \Sigma'$ consists of a set of properly embedded arcs $A$ and a set of embedded loops $L$.

Let $\ell \in L$ and let $N(\ell)$ be a regular neighbourhood of $\ell$ in $X$. Let $T = \partial N(\ell)$. Since $\Sigma$ and $\Sigma'$ are in general position, $\ell$ is either one-sided in both $\Sigma$ and $\Sigma'$, or two-sided in both $\Sigma$ and $\Sigma'$.

Suppose $\ell$ is one-sided in $\Sigma$ and $\Sigma'$. Then $T \cap (\Sigma \cup \Sigma')$ consists of two embedded loops, and $T \setminus (\Sigma \cup \Sigma')$ consists of two annuli. We define a geometric sum of $\Sigma$ and $\Sigma'$ along $\ell$ to be the pseudo 2-complex $S'$, which is formed from $\Sigma \cup \Sigma'$ by deleting $N(\ell)$ and gluing in either of the annuli of $T \setminus (\Sigma \cup \Sigma')$.

If $\ell$ is two-sided in $\Sigma$ and $\Sigma'$, then $T \cap (\Sigma \cup \Sigma')$ consists of four embedded loops, and $T \setminus (\Sigma \cup \Sigma')$ consists of four embedded annuli. We define a geometric sum of $\Sigma$ and $\Sigma'$ along $\ell$ to be the pseudo 2-complex $S'$, which is formed from $\Sigma \cup \Sigma'$ by deleting $N(\ell)$ and gluing in a pair of non-adjacent annuli in $T \setminus (\Sigma \cup \Sigma')$. See Figure 4.1 for a picture of this case.

We define a partial geometric sum $S'$ of $\Sigma$ and $\Sigma'$ to be a choice of geometric sum along each $\ell$ belonging to the collection of loops $L$. We write $S' = \Sigma \bowtie \Sigma'$, where $S'$ is a pseudo 2-complex.

We can also think of $S'$ as the image of an immersion $g : S \hookrightarrow X$, where $S$ is a compact surface, $g(S) = S'$, $g(\partial S) = \partial \Sigma \cup \partial \Sigma'$, and the self-intersections of $g(S)$ are precisely the set of arcs $A$.

**Theorem 4.1.** Let $K$ be a non-trivial knot in $S^3$ with exterior $X$. $K$ has an alternating projection onto $S^2$ if and only if there exist a pair of $\pi_1$-essential spanning surfaces $\Sigma$, $\Sigma'$ for $K$ which satisfy

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial \Sigma, \partial \Sigma') = 2,$$

where $i(\partial \Sigma, \partial \Sigma') \neq 0$.

**Proof.** Suppose that $K$ has an alternating projection $\pi'(K)$ onto $S^2$. If $\pi'(K)$ is not reduced, then reduce it to a projection $\pi(K)$ which is reduced. Then
π(K) has at least 3 crossings since K is a non-trivial knot. K may not be prime but by Theorem 1.1 the prime decomposition is visible in π(K) since it is alternating.

The checkerboard surfaces Σ and Σ' associated to π(K) are both π₁-essential in X by Theorem 1.9. The equation (⋆) follows from Theorem 3.10. Since π(K) contains crossings, $i(\partial \Sigma, \partial \Sigma') > 0$.

For the converse, suppose that Σ and Σ' are a pair of π₁-essential surfaces that satisfy (⋆). Σ and Σ' are spanning surfaces so $[\partial \Sigma] \simeq [\lambda] + 2p[\mu]$ and $[\partial \Sigma'] \simeq [\lambda] + 2p'[\mu]$, where $p, p' \in \mathbb{Z}$ and λ and μ are the preferred longitude and meridian for \( \partial X \). Hence $i(\partial \Sigma, \partial \Sigma') = |2p - 2p'| \in 2\mathbb{N}$, where $p \neq p'$ since $i(\partial \Sigma, \partial \Sigma') \neq 0$.

Isotope Σ and Σ' into general position such that $\partial \Sigma$ and $\partial \Sigma'$ intersect...
minimally on \( \partial X \). This can be done by insisting that \( \partial \Sigma \) and \( \partial \Sigma' \) have least length when \( \partial X \) is given some Euclidean metric. For the rest of the proof, keep \( \partial \Sigma \) and \( \partial \Sigma' \) fixed. This implies that \( \partial \Sigma \) and \( \partial \Sigma' \) form a quadrangulation \( Q \) on \( \partial X \) such that \( Q \) contains \( i(\partial \Sigma, \partial \Sigma') \) vertices, \( 2i(\partial \Sigma, \partial \Sigma') \) edges, and \( i(\partial \Sigma, \partial \Sigma') \) quadrilateral faces.

Let \( \mathcal{A} \) be the collection of arcs of intersection between \( \Sigma \) and \( \Sigma' \). Since \( K \) is non-trivial, neither \( \Sigma \) nor \( \Sigma' \) is a disk. \( \mathcal{A} \) is non-empty because \( i(\partial \Sigma, \partial \Sigma') > 0 \).

Suppose that there is a loop of intersection \( \ell \) between \( \Sigma \) and \( \Sigma' \). Let \( \mathcal{L} \) be the collection of loops of intersection between \( \Sigma \) and \( \Sigma' \). Then \( \ell \) must lie within a region \( R \) of \( \Sigma \setminus \mathcal{A} \) and within a region \( R' \) of \( \Sigma' \setminus \mathcal{A} \) since \( \Sigma \) and \( \Sigma' \) are properly embedded.

If \( \ell \) is trivial in \( \Sigma \), then it must be trivial in \( \Sigma' \) since both surfaces are \( \pi_1 \)-essential. Assume that \( \ell \) is an innermost trivial loop of intersection between \( R \) and \( \Sigma' \), so that \( \ell \) bounds a disk \( D \subset R \) such that \( \hat{D} \cap \Sigma' = \emptyset \). Then \( \ell \) must bound a disk \( \hat{D}' \subset \Sigma' \), and \( D \cup \hat{D}' \) bounds a 3-ball \( B \subset X \) since \( X \) is irreducible. Now suppose \( \hat{D}' \cap \Sigma \) must be a collection of loops. Then \( \hat{D}' \) can be isotopes into \( B \) to remove all loops of \( \hat{D}' \cap \Sigma \). Then \( R \) can be isotopes to remove \( \ell \).

Therefore we can assume that any loop \( \ell \in \mathcal{L} \) is essential in both \( \Sigma \) and \( \Sigma' \), and in particular must be essential in both \( R \) and \( R' \). Hence \( R \) and \( R' \) are not simply connected.

Each region of \( \Sigma \setminus \Sigma' \) and \( \Sigma' \setminus \Sigma \) is orientable. To see this suppose that some region \( R \) of \( \Sigma \setminus \Sigma' \) is non-orientable. Then consider the double cover \( \hat{X} \) of \( X \). \( \hat{X} \) is unique since \( \pi_1(X) \) contains a unique index 2 subgroup, so every non-orientable surface properly embedded in \( X \) must lift to its orientable double cover in \( \hat{X} \). \( \hat{X} \) can be formed by taking two copies of \( X \) cut along \( \Sigma' \) and gluing them together. If \( R \) is non-orientable, then \( \hat{\Sigma} \) is non-orientable, a contradiction.

Let \( S' = \Sigma \pitchfork \Sigma' \), and let \( g : S \rightarrow X \) be an immersion of a twice-punctured surface \( S \) such that \( g(S) = S' \). \( S \) may or may not be connected. Then

\[
\chi(S) = \chi(\Sigma) + \chi(\Sigma'),
\]
and
\[ \chi(S') = \chi(\Sigma) + \chi(\Sigma') - \frac{1}{2}i(\partial\Sigma, \partial\Sigma'), \]
since the self-intersections of \( g(S) \) is the set \( \mathcal{A} \) and \( |\mathcal{A}| = \frac{1}{2}i(\partial\Sigma, \partial\Sigma') \). Assume for now that \( S' \) is connected.

\( X \) is embedded in \( S^3 \) so we can fill \( S^3 \setminus X \) with a solid torus \( T \). \( \partial\Sigma \) and \( \partial\Sigma' \) wrap once along the longitude of \( T \), and can be assumed to do so in a strictly increasing way. Let \( \kappa \) be the core of \( T \). Clearly \( \kappa \) is isotopic to the knot \( K \).

Attach annuli \( A, A' \subset T \) to \( \Sigma, \Sigma' \subset X \) such that \( \partial A = \partial\Sigma \cup \kappa \) and \( \partial A' = \partial\Sigma' \cup \kappa \) respectively, and so that \( \hat{A} \) meets \( \hat{A}' \) in exactly \( i \) open arcs and no loops. Let \( \hat{\Sigma} = \Sigma \cup A \) and \( \hat{\Sigma}' = \Sigma' \cup A' \). If \( \alpha \in \mathcal{A} \), let \( \overline{\mathcal{A}} \) be the collections of arcs \( \overline{\alpha} \) which are extensions of \( \alpha \) by components of \( \hat{A} \cap \hat{A}' \).

Let \( F' = \Sigma \bowtie \hat{\Sigma}' \), where the partial geometric sum is just along the original loops of \( \mathcal{L} \), so that \( F' \) is a pseudo-2-complex. The 1-skeleton of \( F' \) is a trivalent graph \( \Gamma' = \kappa \cup \overline{\mathcal{A}} \), and the pseudo-2-cells are \( F' \setminus \Gamma' \). We call them pseudo-2-cells, since they are not necessarily simply connected, which differs from the usual definition of a 2-complex. In forming \( F' \) from \( S' \), we have added two annuli \( A \) and \( A' \), which have one boundary \( \kappa \) in common, but we have identified \( i(\partial\Sigma, \partial\Sigma') \) pairs of edges together, one from \( A \) and one from \( A' \). Thus \( \chi(F') = \chi(S') + i(\partial\Sigma, \partial\Sigma') \).

Recall that \( R \) and \( R' \) are respectively the regions of \( \Sigma \setminus \mathcal{A} \) and \( \Sigma' \setminus \mathcal{A} \) which contain the essential loop of intersection \( \ell \). \( R \bowtie R' \) consists of either one or two components depending on whether \( \ell \) is one- or two-sided in both \( \Sigma \) and \( \Sigma' \). The loop \( \ell \) cannot be one-sided in one surface and two-sided in the other since \( \Sigma \) and \( \Sigma' \) are in general position in the orientable 3-manifold \( X \). So \( \chi(R \bowtie R') = \chi(R) + \chi(R') \), and since neither \( R \) nor \( R' \) is simply connected, it follows that at least one component of \( R \bowtie R' \) is not simply connected. Call it \( U' \), then \( \chi(U') \leq 0 \), and \( U' \) is a pseudo-2-cell of \( F' \).

Now collapse each arc \( \overline{\alpha}_j \) of \( \overline{\mathcal{A}} \) to a point \( p_j \). This collapses \( F' \) to a new pseudo-2-complex \( F \), while \( \Gamma' \) collapses to a 4-regular graph \( \Gamma \) embedded in \( F \) with vertices \( p_j \) and edges segments of \( \kappa \). The interiors of pseudo-2-cells are homeomorphic to surfaces, and a neighbourhood of a point on
κ \ A is homeomorphic to an open disk, formed by gluing a segment of ∂Σ to a segment of ∂Σ'. A neighbourhood of a point pj is homeomorphic to a disk, since with K a knot, Lemma 3.11 implies that every arc of A is standard. Hence F is homeomorphic to a surface, which is orientable since F is embedded in S3.

It follows that U' must collapse to an orientable surface with boundary U that is homeomorphic to U'. Let β be an essential loop in U. Applying (⋆) and using the relationships derived above between the Euler characteristics of F', S', Σ, and Σ', we get

$$
\chi(F) = \chi(F') = \chi(S') + i(\partial\Sigma, \partial\Sigma') = \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial\Sigma, \partial\Sigma') = 2.
$$

Therefore F is homeomorphic to S2. But β is an essential loop in U ⊂ F, so β must separate S2, which means that K is disconnected, a contradiction. Therefore there are no loops of intersection between Σ and Σ'.

