High Reliability Radar and Communications Based on Random Stepped Frequency Waveforms

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***Abstract*—This paper is on the waveform design of joint radar and communication systems. Focusing on permutation code based random stepped frequency waveforms, we present a new joint radar and communication system that has improved communication error rate performance when compared to existing approaches. More specifically, we propose a subset selection process to improve the Hamming distance between communication waveforms. An efficient encoding scheme is proposed to map the information symbols to selected permutations. Further, an optimal communication receiver based on integer programming followed by a more efficient sub-optimal receiver based on the Hungarian algorithm is also proposed. Considering the optimum maximum likelihood detection, the block error probability is analyzed under both additive white Gaussian noise channels and Rician fading channels. Finally, we discuss the radar performance under the new system and highlight that it has negligible effect on the radar local and global accuracy.**

***Index Terms*—joint radar and communications, error probability, maximum likelihood.**

I. Introduction

In recent literature a considerable amount of research has focused on the topic of joint radar and communications [1]. These works are driven not only by the spectrum scarcity but also by the introduction of millimeter wave (mmWave) frequencies, a band that is already being used for radar sensing, for communications purposes.

In some works, classical communication waveforms such as the orthogonal frequency division multiplexing (OFDM) waveform have been considered for joint radar and communications. In [2], the OFDM waveform is shown to achieve high Doppler tolerance and low sidelobe levels while maintaining the same data transmission capacity. However, due to the high peak-to-average-power ratio (PAPR) of the OFDM waveform the detection range of radar sensing is limited and other approaches such as multiple-input multiple-output (MIMO) OFDM needs to be explored [3]. The new orthogonal time frequency space (OTFS) waveform has also been proposed for the same purposes and in [4], the OTFS waveform is shown to achieve similar radar performance to the OFDM waveform while maintaining a higher communication data rate.

Taking a different approach, in [5], the preamble of the communication frame is exploited to propose a virtual waveform in the mmWave band. This is conceptually similar to the staggered pulse repetition intervals (PRIs) used in long range radar waveforms. In some works, classical radar waveforms have been considered for the same purpose. In [6], [7], a new waveform design is proposed based on the stepped frequency radar waveform and permutation coding. The authors consider all permutations generated by a given set of frequency tones and use the selection of a particular waveform to embed communication data. They also propose efficient methods to encode and decode the data while maintaining good radar performance.

The present work is based on the permutation code based random stepped frequency waveform considered in [7]. When the frequency sequences given by all the permutations are used for transmission, the minimum Hamming distance between two waveforms remains at two. However, according to permutation coding it is possible to improve the communication error rate by selecting a subset of permutations with a larger minimum Hamming distance [8]. Permutation coding for information modulation was first proposed in [9], where the amplitude of the transmitted impulse is modulated according to the permutation corresponding to each code word. This work was later extended in [10] to a communication network with fading channels. In [11], permutation arrays with a specific minimum Hamming distance are presented as a potential coding scheme in powerline communication. Recently, permutation coding based on the inversion vector and the Kendall-Tau distance has been considered for lossless compression in data storage [12].

In contrast to [7], we focus on selecting a subset of permutations such that the communication error rate is improved. This is motivated by low rate applications such as the navigation function in vehicle-to-everything (V2X) communications where high reliability communication with good radar sensing is required. However, designing an efficient encoding scheme and a communication receiver is essential to make any selected subset practical. Noting that the efficient mapping process and the receiver design in [7] are only feasible when the entire set of permutations is used, we make the following contributions.

- We propose a novel encoding process based on the sign of permutations to map incoming data symbols to corresponding waveforms.
- Noting that the communications receiver implementation is not straightforward when a subset of permutations are selected, we design a new numerical method to perform optimal decoding based on integer programming (IP).
- We also propose a novel and efficient sub-optimal receiver based on the Hungarian algorithm which obtains close to optimal error performance.

This work was supported by the Australian Research Council Discovery Project under Grant DP180101205 and Discovery Early Career Researcher Award under Grant DE180100501.

Further, we show that the proposed system maintains a good radar sensing functionality.

