AGT for $\mathcal{N} = 2$ $SU(N)$ Gauge Theories on $\mathbb{C}^2/\mathbb{Z}_n$ and Minimal Models

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Abstract

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Building upon a correspondence between $\mathcal{N} = 2$ $SU(N)$ supersymmetric (SUSY) gauge theories on $\mathbb{C}^2$ and $\mathcal{A}_{N-1}$-Toda conformal field theories (CFTs) known as AGT-$\mathcal{W}$, we study a conjectured correspondence for $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, which we refer to as coset AGT. In this case, the dual CFT is a combined system whose symmetry algebra has 3 factors: A free boson, a $\widehat{\mathfrak{sl}}(n)_N$-Wess-Zumino-Witten (WZW) model, and what is known as an n-th $\mathcal{W}_N$-parafermion model. We specialize this last factor to its minimal models and show that, in this case, both sides of the duality have interesting combinatorics defined in terms of Young diagrams which are coloured.

For the SUSY gauge theories AGT dual to these minimal models, we show that the usual definition of their fundamental object to this conjecture, known as Nekrasov's instanton partition function, is ill-defined and has non-physical poles. We remove these poles by a redefinition of this instanton partition function, encoded by combinatorial conditions known as the Burge conditions.

We use these combinatorial conditions to check our proposal against well-known results for the CFT characters and conformal blocks of $\widehat{\mathfrak{sl}}(n)_N$-WZW models. Having checked our dictionary, we then obtain new conjectural combinatorial expressions for coset branching functions and $\widehat{\mathfrak{sl}}(N)_n$ characters. As a corollary to these, we also obtain new combinatorial relationships between certain pairs of what is known as coloured cylindric partitions. We finish by checking our conjectured combinatorial expressions through explicit computations to a given order.

Declaration of Authorship

I, Nicholas Macleod, declare that this thesis titled, 'AGT for $\mathcal{N} = 2$ $SU(N)$ Gauge Theories on $\mathbb{C}^2/\mathbb{Z}_n$ and Minimal Models' and the work presented in it are my own. I confirm that:

- The thesis comprises only my original work towards the Doctorate of Philosophy except where indicated in the preface;
- due acknowledgement has been made in the text to all other material used; and
- the thesis is fewer than the maximum word limit in length, exclusive of tables, maps, bibliographies and appendices as approved by the Research Higher Degrees Committee.

Signed: Nicholas Macleod

Date: March 29, 2023

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To my collaborators Masahide Manabe and Trevor Welsh, the work presented in here took the combination of all of us. A special mention to Trevor, who stuck with me on our project throughout covid, whether it was on the whiteboards in an empty department, or on the bench in the local park, or simply talking about who would win the next ashes test. A PhD takes the whole school of maths and stats and I also wish to specifically thank Thomas Quella, for both helping me with my questions on CFT (and his great seminar series all those years ago!) and keeping me on track through a really trying time, and David Ridout, for allowing me to audit his Lie Algebra course and helping me with all my questions about their workings. I would also be remiss to not thank Kirsten Hoak, the real engine that keeps this place running and has been helping me since before I started my masters, any question you have she would know the answer or who to ask, no matter what!

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Introduction

0.1 Background and Approach

In 2009, Alday, Gaiotto, and Tachikawa conjectured a correspondence between certain 4D $\mathcal{N} = 2$ $SU(2)$ class S superconformal field theories on $\mathbb{C}^2$, first discovered by Gaiotto [1], and 2D Liouville conformal field theory [2], a duality commonly referred to as the AGT conjecture. The AGT conjecture built on the pioneering work of Seiberg and Witten in obtaining an exact solution for the prepotential in 4D $\mathcal{N} = 2$ supersymmetric Yang-mills theories [3] and Nekrasov's subsequent computation of the instanton partition function Z_{inst} [4, 5], by identifying Z_{inst} with Liouville conformal blocks.

Subsequently, this was generalised to a correspondence known as AGT-$\mathcal{W}$ between $SU(N)$ gauge theories on $\mathbb{C}^2$ and $\mathcal{A}_{N-1}$-Toda CFTs [6], and further again to theories on $\mathbb{C}^2$ with more general gauge groups and CFTs related to more general $\mathcal{W}$-algebras [7, 8, 9]. Another generalization was made in [10], conjecturing a correspondence between $SU(N)$ gauge theories on the ALE spaces $\mathbb{C}^2/\mathbb{Z}_n$ [11] and a combined CFT of a free boson, an $\widehat{\mathfrak{sl}}(n)_N$ Wess-Zumino-Witten model and a so-called n-th $\mathcal{W}_N$-parafermion [12]. In this thesis, we consider this last generalisation which links $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ to CFTs with the symmetry algebra

$$\mathcal{A}(N, n; p) = \frac{\widehat{\mathfrak{gl}}(N)_n}{\widehat{\mathfrak{gl}}(N-p)_n}, \tag{0.1.1}$$

a correspondence we refer to as coset AGT.

Being a correspondence between two different theories, AGT dualities link many distinct areas of mathematics and physics. This includes integrability [13, 14, 15] (of special note, Hitchin systems [16]), topological strings and matrix models [17, 18], combinatorics and symmetric functions [19] (including new special polynomials based on algebras generalizing

$\mathcal{A}(N, n; p)$ [20, 21, 22]), and the geometric Langlands correspondence [23, 24], among the other more obvious links to CFTs and $\mathcal{N} = 2$ gauge theories.

When working on AGT correspondences, one will either work on providing evidence for the conjecture, such as in [19, 25, 26], or use a conjectured correspondence to perform calculations on one side of the duality using the objects and tools of the other, as in [27]. In this thesis, we will take both of these approaches at certain stages.

The initial proposal of the AGT conjecture was supported by explicit term-by-term comparisons of the instanton partition function for the gauge theory and the conformal blocks of Liouville CFT, under the proposed dictionary between parameters. This approach was then extended to $\mathcal{A}_2$-Toda CFT and $SU(3)$ gauge theories on $\mathbb{C}^2$ [25]. Alternatively, evidence was also obtained directly for the original AGT conjecture by proving that the bifundamental multiplet contribution to $Z^{SU(2)}_{inst}$ (which gives all other contributions in special cases) can be obtained in the dual CFT using a special basis of states parameterized by Jack polynomials [19].

This approach where AGT is used to derive special bases for CFTs was generalised to AGT involving $\mathcal{A}(2, n; p)$ in [28, 27]. In this setting, the special basis for $\mathcal{A}(2, n; p)$-modules is described in terms of pairs of checkerboard (2 coloured) Young diagrams. This linked the coset AGT conjecture to quantum toroidal algebras, which can also be approached combinatorially [29, 30, 31]. This work has not been generalized for $n > 2$.

Evidence for the case of AGT involving $\mathcal{A}(2, 4; p)$ has been obtained by comparison of the first few terms of Z_{inst} with conformal blocks of the S_3 parafermion algebra [32]. In this case, the conformal blocks were obtained through an algorithmic computation up to a small level, although these CFT calculations quickly grow cumbersome for higher levels.

In this thesis, we provide first further evidence for coset AGT with a more general procedure. We assume the existence of n-th $\mathcal{W}_N$-minimal models and then follow the approach of [33, 34]. We consider a subset of 4D gauge theories that are conjectured to be AGT dual to CFT minimal models, and show that in these theories Z_{inst} has non-physical poles and must be redefined from a sum over N-tuples of coloured Young diagrams to a sum over coloured cylindric (Burge) multipartitions [35]. The computation of the Burge conditions for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ is found in proposition 3.4.2.1.

Then under the special choice $p = N$, the CFTs with symmetry algebra $\mathcal{A}(N, n; N)$ reduce to a combined CFT of a $\widehat{\mathfrak{sl}}(n)_N$-WZW model and a free boson, allowing us to check Z_{inst} against the KZ differential equation for $\widehat{\mathfrak{sl}}(n)_N$-WZW conformal blocks [36]. The formalism for this idea is found in conjectures 4.3.3.1, 4.3.3.3, and 4.3.3.4.

Subsequently we use this generalized correspondence to calculate character functions for $\mathcal{A}(N, n; p)$-minimal models using the combinatorics implied by the gauge theory in conjecture 5.4.0.1. We consider theories with generic parameter p and use the Burge conditions we have derived to obtain new combinatorial identities for the branching functions of $\mathcal{A}(N, n; p)$ involving coloured cylindric multipartitions. Through this, we also obtain new expressions for $\widehat{\mathfrak{sl}}(N)_n$ string functions in the corollary to the conjecture 5.5.2.1. These expressions can be seen to generalize those obtained from the Kyoto school [37, 38, 39] and those obtained in [40, 41].

0.2 Structure

The thesis is organized as follows: In chapter 1 we will cover all the necessary background material required to understand these results. This will cover basic combinatorics, highest weight representation theory of finite and affine Lie algebras, 2D CFTs, and a generalization of the ADHM construction for instantons on $\mathbb{C}^2/\mathbb{Z}_n$. This material will be mostly standard, except when otherwise noted. Of particular note are the new concepts of Dynkin rings and dual weights in section 1.1.

Chapter 2 will review AGT and AGT-$\mathcal{W}$ for $SU(2)$ and $SU(N)$ gauge theories on $\mathbb{C}^2$ respectively. This will include reviewing the form of $Z_{inst}^{SU(N)}$ obtained in [4], the original AGT conjecture for $SU(2)$ gauge theories and Liouville CFT [2], the generalization of this AGT correspondence to AGT-$\mathcal{W}$ for $SU(N)$ gauge theories and $\mathcal{A}_{N-1}$-Toda CFT [6], and the connection of both of these to CFT minimal models in [33] and [27]. Importantly, this connection to minimal models is the work which we generalize in subsequent chapters.

Chapter 3 describes AGT for $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ (formalized in 3.3.0.1). We begin by reviewing the computation of the instanton partition function for these theories [42]. Then using this dictionary, we find gauge theories AGT dual to minimal models which have ill-defined partition functions. we then remove the poles of these partition function by calculating the Burge conditions.

Chapter 4 focuses on the case of $p = N$, which we use to check this proposed correspondence against known results for $\widehat{\mathfrak{sl}}(n)_N$-WZW models. We prove that the character function of the CFT matches the generating function of the instantons (proposition 4.2.2.2), and show that the first few terms in the series expansions of the instanton partition function and solutions to the KZ equation agree for some simple examples (section 4.4). The work in chapters 3 and 4 is based on [43] which the author co-authored with Omar Foda, Masahide Manabe, and Trevor Welsh.

Chapter 5 then considers the branching functions of CFTs with symmetry algebra $\mathcal{A}(N, n; p)$ for generic p. Inspired by the arguments in [27], we conjecture expressions for the coset branching functions as sums of products of $\mathcal{W}_N$-minimal model characters which we can compute efficiently using newly introduced objects called B-matrices (5.2.39). We then propose that these are equal to the generating functions of coloured Burge multipartition associated to the 4D gauge theory. Finally, we provide evidence for this through explicit calculations in sections 5.5.1 and 5.6.1. This work is based on [44], which is in collaboration with Trevor Welsh. Chapters 4 and 5 rely heavily upon Mathematica, and code is available upon request.

The main novel scientific results of the thesis are contained in chapters 3-5. The author's contribution to these collaborative efforts was to calculate Burge conditions for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ in chapter 3, and their reduction to cylindrinic plane partitions in section 4. The author also wrote code to compute series expansions for the instanton generating function and instanton partition functions up to a given order to both calculate and check the results in section 4.4.

In chapter 5 the author is responsible for calculating the coset branching functions, all calculations involving Dynkin rings, their duals, and their classes, the form of the coset-Burge conjecture, the formalism of sections 5.4.2 and 5.4.3, the proof of how to obtain conjecture 5.5.0.1 from the coset-Burge conjecture, all explicit computed examples in the chapter, and the checks of the cases $(N, n) = (2, 3), (3, 2)$ up to $O(q^{12})$, $(N, n) = (2, 4)$ to $O(q^{10})$, $(N, n) = (3, 4), (4, 3)$ to $O(q^6)$, $(N, n) = (3, 6)$ as described in section 5.6.1, while independently checking selected examples of all other cases. All code used for the computations noted was also due to the author. All results and computations throughout this chapter were developed through collaborative effort with Welsh.

Chapter 1

Preliminaries

This chapter will cover all the background material necessary to understand the content of chapters 2, 3, 4, and 5 for AGT on $\mathbb{C}^2$ and $\mathbb{C}^2/\mathbb{Z}_n$. We will focus on defining all notation and results we will use, while being light on proofs. This material is *not* pedagogic, although references will be provided to learn any of the material covered here.

1.0.0.1 Outline of Chapter

We will begin with the mathematical background required. In section 1.1, we introduce the basic notion of (coloured) partitions and (coloured) Young diagrams, cylindric, Burge Partitions and their generating functions. In sections 1.2 and 1.3, we review the representation theory of finite and affine Lie algebras. We discuss in detail the branching rules, and the Littelwood-Richardson rule for tensor product decompositions. We then define a new object, called a Dynkin ring, which gives a correspondence between the dominant integral weights of $\widehat{\mathfrak{sl}}(n)$ of level N and the dominant integral weights of $\widehat{\mathfrak{sl}}(N)$ of level n. Next we review the technology of crystal graphs for affine Lie algebras, and use them to calculate characters for tensor products of representations. We connect these two areas in section 1.4, where we apply the combinatorics of Young diagrams to the highest weight representation theory of $\widehat{\mathfrak{sl}}(n)$.

The rest of the chapter covers the physics background we will use, mostly from a mathematical viewpoint. In section 1.5, we axiomatically define the main concepts of 2D conformal field theories (CFTs), importantly we define the 4-point conformal blocks. We then define the specific CFT models we will work with, that being the minimal models, Liouville and Toda CFTs, Wess-Zumino-Witten (WZW) models, and coset models. This will include the $\widehat{\mathfrak{sl}}(n)$-WZW fusion rules and the Knizhnik-Zamolodichikov (KZ) differential equation. We will also discuss the relation between WZW models and the theory of integrable affine Lie algebra representations. We will also cover the Coulomb-gas formalism which we will use for AGT involving CFT minimal models. In section 1.6 we construct the asymptotically locally Euclidean (ALE) spaces associated to $\mathbb{C}^2/\mathbb{Z}_n$ and we review the Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction of instantons on $\mathbb{C}^2$ and its generalisation to these ALE spaces.

1.1 Young Diagrams and Colourings

This section will constitute a brief review of the necessary combinatorics required to understand $SU(N)$ AGT for gauge theories on $\mathbb{C}^2$ or $\mathbb{C}^2/\mathbb{Z}_n$ corresponding to CFT minimal models. We will focus on partitions of integers and the equivalent Young diagrams. This material will be standard and can be found in more depth in [45] and [46], and both of these references cover some other common applications of this theory.

We will then extend this material in two ways: First, by colouring our Young diagrams, which will be used in AGT for $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ in chapters 3, 4, 5. This combinatorial material can be found in [45].

Second in section 1.4, we will discuss cylindric partitions (or Burge multipartitions depending on the source), which first appeared in [35]. Cylindric partitions are used to define the partition function for $SU(N)$ gauge theories which are AGT dual to minimal model CFTs [33, 34]. We will first encounter this in sections 2.2.4 and 2.3.3, and this will be fundamental to all subsequent work throughout the thesis. The exposition on cylindric multipartitions is non-standard, but is based on [34] and [47].

1.1.1 Basic Definitions and Notation

A *partition* I of $n \in \mathbb{Z}_{\geq 0}$, is a sequence of weakly decreasing[1] positive integers $I = (p_1, p_2, \dots)$ such that $\sum p_i = n$. As $p_i \in \mathbb{Z}_{\geq 0}$, only finitely many p_i are non-zero. Due to this, we take partitions to have an infinite tail of 0's. A *Young diagram* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N, 0, 0, \dots)$, where $N \in \mathbb{Z}_{\geq 0}$ and $\lambda_N > 0$, is a visual representation of I using left-justified stacked rows of boxes extending down the page, where the number of boxes in the i-th row of λ is defined by $\lambda_i = p_i$. When $N = 0$, we denote the *empty partition* by $\lambda = (0, 0, \dots) = \emptyset$.

We refer to the numbers λ_i as the *parts* of λ so that the i-th part is equal to the number of boxes in the i-th row. A Young diagram has only finitely many parts of non-zero size and infinitely many empty rows stacked at the bottom of the diagram.

We define the number of non-zero parts of a Young diagram to be its *length* so that for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N, 0, 0, \dots)$ with $\lambda_N \neq 0$, we have $l(\lambda) = N$. In the sequel, we will

[1] A sequence $(a_n)_{n\in\mathbb{Z}}$ of weakly decreasing numbers, satisfy $a_{k+1} \leq a_k$.

notate our Young diagrams using only their non-zero parts, that is $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N)$. We will sometimes use the alternate notation for Young diagrams $\lambda = (k_1^{m_1}, k_2^{m_2}, \ldots)$ where $k_1 > k_2 > \ldots$ is a sequence of strictly positive integers, and $m_i \in \mathbb{Z}_{>0}$ is the number of parts $\lambda_j = k_i$. The diagram $\lambda = (k_1^{m_1}, k_2^{m_2}, \ldots, k_{N'}^{m_{N'}})$ can expressed in our original notation as

$$\lambda = (\underbrace{k_1, k_1, \ldots, k_1}_{m_1}, \underbrace{k_2, \ldots, k_2}_{m_2}, \ldots, \underbrace{k_{N'}, \ldots, k_{N'}}_{m_{N'}}), \tag{1.1.1}$$

note that $N = l(\lambda) = \sum_{i=1}^{N'} m_i$. Under this second notation we have $n = \sum_i m_i k_i$, and example 1.1.1.2 below uses both of these notations explicitly. One can always determine the notation we are employing to describe a Young diagram by the presence of exponents on the parts, no exponent means we are employing the original partition-like notation.

We consider a Young diagram to be placed on a semi-infinite lattice. We will label the x axis extending from top to bottom and the y axis from left to right with positive numbers, this is referred to as the *English convention*. Some texts employ the *French convention* where the x axis is labelled from bottom to top and with boxes extending up the page. In figure 1.1 we have attached a visual of the axes for the English convention we employ.

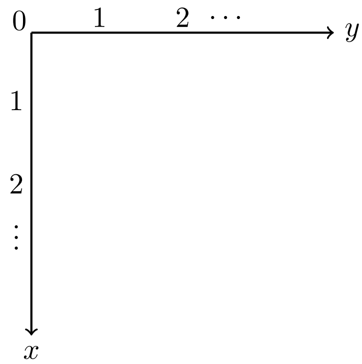


FIGURE 1.1: The x and y axes we place Young diagrams on, and their positive directions.

We refer to a box $\square \in \lambda$ by the (x, y) coordinates of its bottom right corner, so that the j-th box $\square$ in the i-th row is referred to as $\square = (i, j)$. The *transpose Young diagram*[2] λ^T is a Young diagram whose rows are equal to the columns of λ, which is the reflection of λ across the line $x = y$.

We define the *size* of a Young diagram $|\lambda|$ to be its total number of boxes (this is equal to n, the integer being partitioned), and define the *arm length* $A_\lambda(\square)$ and *leg length* $L_\lambda(\square)$ of a box $\square = (i, j)$ to be distance from $\square$ to the end of the i-th row and j-th column respectively,

$$A_\lambda(\square) = \lambda_i - j, \quad L_\lambda(\square) = \lambda_j^T - i. \tag{1.1.2}$$

[2] Some sources refer to this as a *conjugate Young diagram*.

We will sometimes use formulas involving the quantities $A_\lambda(\Box)+1$ and $L_\lambda(\Box)+1$. In these instances we will employ the notation of adding a superscript + sign to $A_\lambda(\Box)$ or $L_\lambda(\Box)$ to indicate this value. Using this notation we have:

$$A_\lambda^+(\Box) = A_\lambda(\Box) + 1 \tag{1.1.3}$$

$$L_\lambda^+(\Box) = L_\lambda(\Box) + 1. \tag{1.1.4}$$

Remark 1.1.1.1. This notation is *not* uniform across the literature. For example in [41], they employ the similar notation of

$$\begin{aligned} A_\lambda^+(\Box) &= A_\lambda(\Box) + \frac{1}{2}, \\ A_\lambda^{++}(\Box) &= A_\lambda(\Box) + 1. \end{aligned}$$

We will never use this notation involving half integers within this thesis, but care must be taken by the reader to check the convention when encountering this notation in the literature.

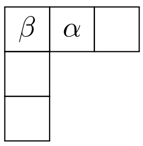
We allow the definition of arm and leg length to extend to boxes $\Box$ that are not in a Young diagram λ. In this case, we have that the arm and leg length of $\Box$ are negative numbers. The converse is also true, if the arm and leg length of a box are negative then $\Box \notin \lambda$. Formally, a box cannot exist on its own without being in a Young diagram, and when we have a negative arm or leg length we are treating the situation informally as a convenient tool to pick specific negative integers.

Finally, we define the *outline* of a Young diagram λ to be a lattice path $\mathcal{L}_\lambda = \{(i_0, j_0), (i_0, j_0), \ldots, (i_k, j_k)\}$, where $k = l(\lambda)+\lambda_1+1$, of south and west steps, which begins at $(i_1, j_1) = (0, \lambda_1)$ and ends at $(i_k, j_k) = (0, l(\lambda))$ and traces the outline of λ. Formally, $(i_m, j_m)-(i_{m+1}, j_{m+1}) = (-1, 0)$ if $(i_{m+1}, j_{m+1}) = (i', \lambda_{i'})$ for some $0 \le i' \le l(\lambda)$, and $(i_m, j_m) - (i_{m+1}, j_{m+1}) = (0, 1)$ if $(i_{m+1}, j_{m+1}) \neq (i', \lambda_{i'})$ for some i'. Note that it is perfectly reasonable to extend these outlines infinitely along the x- and y-axes, although we will not need this. We can easily obtain the outline of the transpose diagram by reversing the i and j-coordinates and order of steps, so that $\mathcal{L}_{\lambda^T} = \{(j_k, i_k), (j_{k-1}, i_{k-1}), \ldots, (j_0, i_0)\}$.

Example 1.1.1.2. If we consider all the possible ways of partitioning 5 and their corresponding representations as Young diagrams we have:

$$
\begin{aligned}
&(5) = (5)\\
&(4,1) = (4,1)\\
&(3,2) = (3,2)\\
&(3,1,1) = (3,1^2)\\
&(2,2,1) = (2^2,1)\\
&(2,1,1,1) = (2,1^3)\\
&(1,1,1,1,1) = (1^5)
\end{aligned}
$$

We now focus on visualising the partition $(3,1,1,0,\dots)$ of 5 by the following Young diagram $\lambda = (3,1,1) = (3,1^2)$, in which we have labelled two boxes as α and β. Remember that the positive y axis points along the first row left to right and the positive x axis along the first column top to bottom.