If S' was not connected, then the partial geometric sum must have split off a set of closed orientable surfaces C. If C ∈ C, then C is formed by subsurfaces of Σ and Σ' joined by annuli. If C is a 2-sphere, then C must be constructed from planar subsurfaces from each of Σ and Σ' joined by annuli coming from some loops in L. But at least one of these planar subsurfaces must be a disk, and that is impossible since all the loops in L are essential in each of Σ and Σ'. Thus \(\chi(C) \leq 0\).

If C ≠ ∅, then the associated F' contains the pseudo-2-complex described above as well as C, and F contains multiple closed surfaces. Let F* be the one that contains Γ. There can be only one since K is a connected knot. Then

$$
\chi(F^*) + \sum_{C \in C} \chi(C) = \chi(F) = 2.
$$

If some C has \(\chi(C) < 0\), this forces \(\chi(F^*) > 2\) which is impossible for a
4.1. CHARACTERISATION OF ALTERNATING KNOTS

connected closed surface. Thus all components of $C$ are tori. In this case, the preceding argument shows that some loop in $U$ is separating in the 2-sphere $F^*$, a contradiction.

Suppose that $L = \emptyset$ but some region $R$ is not simply connected. In this case $S' = \Sigma \cup \Sigma'$. Then collapsing the arcs of $\overline{A}$ in $F'$ gives a projection onto $S^2$, where $F \setminus \Gamma$ contains a separating loop. This is impossible since $K$ is a knot. Therefore every region of $\Sigma \setminus \Sigma'$ and $\Sigma' \setminus \Sigma$ is a disk.

If we place an orientation on $K$ and hence $\kappa$, this orientation is inherited by $\partial \Sigma$ and $\partial \Sigma'$. Since $\partial \Sigma$ and $\partial \Sigma'$ have been isotoped to have least length, they form the quadrangulation $Q$ on $\partial X$. Each crossing of $\partial \Sigma$ with $\partial \Sigma'$ has the same sign. This allows us to obtain crossing information for each vertex $p_j$ of $\Gamma$, producing a projection $\pi(K)$ on $S^2$. The neighbourhood of each crossing looks locally like a checkerboard pattern, with the two colours associated to $\Sigma$ and $\Sigma'$ respectively.

If $\pi(K)$ was not alternating, then there would be a bigon between $\partial \Sigma$ and $\partial \Sigma'$ on $\partial X$, a contradiction since $\partial \Sigma$ and $\partial \Sigma'$ are least length.

Furthermore, $\pi(K)$ is a reduced alternating projection, since if it was not reduced, then one of $\Sigma$ or $\Sigma'$ would be boundary-compressible. \hfill \Box

It turns out that we can drop the requirement that the spanning surfaces are $\pi_1$-essential. This will improve the efficiency of the algorithm in Section 5.3. And we get a second simpler proof of the above theorem. We leave the above original proof of the theorem in this thesis, since the techniques are useful when considering projections onto higher genus surfaces in the Section 4.3.

Both Theorem 4.1 and Theorem 4.2 are the promised non-diagrammatic characterisation of alternating knot exteriors. Note that all the conditions on $X$ are topological in nature, as they only involve embedded surfaces, Euler characteristics and algebraic intersection numbers.

**Theorem 4.2.** Let $K$ be a non-trivial knot in $S^3$ with exterior $X$. $K$ has an alternating projection onto $S^2$ if and only if there exist a pair of spanning
surfaces $\Sigma, \Sigma'$ for $K$ which satisfy
\[ \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = 2, \]
where $i(\partial \Sigma, \partial \Sigma') \neq 0$.

Proof. One direction follows from Theorem 4.1. For the converse, let $\Sigma_0$ and $\Sigma'_0$ be a pair of connected spanning surfaces for $K$ which satisfy ($\ast$).

As above, isotope $\Sigma_0$ and $\Sigma'_0$ in $X$ so that their boundaries have least length and realise the intersection number $i$. We may assume that the interiors of $\Sigma_0$ and $\Sigma'_0$ are in general position, so that they intersect in a set of proper arcs $\mathcal{A}$ and a set of embedded loops $\mathcal{L}_0$.

Recall that $\Sigma_0$ is the extension of $\Sigma_0$ to $S^3$ so that $\partial \Sigma_0 = L$. Let $F'_0 = \Sigma_0 \cup \Sigma'_0$ and $\Gamma' = (\Sigma_0 \cap \Sigma'_0) \setminus \mathcal{L}_0$. Let $\overline{\mathcal{A}}$ be the extension of $\mathcal{A}$ to $S^3$, or in other words, let $\overline{\mathcal{A}}$ be the closure of $\Gamma' \setminus L$.

If we collapse each component of $\overline{\mathcal{A}}$ to a point, then $F'_0$ collapses to an immersed surface $F_0$ since $K$ is a knot and every arc of $\mathcal{A}$ is standard. Since $\chi(F_0) = \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma')$, the equation ($\ast$) tells us that $\chi(F_0) = 2$. $\Gamma'$ collapses to a 4-regular graph $\Gamma$. Let
\[ f_0 : S_0 \hookrightarrow S^3 \]
be the immersion of a 2-sphere $S_0$ such that $f_0(S_0) = F_0$. There are no triple points of self-intersection since $\Sigma_0$ and $\Sigma'_0$ are embedded, and the only double curves of self-intersection are precisely the set $\mathcal{L}_0$.

Suppose $\mathcal{L}_0 \neq \emptyset$ and let $\mathcal{B}_0$ be the collection of loops $f_0^{-1}(\mathcal{L}_0)$ on $S_0$. It is easy to see that $\mathcal{B}_0$ has an even number of elements. Each loop $\beta \in \mathcal{B}_0$ is separating, and $\mathcal{B}_0$ cuts $S_0$ into a collection of planar surfaces with boundary. Let $p_h$ be the number of planar surfaces in $S_0 \setminus \mathcal{B}_0$ which have $h$ boundary components.

Using an Euler characteristic argument, Nowik [72] points out that
\[ \sum_{h \geq 1} (2 - h)p_h = 2, \]
which in particular implies that $p_1$, the number of disk regions in $S_0 \setminus B_0$ is at least 2. Thus there is at least one loop in $L_0$ which bounds a disk in $F_0$.

Let $\ell_0$ be a loop in $L_0$ which bounds a disk $D_0$ in either $\Sigma_0$ or $\Sigma'_0$. Without loss of generality assume $D_0 \subset \Sigma_0$. Let $\{\beta_0, \beta'_0\} = f_0^{-1}(\ell_0)$ where $f_0(N(\beta_0)) \subset \Sigma_0$ and $f_0(N(\beta'_0)) \subset \Sigma'_0$. Notice that $f_0^{-1}$ restricted to the interior of $D_0$ is a homeomorphism onto the interior of a disk in $S_0 \setminus B_0$.

Let $A_0 = \ell_0 \times \{-1, 1\}$ be a regular neighbourhood of $\ell_0$ in $\Sigma'_0$. We perform surgery on $\Sigma'_0$ along $D_0$, by deleting $A_0$ in $\Sigma'_0$ and gluing in the two disks $D_0 \times \{-1\}$ and $D_0 \times \{1\}$.

![Figure 4.2: Surgery on $\Sigma'_0$ along the disk $D_0 \subset \Sigma_0$.](image)

Let $(\Sigma_1, \Sigma'_1)$ be the result of performing surgery along $D_0$ on $(\Sigma_0, \Sigma'_0)$.

Define $S_1$ to be the result doing surgery along $f_0^{-1}(D_0)$ in $S_0$ and deleting the curve $\beta_0$. Let $L_1 = L_0 \setminus \ell_0$ so that $L_1$ is the set of loops of intersection of $\Sigma_1$ and $\Sigma'_1$.

Suppose $|L_0| = k \geq 0$. For $0 \leq j \leq k - 1$, inductively define $(\Sigma_{j+1}, \Sigma'_{j+1})$ to be the result of performing surgery along the disk $D_j$, where $\partial D_j = \ell_j \subset L_j$ and $D_j$ is a sub disk of either $\Sigma_j$ or $\Sigma'_j$. $F'_j$ is the corresponding surgery on the pseudo 2-complex $F'_j$. Define $S_{j+1}$ to be surgery along $f_j^{-1}(D_j)$ in $S_j$ and deleting the curve $\beta_j$ or $\beta'_j$.

A similar calculation to Nowik’s shows that

$$\sum_{h \geq 0} (2 - h)p_h = \chi(S_j) = 2 + 2j,$$

where $S_j$ is a collection of $(j + 1)$ closed 2-spheres since $\beta_j$ and $\beta'_j$ are sepa-
rating in $S_j$. Since $\Gamma$ is connected and disjoint from $L_0$, $f_j^{-1}(\Gamma)$ is contained in exactly one component of $S_j$. Hence $p_0 \leq j$, so that $p_1 \geq 2$, and therefore the disk $D_j$ exists.

Continue this inductive surgery process until we have constructed $\Sigma_k$ and $\Sigma'_k$. At this stage $L_k$ is empty, and $S_k$ consists of $(k+1)$ 2-spheres. Hence $F'_k$ consists of $k$ embedded 2-spheres and one immersed 2-sphere which contains $\Gamma'$. Call it $F''$.

Define $\Sigma$ and $\Sigma'$ to be the components of $\Sigma_k$ and $\Sigma'_k$ respectively which meet $F'$. Then $\Sigma$ and $\Sigma'$ are connected spanning surfaces for $K$ which satisfy $(\ast)$, and whose intersection is exactly $A$. Collapsing the arcs of $\mathcal{A}$ to points, collapses $F'$ and $\Gamma'$ to $F$ and $\Gamma$ respectively.

Since $\partial \Sigma$ and $\partial \Sigma'$ realise $i(\partial \Sigma, \partial \Sigma') = i(\partial \Sigma_0, \partial \Sigma'_0)$, we can recover the information of $\pi(K)$ from $\Gamma$. $\pi(K)$ must be alternating since otherwise, there would be a bigon between $\partial \Sigma$ and $\partial \Sigma'$ on $\partial X$. Note that $\pi(K)$ is not necessarily reduced.

Note that this theorem does not detect primeness. To do that we can use the relative 1-line property.

**Theorem 4.3.** A knot $K$ is prime and alternating if and only if $K$ has a pair of $\pi_1$-essential spanning surfaces $\Sigma$ and $\Sigma'$, which have the relative 1-line property, and satisfy

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial \Sigma, \partial \Sigma') = 2,$$

where $i(\partial \Sigma, \partial \Sigma') \neq 0$.

**Proof.** This follows from Theorems 4.1 and 3.33, because the definitions of prime and weakly prime coincide for diagrams on $S^2$.

We have not stated any of the theorems in this section for links. The issue is that given two spanning surfaces, there could be parallel arcs between different boundary components of $X$. If there are parallel arcs, then the complex $F'$ does not collapse to a surface.

In Figure 4.3, we give an example of a pair of spanning surfaces for a 2-component link which intersect in parallel arcs, yet still satisfy equation $(\ast)$.
Note that the parallel arcs run between different components of $L$. In this case, the link is in fact alternating, however it is possible to construct more elaborate examples where this does not happen. Note also that the link is not prime. Both spanning surfaces are $\pi_1$-essential by [74] since they are Murasugi sums of $\pi_1$-essential spanning surfaces for alternating knots and a spanning annulus for a torus link.

However, if we assume that all arcs of intersection are standard, then we have the following theorem.

