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Probabilistic hesitant fuzzy multiple criteria decision-making with triangular norm based similarity and entropy measures

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ABSTRACT

Existing probabilistic hesitant fuzzy set (PHFS) measures are constructed using two information measures: hesitancy and unwrapped probabilities. We argue that unifying these semantic terms in PHFS information theory is not logical. We introduce a new class of information measures for PHFSs, which address the logical wrapping of hesitant fuzzy sets (HFS) and probability. We propose several similarity measures for these sets that use the Triangular norm operator. We consider the relationship between measures of entropy and similarity and represent the axiomatic definition of PHFS entropy measures. Finally, we use case studies to demonstrate applications of these information measures. We describe two multiple-criteria decision-making algorithms. The last step is devoted to PHFS ranking procedures: one based on the score function of alternatives and the other based on the relative closeness of alternatives. This contribution describes new information measures and uses case studies to illustrate how they can be applied to decision-making processes.

1. Introduction

In most multiple criteria decision making situations, qualitative values are necessary to aid decision making. In a recent and thorough investigation, Wieckowski et al. [1] made substantial contributions to both the theoretical comprehension and practical implementation of economic principles. Their paper proposes various methodological approaches to problem-solving and offers practical insights into individual decision models. It provides readily applicable decision models and conducts a comprehensive evaluation of recent advancements in multiple criteria decision analysis, encouraging well-informed and reliable decision-making processes. Additionally, this examination highlights the most recent advancements in multiple criteria decision analysis, with a particular emphasis on emerging trends in healthcare [2–9], energy development [10–14], supplier selection [15–22], transportation [23–27], sustainable development [28–31], and other related areas.

Any way, the specification of such values gets more difficult when it is required to evaluate alternatives by taking different backgrounds or experiences into account. In such cases, alternatives are represented by various possible values instead of consistent information.

Hesitant fuzzy set (HFS) [32] describes the case where we assign possible values to the membership of an element in a given set.

In recent years, studies in hesitant fuzzy environment and its different extensions have mainly been devoted to those concerning hesitant fuzzy information measures. This class of measures covers mostly the concepts of distance, similarity and entropy, and plays an indispensable role in the multiple criteria decision making context. The two latter measures, similarity and entropy, have the greatest impact. The similarity measure is employed to discriminate between different information, while the entropy measure indicates information quantity. So far, the concept of HFS information measure has been studied from various perspectives. Xu and Xia [33] firstly developed a set of hesitant fuzzy aggregation operators on the basis of algebraic t-norms and t-conorms, and then suggested a number of HFS distance and similarity measures. By emphasising the application of information measures in pattern recognition, Zeng et al. [34] presented a variety of HFS similarity measures together with entropy measures. By fusing a class of hybrid weighted aggregation operators with hesitant fuzzy entropy and correlation measures, Liao and Xu [35] developed several extended hesitant fuzzy hybrid weighted aggregation operators for solving HFS-based decision-making problems. Farhadinia [36] investigated and studied the systematic transformation of HFS information measures into each other, and subsequently presented a set of information measures for interval-valued hesitant fuzzy sets by emphasising that they are to be used for data analysis and classification.

These information measures have been developed and applied to other extensions of HFS. To take a further instance, we may refer to the distance measure for dual hesitant fuzzy sets [37] and hesitant fuzzy linguistic sets [38], the similarity measure for interval-valued hesitant

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fuzzy sets [39], and the entropy measure for generalized hesitant fuzzy sets [40].

As in the case of real-world applications, different degrees of membership in a hesitant fuzzy element (HFE) may have different importance values depending on the individual preferences of experts. By combining the experts' hesitation term to be accompanied with the corresponding probabilities, Zhu and Xu's [41] definition of *probabilistic hesitant fuzzy set* (PHFS) provides a more comprehensible approach to HFS situations.

Following the spirit of this theory, a remarkable variety of studies have been dedicated to the implementation of PHFS information measure applications. Li and Wang [42] enhanced the PHFS-based contributions by proposing the Hausdorff distance measure in the framework of QUALItative FLEXible multiple technique to adequately evaluate green suppliers' terms. Zeng et al. [43] made a unification framework of ordered weighted averaging operators with the concept of probability to present the uncertain probabilistic ordered weighted averaging distance. Ding et al. [44] established a variety of PHFS distance measures with the help of axiomatic design theory, then applied them into multiple attribute group decision making where the information of weights is incomplete. Gao et al. [45] developed a dynamic decision making technique concerning emergency response by using Hamming and Euclidean PHFS distance measures. Wu et al. [46] initially introduced the hesitant degree of PHFSs, and accordingly proposed a distance measure. Liu et al. [47] generalised the notion of PHFS distance measure using the combination of three terms: hesitancy degree, probability, and hesitancy degree multiplied by its corresponding probability. Su et al. [48] initially developed a number of PHFS entropy measures which are inversely proportional to PHFS distance measures, then introduced a like-distance measure to describe another kind of PHFS entropy measure. Liu et al. [49] specified the criteria weight of multiple criteria decision making in an effective way which is improved by taking the mixing entropy into account. Zhu et al. [50] developed an improved multiple criteria decision making technique by adopting both a cross-entropy measure and a PHFS symbol distance measure. In the domain of probabilistic hesitant fuzzy decision-making, entropy and cross-entropy [51] play crucial roles as metrics for assessing uncertainty in decision-making processes. They have widespread applications and hold significant theoretical importance as referenced in [52,53]. Nevertheless, research on entropy measures for probabilistic hesitant fuzzy information is still limited due to its inherent complexity [54, 55]. This complexity stems from the intricate nature of quantifying probabilistic hesitant fuzzy information compared to hesitant fuzzy information. The latter involves two layers of information: membership values and corresponding probabilistic data, making it more intricate than hesitant fuzzy information. Consequently, addressing the handling of probabilistic information becomes a critical challenge that needs to be tackled before developing any measure for probabilistic hesitant fuzzy information.

Existing measures of probabilistic hesitant fuzzy sets (PHFS) have typically been developed based on the concepts of hesitancy and unwrapped probabilities. However, there is a notable absence of a unified approach that effectively integrates these two distinct semantic terms within the framework of PHFS information theory. Moreover, the primary limitation of current information measures lies in the challenge of accurately distinguishing between hesitancy and probability, highlighting the need for exploring logical PHFS information measures from a practical standpoint. To address these limitations, we propose a new class of information measures for PHFSs that aim to reconcile the logical interplay between hesitant fuzzy sets (HFS) and probabilities. Within this proposed framework, our contribution introduces a set of innovative similarity and similarity-based entropy measures specifically tailored for PHFSs. This approach seeks to provide a more comprehensive understanding of the underlying uncertainty within PHFSs by capturing the nuanced relationship between hesitancy and probability, thereby advancing the field of PHFS information theory.

The organisation of this study is as follows. We review some preliminaries about PHFSs, and provide a necessary foundation of developing innovative PHFS information measures in Section 2. Then, Section 3 deals with the main focus of this article: logical and substantive PHFS similarity and PHFS entropy measures. Section 4 is dedicated to the first and second algorithms for PHFS-based multiple criteria decision making. The first algorithm is based on the score function of PHFSs, and the second algorithm is established by encountering of Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS). Section 5 ends the article with some final remarks and provides some perspectives.

2. Preliminaries

This section gives an introduction to the basic concepts which will provide the foundation for main results of this contribution.

Hesitant fuzzy set [32] (HFS) is a concept on the universe of discourse X in terms of a function that returns a subset of $[0,1]$. Symbolically, this concept was presented by Xia and Xu [33] as

$$\begin{aligned} H &= \{\langle x, h(x)\rangle : x \in X\} \\ &= \{\langle x, \bigcup_{h^{(j)}(x)\in h(x)} \{h^{(1)}(x), \ldots, h^{(j)}(x), \ldots, h^{(l)}(x)\}\rangle : x \in X\}, \end{aligned}$$

in which the term $h(x)$ is referred to as *hesitant fuzzy element* (HFE) containing all possible membership degrees of $x \in X$ to the set H.

In this contribution, the notation h stands for the set of HFE $h(x)$ with the length of $|h| = l$, that is,

$$h = \bigcup_{h^{(j)}\in h} \{h^{(j)}\} = \{h^{(1)}, \ldots, h^{(j)}, \ldots, h^{(l)}\},$$

whose elements are arranged in *increasing order*.

Probabilistic hesitant fuzzy set (PHFS) was proposed by Zhang et al. [56] to indicate that each possible value of HFE may have a probability value. It is roughly represented by

$${}^pH = \{\langle x, {}^ph(x) := \bigcup_{\langle h,p\rangle\in {}^ph(x)} \{\langle h, p\rangle\}\rangle : x \in X\}.$$

where ${}^ph = \bigcup_{\langle h,p\rangle\in {}^ph}\{\langle h, p\rangle\} = \{\langle h^{(1)}, p^{(1)}\rangle, \ldots, \langle h^{(l)}, p^{(l)}\rangle\}$ indicates a *probabilistic hesitant fuzzy element* (PHFE). In such a notation, $p^{(j)}$ is a probability associated with the element $h^{(j)}$, for $j = 1, 2, \ldots, l$.

