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Children's Mathematical Thinking in Different Classroom Cultures

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The relationship between normative patterns of social interaction and children's mathematical thinking was investigated in 5 classes (4 reform and 1 conventional) of 7- to 8-year-olds. In earlier studies, lessons from these classes had been analyzed for the nature of interaction broadly defined; the results indicated the existence of 4 types of classroom cultures (conventional textbook, conventional problem solving, strategy reporting, and inquiry/argument). In the current study, 42 lessons from this data resource were analyzed for children's mathematical thinking as verbalized in class discussions and for interaction patterns. These analyses were then combined to explore the relationship between interaction types and expressed mathematical thinking. The results suggest that increased complexity in children's expressed mathematical thinking was closely related to the types of interaction patterns that differentiated class discussions among the 4 classroom cultures.

Key words: Classroom interaction; Cognitive development; Elementary, K–8; Higher-order thinking; Reasoning; Teaching practice

Over the past 15 years, research has provided substantial evidence that the social processes for doing mathematics differ in conventional classrooms and in classes oriented to ideas advocated in U.S. reform documents (e.g., NCTM, 1989, 2000). Such research has focused on social norms, sociomathematical norms, forms of interaction, and discourse that occur in classes as a means for examining the influence of mathematics classroom practice on, for example, the nature of students' explanations and justifications (Cobb, Wood, Yackel, & McNeal, 1992) and argumentation (Krummheuer, 1995). An underlying assumption of this research and of the goals of reform more generally is the importance of mathematical thinking and

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reasoning in the development of conceptual understanding and the central role that abstraction and generalization play in the domain of mathematics. Moreover, it is widely believed that when students learn mathematics through thinking and reasoning, they understand conceptually. Further, theoretically, it is commonly assumed that there is a link among social interaction, the development of human thought, and the construction of knowledge (e.g., Bruner, 1996; Hobson, 2004; Tomasello, 2001).

One strand of research on the differences among mathematics classes focuses on the social nature of children's learning; this research assumes that knowledge is socially constructed. This research on *mathematics classroom practices* therefore focuses on either the classroom culture in relation to children's opportunities for learning (e.g., Cobb & Bauersfeld, 1995), the processes involved in acculturating children into the practices of the classroom culture (e.g., Bowers, Cobb, & McClain, 1999), or the role that discipline- specific social norms (i.e., sociomathematical norms) play in learning mathematics in the classroom (Yackel & Cobb, 1996). This research draws on an interactionist perspective, following Blumer (1969), of the important role of social interaction in meaning making. The research is also derived from sociologists such as Goffman (1959), who claimed that the social structures in everyday life consist of normative patterns of interaction and discourse, and Vygotsky (1978), who focused on children's participation in collective practices as the source of mathematical knowledge.

Another strand of research investigates differences among mathematics classroom practices in relation to students' knowledge of mathematics. This research recognizes that, in reforming classroom mathematics practice, one must consider the interrelationship between children's developing cognition and the structure of mathematics. For that reason, the investigations focus on such topics as the development of individual children's invented mental strategies (e.g., Carpenter, Franke, Jacobs, & Fennema, 1998) or the acquisition of specific content knowledge (e.g., Saxe, Gearhart, & Seltzer, 1999). Thus, studies such as these show that the social structure and children's conceptual understanding differ between conventional and reform-oriented classes. Additionally, several larger-scale studies support these findings (e.g., Askew, Brown, Rhodes, Johnson, & Wiliam, 1997).

In contrast to research on social and psychological conditions, others such as Ball and Bass (2000) contend that there is another set of *practices* that is fundamentally important and should be considered when investigating classrooms; these are the mathematical practices. Ball and Bass define *mathematical practices* as the ways in which inquiry and validation of mathematical knowledge occur in the classroom; mathematical practices represent the practice of mathematics in the same way that the scientific method is considered a scientific practice. In looking at how children construct mathematical knowledge, they argue that mathematical practices are encapsulated in mathematics and are not derived from or viewed as classroom mathematics practices. Added to this assertion of Ball and Bass is the widely held belief that it is the particular characteristics of domain-specific thinking (e.g., deductive reasoning) in mathematics, essential to abstraction and generalization in the construction of mathematical knowledge, that differentiate mathematical practices from other classroom practices (Reid, 2002; Russell, 1999).

These lines of research provide fundamental information relating mathematics classroom processes and mathematics practices to student learning, but a clear understanding of the connection between social interaction and children's development of mathematical thinking is still not well understood. In this article, we begin to address this relationship by examining the nature of students' verbalized mathematical thinking within classroom cultures characterized by differing patterns of social interaction. This research provides initial insight into the relationship between different types of interaction patterns and children's mathematical thinking and reasoning and thus contributes to our understanding of how mathematical thinking may develop within classrooms.

BACKGROUND TO STUDY

Wood (1994, 1996, 1998b) and Wood and Turner-Vorbeck (1999, 2001) empirically established that differences exist among reform-oriented classes in terms of the social features and quality of students' thinking. The analysis of interaction and discourse in several reform-oriented classes yielded broadly defined patterns of interactive and communicative exchanges. These broad patterns of interaction served as the basis from which the similarities and, more important, differences in the mathematics classroom culture across reform-oriented classrooms were identified. Analysis of the data revealed that reform-oriented classes fell into two major types, *strategy reporting* and *inquiry/argument* classrooms. These types were assumed to represent the ways in which the culture of the classrooms differed. The main focus in a strategy reporting classroom culture is on children's presentation of different strategies for the problems solved. Children presenting their solutions may be asked to provide more information about *how* they solved the problem by the teacher but rarely by other student listeners. Classes classified as inquiry/argument are those in which children offer different solution methods, as in the strategy reporting classes, but also *why*; they provide reasons for their thinking in order to make sense to others. In addition, student listeners and teachers in these classes ask questions for further clarification and understanding. These discussions also often include a challenge or disagreement from student listeners or teachers, which initiates an exchange that in turn prompts the thinking of justification in support of children's ideas.

Kazemi and Stipek (2001) reported similar findings of differences among reform-oriented mathematics classes. Their study focused on the social norms, sociomathematical norms, and teaching practices that existed in classes. They concluded that, although social norms were the same across different classes, the sociomathematical norms and teaching practices that promoted conceptual understanding were different. They described differences in student explanations and teacher questioning similar to those that Wood (1994, 1996, 1998b) and Wood and Turner-Vorbeck (1999, 2001) found in their earlier studies of strategy reporting and inquiry/argument classes but attributed the distinction among classes to differences in sociomathematical norms.

Having previously established that differences exist between conventional and reform-oriented classes (Wood, 1998a) as well as among reform-oriented classes, primarily based on social normative characteristics of the classroom practices (e.g., Wood & Turner-Vorbeck, 1999, 2001), this research study looked specifically at the nature of children's mathematical thinking and reasoning as it occurs within these social situations. Although it is generally accepted that differences in classroom interaction influence students' mathematical thinking, there is little research that examines the relationship between specific interaction patterns and student thinking and few studies that examine students' discipline-specific thinking within different interactive situations. The research we report in this article analyzed the specific types of interaction patterns that underlie the broad classification categories of classroom cultures (conventional, conventional problem solving, strategy reporting, and inquiry/argument) coupled with an analysis of children's mathematical thinking within these interaction patterns. The intention was to explore the generally accepted intuition that children in reform-oriented classes develop essential modes of mathematical thought and early forms of mathematical reasoning. We did this by investigating the interrelationship between types of interaction patterns and the nature of children's mathematical thinking as verbalized during class discussion—the reporting phase of reform instruction. To clarify, we analyzed children's mathematical *thinking as verbalized within a group* or public thought, as opposed to *thinking alone* or private thought.[1]

CONCEPTUAL FRAMEWORKS

In this study, we used two conceptual frameworks in order to describe and interpret the relationship between children's verbalized thinking and specific interaction patterns. One conceptual framework was used to investigate specific interaction patterns and the other conceptual framework was employed to examine the quality of students' expressed thinking.

Classroom Culture

Wood and Turner-Vorbeck (1999, 2001) generalized the differences they found in classroom cultures on two dimensions: student participation and student thinking. On the one hand, the participation dimension consisted of the extent to which teachers, in establishing the social norms for interaction with students, made it possible for all pupils to participate actively in the interaction and discourse. Student participation differed between the two reform-oriented classroom cultures in terms of the teacher's expectations for students, both in giving their explanations and in asking questions. From these empirical findings, a theoretical connection was made between the social norms constituted in a class and the social interaction patterns that evolve. On the other hand, the student thinking dimension was related

[1] The italicized terms are taken from Rochat (2001).

to an increasing quality of student thinking that included reasoning and justification of mathematical ideas. Theoretically, these levels of thinking were hypothesized to differentiate among classroom cultures in terms of a deepening in thought processes and as a means to particular kinds of knowledge outcomes.