**Theorem 4.4.** Let $L$ be a non-split link in $S^3$ with exterior $X$. $L$ has an alternating projection onto $S^2$ if and only if there exist a pair of connected spanning surfaces $\Sigma$, $\Sigma'$ for $L$ which satisfy

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} l(\partial \Sigma, \partial \Sigma') = 2,$$

(\star)
and
\[ i(\partial \Sigma, \partial \Sigma') = |\sum_{j=1}^{m} i(\sigma_j, \sigma'_j)|. \]

Proof. Let \( \pi(L) \) be a reduced non-split alternating projection of \( L \) onto \( S^2 \), and let \( \Sigma \) and \( \Sigma' \) be the associated checkerboard surfaces in standard position. Then \( \pi(L) \), \( \Sigma \), and \( \Sigma' \) are connected, so every arc of intersection between \( \Sigma \) and \( \Sigma' \) is standard. This means every intersection number \( i(\sigma_j, \sigma'_j) \) is either positive or every \( i(\sigma_j, \sigma'_j) \) is negative, which implies the second equation. The first equation follows from Theorem 3.10.

For the converse, Lemma 3.11 ensures that every arc of intersection is standard. Then the rest of the proof goes through as in Theorem 4.2. Since \( \Sigma \), and \( \Sigma' \) are connected, \( \pi(L) \) must be non-split.

By a theorem of Gordon and Luecke [29], knot exteriors uniquely determine the knot. The corresponding result is not true for links. Links are however uniquely determined by the link exterior and a prescribed meridian curve on each boundary component. The hypothesis of the existence a spanning surface as in Theorem 4.4 implies that we have been given a meridional slope on each component of \( \partial X \). This is important, because there exist link exteriors, which are the exterior of both alternating and non-alternating links in \( S^3 \).

## 4.2 Definition of Essentially Alternating

A link \( L \) is **essentially alternating** if it has a projection onto a generalised projection surface \( F \),
\[
\pi : F \times I \to F,
\]
where \( F \times I \subset S^3 \) which satisfies all of the following:

1. \( \pi(L) \) is alternating on \( F \).
2. \( \pi(L) \) is separating on \( F \).
3. Both checkerboard surfaces associated to \( \pi(L) \) are \( \pi_1 \)-essential in \( X \).
The set of essentially alternating links includes all weakly generalised alternating links. This follows from the first part of the following lemma:

**Lemma 4.5.** Let \( \pi(L) \) be an essentially alternating projection onto a generalised projection surface \( F \). Then:

1. \( r(\pi(L), F) \geq 2 \).

2. \( L \) is non-split and non-trivial.

**Proof.** If \( r(\pi(L), F) = 0 \), then one of the checkerboard surfaces is compressible. Since \( \pi(L) \) separates \( F \), every loop \( \ell \subset F \) meets \( \pi(L) \) an even number of times so \( r(\pi(L), F) \geq 2 \).

Suppose that \( S \) is a 2-sphere embedded in \( X \). Put \( S \) into general position with respect to the checkerboard surfaces \( \Sigma \) and \( \Sigma' \) which are assumed to be in standard position. Suppose \( \alpha \) is an innermost loop of intersection between \( S \) and \( \Sigma \). Then \( \alpha \) bounds a disk \( D \) in \( \Sigma \) since \( \Sigma \) is \( \pi_1 \)-essential. Thus we can isotope \( S \) to remove \( \alpha \). Similarly all other loops of intersection between \( S \) and \( \Sigma \) or \( \Sigma' \) can be removed.

Hence \( S \) can be isotoped to be disjoint from \( F \times I \). But a generalised projection surface is non-split, so \( S \) must bound a 3-ball in \( X \), and therefore \( L \) is non-split.

Also, \( L \) is non-trivial since the unknot does not bound any non-orientable \( \pi_1 \)-essential spanning surfaces. One of the checkerboard surfaces associated to an essentially alternating projection \( \pi(K) \) of a knot \( K \) must be non-orientable since the checkerboard surfaces have different slopes.

Theorem 1.15 shows us that all adequate links are essentially alternating, where the projection surface is the Turaev surface associated to an adequate planar diagram. Figure 1.16 illustrates such an essentially alternating projection. It is also easy to see that the connected sum of two essentially alternating links is essentially alternating.

Perhaps the simplest example of an essentially alternating projection which does not belong to either of the classes mentioned in the previous paragraph is the two crossing projection of the figure 8 knot onto a Heegaard
torus as seen in Figure 4.4. The two checkerboard surfaces are easily seen to be $\pi_1$-essential since they are a once-punctured torus and a once-punctured Klein bottle at slope 4 respectively.

The non-alternating knots $8_{20}$ and $8_{21}$ are essentially alternating, but are not generalised alternating by Lemma 2.31. It is unknown if these two knots are weakly generalised alternating.

Figure 4.5 shows an essentially alternating projection with 5 crossings of the knot $8_{20}$ onto a Heegaard surface of genus two. The white surface shown is isotopic to the $+$-state surface of the associated planar projection onto the page which has 8 crossings. Theorem 1.15 guarantees that this white surface is $\pi_1$-essential in $X$. The black surface is also $\pi_1$-essential since it is an orientable spanning surface of minimal genus for the given knot.

Figure 4.6 shows an essentially alternating projection with 6 crossings of the knot $8_{21}$ onto a Heegaard torus. The white surface shown is isotopic to the $-$-state surface of the induced planar projection onto the page which has 8 crossings. Theorem 1.15 ensures that this white surface is $\pi_1$-essential in
4.2. DEFINITION OF ESSENTIALLY ALTERNATING

Figure 4.5: An essentially alternating projection of the knot $8_{20}$ onto the Heegaard surface of genus 2.

Figure 4.6: An essentially alternating projection of $8_{21}$ onto the Heegaard torus.
The black surface is also $\pi_1$-essential since it is an orientable spanning surface of minimal genus for the given knot.

The torus knot $8_{19}$ is not essentially alternating, a fact which will follow from Theorem 4.10.

It is not easy to give a diagrammatic characterisation of essentially alternating links. One necessary condition is vertex-representativity.

Define the vertex-representativity $v(\pi(L), F)$ of $\pi(L)$ in $F$ to be the minimum number of intersections between $\Gamma$ and $\ell$, where $\ell$ ranges over the boundaries of all compressing disks for $F$ such that $\ell$ does not meet the interior of any edge of $\Gamma$. This means that $\ell$ only passes through vertices and faces of $\Gamma$, where $\Gamma$ is the projection graph obtained by forgetting the crossing information in $\pi(L)$.

**Lemma 4.6.** Let $\pi(L)$ be an essentially alternating projection of $L$ onto a generalised projection surface $F$. Then $v(\pi(L), F) \geq 2$.

**Proof.** Let $\Sigma$ and $\Sigma'$ be the checkerboard surfaces associated to $\pi(L)$. If $v(\pi(L), F) = 0$, then one of $\Sigma$ or $\Sigma'$ is compressible. If $v(\pi(L), F) = 1$, then one of $\Sigma$ or $\Sigma'$ is boundary-compressible. \qed

However, vertex-representativity of a diagram is not sufficient to guarantee an essentially alternating diagram. Let $F$ be a Heegaard surface for $S^3$ of genus at least one, and let $\pi(L)$ be alternating on $F$, with checkerboard surfaces $\Sigma$ and $\Sigma'$. Let $B$ be a bigon on one side of $F$ with $\partial B = \beta \cup \beta'$, and let $C$ be a bigon on the other side of $F$ with $\partial C = \gamma \cup \gamma'$, where $\beta, \gamma \subset \Sigma$ and $\beta', \gamma' \subset \Sigma'$. Assume both $\partial B$ and $\partial C$ are essential in $F$. If it happens, that $\beta'$ coincides with $\gamma'$, then $B \cup C$ is a compressing disk for $\Sigma$.

More generally, we may allow the arc $\beta$ to run across $\Sigma$, then $\partial X$, then $\Sigma$ again. In that case, $B \cup C$ is a boundary-compressing disk for $\Sigma$. We illustrate this in Figure 4.7 with a diagram of the knot $9_{45}$ which has $v(\pi(K), F) = 2$, but one of the checkerboard surfaces is boundary-compressible.

It is also difficult to find a topological characterisation of the exterior of an essentially alternating knot. One might hope that the analogous condition to Theorem 4.1 would hold for higher genus projection surfaces. On the
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Figure 4.7: A surface-alternating diagram of $9_{45}$ with $v(\pi(L), F) = 2$ which is not essentially alternating. $B \cup C$ is a boundary-compressing disk for one of the checkerboard surfaces.

torus, this condition would be $\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = 0$, but a pair of $\pi_1$-essential spanning surfaces satisfying this equation does not guarantee an essentially alternating projection onto the torus. It is possible that $\ell \in \mathcal{L}$ is one-sided in both $\Sigma$ and $\Sigma'$, which means $R \bowtie R'$ consists of a single component, so it is not actually possible to use the techniques of Theorem 4.1 to show $\mathcal{L} = \emptyset$ here.

Let $B$ be a closed alternating braid on 3 strands. It sits naturally inside an annulus $A$. Embed this annulus in a torus $F$ such that its core $\alpha$ lies along a $(2, 1)$ curve. This means that $\alpha$ bounds a mobius band in the solid torus $T$, where $\partial T = F$. Let the image of $B$ be the knot $K$. Then $\pi(K)$ does not separate $F$. The edges of $A$ each bound a Mobius band in $T$, and these two
mobius bands intersect in a closed loop $\ell \subset \hat{T}$ which is isotopic to the core of $T$. One geometric sum along $\ell$ produces a single annular region in $F \setminus A$ and $\pi(K)$ is non-separating. The other geometric sum along $\ell$ produces a non-separating surface-alternating projection onto another torus $F'$.

This is actually a projection onto the figure 8 immersion of the Klein bottle such that $\pi(K)$ misses the loop of self-intersection. This projection is surface-alternating, however it is not separating in either geometric sum along $\ell$. See Figure 4.8 for an example in the case of the knot $12n_{725}$, which shows the result of one of the geometric sums. Replacing the annular region with a pair of Mobius bands which intersect in the core of the solid torus, gives the desired projection onto an immersed Klein bottle.

Figure 4.8: The knot $12n_{725}$ has an alternating projection onto an immersed Klein bottle.

It is an interesting question to ask if every non-torus knot has an essentially alternating projection onto some closed orientable surface. This would imply Conjecture 1.21. However, Dunfield [23] has a recent example which
shows that there are hyperbolic knots which do not bound any essential non-orientable surface. Therefore there exist hyperbolic knots which are not weakly generalised alternating.

4.3 Relative Acyclicity

Let $\Sigma$ and $\Sigma'$ be $\pi_1$-essential spanning surfaces for a link $L$. We say that $\Sigma$ and $\Sigma'$ are relatively acyclic if

$$\pi_1(\Sigma) \cap x\pi_1(\Sigma')x^{-1}$$

is trivial for all $x \in \pi_1(X)$.

This means that we cannot find a pair of essential loops $\alpha \subset \Sigma$ and $\alpha' \subset \Sigma'$ which are freely homotopic in $X$.

**Lemma 4.7.** Suppose two spanning surfaces $\Sigma$ and $\Sigma'$ are relatively acyclic. Then $\Sigma$ and $\Sigma'$ do not intersect in an essential loop.

**Proof.** Suppose $\Sigma$ and $\Sigma'$ intersect in an essential loop $\ell$. Choose a basepoint contained in $\ell$. Then $\ell$ represents a non-trivial element of $\pi_1(\Sigma) \cap \pi_1(\Sigma')$. 

Two spanning surfaces, at least one of them non-orientable, which have the same boundary slope cannot be relatively acyclic, since Lemma 3.6 shows that they must intersect in a loop.

**Theorem 4.8.** Suppose that $\pi(L)$ is a weakly generalised alternating projection, where all the regions of $\Sigma \setminus \Sigma'$ are disks. Then $\Sigma$ and $\Sigma'$ are relatively acyclic.