Operational laws of PHFEs have been investigated from different perspectives [36,56,57]. Here, the operational laws are described following Farhadinia and Viedma's [57] definition:

$${}^ph^c = \bigcup_{\langle h,p\rangle\in {}^ph} \{\langle 1-h, p\rangle\} = \{\langle 1-h^{(1)}, p^{(1)}\rangle, \ldots, \langle 1-h^{(l)}, p^{(l)}\rangle\}; \tag{1}$$

$$\begin{aligned} \lambda\, {}^ph &= \bigcup_{\langle h,p\rangle\in {}^ph} \{\langle 1-(1-h)^\lambda, p\rangle\} \\ &= \{\langle 1-(1-h^{(1)})^\lambda, p^{(1)}\rangle, \ldots, \langle 1-(1-h^{(l)})^\lambda, p^{(l)}\rangle\},\ \ \lambda > 0; \end{aligned} \tag{2}$$

$$[{}^ph]^\lambda = \bigcup_{\langle h,p\rangle\in {}^ph} \{\langle h^\lambda, p\rangle\} = \{\langle (h^{(1)})^\lambda, p^{(1)}\rangle, \ldots, \langle (h^{(l)})^\lambda, p^{(l)}\rangle\},\ \ \lambda > 0; \tag{3}$$

$$\begin{aligned} {}^ph_1 \oplus {}^ph_2 &= \bigcup_{\langle h_1,p_1\rangle\in {}^ph_1, \langle h_2,p_2\rangle\in {}^ph_2} \{\langle h_1 + h_2 - h_1h_2, p_1 + p_2 - p_1p_2\rangle\} = \\ &\{\langle h_1^{(1)} + h_2^{(1)} - h_1^{(1)}h_2^{(1)}, p_1^{(1)} + p_2^{(1)} - p_1^{(1)}p_2^{(1)}\rangle, \ldots, \\ &\langle h_1^{(l)} + h_2^{(l)} - h_1^{(l)}h_2^{(l)}, p_1^{(l)} + p_2^{(l)} - p_1^{(l)}p_2^{(l)}\rangle\}; \end{aligned} \tag{4}$$

$$\begin{aligned} {}^ph_1 \otimes {}^ph_2 &= \bigcup_{\langle h_1,p_1\rangle\in {}^ph_1, \langle h_2,p_2\rangle\in {}^ph_2} \{\langle h_1h_2, p_1p_2\rangle\} = \\ &\{\langle h_1^{(1)}h_2^{(1)}, p_1^{(1)}p_2^{(1)}\rangle, \ldots, \langle h_1^{(l)}h_2^{(l)}, p_1^{(l)}p_2^{(l)}\rangle\}, \end{aligned} \tag{5}$$

where ${}^ph = \bigcup_{\langle h,p\rangle\in {}^ph}\{\langle h, p\rangle\} = \{\langle h^{(1)}, p^{(1)}\rangle, \ldots, \langle h^{(l)}, p^{(l)}\rangle\}$, ${}^ph_1 = \bigcup_{\langle h_1,p_1\rangle\in {}^ph_1}\{\langle h_1, p_1\rangle\} = \{\langle h_1^{(1)}, p_1^{(1)}\rangle, \ldots, \langle h_1^{(l_1)}, p_1^{(l_1)}\rangle\}$ and ${}^ph_2 = \bigcup_{\langle h_2,p_2\rangle\in {}^ph_2}\{\langle h_2, p_2\rangle\} = \{\langle h_2^{(1)}, p_2^{(1)}\rangle, \ldots, \langle h_2^{(l_2)}, p_2^{(l_2)}\rangle\}$, and also, $l = \max\{l_1, l_2\}$. Note that the PHFE with fewer elements is extended by repeating its maximum element with the probability $p = 0$ until it has the same length with the other PHFE.

Definition 2.1. The PHFE environment can be given a partial ordering under *component-wise subset inclusion* as:

$$ {}^{p}h_1 \subseteq {}^{p}h_2 \quad if \quad \langle h_1^{(j)}, p_1^{(j)} \rangle \subseteq \langle h_2^{(j)}, p_2^{(j)} \rangle, \text{ that is, } h_1^{(j)} \le h_2^{(j)} \text{ and } p_1^{(j)} \le p_2^{(j)} \tag{6} $$

where $j = 1, 2, \ldots, l = \max\{l_1, l_2\}$.

In reality, because of increasing complexity and uncertainty involved in practice, and the incorrect wrapping of the components of PHFE (namely, the HFE and probability) in the existing information measures, it is necessary to determine how to correctly wrap two different semantic terms of HFE and probability. Before dealing with this issue, it is appropriate to present some basic notions below.

Suppose that $\psi : [0,1] \to [0,1]$ is a strictly monotone decreasing function that can be given by

$$ \psi_1(x) = 1 - x; \tag{7} $$
$$ \psi_2(x) = 1 - x^2; \tag{8} $$
$$ \psi_3(x) = \frac{1}{1+x}; \tag{9} $$
$$ \psi_4(x) = \frac{1-x}{1+x}; \tag{10} $$
$$ \psi_5(x) = e^{-x}; \tag{11} $$
$$ \psi_6(x) = 1 - xe^{x-1}. \tag{12} $$

Keeping ${}^{p}h_1 = \bigcup_{\langle h_1,p_1\rangle \in {}^{p}h_1} \{\langle h_1, p_1\rangle\} = \{\langle h_1^{(1)}, p_1^{(1)}\rangle, \ldots, \langle h_1^{(l_1)}, p_1^{(l_1)}\rangle\}$ and ${}^{p}h_2 = \bigcup_{\langle h_2,p_2\rangle \in {}^{p}h_2} \{\langle h_2, p_2\rangle\} = \{\langle h_2^{(1)}, p_2^{(1)}\rangle, \ldots, \langle h_2^{(l_2)}, p_2^{(l_2)}\rangle\}$ in mind, we are now able to define the distance between HFEs (i.e., $|h_1^{(j)} - h_2^{(j)}|$) and the distance between probabilities (i.e., $|p_1^{(j)} - p_2^{(j)}|$) by the help of (7)–(12) as follows:

$$ \psi_{1h}(h_1^{(j)}, h_2^{(j)}) := \psi_1(x = |h_1^{(j)} - h_2^{(j)}|) = 1 - |h_1^{(j)} - h_2^{(j)}|; \tag{13} $$
$$ \psi_{2h}(h_1^{(j)}, h_2^{(j)}) := \psi_2(x = |h_1^{(j)} - h_2^{(j)}|) = 1 - (|h_1^{(j)} - h_2^{(j)}|)^2; \tag{14} $$
$$ \psi_{3h}(h_1^{(j)}, h_2^{(j)}) := \psi_3(x = |h_1^{(j)} - h_2^{(j)}|) = \frac{1}{1 + |h_1^{(j)} - h_2^{(j)}|}; \tag{15} $$
$$ \psi_{4h}(h_1^{(j)}, h_2^{(j)}) := \psi_4(x = |h_1^{(j)} - h_2^{(j)}|) = \frac{1 - |h_1^{(j)} - h_2^{(j)}|}{1 + |h_1^{(j)} - h_2^{(j)}|}; \tag{16} $$
$$ \psi_{5h}(h_1^{(j)}, h_2^{(j)}) := \psi_5(x = |h_1^{(j)} - h_2^{(j)}|) = e^{-|h_1^{(j)} - h_2^{(j)}|}; \tag{17} $$
$$ \psi_{6h}(h_1^{(j)}, h_2^{(j)}) := \psi_6(x = |h_1^{(j)} - h_2^{(j)}|) = 1 - |h_1^{(j)} - h_2^{(j)}| e^{|h_1^{(j)} - h_2^{(j)}| - 1}, \tag{18} $$

and

$$ \psi_{1p}(p_1^{(j)}, p_2^{(j)}) := \psi_1(x = |p_1^{(j)} - p_2^{(j)}|) = 1 - |p_1^{(j)} - p_2^{(j)}|; \tag{19} $$
$$ \psi_{2p}(p_1^{(j)}, p_2^{(j)}) := \psi_2(x = |p_1^{(j)} - p_2^{(j)}|) = 1 - (|p_1^{(j)} - p_2^{(j)}|)^2; \tag{20} $$
$$ \psi_{3p}(p_1^{(j)}, p_2^{(j)}) := \psi_3(x = |p_1^{(j)} - p_2^{(j)}|) = \frac{1}{1 + |p_1^{(j)} - p_2^{(j)}|}; \tag{21} $$
$$ \psi_{4p}(p_1^{(j)}, p_2^{(j)}) := \psi_4(x = |p_1^{(j)} - p_2^{(j)}|) = \frac{1 - |p_1^{(j)} - p_2^{(j)}|}{1 + |p_1^{(j)} - p_2^{(j)}|}; \tag{22} $$
$$ \psi_{5p}(p_1^{(j)}, p_2^{(j)}) := \psi_5(x = |p_1^{(j)} - p_2^{(j)}|) = e^{-|p_1^{(j)} - p_2^{(j)}|}; \tag{23} $$
$$ \psi_{6p}(p_1^{(j)}, p_2^{(j)}) := \psi_6(x = |p_1^{(j)} - p_2^{(j)}|) = 1 - |p_1^{(j)} - p_2^{(j)}| e^{|p_1^{(j)} - p_2^{(j)}| - 1}, \tag{24} $$

where $j = 1, 2, \ldots, l = \max\{l_1, l_2\}$.

3. Information measures for PHFSs

What we should expect from this section is finding the procedure of how we can construct innovative similarity and entropy measures for PHFSs. We first will introduce a class of measures that incorporates the concepts of distances between HFEs and the distances between probabilities together with the concept of t-norm. Then, the proposed similarity measures for the elements of PHFEs are extended to those for PHFSs. The same procedure will be performed for constructing entropy measures for PHFEs and then extending to that for PHFSs.

Now, let us discuss the characteristics of mappings $\psi_h : [0,1] \times [0,1] \to [0,1]$ and $\psi_p : [0,1] \times [0,1] \to [0,1]$ which are derived above by using $\psi : [0,1] \to [0,1]$. To do that, we assume ${}^{p}h_1 = \bigcup_{\langle h_1,p_1\rangle \in {}^{p}h_1} \{\langle h_1, p_1\rangle\} = \{\langle h_1^{(1)}, p_1^{(1)}\rangle, \ldots, \langle h_1^{(l_1)}, p_1^{(l_1)}\rangle\}$, ${}^{p}h_2 = \bigcup_{\langle h_2,p_2\rangle \in {}^{p}h_2} \{\langle h_2, p_2\rangle\} = \{\langle h_2^{(1)}, p_2^{(1)}\rangle, \ldots, \langle h_2^{(l_2)}, p_2^{(l_2)}\rangle\}$ and ${}^{p}h_3 = \bigcup_{\langle h_3,p_3\rangle \in {}^{p}h_3} \{\langle h_3, p_3\rangle\} = \{\langle h_3^{(1)}, p_3^{(1)}\rangle, \ldots, \langle h_3^{(l_3)}, p_3^{(l_3)}\rangle\}$ with $l = \max\{l_1, l_2, l_3\}$. If we focus on the jth element of ${}^{p}h_1$, ${}^{p}h_2$ and ${}^{p}h_3$, we can draw the following conclusion:

Theorem 3.1. *The mappings ψ_h and ψ_p given respectively by formulas (13)–(18) and (19)–(24) satisfy the following properties:*

(S0) $0 \le \psi_h(h_1^{(j)}, h_2^{(j)}),\ \psi_p(p_1^{(j)}, p_2^{(j)}) \le 1$;
(S1) $\psi_h(h_1^{(j)}, h_2^{(j)}) = \psi_h(h_2^{(j)}, h_1^{(j)})$ *and* $\psi_p(p_1^{(j)}, p_2^{(j)}) = \psi_p(p_2^{(j)}, p_1^{(j)})$;
(S2) $\psi_h(h_1^{(j)}, h_2^{(j)}) = 1$ *if and only if* $h_1^{(j)} = h_2^{(j)}$; *and* $\psi_p(p_1^{(j)}, p_2^{(j)}) = 1$ *if and only if* $p_1^{(j)} = p_2^{(j)}$;
(S3) *If* $h_1^{(j)} \le h_2^{(j)} \le h_3^{(j)}$ *and* $p_1^{(j)} \le p_2^{(j)} \le p_3^{(j)}$, *then*

$$ \psi_h(h_1^{(j)}, h_3^{(j)}) \le \psi_h(h_1^{(j)}, h_2^{(j)}) \text{ and } \psi_h(h_1^{(j)}, h_3^{(j)}) \le \psi_h(h_2^{(j)}, h_3^{(j)}); $$
$$ \psi_p(p_1^{(j)}, p_3^{(j)}) \le \psi_p(p_1^{(j)}, p_2^{(j)}) \text{ and } \psi_p(p_1^{(j)}, p_3^{(j)}) \le \psi_p(p_2^{(j)}, p_3^{(j)}). $$

Proof. Taking any jth element $\langle h_1^{(j)}, p_1^{(j)}\rangle$, $\langle h_2^{(j)}, p_2^{(j)}\rangle$ and $\langle h_3^{(j)}, p_3^{(j)}\rangle$ of PHFEs ${}^{p}h_1$, ${}^{p}h_2$ and ${}^{p}h_3$, we conclude that:

Proof of (S0): The strictly monotone decreasing property of mappings $\psi_h : [0,1] \times [0,1] \to [0,1]$ and $\psi_p : [0,1] \times [0,1] \to [0,1]$ lead to the satisfaction of property (S0).