II. Joint Radar and Communication System

The new joint radar and communication approach proposed in this paper is based on the random stepped frequency waveform considered in [7]. For completeness, we introduce only the necessary details of this waveform in the following. A random stepped frequency waveform is generated using M pulses, each with a duration of T seconds. A given waveform consists of M equally spaced frequency tones $f_0, f_1, \ldots, f_{M-1}$ such that each frequency is used only once. Therefore, there can be $M!$ potential waveforms which corresponds to all the permutations generated from the M frequency tones. As such, the complex baseband signal of the i-th waveform can be expressed as,

$$s_i(t) = \sqrt{\frac{E}{MT}} \sum_{m=0}^{M-1} s_p(t-mT) \exp\bigg(2\pi f_m^i (t-mT)\bigg), \quad (1)$$

where $s_p(t)$ is the unit pulse function which is zero outside $0 \le t \le T$ with f_m^i denoting the frequency relating to the m-th index of the i-th permutation given as $[f_0^i, f_1^i, \ldots, f_{M-1}^i]$ and E denoting the signal energy satisfying $\int_0^{MT} s_i^2(t)dt = E, \forall i = 0, 1, \ldots, M!-1$. It is also assumed that all M frequencies are orthogonal with the separation between two tones being $\Delta f = q/T$ where q is an integer. As the new waveform design is based on the well established stepped frequency radar waveform, it results in good radar performance.

A. Subset Selection

We note that under the universal set of permutations considered in [7], the minimum Hamming distance d_m between any two waveforms is two and the number of data bits sent is given by $\log_2(M!)$. According to permutation coding, the communication error rate is inversely proportional to d_m. On the other hand, the communication data rate is proportional to the number of selected waveforms. As such, by selecting a subset of permutations with a larger d_m, we can improve the communication error rate at the expense of lowering the data rate. Motivated by applications such as administrative functions and navigation function in V2X communication that demand for high reliability communication but with low data rate, it is desirable to select a subset of waveforms to increase the resulting minimum Hamming distance [1]. Thus, in this work, we consider the selection of a subset of permutations from the $M!$ universal set to improve the communication reliability. We define the set of waveforms obtained from the selected permutation subset as $\mathcal{S}$.

In the study of permutation arrays, it is well known that the maximum size of the permutation array with a given d_m decreases significantly with increasing d_m [11]. Therefore, while the block error rate can be reduced by increasing d_m, the loss of communication data rate can become significant. Therefore, in this work we consider the special case of $d_m = 3$ which is the next best d_m we can achieve after $d_m = 2$.

To introduce subset selection, first let us define some preliminary terms used with permutations. Let $\chi_i = [\chi_i(1), \ldots, \chi_i(M)]$ denote a random permutation of integers $1, 2, \ldots, M$, with $\chi_i(m)$ denoting the m-th integer, then,

- the number of inversions in χ_i is defined as,
$$N(\chi_i) = \{(m,n) : m < n \text{ and } \chi_i(m) > \chi_i(n)\}.$$
- the sign of permutation χ_i is positive $(+1)$ if $N(\chi_i)$ is even and negative (-1) if $N(\chi_i)$ is odd [13].

In [11], the authors show that the alternative group consisting of the permutations with the same sign has $d_m = 3$ and that the size of this subset is given by $M!/2$. Therefore, we propose the selection of the alternative group that consists of all the permutations with a positive sign and define the resulting set of waveforms as $\mathcal{S} = \{s_0(t), s_1(t), \ldots, s_{M!/2-1}(t)\}$. We also note that under the permutation coding, it is possible to detect up to two errors and correct any single error when the subset of permutations are selected such that $d_m = 3$. Thus, this subset of permutations provides a coding scheme equivalent to single parity-bit coding in communication networks.

We note that to make the implementation of any selected subset feasible, an efficient encoding scheme and a receiver design is essential. Thus, while the concept of Hamming distance based subset selection as a method to reduce the error rate is well known, as the main contributions of this work we design an efficient encoding scheme and a communication receiver implementation for the selected subset.

B. Encoding of communication data

Under the universal set of permutations, the incoming data symbols can be efficiently mapped to their corresponding waveforms using a combinatorial transform called the Lehmer code [7]. Under this approach, the incoming data symbol is first transformed to its natural number according to the factorial number system. Then each natural number is mapped to the rank of the corresponding unique permutation in the lexicographic order. However, when we only select half of the permutations, this direct mapping between the natural number and the lexicographic rank fails. Thus, the mapping process proposed in [7] is no longer feasible. Therefore, in this work we take a different approach and exploit the special structure of the alternative group to design an efficient mapping between the incoming data symbols and the corresponding waveforms in the selected subset. We first provide the following Lemma.

Lemma 1. *According to the lexicographic order, the two adjacent permutations given by $2i$ and $(2i+1)$ have different signs.*

Proof: See Appendix A.