The box α has coordinates $(1,2)$ and the box β has coordinates $(1,1)$. We have $A_\lambda(\alpha) = 1$, $L_\lambda(\alpha) = 0$, and $L_\lambda(\beta) = A_\lambda(\beta) = 2$. We also have $A^+_\lambda(\alpha) = 2$. Note, that we do not visually represent the infinite empty rows in any way. Finally, the outline of λ is the lattice path $\mathcal{L}_\lambda = \{(0,3),(1,3),(1,2),(1,1),(2,1),(3,1),(3,0)\}$.

1.1.2 Coloured and Cylindric Partitions

The combinatorics of the AGT correspondences considered within this thesis will involve two generalizations of the basic Young diagrams described above. In this section, we will describe coloured Young diagrams and cylindric partitions. In chapter 2, when considering $SU(N)$ gauge theories that are AGT dual to minimal model CFTs we will see that instanton partition function is defined as a sum over cylindric partitions. While in chapters 3, 4, and 5 the partition function will be a sum over coloured cylindric partitions.

Let λ be a Young diagram. Denote by $Y(\lambda)$ the set of ordered pairs $(x,y) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ corresponding to the coordinates of the boxes in λ. A *coloured Young diagram* with charge $\sigma \in \mathbb{Z}_n$ is the data of the Young diagram λ together with a colouring map[3] $\phi : Y(\lambda) \longrightarrow \mathbb{Z}_n$ which associates to a box $\square = (x,y)$ a colour $\phi(\square) \equiv \sigma + y - x \bmod n$. Under this colouring,

[3] This definition can be generalised to arbitrary colouring maps, although we will never consider them in this thesis.

the colours increase from left to right in the rows and from bottom to top in the columns, and stay constant along the diagonals.

Denote the coloured Young diagram by λ^σ. We let the total number of boxes coloured $i \in \{0, 1, \ldots, n-1\}$ in λ^σ to be $k_i \in \mathbb{Z}_{\geq 0}$, and we have $|\lambda^\sigma| = \sum_{i=0}^{n-1} k_i$. We will alternatively characterise the colour content of λ^σ by the differences $\delta k_i = k_i - k_0$ for $i = 1, \ldots, n-1$ and the size $|\lambda^\sigma|$, it is clear that if we included $i = 0$ we would have $\delta k_0 = 0$.

Define the *boundary* of a Young diagram λ to be all $\square = (i, j) \in \lambda$ such that $(i+1, j+1) \notin \lambda$. A *boundary n-strip* B is a set of n contiguous boxes in the boundary of λ such that if we remove B from λ we are left with another Young diagram.

If λ is coloured with n colours, a boundary n-strip contains exactly 1 box of each colour 0 through $n-1$. To see this, we note that if we begin at the top right box in B and move through to the bottom left box in a continuous fashion, each consecutive box in B lies either directly to the left or below the previous one. By definition of the colouring, this means that if a box is coloured $i \in \{0, \ldots, n-1\}$, the next box obtained in this process is coloured $i-1$ mod n. As there are n boxes in B, it must then contain one box of each colour.

Let λ be a Young diagram coloured with n colours, with colour data specified by $\delta\mathbf{k} = (\delta k_1, \ldots, \delta k_{n-1})$, and B be a boundary n-strip in λ. Since B contains 1 box of each colour, the Young diagram λ' obtained by removing B has colour data also defined by the vector $\delta\mathbf{k}$ with $|\lambda| - n = |\lambda'|$.

We can then repeat this process successively until we are left with a Young diagram ${}^n\lambda^\sigma$ that contains no boundary n-strips to remove and has the same colour difference vector as the original Young diagram. A Young diagram that has no boundary n-strips is called an *n-core*, and we say that ${}^n\lambda^\sigma$ is the n-core of λ. It is known (see for example [45]) that the n-core of Young diagram is unique, and is independent of the choice of boundary n-strip removed at each step. We also note that for each n and each $\delta\mathbf{k}$ there exists an unique n-core[4]. Sometimes it is convenient to think of a coloured Young diagram as constructed from the reverse of this process, where we begin with an n-core and add boundary n-strips.

Example 1.1.2.1. If we consider the Young diagram $\lambda = (3, 1, 1)$ of the example 1.1.1.2 and assign it a charge $\sigma = 1 \in \mathbb{Z}_3$. We obtain the coloured charged Young diagram $\lambda^{(\sigma=1)}$:

1	2	0
0		
2		

[4]We will not prove this, and to do so is not clear from what we have covered. The simplest proof involves the use of James' n-abacus.

For $\lambda^{(\sigma=1)}$ we have, $k_0 = 2$, $k_1 = 1$ and $k_2 = 2$. Alternatively, we have $\delta k_1 = -1$, $\delta k_2 = 0$, and $|\lambda^{(\sigma=1)}| = |\lambda| = 5$. Note that the colours are all defined modulo 3. There are no boundary 3-strips in λ so it is a 3-core.

In the sequel, we will deal with objects which are described by N-tuples of coloured Young diagrams subject to certain inequalities between parts. We will formalise this using *cylindric partitions* [35], closely following the exposition in [47]. We define a partial ordering on the set of Young diagrams of length N or less, where $\mu = (\mu_1, \ldots, \mu_N)$ and $\mu' = (\mu'_1, \ldots, \mu'_N)$ (where we allow some amount of parts equal to 0) are ordered $\mu \leq \mu'$ if

$$\mu_i \leq \mu'_i, \quad 1 \leq i \leq N. \tag{1.1.5}$$

Using this can define *cylindric partitions*[5].

Definition 1.1.2.2. Given $N \in \mathbb{Z}_{>0}$, two partitions $\mu = (\mu_1, \ldots, \mu_N)$ and $\mu' = (\mu'_1, \ldots, \mu'_N)$ such that $\mu \leq \mu'$, and an integer $d \geq \mu_1$, a *cylindric partition* π of shape $\mu'/\mu/d$ is an array $\pi_{i,j}$, $1 \leq i \leq N, \mu_i + 1 \leq j \leq \mu'_i$, of non-negative integers of the form

$$\begin{array}{cccccccccc}
 & & & \pi_{1,\mu_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{1,\mu'_1} \\
 & \pi_{2,\mu_2+1} & \cdots & \pi_{2,\mu_1+1} & \cdots & \cdots & \cdots & \cdots & \pi_{2,\mu'_2} & \\
 & \vdots & & \vdots & \ddots & & & & & \\
\pi_{N,\mu_N+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{N,\mu'_N} & & &
\end{array}$$

such that the entries are weakly decreasing across rows and down columns, that is

$$\begin{aligned}
\pi_{i,j} \geq& \pi_{i+1,j}, \quad 1 \leq i \leq N-1, \ \ \mu_i + 1 \leq j \leq \mu'_{i+1} && (1.1.8)\\
\pi_{i,j} \geq& \pi_{i,j+1}, \quad 1 \leq i \leq N, \ \ \mu_i + 1 \leq j \leq \mu'_i - 1 && (1.1.9)
\end{aligned}$$

while also satisfying the cyclic condition

$$\pi_{N,j} \geq \pi_{1,j+d}, \quad \max(\mu_1 + 1 - d, \mu_N + 1) \leq j \leq \min(\mu'_1 - d, \mu'_N) \tag{1.1.10}$$

[5]This is not strictly true, and is only a subset of all cylindric paritions as described in [35]. Fully generalized cylindric partitions allow for a weakening of the weakly decreasing conditions by a further set of positive integral numbers $\alpha = (\alpha_1, \alpha_2, \ldots)$. This allows successive elements in the array to be larger by at most one of these integers. For example when we enforce the weakly-decreasing conditions on π_{i,μ_1+1} and π_{i,μ_1+2} we have

$$\pi_{i,\mu_1+1} \geq \pi_{i,\mu_1+1}, \tag{1.1.6}$$

for cylindric partitions in full generality this condition is modified to

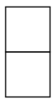
$$\pi_{i,\mu_1+1} + \alpha_1 \geq \pi_{i,\mu_1+1}. \tag{1.1.7}$$

When necessary to refer to fully general cylindric partitions in this thesis we will notate them as (α, β)-cylindric partitions.

We define the *size* of a cylindric partition π to be the sum of its entries, $|\pi| = \sum_{i,j} \pi_{i,j}$. Note that when using a Young diagram $\mu = (\mu_1, \ldots, \mu_N)$ of length $l(\mu) = m < N$ to define a cylindric partition, one must ensure to add $N - m$ parts of size 0 to μ, as these are relevant to the cyclic condition. The cyclic condition can be interpreted as placing a copy of the array below itself such that the row $(\pi_{1,j})_{j=\mu_1+1,\ldots,\mu'_1}$ is directly below the row $(\pi_{N,j})_{j=\mu_N+1,\ldots,\mu'_N}$ and the entry π_{1,μ_1+1} in the copy is d spots to the left of π_{1,μ_1+1} in the original diagram, and repeating this process ad infinitum. Using this point of view, the cyclic condition then amounts to a weakly non-decreasing condition for the columns on this infinite size array, and we can imagine that our array is wrapped around an infinite cylinder. Viewing cylindric partitions as arrays wrapped on an infinite cylinder give the cylindric interpretation of their data.

In subsequent sections, we will only consider cylindric partitions π whose rows can be arbitrarily long. As such, we will take $\mu' = (\underbrace{k, k, \ldots, k}_{N})$ for $k >> 0$, which we will denote by $\mu' = (\infty^N)$. In practice, this means that we ignore μ' in the definition of a cylindric partition's shape and consider all arrays whose rows are displaced according to μ that satisfy the weakly-decreasing and cyclic conditions.

Example 1.1.2.3. Consider the following cylindric partition shape $(\infty^3)/(1,1,0)/4$. The Young diagram $\mu = (1,1,0)$ is visualized as



and we note that for the purposes of cylindric partitions we consider μ to have one zero length row. The following array π satisfies the weakly-decreasing and cyclic conditions as described above and is an example of a cylindric partition of shape $(\infty^3)/(1,1,0)/4$:

$$\begin{array}{ccccccccc} & 5 & 4 & 4 & 2 & 2 & 1 & 1 & 0 \; \ldots \\ & 3 & 2 & 2 & 2 & 1 & 0 & 0 \; \ldots & \\ 4 & 2 & 2 & 1 & 0 & 0 \; \ldots & & & \end{array}$$

The reader should note that we can see the Young diagram for $\mu = (1,1,0)$ in the relative shifts in the first element of each row. We can visualise the cyclic condition, using the following infinite size array

$$
\begin{array}{ccccccccccccccccccccccc}
&&&&&&&&&&&&&& 5 & 4 & 4 & 2 & 2 & 1 & 1 & 0 & \dots \\
&&&&&&&&&&&&&& 3 & 2 & 2 & 2 & 1 & 0 & 0 & \dots & \\
&&&&&&&&&&&&& 4 & 2 & 2 & 1 & 0 & 0 & \dots &&& \\
&&&&&&&&&& 5 & 4 & 4 & 2 & 2 & 1 & 1 & 0 & \dots &&&& \\
&&&&&&&&&& 3 & 2 & 2 & 2 & 1 & 0 & 0 & \dots &&&&& \\
&&&&&&&&& 4 & 2 & 2 & 1 & 0 & 0 & \dots &&&&&&& \\
&&&&&& 5 & 4 & 4 & 2 & 2 & 1 & 1 & 0 & \dots &&&&&&&& \\
&&&&&& 3 & 2 & 2 & 2 & 1 & 0 & 0 & \dots &&&&&&&&& \\
&&&&& 4 & 2 & 2 & 1 & 0 & 0 & \dots &&&&&&&&&&& \\
&& 5 & 4 & 4 & 2 & 2 & 1 & 1 & 0 & \dots &&&&&&&&&&&& \\
&& 3 & 2 & 2 & 2 & 1 & 0 & 0 & \dots &&&&&&&&&&&&& \\
& 4 & 2 & 2 & 1 & 0 & 0 & \dots &&&&&&&&&&&&&&& \\
\iddots & \vdots & \vdots & \vdots & \vdots & \vdots &&&&&&&&&&&&&&&&&
\end{array}
$$

where it is understood that the array repeats infinitely above and below the top and bottom rows in the same pattern. Note that the weakly-decreasing condition is satisfied along the rows and down the columns of this infinite array.

We can also think about cylindric partitions from the perspective of their composite rows, which leads us to an equivalent formulation. Given a cylindric partition $(\infty^N)/\mu/d$, define N separate vectors of integers $(\lambda^{(0)}, \dots, \lambda^{(N-1)})$, where the i-th vector is defined by $\lambda^{(i)} = (\pi_{i+1,\mu_i+1}, \pi_{i+1,\mu_i+2}, \dots)$. The condition (1.1.9) for each row of non-negative integers to be weakly decreasing implies $\lambda^{(i)}$ is a Young diagram. Using this perspective, we stack the Young diagrams adjacent to each other in a newly introduced z-axis, such that the i-th diagram has z-coordinate i and the difference in y-coordinate of the top left boxes (that is $\square = (1,1)$ when considering each diagram individually on its own x and y axes) of the $(i-1)$-th and i-th diagrams is defined by the difference $\mu_i - \mu_{i+1}$ for $1 \leq i < N$, and $d - \mu_1$ for the $(N-1)$-th and 0th diagrams.

The condition (1.1.8) corresponds to demanding there is a weakly decreasing condition across adjacent rows of these Young diagrams along the z-axis. If we define all superscript labels modulo N, this is formalised by the following *cylindric inequalities*

$$\lambda_j^{(i)} \geq \lambda_{j+\zeta_i}^{(i+1)}, \tag{1.1.11}$$

where $\zeta_i = \mu_{i+1} - \mu_{i+2}$ for $0 \leq i \leq N-2$ and $\zeta_{N-1} = d - \mu_1 + \mu_N$. We can then capture the numbers in this array as one N-tuple of Young diagrams, which we denote by $\lambda = (\lambda^{(0)}, \dots, \lambda^{(N-1)})$, and a vector of positive integers $\zeta = (\zeta_0, \dots, \zeta_{N-1})$. This allows us to think about cylindric partitions as tuples of regular Young diagrams subject to certain relations. This is how we will approach cylindric partitions in the sequel.

Remark 1.1.2.4. By introducing a z-axis to stack Young diagrams which live in the xy-plane, we have introduced a notion that cylindric partitions live in 3 dimensional space. Some authors have referred to cylindric partitions as cylindric plane partitions (CPPs) in this sense, as an analogy to plane partitions which are 3D generalizations of Young diagrams.

Example 1.1.2.5. We consider the cylindric partition π of shape $(\infty^3)/(1,1,0)/4$ from the example 1.1.2.3 above. It corresponds to the triple of Young diagrams $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)})$ where

$$\lambda^{(0)} =(5,4,4,2,2,1,1)$$
$$\lambda^{(1)} =(3,2,2,2,1)$$
$$\lambda^{(2)} =(4,2,2,1)$$

and $\zeta_0 = 0$, $\zeta_1 = 1$, and $\zeta_2 = 3$, giving the 3 inequalities

$$\lambda_j^{(0)} \geq \lambda_j^{(1)} \tag{1.1.12}$$
$$\lambda_j^{(1)} \geq \lambda_{j+1}^{(2)} \tag{1.1.13}$$
$$\lambda_j^{(2)} \geq \lambda_{j+3}^{(0)} \tag{1.1.14}$$

When discussing cylindric partitions in the sequel, we will refer to the formal *finite* array as the *fundamental domain*, and we will refer to the data of μ and d (and equivalently the cylindric inequalities) as the *fundamental domain shape*. Note that the cylindric inequalities, and hence fundamental domain shapes, are determined by *one* vector of positive integers $\zeta = (\zeta_0, \dots, \zeta_{N-1})$, as this tells us both the number of Young diagrams and their relative heights.

When visualising the fundamental domain of a cylindric partition in terms of N Young diagrams in 3D space, one should view the array π as a projection along the y axis onto the xz plane. This means that the Young diagram y-axis extends into and out of the page and we choose to view the boxes of the component Young diagrams as coming *out* perpendicular to the page. The entries π_{ij} then tell one the number of boxes extending out of the page along the y axis.

1.2 Review of Finite Semi-Simple Lie Algebras and Their Representations

In this section, we review the basic notion of finite dimensional Lie algebras and their representation theory. We will assume familiarity with the content and only fix notation. The content and notation here is drawn from [48, 49] and [50]. A standard reference such as [51] or [52] will be useful.

1.2.1 Basic Definitions and Notation

Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{h}$ its corresponding Cartan subalgebra. The *rank* of $\mathfrak{g}$ is equal to $\dim(\mathfrak{h})$ and we will often denote it by $r \in \mathbb{Z}_{>0}$. Denote by $\Delta \subset \mathfrak{h}^*$ the set of roots of $\mathfrak{g}$ and Δ_+ the set of *positive roots* (see [50, §13] for a definition). Let $\{\alpha_1, \cdots, \alpha_r\}$ be a set of simple roots, and let $(\, , \,)$ be the Killing form of $\mathfrak{g}$. Let $[X, -] \in \mathrm{End}(\mathfrak{g})$ be the element $Y \mapsto [X, Y]$. We can define the Killing form explicitly as

$$(X, Y) := \mathrm{tr}([X, [Y, -]]), \quad X, Y \in \mathfrak{g}, \tag{1.2.1}$$

note that this is a trace over the endomorphism

Remark 1.2.1.1. The map $Y \mapsto [X, Y]$ above defines a representation of $\mathfrak{g}$ which we call the adjoint representation $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. This adjoint action is sometimes notated as $ad(X)(Y) = [X, Y]$.

We can identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$ via $(\, , \,)$ and this induces a positive-definite scalar product on $\mathfrak{h}^*$, which we also denote by $(\, , \,)$. We use this scalar product as a normalisation to define the *coroot* $\alpha^\vee \in \mathfrak{h}$ associated to $\alpha \in \Delta$ by

$$\alpha^\vee = 2\frac{\alpha}{(\alpha, \alpha)}. \tag{1.2.2}$$

The Cartan matrix $\bar{A} = (A_{ij})_{1 \leq i,j \leq r}$ of $\mathfrak{g}$ is defined to be

$$A_{ij} = 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{1.2.3}$$

One can classify the allowed configurations of simple roots by *Dynkin diagrams*, which are directed graphs with vertices labelled $1, \ldots, r$ and where the number of arrows from vertex i to vertex j is equal to the entry A_{ij} of the Cartan matrix. There is a highest root $\theta \in \Delta$ defined by the property that $\alpha_i + \theta \notin \Delta_+$ for each α_i.

Let $Q(\mathfrak{g}) = \mathbb{Z}_{span}\{\alpha_1, \ldots, \alpha_r\} \subset \mathfrak{h}^*$ be the *root lattice* of $\mathfrak{g}$ and $Q^\vee(\mathfrak{g}) = \mathbb{Z}_{span}\{\alpha_1^\vee, \ldots, \alpha_r^\vee\} \subset \mathfrak{h}$ the coroot lattice. Denote by $P(\mathfrak{g}) \subset \mathfrak{h}^*$ the *weight lattice* which is dual to the coroot lattice. Let $\{\bar{\Lambda}_1, \ldots, \bar{\Lambda}_r\}$ be the set of fundamental weights which is the dual basis of the simple coroots $\{\alpha_1^\vee, \ldots, \alpha_r^\vee\}$, so that we have

$$P(\mathfrak{g}) = \mathbb{Z}\bar{\Lambda}_1 \oplus \cdots \oplus \mathbb{Z}\bar{\Lambda}_r. \tag{1.2.4}$$

A *Weyl reflection* $s_\alpha : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ is defined to be

$$s_\alpha \bar{\Lambda} = \bar{\Lambda} - (\alpha^\vee, \bar{\Lambda})\alpha, \tag{1.2.5}$$

and the *Weyl group* W of $\mathfrak{g}$ is the group generated by the *simple Weyl reflections* s_α, $\alpha \in \Delta$.

To finish this brief review, here we show a presentation of $\mathfrak{g}$ using the *Cartan-Weyl* basis with the generators H^i, $i = 1, \ldots, r$ (which generate $\mathfrak{h}$), E^α and $E^{-\alpha}$ for $\alpha \in \Delta_+$. We also define the additional notation

$$\alpha \cdot H = \sum_{i=1}^{r} (\alpha, H_i) H_i. \tag{1.2.6}$$

In this presentation, the defining Lie brackets are

$$[H^i, H^j] = 0 \tag{1.2.7}$$

$$[H^i, E^\alpha] = (\alpha, H_i) E^\alpha \tag{1.2.8}$$

$$[E^\alpha, E^\beta] = \begin{cases} N_{\alpha,\beta} E^{\alpha+\beta}, & \alpha + \beta \in \Delta \\ (E^\alpha, E^{-\alpha}) \alpha \cdot H, & \alpha = -\beta \\ 0, & \text{else} \end{cases} \tag{1.2.9}$$

together the Serre relations

$$\mathrm{ad}(E^{\alpha_i})^{1-A_{ij}}(E^{\alpha_i}) = \mathrm{ad}(E^{-\alpha_i})^{1-A_{ij}}(E^{-\alpha_i}) = 0, \quad i \neq j, \tag{1.2.10}$$

where $N_{\alpha,\beta}$ is a constant.

1.2.2 Highest Weight Representations of Semi-Simple Lie Algebras

A representation of $\mathfrak{g}$ is a complex vector space V together with a Lie algebra homomorphism $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$. We define a *weight* $\bar{\Lambda}$ of V to be a linear functional on $\mathfrak{h}$ such that the following *weight space* $V_{\bar{\Lambda}}$ has nonzero dimension

$$V_{\bar{\Lambda}} = \{v \in V | \forall h \in \mathfrak{h}, \;\; h \cdot v = \bar{\Lambda}(h) v\}, \tag{1.2.11}$$

and we define the *space of weights* Ω_V to be all $\bar{\Lambda}$ satisfying the above. In the sequel, we will say a vector $v \in V$ is a weight vector of weight $\bar{\Lambda}$, if $h \cdot v = \bar{\Lambda}(h)v$.

Remark 1.2.2.1. The weights of the adjoint representation are the *roots* of $\mathfrak{g}$.