Proof of (S1): From the formulas (13)–(18) and (19)–(24), we easily find that ψ_h and ψ_p are symmetric.

Proof of (S2): According to the formulas (13)–(18) and (19)–(24), it can be obviously seen that $\psi_h(h_1^{(j)}, h_2^{(j)}) = 1$ and $\psi_p(p_1^{(j)}, p_2^{(j)}) = 1$ if and only if $h_1^{(j)} = h_2^{(j)}$ and $p_1^{(j)} = p_2^{(j)}$ which give rise to $\langle h_1^{(j)}, p_1^{(j)}\rangle = \langle h_2^{(j)}, p_2^{(j)}\rangle$.

Proof of (S3): In the case where ${}^{p}h_1 \subseteq {}^{p}h_2 \subseteq {}^{p}h_2$, the relation (6) implies that $\langle h_1^{(j)}, p_1^{(j)}\rangle \subseteq \langle h_2^{(j)}, p_2^{(j)}\rangle \subseteq \langle h_3^{(j)}, p_3^{(j)}\rangle$ which gives that $h_1^{(j)} \le h_2^{(j)} \le h_3^{(j)}$ together with $p_1^{(j)} \le p_2^{(j)} \le p_3^{(j)}$. Therefore, it is deduced that

$$ |h_1^{(j)} - h_3^{(j)}| \ge |h_1^{(j)} - h_2^{(j)}| \text{ and } |h_1^{(j)} - h_3^{(j)}| \ge |h_2^{(j)} - h_3^{(j)}|; \tag{25} $$
$$ |p_1^{(j)} - p_3^{(j)}| \ge |p_1^{(j)} - p_2^{(j)}| \text{ and } |p_1^{(j)} - p_3^{(j)}| \ge |p_2^{(j)} - p_3^{(j)}|. \tag{26} $$

If we feed the Eqs. (25) and (26) into the strictly monotone decreasing mappings of ψ_h and ψ_p, we would obtain

$$ \psi_h(h_1^{(j)}, h_3^{(j)}) \le \psi_h(h_1^{(j)}, h_2^{(j)}) \text{ and } \psi_h(h_1^{(j)}, h_3^{(j)}) \le \psi_h(h_2^{(j)}, h_3^{(j)}); $$
$$ \psi_p(p_1^{(j)}, p_3^{(j)}) \le \psi_p(p_1^{(j)}, p_2^{(j)}) \text{ and } \psi_p(p_1^{(j)}, p_3^{(j)}) \le \psi_p(p_2^{(j)}, p_3^{(j)}). \quad \square $$

Beside the above-mentioned results, the concept of *t-norm* is also applied for constructing the proposed PHFS information measures. Let us now review the well-known definition of t-norm $\Gamma : [0,1] \times [0,1] \to [0,1]$ (see e.g., [58]) which fulfils

- (Γ1) $\Gamma(x, 1) = x$ (boundary condition);
- (Γ2) If $y \le z$, then $\Gamma(x, y) \le \Gamma(x, z)$ (monotonicity property);
- (Γ3) $\Gamma(x, y) = \Gamma(y, x)$ (commutativity property);
- (Γ4) $\Gamma(x, \Gamma(y, z)) = \Gamma(\Gamma(x, y), z)$ (associativity property).

Below, we present the four most frequently used t-norms, namely, Algebraic, Einstein, Hamacher and Frank:

Algebraic t-norm:

$$ \Gamma_1(x, y) = xy; \tag{27} $$

Einstein t-norm:

$$ \Gamma_2(x, y) = \frac{xy}{1 + (1-x)(1-y)}; \tag{28} $$

Hamacher t-norm:

$$ \Gamma_3^{\epsilon}(x, y) = \frac{xy}{\epsilon + (1-\epsilon)(x + y - xy)}, \quad \epsilon > 0; \tag{29} $$

Frank t-norm:

$$\Gamma_4^{\epsilon}(x,y) = log_{\epsilon}(1 + \frac{(\epsilon^x - 1)(\epsilon^y - 1)}{\epsilon - 1}), \quad \epsilon > 1. \tag{30}$$

Before we dive deeper, let us weigh the pros and cons of these t-norms. The Algebraic t-norm is simple to calculate and useful for many applications, but it might not capture subtle differences in fuzzy sets or complex relationships between variables. The Einstein t-norm strikes a balance between intersecting and complementing fuzzy sets, suitable for scenarios requiring equilibrium, yet it is more challenging to calculate and may not always give straightforward results. The Hamacher t-norm allows for flexibility with its parameter, but this adds complexity and requires careful adjustment for meaningful outcomes. As for the Frank t-norm, it offers flexibility with parameter and can handle non-linear relationships, yet its lack of certain mathematical properties may limit its use in some cases, and its formula can be complex to interpret and calculate.

In the following section, we describe the innovative class of information measures between PHFEs including similarity (equivalently, distance) and entropy measures.

3.1. PHFS similarity measure

Coping with all the mentioned requirements in Section 2, a novel class of PHFS similarity measures is described below. Let ${}^ph_1 = \bigcup_{\langle h_1,p_1\rangle \in {}^ph_1}\{\langle h_1,p_1\rangle\} = \{\langle h_1^{(1)},p_1^{(1)}\rangle,\ldots,\langle h_1^{(l_1)},p_1^{(l_1)}\rangle\}$, ${}^ph_2 = \bigcup_{\langle h_2,p_2\rangle \in {}^ph_2}\{\langle h_2,p_2\rangle\} = \{\langle h_2^{(1)},p_2^{(1)}\rangle,\ldots,\langle h_2^{(l_2)},p_2^{(l_2)}\rangle\}$ and ${}^ph_3 = \bigcup_{\langle h_3,p_3\rangle \in {}^ph_3}\{\langle h_3,p_3\rangle\} = \{\langle h_3^{(1)},p_3^{(1)}\rangle,\ldots,\langle h_3^{(l_3)},p_3^{(l_3)}\rangle\}$ with $l = \max\{l_1,l_2,l_3\}$. In the case where $\langle h_1^{(j)},p_1^{(j)}\rangle$, $\langle h_2^{(j)},p_2^{(j)}\rangle$ and $\langle h_3^{(j)},p_3^{(j)}\rangle$ denote the jth element of PHFEs ${}^ph_1$, ${}^ph_2$ and ${}^ph_3$, we may deduce the following theorem.

Theorem 3.2. *Let ψ_h and ψ_p be given respectively by formulas (13)–(18) and (19)–(24), and Γ is that presented by (27)–(30). We define*

$$S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) = \Gamma(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \tag{31}$$

which satisfies

- $(S_{\Gamma}^{\psi}0)$ $0 \le S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) \le 1$;
- $(S_{\Gamma}^{\psi}1)$ $S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) = S_{\Gamma}^{\psi}(\langle h_2^{(j)},p_2^{(j)}\rangle,\langle h_1^{(j)},p_1^{(j)}\rangle)$;
- $(S_{\Gamma}^{\psi}2)$ $S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) = 1$ *if and only if* $h_1^{(j)} = h_2^{(j)}$ *together with* $p_1^{(j)} = p_2^{(j)}$;
- $(S_{\Gamma}^{\psi}3)$ *If* $h_1^{(j)} \le h_2^{(j)} \le h_3^{(j)}$ *and* $p_1^{(j)} \le p_2^{(j)} \le p_3^{(j)}$, *then*

$$S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_3^{(j)},p_3^{(j)}\rangle) \le S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle);$$
$$S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_3^{(j)},p_3^{(j)}\rangle) \le S_{\Gamma}^{\psi}(\langle h_2^{(j)},p_2^{(j)}\rangle,\langle h_3^{(j)},p_3^{(j)}\rangle).$$

Proof. Proof of $(S_{\Gamma}^{\psi}0)$: From the boundedness property of t-norm $\Gamma : [0,1]\times[0,1] \to [0,1]$ and the definition of (31), we find that $0 \le S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) \le 1$.

Proof of $(S_{\Gamma}^{\psi}1)$: The proof is a straightforward application of axiom $(S1)$ in Theorem 3.1.

Proof of $(S_{\Gamma}^{\psi}2)$: Assume that $S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) = 1$ holds true. Then, by employing Eq. (31), we find that

$$S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) = \Gamma(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) = 1.$$

Now, by keeping the boundary condition $(\Gamma 1)$ in mind, the latter relation holds true if and only if

$$\psi_h(h_1^{(j)},h_2^{(j)}) = 1, \quad and \quad \psi_p(p_1^{(j)},p_2^{(j)}) = 1$$

if and only if

$$|h_1^{(j)} - h_2^{(j)}| = 0, \quad and \quad |p_1^{(j)} - p_2^{(j)}| = 0,$$

which imply that $\langle h_1^{(j)},p_1^{(j)}\rangle = \langle h_2^{(j)},p_2^{(j)}\rangle$.