Together the two dimensions provide a conceptual framework for describing the differences among classroom cultures in terms of the relationship between social interaction (participation) and student thinking (cognition). Student participation is believed to become more frequent and sophisticated in relation to the increased complexity of interaction within the classroom cultures. This research draws from Bruner's (1996) long-standing belief in the need for children to participate in social interaction in order to acquire the shared meanings of their culture and to develop in their capacity for thinking. The interaction dimension also draws on the interests of theorists Goffman (1959) and Garfinkel (1967) in processes involved in the interactive constitution of shared meanings. They claimed that the social structures that exist in everyday life consist of normative patterns of interaction and discourse that, once established, become reliable routines found in interactive situations. In order to interact and communicate with one another, individuals need to hold common understandings, which they take as an implicit basis of reference when speaking to each other. From these theories, the constructs *social norms* and *interaction patterns* help to delineate the participation structures created in the different classroom environments and the processes by which mathematical meaning is developed during social interaction.

The student thinking dimension initially drew on the processes involved in abstract reflective thought developed by theorists following Piaget (1985). In the conceptual framework described previously, it was hypothesized that children's increasing responsibility to engage in higher levels of thinking was connected to the type of classroom culture (strategy reporting or inquiry/argument). However, the theoretical constructs underlying this dimension were only broadly conceived (e.g., reflection through contrast/comparison) and lacked both a specificity and an essential connection to mathematics. Therefore, a conceptual framework that could better describe children's mathematical thinking was needed. We chose to draw from the work of Williams (2000) to define mathematical thinking and to provide constructs for analysis.

Mathematical Thinking

In this study, we define *mathematical thinking* as the mental activity involved in the abstraction and generalization of mathematical ideas. This definition draws on the research of Krutetskii (1976) on levels of mathematical activity that are central to the construction of mathematical knowledge and on the observable epistemic actions that result in abstraction and generalization, as conceptualized by Dreyfus, Hershkowitz, and Schwarz (2001).

Williams (2000) created a framework to describe mathematical thinking of students that initially drew on the work of Krutetskii (1976) and used his empirical data on "mental activities" to systematically classify cognitive complexity in order to develop a hierarchy of the cognitive activities students used when solving mathematical problems. The development of this hierarchy was further supported through its consistency with Bloom's (1956) hierarchy of learning objectives, which have since been used to describe cognitive activity (Van Tassel-Baska, 1993). The cognitive activities in the hierarchy, starting with the least demanding, consist of the following: *comprehending*, *applying*, *analyzing*, *synthetic-analyzing*, *evaluative-analyzing*, *synthesizing*, and *evaluating*. These cognitive activities are assumed to be cumulative, with each activity of the system building on successful completion of the previous activity.

Williams then integrated these cognitive activities as a nested set of subcategories with the view of Dreyfus, Hershkowitz, and Schwarz (2001) that the construction of mathematical knowledge consists of three observable epistemic actions that occur during the cognitive processes of abstraction and generalization: *recognizing*, *building-with,* and *constructing*. The identification of epistemic actions during the process of abstraction was empirically derived and theoretically justified by Dreyfus, Hershkowitz, and Schwarz and provides a way to consider mathematical thinking through "observable" cognitive elements. Williams' hierarchy thus consists of specific types of mathematical thinking connected to observable epistemic actions. The cognitive activities for each epistemic action are as follows: *Recognizing*—comprehending and applying; *Building-with*—analyzing, synthetic-analyzing, and evaluative-analyzing; and *Constructing*—constructing and synthesizing. These thinking categories have previously been used to study the individual thinking of students in classroom research in middle and high school mathematics classrooms (Williams, 2002a, 2002b). The categories of *synthetic-analyzing* and *evaluative-analyzing* have been found useful in providing additional detail about the increasing complexity of thinking that occurs as students' progress from *Building-with* to *Constructing*.

Williams' (2004) analysis of the thinking of a middle school student, Leon, illustrates the subcategories of thinking within this framework. Leon *analyzed* the figure formed when he juxtaposed two right-angled triangles and *Recognized* the properties he identified (*Building-with*) as those of a rectangle (*Recognizing: comprehending*). By halving the area of the rectangle, he found the area of the right-triangle (*Building-with*: *analyzing*). He *Recognized* two aspects of this approach in developing two pathways to explore areas of acute-angled triangles: juxtaposing acute-angled triangles and using rectangles. He compared these approaches (*synthetic-analysis*) and decided the latter was likely to produce an "easier way" (*evaluative-analyzing*; *synthetic-analyzing* for the purpose of judgment). He *synthesized* (*Constructing*) by subsuming the relevant attributes of rectangles (length, width) into attributes of acute angled triangles (base, perpendicular height) so he could find areas of triangles without explicit reference to rectangles. He *evaluated* (*Constructing*) his new insight by *Recognizing* its usefulness for another purpose: equating the areas of triangles with the same base and perpendicular height.

These categories in Williams' hierarchy provided a conceptual framework needed to better depict the nature of students' mathematical thinking for the student thinking dimension of Wood and Turner-Vorbeck's conceptual framework. For our purposes, we expanded Williams' hierarchy developed for studying individual cognitive activity to consider mathematical thinking in a broader sense, that is, to investigate the mathematical thinking that children expressed in public situations (their "group thinking") rather than to examine their individual thinking in a setting such as an interview. We took this approach because we believed thinking with others was an important aspect of children's cognitive progress and thus their knowledge construction; moreover, thinking with others was generally viewed as a necessary facet in developing virtual dialogue (thinking alone) and thinking as higher functions (Bruner, 1996; Vygotsky, 1978). It is by and large agreed that engaging in thinking with others allows "individuals to bypass their own cognitive limitations" (Rochat, 2001, p. 139) in the reconstruction and mutual consolidation of thoughts and ideas. Moreover, problem resolution through thinking with others is the way in which young children develop the capacity for thinking via virtual dialogue, an important aspect of adult thinking (Rochat, 2001). Thus, these two conceptual frameworks provide the means for examining the relationship between the patterns of interaction that exist in the classroom and children's expressed mathematical thinking.

METHODOLOGY

Background of Classroom Data

The reform-oriented primary classes used in the empirical analysis had been in existence for approximately 5 years. In these classes, children were encouraged to develop and use mental and invented strategies to solve problems; procedures such as the standard algorithms for addition and subtraction with regrouping were not explicitly taught. The teachers involved in the investigation participated in a 1-week professional development session followed by visits of the project staff to their classrooms (see Cobb, Wood, & Yackel, 1990, for more detail). The classes existed in five elementary schools in the same school district of a medium-sized Midwestern city in the United States[2] The analysis of the data resource initially identified the strategy reporting and inquiry/argument classes that distinguished the reform classroom cultures as discussed previously. For this study, we selected lessons from two classes previously identified as *strategy reporting* and two classes previously identified as *inquiry/argument*.

One additional class was included in our data resource for comparison with the reform-oriented classes. This was a textbook-based class in which the teacher did not participate in the professional development sessions but was from one of the schools in the same school district. An initial analysis of lessons from this class indicated that

[2] The research was sponsored by the National Science Foundation, RED 9254939.

two "cultures" existed in this class, one that consisted of *conventional textbook* lessons and another that consisted of *conventional problem-solving* lessons (McNeal, 1991).

Lesson Selection

For each of the five classes described above, 30 lessons conducted during the first and second semesters were screened. For the reform-oriented classes, we used lessons that occurred in the second semester of second grade and for the conventional class the first semester of third grade because the mathematical topics (place value and addition and subtraction with regrouping) were the common focus of these lessons. The conventional class consisted of textbook-only lessons and lessons that consisted of both problem solving and textbook instruction, with each lesson divided into a section consisting of nonroutine problem solving and a section of solving textbook problems. This combination provided the unique opportunity to examine both textbook and problem- solving cultures within the same classroom.