**Proof.** Let $A$ be an annulus with $\partial A = \alpha \cup \alpha'$. Suppose that $f : A \to X$ is an essential map such that $f(\alpha) \subset \Sigma$ and $f(\alpha') \subset \Sigma'$. Recall that $\Sigma$ and $\Sigma'$ are both $\pi_1$-essential in $X$ by Theorem 3.15. Let $\Sigma$ and $\Sigma'$ be in standard position.

Homotope $f$ to be in general position relative to $\Sigma$ and $\Sigma'$. Consider the intersection of $A$ with $f^{-1}(\Sigma)$ and $f^{-1}(\Sigma')$. This consists of $\partial A$ as well as proper arcs and loops on the interior of $A$. 
We can assume there are no loops of intersection between $\hat{A}$ and $f^{-1}(\Sigma)$, nor between $\hat{A}$ and $f^{-1}(\Sigma')$, where the loop is essential in $A$. If such intersections did occur, we can replace $f$ by $f' : A' \to X$, where $A'$ is an innermost subannulus of $A$.

Suppose $B \subset A$ is an innermost bigon between $f^{-1}(\Sigma)$ and $f^{-1}(\Sigma')$. Then $f(B)$ is a (possibly singular) bigon between $\Sigma$ and $\Sigma'$. But Theorem 3.17 shows that $f(B)$ cannot be an essential singular bigon. Hence it is possible to homotope $f$ to remove all bigons in $A$.

We can also use Theorem 3.15 to homotope away any innermost loops of intersection on the interior of $A$, but this is not important for this proof, as we just need to homotope $f$ so that $\alpha$ is disjoint from $f^{-1}(\Sigma')$, and $\alpha'$ is disjoint from $f^{-1}(\Sigma)$.

Therefore $f(A)$ can be homotoped so that $\alpha$ lies completely within one region of $\Sigma \setminus \Sigma'$, and $\alpha'$ lies completely within one region of $\Sigma' \setminus \Sigma$. But all the regions of $\Sigma \setminus \Sigma'$ are disks, so $\alpha$ must represent a trivial loop in $\Sigma$. This is a contradiction to $f$ being essential.

The proof of Theorem 4.1 shows us that the checkerboard surfaces associated to a reduced planar alternating projection are relatively acyclic. Theorem 4.8 shows that the same thing happens for weakly generalised alternating knot projections onto the torus. This is because at most one region of $F \setminus \pi(K)$ can be non-simply connected (in which case it would be an annulus), so the regions of at least one checkerboard surface are all disks. This allows us to prove a characterisation of knots which have weakly generalised alternating projections onto the torus.

**Theorem 4.9.** A knot $K$ has a weakly generalised alternating projection onto a torus if and only if $K$ has a pair of $\pi_1$-essential spanning surfaces $\Sigma$ and $\Sigma'$, which have the relative 1-line property, are relatively acyclic, and satisfy

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = 0.$$  \(\diamondsuit\)

**Proof.** Suppose that $\pi(K)$ is a weakly generalised alternating projection onto a torus, with associated checkerboard surfaces $\Sigma$ and $\Sigma'$. Then Theorem 3.33
shows that Σ and Σ' are π₁-essential and have the relative 1-line property, Theorem 3.10 shows that Σ and Σ' satisfy (○), and Theorem 4.8 shows that Σ and Σ' are relatively acyclic.

Conversely, suppose that Σ and Σ' are spanning surfaces satisfying all the above conditions. Since Σ and Σ' are relatively acyclic, Lemma 4.7 allows us to isotope them so that they do not intersect in any loops. It then follows from Theorem 3.33 that K is weakly generalised alternating onto some closed orientable surface F. Using a process analagous to the one in the proof of Theorem 4.1 along with (○) ensures that F is a torus.

Note that we do not need to assume that \( i(\partial \Sigma, \partial \Sigma') \neq 0 \). This is because Lemma 3.6 shows that if both Σ and Σ' have the same boundary slope and are relatively acyclic, then they must both be orientable. But then (○) forces one of Σ or Σ' to be a disk and the other to be a once-punctured torus. This means that K is the unknot, in which case the once-punctured torus is compressible.

\[\square\]

Figure 4.9: A weakly generalised alternating projection where the checkerboard surfaces fail to be relatively acyclic.
If a knot $K$ has a weakly generalised alternating projection onto a closed orientable surface $F$, where $\chi(F) < 0$, then it is possible that the checkerboard surfaces fail to be relatively acyclic. Such an example where the projection surface is a double-torus is shown in Figure 4.9. Here each checkerboard surface contains a single annular region, whose cores happen to be freely homotopic in $X$.

4.4 Hyperbolicity

In this section, we give several criteria to decide whether a given essentially alternating projection represents a hyperbolic knot. Recall Theorem 1.11 which describes Thurston’s classification of knots into the three classes: hyperbolic, satellite, or torus.

As we saw in Section 4.2, it is not possible for us to give a diagrammatic characterisation of essentially alternating links. However we do have a diagrammatic definition of weakly generalised alternating link projections, so we will state our results for them.

**Theorem 4.10.** Let $K$ be a $(p,q)$ torus knot with $p > q \geq 2$. If $q \geq 3$, then $K$ does not have an essentially alternating projection onto any generalised projection surface $F$. If $q = 2$, then the only essentially alternating projection of $K$ is onto $S^2$.

**Proof.** Moser [62] showed that if $K$ is a $(p,q)$ torus knot, then it has exactly two boundary slopes, which appear at slopes 0 and $pq$. In fact $K$ only bounds two isotopy classes of essential orientable surfaces. These are the Seifert surface of genus $\frac{1}{2}(p-1)(q-1)$ at slope 0, and the winding annulus at slope $pq$.

If $q \geq 3$, then $K$ does not bound any essential non-orientable surface. If $q = 2$, then $K$ bounds an essential Möbius band at slope $2p$. Its double cover is the winding annulus.

In Lemma 4.5, we showed that at least one of the checkerboard surfaces coming from an essentially alternating projection must be non-orientable, and such a surface must be $\pi_1$-essential by definition. It follows that if $q \geq 3$,
then $K$ does not have a weakly generalised alternating projection onto any $F$.

If $q = 2$, then $K$ is alternating onto $S^2$. Suppose that $K$ has an essentially alternating projection onto the surface of genus $g > 0$. Then the two checkerboard surfaces coming from this projection must be the Seifert surface of genus $\frac{1}{2}(p - 1)(q - 1)$ and the Möbius band. It can be seen in Hatcher and Thurston [34] that an essential Seifert surface for a rational knot has a unique genus. Theorem 3.10 tells us that such a projection must have $p - 2g$ crossings on $F$. This means that the Möbius band would have slope $2p - 4g$, a contradiction.

**Theorem 4.11.** Suppose that $K$ has an essentially alternating projection onto a non-Heegaard torus. Then $K$ is a satellite knot.

**Proof.** Isotope $K$ into $F \times I$ so that $\pi(K)$ is an essentially alternating projection onto $F = F \times \{0\}$.

Since $F$ is non-Heegaard, one component of $S^3 \setminus (F \times I)$ is a solid torus $T$, while the other component is a non-trivial knot exterior $Y$. Parameterise $I$ such that $\partial T = F \times \{-1\}$ and $\partial Y = F \times \{1\}$. Let $T'$ be the solid torus containing $T$ such that $\partial T' = \partial Y$.

Let $D$ be a compressing disk for $\partial Y$ in $S^3$ and put $D$ into general position so that any intersection with $F$ or $K$ is transverse. $\partial Y$ is two-sided so we may assume that $D$ is embedded in $T'$. Then $D$ contains a subdisk $D'$ which is a compressing disk for $F$. But then $D'$, and hence $D$ must intersect $K$ at least twice since by Lemma 4.5, we have $r(\pi(K), F) \geq 2$.

Hence $\partial Y$ is incompressible and meridionally incompressible in $X$, which implies that $\partial Y$ is an essential torus embedded in $X$. Therefore $K$ is a satellite knot.

**Theorem 4.12.** Let $\pi(K)$ be a weakly generalised alternating diagram onto a Heegaard torus $F$, where every region of $F \setminus \pi(K)$ is a disk. Then $K$ is hyperbolic.

**Proof.** Theorem 3.21 shows that $K$ is prime, and Theorem 4.10 shows that $K$ is not a torus knot. Therefore, Theorem 1.14 shows that $K$ is hyperbolic.
Theorem 4.13. Let $\pi(K)$ be an essentially alternating diagram onto a Heegaard torus $F$, where one region of $F \setminus \pi(K)$ is an annulus $A$. If the core of $A$ forms a non-trivial knot is $S^3$, then $K$ is satellite.

Proof. Let $\Sigma$ and $\Sigma'$ be the checkerboard surfaces associated to $\pi(K)$, and assume that they are in standard position. Let $B$ be the set of bubbles at the crossings of $\pi(K)$. Assume without loss of generality, that $A$ is a region of $\Sigma$.

Let $\alpha$ be the core of $A$. If $\alpha$ forms a non-trivial knot in $S^3$, then this knot is a torus knot since it is embedded in the torus $F$. Let $A' = F \setminus (A \setminus B)$, so that $A'$ is an annulus embedded in $F$. Let $T$ be a regular neighbourhood of $A'$ in $S^3$, so that $T$ is a solid torus. We may assume that $K$ is contained in $T$, and that $\partial T$ intersects $A$ in a pair of loops $\beta$ and $\beta'$. Both $\beta$ and $\beta'$ are isotopic to $\alpha$.

Since $\alpha$ is isotopic to the core of $A'$ in $F$, it follows that $\partial T$ bounds a non-trivial torus knot exterior $Y$ on one side and the solid torus $T$ on the other. Hence $\partial T$ is incompressible in $X$.

Suppose that $D$ is a compressing disk for $\partial T$ in $\partial X$. Then $D$ must be contained in $T$, and in fact be meridional in $T$. Thus $\partial D$ has intersection number 1 with each of $\beta$ and $\beta'$, and we can isotope $D$ to realise these intersection numbers and to be in general position with respect to $\Sigma$ and $\Sigma'$.

Theorem 3.14 shows that $\Sigma$ and $\Sigma'$ are essential in $X$ so we can isotope $D$ to remove any loops of intersection with either $\Sigma$ or $\Sigma'$.

Let $\gamma$ be an arc properly embedded in $D \cap F$ so that $\gamma$ connects $\partial D \cap \beta$ to $\partial D \cap \beta'$. Isotope $\alpha$ within $A$ so that $\alpha \cap \gamma = \partial \gamma$. Let $U$ be a regular neighbourhood of $\alpha \cup \gamma$ in $F$. Then $U$ is a punctured torus which is embedded in $\Sigma \setminus \Sigma' \subset F$. But $F$ is a Heegaard torus, so $U$ must be compressible in $X$, contradicting that $\Sigma$ is $\pi_1$-injective in $X$. Hence no compressing disk $D$ exists and $\partial T$ is incompressible in $X$.

Suppose that $D'$ is a meridional compressing disk for $\partial T$ which is in general position with respect to $\Sigma$, $\Sigma'$, and $\partial X$, and let $\delta = D \cap \partial X$. Then $\delta$ has intersection number 1 with $\partial \Sigma'$, but $\Sigma'$ does not meet $\partial T$. This is impossible. Hence $\partial T$ is meridionally incompressible in $X$, and thus not boundary parallel.
Therefore $\partial T$ is an essential torus in $X$ which means that $K$ is a satellite knot.

Figure 4.10: A weakly generalised alternating diagram onto a Heegaard torus of the satellite knot $13n_{4587}$.

This includes the satellite knots $13n_{4587}$ and $13n_{4639}$, which have the smallest crossing number amongst all prime satellite knots [42]. See Figure 4.10.