Proof of $(S_{\Gamma}^{\psi}3)$: If we consider $h_1^{(j)} \le h_2^{(j)} \le h_3^{(j)}$ and $p_1^{(j)} \le p_2^{(j)} \le p_3^{(j)}$, then

$$|h_1^{(j)} - h_3^{(j)}| \ge |h_1^{(j)} - h_2^{(j)}| \ and \ |h_1^{(j)} - h_3^{(j)}| \ge |h_2^{(j)} - h_3^{(j)}|;$$
$$|p_1^{(j)} - p_3^{(j)}| \ge |p_1^{(j)} - p_2^{(j)}| \ and \ |p_1^{(j)} - p_3^{(j)}| \ge |p_2^{(j)} - p_3^{(j)}|$$

and hence,

$$\psi_h(h_1^{(j)},h_3^{(j)}) \le \psi_h(h_1^{(j)},h_2^{(j)}) \ and \ \psi_h(h_1^{(j)},h_3^{(j)}) \le \psi_h(h_2^{(j)},h_3^{(j)}); \tag{32}$$
$$\psi_p(p_1^{(j)},p_3^{(j)}) \le \psi_p(p_1^{(j)},p_2^{(j)}) \ and \ \psi_p(p_1^{(j)},p_3^{(j)}) \le \psi_p(p_2^{(j)},p_3^{(j)}) \tag{33}$$

which are drawn from property of $(S3)$ in Theorem 3.1. Consequently, we find that

$$\begin{aligned} S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_3^{(j)},p_3^{(j)}\rangle) &= \Gamma(\psi_h(h_1^{(j)},h_3^{(j)}),\psi_p(p_1^{(j)},p_3^{(j)})) \\ &\le \Gamma(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \quad (by\ (\Gamma 2)\ and\ (32)–(33)) \\ &= S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle); \end{aligned}$$

and moreover,

$$\begin{aligned} S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_3^{(j)},p_3^{(j)}\rangle) &= \Gamma(\psi_h(h_1^{(j)},h_3^{(j)}),\psi_p(p_1^{(j)},p_3^{(j)})) \\ &\le \Gamma(\psi_h(h_2^{(j)},h_3^{(j)}),\psi_p(p_2^{(j)},p_3^{(j)})) \quad (by\ (\Gamma 2)\ and\ (32)–(33)) \\ &= S_{\Gamma}^{\psi}(\langle h_2^{(j)},p_2^{(j)}\rangle,\langle h_3^{(j)},p_3^{(j)}\rangle). \quad \square \end{aligned}$$

We now indicate how the above similarity measures can be extended to those in form of t-norm-based formulas. Given the above-mentioned t-norms, we define

- Algebraic norm-based similarity measure:

$$\begin{aligned} S_{\Gamma_1}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) &= \Gamma_1(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \\ &= \psi_h(h_1^{(j)},h_2^{(j)}) \times \psi_p(p_1^{(j)},p_2^{(j)}); \end{aligned} \tag{34}$$

- Einstein norm-based similarity measure:

$$\begin{aligned} S_{\Gamma_2}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) &= \Gamma_2(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \\ &= \frac{\psi_h(h_1^{(j)},h_2^{(j)}) \times \psi_p(p_1^{(j)},p_2^{(j)})}{1 + (1 - \psi_h(h_1^{(j)},h_2^{(j)})) \times (1 - \psi_p(p_1^{(j)},p_2^{(j)}))}; \end{aligned} \tag{35}$$

- Hamacher norm-based similarity measure:

$$\begin{aligned} & S_{\Gamma_3^{\epsilon}}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) \\ & = \Gamma_3^{\epsilon}(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \\ & = \frac{\psi_h(h_1^{(j)},h_2^{(j)}) \times \psi_p(p_1^{(j)},p_2^{(j)})}{\epsilon + (1-\epsilon)(\psi_h(h_1^{(j)},h_2^{(j)}) + \psi_p(p_1^{(j)},p_2^{(j)}) - \psi_h(h_1^{(j)},h_2^{(j)}) \times \psi_p(p_1^{(j)},p_2^{(j)}))}; \end{aligned} \tag{36}$$

- Frank norm-based similarity measure:

$$\begin{aligned} S_{\Gamma_4^{\epsilon}}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) &= \Gamma_4^{\epsilon}(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \\ &= log_{\epsilon}(1 + \frac{(\epsilon^{\psi_h(h_1^{(j)},h_2^{(j)})} - 1)(\epsilon^{\psi_p(p_1^{(j)},p_2^{(j)})} - 1)}{\epsilon - 1}). \end{aligned} \tag{37}$$

The above formulas can be made more specific if we replace ψ_h and ψ_p by those given in (13)–(18) and (19)–(24). For instance, by taking Algebraic norm-based similarity measure, we are able to construct the following similarity measure:

$$\begin{aligned} S_{\Gamma_1}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) &= \Gamma_1(\psi_h(h_1^{(j)},h_2^{(j)}),\psi_p(p_1^{(j)},p_2^{(j)})) \\ &= \psi_h(h_1^{(j)},h_2^{(j)}) \times \psi_p(p_1^{(j)},p_2^{(j)}); \\ & \quad (for\ \psi_h := \psi_{1h}\ and\ \psi_p := \psi_{1p}) \\ &= (1 - |h_1^{(j)} - h_2^{(j)}|) \times (1 - |p_1^{(j)} - p_2^{(j)}|). \end{aligned}$$

We are now in a position to extend the proposed similarity measures for the elements of PHFEs to those for PHFEs, and then for PHFSs as the following:

$$S_{\Gamma}^{\psi}({}^ph_1,{}^ph_2) = \frac{1}{l}\sum_{j=1}^{l} S_{\Gamma}^{\psi}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) \tag{38}$$

$$S_{\Gamma}^{\psi}({}^pH_1,{}^pH_2) = \frac{1}{N}\sum_{i=1}^{N} S_{\Gamma}^{\psi}({}^ph_1(x_i),{}^ph_2(x_i)), \tag{39}$$

in which ${}^{p}h_1 = \bigcup_{\langle h_1,p_1\rangle\in {}^{p}h_1}\{\langle h_1,p_1\rangle\} = \{\langle h_1^{(1)},p_1^{(1)}\rangle,\ldots,\langle h_1^{(l_1)},p_1^{(l_1)}\rangle\}$ and ${}^{p}h_2 = \bigcup_{\langle h_2,p_2\rangle\in {}^{p}h_2}\{\langle h_2,p_2\rangle\} = \{\langle h_2^{(1)},p_2^{(1)}\rangle,\ldots,\langle h_2^{(l_2)},p_2^{(l_2)}\rangle\}$ are PHFEs with $l=\max\{l_1,l_2\}$.

In conclusion, we have outlined a procedure that allows us to extend the result of Theorem 3.2 to that for PHFSs.

3.2. PHFS entropy measure

Entropy is an important measure of information theory that is used to specify the degree of uncertainty of any data. Specifically, to specify the criterion weights involved in a multiple criteria decision making problem. Indeed, the uncertainty of PHFSs is due to the fact that the information of criterion weights is partially or completely unknown. In such a case, the entropy measure becomes a common acceptable technique.

In the case where $\langle h_1^{(j)},p_1^{(j)}\rangle$, $\langle h_2^{(j)},p_2^{(j)}\rangle$ and $\langle h_3^{(j)},p_3^{(j)}\rangle$ denote the jth element of PHFEs ${}^{p}h_1$, ${}^{p}h_2$ and ${}^{p}h_3$, we may deduce the following theorem.

Theorem 3.3. *Let ψ_h and ψ_p be given respectively by formulas (13)–(18) and (19)–(24), and $\Delta : [0,1]\to[0,1]$ is a strictly increasing continuous function with $\Delta(0)=0$ and $\Delta(1)=1$. We define*

$$E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle) = \Delta(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))$$
$$= \Delta(\Gamma(\psi_h(h_1^{(j)},1-h_1^{(j)}),\psi_p(p_1^{(j)},p_1^{(j)}))) \tag{40}$$

which satisfies

($E^{\psi}_{\Delta,\Gamma}0$) $0\le E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)\le 1$;
($E^{\psi}_{\Delta,\Gamma}1$) $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=0$ if and only if $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle 0,1\rangle$ or $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle 1,1\rangle$;
($E^{\psi}_{\Delta,\Gamma}2$) $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=1$ if $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle \frac{1}{2},1\rangle$;
($E^{\psi}_{\Delta,\Gamma}3$) $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)\le E^{\psi}_{\Delta,\Gamma}(\langle h_2^{(j)},p_2^{(j)}\rangle)$ if $S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)\le S^{\psi}_{\Gamma}(\langle h_2^{(j)},p_2^{(j)}\rangle,\langle h_2^{(j)c},p_2^{(j)c}\rangle)$.

Proof. Proof of ($E^{\psi}_{\Delta,\Gamma}0$): From definition of t-norm $\Gamma : [0,1]\times[0,1]\to[0,1]$ and strictly increasing continuous function of $\Delta : [0,1]\to[0,1]$, we find that $0\le E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)\le 1$.

Proof of ($E^{\psi}_{\Delta,\Gamma}1$): The relation $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=0$ implies that $\Delta(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))=0$, and under the increasing property of Δ, we find that $S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)=0$. This holds true if and only if $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle 0,1\rangle$, and thus $\langle h_1^{(j)c},p_1^{(j)c}\rangle=\langle 1,1\rangle$, or $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle 1,1\rangle$, and thus $\langle h_1^{(j)c},p_1^{(j)c}\rangle=\langle 0,1\rangle$.

Proof of ($E^{\psi}_{\Delta,\Gamma}2$): The relation $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=1$ directly results in the relation $\Delta(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))=1$, and under the increasing property of Δ, we find that $S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)=1$. This holds true if $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle h_1^{(j)c},p_1^{(j)c}\rangle$. That is, $h_1^{(j)}=h_1^{(j)c}$ and $p_1^{(j)}=p_1^{(j)c}$ which imply that $h_1^{(j)}=1-h_1^{(j)}$ and $p_1^{(j)}=p_1^{(j)}$, or equivalently, $h_1^{(j)}=\frac{1}{2}$ and $p_1^{(j)}=1$. Hence, $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=1$ if and only if $\langle h_1^{(j)},p_1^{(j)}\rangle=\langle\frac{1}{2},1\rangle$.