From each of the four reform-oriented classes, a subset of 8 lessons (for a total of 32 lessons for all four classes) was selected as representative of that particular classroom's culture (strategy reporting, inquiry/argument). From these lessons, 5 lessons were selected from the subset of 8 lessons for intensive analysis. These 5 lessons were the base data from which interaction patterns and children's verbalized thinking were identified. The remaining 3 lessons from each class were used to confirm the identification of the patterns of interaction and children's expressed thinking.

For the conventional class, a subset of 8 lessons was selected that consisted of both problem solving and textbook instruction and 2 lessons that consisted of textbook instruction only for a total of 10 lessons from the conventional class. Five lessons were selected from the subset of the 8 combined problem solving and textbook lessons for analysis. These 5 lessons were the base data from which interaction patterns and children's verbalized thinking were identified. The remaining 3 lessons from the combined problem solving and textbook instruction and 2 textbook-only lessons were used to confirm the identification of the patterns of interaction and children's verbalized thinking.

The lessons focused on the concept of 10[3] and on methods of two-digit addition and subtraction, with the exception of the problem-solving lessons in the conventional class. In the conventional class, the concept of 10 was addressed with traditional textbook place value activities. Using lessons that focus on common content permitted detailed examination and comparison of interaction patterns and student expressed thinking across these classes.

Analyses

Our method of analysis was based on a quantitative-qualitative research paradigm in which two coding schemes, one for analysis of interaction patterns and the other

[3] We follow the definition of this understanding given in Cobb and Wheatley (1988).

for children's mathematical thinking, were used to interpret each of the transcribed videotaped lessons. This analysis was followed by more intensive microanalytic interpretative procedures (similar to those described by Voigt, 1990) of the lessons selected for intensive analysis from each classroom culture.

The procedure for analysis of the interaction patterns consisted of (1) initially segmenting the transcript by each mathematical problem discussed; (2) conducting a line-by-line coding of the dialogue for all the segments in the transcript using a coding scheme described in Wood et al. (1999), and then (3) identifying and sectioning the distinct patterns of the interaction that existed within each segment; and (4) identifying the consistent and repeatable interaction patterns across the lessons and assigning a label or name to these patterns. The interaction patterns were given labels that corresponded to the perceived function, intention, or goal of the interaction. For example, Exploring Methods was a pattern that consisted of different children presenting the various ways in which they solved the problem (see the Appendix for detailed descriptions and labels of the specific patterns of interaction). In some cases, an identified pattern fit a form of interaction previously described in the literature; for example, Hoetker and Ahlbrandt's (1969) IRE, or Initiate Respond Evaluate. It is important to note that all segments of the class discussion were categorized as consisting of some type of interaction pattern or patterns. Once all the class discussion data were coded and the patterns of interaction labeled, the types of interaction patterns were counted within each of the four classroom cultures (conventional textbook, conventional problem solving, report strategies, and inquiry/argument).

The same transcripts of class discussion for each lesson used to code the interaction patterns were used to code and analyze children's thinking, but the coding was done separately from the coding of the interaction patterns. The coding scheme described previously for examining students' mathematical thinking was used to categorize individual children's verbalized statements during class discussion (see Figure 1). Each line of children's expressed thinking was coded and assigned a single code for mathematical thinking (e.g., comprehension).[4] Once all the class discussion data were coded, the categories of mathematical thinking (e.g., *Recognizing: comprehending*) were counted within each of the four classroom cultures.

Following this analysis, the coded interaction patterns and the coded children's statements were combined to recreate each class discussion in its entirety. Each coded statement of children's mathematical thinking was re-examined within the interaction pattern it occurred. The number of occurrences of each category of mathematical thinking was recorded for each interaction pattern. For example, a given instance of the Exploring Methods pattern might consist of four occurrences of mathematical thinking coded respectively as one of *Recognizing: applying,* two of *Building-with: analyzing,* and one of *Building-with: synthetic-analyzing.*

[4] Teacher's questions and statements were also analyzed, but these results are not reported in this study. Information on teacher prompting questions is found in Wood and McNeal (2003).

Mathematical Thinking	Examples of Cognitive Activity	Examples of Mathematical Thinking Revealed in Class Discussion
Recognizing comprehending	^+ Understand concepts behind taught idea or known strategy.	*Problem: 72 – 39 =* *Tracey:* It was 70 minus 30, but first I took the total, that 70. Then I took 30. And I subtracted um, 30 from 70 and I got 40. Then I subtracted 9, [pause] and I got 31. Then I subtracted [shakes her head no], I added 2. (*Apply known mathematical procedures in a new context.*)
Recognizing applying	+ Know when to use a known mathematical idea.	*Problem: 29 + 10 =; then 29 + 20 =* *Jack:* We got 49. We got that because the last was 39 and then you're just adding 10 more. (*Use a known mathematical idea*—Jack applies the mathematical idea of thinking strategies.)
Building-with analyzing	^ Apply known mathematical procedures in a new context. + Solve using a problem with a slight twist. + Familiarize self with problem using specific numerical examples. + Systematize the numerical results and search for patterns.	*Problem: 50 – 20 =* *Mark:* When you add 3 plus 2, you got 5 and 20 plus 30 is just like 3 plus 2. So if you take away 2 from 5 you're gonna [going to] have 3 and if you take away 20 from 50 you're gonna have 30. (*Apply known mathematical procedures in a new context.*)
Building-with synthetic-analyzing	+ In contrast and comparison of two methods for the difference. + Interconnect various representations, operations, and assumptions. * + Use more than one pathway to solve a problem. + Produce an independent generalization-"small discovery." + Analyze one case, or form a guiding principle to formulate a new rule.	*Problem: 72 – 39 =* *Sara:* Um, I almost did it like Gregg, except I did 72 minus 30. And then I um, then I took off 10 and got, uh, 40. (*In contrast and comparison of two methods for the difference.*)
Building-with evaluative-analyzing	+ Interconnect solution pathways for the purpose of identifying flaws and strengthening arguments. + Pull together ideas for making a judgment. + Evaluate whether a method or result is reasonable, efficient, or elegant.	*Problem: 72 – 39 =* *Kari:* I got 47. So I did it this way [vertically] and I got 7 minus 3 is 4 and 9 minus 2 is 7. *Carl:* Hold on! The 7? It's just that you can't subtract the 7 from . . . *Matt:* [interrupts] Kari. It can't be that way because if it were, it would be 40 something. (*Evaluating if result is reasonable through a fast logical argument.*) *Kari:* Yes, but this [7] take away this [3] is 4. Seven take away 3 is 4. *Matt:* When she went like this [7 – 3] that's okay. But when she went like this [2 – 9], you can't do that. That would be nothing or negative 7. That's less than zero. (*Evaluating if result is reasonable.*)

Mathematical Thinking	Examples of Cognitive Activity	Examples of Mathematical Thinking Revealed in Class Discussion
Constructing synthesizing	+ Formulate mathematical arguments to explain discovered patterns. * + Explore the problem from many perspectives rather than just work towards a solution. * + Integrate concepts to create new thought or ideas (new insight). Could vary in: + Number of concepts involved. + Diversity of the domains concepts were drawn from. + Size of the conceptual leap. + Spontaneity with which the process is undertaken. + Progressively explore the problem to continually develop new insights.	*Problem: 72 – 39 =* *Linda:* [writes problem vertically] Well, if you try to take 9 away from 2 you can't do it. I was going to do this but that would be changing the problem [writes 39 – 72 vertically]. *Teacher:* Does anyone have an idea for Linda? *John:* Ah! I have an idea. Listen to this! Any time you have not got enough ones, you can take one 10 and put it with the ones—sort of break it up differently. (*A generalization showing a new insight.*) See—you can't take 9 from 2 but you can take 9 from 12. *Teacher:* But why 12? *John:* You take 10 of the 70 so it is 60 now. See 12 take away 9 is 3 and 60 take away 30 is 30, so it's 33.
Constructing evaluating	* + Progressively reflect on the situation as a whole for the purpose of recognizing inconsistent information and/or finding a more elegant solution pathway. + Reflect upon the process of problem solution for the purpose of recognizing its limitations and its application to other contexts. ^ Reflect upon the solution pathway developed and its possible contribution to generic mathematical processes to employ in the future.	*Problem: 72 – 39 =* *Gabriel:* Well it is close to 30 because it is about 40 less than 70—60—that's 10—50—that's 20—40 30 that's the other 20. Now—72 minus 39 is 62—52—42, and take 9 more—10 more is 32 so 9 more is 33. Yes that's close to 30. Now just one more check—another way—33 and 39 would be around 72 because 30 and 30 are 60 and 3 and 9 are 10 and a bit more. We have 60 and 10 and a bit more—so 72 is okay. (*Progressively reflect on the situation as a whole for the purpose of recognizing inconsistent information —evaluation.*) *John:* I have been thinking [(pause]) what am I going to do with my way if there are no tens? [see synthesis example above]. *Teacher:* Tell us more about what you are thinking John.? *John:* What if I had 702 minus 39—no tens to put with the two? (*Reflect upon the process of problem solution for the purpose of recognizing its limitations and its application to other contexts.*)

Note. Column 1 represents categories drawn from Dreyfus, Hershkowitz, and Schwarz (2001). Column 2 represents categories of Krutetskii (1976)*; Williams (2000)+; and Dreyfus, Hershkowitz, and Schwarz (2001) ^. Column 3 provides examples of children's generated and invented methods.