According to Lemma 1, incoming data symbols can be mapped into permutations as follows. If the natural number of the incoming data symbol is given by i, according to the factorial number system, we compute the sign of the $2i$-th permutation and the $(2i+1)$-th permutation in the lexicographic order. Then, the permutation with the positive sign is selected for transmission. This process of generating two specific permutations and computing their sign can be implemented with a linear complexity in M. At the communication receiver, we first detect the permutation corresponding to the received signal.

If the i-th permutation is detected, then the natural number of the received data symbol is taken as $\lfloor i/2 \rfloor$. Therefore, even if a subset is selected the mapping from incoming data symbols to corresponding waveforms can be implemented very efficiently without the use of a large look-up table.

III. Communication Block Error Rate Analysis

In this section, we analyse the communication block error rate (BLER) of the proposed system. We consider a communication receiver with N receive antennas. As such, when the waveform $s_i(t)$ is transmitted, the $N \times 1$ received signal vector at the communication receiver can be expressed as,

$$\mathbf{r}(t) = \mathbf{h}\, s_i(t) + \mathbf{n}(t), \tag{2}$$

where $\mathbf{h}$ is the small-scale fading channel vector and $\mathbf{n}(t)$ is an additive white Gaussian noise (AWGN) vector where each element is complex Gaussian with zero mean and variance N_0. We consider the optimum maximum likelihood (ML) detection at the communications receiver. Assuming that the channel vector $\mathbf{h}$ is known at the receiver we can apply the ML detection rule and write the detected symbol as,

$$\hat{s_i}(t) = \underset{s_j(t)\in\mathcal{S}}{\arg\max}\ \mathrm{Re}\bigg(\int_0^{MT} s_j^*(t)\mathbf{h}^H\mathbf{r}(t)\,dt\bigg), \tag{3}$$

where Re(.) denotes the real part of the argument and the maximization is over all possible waveforms in the subset $\mathcal{S}$. Given that the transmitted data is assumed to be equally likely, the BLER of detecting the received waveform as a different waveform can be expressed as,

$$P_e = \frac{1}{|\mathcal{S}|}\sum_{i=0}^{|\mathcal{S}|-1}[1 - P_c(i)], \tag{4}$$

where $P_c(i)$ denotes the probability of a correct decision when waveform $s_i(t)$ is transmitted and $|\mathcal{S}|$ is the cardinality of $\mathcal{S}$. Note that the exact computation of P_e requires complex multi-dimensional integrals over the multivariate Gaussian density [7]. Therefore, we take a more tractable approach by considering the union bound which is given by,

$$P_e \le P_e^{UB} = \frac{1}{|\mathcal{S}|}\sum_{i=0}^{|\mathcal{S}|-1}\sum_{j=0,j\neq i}^{|\mathcal{S}|-1} P_{ij}, \tag{5}$$

where P_{ij} denotes the pairwise error probability (PEP) of detecting $s_j(t)$ when $s_i(t)$ is transmitted. Using the ML detection rule, the probability of detecting $s_j(t)$ when $s_i(t)$ is transmitted can be expressed as,

$$\begin{aligned} P_{ij} = \Pr\bigg[&\mathrm{Re}\bigg(\int_0^{MT} s_i^*(t)\mathbf{h}^H\mathbf{r}(t)\,dt\bigg) \\ &< \mathrm{Re}\bigg(\int_0^{MT} s_j^*(t)\mathbf{h}^H\mathbf{r}(t)\,dt\bigg)\bigg]. \end{aligned} \tag{6}$$

In the following, we present analytical expressions for P_e^{UB} under two traditional channel models, namely, the AWGN channel and the Rician channel models.

A. AWGN Channel

Without loss of generality we assume a unit channel gain so that $\mathbf{h}^H\mathbf{h} = N$. Whilst not included due to page limitations, exploiting the symmetric structure in PEPs in the alternative group, the union bound given in (5) can be simplified to,

$$P_e^{UB} = \sum_{l=3}^{M} A_l\,\mathcal{Q}\bigg(\sqrt{\frac{NEl}{N_0M}}\bigg), \tag{7}$$

where A_l represents the number of permutations within the alternative group with Hamming distance l from a given permutation and $\mathcal{Q}(.)$ is the Gaussian Q-function.