For weakly generalised alternating projections onto the torus, the only remaining case to deal with is when $F$ is Heegaard, and one region of $F \setminus \pi(K)$ is an annulus $A$, but the core of $A$ forms a trivial knot in $S^3$. In this case, we would expect $K$ to be hyperbolic, since the torus constructed in the proof of Theorem 4.13 is now compressible, but we have not been able to prove this.
An example of this type of hyperbolic knot is illustrated in Figure 3.18.

In the case of generalised alternating knots, we have the following corollary:

**Theorem 4.14.** Suppose that $K$ has a generalised alternating projection onto a torus $F$. Then $K$ is hyperbolic if and only if $F$ is Heegaard.

*Proof.* By Theorem 2.1, all the regions of $F \setminus \pi(K)$ are disks. Hence the result follows from Theorems 4.11 and 4.12. \qed
Chapter 5

Recognition Algorithms

In this chapter, we will give several normal surface algorithms. In the first two sections we give the background on the relevant results from normal surface theory for compact manifolds with boundary.

In Section 5.3, we give a normal surface algorithm which can decide if a knot manifold is the exterior of a prime and alternating knot in $S^3$. More generally, given a non-alternating planar diagram of a knot, we have a finite algorithm to decide if the knot admits a prime alternating planar diagram. If so, we can construct a prime alternating diagram.

In Section 5.4 we detail an algorithm to decide if a pair of normal surfaces satisfy the relative 1-line property. While we are not able to give a general algorithm which can decide if a knot is weakly generalised alternating, we are able to give several algorithms which work in restricted cases.
5.1 Normal Surface Theory

In this section we provide the background material necessary to construct our algorithms. Kneser introduced the concept of a normal surface, before Haken [30] developed normal surface theory into an important tool for algorithmic topology. We will give a brief outline of the theory, for full details the reader is referred to [52].

Let $M$ be a compact 3-manifold. A triangulation $\mathcal{T}$ is a set of tetrahedra $\{\Delta_i | 1 \leq i \leq t\}$, and a set of equations which identify the faces of the tetrahedra pairwise. If the link of every vertex is a 2-sphere, then $\mathcal{T}$ is a triangulation of a closed 3-manifold $M$. If we choose not to glue up some faces, and the link of any vertex in one of the unglued faces is a disk, then $\mathcal{T}$ is a triangulation of a compact 3-manifold with boundary.

A normal surface $S$ is an embedded surface in $M$ which is transverse to the 2-skeleton of $\mathcal{T}$, and such that $S \cap \Delta$ is a collection of triangular or quadrilateral disks, where $\Delta$ is one tetrahedron of $\mathcal{T}$, and each disk intersects each edge of $\Delta$ in at most one point. There are seven normal isotopy classes of normal disks, four are triangular and three are quadrilateral, and each is these is known as a disk type. $S \cap \phi$ is a collection of arcs, where $\phi$ is one

Figure 5.1: The four normal triangular disk types.
Figure 5.2: The three normal quadrilateral disk types. Note that any two quadrilateral disks of different type must intersect.

2-simplex of $T$.

If we fix an ordering of the disk types $d_1, d_2, ..., d_7$, then a normal surface $S$ can be represented uniquely up to normal isotopy by a 7$t$-tuple of non-negative integers $n(S) = (x_1, x_2, ..., x_7)$, where $x_i$ is the number of disks of type $d_i$, and $t$ is the number of tetrahedra in $T$.

Conversely, given a 7$t$-tuple of non-negative integers $n$, we can impose restrictions on the $x_i$ so that $n$ represents an embedded normal surface. We require that at least two of the three quadrilateral disk types are not present in each tetrahedra. This ensures that the surface is embedded. We also need to make sure that the disk types match up with the disk types in neighbouring tetrahedra.

An arc type is the normal isotopy class of the intersection of a normal surface with a face of a tetrahedron. There are three arc types in each face of each tetrahedron, and each arc type is contributed to by two different disk types, one triangular, the other quadrilateral. We require that the number of each arc type in each face agrees with the number of arc types in each face of the tetrahedron which is glued to it. This gives for each arc type, the linear equation

$$x_i + x_j = x_k + x_l,$$

where the corresponding $d_i, d_j$ are disc types in $\triangle$ and $d_k, d_l$ are disc types in $\triangle'$. Note that $\triangle$ and $\triangle'$ may be the same tetrahedron. These are called the matching equations for the normal surface $S$. In a triangulation of a compact manifold $M$, there are $6t - \frac{3}{2}s$ matching equations, where $t$ is the number of
tetrahedra and $s$ is the number of 2-simplices in the boundary.

The set of non-negative integer solutions to the matching equations lie within an infinite linear cone $\mathcal{S}_T \subset \mathbb{R}^{7_t}$. The linear cone $\mathcal{S}_T$ is called the solution space.

The additional condition that

$$\sum_{i=1}^{7_t} x_i = 1,$$

which is known as the normalising equation, turns the solution space into a compact, convex, linear cell $\mathcal{P}_T \subset \mathcal{S}_T$. We call $\mathcal{P}_T$ the projective solution space, and we let $\hat{n}(S)$ represent the projective class of the normal surface $S$. The carrier of a normal surface $S$, denoted $\mathcal{C}_T(S)$, is defined to be the unique minimal face of $\mathcal{P}_T$ which contains $\hat{n}(S)$.

Let $S$ be a properly embedded surface in a 3-manifold $M$ with triangulation $\mathcal{T}$. Then after a series of isotopies, compressions, boundary-compressions and the removal of trivial 2-spheres and disks, $S'$ can be represented as the union of properly embedded normal surfaces with respect to $\mathcal{T}$. In particular, if $S$ is essential in $M$, then $S$ can be isotoped to be normal with respect to $\mathcal{T}$.

**Theorem 5.1** (Haken [30]). Let $S$ be an incompressible and boundary incompressible surface properly embedded in a 3-manifold $M$. Then $S$ has a normal surface representative in any triangulation of $M$.

Two normal surfaces $S$ and $S'$ are compatible if for each tetrahedron $\triangle$ of $\mathcal{T}$, $S$ and $S'$ do not have quadrilateral disks of different types. If $S$ and $S'$ are compatible, then we can form the Haken sum of $S$ and $S'$, which we denote $S \oplus S'$. The Haken sum is a geometric sum along each arc and loop of intersection between $S$ and $S'$, which is uniquely defined by the requirement that $S \oplus S'$ is a normal surface. Any other choice of geometric sums would produce a surface with folds. If $\mathbf{n}(S) = (x_1, x_2, \ldots, x_{7_t})$ and $\mathbf{n}(S') = (x'_1, x'_2, \ldots, x'_{7_t})$ are representatives of compatible normal surfaces $S$ and $S'$ in a triangulation $\mathcal{T}$ of a 3-manifold $M$, then $\mathbf{n}(S \oplus S') = \mathbf{n}(S) + \mathbf{n}(S') = (x_1 + x'_1, x_2 + x'_2, \ldots, x_{7_t} + x'_{7_t})$. 

A normal surface is called a \textit{vertex surface} if $\mathbf{n}(S)$ lies at a vertex of the projective solution space. This means that whenever some multiple of $S$ can be written as a Haken sum of two surfaces, then both the summands are also multiples of $S$.

A normal surface is called a \textit{fundamental surface} if $\mathbf{n}(S)$ cannot be written as the sum of two solutions to the normal surface equations. Every vertex surface is a fundamental surface, but there exist fundamental surfaces which are not vertex surfaces.

Define the \textit{weight} $\omega(S)$ of a normal surface $S$ to be the number of intersections of $S$ with the 1-skeleton of $\mathcal{T}$. A normal surface $S$ is \textit{least weight} if it minimises weight over all normal surfaces which are isotopic to $S$. If $S \oplus S'$ is the Haken sum of two compatible normal surfaces $S$ and $S'$, then $\chi(S \oplus S') = \chi(S) + \chi(S')$ and $\omega(S \oplus S') = \omega(S) + \omega(S')$.

All normal surfaces can be written as a finite sum of fundamental surfaces. There are a finite number of fundamental surfaces. They can be found algorithmically and Haken used this fact to construct his algorithms. Many of these algorithms have been subsequently improved so that they utilise vertex solutions rather than fundamental solutions. This makes the algorithms much faster.

\textbf{Theorem 5.2} (Haken [30]). \textit{Let $S$ be a two-sided normal surface properly embedded in a 3-manifold $M$. There is an algorithm to decide if $S$ is incompressible and boundary-incompressible in $M$.}

We can use this algorithm to test if a one-sided surface is $\pi_1$-essential by testing whether its orientable double cover is incompressible and boundary-incompressible.

Haken was able to show that if a 3-manifold is Haken, then one of the fundamental solutions represents an essential surface. Jaco and Oertel showed that in a closed manifold such a surface can be found at a vertex of $\mathcal{P}_\mathcal{T}$. The importance of this result is that in order to decide if an irreducible 3-manifold is Haken, it is only necessary to test each vertex surface for incompressibility. Jaco and Tollefson subsequently extended the theorem to the case of a compact manifold with boundary.
Theorem 5.3 (Jaco-Oertel [47], Jaco-Tollefson [52]). Let $S$ be a compact, two-sided, incompressible, boundary-incompressible, least weight normal surface properly embedded in a closed, irreducible, boundary-irreducible 3-manifold $M$. Then every rational point in $C_{\mathcal{T}}(S)$ is the projective class of a $\pi_1$-injective, $\pi_1$-boundary-injective normal surface in $M$.

As a corollary of Theorem 5.3, Jaco and Tollefson [52] were able to show that if $A$ is an essential, two-sided, normal, least weight annulus or torus properly embedded in a compact, orientable, irreducible, boundary-irreducible 3-manifold $M$ with triangulation $\mathcal{T}$, then every vertex surface in $C_{\mathcal{T}}(A)$ is either an essential annulus or an essential torus. It will be necessary to use the following version of that statement.

Theorem 5.4. If a knot exterior $X$ contains an essential torus, then there is an essential torus at a vertex of $\mathcal{P}_{\mathcal{T}}$, where $\mathcal{T}$ is a triangulation of $X$.

Proof. Let $T'$ be an essential torus properly embedded in $X$. There is a least weight essential torus $T$ in the isotopy class of $T'$. Then there exists positive integers $m$ and $n$ such that

$$nT = S_1 \oplus \ldots \oplus S_m,$$

where each $S_i$ is a vertex surface. Furthermore, the corollary of Jaco and Tollefson implies that each $S_i$ is an essential annulus or torus. Note that a solid torus does not contain any essential closed surfaces so $X$ can be assumed boundary-irreducible. But if some $S_i$ was an annulus, then $nT$ would have non-empty boundary which cannot happen for a multiple of a closed surface. Thus each $S_i$ is an essential torus at a vertex of $\mathcal{P}_{\mathcal{T}}$. \hfill \square

We now briefly discuss almost normal surfaces. An octagonal disk is an eight-sided disk in a tetrahedron $\triangle$ which meets each face in two distinct arc types. The octagon meets two opposite edges of $\triangle$ twice and the other edges once. There are three octagonal disk types in each tetrahedron. An almost normal surface is a surface made up of triangular disks, quadrilateral disks and exactly one octagon. In general, the definition of an almost normal
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Figure 5.3: One of the three almost normal octagonal disk types.

surface also includes tubes between two normal disks in some tetrahedron, but for the purposes of Theorem 5.5, these tubes are unnecessary. The solution space for octagonal almost normal surfaces is a subset of the positive cone in $\mathbb{R}^{16t}$. At most one of the quadrilateral or octagonal coordinates can be non-zero in each tetrahedron.

The following 3-sphere recognition algorithm works by finding an almost normal 2-sphere in a triangulation of a homology 3-sphere. This almost normal 2-sphere corresponds to an unstable minimal 2-sphere.

