A SUMMATION FORMULA FOR MACDONALD POLYNOMIALS

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ABSTRACT. We derive an explicit sum formula for symmetric Macdonald polynomials. Our expression contains multiple sums over the symmetric group and uses the action of Hecke generators on the ring of polynomials. In the special cases $t = 1$ and $q = 0$, we recover known expressions for the monomial symmetric and Hall–Littlewood polynomials, respectively. Other specializations of our formula give new expressions for the Jack and q–Whittaker polynomials.

1. INTRODUCTION

Symmetric Macdonald polynomials [10, 11] are a family of multivariable orthogonal polynomials indexed by partitions, whose coefficients depend rationally on two parameters q and t. In the case $q = t$ they degenerate to the celebrated Schur polynomials, which are central in the representation theory of both the general linear and symmetric groups. Using the notation m_λ to denote the monomial symmetric polynomial indexed by a partition λ, the standard definition of Macdonald polynomials in the literature is the following.

Definition. *The Macdonald polynomial $P_\lambda(x_1, \ldots, x_n; q, t)$ is the unique homogeneous symmetric polynomial in $(x_1, \ldots, x_n)$ which satisfies*

$$P_\lambda(x_1, \ldots, x_n; q, t) = m_\lambda(x_1, \ldots, x_n) + \sum_{\mu < \lambda} c_{\lambda,\mu}(q,t) m_\mu(x_1, \ldots, x_n), \qquad \langle P_\lambda, P_\mu \rangle = 0, \ \lambda \neq \mu,$$

with respect to the Macdonald inner product on power sum symmetric functions ([11], Chapter VI, Equation (1.5)), and where $<$ denotes the dominance order on partitions ([11], Chapter I, Section 1).

Up to normalization, Macdonald polynomials can alternatively be defined as the unique eigenfunctions of certain linear difference operators acting on the space of all symmetric polynomials [11]. They can also be expressed combinatorially as multivariable generating functions [5, 6, 13].

Opdam [12] and Cherednik [2, 3] generalized the earlier theory of Macdonald to a non-symmetric setting, and defined non-symmetric Macdonald polynomials E_μ that are indexed by compositions μ. Non-symmetric Macdonald polynomials are defined as joint eigenfunctions of a family of commuting operators in the double affine Hecke algebra [2, 3], and the symmetric polynomial P_λ is obtained as the sum over all E_μ whose index μ lies in the orbit of λ under the symmetric group.

The purpose of this article is to state and prove an explicit sum formula for calculating Macdonald polynomials. This result is derived directly from the recent work [1].

2. NOTATION

2.1. **Partitions and compositions.** Fix a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Let $\ell(\lambda)$ denote the length of λ, which is the number of non-trivial parts $\lambda_i > 0$ in λ. We associate a monomial x_λ in the variables $(x_1, \ldots, x_n)$ to any partition, which is simply $x_\lambda = \prod_{i=1}^{\ell(\lambda)} x_i$.

We write $\lambda' = (\lambda_1', \ldots, \lambda_r')$ for the partition conjugate to λ, as is standard in the literature.

Let λ be a partition whose largest part is $\lambda_1 = r$. For all $0 \leqslant k \leqslant r$, we define a partition $\lambda[k]$ which is obtained by replacing all parts in λ of size $\leqslant k$ with 0. For example, for $\lambda = (3, 3, 2, 1, 1, 1, 0)$ we have

$$\lambda[0] = (3, 3, 2, 1, 1, 1, 0), \quad \lambda[1] = (3, 3, 2, 0, 0, 0, 0),$$
$$\lambda[2] = (3, 3, 0, 0, 0, 0, 0), \quad \lambda[3] = (0, 0, 0, 0, 0, 0, 0),$$

and in general $\lambda[0] = \lambda$, $\lambda[r] = 0$.

We will also consider compositions $\mu = (\mu_1, \ldots, \mu_n)$ whose parts satisfy $\mu_i \geqslant 0$, but are not ordered in any particular way. We write μ^+ to denote the unique partition obtained by arranging the parts of μ in weakly decreasing order. For μ an arbitrary composition, the part-multiplicity function $m_i(\mu)^1$ is defined as follows:

$$m_i(\mu) = \#\{\mu_k : \mu_k = i\}.$$

Further to this, two functions which take a pair of compositions as arguments will be used in our main formula:

$$a_i(\lambda, \mu) = \#\{(\lambda_k, \mu_k) : \lambda_k = 0, \ \mu_k = i\}, \quad b_i(\lambda, \mu) = \#\{(\lambda_k, \mu_k) : i = \lambda_k > \mu_k\}.$$

For example, for $\lambda = (4, 4, 3, 3, 3, 2, 0, 0, 0)$ and $\mu = (0, 3, 0, 0, 3, 0, 4, 3, 4)$, we have

$$a_i(\lambda, \mu) \equiv a_i \begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 & 4 & 3 & 4 \end{pmatrix} = \begin{cases} 0, & i = 1 \\ 0, & i = 2 \\ 1, & i = 3 \\ 2, & i = 4 \end{cases}$$

$$b_i(\lambda, \mu) \equiv b_i \begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 & 4 & 3 & 4 \end{pmatrix} = \begin{cases} 0, & i = 1 \\ 1, & i = 2 \\ 2, & i = 3 \\ 2, & i = 4 \end{cases}$$