Note that the union bound considers the summation over a large number of PEPs and can lead to a loose bound especially when M is large. Therefore we also derive another approximation based on the nearest neighbours. A special property of the alternative group is that for any permutation with a positive sign, all of its neighbors with Hamming distance 3 also have a positive sign and vice versa. As such, we can derive the nearest neighbor (NN) approximation [14] by only considering the PEPs corresponding to the nearest neighbors which have a Hamming distance of 3 and obtain

$$P_e^{NN} = \frac{M(M-1)(M-2)}{3}\mathcal{Q}\bigg(\sqrt{\frac{3NE}{N_0M}}\bigg), \tag{8}$$

as a more accurate approximation to the BLER under the AWGN channel.

B. Rician Fading Channel

To gain further insights into the effect of fading, next we focus on the independent Rician fading model where the strength of the line-of-sight (LoS) path is governed by the Rician factor denoted by K. As such, the $N \times 1$ small scale fading channel vector can be written as $\mathbf{h} = \sqrt{\frac{K}{K+1}}\Delta + \sqrt{\frac{1}{K+1}}\mathbf{u}$, where Δ denotes the complex LoS phase vector with the i-th element having the property $|\Delta_i|^2 = 1$, $\mathbf{u}$ denotes the scattered component vector with the i-th element $u_i \sim \mathcal{CN}(0,1)$. Thus, $\mathbf{h}^H\mathbf{h}$ follows a non-central chi-squared distribution with $2N$ degrees of freedom and non-centrality parameter $2NK$. Whilst not show here due to page limitations, we can use this distribution and obtain the union bound and the NN approximation under Rician fading model as [7],

$$\begin{aligned} P_e^{UB} = &\sum_{l=3}^{M}\sum_{m=0}^{\infty} A_l\bigg(\frac{(NK)^m e^{-NK}}{m!}\bigg)\bigg[\frac{1}{2} + \sum_{n=1}^{N+m}(-1)^n \\ &\binom{N+m}{n} V_n\bigg(\frac{2N_0M(K+1)}{El}\bigg)\bigg], \end{aligned} \tag{9}$$

$$\begin{aligned} P_e^{NN} = &\frac{M(M-1)(M-2)}{3}\sum_{m=0}^{\infty}\bigg(\frac{(NK)^m e^{-NK}}{m!}\bigg)\bigg[\frac{1}{2}+ \\ &\sum_{n=1}^{N+m}(-1)^n\binom{N+m}{n}V_n\bigg(\frac{2N_0M(K+1)}{3E}\bigg)\bigg], \end{aligned} \tag{10}$$

where $V_n(x) = \frac{1}{2(1+x)^{n-1/2}}\sum_{q=0}^{n-1}\binom{n-1}{q}\binom{2q}{q}(x/4)^q$. Communication performance of the proposed subset in terms of BLER and the tightness of the computed bounds in (7)-(10) are further discussed under the numerical results.

IV. Communication receiver implementation

In this section, we focus on the communication receiver implementation of the proposed system. When the universal permutation set is considered, the optimal receiver implementation simplifies to an assignment problem. As such, the Hungarian algorithm always results in the optimal solution [7]. However, when only a subset of permutations is selected for transmission, the Hungarian algorithm may result in a permutation which does not belong to the selected set. Therefore, the receiver implementation proposed in [7] is no longer feasible. As such, we focus on designing a communication receiver that can be used under any permutation subset.

Under the ML decision rule, the obvious implementation of the optimal receiver involves computing the correlation between every potential transmit waveform and the corresponding received waveform to obtain the waveform that produces the highest correlation. However, such an implementation has a complexity of $O(M!/2)$, which could be quite significant for large M. Therefore, we reformulate the optimal receiver in (3) as an IP optimization problem.

1) IP based optimal receiver: First let us define the correlation matrix for the received signal as,

$$\mathbf{R} = (r_{uv}) \in \mathbb{R}^{M\times M}, \tag{11}$$

where the uv-th element of $\mathbf{R}$, r_{uv}, denotes the correlation between $\mathbf{h}^H\mathbf{r}(t)$ and the basis function $\psi_v(t-(u-1)T)$ with $\psi_v(t) = \sqrt{2E/T}s_p(t)\cos(2\pi f_v(t))$. We can express r_{uv} as,

$$r_{uv} = \mathrm{Re}\left(\int_{(u-1)T}^{uT} \mathbf{h}^H\mathbf{r}(t)\psi_v(t-(u-1)T)\,dt\right). \tag{12}$$

Let X_{uv} denotes the indicator function for $\hat{s}_i(t)$ such that $X_{uv} = 1$ if the frequency of $\hat{s}_i(t)$ in the u-th pulse is f_v and $X_{uv} = 0$ otherwise. Next, we formulate the optimization problem of the ML receiver as follows.