Figure 1. Categories of mathematical thinking and cognitive activity.

RESULTS

Lesson Structure Across Classroom Cultures

We defined the general organization of the lessons for each classroom culture by the chronology of major events. In the conventional class, the textbook lessons averaged 60 minutes in length. Lessons in this class generally began with an introduction in which the teacher gave students information and instructions for how to proceed with the next segment of the lesson. The next segment varied according to the way in which the class lesson, individual work, pair work, or individual work and small groups working with the teacher were combined. The conventional problem-solving lessons occurred within the usual mathematics lesson and lasted, on average, 16 minutes. These lessons consisted of an introduction in which the teacher guided students in class discussion, short segments of pair work, then class discussion. Removing the problem-solving segments from the conventional lesson times, the average length of textbook-based lessons was 45 minutes. The reform-oriented classroom (strategy reporting and inquiry/argument) lessons lasted, on average, 45 minutes. They consisted primarily of a brief introduction explaining expectations for students, student pair work (approximately 20–25 minutes), followed by class discussion (15–20 minutes).

The structure of the conventional problem solving, strategy reporting, and inquiry/argument classes was highly consistent across the lessons analyzed in comparison with the more variable textbook lessons. This variation in textbook lesson structure and the shift from textbook to problem-solving instruction within the same lesson appear to reflect the teacher's attempts to compensate for the lack of variety in the prescribed textbook lessons and to motivate children's interest in doing mathematics.

Interaction Patterns

The analysis of the data revealed that not only did the number of interaction patterns increase progressively across the four classroom cultures, as shown in Table 1, but the types of interaction patterns also changed across the classroom cultures, with the textbook lessons consisting of the fewest types of interaction patterns and inquiry/argument lessons having the most types of interaction patterns. The nature of these interaction patterns also changed from closed to more open, suggesting increasing opportunities for student discourse and participation.

In the conventional textbook class discussion, the prevalent type of interaction was the IRE pattern (50%), which consisted of teacher "test" questions, student response (correct or incorrect), and teacher evaluation of the student's response. This was followed by Give Expected Information (21%), a less tightly controlled form of the IRE pattern in which students still provided previously taught information in response to teachers' questions, and then by the Funnel pattern (15%) in which the teacher, through a series of questions, led the student to the correct answer (Bauersfeld, 1980). The dominance of these three interaction patterns indicated that children's participation was limited to responding to teachers' questions by giving known answers or predetermined information.

Table 1
Frequency Percent of Interaction Patterns by Class Culture[a]

Interaction pattern	Conventional: Textbook	Conventional: Problem solving	Reform: Strategy reporting[b]	Reform: Inquiry/ argument[c]
Common to all instruction				
Collect answers	9	2	11	14
Conventional instruction				
IRE	50	18	2	1
Funnel	15	2	6	
Give expected information	21	16	1	
Teacher explain	6	6		
Hint to solution		39		
Reform instruction				
Exploring methods		14	44	35
Argument		2	9	16
Inquiry			5	9
Teacher elaboration			11	2
Proof by cubes			6	4
Proof by pupil explanation			5	1
Focus				4
Building consensus			1	5
Checking for consensus				7
Develop conceptual understanding				3
Pupil self-nominate				1
	$n = 34$	$n = 49$	$n = 85$	$n = 110$

[a] All lines of transcripts were included and counted as an interaction pattern; therefore, all interactions were taken into account. In the case of strategy reporting and inquiry/argument, two identified patterns specific to each were not included in the results because they only occurred once in the data. The data were first sectioned by the type of interaction and then the number of segments was counted; *n* represents the total number of segments in the data.

[b] Two patterns not listed.

[c] Two patterns not listed.

New interaction patterns were observed in the conventional problem-solving class discussions. In Hint to Solution, the teacher dominated the participation in the pattern (39%). In this pattern, the teacher "hinted" at the solution method in ways that essentially removed the mathematical challenge or complexity of the problem, which was then easily solved by the children. This pattern, along with the IRE (18%) and Give Expected Information (16%) patterns, made up 73% of the types of interaction identified in the conventional problem-solving portion of the lesson. However, one open pattern of interaction, Exploring Methods (14%), existed, which allowed for more student participation. In this interaction pattern, students were expected to tell others their strategy for solving the problem. These results indicated that in the conventional problem-solving class there were more opportunities for children to participate in the discourse than in the textbook discussions.

In strategy reporting classes, the proportion of interactions involving Exploring Methods (44%) increased substantially, indicating the importance of children's participation in providing their strategies during the discussion. There were differences between the teachers; one teacher seemed to conceive of the discussion as consisting of children giving one strategy for each problem, and the other appeared to view the situation as children giving many strategies for the same problem. This preference for reporting strategies, as either one or many, was consistent *within* each class discussion and across lessons during Exploring Methods interaction patterns.

It can be seen in Table 1 that the range of interaction patterns in the strategy reporting classroom culture is more extensive than that found in both of the conventional classroom cultures. The interaction patterns that define conventional classroom environments (IRE, Funnel, and Give Expected Information) rarely exist in the strategy reporting classroom culture. This difference reflects an important change in the participation structure to one in which the students do the reporting and explaining of their solutions. However, the next most common interaction pattern, Teacher Elaboration (11%), is one dominated by the teacher and reveals the means by which teachers elaborate on and extend a child's explanation to ensure that important ideas are conveyed to the other children. This appears to be a pedagogical routine that the teachers developed as they shifted in their role from teacher as explainer to student as explainer. Finally, two new patterns comprised 14% of strategy reporting interaction patterns involved: Argument (9%), in which children and teacher participate in discourse to resolve their differences or disagreement about strategies or answers, and Inquiry (5%) in which children and teacher ask questions for clarification of the strategy or ideas of the child explaining. However, these two new patterns were attributed to only one of the strategy reporting classes.

In the inquiry/argument classroom culture, the most frequent interaction pattern observed was also Exploring Methods (35%), just as in the strategy reporting classroom culture. However, 25% of the interaction patterns involved Argument (16%) and Inquiry (9%), which was almost twice as many as in the strategy reporting classes. Furthermore, the proportion of interactions involving Teacher Elaboration was considerably less (2%). The changes in these three interaction patterns reflected another major shift in participation from an emphasis on the child reporting her/his different strategies to the children as listeners taking over the role of the teacher in questioning, clarifying, and validating mathematical ideas. By definition, the inquiry/argument classrooms included the two interaction patterns, Building Consensus and Checking for Consensus (12% combined), that were not observed in the other three classroom cultures. In the Building Consensus interaction, the children and teacher participated to develop common meanings. Checking for Consensus initiated by the teacher appeared to be a final attempt to open the discussion so any child could make comments or ask questions before moving on in the discussion.

Taken together, these results reveal that a combination of specific interaction patterns exists within each classroom culture. These specific patterns of interaction reveal the gradation and particular changes in the roles for participation among children and teachers. In the next section, we present the analysis of the children's expressed mathematical thinking found within the different classroom cultures.