2.2. **Permutations.** Let S_n be the group of all permutations of $(1, \ldots, n)$. For any partition λ, let S_λ be the quotient of S_n by the subgroup S_n^λ of permutations which leave λ invariant. $S_\lambda = S_n/S_n^\lambda$ is thus the set of all permutations which have a distinct action on λ. For example, for $\lambda = (2, 2, 0, 0)$ we find

$$S_\lambda = \{(1, 2, 3, 4), \ (1, 3, 2, 4), \ (1, 3, 4, 2), \ (3, 1, 2, 4), \ (3, 1, 4, 2), \ (3, 4, 1, 2)\}$$

as the complete set of permutations with a distinguishable action on λ.

2.3. **Hecke algebra.** We consider polynomial representations of the Hecke algebra of type A_{n-1}, with generators T_i given by

$$(1) \qquad\qquad T_i = t - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i), \qquad 1 \leqslant i \leqslant n - 1,$$

where s_i is the transposition operator with action $s_i f(\ldots, x_i, x_{i+1}, \ldots) = f(\ldots, x_{i+1}, x_i, \ldots)$ on functions in $(x_1, \ldots, x_n)$. It can be verified that the operators (1) indeed give a faithful representation of the Hecke algebra:

$$(2) \qquad (T_i - t)(T_i + 1) = 0, \qquad T_i T_{i\pm1} T_i = T_{i\pm1} T_i T_{i\pm1}, \qquad T_i T_j = T_j T_i, \ |i - j| > 1.$$

In view of the relations for the generators, we can define T_σ unambiguously as any product of simple transpositions T_i which gives the permutation σ. For example, for $\sigma = (3, 2, 1, 4)$ we can write

$$T_\sigma = T_2 T_1 T_2 = T_1 T_2 T_1,$$

with both expressions having an equivalent action on the ring of polynomials in $(x_1, \ldots, x_n)$.

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[1]Not to be confused with the standard notation m_λ for monomial symmetric polynomials.

3. Main formula

Theorem. *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition with largest part $\lambda_1 = r$, and from this define a set of partitions $\lambda[0], \ldots, \lambda[r]$ as in Section 2.1. Then the Macdonald polynomial P_λ can be written in the form*

$$
(3) \qquad P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma \in S_\lambda} T_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} \left(\sum_{\sigma \in S_{\lambda[i]}} C_i \begin{pmatrix} \lambda[i-1] \\ \sigma \circ \lambda[i] \end{pmatrix} T_\sigma \circ x_{\lambda[i]\circ} \right) 1
$$

with coefficients[2] that satisfy $C_i(\lambda, \mu) = 0$ if any $0 < \lambda_k < \mu_k$, and

$$
(4) \qquad C_i(\lambda, \mu) \equiv C_i \begin{pmatrix} \lambda_1 \cdots \lambda_n \\ \mu_1 \cdots \mu_n \end{pmatrix} = \prod_{j=i+1}^{r} \left(q^{(j-i)a_j(\lambda,\mu)} \prod_{k=1}^{b_j(\lambda,\mu)} \frac{1 - t^k}{1 - q^{j-i}t^{\lambda_i' - \lambda_j' + k}} \right),
$$

otherwise.

In order to clarify the structure of the expression (3), we give explicit examples in Appendix A, where we calculate some Macdonald polynomials of sufficiently small size. We give its proof in Section 4.

We wish to point out that (3) has many structural features in common with the work of Kirillov and Noumi [8, 9]. In these papers the authors construct families of raising operators, which act on Macdonald polynomials by adding columns to the indexing Young diagram. In [8] the raising operators have an analogous form to Macdonald q-difference operators, while in [9] the raising operators are constructed in terms of generators of the affine Hecke algebra. In both papers the Macdonald polynomial is obtained by the successive action of such raising operators on 1, the initial state.

It would be very interesting to find a precise connection between the results of [8, 9] and our formula (3), although this is likely to be a difficult task. In Appendix B we give examples of the Kirillov–Noumi construction [9] for the same Macdonald polynomials as those considered in Appendix A. These simple examples suffice to illustrate that our formula (3) is not equivalent to that of [9], and we elaborate the key differences in Appendix C.

4. Proof

In this section we sketch the proof of (3), citing results from our earlier paper [1], where we obtained a matrix product formula for a family of non-symmetric polynomials $f_\lambda(x_1, \ldots, x_n; q, t)$. These polynomials are indexed by compositions λ, and satisfy the following relations with the generators of the Hecke algebra:

$$
T_i f_{\lambda_1, \ldots, \lambda_n}(x_1, \ldots, x_n; q, t) = f_{\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_n}(x_1, \ldots, x_n; q, t), \qquad \text{when } \lambda_i > \lambda_{i+1}.
$$