$$\begin{aligned} \max_{X_{uv}} \; & \sum_{u=1}^{M}\sum_{v=1}^{M} r_{uv}X_{uv} \\ \text{s.t} \; & \sum_{u=1}^{M} X_{uv} = \sum_{v=1}^{M} X_{uv} = 1, \\ & \sum_{u=1}^{M}\sum_{v=1}^{M} X_{uv}Y_{uv}^{(k)} \geq 1, \quad \forall k \notin \mathcal{S}, \end{aligned} \tag{13}$$

where $Y_{uv}^{(k)} = 0$ if the v-th element of the k-th permutation is u and $Y_{uv}^{(k)} = 1$ otherwise. We note that this integer optimization can be solved using any existing IP solver. While the average complexity of this IP based optimal receiver can be better than the basic optimal receiver, the worst case complexity of any IP solver remains exponential in M. Therefore, in the next subsection we propose a low complexity sub-optimal receiver based on the Hungarian algorithm.

2) Proposed sub-optimal receiver: Note that the last constraint in (13) restricts the optimization to the set of permitted permutations. If we remove this constraint, the resultant receiver simplifies to selecting M elements such that only one element is selected from each row and each column of $\mathbf{R}$ such that the sum is maximized. This optimization can be solved as an assignment problem using the well known Hungarian algorithm [15]. Therefore, to propose a sub-optimal receiver we first remove the last constraint in (13) and proceed to solve the resultant optimization using the Hungarian algorithm. Next, we check whether the detected permutation is within the selected subset $\mathcal{S}$. If so, the detection process is complete. If not, that means the detected permutation has a negative sign. As a result, all $M(M-1)/2$ neighbors of that permutation with the Hamming distance $d=2$ have a positive sign and thus belong to $\mathcal{S}$. We note that there is a high probability that one of these nearest neighbors is the highest correlated waveform within $\mathcal{S}$, because the most common errors result in receiving one or two tones erroneously. As such, we generate the neighboring permutations with $d=2$ from the detected permutation and compute the correlation of the received signal with each waveform corresponding to these neighboring permutations. For each detected permutation which does not belong to $\mathcal{S}$, the correlation of the received signal with another $M(M-1)/2$ waveforms needs to be computed. Finally, the waveform with the highest correlation is selected as the detected waveform. In Algorithm 1, we summarize the main steps of our proposed sub-optimal receiver implementation.

We also note that the worst case complexity of a simple binary search to check if a given permutation belongs to $\mathcal{S}$ is $O(\log|\mathcal{S}|)$. As a result, while performing very close to the optimal receiver, the overall complexity of our proposed sub-optimal algorithm remains $O(M^3)$, which is similar to the Hungarian algorithm.

Algorithm 1: Proposed Sub-Optimal Receiver

```
Negate the correlation matrix R to produce (-R)
ŝ_i(t) ← waveform corresponding to the output of the
    Hungarian algorithm for (-R)
if ŝ_i(t) ∉ S then
    construct S_i, the set of waveforms corresponding to
        d = 2 neighboring permutations of ŝ_i(t)
    for s_j(t) ∈ S_i do
        Compute the correlation of s_j(t) with r(t)
    end
    ŝ_i(t) ← s*_j(t), which is the waveform that results in
        the highest correlation with r(t)
end
```

V. Discussion on Radar Performance

In radar sensing, the local accuracy of the radar waveform - which can be measured in terms of the range resolution, Doppler resolution and the efficiency of spectral usage, depends on the properties of the mainlobe in the ambiguity function (AF). The AF represents the matched filter output when the transmitted radar signal is received with a certain time delay and a Doppler shift [16]. In [17], it is proven that the auto-correlation function, which determines the properties of the mainlobe in the AF, does not depend on the frequency order. As such, the local accuracy remains the same for all the permutations and it will not be impacted by selecting a subset of permutations.