Proof of ($E^{\psi}_{\Delta,\Gamma}3$): If it is satisfied that $S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)\le S^{\psi}_{\Gamma}(\langle h_2^{(j)},p_2^{(j)}\rangle,\langle h_2^{(j)c},p_2^{(j)c}\rangle)$, then by the help of increasing property of Δ, we easily show that

$$\Delta(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))\le\Delta(S^{\psi}_{\Gamma}(\langle h_2^{(j)},p_2^{(j)}\rangle,\langle h_2^{(j)c},p_2^{(j)c}\rangle))$$

which implies that $E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)\le E^{\psi}_{\Delta,\Gamma}(\langle h_2^{(j)},p_2^{(j)}\rangle)$. □

It is interesting to note that if $\Delta_1(x)=x$, $\Delta_2(x)=\sin(\frac{\pi}{2}x)$ and $\Delta_3(x)=xe^{x-1}$, then, different formulas of entropy measures are achieved by the use of (40) as follows:

$$E^{\psi}_{\Delta_1,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=\Delta_1(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))$$
$$=S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle); \tag{41}$$
$$E^{\psi}_{\Delta_2,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=\Delta_2(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))$$
$$=\sin(\frac{\pi}{2}S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)); \tag{42}$$
$$E^{\psi}_{\Delta_3,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle)=\Delta_3(S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle))$$
$$=S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)\ e^{S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_1^{(j)c},p_1^{(j)c}\rangle)-1}. \tag{43}$$

The proposed entropy measures for the elements of PHFEs can be extended to those for PHFEs, and then to those for PHFSs as the following:

$$E^{\psi}_{\Delta,\Gamma}({}^{p}h_1,{}^{p}h_2)=\frac{1}{l}\sum_{j=1}^{l}E^{\psi}_{\Delta,\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle h_2^{(j)},p_2^{(j)}\rangle) \tag{44}$$

$$E^{\psi}_{\Delta,\Gamma}({}^{p}H_1,{}^{p}H_2)=\frac{1}{N}\sum_{i=1}^{N}E^{\psi}_{\Delta,\Gamma}({}^{p}h_1(x_i),{}^{p}h_2(x_i)), \tag{45}$$

in which ${}^{p}h_1 = \bigcup_{\langle h_1,p_1\rangle\in {}^{p}h_1}\{\langle h_1,p_1\rangle\} = \{\langle h_1^{(1)},p_1^{(1)}\rangle,\ldots,\langle h_1^{(l_1)},p_1^{(l_1)}\rangle\}$ and ${}^{p}h_2 = \bigcup_{\langle h_2,p_2\rangle\in {}^{p}h_2}\{\langle h_2,p_2\rangle\} = \{\langle h_2^{(1)},p_2^{(1)}\rangle,\ldots,\langle h_2^{(l_2)},p_2^{(l_2)}\rangle\}$ are PHFEs with $l=\max\{l_1,l_2\}$.

Consequently, we have generalised above a procedure that allows us to extend the result of Theorem 3.3 to that for PHFSs.

4. PHFS-based multiple criteria decision making

In this section, we describe two multiple criteria decision making algorithms in which the last step is devoted to PHFS ranking procedure: one is based on the score function of alternatives and the other is based on the relative closeness of alternatives.

In the next two subsections, we denote A_i $(i=1,2,\ldots,N_I)$ as the alternatives which are assessed by the criteria c_j $(j=1,2,\ldots,N_J)$ whose weights are indicated by $w=(w_1,\ldots,w_{N_J})$ satisfying $\sum_{j=1}^{N_J}w_j=1$ and $w_j\ge 0$.

4.1. First algorithm for PHFS-based multiple criteria decision making

Assume that a group of decision makers are invited to evaluate the characteristics of alternatives A_i $(i=1,2,\ldots,N_I)$ with respect to criteria c_j $(j=1,2,\ldots,N_J)$ by the use of PHFEs:

$${}^{p}h_{ij}=\bigcup_{\langle h_{ij},p_{ij}\rangle\in {}^{p}h_{ij}}\{\langle h_{ij},p_{ij}\rangle\}=\{\langle h_{ij}^{(1)},p_{ij}^{(1)}\rangle,\ldots,\langle h_{ij}^{(l_{ij})},p_{ij}^{(l_{ij})}\rangle\}$$

where the associated probabilities satisfy $\sum_{k=1}^{l}p_{ij}^{(k)}=1$ and $l=\max_{i,j}\{l_{ij}\}$ for any $1\le i\le N_I$ and $1\le j\le N_J$.

Keeping the weighting vector $w=(w_1,\ldots,w_{N_J})$ in mind, the following aggregation operator, known as the probabilistic hesitant fuzzy weighted averaging (PHFWA) operator [56], is used to aggregate all the alternatives A_i $(i=1,2,\ldots,N_I)$:

$${}^{p}h_i=PHFWA({}^{p}h_{i1},\ldots,{}^{p}h_{j2},\ldots,{}^{p}h_{iN_J})$$
$$=w_1({}^{p}h_{i1})\oplus\cdots\oplus w_j({}^{p}h_{ij})\oplus\cdots\oplus w_{N_J}({}^{p}h_{iN_J}). \tag{46}$$

Needless to say that in the case where the weighting vector is not known, we are required to utilise an aggregation technique such as the *ordered weighted averaging* operator [59].

Before going more into detail on the algorithm steps, this section provides a class of ranking functions for PHFEs based on the similarity between a PHFE and the full PHFE $\mathbf{1}_{ph}=\{\langle 1,1\rangle\}$ as

$$Sco^{\psi}_{\Gamma}({}^{p}h_1)=Sco^{\psi}_{\Gamma}(\bigcup_{\langle h_1,p_1\rangle\in {}^{p}h_1}\{\langle h_1,p_1\rangle\}=\{\langle h_1^{(1)},p_1^{(1)}\rangle,\ldots,\langle h_1^{(l_1)},p_1^{(l_1)}\rangle\})$$
$$=S^{\psi}_{\Gamma}({}^{p}h_1,\mathbf{1}_{ph})=\frac{1}{l_1}\sum_{j=1}^{l_1}S^{\psi}_{\Gamma}(\langle h_1^{(j)},p_1^{(j)}\rangle,\langle 1,1\rangle). \tag{47}$$

The above formula will be more specific, if we replace ψ_h and ψ_p with those given by (13)–(18) and (19)–(24), and moreover, Γ with those represented by (34)–(37). For instance, by taking ψ_{1h}, ψ_{1p} and

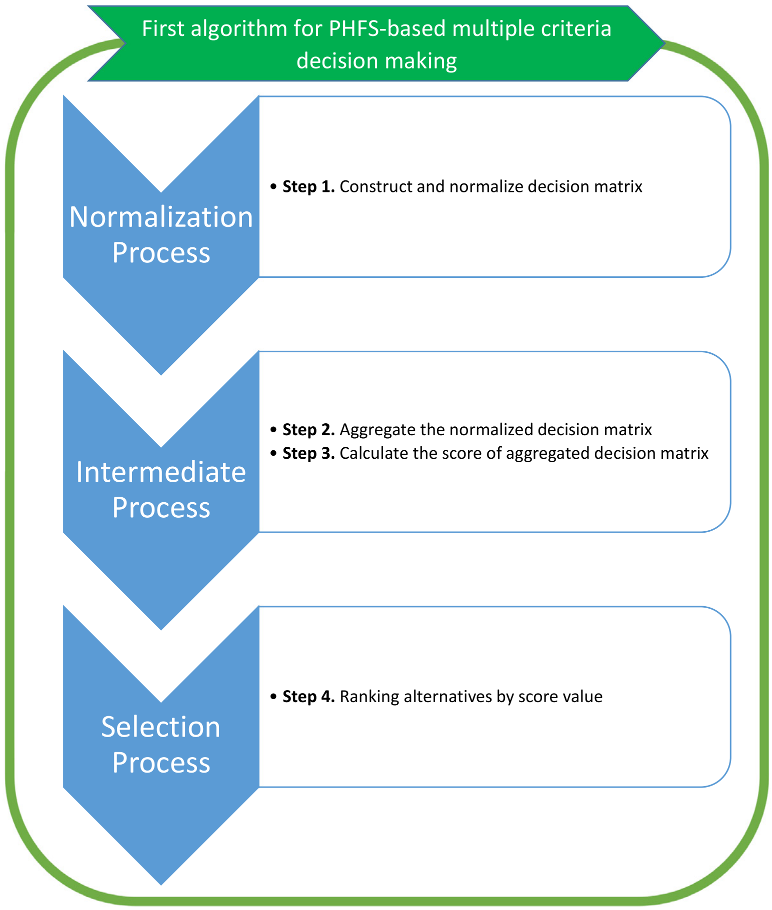

Fig. 1. First algorithm for PHFS-based multiple criteria decision making.

Γ_1 as the Algebraic t-norm, we are able to construct the following similarity-based ranking function for the PHFE ${}^p h_1$:

$$\begin{aligned} Sco_{\Gamma_1}^{\psi_1}({}^p h_1) = \frac{1}{l_1}\sum_{j=1}^{l_1} S_{\Gamma_1}^{\psi_1}(\langle h_1^{(j)}, p_1^{(j)}\rangle, \mathbf{1}_{{}^p h}) &= \frac{1}{l_1}\sum_{j=1}^{l_1} \Gamma_1(\psi_{1h}(h_1^{(j)},1), \psi_{1p}(p_1^{(j)},1)) \\ &= \frac{1}{l_1}\sum_{j=1}^{l_1} \psi_{1h}(h_1^{(j)},1)\times\psi_{1p}(p_1^{(j)},1); \\ &= \frac{1}{l_1}\sum_{j=1}^{l_1}(1-|h_1^{(j)}-1|)\times(1-|p_1^{(j)}-1|) \\ &= \frac{1}{l_1}\sum_{j=1}^{l_1} h_1^{(j)}\times p_1^{(j)}. \end{aligned}$$

In summary, the PHFS-based multiple criteria decision making algorithm is described below:

Algorithm 4.1 (*First algorithm for PHFS-based multiple criteria decision making*).

Step 1. We collect the decision makers' preference of each alternative A_i ($i = 1,2,\dots,N_I$) with respect to each criterion c_j ($j = 1,2,\dots,N_J$) being stated by PHFEs ${}^p h_{ij}$.

Step 2. Using the PHFWA operator given by (46), we aggregate all PHFEs ${}^p h_{ij}$ for any $1 \le j \le N_J$.

Step 3. We apply the comparison technique (47) to compute the score function of ${}^p h_i$ for any $1 \le i \le N_I$.

Step 4. We rank all alternatives A_i ($i = 1,2,\dots,N_I$) in terms of score value of ${}^p h_i$, and then select the best alternative with the highest value.

In the following case study adopted from [60], we apply the methods described above to determine the best Chinese hospital in a context of limited medical resources and an ageing population. Four Chinese hospitals, namely, West China Hospital of Sichuan University (A_1), Huashan Hospital of Fudan University (A_2), Union Medical College Hospital (A_3) and Chinese PLA General Hospital (A_4) are evaluated based on three criteria c_1: the environment of health service; c_2: the treatment optimisation; and c_3: the social resource allocation and health services. These criteria have the weighting values $w_1 = 0.2$, $w_2 = 0.1$ and $w_3 = 0.7$. As is well known, the preference of a group of experts tends to be more accurate than that of an individual expert. Hence, various experts are invited to express their preference for such hospitals based on the above criteria. In order to prevent the loss of information, we denote all the preferences of experts using the following PHFE decision matrix in Table 2 through *Step 1* of Algorithm 4.1.