These relations allow us to express a polynomial f_μ indexed by an arbitrary composition μ in terms of Hecke generators acting on f_{μ^+}. As was demonstrated in [1], the symmetric Macdonald polynomial P_λ is obtained as a sum over all polynomials f_μ, whose composition is a distinct permutation of the partition λ:

$$
(5) \qquad P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\sigma \in S_\lambda} f_{\sigma \circ \lambda}(x_1, \ldots, x_n; q, t) = \sum_{\sigma \in S_\lambda} T_\sigma \circ f_\lambda(x_1, \ldots, x_n; q, t).
$$

We now recall the matrix product expression for f_λ as obtained in [1]. Assume $\lambda \subseteq r^n$ and introduce the following family of $(r - s + 2) \times (r - s + 1)$ operator-valued matrices $L^{(s)}(x)$, $1 \leqslant s \leqslant r$. We

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[2]We will use three notations for the coefficients interchangeably: $C_i(\lambda, \mu) \equiv C_i \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \equiv C_i \begin{pmatrix} \lambda_1 \cdots \lambda_n \\ \mu_1 \cdots \mu_n \end{pmatrix}$.

index rows by $i \in \{0, s, \ldots, r\}$ and columns by $j \in \{0, s+1, \ldots, r\}$[3], and take

$$L_{00}^{(s)} = 1, \qquad L_{0j}^{(s)} = \phi_j, \text{ for } s+1 \leqslant j \leqslant r, \qquad L_{i0}^{(s)}(x) = x \times \begin{cases} \prod_{l=i+1}^{r} k_l, & i = s \\ \\ \phi_i^\dagger \prod_{l=i+1}^{r} k_l, & s < i \leqslant r \end{cases}$$

(6)
$$L_{ij}^{(s)}(x) = x \times \begin{cases} \prod_{l=i+1}^{r} k_l, & i = j \\ \\ \phi_i^\dagger \phi_j \prod_{l=i+1}^{r} k_l, & i > j \\ \\ 0, & i < j \end{cases} \quad \text{for } s \leqslant i \leqslant r, \ s+1 \leqslant j \leqslant r.$$

We also introduce a twist operator

(7)
$$S^{(s)} = \prod_{l=s+1}^{r} k_l^{(l-s)u} \text{ for } 1 \leqslant s \leqslant r-1, \qquad S^{(r)} = 1,$$

where we perform the re-parametrization $q = t^u$. The operators $\{k, \phi, \phi^\dagger\}$ are generators of the t-boson algebra:

(8)
$$\phi k = t k \phi, \qquad t \phi^\dagger k = k \phi^\dagger, \qquad \phi \phi^\dagger - t \phi^\dagger \phi = 1 - t,$$

with subscripts used to denote commuting copies of the algebra. Note that all operators in $L^{(s)}(x)$ implicitly also carry an index (s), as we will assume that operators in $L^{(s)}(x)$ and $L^{(s')}(x)$ (as well as $S^{(s)}$ and $S^{(s')}$) commute for $s \neq s'$.

¿From this we construct an $(r+1)$-dimensional vector $\mathbb{A}(x)$ and a total twist operator $\mathbb{S}$:

$$\mathbb{A}(x) = L^{(1)}(x) \ldots L^{(r)}(x), \qquad \mathbb{S} = S^{(1)} \ldots S^{(r)}.$$

We denote the components of $\mathbb{A}(x)$ by $A_i(x)$. The principal result of [1] was the formula

(9)
$$\Omega_{\lambda^+}(q, t) f_\lambda(x_1, \ldots, x_n; q, t) = \text{Tr}\left[A_{\lambda_1}(x_1) \ldots A_{\lambda_n}(x_n) \mathbb{S}\right],$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is any composition consisting of n parts $\lambda_i \geqslant 0$, $\Omega_{\lambda^+}(q, t)$ is an overall normalization which only depends on the partition λ^+. Here we trace over suitable representations of the (multiple copies of the) t-boson algebra (8). Up to changes of the normalization Ω_{λ^+}, equation (9) is independent of the choice of these representations as long as they are faithful, so we choose the Fock representation for all copies of the t-boson algebra:

(10)
$$\phi |m\rangle = (1 - t^m) |m-1\rangle, \qquad \phi^\dagger |m\rangle = |m+1\rangle, \qquad k |m\rangle = t^m |m\rangle,$$
$$\langle m| \phi = (1 - t^{m+1}) \langle m+1|, \qquad \langle m| \phi^\dagger = \langle m-1|, \qquad \langle m| k = t^m \langle m|,$$

for which the correct normalization[4] in (9) is

(11)
$$\Omega_{\lambda^+}(q, t) = \prod_{i=1}^{r} \prod_{j=i+1}^{r} \frac{1}{1 - q^{j-i} t^{(\lambda^+)'_i - (\lambda^+)'_j}},$$

where r is the largest part of λ.

A useful way of decomposing (9), which is central to our proof of (3), is in terms of the transition matrix elements

$$T_{\lambda, \mu}^{(s)}(x_1, \ldots, x_n) = \text{Tr}\left[L_{\lambda_1, \mu_1}^{(s)}(x_1) \ldots L_{\lambda_n, \mu_n}^{(s)}(x_n) S^{(s)}\right],$$

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[3]This unusual indexing of the entries of the matrix is the most convenient for our purposes, since we ultimately want to identify these matrix elements with the partitions $\lambda[s]$ introduced in Section 2.1.