**Theorem 5.5** (Rubinstein [81], Thompson [89]). There is an algorithm to decide if a 3-manifold is homeomorphic to the 3-sphere.

A triangulation is 0-efficient if the only normal disks or normal spheres are vertex-linking. A triangulation is 1-efficient if it is 0-efficient and the only normal tori or normal annuli are boundary-parallel or edge-linking.

In the case of a compact 3-manifold with torus boundary $M$, the advantage of working with a 1-efficient triangulation of $M$ is that there are only a finite number of normal surfaces of bounded Euler characteristic. If there were an essential torus $T$, and $S$ is any other properly embedded normal surface $S$ which meets $T$, then $S \oplus kT$ is not isotopic to $S$ for any $k > 1$, but
\[ \chi(S \oplus kT) = \chi(S) \text{ for all } k \in \mathbb{N}. \]

**Theorem 5.6** (Jaco-Rubinstein [49]). Let \( M \) be a compact orientable irreducible and boundary-irreducible 3-manifold with non-empty boundary. Then there exists a 0-efficient triangulation of \( M \).

**Corollary 5.7.** Let \( X \) be a knot exterior. Then there exists a one-vertex triangulation of \( X \).

**Proof.** This follows from Theorem 5.6 when \( X \) is the exterior of a non-trivial knot, since the only vertices in a 0-efficient triangulation are a single vertex in each boundary component. There also exists a one-vertex triangulation of the solid torus (see Figure 5.4).

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**5.2 The Boundary Solution Space**

Let \( \mathcal{T} \) be a one-vertex triangulation of a 3-manifold \( M \) with a torus boundary. Then there is an induced one-vertex triangulation of \( \mathcal{T}_\partial \) of \( \partial M \). The boundary triangulation \( \mathcal{T}_\partial \) consists of one vertex, three edges, and two faces. There are
six normal arc types, however a normal curve can is determined by just three of these arc types.

Every curve on $\partial M$ has a unique normal representative. This means that isotopy classes of curves on $\partial M$ correspond to normal isotopy classes of curves on $T_\partial$.

Figure 5.5: Normal arc types in the boundary triangulation.

Fix an ordering of the disk types in $T$ such that $d_1, \ldots, d_7$ represent the disk types in one of the tetrahedra which meets $\partial M$ in a face $\phi$. Furthermore, let $d_1, \ldots, d_4$ represent triangular disk types, and $d_5, d_6, d_7$ represent quadrilateral disks, such that $d_i$ and $d_{i+4}$ meet $\phi$ in the same arc type $a_i$ for $i = 1, 2, 3$. Here $d_4$ is the triangular disk type which is disjoint from $\phi$.

Let $y_i$ be the number of arcs of type $a_i$ in $\phi$. It follows that

$$y_i = x_i + x_{i+4}$$

for $i = 1, 2, 3$. Jaco and Segwick showed that $y_1, y_2, y_3$ and the matching equations for normal curves determine the number of arcs of each type in the other 2-simplex of $T_\partial$. We define the boundary solution space of $\partial M$ to be

$$S_{T_\partial} = \{(y_1, y_2, y_3) | y_i \in \mathbb{N}_0\} \subset \mathbb{R}^3.$$
Similarly, we define the projective boundary solution space to be

\[ \mathcal{P}_{\mathcal{T}_0} = \{(y_1, y_2, y_3) | y_1 + y_2 + y_3 = 1\}, \]

which is a triangle in \( \mathbb{R}^3 \) with vertices at (1, 0, 0), (0, 1, 0), and (0, 0, 1). If \( \partial S \) is the boundary of a properly embedded normal surface, then \( \partial S \) is represented by \( n(\partial S) = (y_1, y_2, y_3) \) in \( \mathcal{S}_{\mathcal{T}_0} \) and by \( \hat{n}(\partial S) \) in \( \mathcal{P}_{\mathcal{T}_0} \).

If each coordinate of \( n(\partial S) \) is non-zero, then \( \partial S \) contains a trivial curve. Hence if \( S \) is an incompressible surface, then at least one of the coordinates of \( n(\partial S) \) is zero.

**Theorem 5.8** (Jaco-Sedgwick [51]). *Let \( S \) be a properly embedded \( \pi_1 \)-essential normal surface with boundary in a compact 3-manifold \( M \) with triangulation \( \mathcal{T} \). Then every surface in \( \mathcal{C}_{\mathcal{T}}(S) \) is either closed, or has the same slope as \( S \).*

Theorem 5.8 shows that if \( \frac{p}{q} \) is a boundary slope of \( X \), then there is a vertex surface \( S \) which has slope \( \frac{p}{q} \). Hence it is only necessary to check the vertices of \( \mathcal{P}_{\mathcal{T}} \) in order to list all boundary slopes of \( X \). In proving this theorem, Jaco and Sedgwick have given another proof of Theorem 2.28. Recall that the set of boundary slopes of a link exterior is not necessarily finite, so our algorithms are only going to be concerned with knots.

**Dehn filling** is a process of creating a closed 3-manifold from a knot-manifold. Let \( M \) be a knot exterior. If \( \ell \) is an essential closed curve on \( \partial M \), then we form \( M(\ell) \), the Dehn filling of \( M \) along \( \ell \), by gluing a solid torus \( T \) to \( M \) along their boundaries, so that the boundary of the meridional disk for \( T \) is glued to \( \ell \).

If \( X \) is a knot exterior in \( S^3 \), then the meridian \( \mu \) is the unique slope such that \( X(\mu) \cong S^3 \). In \( S^3 \), this slope is unique by a result of Gordon and Luecke [29].

**Theorem 5.9** (Jaco-Sedgwick [51]). *Let \( M \) be a knot manifold. There is an algorithm to decide if \( M \) is a knot exterior in \( S^3 \), and an algorithm to find the meridian \( \mu \) of a knot exterior \( X \).*

This algorithm uses Theorem 5.5 and also includes a check that \( X \) is not a solid torus.
Theorem 5.10 (Haken [30]). There is an algorithm to decide if a knot in the 3-sphere is the unknot.

This algorithm has been improved many times, and it is now known that some spanning disk for $K$ can be found as a vertex solution if $K$ is the unknot.

The boundary triangulation $\mathcal{T}_\partial$ consists of two 2-simplices and three edges. We can modify the triangulation $\mathcal{T}$ by gluing a two faces of a tetrahedron $\Delta$ to $\mathcal{T}_\partial$. The resulting triangulation $\mathcal{T}' = \mathcal{T} \cup \Delta$ is another one vertex triangulation of $M$, and $\mathcal{T}'$ is called a layered triangulation. In effect, this is a 2, 2-Pachner move on $\mathcal{T}_\partial$. The other two faces of $\Delta$ form the boundary triangulation $\mathcal{T}'_\partial$.

Layering a tetrahedron changes the slope of one of the edges in the boundary triangulation. It is always possible to layer a triangulation with a sequence of tetrahedra so that the edges of $\mathcal{T}'_\partial$ have slopes $\infty, k, k + 1$, for some $k \in \mathbb{Z}$. We will choose to do this so that $(1, 0, 0) \in S_{\mathcal{T}_\partial}$ represents the meridian $\mu$.

The size of a normal surface $S$, written $\sigma(S)$, is the number of elementary disks in $S$. We write $\sigma_3(S)$ and $\sigma_4(S)$ for the number of triangular and
quadrilateral disks respectively in $S$. Thus
\[
\sigma(S) = \sum_{i=1}^{7t} x_i,
\]
and $\sigma(S) = \sigma_3(S) + \sigma_4(S)$.

The \textit{boundary size} of a normal surface $S$, written $\sigma(\partial S)$, is the number of elementary arcs in $\partial S$. If $S$ is closed, $\sigma(\partial S) = 0$. Since every boundary component of a compact surface is homeomorphic to a circle, it follows that
\[
\sigma(\partial S) = \omega(\partial S) = 2 \sum_{i=1}^{y_i}.
\]

**Lemma 5.11.** There is an algorithm to compute the Euler characteristic of a compact normal surface $S$ properly embedded in a 3-manifold $M$ with triangulation $\mathcal{T}$. In particular,
\[
\chi(S) = \omega(S) - \frac{1}{2} (\sigma(S) + \sigma_4(S) + \sigma(\partial S)).
\]

\textit{Proof.} The normal surface $S$ inherits a cellular structure from its intersection with the triangulation $\mathcal{T}$. The number of vertices in $S$ is equal to $\omega(S)$, and an algorithm to calculate the weight of a normal surface can be found in [52]. The number of faces in given by $\sigma(S)$. For a closed surface $S$, twice the number of edges of $S$ is equal to $3\sigma_3(S) + 4\sigma_4(S)$. Hence the number of edges in a compact surface with boundary is given by $\frac{3}{2}\sigma_3(S) + 2\sigma_4(S) + \frac{1}{2}\sigma(\partial S)$, since we have not double counted the edges in the boundary. The desired equation follows from an application of Euler’s formula. \hfill $\square$

### 5.3 Alternating Algorithm

We now describe a normal surface algorithm to decide if a knot is prime and alternating on $S^2$. For our algorithm the input is a triangulation $\mathcal{T}$ of a knot exterior $X$. If instead we are given a non-alternating planar diagram $\pi(K)$, then there is a method of Petronio [79] to construct a spine of the
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knot complement \( S^3 \setminus K \) from the diagram \( \pi(K) \). Dual to this spine is an ideal triangulation of \( S^3 \setminus K \). We can then use an inflation of Jaco and Rubinstein [50] to construct a one-vertex triangulation of the knot exterior \( X \) from the ideal triangulation.

We note that there already exists an algorithm to decide if a knot is prime and alternating, however it is very impractical. It was described by Haken, and the proof was completed by Hemion [36]. Assume we are given a knot exterior \( X \) with triangulation \( T \). Enumerate all reduced prime alternating knot diagrams in order of crossing number, and for each diagram \( \pi(K') \) construct a triangulation \( T' \) of its exterior \( X' \). Then use the Haken-Waldhausen algorithm [92], which can decide if \( X' \) is homeomorphic to \( X \). Eventually we will encounter a homeomorphism, which by Gordon and Luecke [29] is an equivalence of knot types. If the \( \pi(K') \) corresponding to \( X' \) is alternating, then \( X \) is the exterior of an alternating knot \( K \). If \( \pi(K') \) is non-alternating, then \( K' \) does not have an alternating diagram, since by Part 3 of Theorem 1.2, we would have previously encountered it in our list of reduced prime alternating knot diagrams, so in this case \( X \) is the exterior of a non-alternating knot.

In order to prove that we only need to look at fundamental surfaces in our algorithm, we need one more result on spanning surfaces.

**Lemma 5.12.** Suppose that \( \Sigma \) and \( \Sigma' \) are spanning surfaces for a knot \( K \subset S^3 \). If

\[
\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') > 2,
\]

then at least one of \( \Sigma \) or \( \Sigma' \) is not connected.

**Proof.** Isotope \( \Sigma \) and \( \Sigma' \) so that they are in general position and realise \( i(\partial \Sigma, \partial \Sigma') \) on \( \partial X \). As in the proof for Theorem 4.2, we form the pseudo 2-complex \( F' = \Sigma \cup \Sigma' \) and collapse the arcs of intersection to points to obtain an immersed surface \( F \), where \( \chi(F') = \chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') > 2 \). Let \( f : S \hookrightarrow S^3 \) be an immersion of a closed surface \( S \) such that \( f(S) = F \). The only possible self-intersections of \( f(S) \) are loops, so \( \chi(S) = \chi(F) > 2 \), and therefore \( S \) is not connected. Hence either \( \Sigma \) or \( \Sigma' \) is not connected. \( \square \)

**Theorem 5.13.** Let \( X \) be the exterior of a knot \( K \subset S^3 \). Given \( X \), there is a normal surface algorithm to decide if \( K \) is prime and alternating.
Proof. Let $\mathcal{T}'$ be a one-vertex triangulation of $X$. As shown in [51], any other triangulation of $X$ can be modified to a one-vertex triangulation.