Kinds of Children's Mathematical Thinking Within Classroom Culture

The level of verbalized mathematical thinking varied considerably across the classroom cultures (see Table 2). First, we note that only five incidents of mathematical thinking were observed in all the conventional textbook lessons that were analyzed (averaging about 1 incident per 100 minutes of class time). Of these five, none were observed above the epistemic action of *Recognizing*. Thus, the majority of mathematics expressed by students in the conventional textbook environment was limited to recognition and recall of information, the least complex of cognitive operations (Bloom, 1956).[5]

Table 2
Percent of Mathematical Thinking by Class Culture

	Conventional		Reform	
Mathematical Thinking	*Textbook*	*Problem solving*	*Strategy reporting*	*Inquiry/ argument*
Recognizing				
Comprehending	40	38	22	13
Applying	60	24	36	24
Building-with				
Analyzing		33	27	20
Synthetic-analyzing		0	9	16
Evaluative-analyzing		5	4	24
Constructing				
Synthesizing			0	3
Evaluating			2	0
	$n = 5$	$n = 21$	$n = 90$	$n = 148$

Note. Bold print represents categories drawn from Dreyfus, Hershkowitz, and Schwarz, 2001. Standard print represents Williams' (2000) categories adapted from Krutetskii, 1976. *n* represents the raw score for number of incidents of children's thinking.

In the conventional problem-solving lessons, 21 incidents of mathematical thinking were counted (averaging about 19 per 100 minutes). Most of the students' expressed thinking involved comprehending and applying demonstrating more than simple memorization. Furthermore, 38% of the incidents of verbalized mathematical thinking were at the higher level epistemic action of *Building-with: analyzing,* in which children identify parts and the relationship between the parts.

[5] The total number of incidents reported represents the occurrence of mathematical thinking and not the total of all responses that children made. A low frequency of mathematical thinking reflects a high frequency of the category *recall,* defined as rote or recognition, and not considered in this study as mathematical thinking.

These findings indicate that more mathematics was expressed by children in the conventional problem-solving lessons than in the conventional textbook lessons, and this mathematics was at a higher level. The change in the type of problems that children solved (from closed to open-ended) could support higher-level thinking in these lessons and could help explain these differences. We will return to this issue when we discuss the relationship between interaction patterns and student expressed mathematical thinking.

In the reform strategy reporting classes, 92 incidents of expressed mathematical thinking were counted (averaging about 26 per 100 minutes). Of these incidents of mathematical thinking, 57% were the epistemic action of *Recognizing;* 40% were at the higher level of epistemic action *Building-with;* and 2% were at the highest level of epistemic action, *Constructing*. It is interesting to note that the conventional problem-solving lessons were quite similar to the strategy reporting classes by this broad category analysis (62% at *Recognizing*, 38% at *Building-with*). Looking more closely at the specific types of expressed mathematical thinking observed, however, the incidents of verbalized mathematical thinking in the strategy reporting classes were higher *within* each level; in other words, there were more incidents of *Recognizing: applying* or *Building-with: analyzing*. Therefore, children in these classes were more likely to express thinking involving the application of mathematical ideas to new situations, breaking mathematical tasks into component parts, and analyzing the relationship between the parts. This difference in thinking expressed across the three classroom cultures also reflects a shift in emphasis from mathematics expressed as recall of information to comprehension, application, and then to cognitive operations that involve analysis.

In the inquiry/argument classes, 148 expressions of mathematical thinking were counted (averaging about 41 per 100 minutes). Of these mathematical incidents, 36% were involved in the epistemic action of *Recognizing*, while 61% were at the higher level of epistemic action *Building-with*. This is nearly a complete reversal of the strategy reporting results (57% and 40%, respectively). There were also twice as many instances of mathematics at the highest level of epistemic action *Constructing* (3%) for approximately the same class time. Looking within these levels, the mathematics expressed in the inquiry/argument classes was substantially higher than that expressed in the strategy reporting classes. Children's expressed thought in the inquiry/argument classes consisted principally of *Building-with* mathematical ideas focusing most on evaluative-analyzing, indicating that children during these discussions more frequently expressed thought involving the use of more than one method or representation to judge the reasonableness of the mathematics they generated (26%) than those in strategy reporting classes (4%). Furthermore, children's expressed thought consisted almost equally of analyzing (20%) and synthetic-analyzing (16%) involving cognitive operations of analysis of relationships between parts and comparing and contrasting different ways of interconnecting various representations, operations, or assumptions. Thus, a major difference between strategy reporting and inquiry/argument classroom cultures was the emphasis on verbalized mathematical thinking involving synthetic-analyzing and evaluative-analyzing, which is the foundation for connecting and validating mathematical ideas. Taken together, these findings indicate that not only did the occurrence of expressed mathematical thinking increase across the four classroom cultures but so did the quality of thought.

The results reported so far suggest that differences exist across the four classroom cultures in terms of the specific types of interaction patterns that occur, the mathematics expressed, and the kinds of mathematical thinking that children articulated. The identification and categorization of the specific types of the interaction patterns found in the classroom cultures can now be interconnected with the analysis of verbalized children's mathematical thinking, the point of central interest, and compared across the four classroom cultures. The next step toward our goal of understanding how certain interaction patterns might support higher-order mathematical thinking is to examine which kinds of expressed thinking occurred within each interaction pattern.

Distribution of Types of Mathematical Thinking Within Specific Interaction Patterns

Table 3 provides a comparison of the distribution of the types of mathematical thinking expressed within the specific categories of interaction patterns. We include raw score and percent data for the number of interaction patterns and the number of expressions of mathematical thinking counted in each classroom type in order to assist the reader in making comparisons both within a classroom culture and across classroom cultures. The data for each type of verbalized mathematical thinking are reported as raw score data only. This enables the reader to make his/her own proportional comparisons. For example, one can say that in the strategy reporting classroom culture, 25 of the 32 observed expressions of mathematical thinking involving *Building-with: analysis* occurred within the Exploring Methods pattern of interaction. Or one can say that 25 of the 54 expressions of verbalized mathematical thinking observed in Exploring Methods interaction patterns involved *Building-with: analysis*.

As shown in Table 3, in the conventional textbook culture very few incidents of children's mathematical thinking were expressed and only at the level of *Recognizing* during the IRE pattern. The children expressed no mathematical thinking in any of the other interaction patterns. This indicates that these interactions are situations in which children's participation consists of responding to teacher questioning for the purpose of providing known information. In other words, in the textbook classroom culture, all that is expected of a child is to remember and recall information.

In the conventional problem-solving classroom culture, the most frequent pattern of interaction and the one unique to this culture, Hint to Solution, along with the next most frequently occurring pattern, IRE, both of which are teacher dominated, provided little opportunity for children to actively participate in expressing their mathematical thinking. However, over half (12/21, or 57%) of the incidents of children's expressed thinking occurred during the Exploring

Methods interaction, in which children participate by giving their strategies, with a majority of incidents being at higher levels of thinking (applying and analyzing). Returning to the issue of the influence of open-ended problems, it is likely that the change to a more open-ended type of problem does contribute to the expression of mathematical thinking in these lessons, despite the hindrance of the Hint to Solution interaction (Stein, Grover, & Henningsen, 1996). In addition, as can be seen in Table 3, the pattern of participation also changes in this culture to include the interaction pattern, Exploring Methods, which is also the most frequent pattern found in reform-oriented class cultures. This finding becomes even more evident when we consider the relationship of participation and expressed thinking in strategy reporting and inquiry/argument classroom cultures.

As in the conventional problem-solving classroom culture, in the strategy reporting classroom culture children offered most of their contributions during the Exploring Methods interaction (63%). This interaction pattern was dominated by expressed mathematical thinking of applying (25 incidents) and analyzing (13 incidents). The second most occurring interaction pattern, Teacher Elaboration (9 incidents), yielded almost no student contributions of mathematical thinking. Finally, the Argument and Inquiry interaction patterns together occurred 14% of the time. These interaction patterns consisted of situations in which the teacher, as well as the children who were listening to the explanation, asked questions or made challenges. In these interaction patterns, children provided responses of more complex mathematical thinking, *Building-with;* however, children's expressed thinking during Argument and Inquiry seldom extended beyond analyzing to the higher levels of thinking that involve interconnecting ideas (synthetic-analyzing) or the reasoning of justification (evaluative-analyzing).