We say a weight $\bar{\Lambda}$ of a representation is *integral*, if

$$\bar{\Lambda}_\alpha = (\bar{\Lambda}, \alpha^\vee) \in \mathbb{Z}, \tag{1.2.12}$$

for any root α, and we say it is *dominant* if $\bar{\Lambda}_\alpha \geq 0$.

By acting on a vector v of weight $\bar{\Lambda}$ in a $\mathfrak{g}$ representation with E^α, and using the relation (1.2.8), we have $H^i(E^\alpha v) = (E^\alpha H^i + \alpha^i E^\alpha)v = (\alpha^i + \bar{\Lambda})(E^\alpha v)$. From this we can see that if the action of E^α (resp $E^{-\alpha}$) on a state with weight $\bar{\Lambda}$ is non-zero, it produces a state with weight $\bar{\Lambda} + \alpha$ (resp $\bar{\Lambda} - \alpha$).

Thus for a finite dimensional, irreducible, representation of a simple Lie algebra $\mathfrak{g}$, there must be a vector v of dominant integral weight $\bar{\Lambda}$ such that $E^\alpha v = 0$ for all $\alpha \in \Delta^+$ (as the dimension of the representation is at least equal to the number of distinct weights, and if there were no weight satisfying this property the representation would have infinite dimension). We call this special weight $\bar{\Lambda}$ the *highest weight* and the representation a *highest weight representation*. We will also often use the physics notation of representing a highest weight vector of weight $\bar{\Lambda}$ by a ket $|\bar{\Lambda}\rangle$.

Alternatively, we will consider such a representation as generated by the *highest weight vector* $v_{\bar{\Lambda}}$, by the action of the operators corresponding to the negative roots $E^{-\alpha}$ in the Cartan-Weyl basis for $\alpha \in \Delta_+$. These basis elements are sometimes referred to as *lowering ladder operators*, which is an alternative terminology borrowed from physics (the generators E^α are similarly referred to as the *raising ladder operators*). We will henceforth drop the "ladder", and refer to $E^{\pm\alpha_i}$ as the *raising* and *lowering operators*.

The converse is true, we can generate a finite-dimensional, irreducible $\mathfrak{g}$ representation starting from a dominant integral weight as we now describe: A *Verma module* $\mathcal{V}_{\bar{\Lambda}}$ of highest weight $\bar{\Lambda}$ is constructed by taking a highest weight state $|\bar{\Lambda}\rangle$, and then constructing an infinite dimensional vector space by taking the span of elements

$$E^{-\alpha_{i_1}} E^{-\alpha_{i_2}} \dots E^{-\alpha_{i_k}} |\bar{\Lambda}\rangle, \quad k \in \mathbb{Z}_{\geq 0}, \tag{1.2.13}$$

and imposing the defining Lie bracket relations of $\mathfrak{g}$. This space $\mathcal{V}_{\bar{\Lambda}}$ is obviously a $\mathfrak{g}$-representation.

We can construct a finite-dimensional, irreducible, highest weight module $L_{\bar{\Lambda}}$ of highest weight $\bar{\Lambda}$ by taking a suitable quotient of $\mathcal{V}_{\bar{\Lambda}}$ by a maximal submodule M (see [51, §VI] for the details of this construction, and a proof of the irreducibility and finite-dimensionality of $L_{\bar{\Lambda}}$). Thus states in $L_{\bar{\Lambda}}$ are of the form $(E^{-\alpha_1})^{n_1}\dots(E^{-\alpha_r})^{n_r}\,|\bar{\Lambda}\rangle$.

Importantly, there is an algorithm to calculate all possible $n_i \in \mathbb{Z}_{\geq 0}$, given a specific $\bar{\Lambda}$. The algorithm is phrased in terms of *Dynkin labels*, which we now define. Let $\bar{\Lambda}$ be a highest weight, and write $\bar{\Lambda} = \sum_{i=1}^{r} l_i\bar{\Lambda}_i$, where $\bar{\Lambda}_i$ are the fundamental weights of $\mathfrak{g}$. The coefficients l_i for $1 \leq i \leq r$ are the Dynkin labels of a representation. We will also denote (by an abuse of notation) the weight $\bar{\Lambda}$ by its Dynkin labels as $\bar{\Lambda} = [l_1, \dots, l_r]$.

We are now ready to describe the algorithm to construct the states in $L_{\bar{\Lambda}}$. Beginning from $\bar{\Lambda}$ we form sequences of weights for each i such that $l_i > 0$ that are of the form $\bar{\Lambda} - n\alpha_i$ for $n = 1, \dots, l_i$. Then we repeat this process for each weight $\bar{\Lambda} - n\alpha_i$ in these sequences that we have generated, and iterate until we have no more states that have at least 1 positive Dynkin label. This algorithm always terminates as $\dim(L_{\bar{\Lambda}}) < \infty$, although we will not prove this fact, and $L_{\bar{\Lambda}}$ is the span of all states we have generated in all of these sequences.

Using these definitions Élie Cartan classified the finite dimensional irreducible representations of simple Lie algebras to be in correspondence with dominant integral weights in 1914 (see [53] for a modern distillation of his proof).

Theorem 1.2.2.2. ([54]) There is a one-to-one correspondence between the set of dominant integral weights and the isomorphism classes of finite dimensional irreducible $\mathfrak{g}$-modules, where the pairing is $\bar{\Lambda} \longleftrightarrow L_{\bar{\Lambda}}$.

We define the *multiplicity of a weight* $\bar{\mu}$ in a highest weight representation of weight $\bar{\Lambda}$ to be the dimension of the weight space $(L_{\bar{\Lambda}})_{\bar{\mu}}$ of $\bar{\mu}$ in $L_{\bar{\Lambda}}$. We denote this by $\mathrm{mult}(\bar{\mu}) = \dim((L_{\bar{\Lambda}})_{\bar{\mu}})$. The multiplitictiy of a weight will be equal to the number of unique sequences in the algorithm described above that end at the weight $\bar{\mu}$. For $w \in W$ we have $\mathrm{mult}(w\bar{\mu}) = \mathrm{mult}(\bar{\mu})$. We can summarise the information about all $\bar{\mu} \in \Omega_{L_{\bar{\Lambda}}}$ (the space of weights for $L_{\bar{\Lambda}}$) and their multiplicities into a formal sum $\chi^{\mathfrak{g}}_{\bar{\Lambda}}$, called a *character*, which is defined as

$$\chi^{\mathfrak{g}}_{\bar{\Lambda}} = \sum_{\bar{\mu}} \mathrm{mult}(\bar{\mu})e^{\bar{\mu}} \in \mathbb{Z}[P(\mathfrak{g})], \tag{1.2.14}$$

where we treat $e^{\bar{\mu}}$ to be a formal exponential.

1.2.3 Embeddings and the Littlewood-Richardson Rule

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{p} \subset \mathfrak{g}$ be a Lie subalgebra. In general, an irreducible representation $L_{\bar{\Lambda}}$ of $\mathfrak{g}$ breaks down into a direct sum of irreducible representations of $\mathfrak{p}$ with some multiplicities $b_{\bar{\Lambda}\bar{\mu}} \in \mathbb{Z}_{\geq 0}$. We represent this by

$$L_{\bar{\Lambda}}|_{\mathfrak{p}} = \bigoplus_{\bar{\mu}} b_{\bar{\Lambda}\bar{\mu}} L_{\bar{\mu}}, \quad b_{\bar{\Lambda}\bar{\mu}} \in \mathbb{Z}, \tag{1.2.15}$$

where $L_{\bar{\mu}}$ are representations of $\mathfrak{p}$. The multiplicities $b_{\bar{\Lambda}\bar{\mu}}$ are the *branching coefficients* and the whole system (1.2.15) is called the *branching rules.*

When discussing how to calculate the $\widehat{\mathfrak{sl}}(n)_k$ fusion rules for Wess-Zumino-Witten models, we will need to understand how to combinatorially calculate the branching rules for $\mathfrak{g} = \mathfrak{sl}(n) \otimes \mathfrak{sl}(n)$ and $\mathfrak{p} = \mathfrak{sl}(n) \subset \mathfrak{sl}(n) \otimes \mathfrak{sl}(n)$. That is, determine the coefficients in the decomposition

$$L_{\bar{\Lambda}} \otimes L_{\bar{\Lambda}'} = \bigoplus_{\bar{\mu}} b_{\bar{\Lambda}\otimes\bar{\Lambda}',\bar{\mu}} L_{\bar{\mu}}. \tag{1.2.16}$$

To do this we use an algorithm that is called the *Littelwood-Richardson rule.* This algorithm is defined in terms of Young diagrams associated to highest weights of Lie algebra representations. To a $\mathfrak{sl}(n)$ dominant integral weight $\bar{\Lambda} = [l_1, \ldots, l_{n-1}]$, we associate a Young diagram $\mathrm{par}(\bar{\Lambda})$, with parts

$$\mathrm{par}(\bar{\Lambda})_i = \sum_{j=i}^{n-1} l_j. \tag{1.2.17}$$

To calculate the tensor product multiplicities (branching coefficients) for the tensor product $L_{\bar{\Lambda}} \otimes L_{\bar{\Lambda}'}$, we form the two Young diagrams $\mathrm{par}(\bar{\Lambda})$ and $\mathrm{par}(\bar{\Lambda}')$ and then fill the first row of boxes in $\mathrm{par}(\bar{\Lambda}')$ with 1s, the second with $2s$ etc. Then for each $i = 1, 2, \ldots, n-1$ in order, we add all boxes with an i in them (that is the i-th row of $\mathrm{par}(\bar{\Lambda}')$) to $\mathrm{par}(\bar{\Lambda})$ in all possible ways subject to the following conditions:

1. We must form standard Young diagrams (parts must be weakly decreasing down the rows).
2. Each column can only contain 1 box with an i in it.
3. When counting from right to left and top to bottom (in this order), the number of boxes with an i in it must be less than or equal to the number of boxes with an $i-1$ in them.

When performing step 3 as described above, we count the numbers as they appear in *one* row from right to left, then move onto the next row and repeat the procedure. Once we have formed all possible Young diagrams subject to these conditions we then delete any columns of length n.

We can then reconstruct the dominant integral weights associated to each of the Young diagrams. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a Young diagram produced by this process. We define the

Dynkin labels $[l_1, \ldots, l_{n-1}]$ of the associated weight through the equations $l_i = \lambda_i - \lambda_{i+1}$ for each $i = 1, \ldots, n-1$, note that if $j > k$, we substitute $\lambda_j = 0$.

The weights we obtain from this are the highest weights that appear in the tensor product decomposition. The multiplicities are obtained by counting how many times the same Young diagram is formed.

Example 1.2.3.1. Here we recall the example from [50, §13.5.3], for the tensor product of irreducible $\mathfrak{sl}(3)$-modules. Let $\bar{\Lambda} = [2, 0]$ and $\bar{\Lambda}' = [1, 1]$ so that we have $\mathrm{par}(\bar{\Lambda}) = (2)$ and $\mathrm{par}(\bar{\Lambda}') = (2, 1)$. After filling the boxes of $\mathrm{par}(\bar{\Lambda}')$ we can then represent the tensor product $L_{\bar{\Lambda}} \otimes L_{\bar{\Lambda}'}$ using the two associated Young diagrams as

$$\begin{ytableau} \ & \ \end{ytableau} \otimes \begin{ytableau} 1 & 1 \\ 2 \end{ytableau}. \tag{1.2.18}$$

We now add both boxes labelled 1 to $\mathrm{par}(\bar{\Lambda})$ to obtain

$$\begin{ytableau} \ & \ & 1 & 1 \end{ytableau}, \begin{ytableau} \ & \ & 1 \\ 1 \end{ytableau}, \begin{ytableau} \ & \ \\ 1 & 1 \end{ytableau}. \tag{1.2.19}$$

We then add the box labelled 2 in all possible ways to these diagrams, subject to 3 rules above, to obtain

$$\begin{ytableau} \ & \ & 1 & 1 \\ 2 \end{ytableau}, \begin{ytableau} \ & \ & 1 \\ 1 & 2 \end{ytableau}, \begin{ytableau} \ & \ & 1 \\ 1 \\ 2 \end{ytableau}, \begin{ytableau} \ & \ \\ 1 & 1 \\ 2 \end{ytableau}, \tag{1.2.20}$$

note that rule 3 means that we cannot place the box labelled 2 at the end of the first row. We now delete all columns of length 3, which gives us the diagrams

$$\begin{ytableau} \ & \ & \ & \ \\ \ \end{ytableau}, \begin{ytableau} \ & \ & \ \\ \ & \ \end{ytableau}, \begin{ytableau} \ & \ \end{ytableau}, \begin{ytableau} \ \\ \ \end{ytableau}. \tag{1.2.21}$$

We read off the weights that appear in the branching rules from this as

$$L_{[2,0]} \otimes L_{[1,1]}\big|_{\mathfrak{sl}(3)} = L_{[3,1]} \oplus L_{[1,2]} \oplus L_{[2,0]} \oplus L_{[0,1]} \tag{1.2.22}$$

Remark 1.2.3.2. Note that in this example, all the branching coefficients are equal to 1. This is not true in general, where it is possible to have multiply copies of the same highest-weight module in a tensor product decomposition.

1.3 Affine Lie Algebras

This section constitutes a review of affine Lie algebras and their representations, similarly to the one above for finite Lie algebras. As they are more complicated objects than their finite cousins, and play a central role in this thesis, this review will be longer and more in depth than the previous one for the finite case. Much of the first part of this review will cover standard material, where, as we will mostly encounter affine Lie algebras in connection to conformal field theory, much of the presentation is again based on that in [50]. Discussion of characters and explicit formula for them will follow standard results from the canonical reference [55]. The author also found [48] to be very useful when learning the material presented.

1.3.1 Basic Definitions and Notation

Untwisted affine Kac-Moody Lie algebras (or just affine Lie algebras), are infinite-dimensional Lie algebras constructed from finite Lie algebras. In our case, we will only construct affine Lie algebras from finite semi-simple Lie algebras. We construct the affine Lie algebra $\widehat{\mathfrak{g}}$ from the finite semi-simple Lie algebra $\mathfrak{g}$ in two steps: First we form the algebra of Laurent polynomials $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ (commonly referred to as the *loop algebra* for $\mathfrak{g}$). Then we adjoin a central element $\hat{k}$ and a derivation $\hat{d} = t\frac{d}{dt}$. We say that an element $Xt^n \in \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$, for $X \in \mathfrak{g}$, is of grade $n \in \mathbb{Z}$. The construction (and form) of the algebra $\widehat{\mathfrak{g}}$ is summarized as

$$\mathfrak{g} \mapsto \widehat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\hat{k} \oplus \mathbb{C}\hat{d}. \tag{1.3.1}$$

On this algebra we define the following Lie bracket

$$[X \otimes t^n, Y \otimes t^m] = [X, Y]_{\mathfrak{g}} \otimes t^{n+m} + \hat{k}n(X, Y)_{\mathfrak{g}}\delta_{n+m,0}, \tag{1.3.2}$$

$$[\hat{k}, \widehat{\mathfrak{g}}] = 0, \tag{1.3.3}$$

$$[\hat{d}, X \otimes t^n] = nX \otimes t^n. \tag{1.3.4}$$

Here $[\ ,\ ]_{\mathfrak{g}}$ and $(\ ,\ )_{\mathfrak{g}}$ are the Lie bracket and Killing form on the finite Lie algebra $\mathfrak{g}$. We will denote the elements of the affine algebra $\widehat{\mathfrak{g}}$ by $Xt^n = X_n$. From this construction we can see that $\widehat{\mathfrak{g}}$ has infinite dimension due to the presence of the loop algebra $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$.

We define a Killing form on $\widehat{\mathfrak{g}}$ on such that

$$(X_n, Y_m) = \begin{cases} (X, Y)_{\mathfrak{g}}, & n = -m \\ 0, & \text{else.} \end{cases} \tag{1.3.5}$$

We have that $(X_n, k) = (\hat{k}, \hat{d}) = (X_n, \hat{d}) = 0$ and $(\hat{d}, \hat{k}) = 1$, and supplement $(\hat{d}, \hat{d}) = 0$. As before the Killing form is non-degenerate and identifies $\widehat{\mathfrak{h}}$ with its dual $\widehat{\mathfrak{h}}^*$, and therefore defines a scalar product on $\widehat{\mathfrak{h}}^*$.

We define the *affine roots* $\widehat{\alpha}$ to be the vector of eigenvalues for the adjoint action of H_0^i for $i = 1, \dots, r$, the central element $\hat{k}$, and the derivation $\hat{d}$ on $\widehat{\mathfrak{g}}$.

$$\widehat{\alpha} = (\widehat{\alpha}(H_0^1), \dots, \widehat{\alpha}(H_0^r); \widehat{\alpha}(\hat{k}); \widehat{\alpha}(\hat{d})). \tag{1.3.6}$$

Since $\hat{k}$ commutes with $\widehat{\mathfrak{g}}$, and the eigenvalues of $\hat{d}$ (here we are denoting the eigenvalue of $\hat{d}$ by n) are integers, we can rewrite this as

$$\widehat{\alpha} = (\alpha; 0; n), \tag{1.3.7}$$

for a finite root $\alpha \in \Delta_{\mathfrak{g}}$ and $n \in \mathbb{Z}$. We refer to the eigenvalue n of $\hat{d}$ as the *grade* of a vector. It is convenient to write the affine roots in a nice basis for use in applications. Since the first r components are defined using the finite roots, it is natural to use finite simple roots to describe them. The eigenvalue of $\hat{d}$, is an integer so the natural basis element is $\delta = (0; 0; 1)$, which is called the *imaginary root*. Using this basis, we can write an element $\widehat{\alpha}$ of the space of *affine roots* $\widehat{\Delta}$ as $\widehat{\alpha} = \alpha + n\delta$ for $\alpha \in \Delta_{\mathfrak{g}} \cup \{0\}$ and $n \in \mathbb{Z}$.

The scalar product induced on the space of affine roots by the Killing form is

$$(\widehat{\Lambda}, \widehat{\mu}) = (\Lambda, \mu) + \widehat{\Lambda}(\hat{k})\widehat{\mu}(\hat{d}) + \widehat{\mu}(\hat{k})\widehat{\Lambda}(\hat{d}). \tag{1.3.8}$$

This informs the name imaginary root for δ, as since $(\delta, \delta) = 0$ its length vanishes.

Our next step is to define the set of simple roots of $\widehat{\mathfrak{g}}$, which we take to be the set of finite simple roots α_i for $i = 1, \dots, r$ supplemented with the *affine root* $\alpha_0 = (-\theta; 0; 1) = -\theta + \delta$ (the finite root θ is the highest root as defined in section 1.2.1). Where there is no confusion we will denote the affine simple roots by $\{\alpha_i\}_{i=0,1,\dots,r}$. The *affine simple coroots* are defined to be $\widehat{\alpha}_i^\vee = \frac{2}{(\widehat{\alpha}_i, \widehat{\alpha}_i)}\widehat{\alpha}_i$ for $i = 0, \dots, r$. Associated to an affine Lie algebra we have an *affine Cartan matrix* $\bar{A}_{ij} = (\alpha_i, \alpha_j^\vee)$, $i, j = 0, 1, \dots, r$, and we have $(\bar{A}_{ij})_{1 \le i,j \le r} = A$, where A is the Cartan matrix associated to $\mathfrak{g}$.

It will always be clear from context whether roots in this thesis are affine or finite, as such we shall henceforth remove the hat notation for affine roots. We define the fundamental weights Λ_i to be dual to the simple affine coroots and of grade zero, and we define the *affine root and weight lattices* $Q(\widehat{\mathfrak{g}})$ and $P(\widehat{\mathfrak{g}})$ respectively as

$$Q(\widehat{\mathfrak{g}}) = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_r \tag{1.3.9}$$

$$P(\widehat{\mathfrak{g}}) = \mathbb{Z}\Lambda_0 \oplus \cdots \oplus \mathbb{Z}\Lambda_r \tag{1.3.10}$$

For any $0 \leq i, j \leq r$, we have $(\Lambda_i, \widehat{\alpha}_j^\vee) = \delta_{ij}$. We take the 0-th fundamental weight to be $\Lambda_0 = (0; 1; 0)$, and using this we can write the affine fundamental weights as $\Lambda_i = \bar{\Lambda}_i - a_i^\vee \Lambda_0$, where $a_i^\vee$ are the r coefficients of $\theta = \sum_{i=1}^r a_i^\vee \alpha_i^\vee$ expanded in the basis of coroots, which are referred to as the *comarks*.

The Cartan-Weyl presentation for $\mathfrak{g}$ together with the affine Lie brackets (1.3.2), (1.3.3), and (1.3.4) implies the following *affine Cartan-Weyl* presentation for $\widehat{\mathfrak{g}}$. We use the generators E_n^α, H_n^i for $i = 1, \dots, r$, $\alpha \in \Delta$ and $n \in \mathbb{Z}$, with the derivation $\hat{d}$, and central element $\hat{k}$. The presentation has the following defining relations

$$[H_n^i, H_m^j] = \hat{k} n \delta^{ij} \delta_{n+m,0} \tag{1.3.11}$$

$$[H_n^i, E_m^\alpha] = \alpha^i E_{n+m}^\alpha \tag{1.3.12}$$

$$[E_n^\alpha, E_m^\beta] = \begin{cases} N_{n,m}^{\alpha,\beta} E_{n+m}^{\alpha+\beta}, & \alpha + \beta \in \Delta \\ \frac{2}{|\alpha|^2} \left(\alpha \cdot H_{n+m} + \hat{k} n \delta_{n+m,0} \right), & \alpha = -\beta \\ 0, & \text{else} \end{cases} \tag{1.3.13}$$

$$[k, \widehat{\mathfrak{g}}] = 0 \tag{1.3.14}$$

$$[\hat{d}, H_n^i] = n H_n^i \tag{1.3.15}$$

$$[\hat{d}, E_n^i] = n E_n^i \tag{1.3.16}$$

together with the Serre relations

$$\mathrm{ad}(E_n^{\alpha_i})^{1-A_{ij}}(E_n^{\alpha_i}) = \mathrm{ad}(E_m^{-\alpha_i})^{1-A_{ij}}(E_m^{-\alpha_i}) = 0, \quad i \neq j \tag{1.3.17}$$

Note that in these relations Δ is the set of roots for the finite Lie algebra $\mathfrak{g}$, and $\alpha, \beta \in \Delta$. The affine Cartan subalgebra $\widehat{\mathfrak{h}}$ is generated by $\hat{k}$, $\hat{d}$, and H_0^i for $i = 1, \dots, r$. We will also let $J_m^i = J^i \otimes t^m$ for $i = 1, \dots, \dim(\mathfrak{g})$ and $m \in \mathbb{Z}$ denote a generic basis for $\widehat{\mathfrak{g}}$, where the set $\{J^i\}_m$ denotes a generic basis for the loop algebra of $\mathfrak{g}$.