On the other hand, the global accuracy of the radar waveform can be measured by the sidelobe behavior of the AF. Some permutations have better radar performance compared to others due to their low peak-to-sidelobe ratio (PSLR) [17]. We note that due to the symmetric structure of the universal permutation set, for any given permutation there exists a reverse permutation whose AF is the exact mirror image of the AF of the original permutation. As such, the PSLR of these two permutations remains the same. We also note that the reverse permutation can be obtained by interchanging $\lfloor M/2 \rfloor$ disjoint positions in the original permutation. However, any interchange in two positions changes the sign of a permutation [13]. Therefore, when $\lfloor M/2 \rfloor$ is even the reverse permutation has the same sign as the original permutation but when $\lfloor M/2 \rfloor$ is odd the sign is reversed. When $\lfloor M/2 \rfloor$ is odd, we select half of the permutations in the alternative group. They are selected such that the number of permutations with the same PSLR is halved. As a result, the radar performance of this subset is equivalent to the universal set. On the other hand, when $\lfloor M/2 \rfloor$ is even, we cannot guarantee that the permutations are selected such that the number of permutations with the same PSLR is halved. However, from numerical examples we observe that even in this case, the change in the mean PSLR is negligible, i.e., the proposed subset selection improves the communication performance while not changing the radar performance when $\lfloor M/2 \rfloor$ is odd and having a negligible effect on the radar performance when $\lfloor M/2 \rfloor$ is even.

VI. Numerical Results

In this section, we provide numerical examples illustrating the performance of the proposed joint radar and communication system. We note that due to limited work in the area of the random stepped frequency radar waveform, only the universal set considered in [7] is compared against our system.

Fig. 1 plots the simulated BLER and the analytical bounds versus the received signal-to-noise-ratio (SNR) for a communication receiver with $N = 4$ antennas. We set $M = 6$ and plot the BLER performance for both the AWGN channel and a fading channel with Rician factor $K = 3$. The energy of the waveform $E = 1$ and the frequency separation $\Delta f = 1/T$ with $T = 1$, to maintain the orthogonality between frequencies. Adopting the proposed system, we select $M!/2 = 360$ waveforms such that each selected permutation has a positive sign. We consider the optimal and the sub-optimal receiver implementations using (13) and Algorithm 1, respectively. The analytical approximations are generated using the union bounds in (7), (9) and the NN approximations in (8), (10). From the plot we can observe that under the AWGN channel, the union bound accurately follows the simulation results in the high SNR regime, while in the low SNR regime, the NN approximation is more accurate. However, under the Rician fading channel, the NN approximation is more accurate compared to the union bound even in the high SNR regime. Further, we can observe that the low complex sub-optimal receiver in Algorithm 1 provides a similar BLER compared to the optimal receiver.

Fig. 2 plots the simulated BLER versus the received SNR for an AWGN channel under the universal set considered in [7] and the proposed $d_m = 3$ subset. We set $N = 2$ and $M = 4, 6$. Under the universal set, we consider all $M!$ waveforms, whereas under the $d_m = 3$ subset, we select $M!/2$ waveforms. Given that the communication data rate is proportional to the number of selected waveforms, the maximum data rate drops by $1/\log_2(M!)$ which is around 22% and 10% when $M = 4$

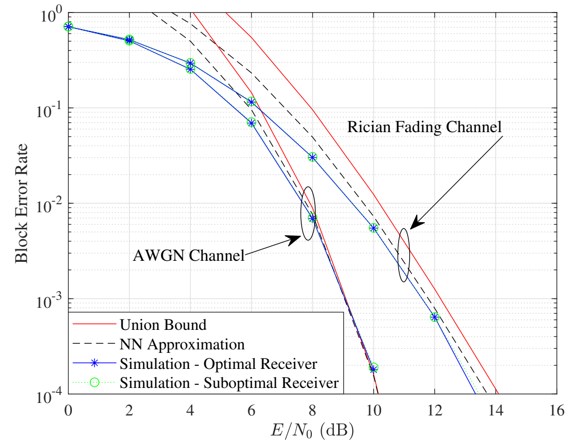

Fig. 1: The BLER versus received SNR under the $d_m = 3$ subset with $N = 4$, $K = 3$ and $M = 6$.

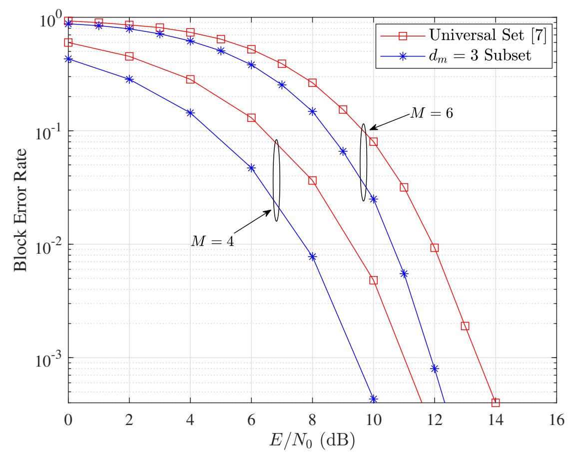

Fig. 2: The BLER versus received SNR under the universal set and the $d_m = 3$ subset with $N = 2$ and $M = 4, 6$.