The data in Table 3 indicate that the Exploring Methods interaction, which distinguishes children's participation in the strategy reporting classroom culture, is also the most frequent interaction pattern in inquiry/argument classrooms and occurs nearly as often as in the strategy reporting; however, 16 of 60 (27%) of the incidents of mathematical thinking that occurred were at higher levels of thinking (synthetic-analyzing and evaluative-analyzing). This is compared to 8 of 54 (15%) found in the strategy reporting classrooms. Examining these results further indicates that in the interaction pattern Argument, a situation involving children participating in challenge and justification, children express more incidents of thinking involving evaluative-analyzing, considering whether a method or result is reasonable. Building Consensus, an interaction pattern unique to this class culture, involves teacher and children participating to establish shared or common meaning for a mathematical idea. There are also more incidents of synthetic-analyzing and evaluative-analyzing. Although Building Consensus occurs less often than Exploring Methods or Argument, the results suggest that this pattern of interaction is one in which children are particularly likely to express higher levels of mathematical thinking. The data indicate that in this classroom culture, children participate more in interaction patterns involving the clarification, justification, and validation of their mathematical

Table 3
Comparison of Children's Levels of Expressed Mathematical Thinking Within Interaction Patterns

	Conventional textbook									Conventional problem solving								
	IP	MT	Mathematical thinking							IP	MT	Mathematical thinking						
Interaction pattern	RS (%)	RS (%)	C	A	AN	SA	EA	SN	E	RS (%)	RS (%)	C	A	AN	SA	EA	SN	E
Collect answers	3(9)	0								1(2)	0							
IRE	17(50)	5(100)	2	3						9(18)	0							
Give expected information	7(21)	0								8(16)	5(24)	4				1		
Funnel	5(15)	0								1(3)	0							
Teacher explain	2(6)	0								3(6)	0							
Hint to solution										19(39)	2(10)	1		1				
Exploring methods										7(14)	12(57)	2	5	5				
Argument										1(2)	2(10)	1		1				
Inquiry																		
Teacher elaboration																		
Proof by cubes																		
Proof by pupil explanation																		
Focus																		
Building consensus																		
Checking consensus																		
Develop concept understanding																		
Pupil self-nominate																		
Totals	34	5	2	3	0	0	0	0	0	49	21	7	5	6	0	1	0	0

Table 3 *(continued)*
Comparison of Children's Levels of Expressed Mathematical Thinking Within Interaction Patterns

	Reform strategy reporting									Reform inquiry/argument								
	IP	MT	Mathematical thinking							IP	MT	Mathematical thinking						
Interaction pattern	RS (%)	RS (%)	C	A	AN	SA	EA	SN	E	RS (%)	RS (%)	C	A	AN	SA	EA	SN	E
Collect answers	9(11)	1(1)		1						15(14)	4(3)							
IRE	2(2)	0								1(1)	0							
Give expected information	1(1)	0																
Funnel	5(6)	0																
Teacher explain																		
Hint to solution																		
Exploring methods	37(44)	54(63)	8	25	13	5	3			38(35)	60(41)	12	16	16	10	6		
Argument	8(9)	15(16)	4	3	5	1	1		1	18(16)	30(20)	3	9	2	4	12		
Inquiry	4(5)	9(10)	1	2	5	1				10(9)	16(11)	2	2	6	2	4		
Teacher elaboration	9(11)	1(1)	1							2(2)	0							
Proof by cubes	5(6)	0								4(4)	3(2)		1			2		
Proof by pupil explanation	4(5)	9(10)	5	1	1	1			1	1(1)	1(1)					1		
Focus	1(1)	1(1)	1							4(4)	6(4)		2		1	2	1	
Building consensus										5(5)	16(11)	2	2	3	3	5	1	
Checking consensus										8(7)	6(4)			1	2	3		
Develop concept understanding										3(3)	2(1)		1					
Pupil self-nominate										1(1)	4(3)					2	2	
Totals	85	92	20	32	24	8	4	0	2	110	148	19	35	29	23	38	4	0

understanding and express higher levels of mathematical thinking than in a strategy reporting classroom culture.

Micro-interpretive Results

The results presented thus far provide a quantitative overview of the fundamental differences in children's patterns of participation and incidents of their expressed mathematical thinking within different classroom cultures. A micro-interpretive analysis of the transcripts provides detailed information about the specific patterns of interaction and the consequent levels of verbalized mathematical thinking that occur in the classroom cultures. We provide examples from the data to illustrate the interrelationship between the interaction patterns and the quality of children's expressed mathematical thinking that characterizes the differences among the classroom cultures of conventional problem solving, strategy reporting, and inquiry/argument. These examples are presented in Figure 2. It is clear from the quantitative results that although some of the same interaction patterns (e.g., Exploring Methods) are found across all three of the classroom cultures, the quality of expressed thinking differs within those patterns. The micro-interpretive analysis of these patterns and the thinking that children verbalize provides further insight into these differences. Each example contains a complete episode wherein a problem is given, children spend time solving the problem, and a complete discussion takes place of the problem; the episode ends when the discussion of the problem is finished.

In the example from the conventional problem-solving classroom culture (see Figure 2), the discussion begins with the teacher reading the problem (line 81) followed by an IRE pattern (lines 81–85), the most frequent interaction pattern found in conventional textbook instruction. The hallmark interaction pattern of this classroom culture (Hint to the Solution) follows (lines 87–89). The teacher begins to eliminate the mathematical challenge of the problem by asking, "Did they give themselves a card?" to which the children need only respond, "No." Next, the teacher asks a rhetorical question that she answers to complete the elimination of the intellectually challenging aspect of the problem: "No. That means Susie gave a card to who? Myra, Sylvia, and Felicia.". It is clear that this pattern diminishes the effectiveness of using an open-ended problem; nevertheless, an Exploring Methods interaction pattern follows (lines 98–103) in which Becky and Karl give explanations expressing thinking at the level of *Recognizing: applying*.

Considering the segments in total, the episode characterizes the way in which the conventional problem-solving discussion contains both conventional and reform types of interaction. It also shows how the Hint to Solution pattern illustrates the tension between conventional and reform types of interaction. As in conventional textbook classes, the teacher wants to ensure that students know how to solve the problem in order to get the correct answer. To the same degree as in reform-oriented classes, the teacher also desires to open the discussion to students' solutions and to engage them in problem solving. But in her efforts to make the

Conventional Problem Solving	**Reform Strategy Reporting**	**Reform Inquiry/ Argument**
Lesson 8 81. *Teacher:* [reads the problem] Sylvia, Myra, Susie, and Felicia exchanged Valentine's Day cards. How many cards were exchanged? **IRE** 81. *Teacher:* Now let's see if we understand the question. How many girls are we talking about? 82. *Class:* 4. *Recall* 83. *Teacher*: 4. Okay. Now. What did they exchange? 84. *Class:* Valentine's Day cards. *Recall* 85. *Teacher:* Valentine's cards. What does exchange mean? **Hint to the Solution** 87. *Teacher*: Trade. Okay. All right. Did they give themselves a card? 88. *Class:* No. 89. *Teacher:* No. That means Susie gave a card to who? Myra, Sylvia, and Felicia. How many cards were exchanged altogether? Okay get with a partner [to solve the problem]. [Solving Time] **Explore Methods** 98. *Teacher:* All right, everybody back to your seats. Wow. That was fantastic. How did you come up with, and most everybody did, that 12 cards were exchanged? Let's talk about the ways. Becky? 99. *Becky:* Um, there were 4 girls, and they didn't give one to themselves, so there was 3 people they would give them to. *Recognizing (comprehending)* 100. *Teacher:* Uh huh. 101. *Becky:* And um, 4 people got 3 cards, that's 3 and 3 for two people is 6, and 3 and 3	Lesson 4 1. *Teacher:* [writes the sentence] The number sentence is 72 minus 39. Would you like to think for a minute and tell me what you think would be a good way to solve that problem? That number sentence. Now remember, I'm asking you for a way of working the problem, aren't I? [Solving Time] **Explore Methods** 2. *Gregg*: I take—um minus um 3 from 7 and that would make it 40. But then you don't have enough to minus a 9 from 2 so you have to go over and get a 10. So then you've got 3 left so that makes it 30. And then you have 12 left and then you minus 2 and then you got 7 from minus that. And then you, you just, you already know that if you minus 7 from 10 you've got 3, so it equals 33. *Building with (synthetic analyzing) [two processes considered simultaneously to arrive at a solution. Recognizing each idea contributed nested within synthetic analyzing].* 22. *Kent:* I just took 72 minus 30 is 42, then I took off 2 from the 9 and it's um, 40 and then I took off the other 7 and that's 33. *Building with (application) [applied a known procedure].* 43. *Sara:* Um, I almost did it like Gregg except I did 72 minus 30. And then I um, then I took off 10 and then I got uh, 40. *Building with (incomplete expression of synthetic analyzing) [synthetic analyzing: comparison of two methods.*	Lesson 4 [Solving Time] 1. *Teacher:* [reads problem]. Okay, it says—this many cherry candies [84] are in the candy shop, you can sell 68 and we want to know how many are left? [Draws a picture of 8 rolls and 4 single candies on overhead and writes 68.] Johnny, do you want to start? **Explore Methods** 31. *Teacher:* I think that's what we've got so far with the green number sentence. [80 – 60 = 20]. Isn't that kind of what you just said? Can that help us do the other one? 33. *Teacher:* How can it help us Mark? 34. *Mark:* You could like have 80 minus 60 equals 20, then you could have, say you added 4, then take away 8, it would be 16. *Explanation* 35. *Teacher*: I'm sorry. Will you say that one more time please? 36. *Mark:* 80 minus 60 would equal 20, then you take 20 minus 4 would be 16. *Building-with (synthetic analyzing) [two processes considered simultaneously to arrive at a solution; nested Recognizing of their relevance].* 37. *Teacher:* So you're saying add 4 on to what? 38. *Mark:* Add 4 on to 20 would be 24, then take away 8. **Inquiry** 39. *Teacher:* Why do you do that? Why do you add 4 onto 20? 40. *Mark*: Well I thought it would be easier. *Building-with (incomplete expression of evaluative-analyzing; reason required but not given).* 41. *Teacher:* [to the class].