1.3.2 Dynkin Labels of Affine Weights and Highest Weight Representations

Analogously to finite Lie algebras, we define a *representation* of $\widehat{\mathfrak{g}}$ to be a vector space V together with a Lie algebra homomorphism $\widehat{\rho} : \widehat{\mathfrak{g}} \longrightarrow \mathrm{End}(V)$. We define an *affine weight* $\widehat{\Lambda}$ of a vector v in V, to be the vector of eigenvalues

$$\widehat{\Lambda} = (\widehat{\Lambda}(H_0^1), \dots, \widehat{\Lambda}(H_0^r); \widehat{\Lambda}(\hat{k}); \widehat{\Lambda}(\hat{d})), \tag{1.3.18}$$

of v. The first r components of $\widehat{\Lambda}$ comprise a finite weight Λ for $\mathfrak{g}$, and we call the last two the *level* and the *grade* respectively. We also define *weight spaces* V_Λ and the *space of weights*, as for finite Lie algebras but with the finite Cartan algebra $\mathfrak{h}$ replaced by the affine Cartan algebra $\widehat{\mathfrak{h}}$. Writing an affine weight $\widehat{\Lambda}$ in terms of the fundamental weights and the imaginary root as $\widehat{\Lambda} = \sum_{i=0}^{r} l_i \Lambda_i + l\delta$ (note that $\bar{\Lambda} = [l_1, \ldots, l_r]$ defines the Dynkin labels of a finite $\mathfrak{g}$ weight) allows us to define the *level* k of a weight by its action on the central element $\hat{k}$ of $\widehat{\mathfrak{g}}$ as

$$k := \widehat{\Lambda}(\hat{k}) = \sum_{i=0}^{r} a_i^{\vee} l_i. \tag{1.3.19}$$

Due to the above relation, if we specify a level k and a finite $\mathfrak{g}$ weight Λ, we have specified an affine weight $\widehat{\Lambda}$ by solving for $l_0 = k - \sum_{i=1}^{r} a_i^{\vee} l_i$. Finally, the affine roots are seen to be of level zero. We will write $\widehat{\mathfrak{g}}_k$ to represent a $\widehat{\mathfrak{g}}$ representation of level k.

We call the coefficients l_i for $i = 0, \ldots, r$ the *affine Dynkin labels*, and we will often write affine weights using their Dynkin labels, for example $\Lambda = [l_0, l_1, \ldots, l_r]$ (note we begin our indexing with zero), but this notation does *not* keep track of the grade of weights.

We will say an affine weight Λ is *dominant* if $l_i \geq 0$ and *integral* if $l_i \in \mathbb{Z}$, and we will denote the set of dominant integral weights of level k for $\widehat{\mathfrak{sl}}(r)$ by $P^+_{r,k}$. We will sometimes denote the subset of these whose Dynkin labels satisfy $l_i > 0$ for all $i = 0, \ldots, r$, by $P^{++}_{r,k}$. We will also use P^+_r and P^{++}_r to denote the set of all dominant integral and strictly positive integral $\widehat{\mathfrak{sl}}(r)$ weights respectively. Associated to a dominant integral weight $\Lambda = [l_0, \ldots, l_r]$, we will define a Young diagram $\mathrm{par}(\Lambda) = (\mathrm{par}(\Lambda)_1, \mathrm{par}(\Lambda)_2, \ldots)$, whose parts are

$$\mathrm{par}(\Lambda)_i = \begin{cases} \sum_{j=i}^{r} l_j & 1 \leq i \leq n-1, \\ 0 & i \geq n, \end{cases} \tag{1.3.20}$$

which is the partition associated to $\bar{\Lambda}$ the finite part of the weight we defined when describing the Littelwood-Richardson rules.

Consider the level $k = l_0 + \sum_{i=1}^{r} a_i^{\vee} l_i$ of a weight Λ, the comarks satisfy $a_i^{\vee} \geq 0$ by the definition of θ so that for dominant, integral weights $k \in \mathbb{Z}_{\geq 0}$. As in the finite case, *highest weight representations* of dominant integral weights are of particular interest. We again denote the *irreducible highest weight module* (produced by a quotient of an affine Verma module analogous to the finite case) by L_Λ for dominant, integral Λ. This module L_Λ is not in general finite-dimensional (and as such our algorithm described above for the finite case will not in general terminate), instead it is *integrable* which we define below.

Definition 1.3.2.1. (**4.2** and **4.8** in [56]) A representation $\rho : \widehat{\mathfrak{g}} \longrightarrow \mathrm{End}(V)$ is said to be integrable if

- V is $\widehat{\mathfrak{h}}$-diagonalizable, that is it decomposes into weight spaces V_Λ of $\widehat{\mathfrak{h}}$ as

$$V = \bigoplus_{\Lambda\in\widehat{\mathfrak{h}}} V_\Lambda. \tag{1.3.21}$$

- For $\Lambda \in \widehat{\mathfrak{h}}$, $\dim(V_\Lambda) < \infty$.

- Recall Ω_V denotes the weight space of V(defined in 1.2.1). For all $\Lambda \in \Omega_V$, there exists $M \geq 0$ such that for any $m \geq M$, $\Lambda + m\alpha_i \notin \Omega_V$ and $\Lambda - m\alpha_i \notin \Omega_V$, for all $i = 0, \ldots, r$.

In chapter 5, we will need to find sequences of raising and lowering operators that when applied on a vector of weight Λ in a representation, gives us a vector of weight Λ. Stated another way, we wish to find sequences of affine roots, that when added together, form the difference between two dominant integral affine weights. Here we describe how we will do this, by mapping Dynkin labels of level 0 weights into sets of affine roots.

Since $\hat{k}$ is central in $\widehat{\mathfrak{g}}$, we see that the level is constant across all vectors in a highest weight module. The Dynkin labels of the simple root α_i are equal to the i-th row of the affine Cartan matrix

$$\alpha_i = [A_{i0}, A_{i1}, \ldots, A_{ir}], \tag{1.3.22}$$

and this gives us an easy way to calculate the weight of the state $E^{\pm\alpha_i}\left|[l_0, \ldots, l_r]\right\rangle$ by $\Lambda \pm \alpha_i = [l_0 \pm \bar{A}_{i0}, \ldots, l_r \pm \bar{A}_{ir}]$, and we can extrapolate this to $(E^{\alpha_0})^{n_0} \ldots (E^{\alpha_r})^{n_r} \left|\Lambda\right\rangle$ by linearity. This is just the matrix multiplication $A \cdot (n_0, \ldots, n_r)^T$.

We can invert this process using the finite Cartan matrix A for the finite Lie algebra $\mathfrak{g}$ associated to $\widehat{\mathfrak{g}}$. In the irreducible module L_Λ, consider the two weights $\Lambda' = [l'_0, \ldots, l'_r]$ and $\Lambda'' = [l''_0, \ldots, l''_r]$ such that $\Lambda', \Lambda'' \in \Omega_{L_\Lambda}$ and Λ'' is a descendant state of Λ', where we then have

$$\Lambda'' = \Lambda' - \sum_{i=0}^{r} n_i \alpha_i. \tag{1.3.23}$$

Note that since Λ'' is a descendant of Λ', we wish to find an expansion in terms of roots such that $n_i \geq 0$ for $i = 0, \ldots, r$. We define the finite weights $\bar{\Lambda}' = [l'_1, \ldots, l'_r]$ and $\bar{\Lambda}'' = [l''_1, \ldots, l''_r]$ associated to Λ' and Λ''. First we calculate

$$\bar{A}^{-1} \cdot (l'_1 - l''_1, \ldots, l'_r - l''_r)^T, \tag{1.3.24}$$

which gives us an r-dimensional vector $R = (R_1, \ldots, R_r)$, whose entries give a root expansion between $\bar{\Lambda}'$ and $\bar{\Lambda}''$. If we have that $R_i \geq 0$ for $i = 1, \ldots, r$, then the weights Λ' and Λ'' first appear at the same grade and

$$\Lambda'' = \Lambda' - \sum_{i=1}^{r} R_i \alpha_i. \tag{1.3.25}$$

In this case, we have $n_0 = 0$ and $n_i = R_i$ for $i = 1, \dots, r$. If one or more $R_i < 0$, then we define $\min_{i=1,\dots,r}\{R_i\} = m$ and $R' = (R_1 + m, R_2 + m, \dots, R_r + m)$, for which $R_i + m \geq 0$ for $i = 1, \dots, r$. Then m is the difference of grades for which Λ' and Λ'' first appear at. We then have that if Λ' first appears at grade $g \in \mathbb{Z}_{\geq 0}$, there exists a vector $v_{\Lambda'}^{g+m}$ of weight Λ' at grade $(g + m)$ for which we have a vector $v_{\Lambda''}^{m}$ of weight Λ'' at grade $(g + m)$ which is obtainable by some sequence of lowering operators where we use exactly R_i applications of $E^{-\alpha_i}$ for each $i = 1, \dots, r$.

Thus to obtain the positive integers n_i for $i = 0, \dots, r$ defining the weight Λ'' as a descendant of Λ' through a sequence of of n_i applications of $E^{-\alpha_i}$ for $i = 0, \dots, r$, we need to first move from Λ' at grade g to Λ' at grade $(m + g)$ and then follow the sequence defined by R. Since the affine root α_0 takes us one grade higher we can then use $\alpha_0 = \sum_{i=1}^{r} \alpha_i$ to see that

$$\Lambda'' = \Lambda' - m\alpha_0 - \sum_{i=1}^{r}(R_i + m)\alpha_i, \tag{1.3.26}$$

and in this situation we have $n_0 = m$ and $n_i = R_i + m$ for $i = 1, \dots, r$. This approach to moving from roots to Dynkin labels and vice versa using the A and $\bar{A}^{-1}$ will be crucial to our conjecture in chapter 5.

Remark 1.3.2.2. These computations rely on the fact that for a fixed level k, we can fully specify $\widehat{\mathfrak{g}}_k$ weights by defining their finite parts. In this case, since roots are of level 0 and we are considering two descendant states, we know the level of their difference will also be 0.

Abstractly, we say that two level k weights Λ_1, Λ_2 are in the same *congruence class*, if $\Lambda_1 = \Lambda_2 + \sum_{i=0}^{r} m_i\alpha_i$ for $m_i \in \mathbb{Z}$, and this means that all weights in a highest weight representation are in the same congruence class. To determine the congruence class of a weight $\Lambda = [l_0, \dots, l_r]$ we define

$$cls(\Lambda) = \sum_{i=1}^{r} b_i l_i \bmod (\det(A)), \quad \widehat{\mathfrak{g}} \neq \widehat{\mathfrak{so}}(4n),^{6} \tag{1.3.27}$$

and if $cls(\Lambda_1) = cls(\Lambda_2)$, they are in the same congruence class. The numbers b_i are specific to $\widehat{\mathfrak{g}}$, and they can be found for all simple affine Lie algebras in [48, **Fig 6.7**]. We define an *affine Weyl reflection* w_α with respect to an affine root $\alpha \neq n\delta$ on a weight Λ (it should always be clear from context if s_α is a finite or affine Weyl reflection) as

$$s_\alpha(\Lambda) = \Lambda - (\Lambda, \alpha^\vee)\alpha, \tag{1.3.28}$$

[6] We will never consider $\widehat{\mathfrak{g}} = \widehat{\mathfrak{so}}(4n)$ in this thesis. The interested reader can find the formula for congruence classes in this case in [48].

and for $\alpha = (\alpha'; 0; m)$ and $\Lambda = (\Lambda'; k; n)$ we can make this explicit as

$$s_\alpha = (s_{\alpha'}(\Lambda' + km\alpha'); k; n - [(\Lambda, \alpha) + km]\tfrac{2m}{(\alpha', \alpha')}). \tag{1.3.29}$$

We define the *affine Weyl group* $\widehat{W}$ to be generated by the set $\{s_\alpha\}_{\alpha \in \Delta \setminus \{n\delta\}_{n \in \mathbb{Z}}}$. $\widehat{W}$ divides the space of affine weights into *affine Weyl chambers* defined by

$$C_w = \{\Lambda | (w\Lambda, \alpha_i) \geq 0, i = 0, 1, \ldots, r\}, \tag{1.3.30}$$

for each $w \in \widehat{W}$. We define the *fundamental chamber* C_e to be the chamber corresponding to the identity e in $\widehat{W}$, and for $\Lambda \in C_e$ we can see that Λ is a dominant weight. Orbits of the Weyl group have infinitely many weights and have a unique dominant weight within the fundamental chamber.

1.3.3 Characters

We define the character $\chi_\Lambda^{\hat{\mathfrak{g}}}$ of the affine highest weight module L_Λ similarly to the finite case, although character functions for affine Lie algebras are more involved to calculate. A central problem being the computation of infinite multiplicities, and as such affine characters will therefore involve infinite series. Recall for a representation V that Ω_V is the set of weights of V. We say a weight $\mu \in \Omega_V$ is *maximal* if $(\mu + \delta) \notin \Omega_V$, and denote the set of all maximal weights by Ω_V^{max}. Note that if $\mu \in \Omega_{L_\Lambda}$, $cls(\mu) = cls(\Lambda)$. We collect the information of states with weight $\mu, \mu - \delta, \mu - 2\delta, \ldots$ into a generating function $\sigma_\mu^\Lambda(q)$ of multiplicities which we call the *string function* of μ in L_Λ, defined by

$$\sigma_\mu^\Lambda(q) = \sum_{n=0}^{\infty} \text{mult}(\mu - n\delta) q^n. \tag{1.3.31}$$

It is clear that knowing all of the string functions for a highest weight representation is equivalent to knowing the character, because if $\mu' \in \Omega_{L_\Lambda}$ we have that μ' is either maximal or of the form $\mu' = \mu - n\delta$ for $\mu \in \Omega_{L_\Lambda}^{max}$, and some $n \in \mathbb{Z}_{>0}$. However, we can calculate the affine character without explicitly calculating each string function $\sigma_\mu^\Lambda(q)$ associated to each $\mu \in \Omega_\Lambda^{max}$. To see this we use the following lemma which is a generalisation of a similar result for finite Lie algebras

Lemma 1.3.3.1. ([50]) Let $w \in \widehat{W}$ and $\mu \in \Omega_{L_\Lambda}$, then $\text{mult}(\mu) = \text{mult}(w(\mu))$.

Thus, the complete information needed for an affine character of L_Λ is contained in the string functions of all the dominant maximal weights in L_Λ, and the full information is obtained by Weyl reflections. We write the affine character for L_Λ with Λ a level k weight as

$$\chi_\Lambda^{\widehat{\mathfrak{g}}} = \sum_{\mu \in \Omega_{L_\Lambda}} \mathrm{mult}(\mu) e^\mu = \sum_{\mu \in \Omega_{L_\Lambda}^{max}} \sigma_\mu^\Lambda(e^{-\delta}) \cdot e^\mu. \tag{1.3.32}$$

In section 1.5.9, and subsequently chapter 4, we will use the *Weyl-Kac character formula* [57, **eq (10.4.5)**]

$$\chi_\Lambda^{\widehat{\mathfrak{g}}} = \sum_{w \in \widehat{W}} \frac{\epsilon(w) e^{w(\Lambda+\rho)}}{\epsilon(w) e^{w(\Lambda)}}. \tag{1.3.33}$$

We now specialise to the case of $\widehat{\mathfrak{sl}}(r)$ and define vectors $\mathbf{e}_i \in \mathbb{C}^r$ by embedding the roots of $\widehat{\mathfrak{sl}}(r)$ into $\mathbb{C}^r$ in the fundamental representation. We then have that $\alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for $i = 1, \ldots, r-1$. with $\alpha_0 = \mathbf{e}_r - \mathbf{e}_1 + \delta$. We then write the character function using the convenient notation $e^{-\delta} = \mathfrak{q}$, $e^{\mathbf{e}_i} = x_i$ for $i = 1, \ldots, r-1$, and define $\mathbf{x} = (x_1, \ldots, x_{r-1})$ so that

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(r)}(\mathfrak{q}; \mathbf{x}) = e^\Lambda \chi_\Lambda^{\widehat{\mathfrak{sl}}(r)}\Big|_{e^{-\delta} \mapsto \mathfrak{q}, e^{\mathbf{e}_1} \mapsto x_1, \ldots, e^{\mathbf{e}_{r-1}} \mapsto x_{r-1}}. \tag{1.3.34}$$

We will also need a different form for the $\widehat{\mathfrak{sl}}(r)$ characters, using a set of variables $\hat{\mathfrak{t}} = (\hat{\mathfrak{t}}_1, \ldots, \hat{\mathfrak{t}}_{r-1})$. The $\hat{\mathfrak{t}}$ parameters will be associated to the fundamental weights instead of the simple roots. The relation $\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}$ then gives us the change of variables

$$\frac{x_i}{x_{i+1}} = \frac{\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_{i+1}}{\hat{\mathfrak{t}}_i^2} \iff x_i = \frac{\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_r}{\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{r-1}} x_r, \tag{1.3.35}$$

where $1 \leq i < r$ and we set $\hat{\mathfrak{t}}_r = \hat{\mathfrak{t}}_0 = 1$. This gives us an explicit form of the $\widehat{\mathfrak{sl}}(r)$ character as [41]

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(r)_R}(q, \hat{\mathfrak{t}}) = q^{h_\Lambda} \frac{\mathcal{N}_\Lambda(q, \hat{\mathfrak{t}})}{(q;q)_\infty^{r-1} \prod_{1 \leq i < j \leq r} (\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_j/\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{j-1}; q)_\infty (q\,\hat{\mathfrak{t}}_i\hat{\mathfrak{t}}_{j-1}/\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_j; q)_\infty} \prod_{i=1}^{r-1} \hat{\mathfrak{t}}_i^{d_i}, \tag{1.3.36}$$

where, with $\lambda = \mathrm{par}(\Lambda)$,

$$\mathcal{N}_\Lambda(q, \hat{\mathfrak{t}}) = \sum_{\substack{k_1, \ldots, k_r \in \mathbb{Z} \\ k_1 + \cdots + k_r = 0}} \det_{1 \leq i,j \leq r} \left((\hat{\mathfrak{t}}_i/\hat{\mathfrak{t}}_{i-1})^{(r+R)k_i + \lambda_j - j - \lambda_i + i} q^{(\lambda_j - j)k_i + \frac{1}{2}(r+R)k_i^2} \right). \tag{1.3.37}$$

We also define the *principal character* $\mathrm{Pr}\chi_\Lambda(q)$ as

$$\mathrm{Pr}\chi_\Lambda^{\widehat{\mathfrak{sl}}(r)}(q) = e^{-\Lambda} \chi_\Lambda^{\widehat{\mathfrak{sl}}(r)}(\mathfrak{q}; \mathbf{x})\Big|_{\substack{\mathfrak{q} \mapsto q^r \\ x_i \mapsto q, \ i=1,\ldots,r}}. \tag{1.3.38}$$

We will refer to the principal character as the $\widehat{\mathfrak{sl}}(r)$ q-character in the sequel, and when it is clear to do so we will write the principally specialised character as a character $\chi_\Lambda^{\widehat{\mathfrak{sl}}(r)}(q)$ which is a function of only q. In principle, the multiplicity of weight $\mu \in \Omega_{L_\Lambda}$ can be calculated using *Freudenthal's recursion formula* (see [51, §22.3])

$$(|\Lambda+\rho|^2 - |\mu+\rho|^2)\mathrm{mult}(\mu) = 2\sum_{\alpha\in\widehat{\Delta}_+}\sum_{j=1}^{\infty}(\mu+j\alpha,\alpha)\mathrm{mult}(\mu+j\alpha), \tag{1.3.39}$$

where $\rho = [1,\ldots,1]$ is the *affine Weyl vector*, and $\widehat{\Delta}_+$ are the *positive affine roots* which we count with their multiplicities (that is 1 for real roots and r for imaginary roots). In practice this calculation is too involved for most highest weight representations, and when we need the coefficients of string functions for computations we will use the tables in [49] (themselves based on the formulas found in [55]).

1.3.4 Integrable $\widehat{\mathfrak{sl}}(n)$ Highest Weight Representations

Central to this thesis is the structure of integrable $\widehat{\mathfrak{sl}}(n)$ highest weight representations, which we will explore in more depth here. The finite Lie algebra $\mathfrak{g} = \mathfrak{sl}(n)$ is of rank $(n-1)$ with highest root $\theta = \alpha_1 + \cdots + \alpha_{n-1}$ [58], so that $\widehat{\mathfrak{sl}}(n)$ has n simple roots $\{\alpha_i\}_{i=0,\ldots,n-1}$ and Cartan matrix $(A_{ij})_{i,j=0,\ldots,n-1}$ defined by[7]

$$A_{ij} = \begin{cases} 2 & i = j, \\ -1 & i \equiv j \pm 1 \mod n, \\ 0 & \text{else.} \end{cases} \tag{1.3.40}$$

We let $\Lambda = [l_0,\ldots,l_{n-1}] \in P^+_{n,k}$ be a dominant integral weight of level k and consider the integrable highest weight representation L_Λ with highest weight vector $|\Lambda\rangle$. When computing congruence classes in $\widehat{\mathfrak{sl}}(n)$ representations (see equation (1.3.27)) we have $b_i = i$ so that for $\mu = [l'_0,\ldots,l'_{n-1}] \in \Omega_{L_\Lambda}$

$$cls(\mu) \equiv \sum_{i=1}^{n-1} i l'_i \equiv \sum_{i=1}^{n-1} i l_i \equiv cls(\Lambda) \mod n \tag{1.3.41}$$

From this we can see that there are n congruence classes of weights for $\widehat{\mathfrak{sl}}(n)$.