[4]This normalization is chosen such that f_λ is monic, or in other words, the coefficient of its leading monomial $x_1^{\lambda_1} \ldots x_n^{\lambda_n}$ is 1. This is clearly the normalization required for (5) to be valid.

where λ and μ are compositions taking values in $\{0, s, \ldots, r\}$ and $\{0, s+1, \ldots, r\}$, respectively. Using this definition we are able to write down the recursion relation

$$(12) \qquad \Omega_{\lambda+}^{(s)}(q,t) f_{\lambda}^{(s)}(x_1, \ldots, x_n; q, t) = \sum_{\mu} T_{\lambda,\mu}^{(s)}(x_1, \ldots, x_n) \Omega_{\mu+}^{(s+1)}(q,t) f_{\mu}^{(s+1)}(x_1, \ldots, x_n; q, t),$$

where the sum is taken over compositions μ such that $m_i(\lambda) = m_i(\mu)$ for all $s+1 \leqslant i \leqslant r$. Here $f_{\lambda}^{(s)}(x_1, \ldots, x_n; q, t)$ denotes a reduced version of (9), in which the operators $A_i(x)$ are components of $\mathbb{A}^{(s)}(x) \equiv L^{(s)}(x) \ldots L^{(r)}(x)$, the twist is given by $\mathbb{S}^{(s)} \equiv S^{(s)} \ldots S^{(r)}$, and

$$\Omega_{\lambda+}^{(s)}(q,t) = \prod_{i=s}^{r} \prod_{j=i+1}^{r} \frac{1}{1 - q^{j-i} t^{(\lambda+)'_i - (\lambda+)'_j}}.$$

To complete the proof of (3), we establish the following result.

Lemma. *Let λ be a partition with parts in $\{0, s, \ldots, r\}$, and μ a composition with parts in $\{0, s+1, \ldots, r\}$, such that $m_i(\lambda) = m_i(\mu)$ for all $s+1 \leqslant i \leqslant r$. Then*

$$(13) \qquad T_{\lambda,\mu}^{(s)}(x_1, \ldots, x_n) = \frac{x_\lambda C_s(\lambda, \mu) \Omega_{\lambda}^{(s)}(q,t)}{\Omega_{\mu+}^{(s+1)}(q,t)},$$

where $C_s(\lambda, \mu)$ is given by (4).

Proof. We begin by noticing that, by virtue of the vanishing of some of the matrix elements (6), $T_{\lambda,\mu}^{(s)} = 0$ if $0 < \lambda_k < \mu_k$ for some k. Hence both sides of (13) have the same vanishing property. This constrains $\lambda_k \geqslant \mu_k$ for all $1 \leqslant k \leqslant \ell(\lambda)$, which we assume from this point on. Given that λ is a partition, using the explicit form of the $L^{(s)}$ matrix entries we find that

$$(14) \quad T_{\lambda,\mu}^{(s)}(x_1, \ldots, x_n) = \prod_{i=1}^{\ell(\lambda)} x_i \times \mathrm{Tr} \left[\left(\prod_{l=s+1}^{r} k_l^{m_s(\lambda)} \right) \left(\prod_{j=s+1}^{r} \phi_j^{a_j(\lambda,\mu)} \right) S^{(s)} \left((\phi_r^\dagger)^{b_r(\lambda,\mu)} \prod_{j=s+1}^{r-1} \phi_j^\# \right) \right.$$

$$\left. \times \left(k_r^{m_{r-1}(\lambda)} (\phi_{r-1}^\dagger)^{b_{r-1}(\lambda,\mu)} \prod_{j=s+1}^{r-2} \phi_j^\# \right) \cdots \left(\prod_{l=s+2}^{r} k_l^{m_{s+1}(\lambda)} (\phi_{s+1}^\dagger)^{b_{s+1}(\lambda,\mu)} \right) \right]$$

where we use $\#$ to denote a multiplicity in the composition μ which does not affect the final answer, so we suppress it. Note that we have used the cyclicity of the trace to reorder the product, so that the leftmost parentheses correspond with parts of λ of size s, the next parentheses with parts of size 0, followed by the twist, followed by parts of size r and decreasing thereafter down to parts of size $s+1$.

The expression (14) is seemingly very complicated. But in view of the fact that operators with different subscripts commute, and the simple commutation (8) between operators from the same copy of the algebra, as well as the factorized form (7) of the twist, we can factorize (14) as follows:

$$(15) \quad T_{\lambda,\mu}^{(s)}(x_1, \ldots, x_n) = \prod_{i=1}^{\ell(\lambda)} x_i \times \prod_{j=s+1}^{r} q^{(j-s)a_j(\lambda,\mu)} \mathrm{Tr} \left[\phi_j^{b_j(\lambda,\mu)} (\phi_j^\dagger)^{b_j(\lambda,\mu)} k_j^{m_s(\lambda) + \cdots + m_{j-1}(\lambda) + (j-s)u} \right].$$