Conventional Problem Solving	Reform Strategy Reporting	Reform Inquiry/ Argument
for two more people is 12. *Recognizing (applying)* 102. *Teacher:* Very good. So you added up 3 four times, right, Becky? [she nods]. Anybody else do it a different way? Karl? 103. *Karl:* Well, I did it the same way, but I did like three times, so like one would, one person would have to, one people, one person would have to give 3, and then another person would have to give 3, and 3 times 4 is 12. *Recognizing (applying)* 104. *Teacher:* Very good. Okay. Anybody else have a different way they got it? [no hands] Okay.	*Recognizing required identification of features of similarity and difference (not given)].* And then, no, I took off 10 and I got 30. Wait. 44. *Teacher:* You took 72 minus 30. Then you got 42, then you took off 10. Why did you take off 10? Do you remember why you took off 10? 45. *Sara*: Yeah, because it was easier. It was close to 9. *Building with (evaluative analyzing). [Synthetic analyzing for the purpose of judgment nested within evaluative analyzing].* 46. *Teacher:* Yeah. Because 9's close to 10. Okay. So when you take off 10 what do you get? 47. *Sara:* 52, no I mean 32. 48. *Teacher*: 32. And then? 49. *Sara:* Then I added 1. 50. *Teacher*: You added 1 because you were only taking off 9 instead of 10. [(Teacher writes on the overhead: (72 – 30 = 42. 42 – 10 = 32. 32 + 1 = 33).] Okay. Did anybody think, in case you didn't have a strategy before, while you heard these other people telling about their strategies did you think of one as we were going along? Is there another way we can do this problem? No more different ways? 53. *Tracey:* I was doing it sort of like Gregg, but not exactly. *Building with (incomplete expression of synthetic analyzing) [as for Sara, 43 above, synthetic analyzing: comparison of two methods; nested Recognizing required feature/s of similarity and different to be identified].* 54. *Teacher:* Gregg's is right here. [(points to number sentence written on overhead)]. You were taking 70 minus 30.	What do you think about that idea? Does adding 4 onto 20 make sense? 42. *Brian*: I don't get it. *Prompt Recognizing (comprehending)* 43. *Teacher:* Why did you add 4—on to 20? You said it's easier. Can you tell us more? 44. *Mark:* 'Cause if you just did that [points to the 8] it would be about 12. *Building-with (evaluative-analyzing) considering another way to check the reasonableness of some mathematics.* 45. *Teacher:* You've already subtracted the tens haven't you? To get 20? 46. *Mark:* Yeah, I used it to get these [points to 4 in 84 and 8 in 68]. 47. *Teacher*: Does that make sense now? [to class] So you're saying we would have 24 minus 8? 48. *Mark:* Yeah. **Checking for Consensus; Building Consensus** 51. *Teacher:* Does anyone have a question for Mark? 52. *Claire:* Why didn't you subtract 4 instead of add 4? *Building-with (analyzing)* 53. *Mark:* If you— I just wanted to take away 8 because it'd be a lot easier . . . *Building-with (incomplete expression of evaluative-analyzing, reasoning required but not given).* 54. *Teacher:* Mark, can you show us in the picture, taking away 60? Can you mark them out or something so we can see it? *[prompt for expression of evaluative analysis]* In the picture, up at the top where the rolls of candy are. [The picture shows 8 rolls of

Conventional Problem Solving	Reform Strategy Reporting	Reform Inquiry/ Argument
	55. *Tracey:* Well just, see I took. Yeah it was 70 minus 30, but first I took the total, that 70. Then I took 30. And I subtracted um, 30 from 70 and I got 40. Then I subtracted 9; (pause) and I got 31. Then, and then I subtracted [shakes her head no]. I added 2. *Recognizing, comprehending* [Teacher writes on the overhead as the student talks: (70 = 30 = 40. 40 – 9 = 31. 31 + 2).] 56. *Teacher:* Tracey's got a small voice. And I heard what she said, but you might not have understood. She took—she took the 2 off of 72 and off the 39. [Teacher is pointing at number on the overhead.] And so she took 70 minus 30 and got 40. Then she subtracted the 9 out of the 39 and got 31. Then she went back and added the 2 that was in 72, and got 33, for her answer. [Writes on the overhead 33.] Okay.	candy and 4 single candies.] 55. *Mark:* 10, 20, 30, 40, 50, 60 [crosses out each of the 6 rolls]. So you had 10, 20, you could add 4 more on to get 24. 56.: *Teacher:* Okay. Why are we going to add 4 more on? [short pause] Does the picture help you see that? [short pause] What do you have left there besides 20? 57. *Mark:* 4. So minus 8, it should be 16. *Building with (synthetic analyzing) [more than one process considered simultaneously to arrive at a solution; Recognizing each idea contributed nested within synthetic analyzing].* 60. *Teacher:* [to the class] So he says it should be 24 minus 8. 61. *Class:* Agree.

Figure 2. Classroom examples of interaction patterns and children's expressed thinking.

problem "understandable," she inadvertently creates a situation in which the students no longer need to engage in thinking of *Building-with: analysis* in order to solve the problem. Thus, when children express their thinking in the Exploring Methods pattern, all that is required of them to solve the problem is to apply learned information.

Turning now to discussions in reform-oriented classroom cultures, we present examples to illustrate important differences. The reader will recall that the standard algorithms for addition and subtraction were not taught in these classrooms; instead, children were encouraged to invent strategies to solve problems. In Figure 2, the example highlights the reform strategy reporting classroom culture. Similar to the previous example, the teacher reads the problem that students are to solve mentally without paper and pencil (line 1), which is the number sentence 72 – 39 = ___. Solving time is given, and the discussion follows. The whole of the episode consists of an Exploring Methods pattern that begins with Gregg (line 2), followed by two other children, and then Kent (line 22). Kent solves the problem as 72 minus 30 is 42. He cancels out the 2 in 42 to get 40 and a 2 in 9 to get 7. Then he subtracts 7 from 40 to get 33.

Continuing (lines 43–55), Sara states that her way is similar to Kent's because she did 72 minus 30, but different because she then took off 10 instead of working with the 9 as Kent did. Reporting her strategy thus far, Sara is expressing thinking that initially involved comparison of two methods for the difference, *Building-with: synthetic-analyzing,* and *Recognizing* some feature of similarity between Kent's method and her own (line 43). As she continues explaining, she becomes confused about her strategy. Here the teacher steps in to support (line 44) by summarizing and asking the question, "Why did you take off 10?" This requires Sara to give a reason (line 45): "Yeah, because it was easier" (evaluative-analysis). "It was close to 9" (synthetic-analysis). She then takes 10 from 42 to get 32, adds the 1 (from the initial move to make ten), and gets 33. To use this solution strategy requires a sense of number relationships, the recognition that "making 10" makes mental computation easier (Recognizing), but then flexibility in thinking to realize that the number 1 needs to be added to the sum to make 33 (synthetic-analyzing, considering the ease of working in 10s in conjunction with remembering to increase by 1 because there were only 9).