and $M = 6$, respectively. As expected BLER reduces with increasing minimum Hamming distance. More specifically, we observe that at 10 dB $d_m = 3$ subset achieves a 10-fold and 4-fold reduction in the BLER with $M = 4$ and $M = 6$, respectively compared to [7]. As such, there is a clear trade-off between the achievable data rate and the BLER. This confirms that the proposed $d_m = 3$ subset is suitable for low rate applications that require high reliability communication with good radar sensing. Further, as the block size increases with M, the SNR per block decreases under constant waveform energy. As such, BLER increases with M.

Next, we proceed to compare the impact of using $d_m = 3$ subset on the radar performance. As discussed in Section V, the selection of a subset does not impact the local accuracy. However, lower PSLR results in higher global accuracy for radar sensing. Therefore, in Fig. 3, we compare the PSLR distribution of the proposed approach with the universal set in [7]. We note that PSLR relies on the concept of the difference triangle which consists of a discrete set of values [17]. As a result, the PSLR distribution in Fig. 3 is limited to a discrete set of values. For

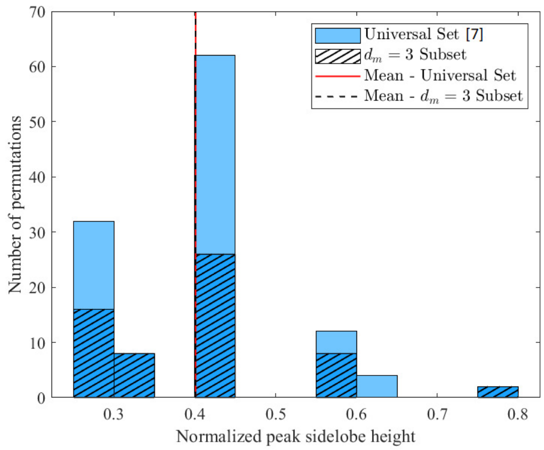

Fig. 3: Normalized PSLR distribution under the universal set and the $d_m = 3$ subset with $M = 5$.

the universal set consisting of all 120 waveforms, the mean PSLR value is 0.4010. Since $\lfloor M/2 \rfloor$ is even, under the $d_m = 3$ subset, the selection of half of the waveforms does not guarantee that the number of permutations with the same PSLR is halved. However, we can observe that the mean PSLR value under the $d_m = 3$ subset is 0.4008 which is very similar to that of the universal set. Therefore, there is no substantial change in the radar performance by using $d_m = 3$ subset.

VII. Conclusion

We proposed a new set of waveforms for the joint radar and communication problem. The proposed approach selects a subset of permutation based random stepped frequency waveforms in such a way that the communication error performance is improved while maintain good radar performance. This is motivated by low rate applications in V2X communications. An efficient encoding process is introduced for the communication transmitter. For the communications receiver, we presented an IP based optimal receiver implementation as well as a low complexity sub-optimal receiver implementation based on the Hungarian algorithm. Based on the ML detectors, we analyzed the BLER performance and derived closed form expressions for the union bound and the nearest neighbor approximation under both the AWGN channel and the Rician channel. Finally, we discussed the impact of our proposed subset selection on the radar performance. We showed that the proposed approach to subset selection can improves communication error rate while having a negligible affect on radar performance.

Appendix A
Proof of Lemma 1

Let us consider the r-th permutation in the lexicographic order where $r \in \{0, 1, ..., M! - 1\}$. We note that the number of inversions in the r-th permutation can be computed as the sum of the Lehmer code using the factorial number system and can be expressed as [18],

$$N(\chi_r) = \sum_{j=1}^{M} \mod \left(r_j, j\right), \tag{14}$$

where $r_j = \lfloor r_{j-1}/(j-1) \rfloor$ with $r_1 = r$. Using the method of induction, it can be shown that $r_j = \lfloor r/(j-1)! \rfloor$. Therefore, the number of inversions given in (14) can be written as,

$$N(\chi_r) = \mod (r, 2) + 2\lfloor r/2 \rfloor + \sum_{j=2}^{n} (1-j) \lfloor r/j! \rfloor, \tag{15}$$

when n is selected such that $n! \leq r < (n+1)!$. Next, we write the sign of the permutation χ_r as [13],

$$\text{sign}(\chi_r) = (-1)^{\text{mod}(r,2)} \times \prod_{j=2}^{n} (-1)^{(1-j)\lfloor r/j! \rfloor}. \tag{16}$$