Before we proceed with the methodology, let us delve into how the decision matrix was developed in more detail. It is important to note that the arrays in Table 1 depict expert preferences. For instance, in the first row and first column, 40% of experts prefer 0.5, while 60% prefer 0.7. Additionally, another array, located in the first row and second column, shows that all experts (100%) prefer 0.9 (see Figs. 1 and 2).

The aggregation operator PHFWA given by (46) allows us to employ the decision matrix of Table 2 to evaluate the information of hospitals A_i ($i = 1,2,3, N_I = 4$) in *Step 2*. This gives rise to

$$\begin{aligned} A_1 &: {}^p h_1 = \{\langle 0.461, 0.08\rangle, \langle 0.514, 0.12\rangle, \langle 0.574, 0.32\rangle, \langle 0.616, 0.48\rangle\}, \\ A_2 &: {}^p h_2 = \{\langle 0.781, 0.12\rangle, \langle 0.809, 0.28\rangle, \langle 0.865, 0.18\rangle, \langle 0.882, 0.42\rangle\}, \\ A_3 &: {}^p h_3 = \{\langle 0.75, 0.3\rangle, \langle 0.776, 0.3\rangle, \langle 0.845, 0.2\rangle, \langle 0.862, 0.2\rangle\}, \\ A_4 &: {}^p h_4 = \{\langle 0.698, 0.25\rangle, \langle 0.715, 0.25\rangle, \langle 0.738, 0.25\rangle, \langle 0.752, 0.25\rangle\}. \end{aligned}$$

As can be seen from the above data, all the aggregated PHFSs ${}^p h_1$, …, ${}^p h_4$ have the same length.

Table 1
Explanation of abbreviation.

Abbreviation	Explanation
HFS	Hesitant fuzzy set
HFE	Hesitant fuzzy element
PHFS	Probabilistic hesitant fuzzy set
PHFE	Probabilistic hesitant fuzzy element
PHFWA	Probabilistic hesitant fuzzy weighted averaging
TOPSIS	Technique for Order of Preference by Similarity to Ideal Solution

Table 2
Probabilistic hesitant fuzzy decision matrix.

	c_1	c_2	c_3
A_1	$\{\langle 0.5,0.4\rangle,\langle 0.7,0.6\rangle\}$	$\{\langle 0.9,1\rangle\}$	$\{\langle 0.3,0.2\rangle,\langle 0.5,0.8\rangle\}$
A_2	$\{\langle 0.8,0.3\rangle,\langle 0.9,0.7\rangle\}$	$\{\langle 0.5,1\rangle\}$	$\{\langle 0.8,0.4\rangle,\langle 0.9,0.6\rangle\}$
A_3	$\{\langle 0.5,1\rangle\}$	$\{\langle 0.7,0.5\rangle,\langle 0.9,0.5\rangle\}$	$\{\langle 0.8,0.6\rangle,\langle 0.9,0.4\rangle\}$
A_4	$\{\langle 0.8,0.5\rangle,\langle 0.9,0.5\rangle\}$	$\{\langle 0.3,0.5\rangle,\langle 0.6,0.5\rangle\}$	$\{\langle 0.7,1\rangle\}$

If we perform *Step 3* and *Step 4*, the ranking orders of the Chinese hospitals with respect to different similarity-based ranking functions will be the same as those given in Table 4. As can be observed from Table 4, the most frequently occurring alternative is A_2, implying that the most appropriate hospital is the Huashan Hospital of Fudan University A_2.

Now, in order to make a comparison with the other existing techniques, we adopt Farhadinia and Xu's [60] contribution in which a two step-based processing for ranking Chinese hospitals was proposed, and its results are compared with that of Zhang et al.'s [56], Song et al.'s [61] and Fang's [51] technique (see all the corresponding results in Table 3).

However, the aforementioned techniques in Table 3 do not always provide a reliable ranking result because of the following limitations:

- The possibilistic-based technique of Song et al. [61], which defines the mutual relationship between alternatives based on the intersecting values, needs to do time consuming work to properly address the calculation of the possibility degrees for each pair of alternatives in accordance with criteria.
- The probabilistic-based technique of Zhang et al. [56] is a two step-based process, and needs to do more tasks such as computing the score and deviation degrees of each PHFE.
- The probabilistic-based technique of Farhadinia and Xu [60] is required to do more tasks to obtain the multiplying and exponential deformation formulas of each pair of possible membership degrees and its associated probability.

4.2. Second algorithm for PHFS-based multiple criteria decision making

On the basis of Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method, Wang et al. [62] depicted an extension methodology under PHFS environment.

Once again, as in the preceding subsection, we assume here that A_i $(i = 1, 2, \dots, N_I)$ denotes the alternatives whose characteristics are evaluated with respect to criteria c_j $(j = 1, 2, \dots, N_J)$ by the use of PHFEs:

$$^{p}h_{ij} = \bigcup_{\langle h_{ij},p_{ij}\rangle \in {}^{p}h_{ij}} \{\langle h_{ij}, p_{ij}\rangle\} = \{\langle h_{ij}^{(1)}, p_{ij}^{(1)}\rangle, \dots, \langle h_{ij}^{(l_{ij})}, p_{ij}^{(l_{ij})}\rangle\}$$

where the associated probabilities satisfy $\sum_{k=1}^{l} p_{ij}^{(k)} = 1$ and $l = \max_{i,j}\{l_{ij}\}$ for any $1 \le i \le N_I$ and $1 \le j \le N_J$.

Now, the following decision making algorithm is offered to rank the alternatives in correspondence with the larger value of their relative closeness.

Algorithm 4.2 (*Second Algorithm For PHFS-based Multiple Criteria Decision Making*)**.**

Step 1. We construct the arrays $^{p}h_{ij}$'s of decision matrix by the help of assessing the alternatives A_i $(i = 1, 2, \dots, N_I)$ on the criteria c_j $(j = 1, 2, \dots, N_J)$.

All the column-PHFEs are assumed to be length-scale unified, and thus, $l_{ij} = \max\{l_{1j}, \dots, l_{N_I j}\}$.

Step 2. We specify the unknown criteria weight using the entropy measure (40) as follows:

$$w_j = \frac{\sum_{i=1}^{N_I}(1 - \frac{1}{l_{ij}}\sum_{k=1}^{l_{ij}} E_{\Delta,\Gamma}^{\psi}(\langle h_{ij}^{(k)}, p_{ij}^{(k)}\rangle))}{N_I N_J - \sum_{i=1}^{N_I}\sum_{j=1}^{N_J}\frac{1}{l_{ij}}\sum_{k=1}^{l_{ij}} E_{\Delta,\Gamma}^{\psi}(\langle h_{ij}^{(k)}, p_{ij}^{(k)}\rangle)}, \quad j = 1, \dots, N_J.$$

Step 3. If we describe the Positive Ideal Solution (PIS) and Negative Ideal Solution (NIS), respectively, by

$$A^{+} = \{^{p^{+}}h_1^{+}, \dots, {}^{p^{+}}h_j^{+}, \dots, {}^{p^{+}}h_{N_J}^{+}\}$$
$$A^{-} = \{^{p^{-}}h_1^{-}, \dots, {}^{p^{-}}h_j^{-}, \dots, {}^{p^{-}}h_{N_J}^{-}\}$$

where

$$^{p^{+}}h_j^{+} = \{\langle \max_{1\le i\le N_I}\{h_{ij}^{(k)}\}, \max_{1\le i\le N_I}\{p_{ij}^{(k)}\}\rangle\}_{k=1}^{l_{ij}},$$
$$^{p^{-}}h_j^{-} = \{\langle \min_{1\le i\le N_I}\{h_{ij}^{(k)}\}, \min_{1\le i\le N_I}\{p_{ij}^{(k)}\}\rangle\}_{k=1}^{l_{ij}},$$

for any $j = 1, \dots, N_J$, then, the similarity measure between alternative A_i and PIS A^{+}, and that between A_i and NIS A^{-} will be computed by means of Eq. (38).

Step 4. We determine the priority of alternatives A_i corresponding to the larger value of relative closeness of A_i which is calculated by

$$RC_{\Delta,\Gamma}^{\psi}(A_i) = \frac{S_{\Gamma}^{\psi}(A_i, A^{+})}{S_{\Gamma}^{\psi}(A_i, A^{+}) + S_{\Gamma}^{\psi}(A_i, A^{-})}, \quad (i = 1, \dots, N_I).$$

Recently, a case study has been carried out based on the protection of Yangtze finless porpoise which has attracted widespread attention. Here, we investigate four areas in China, namely, A_1: Yueyang, Hunan; A_2: Duchang, Jiangxi; A_3: Haining, Zhejiang; and A_4: Tongling, Anhuiin to select the most suitable area for Yangtze finless porpoise survival. To do this end, Wang et al. [62] considered a number of criteria which influence Yangtze finless porpoise survival, including c_1: water pollution, c_2: overfishing, c_3: overexploitation of sand, c_4: overexploitation, and c_5: low reproduction rate. Considering the experts' weights to be known, we express the evaluation information in the form of a PHFE decision matrix given in Table 5 which implies the execution of *Step 1* of Algorithm 4.2.

In accordance with the above characterization and the Eqs. (41)–(43), the entropy measures $E_{\Delta_1,\Gamma_1}^{\psi_1}$, $E_{\Delta_2,\Gamma_1}^{\psi_1}$ and $E_{\Delta_3,\Gamma_1}^{\psi_1}$ transform the data in Table 5 to that of Table 6.

Based on the Eqs. (41)–(43), *Step 2* of Algorithm 4.2 specifies the weight vector of alternatives corresponding to each entropy measure as follows:

$$w_{E_{\Delta_1,\Gamma_1}^{\psi_1}} = (0.1786,\ 0.1339,\ 0.1786,\ 0.1518,\ 0.1429),$$
$$w_{E_{\Delta_2,\Gamma_1}^{\psi_1}} = (0.0894,\ 0.0603,\ 0.0894,\ 0.0752,\ 0.0597),$$
$$w_{E_{\Delta_3,\Gamma_1}^{\psi_1}} = (0.3347,\ 0.2626,\ 0.3347,\ 0.2858,\ 0.2879).$$

This is while, Wang et al.'s [62] result is

$$w_{Wang} = (0.1973,\ 0.2037,\ 0.1969,\ 0.1999,\ 0.2022).$$

Table 3
Comparison of the results based on different techniques in [60].