Unlike in the conventional problem-solving lesson, the teacher does not provide hints for how to solve the problem; instead, the teacher supports students as they explain by asking questions that require further explanation, clarification, and reasoning. Moreover, the Exploring Methods pattern of interaction, as the most frequent pattern in a reform strategy reporting classroom culture, requires a different level of participation. In this interaction pattern, children's participation requires reporting a different way of solving the problem and acknowledging in what way their solution is the same but different (synthetic-analyzing) from a strategy previously given. Confusion is allowed and support is given by the teacher to help resolve the situation. When participating in explaining a strategy, the child knows that further questions might be asked but usually only from the teacher. These teacher questions ask for further clarification and reasoning, which require more complex cognitive operations such as synthetic-analyzing in response. However, as illustrated in this episode, participation in this classroom culture is typically limited to a dyadic exchange between the child explaining and the teacher.

The way of participating and the thinking expressed during these Exploring Methods patterns are more demanding than in the conventional problem-solving classroom culture, with the teacher assuming a supportive role of summarizing and elaborating on the child's explanation (lines 46, 50, 56) to provide or fill in important information and connections that the child explaining did not provide. It appears that the teacher's purpose in providing additional information and relationships is to ensure that those children listening understand the strategy given. This inadvertently creates a situation in which those listening need pay attention but only for the purpose of determining whether their solution method is the same and/or different in order to volunteer to report. Thus, when children express their thinking in this classroom culture, it seldom consists of evaluative analysis as is found in the inquiry/argument classroom culture shown in the next episode (see Figure 2).

In the beginning of the inquiry/argument example (lines 31–38) the interaction pattern of Exploring Methods is similar to that found in strategy reporting discussions; Mark presents his solution strategy to the problem 84 – 68 as "80 minus 60 equals 20," add 4, and "then take away 8, it would be 16" (line 34). However, when asked by the teacher, "Will you say that one more time please?" Mark provides a different strategy, "20 minus 4 would be 16," similar to the canceling strategy of Kent in the previous episode. In the next exchange (lines 37, 38), Mark returns to his original strategy of adding 4 to 20 and subtracting 8. At this point, the episode shifts to an inquiry pattern of interaction (lines 39–48). During this exchange, Mark is asked to give a reason for adding 4 to 20, to which he responds with "it would be easier" but does not provide a reason why it is easier. The teacher, rather than providing further questioning, turns the discussion over to the listeners (line 41), at which point Brian says he does not understand (line 42).

Following Mark's explanation, the teacher again turns the discussion to the listeners (line 51), and the interaction shifts to Checking for Consensus and Building Consensus. Given Mark's different strategy solutions, Claire asks a crucial question: "Why didn't you subtract 4 instead of add 4?" The nature of Claire's question requires Mark to respond with a reason, but his response involves only a partial explanation of his reasoning (lines 52, 53). The teacher, realizing that his explanation does not answer Claire's question, asks Mark to relate his method to the original drawing of rolls and candies on the overhead projector in order to show why he added 4. This is an attempt to interconnect strategies, by asking Mark to show and perhaps discuss how his method and the strategy of drawing single candies and rolls of candies (a strategy known and used in this class) are the same (lines 54–57) (synthetic-analyzing, showing connections between two representations when describing a solution) and the class agrees (lines 60–61). Because Mark still does not give a reason for his two strategies or why one strategy is easier, and the class only responds with "agree," we can only say that the interaction is an attempt at building consensus and that the interactive constitution of shared meaning likely is incomplete.

This episode illustrates the major differences between the two reform-oriented classroom cultures strategy reporting and inquiry/argument. This is illustrated by interaction patterns that provide listeners with opportunities to participate by asking questions of the child giving an explanation. Moreover, the thinking that children expressed consists of incidents of *Building-with: evaluative analysis,* making judgments about the reasonableness of a method or result after simultaneously considering the problem through two different representations.

Summary

The data reported here show the differences between conventional and reform-oriented classrooms in the quality of mathematical thinking expressed and the nature of the interaction patterns that occur during class discussion. Results of the analysis showed that children in the conventional textbook classroom culture were most often engaged in recalling previously taught information. It is generally accepted that recall or recognition only requires remembering the information and does not qualify as a form of higher-order thinking (Resnick, 1987). The results of the data from the conventional problem-solving classroom culture provided insights into the tension between conventional and reform types of interaction. As we discussed previously, although the goal was to create an open discussion of students' solutions and to engage them in problem solving, the teacher's efforts to make the problem "understandable" unintentionally created a situation in which the mathematical challenge of the problem was removed, eliminating the need for students to engage in higher-level thinking.

The most important results of the study were the findings that revealed the nature of the important differences that existed between reform-oriented classroom cultures. As we reported, in the strategy reporting classroom culture the established pattern of interaction is dyadic between the child explaining and the teacher. This provided a situation in which the individual explaining expressed quality thinking but did not foster the development of collaboration that is central to inquiry instruction (Palincsar & Herrenkohl, 2002). Only in the inquiry/argument classroom culture were there opportunities for all children to be involved in meaning making and to develop a common ground on which to build shared understanding (Rommetveit, 1974). In the inquiry/argument classroom culture, children's thinking was extended to include whether a method or result is reasonable (evaluative-analysis), the pulling together of ideas for making a judgment (synthetic-analysis), and identifying flaws (evaluative-analysis) and strengthening arguments by considering the mathematics from a different perspective (synthetic-analysis, evaluative-analysis)—all as a process for establishing shared mathematical meaning.

DISCUSSION AND CONCLUSION

In this article, we undertook to examine the interrelationship between types of interaction patterns and the nature of children's mathematical thinking expressed within these patterns. Our fundamental argument—that social interaction patterns established in the classroom specifically affect how children construct mathematical knowledge in that classroom—is supported by our data. Those interaction patterns that required greater involvement from the participants were related to higher levels of expressed mathematical thinking by children. However, this study has limitations. One limitation is that the results are based on an empirical analysis of a small sample of reform classes selected from one specific approach to reform. Although we attempted to account for this limitation by corroborating the analysis with additional data from both conventional and other reform classes, there is still the possibility that the results are biased and represent only processes found in the classes used in the study. Another limitation is that these results only provide insight into children's mathematical thinking as it is expressed during class discussion and examines neither the reasoning of individual children nor the development of their mathematical thought in the various classroom cultures.

Our primary interest in mathematics classrooms centers on a concern for the intellectual development of children's mathematical thought. In this article, we offer a conceptual framework that consists of interaction and thinking dimensions and analysis of classroom data in an attempt to explain children's development of mathematical thinking and reasoning that capitalizes on the importance of humans learning through each other. Although we provided evidence of the relationship between interaction patterns and children's verbalization of mathematical thinking, we have not provided an explanation for how or why these are related. For this, we return to the work of Bruner, along with Tomasello and Hobson, on the role of social cognition in the development of thought in general and mathematics specifically. Bruner (1996) argued that there is always idiosyncratic thought in humans, but meaning making in culture is the building of common ground found in interpersonal interaction. He emphasized that children must have certain "social cognitive skills" in order to enter into joint attentional interactions with adults and to understand adult communicative intentions that are central in learning through others. The groundwork for meaning making and building of common ground occurs in early child and caretaker interactions, through very highly structured formats and with sensitive adults as support. As children become more skillful at maintaining joint attention and determining adult communicative intentions in a wider variety of interactive situations, highly structured formats become less crucial to the process (Tomasello, 2001).

By applying these ideas to the development of children's mathematical thinking in classrooms, the interaction patterns in which children express their mathematical thinking can be viewed as less structured "formats," which, once established, become reliable repeatable routines that constitute the social structure of the class. The meaning making found in interpersonal interaction relies on these interaction patterns because they facilitate children's social cognitive capacity to determine the intentions of the others, including the teacher (Tomasello, 2001). Therefore, it is not interaction patterns per se that affect the development of mathematical thought but the fact that these patterns represent specific types of interpersonal interactions through which meaning making and experiences are shared (Hobson, 2004). Therefore, the requirements of interpersonal interaction (the turn taking and questioning) affect how children think. It is important, then, to examine children's expressed mathematical thinking within the context of these interaction patterns to better understand the development of mathematical thought. Expressed theoretically, social processes, social norms, including discipline-specific norms, and interaction patterns, help to delineate the interpersonal interaction or participation structures created in different classroom environments; the cognitive processes, epistemic actions, and types of mathematical thinking involved in the mental activity of abstraction and generalization account for the construction of mathematical knowledge. However, it is the social cognitive processes of joint attention and understanding of others' communicative intentionality that is the medium by which mathematical thought develops through meaning making with others. Although all three processes are essential to the development of mathematical thought, it is the uniqueness of human beings' capacity for social cognition (Tomasello, 1999) that enables the connection between social interaction and cognition.

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