Example 1.3.4.1. Consider the possible highest weight $\widehat{\mathfrak{sl}}(2)_2$ representations. We have 3 dominant integral weights $[2,0]$, $[1,1]$, and $[0,2]$ which are of class 0, 1, and 0 respectively. Thus, for $L_{[2,0]}$ and $L_{[0,2]}$ we need the two string functions $\sigma^\Lambda_{[2,0]}$ and $\sigma^\Lambda_{[0,2]}$ for weights in class 0, that is $\Lambda = [2,0],[0,2]$, to calculate the character function χ_Λ, where $[2,0]$ is the highest weight of $L_{[2,0]}$, and $[0,2]$ is obtained by $[2,0]-\alpha_0 = [2,0]-[2,-2]$. Similarly, $[0,2]$ is the highest weight of $L_{[0,2]}$, and $[2,0]$ is obtained by $[0,2]-\alpha_1 = [0,2]-[-2,2]$. Notice that two

[7] It is easy to check that this has the $\mathfrak{sl}(n)$ Cartan matrix as a submatrix for $i,j = 1,\ldots,n-1$, one can then calculate the zeroth row and column using $(\alpha_0,\alpha_j) = (-\theta,\alpha_j)$ and $(\alpha_i,\alpha_0) = (\alpha_i,-\theta)$.

weights $[2,0]$ and $[0,2]$ are not in the same $\widehat{W}$-orbit. For $L_{[1,1]}$, there is only one weight of class 1 so that we only need the one string function $\sigma^{[1,1]}_{[1,1]}$.

For $\Lambda = [l_0, \ldots, l_{n-1}] \in P^+_{n,k}$ we define a *dual weight* $\Lambda^\dagger = [l'_0, \ldots, l'_{k-1}] \in P^+_{k,n}$ as follows. We first define the set $\sigma = (\sigma_1, \ldots, \sigma_n)$ where $\sigma_j = \sum_{i=0}^{j-1} l_i$ (note that $(k-\sigma_1, k-\sigma_2, \ldots, k-\sigma_n) = \text{par}(\Lambda)$), and define

$$\Lambda^\dagger = \sum_{j=1}^{n} \Lambda_{\sigma_j}, \tag{1.3.42}$$

where we define the subscripts modulo k. Note that this ensures that the zeroth label $l'_0 \geq 1$ as $\sigma_n = \sum_{j=0}^{n-1} l_0 = k \equiv 0 \mod k$. We can visually represent a dominant integral weight $\Lambda \in P^+_{n,k}$ and its dual weight $\Lambda^\dagger \in P^+_{k,n}$ using a closed ring with beads placed on it, which we call a *Dynkin ring*. We consider $(n+k)$ equidistant slots for beads which we label $1, 2, \ldots, n+k$, and then fill the slots corresponding to the set

$$\omega(\Lambda) = \{j + \sum_{i=0}^{j-1} l_i | j = 1, 2, \ldots, n\}. \tag{1.3.43}$$

The dual weight $\Lambda^\dagger$ is then obtained by interchanging beads and spaces and then calculating the l_i's, although cyclic permutations of labels are equivalent in this process due to the rotational symmetry of the ring. We fix a convention of labelling the beads in a clockwise manner, beginning below the 3 o'clock position. It should be clear to the reader that the double dual $(\Lambda^\dagger)^\dagger$ is some cyclic permutation of Λ.

Example 1.3.4.2. Consider $\widehat{\mathfrak{sl}}(2)_3$. Let $\Lambda = [2,1] \in P^+_{2,3}$, so that $\sigma = (2,3)$. We obtain the dual weight $\Lambda^\dagger = [1,0,1] \in P^+_{3,2}$ of $\widehat{\mathfrak{sl}}(3)_2$ (note that $l'_0 = 1$) and we see that $\omega(\Lambda) = \{3,5\}$ and $\omega(\Lambda^\dagger) = \{2,3,5\}$. We see that Λ and $\Lambda^\dagger$ are obtained from each other by interchanging beads and empty slots and a cyclic relabelling of slots.

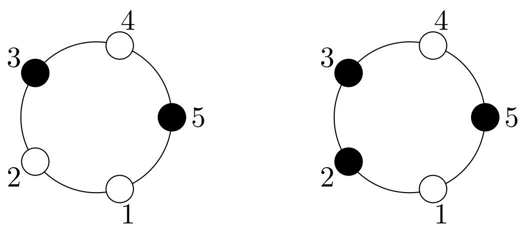


FIGURE 1.2: The Dynkin rings of $\Lambda = [2,1]$ and $\Lambda^\dagger = [1,0,1]$, the labelling convention employed in this thesis is shown in this example.

Later, when discussing minimal model coset characters arising from the AGT conjecture for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, we will use this process of computing dual weights when translating from gauge theoretic to CFT objects and vice versa.

Analogously to the finite case, if we have an affine Lie algebra $\widehat{\mathfrak{g}}$ with $\widehat{\mathfrak{p}} \subset \widehat{\mathfrak{g}}$ an affine subalgebra, the highest weight representation L_Λ has *branching rules* as follows where $k \in \mathbb{Z}_{\geq 0}$

$$L_\Lambda|_{\widehat{\mathfrak{p}}} = \bigoplus_{\mu,k} b_{\Lambda,(\mu-k\delta)} L_{\mu-k\delta}, \quad b_{\Lambda,(\mu-k\delta)} \in \mathbb{Z}_{\geq 0}, \tag{1.3.44}$$

for μ a $\widehat{\mathfrak{p}}$ dominant integral weight. We call the integers $b_{\Lambda,(\mu-k\delta)}$ the *affine branching coefficients*. More care must be taken in affine branching, as the imaginary root does not change the Dynkin labels of an affine weight, we must consider every possible variation of applying δ when calculating branchings. We collect the information of an affine branching into the q-series *branching function* using the branching coefficients as

$$b_\mu^\Lambda(q) = \sum_{k=0}^{\infty} b_{\Lambda,(\mu-k\delta)} q^k. \tag{1.3.45}$$

Branching will be important when discussing coset characters, as these character functions are extremely similar to branching functions. We will exploit this in chapter 5.

1.3.5 Crystal Bases and Paths for $\widehat{\mathfrak{sl}}(n)$

We wish to introduce canonical bases for our integrable $\widehat{\mathfrak{sl}}(n)$ modules that will allow a systematic study of certain module tensor products. As such we will use the crystal (or canonical) bases of the Kyoto school which were introduced independently in [37] and [59]. There is an alternative approach to these structures using paths as a basis for $\widehat{\mathfrak{sl}}(n)$ which originated in [60]. The path model was subsequently further developed by Littelmann (see [61, 62]), and we will tend to refer to the paths in this approach as Littelmann paths.

Here we will describe the data involved in defining the crystal basis and crystal graph associated to an integrable $\widehat{\mathfrak{sl}}(n)$ representation and the structure of these under tensor products. We will not prove any formula, nor the existence and uniqueness of any crystal graph related data for integrable $\widehat{\mathfrak{sl}}(n)$ representations. Aside from the references above, the interested reader is referred to [38, 63, 39, 64] and the more modern review [65].

Before we continue, we must point out that although crystal graphs and bases are associated to modules of the quantum group $\mathcal{U}_q(\mathfrak{sl}(n))$ at $q = 0$, we will treat them as associated to $\widehat{\mathfrak{sl}}(n)$ modules (as is common) due to their role in the Littlemann path formulation[8]. We will focus on crystal graphs associated to the affine Lie algebras $\widehat{\mathfrak{sl}}(n)$, but also note that this same structure exists for finite Lie algebras (such as $\mathfrak{sl}(n)$).

[8] In the Littelmann path formulation there are operators e_α and f_α equivalent to the Kashiwara raising and lowering operators [66], and one can also define the Littelmann coloured graph $\mathcal{G}(\pi)$ which was independently proven to be isomorphic to the crystal graph in [38] and [67], and as a corollary to the work in [68].

Let L_Λ be the irreducible highest weight module of $\widehat{\mathfrak{sl}}(n)$ of highest weight $\Lambda \in P_n^+$. We now describe the crystal graph, denoted by $\mathcal{B}_\Lambda$, of L_Λ. The nodes of $\mathcal{B}_\Lambda$ are labelled by special basis elements $\{b\}_{b\in B_\Lambda}$ for L_Λ, the *crystal (or canonical) basis* B_Λ. We also define the *Kashiwara raising and lowering operators* e_i and f_i for $i = 0, \dots, n-1$ that act on L_Λ. We denote the edge set for $\mathcal{B}_\Lambda$ by E_Λ. For each i-coloured edge $(b \xrightarrow{i} b') \in E_\Lambda$ between two nodes $b, b' \in B_\Lambda$, which we will denote as an ordered pair $(b, b')_i$, there is an action of Kashiwara operators between the corresponding vectors in L_Λ. We have the following action among corresponding basis elements $b, b' \in L_\Lambda$

$$f_i(b) = b', \quad e_i(b') = b, \tag{1.3.46}$$

and $f_i(b) = 0$ (equivalently $e_i(b') = 0$) if there is no $(b, b')_i \in E_\Lambda$ for some $b' \in B_\Lambda$ (respectively $b \in B_\Lambda$). Using this, we identify an action of e_i and f_i on $B_\Lambda \cup \{0\}$ such that $e_i(b') = b$ and $f_i(b) = b'$ if there exists $(b, b')_i \in E_\Lambda$ and $e_i(b') = f_i(b) = 0$ otherwise.

Crystal graphs can also be defined starting from the action of operators f_i and e_i on a set of nodes. In that case, we define the data of a set of nodes B and operators $f_i, e_i : B \longrightarrow B \cup \{0\}$ for $i = 0, 1, \dots, n-1$ and define a crystal graph $\mathcal{B}$ with nodes B and edges $(b, b')_i$ corresponding to $f_i(b) = b'$ (equivalently $e_i(b') = b$).

Example 1.3.5.1. We consider the $\mathfrak{sl}(3)$ highest weight representation V_Λ with highest weight $\Lambda = [1, 0]$. If we denote the highest weight vector by v, this module has a basis composed of v, $f_1 v$, and $f_2 f_1 v$, and we have the following crystal graph $\mathcal{B}$.

$$\overset{v}{\cdot} \xrightarrow{1} \overset{f_1 v}{\cdot} \xrightarrow{2} \overset{f_2 f_1 v}{\cdot}$$

Here the numbers over the edges denote their colours.

Due to the absence of mono-coloured cycles, the graph $\mathcal{B}_\Lambda^i$, with nodes B_Λ and the i-coloured edges which we denote E_Λ^i, is composed of disjoint finite directed paths which we call *i-strings*. For fixed $i = 0, 1, \dots, n-1$, each node b is part of one i-string, which we split into two halves, the sequence of edges $(e_i(b), b)_i$, $(e_i^2(b), e_i(b))_i, \dots, (e_i^{\varepsilon_i(b)}(b), e_i^{\varepsilon_i(b)-1}(b))_i$ and the sequence of edges $(b, f_i(b))$, $(f_i^{\phi_i(b)-1}(b)_i, \dots, f_i^{\phi_i(b)}(b))_i$, where we have defined $\varepsilon_i(b), \phi_i(b) \in \mathbb{Z}_{\geq 0}$ to be the minimum integers such that $e^{\varepsilon_i(b)+1}(b) = f^{\phi_i(b)+1}(b) = 0$. These numbers are always finite since L_Λ is integrable.

Using this, we associate to a node $b \in B_\Lambda$ an *i-signature* $\omega_i(b)$, which is a sequence of $\varepsilon_i(b)$ number of minus signs followed by ϕ_i number of plus signs, and an integer $l_i(b) = \phi_i(b) - \varepsilon_i(b)$ which we will refer to as a label. Below we will see that these labels l_i form the Dynkin labels for states in the module L_Λ. We can visualise the application of e_i on a node b using i-signatures, where applying e_i to b flips the right most minus sign in $\omega_i(b)$ to a plus in

$\omega_i(e(b))$ and the action of $f_i(b)$ as flips the left most plus sign in $\omega_i(b)$ to a minus sign in $\omega_i(f(b))$.

Example 1.3.5.2.

$$\underbrace{\overset{e_i^3(b)}{\cdot} \xrightarrow{-} \overset{e_i^2(b)}{\cdot} \xrightarrow{-} \overset{e_i(b)}{\cdot} \xrightarrow{-}}_{\varepsilon_i(b)=3} \overbrace{\overset{b}{\cdot} \xrightarrow{+} \overset{f_i(b)}{\cdot} \xrightarrow{+} \overset{f_i^2(b)}{\cdot}}^{\phi_i(b)=2} \tag{1.3.47}$$

In the example above, we have an i-string of 5 edges, corresponding to 3 possible applications of e_i and 2 of f_i on the node b. This gives us $\varepsilon_i(b) = 3$ and $\phi_i(b) = 2$, the i-signature as $\omega_i(b) = - - - + +$, and the label $l_i = 2 - 3 = -1$.

We use ε_i, ϕ_i, and l_i to make contact with the representation theory of L_Λ as follows

Definition 1.3.5.3. ([64, **2.1**] specialised to $\widehat{\mathfrak{sl}}(n)$ integrable representations) Let Λ_i for $i = 0, 1, \ldots, n-1$ be the fundamental weights of $\widehat{\mathfrak{sl}}(n)$. For each $b \in B_\Lambda$ define the following 3 elements of the weight lattice $P(\widehat{\mathfrak{sl}}(n))$:

1. $\phi(b) := \sum_{i=0}^{n-1} \phi_i(b)\Lambda_i$
2. $\varepsilon(b) := \sum_{i=0}^{n-1} \varepsilon_i(b)\Lambda_i$
3. $wt(b) := \sum_{i=0}^{n-1} l_i\Lambda_i$

The weight $wt(b)$ is equal to the weight of the corresponding vector $b_\Lambda \in L_\Lambda$, and the l_i form the Dynkin labels of $wt(b)$. Let $\mathcal{B}^1_\Lambda$ be the crystal graph of the highest weight module L_Λ and $\mathcal{B}^2_{\Lambda'}$ the one of $L_{\Lambda'}$ with corresponding nodes B^1_Λ and $B^2_{\Lambda'}$. We define the product crystal graph $\mathcal{B}_{\Lambda,\Lambda'}$ as the graph with nodes $B_{\Lambda,\Lambda'} = \{b_1 \otimes b_2 | b_1 \in B^1_\Lambda,\, b_2 \in B^2_{\Lambda'}\}$ and edges[9] $(b_1 \otimes b_2, b'_1 \otimes b'_2)_i \in E_{\Lambda,\Lambda'}$ corresponding to $f_i(b_1 \otimes b_2) = b'_1 \otimes b'_2$. The crystal graph defined this way is the crystal graph of the tensor product $L_\Lambda \otimes L_{\Lambda'}$ [69]. The action of the Kashiwara operators is defined as follows for the node $(b_1 \otimes b_2) \in B^{1,2}_{\Lambda,\Lambda'}$ where $b_1 \in B^1_\Lambda$ and $b_2 \in B^2_{\Lambda'}$,

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2, & \phi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes e_i(b_2), & \text{else} \end{cases} \tag{1.3.48}$$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i(b_1) \otimes b_2, & \phi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes f_i(b_2), & \text{else} \end{cases} \tag{1.3.49}$$

[9] Here the pairs of nodes in each factor, that is b_1 with b'_1 and b_2 with b'_2 are not necessarily distinct from each other.

Example 1.3.5.4. We consider again the module V_Λ and crystal graph $\mathcal{B}_\Lambda$ from example 1.3.5.1. We have the following values for ϵ_i and ϕ_i for $i = 1, 2$:

	v	$f_1 v$	$f_2 f_1 v$
ϵ_1	0	1	0
ϕ_1	1	0	0
ϵ_2	0	0	1
ϕ_2	0	1	0

Using these we can build the crystal graph $\mathcal{B}_{\Lambda,\Lambda}$ using the crystal graph tensor product rules (1.3.48) and 1.3.49. We borrow this example and diagram from [70] and will match this notation by defining $v \mapsto v_1$, $f_1 v \mapsto v_2$, and $f_2 f_1 v \mapsto v_3$. In this picture we have placed all

$$\begin{tikzcd}
 & & & V & \\
 & & v_1 \arrow[r, "f_1"] & v_2 \arrow[r, "f_2"] & v_3 \\
 & v_1 \arrow[d, "f_1"'] & v_1 \otimes v_1 \arrow[r, "f_1"] & v_2 \otimes v_1 \arrow[r, "f_2"] \arrow[d, "f_1"'] & v_3 \otimes v_1 \arrow[d, "f_1"'] \\
V & v_2 \arrow[d, "f_2"'] & v_1 \otimes v_2 \arrow[d, "f_2"'] & v_2 \otimes v_2 \arrow[r, "f_2"] & v_3 \otimes v_2 \arrow[d, "f_2"'] \\
 & v_3 & v_1 \otimes v_3 \arrow[r, "f_1"] & v_2 \otimes v_3 & v_3 \otimes v_3 \\
 & & & V \otimes V &
\end{tikzcd}$$

the nodes $v_i \otimes v_j \in B_{\Lambda,\Lambda}$ on a lattice and then placed arrows corresponding to the action of f_1 and f_2 according to our rules from above.

Remark 1.3.5.5. Drawing the tensor product in this way, where we have the crystal graphs of the two factors on the outside, allows one to clearly see some features of its structure. For instance, one can only have arrows corresponding to f_i parallel and in line with one of the f_i occurring in one of the factors. Furthermore, the disjoint connected components of the product graph correspond to the tensor product decomposition into highest weight modules. From this simple diagram we can easily see that we have the following tensor product decomposition of $\mathfrak{sl}(3)$ highest weight modules:

$$L_{[1,0]} \otimes L_{[1,0]} = L_{[2,0]} \oplus L_{[0,1]}. \tag{1.3.50}$$

The utility of this construction is much clearer when considering the i-signatures in the tensor product. The i-signature $\omega_i(b_1 \otimes b_2)$ of a node $b_1 \otimes b_2 \in B^{1,2}_{\Lambda,\mu}$ is obtained by placing the i-signature $\omega_i(b_1)$ to the left of $\omega_i(b_2)$ and successively eliminating pairs of signs where a $+$ sign is to the left of a $-$ sign, a process we will denote by bracketing.

Example 1.3.5.6. Consider two nodes $b_1 \in B^1_\Lambda$ and $b_2 \in B^2_\mu$ with i-signatures

$$\omega_i(b_1) = ----++, \tag{1.3.51}$$
$$\omega_i(b_2) = ---+, \tag{1.3.52}$$

the i-signature of the node $b_1 \otimes b_2 \in B^{1,2}_{\Lambda,\mu}$ is

$$\omega_i(b_1 \otimes b_2) = ----(+(+-)-)-+ = ------+\,. \tag{1.3.53}$$

We will use notion of the generalised i-signatures later. A generalised i-signature is a sequence of $+$ signs, $-$ signs and empty slots $\square$ with an infinite cyclic tail condition. For a highest weight $\widehat{\mathfrak{sl}}(n)_k$-module with highest weight $\Lambda = [d_0, \dots, d_{n-1}]$, the infinite cyclic tail condition is such that for sufficiently large m the m-th slot, denoted by $\omega_{i,m}$, in the generalised i-signature is defined to be

$$\omega_{i,m} = \begin{cases} \underbrace{-\cdots-}_{d_i}\underbrace{+\cdots+}_{d_{i+1}}\ \underbrace{\square\dots\square}_{k-d_i-d_{i+1}}, & m \equiv i \bmod n \\ \underbrace{\square\dots\square}_{k}, & \text{else} \end{cases} \tag{1.3.54}$$

Remark 1.3.5.7. These generalised i-signatures are equivalent to the Kyoto (and Littelmann) path models for integrable highest weight $\widehat{\mathfrak{sl}}(n)_k$ modules, see [37, 38] and [66, 61, 62]. We use them as a way to capture the combinatorics of tensor products of highest weight $\widehat{\mathfrak{sl}}(n)$-modules.

For us, the utility of the crystal graph construction comes from the interpretation of the i-signatures combinatorially. Having done so, we can easily calculate characters for tensor products of integrable $\widehat{\mathfrak{sl}}(n)$-modules by taking products of the generating functions for i-signatures. These generating functions have been shown to be the characters of CFT minimal models (explained in section 1.5), building on the ABF path approach to restricted solid-on-solid integrable models [71]. This forms the basis for our arguments in chapter 5.

1.4 Young Diagrams and $\widehat{\mathfrak{sl}}(n)$

Here we will describe a connection between highest weights of $\widehat{\mathfrak{sl}}(n)_N$-representations and cylindric partitions. This section builds on the content of sections 1.1, 1.2, and 1.3.

1.4.1 Cylindric and Burge Partitions and $\widehat{\mathfrak{sl}}(n)$ Representations

Let $\{\Lambda_0, \ldots, \Lambda_{n-1}\}$ be the fundamental weights of $\widehat{\mathfrak{sl}}(n)$, so that they form a basis of the weight lattice $P(\widehat{\mathfrak{sl}}(n))$ of $\widehat{\mathfrak{sl}}(n)$. Let $\Lambda = \sum_{i=0}^{n-1} a_i\Lambda_i$, $a_i \in \mathbb{Z}_{\geq 0}$, be a dominant integral weight. We can rewrite Λ as $\Lambda = \sum_{i=0}^{N-1} \Lambda_{\sigma_i}$, where $0 \leq \sigma_0 \leq \cdots \leq \sigma_{N-1} < n$, uniquely. The number N is referred as the level of Λ. Under this prescription we have explicitly $\sigma = (\sigma_0, \ldots, \sigma_{N-1}) = (\underbrace{0, \ldots, 0}_{a_0}, \ldots, \underbrace{n-1, \ldots, n-1}_{a_{n-1}})$. We define $\zeta_i = \sigma_{i+1} - \sigma_i$ for $0 \leq i < N-1$ and $\zeta_{N-1} = n + \sigma_{N-1}$ and use these to specify the cylindric inequalities (1.1.11) for $\zeta = (\zeta_0, \ldots, \zeta_{N-1})$ that are equivalent to the fundamental domain shape $(\infty^N)/(n^{a_0}, (n-1)^{a_1}, \ldots, 1^{a_{n-1}})/n$.

This process also works in reverse, given a set of cylindric inequalities we can associate a dominant integral $\widehat{\mathfrak{sl}}(n)_N$ weight. We will denote the set of all cylindric partitions whose cylindric inequalities are associated to an $\widehat{\mathfrak{sl}}(n)$ weight Λ in this way by $\mathcal{C}_\Lambda$. Note that the level of the affine weight is equal to the number of rows of the fundamental domain shape (the parameter N), and the rank is the cyclic shift (the parameter n).