Technique	Ranking order	Best alternative
Zhang et al.'s [56]	$A_2 > A_3 > A_4 > A_1$	A_2
Song et al.'s [61]	$A_2 >^{0.838} A_3 >^{0.819} A_4 >^{1} A_1$	A_2
Farhadinia and Xu's [60]		
Multiplying deformation formula: $\overline{R}_3(A_1) = (0.1445, 0.1376)$, $\overline{R}_3(A_2) = (0.2116, 0.1171)$, $\overline{R}_3(A_3) = (0.1998, 0.0428)$, $\overline{R}_3(A_4) = (0.1814, 0.1824)$	$\overline{R}_3(A_2) \geq_{lex} \overline{R}_3(A_3) \geq_{lex} \overline{R}_3(A_4) \geq_{lex} \overline{R}_3(A_1)$ $A_2 > A_3 > A_4 > A_1$	A_2
Exponential deformation formula: $\underline{R}_3(A_1) = (0.8732, 0.1499)$, $\underline{R}_3(A_2) = (0.9590, 0.1114)$, $\underline{R}_3(A_3) = (0.9454, 0.0475)$, $\underline{R}_3(A_4) = (0.9229, 0.0430)$,	$\underline{R}_3(A_2) \geq_{lex} \underline{R}_3(A_3) \geq_{lex} \underline{R}_3(A_4) \geq_{lex} \underline{R}_3(A_1)$ $A_2 > A_3 > A_4 > A_1$	A_2
Fang's [51] ($\theta = 0.5$) $g_1(x, y) = \|x - y\|$	$A_2 > A_3 > A_4 > A_1$ 0.8464 0.7992 0.7257 0.5779	A_2
$g_2(x, y) = 1 - cos(\frac{\pi}{2}(x - y))$	$A_2 > A_3 > A_4 > A_1$ 0.2351 0.2247 0.2042 0.1435	A_2
$g_3(x, y) = 1 + \frac{1}{2}(1 + x - y)log\frac{1}{2}(1 + x - y) + \frac{1}{2}(1 - x + y)log\frac{1}{2}(1 - x + y)$	$A_2 > A_3 > A_4 > A_1$	A_2
	0.6809 0.6723 0.6549 0.6005	

Table 4
The ranking results on the basis of the proposed techniques.

Technique	Score of hospitals	Ranking order	Best Alternative
Γ_1			
$S^{\psi_1}_{\Gamma_1}$	0.1445 0.2116 0.1998 0.1814	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_2}_{\Gamma_1}$	0.3358 0.4137 0.4168 0.4044	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_3}_{\Gamma_1}$	0.3973 0.4937 0.4801 0.4486	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_4}_{\Gamma_1}$	0.0629 0.1091 0.0958 0.0814	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_5}_{\Gamma_1}$	0.3059 0.4044 0.3900 0.3591	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_6}_{\Gamma_1}$	0.2944 0.3751 0.3758 0.3606	$A_2 > A_4 > A_3 > A_1$	A_2
Γ_2			
$S^{\psi_1}_{\Gamma_2}$	0.1142 0.1912 0.1746 0.1506	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_2}_{\Gamma_2}$	0.3093 0.4079 0.4079 0.3879	$A_2 > A_4 > A_3 > A_1$	A_2
$S^{\psi_3}_{\Gamma_2}$	0.3517 0.4661 0.4497 0.4108	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_4}_{\Gamma_2}$	0.0437 0.0904 0.0751 0.0596	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_5}_{\Gamma_2}$	0.2588 0.3754 0.3576 0.3189	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_6}_{\Gamma_2}$	0.2630 0.3614 0.3579 0.3347	$A_2 > A_3 > A_4 > A_1$	A_2
$\Gamma_3,\ \epsilon = \frac{1}{2}$			
$S^{\psi_1}_{\Gamma_3}$	0.1677 0.2238 0.2155 0.2022	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_2}_{\Gamma_3}$	0.4132 0.4167 0.4213 0.4132	$A_2 > A_4 > A_3 > A_1$	A_2
$S^{\psi_3}_{\Gamma_3}$	0.4702 0.5088 0.4969 0.4702	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_4}_{\Gamma_3}$	0.0998 0.1222 0.1113 0.0998	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_5}_{\Gamma_3}$	0.3834 0.4209 0.4085 0.3834	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_6}_{\Gamma_3}$	0.3752 0.3824 0.3855 0.3752	$A_2 > A_4 > A_3 > A_1$	A_2
$\Gamma_4,\ \epsilon = e^1$			
$S^{\psi_1}_{\Gamma_4}$	0.1624 0.1997 0.1848 0.1624	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_2}_{\Gamma_4}$	0.2681 0.2789 0.2765 0.2681	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_3}_{\Gamma_4}$	0.4285 0.4795 0.4643 0.4285	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_4}_{\Gamma_4}$	0.0664 0.0973 0.0822 0.0664	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_5}_{\Gamma_4}$	0.3367 0.3888 0.3725 0.3367	$A_2 > A_3 > A_4 > A_1$	A_2
$S^{\psi_6}_{\Gamma_4}$	0.3466 0.3679 0.3664 0.3466	$A_2 > A_3 > A_4 > A_1$	A_2

Table 5
Decision information for areas A_i $(i = 1, 2, 3, 4)$.

	c_1	c_2	c_3	c_4	c_5
A_1	$\{\langle 0.8, 1.0\rangle\}$	$\{\langle 0.6, 0.7\rangle, \langle 0.7, 0.3\rangle\}$	$\{\langle 0.7, 0.2\rangle, \langle 0.8, 0.8\rangle\}$	$\{\langle 0.8, 0.7\rangle, \langle 0.9, 0.3\rangle\}$	$\{\langle 0.6, 0.8\rangle, \langle 0.8, 0.2\rangle\}$
A_2	$\{\langle 0.6, 0.8\rangle, \langle 0.7, 0.2\rangle\}$	$\{\langle 0.5, 0.3\rangle, \langle 0.6, 0.7\rangle\}$	$\{\langle 0.6, 0.1\rangle, \langle 0.7, 0.6\rangle, \langle 0.8, 0.3\rangle\}$	$\{\langle 0.5, 0.6\rangle, \langle 0.6, 0.4\rangle\}$	$\{\langle 0.6, 0.3\rangle, \langle 0.7, 0.6\rangle, \langle 0.8, 0.1\rangle\}$
A_3	$\{\langle 0.6, 0.1\rangle, \langle 0.7, 0.2\rangle, \langle 0.8, 0.7\rangle\}$	$\{\langle 0.7, 0.5\rangle, \langle 0.8, 0.5\rangle\}$	$\{\langle 0.8, 0.8\rangle, \langle 0.9, 0.2\rangle\}$	$\{\langle 0.6, 0.2\rangle, \langle 0.7, 0.2\rangle, \langle 0.8, 0.6\rangle\}$	$\{\langle 0.6, 0.1\rangle, \langle 0.7, 0.3\rangle, \langle 0.8, 0.6\rangle\}$
A_4	$\{\langle 0.8, 0.2\rangle, \langle 0.9, 0.8\rangle\}$	$\{\langle 0.7, 0.2\rangle, \langle 0.8, 0.7\rangle, \langle 0.9, 0.1\rangle\}$	$\{\langle 0.6, 0.9\rangle, \langle 0.8, 0.1\rangle\}$	$\{\langle 0.7, 0.9\rangle, \langle 0.8, 0.1\rangle\}$	$\{\langle 0.6, 0.1\rangle, \langle 0.7, 0.5\rangle, \langle 0.8, 0.5\rangle\}$

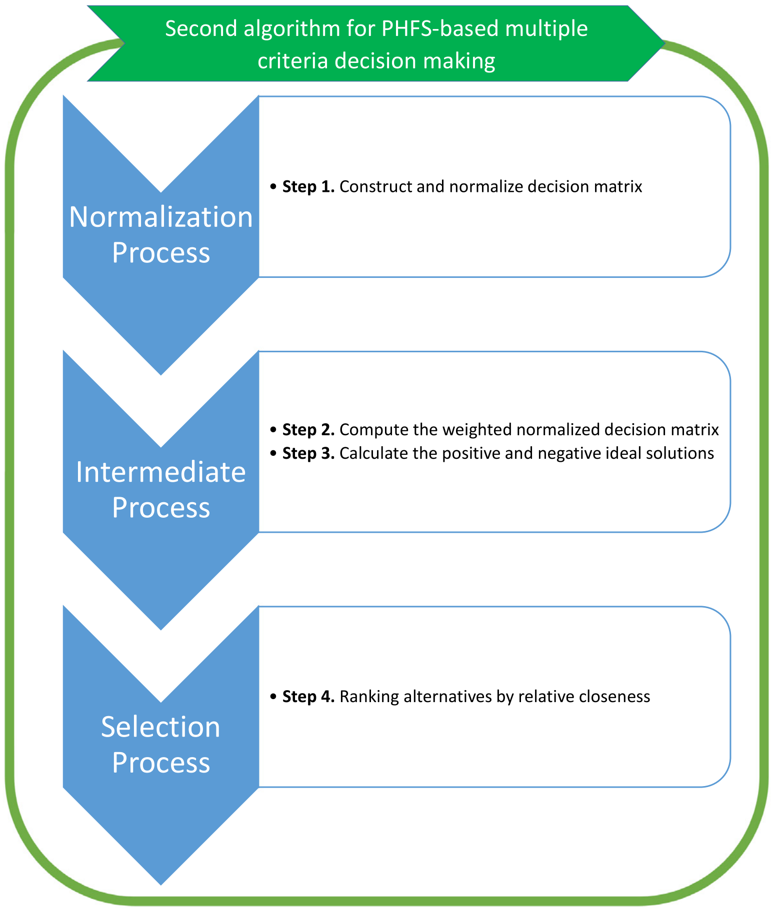

Fig. 2. Second algorithm for PHFS-based multiple criteria decision making.