Use the algorithm of Theorem 5.9 to find the meridian $\mu$ of $X$. Included in this process is a check whether $X$ is a solid torus. If $X$ is a solid torus, then $K$ is the unknot and is not alternating. If $X$ is not a solid torus, then Theorem 5.9 finds the unique meridian $\mu$.

Theorem 5.4 tells us that if $X$ contains an essential torus $T$, then some vertex of $\mathcal{P}_{\mathcal{T}'}$ must represent an essential torus. Check whether any of the vertices of $\mathcal{P}_{\mathcal{T}'}$ are essential tori. If any vertex tori are incompressible and not boundary parallel, then $K$ is a satellite knot, but Theorem 1.12 tells us that a prime alternating knot cannot be a satellite knot. Thus we can assume that the only incompressible torus is boundary parallel.

Layer the triangulation until one of the edges in the boundary is parallel to $\mu$. Then the other edges in the boundary are parallel to $\lambda + k\mu$ and $\lambda + (k + 1)\mu$ for some $k \in \mathbb{Z}$. Call this triangulation $\mathcal{T}$.

Let $\Delta$ be one of the two tetrahedra that meets the boundary and let $\phi$ be a face of $\Delta$ which lies in the boundary. Let $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$ describe the normal coordinates of $S \cap \Delta$ where $S$ is a properly embedded surface with boundary in $X$. Let $(y_1, y_2, y_3)$ describe the normal coordinates of $\partial S \cap \phi$. We label the arc and disc types such that $y_i = x_i + x_{i+4}$ for each $i = 1, 2, 3$, and such that $(1, 0, 0)$ represents $\mu$ in $\partial X$. Let $(0, 1, 0)$ represent $\lambda + k\mu$ so that $(0, 0, 1)$ represents $\lambda + (k + 1)\mu$ for some $k \in \mathbb{Z}$.

Any spanning surface for $X$ meets $\mu$ exactly once. It follows that if $(y_1, y_2, y_3)$ represents a spanning surface, then $y_2 + y_3 = 1$. So there are two types of coordinates in $\mathcal{S}_{\mathcal{T}_0}$ which can represent spanning surfaces: $(y, 1, 0)$ and $(y, 0, 1)$ for some $y \in \mathbb{N}_0$.

Let $\Sigma$ and $\Sigma'$ be normal spanning surfaces in $X$. We can read the intersection number of their boundaries off their coordinates in $\mathcal{S}_{\mathcal{T}_0}$. Let $y, y' \in \mathbb{N}_0$. There are three cases:

1. If $\partial \Sigma$ and $\partial \Sigma'$ are represented by $(y, 1, 0)$ and $(y', 1, 0)$ respectively, then $i(\partial \Sigma, \partial \Sigma') = |y - y'|$.

2. If $\partial \Sigma$ and $\partial \Sigma'$ are represented by $(y, 0, 1)$ and $(y', 0, 1)$ respectively,
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then \( i(\partial \Sigma, \partial \Sigma') = |y - y'| \).

3. If \( \partial \Sigma \) and \( \partial \Sigma' \) are represented by \( (y, 1, 0) \) and \( (y', 0, 1) \) respectively, then \( i(\partial \Sigma, \partial \Sigma') = y + y' + 1 \). See Figure 5.6 for an example of this case.

Note that we could continue layering the triangulation until \( k = 0 \), which would require detection of a Seifert surface, but this is not necessary since we are only interested in the differences of spanning slopes, and not the boundary slopes themselves.

Theorem 4.1 tells us that we need to find a pair of essential spanning surfaces at even boundary slope, which satisfy

\[
\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = 2, \tag{*}
\]

Let \( \Sigma \) be such a surface. Now suppose that \( \mathbf{n}(\Sigma) \) is not a fundamental solution. Then

\[\Sigma = \Sigma_1 \oplus \ldots \oplus \Sigma_a \oplus S_1 \oplus \ldots \oplus S_b,\]

where each \( \Sigma_i \) is a properly embedded compact surface with boundary, and each \( S_j \) is a properly embedded closed surface.

Theorem 5.8 tells us that each \( \Sigma_i \) must have the same slope as \( \Sigma \). Since \( \Sigma \) is a spanning surface, then \( a \) must equal one, and \( \Sigma_1 \) is also a spanning surface at the same slope as \( \Sigma \). In fact, \( \Sigma_1 \) must be fundamental.

Since \( X \) is orientable, irreducible, atoroidal and embedded in \( S^3 \), it follows that \( \chi(S_i) \leq -2 \) for each \( i = 1, \ldots, b \). Here we should note that if \( S \) is a normal surface, and \( T \) is a boundary-parallel torus, then \( S \oplus T \) is isotopic in \( X \) to \( S \), so Haken sum with \( T \) can be ignored. But then \( \chi(\Sigma) < \chi(\Sigma_1) \) which implies that

\[\chi(\Sigma_1) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma_1, \partial \Sigma') > 2.\]

Hence, by Lemma 5.12, at least one of \( \Sigma_1 \) or \( \Sigma' \) must be disconnected. Every fundamental surface is connected, so \( \Sigma' \) must be disconnected.

In that case,

\[\Sigma' = \Sigma'_1 \oplus S'_1 \oplus \ldots \oplus S'_c,\]

where \( \Sigma'_1 \) is fundamental and each \( S'_i \) is a closed embedded normal surface.
Repeating the previous argument shows that one of \( \Sigma_1 \) or \( \Sigma_1' \) is not connected, contradicting that they are both fundamental. Hence \( \Sigma \) and \( \Sigma' \) are fundamental surfaces.

Let \( \mathcal{F} \) be the set of all fundamental spanning surfaces in \( X \). For each pair of surfaces \( \Sigma, \Sigma' \in \mathcal{F} \), calculate the intersection number \( i(\partial \Sigma, \partial \Sigma') \), and using Lemma 5.11, calculate \( \chi(\Sigma) \) and \( \chi(\Sigma') \). If \( \Sigma \) and \( \Sigma' \) satisfy Equation (\( \star \)), then \( K \) is alternating by Theorem 4.2. Since we know \( K \) is not a satellite knot, \( K \) is prime. If no pair of surfaces from \( \mathcal{F} \) satisfy Equation (\( \star \)), then \( K \) is not prime and alternating.

Let \( \pi(K) \) be an alternating diagram of the prime knot \( K \) with associated checkerboard surfaces \( \Sigma \) and \( \Sigma' \). Let \( \pi_*(K) \) be a different alternating diagram of \( K \) with associated checkerboard surfaces \( \Sigma_* \) and \( \Sigma'_* \). If \( \pi(K) \) and \( \pi_*(K) \) are both reduced, then we know from Theorem 1.2, that \( \pi(K) \) and \( \pi_*(K) \) are related by a sequence of flypes. In that case, \( \Sigma \) is isotopic to \( \Sigma_* \), and while \( \Sigma' \) and \( \Sigma'_* \) are homeomorphic and have the same boundary slope, \( \Sigma' \) and \( \Sigma'_* \) are not isotopic.

However, every checkerboard surface for a reduced alternating diagram is \( \pi_1 \)-essential, and thus will appear amongst our collection of fundamental spanning surfaces \( \mathcal{F} \). The collection \( \mathcal{F} \) may also contain some pairs of surfaces which correspond to an alternating diagram which is not reduced. In this case, at least one of the checkerboard surfaces fails to be \( \pi_1 \)-essential.

Let \( \Sigma \) and \( \Sigma' \) have minimal intersection number amongst all surfaces from \( \mathcal{F} \) which satisfy (\( \star \)). Place an orientation on \( \partial \Sigma \), and Label the vertices of \( \partial \Sigma \cap \partial \Sigma' \) in the order they are encountered as one traverses \( \partial \Sigma \) by 1, ..., \( i \) where \( i = (\partial \Sigma, \partial \Sigma') \). Then each arc of intersection between \( \Sigma \) and \( \Sigma' \) is labelled by the two numbers at its two ends. These pairs of numbers correspond to the Dowker-Thistlethwaite notation [20] of a planar alternating diagram.

Thus given a non-alternating planar diagram of a knot \( K \), there is an algorithm to decide if \( K \) is prime and alternating, and if so, there is an algorithm to produce an alternating diagram of \( K \) up to chirality.
5.4 Weakly Generalised Alternating Algorithms

In this section, we will show that in some cases there is a normal surface algorithm to recognise a weakly generalised alternating knot from its exterior. According to Theorem 3.33, in order to check that a knot \( K \) is weakly generalised alternating, it is enough to find a pair of \( \pi_1 \)-essential spanning surfaces \( \Sigma \) and \( \Sigma' \) properly embedded in the knot exterior \( X \) such that \( \Sigma \) and \( \Sigma' \) have the relative 1-line property and can be isotoped to intersect only in standard arcs.

**Lemma 5.14.** Let \( N_\chi \) be the collection of normal \( \pi_1 \)-essential spanning surfaces of Euler characteristic at least \( \chi \) which are properly embedded in a knot exterior \( X \) with triangulation \( T \). There is an algorithm to decide if a pair of surfaces \( \Sigma, \Sigma' \in N_\chi \) can be isotoped to \( \Sigma_0 \) and \( \Sigma'_0 \) respectively so that \( \Sigma_0 \) and \( \Sigma'_0 \) have the relative 1-line property and intersect only in standard arcs.

**Proof.** Suppose that we can find two \( \pi_1 \)-essential spanning surfaces \( \Sigma \) and \( \Sigma' \) which have the relative 1-line property. Lemma 3.25 states that if \( \Sigma \) and \( \Sigma' \) are least area in their isotopy classes, then there are no product regions between them. Jaco and Rubinstein [48] tell us that every least area \( \pi_1 \)-essential surface has a normal representative.

As normal surfaces \( \Sigma \) and \( \Sigma' \) may in fact intersect in product regions, however there are always normal representatives such that the product region \( P \) is 0-weight. This means that \( P \), which is topologically a handlebody, is disjoint from the 1-skeleton of \( T \).

First we check that \( \Sigma \) and \( \Sigma' \) can be isotoped to intersect only in arcs. If \( \partial \Sigma \) and \( \partial \Sigma' \) have the same slope on \( \partial X \), then by Lemma 3.6, either \( \Sigma \) and \( \Sigma' \) intersect in an essential loop or \( \Sigma \) and \( \Sigma' \) are both orientable. We are not interested in either case, so we may assume that \( \partial \Sigma \) and \( \partial \Sigma' \) have different slopes.

Then there is a normal isotopy of \( \Sigma \) so that \( \partial \Sigma \) and \( \partial \Sigma' \) form a quadrangulation of \( \partial X \). This is because Jaco and Sedgwick [51] show that normal
isotopy classes of curves on a one-vertex triangulation of a torus \( \partial X \) correspond to isotopy classes of curves on \( \partial X \).

Let \( \mathcal{L} \) be the collection of loops of intersection between \( \Sigma \) and \( \Sigma' \). We will assume \( \mathcal{L} \) is non-empty as otherwise there is nothing to do. Split \( \Sigma \) and \( \Sigma' \) along \( \mathcal{L} \). Search amongst the pieces for regions \( R \subset \Sigma \setminus \mathcal{L} \) and \( R' \subset \Sigma' \setminus \mathcal{L} \) such that \( \partial R = \partial R' \subset \mathcal{L} \) and \( R \cong R' \). We can then check if \( R \cup R' \) bounds a product region. If there are no product regions but \( \mathcal{L} \) is non-empty, then \( \mathcal{L} \) contains a loop of intersection that cannot be isotoped away, and we can discard that pair of surfaces.