Example 1.4.1.1. Consider again the cylindric partition π with fundamental domain shape $(\infty^3)/\mu/4$, where $\mu = (1, 1, 0)$, from examples 1.1.2.3 and 1.1.2.5 above. We can read off the level and rank of the associated weight using $d = 4 = n$ and $N = 3$, which is the number of rows of μ. Using the above prescription, we can use $\zeta_0 = \mu_1 - \mu_2 = 1 - 1 = 0$, $\zeta_1 = \mu_2 - \mu_3 = 1 - 0 = 1$ and $\zeta_2 = d - \mu_1 + \mu_3 = 4 - 1 + 0 = 3$. Therefore, the associated $\widehat{\mathfrak{sl}}(4)_3$ weight corresponding to these ζ_i is $\Lambda = \Lambda_0 + \Lambda_0 + \Lambda_1 = [2, 1, 0, 0]$. The interested reader can easily check that this gives the cylindric inequalities from the previous example.

Example 1.4.1.2. We now work in the reverse direction, and begin with an $\widehat{\mathfrak{sl}}(n)$ weight to obtain the associated cylindric inequalities. Consider the fundamental domain shape associated to the $\widehat{\mathfrak{sl}}(3)_4$ weight $\Lambda = \Lambda_0 + \Lambda_0 + \Lambda_1 + \Lambda_2 = [2, 1, 1]$. Using the prescription above we find that $\zeta_0 = 0$, $\zeta_1 = 1$, $\zeta_2 = 1$, and $\zeta_3 = 3 - 2 = 1$ so that we have the following cylindric inequalities

$$\begin{aligned}
\lambda_i^{(0)} &\geq \lambda_i^{(1)},\\
\lambda_i^{(1)} &\geq \lambda_{i+1}^{(2)},\\
\lambda_i^{(2)} &\geq \lambda_{i+1}^{(3)},\\
\lambda_i^{(3)} &\geq \lambda_{i+1}^{(0)}.
\end{aligned}$$

We then choose a specific example of this fundamental domain by using the following quadruple of Young diagrams $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$, where $\lambda^{(0)} = (6, 4, 1)$, $\lambda^{(1)} = (4, 2, 1)$, $\lambda^{(2)} = (3, 2, 1)$ and $\lambda^{(3)} = (5, 1, 1)$. One can easily check the parts of these satisfy the cylindric inequalities above and visually we can represent this quadruple of Young diagrams and cylindric inequalities by the cylindric partition:

$$\begin{array}{cccccccc}
 & & & & \vdots & \vdots & \vdots & \iddots \\
 & & & & 6 & 4 & 1 & \\
 & & & & 4 & 2 & 1 & \\
 & & & 3 & 2 & 1 & & \\
 & & 5 & 1 & 1 & & & \\
 & 6 & 4 & 1 & & & & \\
\iddots & \vdots & \vdots & \vdots & & & &
\end{array}$$

Remark 1.4.1.3. The fundamental domain in the infinite array picture could begin on any row, thus domains defined by a cyclic permutation of the labels and ζ_i's are equivalent. This changes the associated weight by a cyclic permutation of the labels σ_1 through σ_{N-1} and relabelling $\sigma_0 = n - \sigma_{N-1}$. Moreover we see that the weights $\Lambda = [d_0, d_1, \ldots, d_{n-1}]$ and $\Lambda' = [d_1, d_2, \ldots, d_{n-1}, d_0]$ have equivalent associated fundamental domain shapes.

We can define a natural colouring on a cylindric partition $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$ using its associated $\widehat{\mathfrak{sl}}(n)_N$ weight. We colour its component Young diagrams with n-colours such that the diagram $\lambda^{(i)}$ has a charge σ_i.

There is no reason to restrict only to this natural colouring, and we can also define other colourings on a cylindric partition λ in m colours by the charge maps $\sigma : \{0, 1, \ldots, N-1\} \longrightarrow \mathbb{Z}_m$, where we denote $\sigma_i = \sigma(i)$, and define $\sigma = (\sigma_0, \ldots, \sigma_{N-1}) \in (\mathbb{Z}_m)^N$ to be the vector of charges. Note that this general colouring works on any N-tuple of Young diagrams, and we will not restrict this process to apply to only cylindric partitions. Once we have assigned a $\mathbb{Z}_m$-charge to each component Young diagram, we colour them as described above and this defines a colouring on the N-tuple λ. We will sometimes denote the m-coloured N-tuple by $\lambda^\sigma = \big((\lambda^{(0)})^{\sigma_0}, \ldots, (\lambda^{(N-1)})^{\sigma_{N-1}}\big)$, although it is usually obvious if we are discussing an N-tuple λ that is coloured or not.

We will let $k_j^{(i)}$ denote the number of j-coloured boxes in $\lambda^{(i)}$ and $\delta k_j^{(i)} = k_j^{(i)} - k_0^{(i)}$, we then define $k_j = \sum_{i=0}^{N-1} k_j^{(i)}$ and $\delta k_j = \sum_{i=0}^{N-1} \delta k_j^{(i)}$ for $j = 0, 1, \ldots, m-1$, and let $k = (k_0, \ldots, k_{m-1})$ and $\delta k = (\delta k_0, \ldots, \delta k_{m-1})$. By defining the size of a N-tuple of Young diagrams to be $|\lambda| = \sum_{i=0}^{N-1} |\lambda^{(i)}|$, we have that $|\lambda| = \sum_{j=0}^{m-1} k_j$.

We will also consider more general cylindric style partitions by introducing a second vector of positive integer parameters $\xi = (\xi_0, \ldots, \xi_{N-1}) \in (\mathbb{Z}_{\geq 0})^{N-1}$.

Definition 1.4.1.4. Let $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$ be an N-tuple of Young diagrams and $\zeta = (\zeta_0, \ldots, \zeta_{N-1})$ $\xi = (\xi_0, \ldots, \xi_{N-1})$ be N-tuples of positive integers. If λ satisfies

$$\lambda_j^{(i)} \geq \lambda_{j+\zeta_i}^{(i+1)} - \xi_i, \tag{1.4.1}$$

we say it is a *Burge multipartition* of weight (ζ, ξ)

In the spirit of the connection between cylindric partitions and $\widehat{\mathfrak{sl}}(n)$-representations, we choose to take both ζ and ξ as defining $\widehat{\mathfrak{sl}}(n)$ weights. In this case, we have that $\zeta \in P^+_{N,n}$ and $\xi \in P^+_{N,k}$ for some level k.

Remark 1.4.1.5. Building on our discussion of the 3D interpretation of cylindric partitions, we can think of the presence of non-zero ξ as a 3D shift of Young diagrams. Cylindric multipartitions only depend on ζ, which was visualised as a shift in the 2-dimensional xy-plane, as demonstrated in our visualization of them as arrays of numbers. The extra shift introduced by ξ can be thought of as an additional shift along the z-axis.

We consider Burge multipartitions to be generalisations of restricted partition pairs [72], as considered by Burge hence the name, which are $N = 2$ Burge multipartitions in this language. The cylindric partitions and Burge multipartitions themselves are special cases of (α, β)-cylindric partitions. In the language of [35], Burge multipartitions are $(0, \beta)$-cylindric partitions, where β carries the information of both ζ and ξ.

1.4.2 Generating Functions

From an AGT perspective, the combinatorial objects we have just defined will be taken to represent physical states that we will sum over when defining the instanton partition function. For our purposes it will be useful to count these states using a generating function.

Recall that $\mathcal{C}_\Lambda$ denotes the set of all cylindric partitions associated to a weight $\Lambda \in P^+_{n,N}$. We define the generating function X^n_Λ of coloured cylindric partitions associated to Λ by

$$X^n_\Lambda(\mathfrak{q}; \mathbf{z}) = \sum_{\lambda \in \mathcal{C}_\Lambda} \mathfrak{q}^{k_0} \prod_{i=1}^{n-1} z_i^{\delta k_i}. \tag{1.4.2}$$

Here the exponent of $\mathfrak{q}$ counts the 0-coloured boxes and the exponent of z_i counts the difference between i-coloured boxes and 0-coloured boxes. We will also define a specialised version of the generating function depending on only the parameter q by first sending $\mathfrak{q} \mapsto q^n$ and then $z_i \mapsto q$

$$X_\Lambda(q) = \sum_{\lambda \in \mathcal{C}_\Lambda} q^{\sum_{i=0}^{n-1} k_i} = \sum_{\lambda \in \mathcal{C}_\Lambda} q^{|\lambda|}, \tag{1.4.3}$$

this specialised version ignores the colouring data, and thus counts *uncoloured* cylindric partitions.

Lemma 1.4.2.1. ([47]) For an $\widehat{\mathfrak{sl}}(n)$ weight Λ we have the following explicit expression for the specialised generating function $X_\Lambda(q)$,

$$X_\Lambda(q) = \sum_{\lambda \in \mathcal{C}_\Lambda} q^{|\lambda|} = \frac{1}{(q;q)_\infty^n} \sum_{K_1+\cdots+K_n=0} \det_{1\leq s,t\leq n} \left(q^{(\mu_t - t)(nK_s+s-t)+K_s(\frac{1}{2}nK_s+s)(n+N)} \right), \tag{1.4.4}$$

where $\mu = (n^{a_0}, (n-1)^{a_1}, \ldots, 1^{a_{n-1}})$, is the partition $\mathrm{par}(\Lambda)$ associated to Λ (see 1.3.20 for a definition).

In this lemma we have used the following notation

$$(a;q)_k = \prod_{i=1}^{k}(1-aq^i), \tag{1.4.5}$$

for the q-Pochammer symbol. We will frequently make use of the q-Pochammer symbol throughout this thesis in the guise of

$$\frac{1}{(q;q)_\infty} = 1 + q + 2q^2 + 3q^3 + 5q^4 + \ldots, \tag{1.4.6}$$

which is the generating function for partitions

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} = \frac{1}{(q;q)_\infty}. \tag{1.4.7}$$

Remark 1.4.2.2. As explained in [47], this expression for the generating function $X_\Lambda(q)$ is a special case of [35, **Th 2**], where a more general expression is obtained in terms of the parameters notated as b_s and a_t. We can then obtain (1.4.4) by substituting the value $b_s = 0$ and sending $a_t \mapsto \infty$ for all t and s.

Consider an N-tuple of Young diagrams $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$. Define $k_j^i(\lambda)$ to be the number of rows with i number of boxes (that is $\lambda_m^{(l)} = i$ for some $1 \leq m \leq l(\lambda^{(l)})$ and $0 \leq l \leq N-1$) such that the box at the end of the row is j-coloured, in all diagrams[10].

Definition 1.4.2.3. ([73]) λ is said to be highest lift if for each $i > 0$, there exists at least one $j > 0$ such that $k_j^i(\lambda) = 0$. If λ is also a cylindric partition, we call it a FLOTW[11] multipartition.

Example 1.4.2.4. In the case of $N = 1$ and arbitrary n, every Young diagram $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a cylindric partition by the weakly decreasing property on its parts. In this case, $k_j^i(\lambda) = 0$ implies that no more than $(n-1)$ parts can be equal to each other. This condition defines the n-regular partitions and they are the highest lift FLOTW multipartitions for $N = 1$.

[10] Note that this is a different object to $k_j^{(i)}$ which we defined above. These two notations are differentiated by the presence of the brackets on the superscript i.

[11] FLOTW is an acronym for the first letter of the surnames of the authors in the referenced paper.

The following lemma shows how the concept of highest lift multipartitions is useful when computing generating functions of cylindric partitions.

Lemma 1.4.2.5. ([47]) The generating function for FLOTW multipartitions $(X_\Lambda^{\mathbf{k}})^*(\mathfrak{q};\mathbf{z})$ is related to the generating function $X_\Lambda^{\mathbf{k}}(\mathfrak{q};\mathbf{z})$ for cylindric partitions by

$$X_\Lambda^{\mathbf{k}}(\mathfrak{q};\mathbf{z}) = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty}(X_\Lambda^{\mathbf{k}})^*(\mathfrak{q};\mathbf{z}). \tag{1.4.8}$$

Importantly, the generating function of FLOTW multipartitions gives the character (1.3.34) for irreducible integrable $\widehat{\mathfrak{sl}}(n)_N$ highest weight modules, denoted by L_Λ, as

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(\mathfrak{q};\mathbf{z}) = e^\Lambda (X_\Lambda^k)^*(\mathfrak{q};\mathbf{z}), \tag{1.4.9}$$

where $\mathfrak{q} = e^\delta$ and $z_i = e^{\alpha_i}$ for $i = 1,\dots,n-1$. Explicitly, let $\Lambda = \sum_{i=0}^{N-1}\Lambda_{\sigma_i}$, where $\sigma_0 \geq \cdots \geq \sigma_{N-1}$, and define the partition $\sigma = (\sigma_0,\dots,\sigma_{N-1})$. In chapter 4, we will match the character for $\widehat{\mathfrak{sl}}(n)_N$ highest weight modules with a generating function for coloured cylindric partitions with charges defined by σ. Due to this, it will be convenient to consider a weight Λ to be defined using σ.

We define $\mathcal{M}^\sigma$ to be the set of N-tuples of coloured Young diagrams $\lambda = (\lambda^{(0)},\dots,\lambda^{(N-1)})$ that satisfy the following inequalities between their rows

$$\lambda_j^{(i)} \geq \lambda_{j+\sigma_i-\sigma_{i+1}}^{(i+1)} \text{ for } j \geq 1,\ \ 0 \leq i \leq N-2; \tag{1.4.10}$$

$$\lambda_j^{(N-1)} \geq \lambda_{j+\sigma_{N-1}-\sigma_0+n}^{(0)},\ \text{ for } j \geq 1, \tag{1.4.11}$$

it is clear that these are the cylindric partitions associated to Λ so that $\mathcal{M}^\sigma = \mathcal{C}_\Lambda$, but here we think of them as defined by the charges σ. We take each σ_i to be the natural charge described above to colour $\lambda^{(i)}$ with n colours and denote the resulting coloured Young diagram by $(\lambda^{(i)})^{\sigma_i}$. We will identify $(\lambda^{(i)})^{\sigma_i}$ with $\lambda^{(i)}$ when it is clear to do so.

We define $\mathcal{M}_*^\sigma \subset \mathcal{M}^\sigma$ to be the FLOTW multipartitions and have the following explicit expressions for the character formula $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}$ (which we defined in (1.3.34)).

Lemma 1.4.2.6. ([60]) The character formula $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}$ has the following forms as sums over $\mathcal{M}_*^\sigma$ for $q = e^{-\delta}$ and $\mathfrak{t}_i = e^{\mathbf{e}_i}$ and $\mathfrak{t} = (\mathfrak{t}_1,\dots,\mathfrak{t}_{N-1})$

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q,\mathfrak{t}) = e^\Lambda \sum_{\lambda\in\mathcal{M}_*^\sigma} q^{k_0(\lambda)} \prod_{i=0}^{N-1}\left(\frac{\mathfrak{t}_i}{\mathfrak{t}_{i+1}}\right)^{k_i(\lambda)} \tag{1.4.12}$$

$$= e^\Lambda \sum_{\lambda\in\mathcal{M}_*^\sigma} q^{k_0(\lambda)} \prod_{i=1}^{N-1}\left(\frac{\mathfrak{t}_i}{\mathfrak{t}_{i+1}}\right)^{\delta k_i(\lambda)}. \tag{1.4.13}$$

Where $\mathbf{e}_i - \mathbf{e}_{i+1} = \alpha_i$ are the weights of the fundamental representation L_{Λ_1} of $\mathfrak{sl}(n)$.

As described in [47], the set $\mathcal{M}^\sigma$ of *all* cylindric multipartitions is isomorphic to the product of $\mathcal{M}^\sigma_*$ and the set of partitions

$$\mathcal{M}^\sigma \leftrightarrow \mathcal{M}^\sigma_* \times \text{Par}. \tag{1.4.14}$$

Under this isomorphism, an N-tuple of coloured Young diagrams λ is mapped to (λ^*, μ) where $\delta k_i(\lambda) = \delta k_i(\lambda^*)$ and $k_0(\lambda) = k_0(\lambda^*) + |\mu|$. We can then use the generating function for partitions to obtain the following expression for the character of L_Λ

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q, \mathfrak{t}) = (q; q)_\infty \left(e^\Lambda \sum_{\lambda \in \mathcal{M}^\sigma} q^{k_0(\lambda)} \prod_{i=1}^{n-1} \left(\frac{\mathfrak{t}_i}{\mathfrak{t}_{i+1}} \right)^{\delta k_i(\lambda)} \right), \tag{1.4.15}$$

and it is this form of the character that we will compare against in chapter 4.

1.5 2D Conformal Field Theory

In this section we will review the necessary 2D conformal field theory notation, objects, and results for studying an AGT style correspondence. As such, this will not form an in-depth or comprehensive review of 2D conformal field theory as a whole. We will focus mostly on collecting results and fixing notation except where we deem it enlightening to expand calculations for the subsequent work presented in chapters 3, 4, and 5.

Most of this material is standard and taken from [50] with some sections, especially our discussion of the free boson in section 1.5.6, supplemented by [74]. Our discussion of Liouville theory in section 1.5.7 also draws on the notes [75], and we have also used this when writing about axiomatic foundations for CFTs in section 1.5.1. Each of these three references are excellent resources to learn CFT, although we single out [50] as the bible of CFT and [75] for the reader without much of a background in quantum field theory.

This section is large, as most of the thesis centres on the CFT side of AGT correspondences. We will begin in sections 1.5.1 and 1.5.2 by briefly reviewing two separate formalisms for conformal field theory: the algebraic approach which is similar to what we have already covered in sections 1.2 and 1.3, and the operator formalism. They are equivalent, as we will describe. In sections 1.5.3 and 1.5.4 we will discuss the main objects of CFT in the AGT context, the correlation functions and especially conformal blocks. The rest of the sections focus on all the specific CFTs we will encounter in the context of AGT within this thesis including: the free boson, minimal models (for both the Vir and $\mathcal{W}_N$ algebras), Wess-Zumino-Witten models, coset models, Liouville CFT, and Toda CFT.

1.5.1 Basic Definitions and Notation - Algebraic Formalism

We will begin by reviewing what we mean by conformal field theory (CFT) in this thesis, and give a formal definition. We will then discuss some of the basic objects, definitions, and algebraic structure of CFTs. Within this thesis, we will always take the word algebra to mean a Lie algebra or a vertex operator algebra (VOA) unless otherwise stated.

Let (M, g) be a pseudo-Riemannian manifold referred to as *space-time* where $g : TM \times TM \to \mathbb{R}$ is the pseudo-Riemannian metric. Throughout this section, we will always work locally, unless otherwise stated, and take a function $\phi(x)$ at the point $x \in M$ to be a function defined for a local coordinate on M. Also let $\mathcal{H}$ be a vector space [12] equipped with a scalar product that is called the *space of states* or *spectrum* of the theory. We will sometimes notate an element $\sigma \in \mathcal{H}$ as a ket $|\sigma\rangle$. As of now, this ket notation is *different* from used in sections 1.2 and 1.3 as it represents *any* element in $\mathcal{H}$ and there is no Lie algebra action yet defined on it. Below, we will link these two notations together.

Next, we recall the definition of a *conformal transformation*. Let $C : M \to M$ be an invertible transformation on (M, g), and denote by g_x the metric tensor on the tangent space T_xM to the point $x \in M$. Denote by C^*g the pullback of g, and $(C^*g)_x$ the pullback at the point x. The transformation C is conformal if the pullback of g is locally invariant up to a scale factor, that is

$$g_{C(x)} = \Lambda(x)(C^*g)_x, \tag{1.5.1}$$

where $\Lambda(x)$ is a scalar, dependent only on the local coordinates for M. Here we note that this ensures that angles are locally preserved, hence the name conformal.

In the case where $M = \mathbb{R}^d$ and $g_{\mu\nu} = \eta_{\mu\nu}$ is a metric of signature (p, q), the group of global conformal transformations is the Lie group $SO(p + 1, q + 1)$. Its Lie algebra $\mathfrak{so}(d + 1, 1)$ is generated by: translations P_μ, dilatations D, rotations $L_{\mu\nu}$, and special conformal transformations K_μ. We will never need the explicit form or brackets of these generators, although the reader can find them in [50, §4] or [74, §1]. For the rest of this section, we will focus on the algebra of *local* conformal transformations for the case where $M = \Sigma$ is a Riemann surface.

We now discuss the algebraic structure of CFTs before providing a more formal definition. The generating algebra of local conformal transformations in 2-dimensions is a tensor product of analytic and anti-analytic transformations, which are generated by the differential operators $l_n = -z^{n+1}\partial_z$ and $\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$ for $n \in \mathbb{Z}$ respectively. This is isomorphic to two commuting copies of the *Witt algebra*, a Lie algebra with generators l_n for $n \in \mathbb{Z}$ and

[12] Typically in a quantum field theory, $\mathcal{H}$ is a Hilbert space. We relax this property to allow us to consider CFTs that could be non-unitary.

defining brackets

$$[l_m, l_n] = (m-n)l_{m+n}. \tag{1.5.2}$$

In a 2D theory with conformal symmetry, it is then natural to choose local coordinates $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$, holomorphic and anti-holomorphic coordinates, and we will employ this in the sequel.

Due to the commutativity of the two isomorphic algebras, we will focus all our subsequent discussion on just the holomorphic sector and notate objects as only depending on the holomorphic coordinate z. The reader should note that although most objects in the sequel will be notated as depending on z, they will also have an anti-holomorphic factor dependent on $\bar{z}$, unless otherwise stated, which we will not notate. Similarly, for any result or computation in the holomorphic sector there is usually an analogous result or computation for the anti-holomorphic sector.

We define the *Virasoro algebra* (which we will denote by Vir) as the unique central extension[13] of the Witt algebra with generators L_m for $m \in \mathbb{Z}$ that satisfy the Lie bracket

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\tilde{c}}{12}(m^3 - m)\delta_{n+m,0}, \tag{1.5.3}$$

where $\tilde{c}$ is central in the algebra. When considering a specific representation of Vir, the eigenvalue $c \in \mathbb{C}$ of the central element $\tilde{c}$ on that representation will be called the *central charge*.

We denote a set of functions $F = \{\phi_1, \phi_2, ...\}$ on Σ which are referred to as the *fields* of the theory. In general, there are infinitely many such fields in a CFT. We also note that if $\phi \in F$ we also have $\partial\phi \in F$. These fields generate a vector space, called the *space of fields*. We take as an axiom the *state-field correspondence*:

Axiom 1.5.1.1. ([75, **1.1**]) There is an injective linear map from the spectrum $\mathcal{H}$ to the space of fields which we notate as

$$|\sigma\rangle \mapsto V(|\sigma\rangle, z). \tag{1.5.4}$$

Remark 1.5.1.2. This is not true for other fields theories, so that the state-field correspondence is a special property of conformal field theories. We also note that while this axiom usually goes by the name of the state-field correspondence in the literature it is also sometimes referred to as the operator-field correspondence, for example in [50].