Each trace in this product can be calculated using the general relation

$$\mathrm{Tr} \left[\phi^b (\phi^\dagger)^c k^d \right] = \frac{\delta_{b,c} \prod_{i=1}^{b} (1 - t^i)}{\prod_{i=0}^{b} (1 - t^{d+i})}, \quad \text{for } b, c \geqslant 0, \ d \in \mathbb{C},$$

valid for the representation (10) which we have used. Applying this relation to (15) and using the fact that $\sum_{s \leqslant i < j} m_i(\lambda) = \lambda'_s - \lambda'_j$, we obtain

$$
(16) \qquad T^{(s)}_{\lambda,\mu}(x_1, \ldots, x_n) = x_\lambda \times \prod_{j=s+1}^{r} \left(\frac{q^{(j-s)a_j(\lambda,\mu)} \prod_{k=1}^{b_j(\lambda,\mu)} (1 - t^k)}{\prod_{k=0}^{b_j(\lambda,\mu)} (1 - q^{j-s} t^{\lambda'_s - \lambda'_j + k})} \right).
$$

Finally, in view of the fact that $m_i(\lambda) = m_i(\mu)$ for all $s + 1 \leqslant i \leqslant r$, we find that

$$
\Omega^{(s)}_\lambda(q,t) \Big/ \Omega^{(s+1)}_{\mu^+}(q,t) = \prod_{j=s+1}^{r} \frac{1}{1 - q^{j-s} t^{\lambda'_s - \lambda'_j}},
$$

allowing us to extract the $k = 0$ term from the denominator of (16), recovering exactly (13). □

Returning to equation (12), we have thus shown that when λ is a partition,

$$
(17) \qquad f^{(s)}_\lambda(x_1, \ldots, x_n; q, t) = x_\lambda \times \sum_\mu C_s(\lambda, \mu) f^{(s+1)}_\mu(x_1, \ldots, x_n; q, t), \qquad \forall \, 1 \leqslant s \leqslant r - 1.
$$

To complete the proof of equation (3) it remains only to combine (5) with the recursion relation (17), iterated $r - 1$ times, with the final value $f^{(r)}_\mu = x_\mu$. At each step in the iteration, we need to order the composition μ indexing $f^{(s+1)}_\mu$ so that it becomes a partition (otherwise the result (17) is not valid at the next stage). This ordering is achieved by the action of T_σ operators at each step.

5. Specializations

5.1. **Monomial symmetric polynomials.** By specializing $t = 1$, the Macdonald polynomial P_λ reduces to a monomial symmetric polynomial $m_\lambda(x_1, \ldots, x_n)$. It is especially simple to evaluate (3) at $t = 1$. Due to the factors $1 - t^k$ in (4), we find that each coefficient C_i vanishes, unless $b_j(\lambda[i-1], \sigma \circ \lambda[i]) = 0$ for all $i + 1 \leqslant j \leqslant r$. This constrains each σ (apart from that of the leftmost sum) to be the identity. Furthermore, because $a_j(\lambda[i-1], \lambda[i]) = 0$ for all $i + 1 \leqslant j \leqslant r$, all coefficients $C_i(\lambda[i-1], \lambda[i]) = 1$.

Finally, at $t = 1$ the Hecke generators (1) reduce to simple transpositions s_i. Putting all of this together, equation (3) reduces to

$$
P_\lambda(x_1, \ldots, x_n; q, 1) = \sum_{\sigma \in S_\lambda} s_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} x_{\lambda[i]} = \sum_{\sigma \in S_\lambda} \sigma \circ \left(\prod_{i=1}^{n} x_i^{\lambda_i} \right) = m_\lambda(x_1, \ldots, x_n),
$$

as required.

5.2. **Hall–Littlewood polynomials.** The next specialization of interest is when $q = 0$, which gives the Hall–Littlewood polynomial $P_\lambda(x_1, \ldots, x_n; t)$. In fact the formula (3) simplifies greatly when $q = 0$. Clearly

$$
C_i \left. \begin{pmatrix} \lambda[i-1] \\ \sigma \circ \lambda[i] \end{pmatrix} \right|_{q=0} = 0, \quad \text{unless} \quad a_j \left(\lambda[i-1], \sigma \circ \lambda[i] \right) = 0 \ \text{ for all } j > i.
$$

This constrains each permutation σ (apart from that of the leftmost sum) to be the identity. Furthermore, in view of the fact that $b_j(\lambda[i-1], \lambda[i]) = 0$ for all $j > i$, we have $C_i(\lambda[i-1], \lambda[i]) = 1$

for all $1 \leqslant i \leqslant r - 1$. Hence the formula (3) reduces to

$$(18) \qquad P_\lambda(x_1, \ldots, x_n; t) = \sum_{\sigma \in S_\lambda} T_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} x_{\lambda[i]} = \sum_{\sigma \in S_\lambda} T_\sigma \circ \left(\prod_{i=1}^{n} x_i^{\lambda_i} \right).$$

Equation (18) is a known expression for Hall–Littlewood polynomials, and seems to have been first pointed out in [4] (Remark 3, following Theorem 3.1). For more details we refer the reader to Section 6.2 of [7], where the correspondence between (18) and the standard sum expression

$$(19) \qquad P_\lambda(x_1, \ldots, x_n; t) = \sum_{\sigma \in S_\lambda} \sigma \circ \left(\prod_{i=1}^{n} x_i^{\lambda_i} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

([11], Chapter III, Equation (2.2)) is explained in greater detail.