Table 6
The result of entropy measures $E^{\psi_1}_{\Delta_1,\Gamma_1}$, $E^{\psi_1}_{\Delta_2,\Gamma_1}$ and $E^{\psi_1}_{\Delta_3,\Gamma_1}$.

	c_1	c_2	c_3	c_4	c_5
$E^{\psi_1}_{\Delta_1,\Gamma_1}$					
A_1	0.4000	0.7000	0.5000	0.3000	0.6000
A_2	0.7000	0.9000	0.6000	0.9000	0.7000
A_3	0.6000	0.5000	0.3000	0.6000	0.6000
A_4	0.3000	0.4000	0.6000	0.5000	0.5000
$E^{\psi_1}_{\Delta_2,\Gamma_1}$					
A_1	0.5878	0.8800	0.6984	0.4484	0.7694
A_2	0.8800	0.9755	0.7826	0.9755	0.8800
A_3	0.7826	0.6984	0.4484	0.7826	0.7826
A_4	0.4484	0.5686	0.7694	0.6984	0.6984
$E^{\psi_1}_{\Delta_3,\Gamma_1}$					
A_1	0.2195	0.5286	0.3109	0.1547	0.4373
A_2	0.5286	0.8275	0.4256	0.8275	0.5286
A_3	0.4256	0.3109	0.1547	0.4256	0.4256
A_4	0.1547	0.2372	0.4373	0.3109	0.3109

Step 3 determines PIS A^+ and NIS A^- in the form of

$$\begin{aligned} A^+ = \{&\{\langle 0.8000, 0.4000\rangle, \langle 0.9000, 0.3200\rangle, \langle 0.9000, 0.2800\rangle\},\\ &\{\langle 0.7000, 0.4667\rangle, \langle 0.8000, 0.4667\rangle, \langle 0.9000, 0.0667\rangle\},\\ &\{\langle 0.8000, 0.4500\rangle, \langle 0.9000, 0.4000\rangle, \langle 0.9000, 0.1500\rangle\},\\ &\{\langle 0.8000, 0.4737\rangle, \langle 0.9000, 0.2105\rangle, \langle 0.9000, 0.3158\rangle\},\\ &\{\langle 0.7000, 0.4000\rangle, \langle 0.8000, 0.3000\rangle, \langle 0.8000, 0.3000\rangle\}\}, \end{aligned}$$

and

$$\begin{aligned} A^- = \{&\{\langle 0.6000, 1.0000\rangle, \langle 0.7000, 0.0000\rangle, \langle 0.7000, 0.0000\rangle\},\\ &\{\langle 0.5000, 0.4000\rangle, \langle 0.6000, 0.6000\rangle, \langle 0.6000, 0.0000\rangle\},\\ &\{\langle 0.6000, 0.5000\rangle, \langle 0.7000, 0.5000\rangle, \langle 0.8000, 0.0000\rangle\},\\ &\{\langle 0.5000, 0.6667\rangle, \langle 0.6000, 0.3333\rangle, \langle 0.6000, 0.0000\rangle\},\\ &\{\langle 0.6000, 0.3333\rangle, \langle 0.7000, 0.6667\rangle, \langle 0.8000, 0.0000\rangle\}\}. \end{aligned}$$

Step 4 determines the priority of alternatives A_i with respect to their larger value of relative closeness. The relative closeness values corresponding to entropies $E^{\psi_1}_{\Delta_1,\Gamma_1}$, $E^{\psi_1}_{\Delta_2,\Gamma_1}$ and $E^{\psi_1}_{\Delta_3,\Gamma_1}$ are given in Table 7.

In this section, a comparative study is also conducted to compare the performance of proposed similarity measures concerning different entropies $E^{\psi_1}_{\Delta_1,\Gamma_1}$, $E^{\psi_1}_{\Delta_2,\Gamma_1}$ and $E^{\psi_1}_{\Delta_3,\Gamma_1}$ with the performance of some existing PHFS distance measures which can be easily transformed to similarity measures as below:

$$S_D({}^p h_1, {}^p h_2) = \frac{\pi[D({}^p h_1, {}^p h_2)] - \pi[1]}{\pi[0] - \pi[1]} \tag{48}$$

where π denotes a real-valued and strictly monotone decreasing function, D and S_D stand respectively for PHFS distance measure and PHFS similarity measure.

With the help of the above transformation, we are able to generate different formulas for calculating a PHFS similarity measure using the strictly monotone decreasing function $\pi : [0,1] \to [0,1]$ in the forms of (1) $\pi(x) = 1 - x$; (2) $\pi(x) = \frac{1-x}{1+x}$; (3) $\pi(x) = 1 - xe^{x-1}$; and (4) $\pi(x) = 1 - x^2$.

The following are a number of well-known PHFS distance measures:

- Din et al.'s [44] distance measure

$$\mathcal{D}_{Din}({}^p h_1, {}^p h_2) = \sum_{i=1}^{N_I} w_i \Big(\sum_{k=1}^{l_{1,2}} |h_1^{(k)} p_1^{(k)} - h_2^{(k)} p_2^{(k)}|\Big); \tag{49}$$

Table 7
The relative closeness of A_i $(i = 1, 2, 3, 4)$.

$CR^{\psi}_{\Delta,\Gamma}$					Ranking order
$E^{\psi_1}_{\Delta_1,\Gamma_1}$	0.6506	0.3976	0.6240	0.6388	$A_1 > A_4 > A_3 > A_2$
$E^{\psi_1}_{\Delta_2,\Gamma_1}$	0.5991	0.4444	0.6351	0.6319	$A_3 > A_4 > A_1 > A_2$
$E^{\psi_1}_{\Delta_3,\Gamma_1}$	0.7251	0.3310	0.6153	0.6510	$A_1 > A_4 > A_3 > A_2$

Table 8
The relative closeness of A_i $(i = 1, 2, 3, 4)$.

	$RC(A_1)$	$RC(A_2)$	$RC(A_3)$	$RC(A_4)$	
$\mathcal{D}_{Din}$	0.5584	0.2163	0.4780	0.7730	$A_4 > A_1 > A_3 > A_2$
$\mathcal{D}_{Su}$	0.6529	0.0946	0.6853	0.8112	$A_4 > A_3 > A_1 > A_2$
$\mathcal{D}_{Fang}$	0.3352	0.0955	0.3410	0.3490	$A_4 > A_3 > A_1 > A_2$
$\mathcal{D}_{Wang}$	0.5027	0.1741	0.4756	0.7431	$A_4 > A_1 > A_3 > A_2$
Proposed similarity by					
$E^{\psi_1}_{\Delta_1,\Gamma_1}$	0.6790	0.3993	0.6461	0.7042	$A_4 > A_1 > A_3 > A_2$
$E^{\psi_1}_{\Delta_2,\Gamma_1}$	0.6218	0.4422	0.6525	0.6882	$A_4 > A_3 > A_1 > A_2$
$E^{\psi_1}_{\Delta_3,\Gamma_1}$	0.7597	0.3358	0.6390	0.7355	$A_1 > A_4 > A_3 > A_2$

- Su et al.'s [48] distance measure

$$\mathcal{D}_{Su}({}^p h_1, {}^p h_2) = \sum_{i=1}^{N_I} w_i (|\sum_{k=1}^{l_1} h_1^{(k)} p_1^{(k)} - \sum_{k=1}^{l_2} h_2^{(k)} p_2^{(k)}|); \tag{50}$$

- Fang et al.'s [63] distance measure

$$\mathcal{D}_{Fang}({}^p h_1, {}^p h_2) = \sum_{i=1}^{N_I} w_i (\frac{1}{2} \sum_{k=1}^{l_{1,2}} (|h_1^{(k)} p_1^{(k)} - h_2^{(k)} p_2^{(k)}| + |h_1^{(k)} - h_2^{(k)}| p_1^{(k)} p_2^{(k)})); \tag{51}$$

- Wang et al.'s [62] distance measure

$$\mathcal{D}_{Wang}({}^p h_1, {}^p h_2) = \sum_{i=1}^{N_I} w_i (\frac{1}{2} (\frac{1}{l_{1,2}} \sum_{k=1}^{l_{1,2}} |h_1^{(k)} p_1^{(k)} - h_2^{(k)} p_2^{(k)}| + |\frac{1}{l_1} \sum_{k=1}^{l_1} p_1^{(k)} |h_1^{(k)} - \frac{1}{l_1} \sum_{k=1}^{l_1} h_1^{(k)} p_1^{(k)}| - \frac{1}{l_2} \sum_{k=1}^{l_2} p_2^{(k)} |h_2^{(k)} - \frac{1}{l_2} \sum_{k=1}^{l_2} h_2^{(k)} p_2^{(k)}||)). \tag{52}$$

In the above formulas, we set $l_{1,2} = \max\{l_1, l_2\}$.

Now, we are in a position to proceed with

$w_{Wang} = (0.29716,\ 0.15306,\ 0.12319,\ 0.21329,\ 0.21329)$

as that considered by Wang et al. [62], and derive the priority of alternatives by implementing Algorithm 4.2. The results are shown in Table 8.

There are still difficulties concerning the deficiency of existing distance-based techniques:

- None of the distance (or similarity) measures of Din et al. [44], Su et al. [48], Fang et al. [63] and Wang et al. [62] distinguish between the two different conceptual terms of hesitancy and probability. The evidence to this fact is the presence of $h_1^{(k)} \times p_1^{(k)}$ in all their formulas.
- Although in the definition of Fang et al.'s [63] distance, there are separated terms $|h_1^{(k)} - h_2^{(k)}|$ and $p_1^{(k)} p_2^{(k)}$, the latter relation does not refer to the difference of probabilities.

5. Conclusions and future works

In this paper, we have presented innovative axiomatic frameworks for similarity and entropy measures specifically designed for PHFEs. These frameworks address the limitations of existing measures by incorporating considerations of both hesitancy and wrapped probabilities. As a result, we have established a comprehensive suite of similarity and entropy measures that advance the field of PHFE information theory, with the underlying concept of t-norms playing a crucial role in their formulation. Additionally, we have introduced similarity-based entropy measures tailored for PHFEs, enriching the existing framework and providing deeper insights into the uncertainty inherent in PHFEs.

To apply these novel uncertainty measures effectively, we have developed a Multi-Attribute Decision Making (MADM) method within the TOPSIS framework. This method leverages the developed similarity and entropy measures to determine attribute weights and assess the discrimination degree between alternatives and reference ideal solutions. To validate the efficacy of our proposed measures and ranking method, we have illustrated their application in a real-world scenario involving the protection of the Yangtze finless porpoise.

In summary, the strengths and limitations of our study are outlined as follows:

Advantages:

- Our proposed similarity and entropy measures for PHFEs consider both hesitancy and wrapped probabilities, resulting in more accurate outcomes and facilitating practical decision-making.
- These measures inherently complement each other as uncertainty measures for PHFEs.
- The developed uncertainty measures and MADM method for PHFEs can be extended to Probabilistic Linguistic Term Sets (PLTSs), as PHFEs are an extension of PLTSs.

Limitation:

- The tradeoff parameters of our proposed similarity and entropy measures for PHFEs are subject to subjective determination.

The uncertainty surrounding probabilistic hesitant fuzzy information remains still an ongoing challenge. This implies that our methodologies can be expanded to explore other types of information measures, such as knowledge measures and accuracy measures, to address various practical problems including group decision making, information retrieval, and data mining. This broader application aims to enhance the relevance of PHFE information theory in real-world contexts, contributing to the development of more robust and adaptable approaches for managing uncertainty in decision-making processes.

Human and animal rights

This article does not contain any studies with human participants or animals performed by the authors. Moreover, there is no financial interest to report.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

Data will be made available on request.

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