[13] The appearance of the Virasoro algebra instead of the Witt algebra in CFT comes from the concept of quantization in physics. We only ever consider quantum (quantised) CFTs in this thesis so will always deal with Virasoro not the Witt algebra.

A CFT on (Σ, g) with spectrum $\mathcal{H}$, is a collection of functions called the n-point *correlation functions* for all $n \in \mathbb{Z}_{>0}$

$$\langle \prod_{i=1}^{n} V(|\sigma_i\rangle, z_i) \rangle, \tag{1.5.5}$$

where $z_1, \cdots, z_n \in \Sigma$ and $\sigma_i \in \mathcal{H}$, that are linear on the set F of fields, associated to $\mathcal{H}$ through the state-field correspondence, and invariant under conformal transformations. In section 1.5.4, we will discuss more of the structure of the correlation functions. Our goal is to solve (or partially solve) CFTs, which is defined to be a determination of the spectrum and all the correlation functions.

We collect all the data contained in this informal discussion into one formal definition for a CFT.

Definition 1.5.1.3. (Paraphrased slightly from [12]) A meromorphic CFT (mCFT) defined on a compact Riemann surface Σ is composed of a vector space $\mathcal{H}$ and a map $V(|\sigma\rangle, z)$, that maps a state $|\sigma\rangle \in \mathcal{H}$ and point $z \in \Sigma$ to the space of fields on Σ and $\mathcal{H}$. Furthermore, there is a distinguished state $|L\rangle$ which has an operator valued Laurent expansion

$$V(|L\rangle, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = T(z), \tag{1.5.6}$$

whose modes $\{L_n\}_{n\in\mathbb{Z}}$ generate the Virasoro algebra for some central charge $c \in \mathbb{C}$.
The map V satisfies the following properties:

1. There exists a unique state $|\Omega\rangle \in \mathcal{H}$, called the vacuum state, such that $V(|\sigma\rangle, z)|\Omega\rangle = e^{zL_{-1}}|\sigma\rangle$.
2. The scalar product[14] $\langle\sigma_1| V(|\sigma\rangle, z)|\sigma_2\rangle$ for $\sigma_1, \sigma_2 \in \mathcal{H}$ is a meromorphic function of z.
3. $\langle\sigma_1| V(|\sigma\rangle, z)V(|\phi\rangle, w)|\sigma_2\rangle$ is a meromorphic function for $|z| > |w|$.
4. $\langle\sigma_1| V(|\sigma\rangle, z)V(|\phi\rangle, w)|\sigma_2\rangle = \epsilon_{\sigma\phi}\langle\sigma_1| V(|\phi\rangle, z)V(|\sigma\rangle, w)|\sigma_2\rangle$ where $\epsilon_{\sigma\phi} = \pm 1$. If $\epsilon_{\sigma\phi} = 1$ we say the fields are bosonic and if $\epsilon_{\sigma\phi} = -1$ we say the fields are fermionic.

Of note, property one of this definition gives us the state-field correspondence. By taking $z \to 0$ we obtain

$$V(|\sigma\rangle, 0)|\Omega\rangle = |\sigma\rangle, \tag{1.5.7}$$

which is the usual form of the state-field correspondence encountered in textbooks.

The map V is called the *vertex operator map*. Although we have notated fields as depending on only z, in general most fields depend on z and $\bar{z}$ (this is an example of notating only the

[14] This is physics notation for $\langle\sigma_1, (V(|\sigma\rangle, z)\cdot\sigma_2)\rangle = \langle(\sigma_1 \cdot V^\dagger(|\sigma\rangle, z)), \psi_2\rangle$, where $\langle\ ,\ \rangle =$ is the scalar product on $\mathcal{H}$. We will use it throughout this thesis. In section 1.5.4, we will discuss this scalar product for the Verma modules of CFTs and its relation to the correlation functions in more depth.

holomorphic sectors as we noted above). Any fields that *only* depend on z or $\bar{z}$ are referred to as *chiral* or *anti-chiral* respectively.

$T(z)$ is refered to as the (holomorphic) *Virasoro energy-momentum tensor* and the existence of $T(z)$ ensures that $\mathcal{H}$ forms a representation of Vir. When clear, we will sometimes notate $V(|\psi\rangle, z) = V_\psi(z)$ or $V(|\psi\rangle, z) = \psi(z)$ depending on context. This is the state-field map from axiom 1.5.1.1. While this notation is useful when developing the theory axiomatically and using the state-field correspondence, we will usually explicitly specify fields and their corresponding states separately for clarity in actual computations. Finally, we will drop the meromorphic prefix and henceforth refer to mCFTs as CFTs.

We will call $\{L_{-n}\}$ and $\{L_n\}$ for $n > 0$ the *raising* and *lowering* operators[15] respectively, analogous to what was described for semi-simple Lie algebras in sections 1.2 and 1.3. We will only consider the case where the spectrum $\mathcal{H}$ will be a product of highest weight *Verma modules* for Vir, where each Verma module will form a representation of the same central charge c.

As for simple Lie algebras, we define a Vir highest weight state $|h\rangle$ by the following two properties

$$L_0 |h\rangle = h |h\rangle, \quad L_n |h\rangle = 0, \quad n > 0. \tag{1.5.8}$$

The eigenvalue h of the L_0 operator is what is known as the *conformal dimension* [16]. We can then create the *descendant states* of $|h\rangle$ by the raising operators. Explicitly, a descendant state $|\psi\rangle$ is of the form

$$|\psi\rangle = L_{-n_1} L_{-n_2} \dots L_{-n_k} |h\rangle, \quad n_i \in \mathbb{Z}_{>0}. \tag{1.5.9}$$

Through the state-field correspondence, we can define analogous descendant fields. In CFT, highest weight states are usually referred to as *primary states*, and their corresponding fields as *primary fields*. We will only consider the case where $h \geq 0$. Analogously, the descendant states correspond to *descendant fields*.

We define the Verma module $\mathcal{V}_{c,h}$ to be the Vir-module generated by the highest weight state $|h\rangle$ and its descendant states, with central charge c. Generally, the highest weight Verma modules generated by primary fields may form reducible Vir representations and will contain infinitely many subrepresentations. Physicists refer to the set of a primary field and all its descendant fields as a *conformal family*. If we let $\phi(z)$ be a primary field, the conformal

[15] Physicists refer to these as creation and annihilation operators, due to their action on highest weight states and the vacuum.

[16] The notation h for conformal dimension is not uniform across the literature. Throughout much of this review of material we will defer to the canonical sources we have cited in the introduction and use h. When discussing AGT correspondences, we will instead tend to use Δ, in line with the bulk of the AGT literature.

family of $\phi(z)$ is notated as $[\phi]$. Finally, non-primary, or descendant fields in a CFT are also referred to as *secondary fields* by physicists.

From the Vir commutation relations (1.5.3), we see that the raising operators, L_n for $n < 0$, increase the conformal dimension (L_0 eigenvalue) of a state and the lowering operators, L_n for $n > 0$, reduce it. We also note that even though L_0 is not central in the algebra Vir, it acts diagonally (and is the *only* non central generator that does so) on highest weight representations. We will refer to $\sum_i n_i$ as the *level* of a descendant state, and all states of level l will form the l-th level $\mathcal{V}_{c,h}^{(l)}$ of a Vir Verma module $\mathcal{V}_{c,h}$.

Analogously to Lie algebras, CFTs have their own character functions which count the states in a Vir-module. We will calculate these using AGT combinatorics in chapters 2, 4 and 5.

Definition 1.5.1.4. The character $\chi_{c,h}(q)$ of a Vir Verma module $\mathcal{V}_{c,h}$ with central charge c acting on a highest weight state $|h\rangle$ of conformal dimension h is defined to be the trace over the module of states weighted by their conformal dimension

$$\chi_{c,h}(q) := \mathrm{Tr}_{\mathcal{V}_{c,h}} q^{L_0 - c/24} \tag{1.5.10}$$

$$= \sum_{n=0}^{\infty} \dim(\mathcal{V}_{c,h}^{(n)}) q^{n+h-c/24}. \tag{1.5.11}$$

In the second line of this definition we have defined $\dim \mathcal{V}_{c,h}^{(n)}$ as the number of linearly independent states in $\mathcal{V}_{c,h}$ at level n.

Finally, we say a field is a *current* if it is a chiral field of conformal dimension $h = 1$. We will sometimes refer to the conformal dimension of a current as its *spin*[17].

1.5.2 Basic Definitions and Notation - Operator Formalism

Above, we discussed the basic objects and structure of CFTs from an algebraic perspective, focusing on the states. We now look at the fields of a theory, taking the operator perspective. This discussion follows the necessary material in [50, §6].

Let $\phi(z)$ be a primary field associated to a primary state $|h\rangle$ of conformal dimension h. We define a mode expansion of $\phi(z)$ by

$$\phi(z) := V(|h\rangle, z) = \sum_n \phi_n z^{-n-h}, \tag{1.5.12}$$

[17]Spin in a CFT is defined to be the difference $L_0 - \bar{L}_0$ of the two chiral L_0 operators. For a chiral field, one of these operators (which we take to be the anti-holomorphic $\bar{L}_0$) must act trivially so that spin in this case is simply equal to the eigenvalue of the other (L_0).

where the sum runs over $n \in \mathbb{Z}$ (*bosonic* fields) or $n \in \frac{1}{2}\mathbb{Z}$ (*fermionic* fields). We can invert this expansion to obtain an expression for the modes

$$\phi_n = \frac{1}{2\pi i} \oint_{z=0} dz\, z^{n+h-1} \phi(z). \tag{1.5.13}$$

We now postulate two more axioms, that are essential to our operator approach to CFT. In the sequel, we only consider theories containing bosonic fields.

Axiom 1.5.2.1. (Radial Ordering) The fields appearing within an N-point correlation are radially ordered. Here the radial ordering operator is defined to act on pairs of fields as

$$\mathcal{R}\left(A(z)B(w)\right) = \begin{cases} A(z)B(w) \text{ if } |z| > |w|, \\ B(w)A(z) \text{ if } |z| < |w|. \end{cases} \tag{1.5.14}$$

Axiom 1.5.2.2. There exists an *operator product expansion* (OPE) between two fields $A(z)$ and $B(w)$

$$A(z)B(w) = \sum_{r=-\infty}^{r_0} \frac{(AB)_r(w)}{(z-w)^r} \tag{1.5.15}$$

for some $r_0 \in \mathbb{Z}$, and where the coefficients $(AB)_r$ are well defined as $z \to w$. The OPE is understood to only hold within correlations functions, and as such radial ordering is always assumed to hold. Usually we will deal with OPEs where $r_0 > 0$ so that there are singular terms in the expansion.

We define the *normal ordered product* at the point w as the non-singular terms $(AB)(w) = \sum_{r=0}^{\infty}(AB)_r(w)(w-w)^r = (AB)_0(w)$. This is the CFT generalisation of the usual *normal ordering* from physics, which we define now. The normal ordering of two free fields[18] $\phi_1(z)$ and $\phi_2(w)$ is notated as $:\phi_1\phi_2:$ and means we place all raising operators involved in the product to the left and all lowering operators to the right. Note that while normal ordering and the normal ordered product are similar, they are not identical.

We define one final piece of notation for OPEs, where we take the singular terms to be called a *contraction*, which we notate as

$$\overbracket{A(z)B}(w) = \sum_{r=1}^{r_0} \frac{(AB)_r(w)}{(z-w)^r}. \tag{1.5.16}$$

Using this notation, the normal ordered product at w is the difference of the OPE and the contraction of a product of fields

$$(AB)(w) = \left(A(z)B(w) - \overbracket{A(z)B}(w) \right) \Big|_{z \to w} \tag{1.5.17}$$

[18] A *free field* in a CFT is taken to be a field whose OPE has only *one* singular term.

The equivalence between the algebraic and operator formalism goes as follows: Consider two fields A and B with mode expansions $A(z) = \sum_n a_n z^{-n-h_A}$ and $B(z) = \sum_n b_n z^{-n-h_B}$, respectively. We can calculate the commutator of the modes a_n and b_m for $n, m \in \mathbb{Z}$ using contour integrals as

$$
\begin{aligned}
&[a_n, b_m] \\
&= \left(\left(\oint_{z=0} dz \oint_{w=0} dw \right)_{|z|>|w|} - \left(\oint_{w=0} dw \oint_{z=0} dz \right) \right)_{|z|<|w|} z^{n-1+h_A} w^{m-1+h_B} \mathcal{R}(A(z)B(w))
\end{aligned} \tag{1.5.18}
$$

$$
= \oint_0 dw \oint_w dz \left(w^{m-1+h_B} z^{n-1+h_A} A(z)B(w) \right), \tag{1.5.19}
$$

where we have deformed the contours in between the first and second lines. Conversely, we can reverse this and begin by calculating the right-hand side above for two fields $A(z)$ and $B(w)$. Note that it is standard to drop the notation for radial ordering in such integrals, as radial ordering is always assumed.

We have the following two examples, which we will need in the later subsections of this section.

Example 1.5.2.3. The Virasoro energy-momentum tensor $T(z)$ has the following OPE with itself

$$
T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + O((z-w)^0). \tag{1.5.20}
$$

This is equivalent to the modes L_n of $T(z)$ generating a Vir representation with central charge c, so that a theory containing any field with this singular OPE possesses Vir symmetry and is a CFT.

Example 1.5.2.4. The OPE of $T(z)$ with a primary field $\phi(w)$ of conformal dimension h is

$$
T(z)\phi(w) = \frac{h\phi(z)}{(z-w)^2} + \frac{\partial\phi(z)}{z-w} + O((z-w)^0), \tag{1.5.21}
$$

This OPE gives an action of the Vir modes on primary fields.

We can perform a consistency check on the OPEs given in these two examples, by checking the consistency of the operator formalism with the algebraic formalism for Vir and primary fields discussed in the previous section. In this case, we have that

$$
\begin{aligned}
[L_m, L_n] &= \left(\left(\oint_{z=0} dz \oint_{w=0} dw \right)_{|z|>|w|} - \left(\oint_{w=0} dw \oint_{z=0} dz \right) \right)_{|z|<|w|} \frac{z^{m+1} w^{n+1}}{(2\pi i)^2} T(z)T(w) \\
&= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{n+m,0},
\end{aligned} \tag{1.5.22}
$$

by using the OPE (1.5.2.3) inside the integral. For the second, we again take $\phi(z)$ to be a primary field of conformal dimension h and calculate

$$
\begin{aligned}
L_n\phi(w) &= \frac{1}{2\pi i}\oint_{z=0} dz\, z^{n+1}T(z)\phi(w) \\
&= h(n+1)w^n\phi(w) + w^{n+1}(h+1)\partial\phi(w), \quad n > -2
\end{aligned}
\tag{1.5.23}
$$

here we reverse the contour and calculate the residue at $z = w$ instead of $z = 0$. We see from this that indeed $L_0\phi(0) = h\phi(0)$ and $L_n\phi(0) = 0$ for $n > 0$, and that both OPEs are equivalent contain the same information contained within the representatinon theory and Lie brackets we previously discussed. We also see that $L_{-1}\phi(w) = \partial\phi(w)$, and in section 1.5.10 we will use this to derive a differential equation satisfied by n-point correlation functions in WZW models.

We now focus on the case where $\Sigma = \mathbb{P}^1$ is the Riemann sphere, and discuss the structure of the OPE between primary fields. Let $|h\rangle \in \mathcal{H}$ be a primary state that generates a highest weight representation of Vir, with associated primary field $\phi(z)$. We define the notation $\phi^{\vec{k}}$ for a vector $\vec{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r_{>0}$ of r positive integers, to denote a descendant field in the Vir Verma module generated by ϕ as

$$
\phi^{\vec{k}}(z) = L_{-k_1}L_{-k_2}\ldots L_{-k_r}\phi(z). \tag{1.5.24}
$$

We can use this notation to write the OPE for two primary fields $\phi^{(1)}(z)$ and $\phi^{(2)}(0)$, with conformal dimensions h_1 and h_2 respectively. Let $\phi^{(p)}$ be a primary field with coformal weight h_p, and denote by $\phi^{(p;\vec{k})}(0)$ the descendant field $\phi^{(p;\vec{k})}(0) = L_{-k_1}L_{-k_2}\ldots L_{-k_r}\phi^{(p)}(0)$. Then

$$
\phi^{(1)}(z)\phi^{(2)}(0) = \sum_{\vec{k}}\sum_{p} C_{12}^{p;\vec{k}} z^{h_p-h_1-h_2+\sum_i k_i}\phi^{(p;\vec{k})}(0), \tag{1.5.25}
$$

where p runs over the labelling set of primary fields in the theory, $\vec{k}$ runs over vectors of positive integers of arbitrary size, and $C_{12}^{p;\vec{k}}$ are some coefficients to be determined. This OPE structure is referred to as the *operator algebra* of the CFT.

Obtaining the coefficients in the OPE above is equivalent to determining the spectrum of the model and allows one to reduce all higher point correlation functions to 2-point functions. In the next section, we will discuss the correlation functions in CFTs and, specifically, the conformal blocks which are the subject of our focus in chapters 2, 3, and 4.

Within the context of CFTs we have one more product to discuss. We call the process of taking an OPE of two families of local fields in a CFT *fusion*. We say the *fusion rules* are conditions on the possible conformal dimensions of conformal families that appear with non-zero coefficients in an OPE between two fields in a CFT. Schematically we represent the rules by

$$[\phi_i] \times [\phi_j] = \sum_k \mathcal{N}^{ij}{}_k [\phi_k], \tag{1.5.26}$$

where ϕ_i, ϕ_j, and ϕ_k are primary fields, the coefficients $\mathcal{N}^{ij}{}_k$ are integers, and the $\times$ symbol means to take an OPE. Note that the summation range is over *all* primary fields in the theory, and as such the coefficients can vanish.

We note here that fusion rules are similar to the operator algebra defined in 1.5.25 above, although not precisely the same. Fusion works at the level of conformal families, and only states the possibility of fields appearing in an OPE. It has information about which Verma modules can be included in the spectrum of the theory, and how they can be obtained through OPEs. The full operator algebra can be used in the context of correlation functions for the purpose of computation, it not only has information about which families appear but also differentiates each individual state within the Verma modules obtained through the OPE.

1.5.3 Correlation Functions and Conformal Blocks

We begin by briefly reviewing the 2- and 3-point functions of CFTs, this material is a summary of [74, §2]. We will denote a primary field of conformal dimension h_i by $\phi^{(i)}$ throughout this section. First, consider the 2-point correlation function of two primary fields $\phi^{(i)}(z)$ and $\phi^{(j)}(w)$, with conformal dimensions h_i and h_j, located at z and w respectively. Conformal symmetry restricts this to be of the form

$$\langle \phi^{(i)}(z)\phi^{(j)}(w)\rangle = \frac{C_{ij}}{(z-w)^{2h_i}}\delta_{h_i h_j}, \tag{1.5.27}$$

where C_{ij} is some constant. This is obtained by acting on the theory with the generators of the global conformal group and demanding the correlation function be invariant. For example, invariance under translations and rotations implies the 2-point function can only have dependence on the distance between the two coordinates. As noted in [50, §6], the coefficients C_{ij} are symmetric $C_{ij} = C_{ji}$, and as such we are free to pick a normalized basis for the primary fields of the theory such that $C_{ij} = \delta_{ij}$.

Similarly, the 3-point correlation functions of primary fields $\phi^{(i)}(z_i)$, $\phi^{(j)}(z_j)$, and $\phi^{(k)}(z_k)$ are constrained to be of the form

$$\langle \phi^{(i)}(z_i)\phi^{(j)}(z_j)\phi^{(k)}(z_k)\rangle = \frac{C_{ijk}}{z_{ij}^{h_i+h_j-h_k} z_{jk}^{h_j+h_k-h_i} z_{ki}^{h_k+h_i-h_j}}, \tag{1.5.28}$$

where we have employed the notation $z_{ij} = (z_i - z_j)$. The constants C_{ijk} are called the *3-point structure constants*, and can be used to calculate the coefficients $C_{12}^{p;\vec{k}}$ of the operator algebra from (1.5.25). Of note, the 3-point structure constants are *not* fixed by conformal invariance, they are model dependent.

Remark 1.5.3.1. The fusion coefficients, (1.5.26) above, can be thought of as selection rules for the 3-point correlation function. Fusion of two primary fields $\phi^{(1)}$ and $\phi^{(2)}$ onto a third $\phi^{(3)}$ is possible if the correlation function $\langle\phi^{(1)}\phi^{(2)}\phi^{(3)}\rangle$ does not vanish.

After considering conformal invariance, N-point functions, where $N > 3$, for fields located at points z_i for $i = 1, \ldots, N$ have a general dependence on the *cross-ratios* of coordinates of the form

$$\frac{|z_{ij}||z_{kl}|}{|z_{ik}||z_{jl}|}, \quad 1 \leq i, j, k, l \leq N, \tag{1.5.29}$$

where i, j, k, l are distinct. The number of independent cross-ratios for N coordinates for theories of dimension greater than 2 is $N(N-3)/2$ [76]. We now use this knowledge and turn to the 4-point function of four primary fields $\phi^{(i)}(z_i)$ for $i = 1, \ldots, 4$, which we write as

$$\langle\phi^{(1)}(z_1)\phi^{(2)}(z_2)\phi^{(3)}(z_3)\phi^{(4)}(z_4)\rangle. \tag{1.5.30}$$

In this case, there is only one independent cross-ratio[19]

$$q = \frac{z_{12}z_{34}}{z_{13}z_{24}}. \tag{1.5.31}$$

To make the dependence of the 4-point function on the cross ratio explicit, we perform a global conformal transformation on the sphere. This allows us to fix any 3 points and we choose a transformation such that $z_1 = \infty$, $z_2 = 1$, $z_3 = q$, and $z_4 = 0$. The 4-point correlation function is now of the form

$$G^{21}_{34}(q) = \lim_{z_1\to\infty} z_1^{2h_1}\langle\phi^{(1)}(\infty)\phi^{(2)}(1)\phi^{(3)}(q)\phi^{(4)}(0)\rangle. \tag{1.5.32}$$

We can now use the OPE of primary fields within this correlation function form a sum over 3-point functions using the operator algebra (1.5.25). In this case, we use an OPE involving $\phi^{(3)}(q)$ and $\phi^{(4)}(0)$, which we write as a sum over the conformal families, represented by the subscript p, as

$$\phi^{(3)}(z)\phi^{(4)}(0) = \sum_p C^p_{34} z^{h_p-h_3-h_4}\Psi_p(q|0), \tag{1.5.33}$$

where $\Psi_p(q|0) = \sum_{\vec{k}} z^{\sum_i k_i}\phi^{(p;\vec{k})}(0)$. In this equation, the coefficients C^p_{34} only encode the relationship of the operator algebra between $\phi^{(3)}$, $\phi^{(4)}$, and the conformal families represented by the subscript p and as such are the 3-point structure constants. All other information relating to the descendant fields within the operator algebra is contained within the function

[19] In this case we have $|z_{ij}| = z_{ij}$, as we are considering 1-dimensional complex vectors of coordinates.