5.3. Jack polynomials.

The Jack polynomial $P_\lambda^{(\alpha)}(x_1, \ldots, x_n)$ is obtained as the $q = t^\alpha, t \to 1$ limit of the Macdonald polynomial P_λ ([11], Chapter VI, Section 10). Let us first examine the effect of this limit on the coefficients (4). We find that

$$C_i(\lambda, \mu) = 0, \text{ if any } 0 < \lambda_k < \mu_k, \quad C_i(\lambda, \mu) = \prod_{j=i+1}^{r} \prod_{k=1}^{b_j(\lambda,\mu)} \left(\frac{k}{(j-i)\alpha + \lambda_i' - \lambda_j' + k} \right), \text{ otherwise.}$$

Further to this, in the limit $q = t^\alpha, t \to 1$ the Hecke generators (1) are again just simple transpositions s_i. It follows that

$$P_\lambda^{(\alpha)}(x_1, \ldots, x_n) = \sum_{\sigma \in S_\lambda} s_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} \left(\sum_{\sigma \in S_{\lambda[i]}} C_i \binom{\lambda[i-1]}{\sigma \circ \lambda[i]} s_\sigma \circ x_{\lambda[i]} \circ \right) 1,$$

where it is trivial to calculate the action of s_σ at each step, since it simply acts on monomials in $(x_1, \ldots, x_n)$ by the permutation σ. We therefore obtain

$$P_\lambda^{(\alpha)}(x_1, \ldots, x_n) = \sum_{\sigma[0] \in S_{\lambda[0]}} \sum_{\sigma[1] \in S_{\lambda[1]}} \cdots \sum_{\sigma[\bar{r}] \in S_{\lambda[\bar{r}]}} \prod_{i=1}^{\bar{r}} C_i \binom{\lambda[i-1]}{\sigma[i] \circ \lambda[i]}$$

$$\times \left\{ \sigma[0] \circ x_{\lambda[0]} \right\} \left\{ \sigma[0] \circ \sigma[1] \circ x_{\lambda[1]} \right\} \cdots \left\{ \sigma[0] \circ \sigma[1] \circ \cdots \circ \sigma[\bar{r}] \circ x_{\lambda[\bar{r}]} \right\},$$

where we have set $r - 1 \equiv \bar{r}$ to make the formula more compact.

5.4. q–Whittaker polynomials.

Another special case of (3) is when $t = 0$, giving q–Whittaker polynomials $P_\lambda(x_1, \ldots, x_n; q, 0)$. Taking this specialization of (3), we obtain

$$P_\lambda(x_1, \ldots, x_n; q, 0) = \sum_{\sigma \in S_\lambda} D_\sigma \circ x_\lambda \circ \prod_{i=1}^{r-1} \left(\sum_{\sigma \in S_{\lambda[i]}} C_i \binom{\lambda[i-1]}{\sigma \circ \lambda[i]} D_\sigma \circ x_{\lambda[i]} \circ \right) 1$$

with coefficients that satisfy $C_i(\lambda, \mu) = 0$ if any $0 < \lambda_k < \mu_k$, and $C_i(\lambda, \mu) = \prod_{j=i+1}^{r} q^{(j-i)a_j(\lambda,\mu)}$ otherwise, and where each D_σ is now composed of the divided-difference operators

$$D_i = (x_i/x_{i+1} - 1)^{-1}(1 - s_i), \qquad 1 \leqslant i \leqslant n - 1.$$

Appendix A. Explicit examples

A.1. **Two variable example.** In the case $\lambda = (3, 1)$, we have

$$\lambda[0] = (3,1), \ \lambda[1] = (3,0), \ \lambda[2] = (3,0), \ \lambda[3] = (0,0),$$

and $S_{\lambda[0]} = S_{\lambda[1]} = S_{\lambda[2]} = S_2$. Hence

$$P_{(3,1)}(x_1, x_2; q, t) = \sum_{\sigma \in S_2} T_\sigma \circ x_1 x_2 \circ \sum_{\rho \in S_2} C_1 \begin{pmatrix} \lambda[0]_1 & \lambda[0]_2 \\ \lambda[1]_{\rho_1} & \lambda[1]_{\rho_2} \end{pmatrix} T_\rho \circ x_1 \circ \sum_{\pi \in S_2} C_2 \begin{pmatrix} \lambda[1]_1 & \lambda[1]_2 \\ \lambda[2]_{\pi_1} & \lambda[2]_{\pi_2} \end{pmatrix} T_\pi \circ x_1.$$

We compute each sum in turn, starting with the rightmost:

$$\sum_{\pi \in S_2} C_2 \begin{pmatrix} \lambda[1]_1 & \lambda[1]_2 \\ \lambda[2]_{\pi_1} & \lambda[2]_{\pi_2} \end{pmatrix} T_\pi \circ x_1 = x_1 + C_2 \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} T_1 \circ x_1 = x_1 + q \left(\frac{1-t}{1-qt} \right) x_2.$$