Let us now consider the two adjacent permutations given by the $(2i)$-th permutation and the $(2i+1)$-th permutation. Given that $n!$ is even for $n \geq 2$, we have $n! \leq 2i, 2i+1 < (n+1)!\,, \forall i$. Therefore, the signs of these two permutations can be computed using (16) as

$$\text{sign}(\chi_{2i}) = \prod_{j=2}^{n} (-1)^{k_j}, \qquad \text{sign}(\chi_{2i+1}) = (-1) \prod_{j=2}^{n} (-1)^{k_j},$$

where $k_j = (1-j) \lfloor (2i)/j! \rfloor = (1-j) \lfloor (2i+1)/j! \rfloor, \forall j \leq n$. Thus, $\text{sign}(\chi_{2i+1}) = (-1) \times \text{sign}(\chi_{2i})$. This completes the proof of Lemma 1.

References

[1] Z. Feng, Z. Fang, Z. Wei, X. Chen, Z. Quan, and D. Ji, "Joint radar and communication: A survey," *China Communications*, vol. 17, no. 1, pp. 1 – 27, January 2020.

[2] D. Garmatyuk, J. Schuerger, K. Kauffman, and S. Spalding, "Wideband OFDM system for radar and communications," in *IEEE Radar Conf.*, Pasadena, CA, USA, May 2009.

[3] T. Bai, H. Hu, and X. Song, "OFDM MIMO radar waveform design with high range resolution and low sidelobe level," in *IEEE Int. Conf. Commun. Technol.*, Chengdu, China, October 2017.

[4] L. Gaudio, M. Kobayashi, G. Caire, and G. Colavolpe, "On the effectiveness of OTFS for joint radar parameter estimation and communication," *IEEE Trans. Wireless Commun.*, vol. 19, no. 9, pp. 5951–5965, September 2020.

[5] P. Kumari, S. A. Vorobyov, and R. W. Heath, "Adaptive virtual waveform design for millimeter-wave joint communication–radar," *IEEE Trans. Signal Process.*, vol. 68, pp. 715–730, November 2019.

[6] R. Senanayake, P. Smith, J. Evans, B. Moran, and R. J. Evans, "A novel joint radar and communications technique based on frequency permutations," in *IEEE Veh. Technol. Conf.*, Norman, OK, USA, September 2021.

[7] R. Senanayake, P. Smith, T. Han, J. Evans, W. Moran, and R. Evans, "Frequency permutations for joint radar and communications," *IEEE Trans. Wireless Commun.*, vol. 21, no. 11, pp. 9025–9040, May 2022.

[8] I. F. Blake, G. Cohen, and M. Deza, "Coding with permutations," *Information and Control*, vol. 43, no. 1, pp. 1–19, 1979.

[9] D. Slepian, "Permutation modulation," *Proc. IEEE*, vol. 53, no. 3, pp. 228–236, March 1965.

[10] A. Nordio and E. Viterbo, "Permutation modulation for fading channels," in *Int. Conf. Telecommun.*, Papeete, France, March 2003.

[11] W. Chu, C. J. Colbourn, and P. Dukes, "Constructions for permutation codes in powerline communications," *Designs, Codes and Cryptography*, no. 32, pp. 51–64, May 2004.

[12] D. Wang, A. Mazumdar, and G. W. Wornell, "Compression in the space of permutations," *IEEE Trans. Inf. Theory*, vol. 61, no. 12, pp. 6417–6431, December 2015.

[13] P. J. Cameron, *Permutation Groups*. Cambridge University Press, 1999.

[14] J. G. Proakis and M. Salehi, *Digital Communications*. New York, USA: McGraw-Hill, 2008.

[15] H. W. Kuhn, *The Hungarian Method for the Assignment Problem*. Wiley Online Library, 1955.

[16] N. Levanon and E. Mozeson, *Radar Signals*. Wiley Online Library, 2004.

[17] J. Costas, "A study of a class of detection waveforms having nearly ideal range—Doppler ambiguity properties," *Proc. IEEE*, vol. 72, no. 8, pp. 996–1009, August 1984.

[18] D. E. Knuth, *Volume 2: Seminumerical Algorithms, The Art of Computer Programming*. Boston, MA, USA: Addison-Wesley, 1998.