Otherwise let \( R \cup R' \) be an innermost product region \( P \). If \( \omega(P) > 0 \), then we can discard this pair of surfaces. If \( \omega(P) = 0 \), then there is a normal isotopy of \( \Sigma \) which pushes \( R \) through \( R' \) and removes the loops of \( \partial R \). Repeat this process until there are no loops of intersection, or there are no product regions. In the latter case, we discard that pair of surfaces. It is now safe to assume that \( \Sigma \) and \( \Sigma' \) are normal and \( \mathcal{L} = \emptyset \).

Now we can check whether \( \Sigma \) and \( \Sigma' \) satisfy the relative 1-line property. From above, there is a normal isotopy so that \( \Sigma \) and \( \Sigma' \) intersect only in standard arcs. Retriangulate \( X \) with a triangulation \( \mathcal{T}' \) so that \( \Sigma' \) lies in the 2-skeleton of \( \mathcal{T}' \) and such that \( \Sigma \) is normal in \( \mathcal{T}' \). Let \( Y \) be the 3-manifold obtained by cutting \( X \) along \( \Sigma' \). Let \( Z \) be the 3-manifold obtained by doubling \( Y \) across \( \Sigma' \), and let \( \mathcal{U} \) be the induced triangulation of \( Z \).

This construction is a little different from doubling along the boundary described in Section 3.7. There we doubled \( X \) along the entirety of \( \partial X \), but here we double \( Y \) along \( \Sigma' \) where \( \Sigma' \) is a subsurface of \( \partial Y \). Let \( 2\Sigma \) be the result of doubling \( \Sigma \) along the arcs \( \Sigma \cap \Sigma' \). As we construct the triangulation \( \mathcal{U} \) from \( \mathcal{T}' \), we are able to build the normal surface \( 2\Sigma \).

Now we use Theorem 5.2 to check whether \( 2\Sigma \) is incompressible in \( Z \). If there is a compressing disk \( D \) for \( 2\Sigma \) in \( Z \), then there is a subdisk \( B \subset D \) such that \( B \) is an essential bigon between \( \Sigma \) and \( \Sigma' \). Conversely, if \( B \) is an essential bigon between \( \Sigma \) and \( \Sigma' \), then \( 2B \) is a compressing disk for \( 2\Sigma \) in \( Z \). From Theorem 3.31 we know that \( \Sigma \) and \( \Sigma' \) have the relative 1-line property if and only if there are no essential bigons between \( \Sigma \) and \( \Sigma' \).

Repeat this algorithm on every pair of surfaces in \( \mathcal{N}_X \). If we find a pair
(Σ, Σ') which can be isotoped to a pair (Σ₀, Σ₀') which intersect only in standard arcs, and such that 2Σ₀ is incompressible in Z, then Σ and Σ' have the relative 1-line property. If 2Σ₀ is compressible in Z, then Σ and Σ' do not have the relative 1-line property.

Now we are ready to give algorithms which can decide if a knot is weakly generalised alternating from its exterior. In order to apply Lemma 5.14, we need to be sure that \( \mathcal{N}_x \) is finite. This is not true if \( K \) is a satellite knot, since in the presence of an essential torus \( T \), there are infinitely many spanning surfaces of a given Euler characteristic found by taking Haken sums with multiples of \( T \). If \( K \) is hyperbolic, then we can assume that the triangulation \( \mathcal{T} \) of \( X \) is 1-efficient, and there are a finite number of normal surfaces of a given Euler characteristic.

A 1-efficient triangulation can be constructed by inflating an ideal triangulation of a knot complement \( S³ \setminus K \) which has an angle structure or a taut structure. Taut structures on ideal triangulations can be found using work of Lackenby [55] and angle structures on ideal triangulations can be found using work of Hodgson, Rubinstein and Segerman [41].

**Theorem 5.15.** Let \( X \) be the exterior of a knot \( K \). Given \( X \), there is an algorithm to decide if \( K \) has a weakly generalised alternating projection \( \pi(K) \) onto a Heegaard torus \( F \), such that all the regions of \( F \setminus \pi(K) \) are disks.

**Proof.** Let \( \mathcal{T} \) be a one-vertex triangulation of \( X \). We first check that \( X \) is not a solid torus. Theorem 3.19 ensures that the unknot is not weakly generalised alternating.

If a knot \( K \) has a weakly generalised alternating projection \( \pi(K) \) onto a Heegaard torus \( F \), such that all the regions of \( F \setminus \pi(K) \) are disks, then Theorem 4.12 tells us that \( K \) is hyperbolic. So we can use Theorem 5.4 to check for any essential tori or essential annuli amongst the vertex solutions. If there is an incompressible torus which is not boundary parallel, then \( K \) is a satellite knot, and if there is an incompressible and boundary incompressible annulus, then \( K \) is either a satellite knot or a torus knot. If there are no essential tori or annuli in \( X \), we may assume that \( \mathcal{T} \) is in fact a 1-efficient triangulation of \( X \).
Use Theorem 5.9 to find the meridian $\mu$ and layer $T$ so that $(1,0,0) \in S_{T_0}$ represents $\mu$. List the surfaces which correspond to the vertices of $\mathcal{P}_T$ and list their slopes. This list includes all boundary slopes of $K$ by Theorem 5.8. Let $d$ be the largest difference in slopes of listed surfaces which are spanning surfaces. Then $d \in 2\mathbb{N}_0$ and $d$ is an upper bound for the spanning slope diameter.

If $d = 0$, then $K$ is not weakly generalised alternating. In fact, since $K$ is not a torus knot, $K$ would be a counterexample to Conjecture 1.21.

Then $\chi(\Sigma) \geq 1 - \frac{d}{2}$ since every region is a disk and $\Sigma$ is made up of some number of disks connected by at most $\frac{d}{2}$ half twisted bands. This gives us a bound on the Euler characteristic for each spanning surface.

Let $\mathcal{F}$ be the collection of fundamental surfaces for $X$ which are not boundary-parallel tori. Let $\mathcal{N}_\chi$ be the collection of all Haken sums of surfaces from $\mathcal{F}$ such that if $\Sigma \in \mathcal{N}_\chi$, then $\Sigma$ is a $\pi_1$-essential spanning surface and $\chi(\Sigma) \geq 1 - \frac{d}{2}$. We can see if $\Sigma$ is a spanning surface by looking at the coordinates of $n(\partial \Sigma)$, we can test whether $\Sigma$ is $\pi_1$-essential with Theorem 5.2, and we can calculate $\chi(\Sigma)$ with Lemma 5.11.

Now apply Lemma 5.14 to see if some pair of surfaces from $\mathcal{N}_\chi$ can be isotoped to intersect only in standard arcs and satisfy the relative 1-line property. Keep all pairs which pass this test.

For any remaining pair of surfaces $(\Sigma, \Sigma')$, we can use

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2} i(\partial \Sigma, \partial \Sigma') = 0,$$

which follows from Theorem 3.10 to check that $F$ is a torus.

Finally for any remaining pair of surfaces $(\Sigma, \Sigma')$, we can check that every component of $\Sigma$ cut along $\Sigma'$ is a disk, and also that every component of $\Sigma'$ cut along $\Sigma$ is a disk. If not, then $K$ is weakly generalised alternating on the torus, but the regions of $F \setminus \pi(K)$ are not all disks.

This surface $F$ must be Heegaard since otherwise Theorem 4.11 forces $K$ to be a satellite knot, which has already been ruled out earlier in the algorithm.

For projections onto higher genus surfaces, we do not have as much control...
over when a diagram represents either a satellite knot or a hyperbolic knot. Hence for our final two algorithms we assume that we are given a hyperbolic knot exterior.

**Theorem 5.16.** Let $X$ be the exterior of a hyperbolic knot $K$. Given $X$, there is an algorithm to decide if $K$ has a weakly generalised alternating projection $\pi(K)$ onto a closed orientable surface $F$ such that all the regions of $F \setminus \pi(K)$ are disks.

*Proof.* Since $K$ is hyperbolic and we are not looking for the unknot, we may assume as in the previous algorithm that $X$ can be triangulated with a layered 1-efficient triangulation $T$, where $(1, 0, 0) \in S_{T_0}$ represents the meridian $\mu$.

List the surfaces which correspond to the vertices of $P_T$ and list their slopes. This list includes all boundary slopes of $K$ by Theorem 5.8. As before, let $d \in 2\mathbb{N}$ be the largest difference in slopes of listed surfaces which are spanning surfaces. If $d = 0$, then $K$ is not weakly generalised alternating.

Then $\chi(\Sigma) \geq 1 - \frac{d}{2}$ since every region is a disk and $\Sigma$ is made up of some number of disks connected by at most $\frac{d}{2}$ half twisted bands. This gives us a bound on the Euler characteristic for each spanning surface.

Let $\mathcal{N}_x$ be the finite collection of spanning surfaces which are Haken sums of non-boundary-parallel fundamental surfaces from $S_T$ such that if $\Sigma \in \mathcal{N}_x$, then $\Sigma$ is a $\pi_1$-essential normal spanning surface and $\chi(\Sigma) \geq 1 - \frac{d}{2}$.

Now apply Lemma 5.14 to see if some pair of surfaces from $\mathcal{N}_x$ can be isotoped to intersect only in standard arcs and satisfy the relative 1-line property. Keep all pairs which pass this test.

Finally for any remaining pair of surfaces $(\Sigma, \Sigma')$, we can check that every component of $\Sigma$ cut along $\Sigma'$ is a disk, and also that every component of $\Sigma'$ cut along $\Sigma$ is a disk. If not, then $K$ is weakly generalised alternating, but the regions of $F \setminus \pi(K)$ are not all disks.

Once we have found the two surfaces $\Sigma$ and $\Sigma'$, we can use Theorem 3.10 to determine the genus of the projection surface $F$. \qed

**Theorem 5.17.** Let $X$ be the exterior of a hyperbolic knot $K$. Given $X$ and $g \in \mathbb{N}$, there is an algorithm to decide if $K$ has a weakly generalised
alternating projection $\pi(K)$ onto a closed orientable surface $F$ such that the genus of $F$ is at most $g$.

Proof. Since $K$ is hyperbolic and we are not looking for the unknot, we may assume as in the previous algorithm that $X$ can be triangulated with a layered 1-efficient triangulation $T$, where $(1, 0, 0) \in S_{r_0}$ represents the meridian $\mu$.

If $\pi(K)$ is weakly generalised alternating on $F$ with associated checkerboard surfaces $\Sigma$ and $\Sigma'$, then

$$\chi(\Sigma) + \chi(\Sigma') + \frac{1}{2}i(\partial \Sigma, \partial \Sigma') = \chi(F) \geq 2 - 2g, \tag{*}$$

by Theorem 3.10. Use Theorem 5.8 to list all slopes of spanning surfaces in $X$, and let $d \in \mathbb{N}$ be the diameter of this set. Then $d$ is an upper bound for $i(\partial \Sigma, \partial \Sigma')$.

Since $K$ is hyperbolic, no spanning surface is a disk or Mobius band, so $\chi(\Sigma') \leq -1$. Hence, Equation \((*)\) implies that

$$\chi(\Sigma) \geq 3 - 2g - \frac{d}{2}.$$

Let $\mathcal{N}_\chi$ be the finite collection of $\pi_1$-spanning surfaces which are Haken sums of fundamental surfaces from $T$ and so that every $\Sigma \in \mathcal{N}_\chi$ satisfies $\chi(\Sigma) \geq 3 - 2g - \frac{d}{2}$.

Use Lemma 5.14 to search for any pair from $\mathcal{N}_\chi$ which have the relative 1-line property, and can be isotoped to intersect only in standard arcs. If such a pair also satisfies \((*)\), then $K$ is weakly generalised alternating onto a surface of genus at most $g$.

We do not currently have an algorithm which can decide if a knot is weakly generalised alternating in general. One way to do that, would be to show that if two surfaces have the relative 1-line property, then they are vertex surfaces.
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