Combining with the middle sum, we find that

$$\sum_{\rho \in S_2} C_1 \begin{pmatrix} \lambda[0]_1 & \lambda[0]_2 \\ \lambda[1]_{\rho_1} & \lambda[1]_{\rho_2} \end{pmatrix} T_\rho \circ \left(x_1^2 + q \frac{1-t}{1-qt} x_1 x_2 \right) = x_1^2 + q \frac{1-t}{1-qt} x_1 x_2,$$

where we have used the fact that $C_1 \begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix} = 1$ and $C_1 \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = 0$. Combining everything and passing to the leftmost sum, we have

$$P_{(3,1)}(x_1, x_2; q, t) = \sum_{\sigma \in S_2} T_\sigma \circ \left(x_1^3 x_2 + q \frac{1-t}{1-qt} x_1^2 x_2^2 \right)$$

$$= (1 + T_1) \circ \left(x_1^3 x_2 + q \frac{1-t}{1-qt} x_1^2 x_2^2 \right) = x_1^3 x_2 + \frac{1 - t + q - qt}{1 - qt} x_1^2 x_2^2 + x_1 x_2^3.$$

A.2. **Three variable example.** In the case $\lambda = (3, 2, 1)$, we have

$$\lambda[0] = (3,2,1), \ \lambda[1] = (3,2,0), \ \lambda[2] = (3,0,0), \ \lambda[3] = (0,0,0),$$

and $S_{\lambda[0]} = S_{\lambda[1]} = S_3$, $S_{\lambda[2]} = \{(1,2,3), (2,1,3), (2,3,1)\}$. Hence

$$P_{(3,2,1)}(x_1, x_2, x_3; q, t) =$$

$$\sum_{\sigma \in S_3} T_\sigma \circ x_1 x_2 x_3 \circ \sum_{\rho \in S_3} C_1 \begin{pmatrix} \lambda[0]_1 & \lambda[0]_2 & \lambda[0]_3 \\ \lambda[1]_{\rho_1} & \lambda[1]_{\rho_2} & \lambda[1]_{\rho_3} \end{pmatrix} T_\rho \circ x_1 x_2 \circ \sum_{\pi \in S_{\lambda[2]}} C_2 \begin{pmatrix} \lambda[1]_1 & \lambda[1]_2 & \lambda[1]_3 \\ \lambda[2]_{\pi_1} & \lambda[2]_{\pi_2} & \lambda[2]_{\pi_3} \end{pmatrix} T_\pi \circ x_1.$$

Starting with the rightmost sum, we have

$$\sum_{\pi \in S_{\lambda[2]}} C_2 \begin{pmatrix} \lambda[1]_1 & \lambda[1]_2 & \lambda[1]_3 \\ \lambda[2]_{\pi_1} & \lambda[2]_{\pi_2} & \lambda[2]_{\pi_3} \end{pmatrix} T_\pi \circ x_1 = x_1 + C_2 \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} T_2 T_1 \circ x_1 = x_1 + q \left(\frac{1-t}{1-qt^2} \right) x_3,$$

where we have used that $C_2\begin{pmatrix} 3 & 2 & 0 \\ 0 & 3 & 0 \end{pmatrix} = 0$. Combining with the middle sum and using the fact

that $C_1\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix} = 1$ and all other $C_1 = 0$, we find

$$\sum_{\rho \in S_3} C_1\begin{pmatrix} \lambda[0]_1 & \lambda[0]_2 & \lambda[0]_3 \\ \lambda[1]_{\rho_1} & \lambda[1]_{\rho_2} & \lambda[1]_{\rho_3} \end{pmatrix} T_\rho \circ \left(x_1^2 x_2 + q\frac{1-t}{1-qt^2}x_1 x_2 x_3 \right) = x_1^2 x_2 + q\frac{1-t}{1-qt^2}x_1 x_2 x_3.$$

Combining everything, the leftmost sum finally gives

$$\sum_{\sigma \in S_3} T_\sigma \circ \left(x_1^3 x_2^2 x_3 + q\frac{1-t}{1-qt^2}x_1^2 x_2^2 x_3^2 \right) = \sum_{\sigma \in S_3} x_{\sigma_1}^3 x_{\sigma_2}^2 x_{\sigma_3} + (2 + q + t + 2qt)\frac{1-t}{1-qt^2}x_1^2 x_2^2 x_3^2.$$

APPENDIX B. KIRILLOV–NOUMI FORMULA

B.1. **The Kirillov–Noumi raising operators.** Introduce the Dunkl operators Y_i, which are central elements of the affine Hecke algebra:

$$Y_i = t^{2i-n-1}T_i \dots T_{n-1}\omega T_1^{-1} \dots T_{i-1}^{-1}$$

where in the polynomial representation the inverse Hecke generators are given explicitly by

$$T_i^{-1} = t^{-1}\left(1 - \frac{tx_i - x_{i+1}}{x_i - x_{i+1}}(1 - s_i) \right),$$

and where ω has the cyclic action $\omega f(x_1, \dots, x_n) = f(qx_n, x_1, \dots, x_{n-1})$. The raising operators introduced in [9] are given by

$$(20) \qquad B_m^{(n)} = \sum_{1 \leqslant i_1 < \dots < i_m \leqslant n} x_{i_1} \dots x_{i_m}(1 - t^m Y_{i_1}) \dots (1 - tY_{i_m}), \qquad 1 \leqslant m \leqslant n,$$

and have the following action on Macdonald polynomials:

$$(21) \qquad B_m^{(n)} P_\lambda(x_1, \dots, x_n; q, t) = \prod_{i=1}^m (1 - t^{m-i+1}q^{\lambda_i})P_{\lambda+(1^m)}(x_1, \dots, x_n; q, t),$$

where $\lambda + (1^m)$ indicates adding a single box to the first m rows of the Young diagram of λ. This action suffices to build up any Macdonald polynomial starting from $P_0 \equiv 1$, since any Young diagram can be decomposed uniquely into its columns:

$$(22) \qquad B_{\lambda_1'}^{(n)} \cdots B_{\lambda_r'}^{(n)} \cdot 1 = c_\lambda(q, t)P_\lambda(x_1, \dots, x_n; q, t), \qquad c_\lambda(q, t) = \prod_{s \in \lambda}(1 - q^{a(s)}t^{l(s)+1}),$$

where $(\lambda_1', \dots, \lambda_r')$ is the conjugate partition of λ, and $c_\lambda(q, t)$ is the multiplicative factor which converts a Macdonald polynomial P_λ to its integral form J_λ ([11], Chapter VI, Section 8).

B.2. **Two variable example.** In the case $\lambda = (3, 1)$, we have $\lambda' = (2, 1, 1)$, and

$$B_1^{(2)} \cdot 1 = (1-t)(x_1 + x_2) = c_{(1)}(q, t)P_{(1)}(x_1, x_2; q, t),$$

$$B_1^{(2)} B_1^{(2)} \cdot 1 = (1-t)(1-qt)\left(x_1^2 + \frac{1-t+q-qt}{1-qt}x_1 x_2 + x_2^2 \right) = c_{(2)}(q, t)P_{(2)}(x_1, x_2; q, t),$$

$$B_2^{(2)} B_1^{(2)} B_1^{(2)} \cdot 1 = (1-t)^2(1-qt)(1-q^2t^2)\left(x_1^3 x_2 + \frac{1-t+q-qt}{1-qt}x_1^2 x_2^2 + x_1 x_2^3 \right)$$

$$= c_{(3,1)}(q, t)P_{(3,1)}(x_1, x_2; q, t),$$

which can be easily verified by implementing the raising operators (20) in a symbolic computation package.

B.3. **Three variable example.** In the case $\lambda = (3, 2, 1)$, we have $\lambda' = (3, 2, 1)$, and

$$B_1^{(3)} \cdot 1 = (1 - t)(x_1 + x_2 + x_3) = c_{(1)}(q, t)P_{(1)}(x_1, x_2, x_3; q, t),$$

$$B_2^{(3)}B_1^{(3)} \cdot 1 = (1 - t)^2(1 - qt^2)\left(x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2\right.$$

$$\left. + \frac{(1 - t)(2 + q + t + 2qt)}{1 - qt^2} x_1 x_2 x_3 \right) = c_{(2,1)}(q, t)P_{(2,1)}(x_1, x_2, x_3; q, t),$$

$$B_3^{(3)}B_2^{(3)}B_1^{(3)} \cdot 1 = (1 - t)^3(1 - qt^2)^2(1 - q^2 t^3)P_{(3,2,1)}(x_1, x_2, x_3; q, t)$$

$$= c_{(3,2,1)}(q, t)P_{(3,2,1)}(x_1, x_2, x_3; q, t).$$

Appendix C. Comparison of formulae

The examples in the previous two appendices allow us to directly compare our summation formula (3) with the raising-operator expression (22) obtained in [9]. A number of differences should be pointed out:

1. While both (3) and (22) feature a decomposition of the partition λ, the decomposition is different in the two equations. In (3), the sequence of partitions $\lambda[0] \supseteq \lambda[1] \supseteq \cdots \supseteq \lambda[r]$ is a decomposition of λ into its rows, whereas the raising operators in (22) build λ up by its columns.

2. By virtue of their action (21), the Kirillov–Noumi raising operators $B_m^{(n)}$ map one Macdonald polynomial to another. This means that one obtains a Macdonald polynomial at all intermediate steps on the way to constructing λ, as can be seen in the examples of Appendices B.2 and B.3. In contrast, if one truncates the summation (3) at any intermediate point k (meaning that we only perform the sums $\sum_{\sigma \in S_{\lambda[i]}}$ for $k \leqslant i \leqslant r - 1$), the result will not be a Macdonald polynomial. In fact, these intermediate polynomials will not in general even be symmetric in $(x_1, \ldots, x_n)$, as can be seen from the examples in Appendices A.1 and A.2. It is only at the final sum $\sum_{\sigma \in S_\lambda}$ that symmetry is restored and the Macdonald polynomial P_λ is recovered.

3. In (22), the q parameter is introduced via the operation ω which appears in each Dunkl operator. In contrast, (3) does not feature ω at all, and q is introduced explicitly in the combinatorial factors $C_i(\lambda, \mu)$. This is quite advantageous, since it easily allows specializations of q to be taken.

¿From this, one can see that while equations (3) and (22) are related at the level of ideas, it is challenging to find a direct link between the two.

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