$\Psi_p(q|0)$. This allows us to reduce the calculation of the 4-point correlation function to a sum of 3-point correlation functions.

As discussed previously, the 3-point correlation functions have two factors: the 3-point structure constants, which are model dependent, and what is left that is fixed by conformal invariance. We choose to write the 4-point correlation function in a new form, where we separate the model dependence from the factors that are fixed by conformal invariance as

$$G_{34}^{21}(q) = \sum_p C_{34}^p C_{12}^p \mathcal{F}_{34}^{21}(p|q). \tag{1.5.34}$$

In this equation, we have a function $\mathcal{F}_{34}^{21}(p|q)$ which contains all the dependence of the 4-point function fixed by conformal invariance, and this function is called a *conformal block*. Note that this contains only the holomorphic dependence, and for the full correlation function between non-chiral fields there is also an anti-holomorphic conformal block factor. Throughout this thesis we will write the conformal blocks as a power series in q as

$$\mathcal{F}_{34}^{21}(p|q) = q^{h_p - h_3 - h_4} \sum_{i=0}^{\infty} \mathcal{F}_i q^i. \tag{1.5.35}$$

The approach to 4-point functions we have just described is dependent on the conformal transformation we picked. We instead could have chosen a conformal transformation which fixed $z_2 = 0$ and $z_4 = 1$, and in this case we would have had $z_3 = 1 - q$. As the correlation functions of a theory are required to be invariant under conformal transformations, we have that

$$G_{34}^{21}(q) = G_{32}^{41}(1 - q). \tag{1.5.36}$$

We can then repeat our arguments but interchange the roles of $\phi^{(2)}$ and $\phi^{(4)}$ and obtain a different conformal block $\mathcal{F}_{32}^{41}(l|1 - q)$.

We can represent the specific conformal block (or equivalently, OPE used when calculating 4-point correlation functions) diagrammatically. In figure 1.3 we represent the calculation involving the conformal block $\mathcal{F}_{34}^{21}(p|q)$. We also represent the calculation involving the

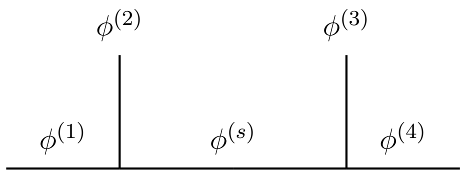


FIGURE 1.3: A 4-point correlation function between primary fields represented pictorially. In this case we are considering the s-channel.

conformal block $\mathcal{F}_{32}^{41}(l|1-q)$ in figure 1.4. These diagrams form an analogy to the perturbation theory of quantum field theory, and due to this we refer to the calculation involving $F_{34}^{21}(p|q)$

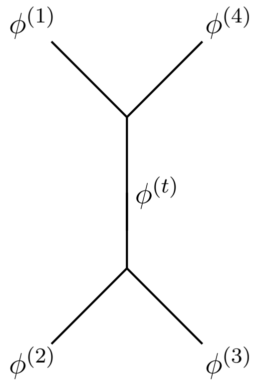


FIGURE 1.4: A 4-point correlation function between primary fields represented pictorially. In this case we are considering the t-channel.

as the s-channel, and the one involving $F_{32}^{41}(l|1-q)$ as the t-channel. In both diagrams, we have notated the conformal families that result from the OPE expansions as $\phi^{(s)}$ and $\phi^{(t)}$ respectively. We will sometimes refer to these families as *flowing in the channel*.

The invariance of the correlation functions in a CFT under conformal transformations guarantees that the result of the calculation in the s- and t-channel agree. Thus we obtain the consistency condition

$$\sum_p C_{34}^{p} C_{12}^{p} \mathcal{F}_{34}^{21}(p|q) = \sum_l C_{32}^{l} C_{14}^{l} \mathcal{F}_{32}^{41}(l|1-q), \tag{1.5.37}$$

for the conformal blocks. In principle, we can solve these equations to determine the conformal blocks and solve all the conformal dependence of the correlation functions in the theory, although in practice the algorithm to do so is tedious. This method is called the *conformal bootstrap*, and is a historic alternative to what is presented within this thesis. The AGT correspondences provide a direct way to compute these conformal blocks from 4D supersymmetric gauge theories without the need for such an algorithm.

1.5.4 Scalar Product on the Space of States

Before discussing minimal models, the main family of CFTs we will consider within this thesis, we will need to briefly touch on the structure of the scalar product on the spectrum $\mathcal{H}$. In particular, this scalar product between two states is the 2-point correlation function between their respective fields. The N-point correlation functions are similarly defined, where one first acts on a state with $(N-2)$ field operators and then takes the scalar product. In the section below, we will use this scalar product to determine which CFTs are Vir-minimal models.

Let $|h\rangle \in \mathcal{H}$ be a primary state of conformal dimension h that generates a Verma module $\mathcal{V}_{c,h}$ for a representation of Vir with central charge c. This may or may not be a reducible Vir-module. We define the normalized pairing of $|h\rangle$ with itself as

$$\langle h|h\rangle = 1. \tag{1.5.38}$$

We extend this to a scalar product on $\mathcal{H}$ as follows. For a descendant state

$$L_{-n_1}L_{-n_2}\dots L_{-n_k}|h\rangle = |\psi\rangle \in \mathcal{V}_{c,h}, \quad n_1,\dots,n_k > 0, \quad k > 0, \tag{1.5.39}$$

we define the *dual state*, which we notate as a bra, using the Hermitian dual Vir operators $(L_{-n})^\dagger = L_n$ by

$$\langle\psi| = \langle h| L_{n_k}\dots L_{n_2}L_{n_1}. \tag{1.5.40}$$

Note that this allows us to define a highest weight state with its dual definition

$$L_n|h\rangle = \langle h| L_{-n} = 0, \quad n > 0. \tag{1.5.41}$$

We first extend this to a scalar product on $\mathcal{V}_{c,h}$ space by bilinearity. As we assume that the $\mathcal{H}$ is a direct sum of Verma modules and their quotients[20], we can extend this scalar product again to the full spectrum $\mathcal{H}$ through bilinearity. This defines the scalar product on $\mathcal{H}$. We will not need this full formalism, as we will only consider scalar products between single primary and descendant states within one Verma module, together with any operators acting on them.

By convention, when there is an operator inside the bra and ket states, we take it to act on the ket $|\psi\rangle$. We can then calculate the scalar products of the descendant states from $|h\rangle$ using the commutation rules obtained from the Lie bracket of Vir (1.5.3). From this, we note that two states $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{V}_{c,h}$ have a non-zero scalar product only if they are the same level l in the same Verma module.

Example 1.5.4.1. Let $|\psi_1\rangle = L_{-1}^2|h\rangle$ and $|\psi_2\rangle = L_{-2}|h\rangle$. We can calculate the scalar product $\langle\psi_1|\psi_2\rangle$ by

$$\begin{aligned}\langle\psi_1|\psi_2\rangle &= \langle h| L_1L_1L_{-2}|h\rangle\\ &= \langle h| L_1([L_1, L_{-2}] + L_{-2}L_1)|h\rangle\\ &= \langle h| L_1(3L_{-1})|h\rangle\\ &= 3\langle h| 2L_0|h\rangle = 6h.\end{aligned}$$

[20] We will explain these quotients in the next section.

1.5.5 Minimal Models

In general, CFTs tend to necessitate an infinite number of conformal families, although this is not clear from what we have covered so far. Minimal models are CFTs composed of a finite number of conformal families. Later in this section, we will give an idea why this occurs within certain CFTs.

The primary fields in a minimal model are formed as an irreducible quotient of a full Verma module by the maximal non-trivial highest weight submodule[21]. We will first begin by discussing how to identify the primary states of submodules[22]. Using this identification, we will state which highest weight Vir-representations (defined by their central charge) have Verma modules with these submodules, and state the conformal dimensions of their primary states and fields. We will construct the minimal models using these primary fields. Finally, we will briefly describe why the minimal models have a spectrum composed of finitely many conformal families.

Let $|h\rangle$ be a primary state that generates a Verma module $\mathcal{V}_{c,h}$. A descendant state $|\chi\rangle \in \mathcal{V}_{c,h}$ of $|h\rangle$ that satisfies the highest weight condition (1.5.41) is called a *singular vector*. Singular vectors generate non-trivial Verma submodules. To construct an irreducible Vir representation, we must identify all singular vectors and form a quotient that identifies them and their descendants with the zero vector. This ensures that all singular states and their descendants decouple from the theory.

Not all Vir representations have singular vectors, so we must first establish which representations do. We begin by noting that singular vectors have a vanishing scalar product with themselves. We do so by representing a singular vector $|\chi\rangle$ as a descendant state of $|h\rangle$

$$\langle\chi| = \langle h| \, L_{n'_{k'}} L_{n'_{k'-1}} \dots L_{n'_1}, \quad n'_1, \dots, n'_{k'} > 0, \quad k' > 0, \tag{1.5.42}$$

where we have defined $|\chi\rangle$ using its dual definition, and note that

$$\langle\chi|\chi\rangle = \langle h| \, L_{n'_{k'}} L_{n'_{k'-1}} \dots L_{n'_1} \, |\chi\rangle = 0,$$

as $n'_1 > 0$. Moreover, by repeating this argument for the descendant state (1.5.39) of $|h\rangle$, we see that $|\psi\rangle$ has a vanishing scalar product with $|\chi\rangle$ since

$$\langle\psi|\chi\rangle = \langle h| \, L_{n_k} L_{n_{k-1}} \dots L_{n_1} \, |\chi\rangle = 0. \tag{1.5.43}$$

[21] In fact this is similar to the process to construct the highest weight module L_λ for simple Lie algebras.

[22] In the language of Lie algebras, this is a vector obtained from a highest weight vector in a Verma module, that generates its own highest weight submodule.

By similar arguments, we also note that any descendant states of $|\chi\rangle$, at level l, have a vanishing scalar product with themselves and other level l states in $\mathcal{V}_{c,h}$.

We fix a basis $\{|i\rangle\}$ for $\mathcal{V}_{c,h}$, using an index set $i \in \mathcal{I}$. Here we have introduced new notation where i is a formal label, not in anyway related to conformal dimension of states. Note that these states can be primary or descendants. Using this notation, we define the *Gram matrix* $M = (M_{ij})$ by

$$M_{ij} = \langle i|j\rangle \,, \tag{1.5.44}$$

which, by the commutation relations (1.5.3), is block diagonal. We will denote the blocks by $M^{(l)}$ where $l \in \mathbb{Z}_{>0}$ refers to the states $\mathcal{V}^{(l)}_{c,h} \subset \mathcal{V}_{c,h}$ of level l in the Verma module. Since M is Hermitian (by definition of the dual states), it is diagonalizable by a unitary matrix U.

If the state $|i\rangle$ is a singular vector in the l-th block, we know that $\langle i|j\rangle = 0$ for all j in the l-th block. Therefore the i-th row in Gram matrix vanishes. Hence, the question of reducibility of $\mathcal{V}_{c,h}$ as a Vir representation can be reduced to the existence of a zero eigenvalue for some $M^{(l)}$. As the determinant of the block $M^{(l)}$ is equal to the product of its eigenvalues, an eigenvalue of zero is equivalent to the determinant vanishing. Therefore $\mathcal{V}_{c,h}$ contains a singular vector and is reducible if and only if $\det(M^{(l)}) = 0$ for some $l \in \mathbb{Z}_{>0}$.

Lemma 1.5.5.1. ([50, **eq (7.28)**]) The determinant of the Gram matrix has the following formula, the so-called Kac determinant

$$\det(M^{(l)}) = \alpha_l \prod_{\substack{r,s\geq 1\\ rs\leq l}} (h - h_{r,s}(c))^{\mathbf{p}(l-rs)} \,. \tag{1.5.45}$$

Remark 1.5.5.2. The Kac determinant was first proposed in [77], and its first published proof is in [78].

In the formula for the Kac-determinant, α_l is a non-zero constant, $r, s \in \mathbb{Z}_{>0}$ are parameters that we will fix further below, $h_{r,s}(c)$ is a function of the central charge c that will be the conformal dimension of minimal model primaries, and $\mathbf{p}(l - rs)$ is the number of partitions of the number $(l - rs)$.

We give three forms for the central charge and conformal dimensions of minimal models. These are all useful in different applications, and we will use each of them in subsequent sections.

For unitary CFTs,the functions $h_{r,s}(c)$ have the following form

$$h_{r,s}(c) = \frac{1}{24}(c-1) + \frac{1}{4}(r\alpha_+ + s\alpha_-), \tag{1.5.46}$$

where we have parameterised

$$\alpha_{\pm} = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}. \tag{1.5.47}$$

We note that for $c > 1$, this parameterization implies that $h_{r,s}$ is a complex number. Thus, we only consider unitary Vir-minimal models where $c \leq 1$.

Equivalently, we can introduce the parameter

$$p = -\frac{1}{2}\left(1 \mp \sqrt{\frac{25-c}{1-c}}\right), \tag{1.5.48}$$

and write the conformal dimensions as

$$h_{r,s}(c) = \frac{((p+1)s - pr)^2 - 1}{4p(p+1)}, \tag{1.5.49}$$

where r and s are positive integers such that $rs \leq l$. We will only consider the case where $p \in (0, \infty)$. We will mostly consider the *unitary minimal models*[23], whose characters we will use in chapter 4. The unitary minimal models are a discrete subset of these Vir-minimal models where $p \in \mathbb{Z}_{>2}$.

Using the Kac determinant we can now follow the arguments made above. By noting the form of (1.5.45), we can see that the Gram matrix has vanishing eigenvalues if and only if $h = h_{r,s}(c)$. This fixes the necessary form of h and c so that $\mathcal{V}_{c,h}$ contains singular vectors. Thus, if we have a unitary CFT which forms a representation of Vir with central charge $c_{p,p+1}$, whose primary fields have conformal dimensions $h_{r,s}(c)$ we have a CFT with singular vectors.

We have one final parameterization, which will be useful for the AGT correspondences involving minimal models in chapter 2. We let $b = i\sqrt{\frac{p}{p'}}$ and write the conformal dimensions and central charge as

$$c_{p,p'} = 1 + 6(b + b^{-1})^2, \tag{1.5.50}$$

$$h_{r,s}(c) = \frac{1}{4}\left((b + b^{-1})^2 - (rb + sb^{-1})^2\right). \tag{1.5.51}$$

Relaxing our focus from unitary minimal models, we note that the CFTs described above are a special case of a more general family of minimal models. These are described by two coprime integers p and p', and have central charge and conformal dimensions which generalize

[23] In unitary CFTs, the scalar product on $\mathcal{H}$ is a Hermitian inner product and makes $\mathcal{H}$ a Hilbert space.

our second parameterization for minimal models by

$$c_{p,p'} =1-\frac{6(p-p')^2}{pp'}, \tag{1.5.52}$$

$$h_{r,s}(c) =\frac{(p'r-ps)^2-(p-p')^2}{4pp'}, \tag{1.5.53}$$

where $1 \leq r < p'$, and $1 \leq s < p$. We note that $h_{r,s} = h_{p-r,p'-s} = h_{p+r,p'+s}$, and we will use this fact, referred to as a *periodicity property*, below. This class of minimal model CFTs was shown to be rational (have finitely many primary fields) in [79]. The unitary minimal models above correspond to the case $p' = p + 1$.

We will label all minimal model by the coprime integers $p, p' \in \mathbb{Z}_{>0}$ that parameterize its central charge (for unitary minimal models we take $p' = p + 1$). The primary fields of the Vir-minimal models are the highest weights states of the irreducible Vir-modules created by taking the quotient of $\mathcal{V}_{c_{p,p'},h_{r,s}}$ with the maximal non-trivial submodule. These highest weight modules are sometimes referred to as *degenerate representations*. We will also label the primary fields of a minimal model by the integers r and s parameterizing their conformal dimension as $\phi_{r,s}$. We notate the Vir-minimal model, which is the CFT of central charge $c_{p,p'}$ with the primary fields $\phi_{r,s}$, for p and p' by $M(p, p'; 2)$.

To finish this section on minimal models, we will briefly talk about their fusion rules and how this impacts their spectrum. It provides our first example of fusion, a process we will do in chapter 4 for WZW models, and allows us to see why the spectrum of minimal models only contains finitely many irreducible Vir representations.

The fusion rules (1.5.26) for the minimal model $M(p, p'; 2)$, which first appeared in [80], are

$$[\phi_{r,s}] \times [\phi_{m,n}] = \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \bmod 2}}^{\min(r+m-1,2p'-1-r-m)} \sum_{\substack{l=1+|s-n| \\ k+s+n=1 \bmod 2}}^{\min(s+n-1,2p-1-s-n)} [\phi_{k,l}], \tag{1.5.54}$$

where $\phi_{r,s}$ and $\phi_{m,n}$ represent minimal model primary fields of dimension $h_{r,s}$ and $h_{m,n}$ respectively.

We are now ready to understand where the name minimal modes come from. Fusion rules of the form above, imply that there is an infinite number of fields in the spectrum of a theory, as we could successively fuse $\phi_{r,s}$ with itself and keep generating fields of new conformal dimension. In the case of $M(p, p'; 2)$, the periodicity property $h_{r,s} = h_{r+p,s+p'}$ ensures that this process is cyclic and there are only a finite number of fields created from this iterative fusion process.

1.5.6 The Free Boson and Vertex Operators

Here we review the free boson in the setting of CFTs. The purpose of this is twofold, it is the simplest example we can use to illustrate the machinery of CFTs, and we will use it in sections 1.5.7 and 1.5.14 to build the CFTs that form the 2D side of the AGT correspondences we will consider in chapter 2. This section will just collect results we need, and show how they fit in with the general theory developed above.

We define the free boson to be a *scalar field* φ on a Riemann surface. For simplicity, we restrict to the complex plane[24], so that $\varphi : \mathbb{C} \longrightarrow \mathbb{C}$. The mode expansion of the free boson reads

$$\varphi(z) = \varphi_0 - ia_0 \log(z) + i \sum_{n \neq 0} \frac{1}{n} a_n z^{-n}. \tag{1.5.55}$$

The free boson itself is not a conformal field, but its derivative[25] $\partial\varphi$ is. For our purposes, we take the singular OPE of $\partial\varphi$ with itself to be

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z-w)^2} + O((z-w)^0), \tag{1.5.56}$$

as an axiom. We can use the field $\partial\varphi$ to construct the energy-momentum tensor

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\partial\varphi(z) :, \tag{1.5.57}$$

whose OPE with itself can be shown to be

$$\begin{aligned} T(z)T(w) =& \frac{1}{4} : \partial\varphi(z)\partial\varphi(z) :: \partial\varphi(w)\partial\varphi(w) : \\ =& \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + O((z-w)^0) \end{aligned} \tag{1.5.58}$$

Note that to obtain this result, one needs to use Wick's theorem (see [50, §2.3.5]).

By comparing the OPE of $T(z)$ with (1.5.20), we see that the theory of the free field (defined in footnote 18) $\partial\varphi$ is a conformal field theory with central charge $c = 1$. We can then calculate the OPE of $T(z)$ with $\partial\varphi$, by again invoking Wick's theorem, as

$$T(z)\partial\varphi(w) = \frac{\partial\varphi(w)}{(z-w)^2} + \frac{\partial_w^2\varphi(w)}{(z-w)} + O((z-w)^0). \tag{1.5.59}$$

[24] It is perfectly reasonable to define the free boson on other Riemann surfaces as a CFT, or general pseudo-Riemannian manifolds as a field theory.

[25] To see this would take us outside the scope of this thesis. For the reader who is familiar with quantum field theory, this is explained in [50] and [74]. In both references this is achieved by calculating the OPE using Lagrangian field theory.

This shows that indeed $\partial\varphi$ is a primary field with conformal dimension $h = 1$, with mode expansion

$$i\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}. \tag{1.5.60}$$

The Vir modes for $T(z)$ can be calculated in terms of the modes of $\partial\varphi(z)$ as

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m, \quad n \neq 0, \tag{1.5.61}$$

$$L_0 = \frac{1}{2}a_0^2 + \sum_{n>0} a_{-n}a_n. \tag{1.5.62}$$

Remark 1.5.6.1. The free boson modes generate the Heisenberg ($\widehat{\mathfrak{u}}(1)$) algebra $\mathcal{H}$. That is, they satisfy the commutation relations

$$[a_n, a_m] = n\delta_{n+m}. \tag{1.5.63}$$

The additional factor of i used when defining the mode expansion (1.5.60) of $\partial\varphi$ makes this $\mathcal{H}$-symmetry apparent.

Finally, we can define another primary field using the exponential of a free boson. Using a parameter $\alpha \in \mathbb{C}$, which we call the *conformal charge*[26] we define the *vertex operators*

$$V_\alpha(z) :=: \exp(\sqrt{2}i\alpha\varphi(z)) :, \tag{1.5.64}$$

which are fields in the CFT of the free boson. The normal ordering for the exponential reminds us that when we calculate OPE's using Wick's theorem, we are not to contract the fields appearing within its series form

$$: e^{\sqrt{2}i\alpha\varphi(z)} := \sum_{n=0}^{\infty} \frac{: (\sqrt{2}i\alpha\varphi(z))^n :}{n!}, \tag{1.5.65}$$

with each other. When writing these vertex operators, we will adopt the standard convention that the normal ordering will always be assumed, and therefore will not be notated.

We claim the vertex operators V_α, are primary fields of the CFT defined using the free boson. To show this we calculate the OPE of $V_\alpha(w)$ with $T(z)$, again using Wick's theorem, as

$$T(z)e^{\sqrt{2}i\alpha\varphi(w)} = -\frac{1}{2}\left([\partial\varphi(z)2i\alpha\varphi(w)]^2\, e^{\sqrt{2}i\alpha\varphi(w)} + \partial\varphi\partial\varphi(z)2i\alpha\varphi(w)e^{\sqrt{2}i\alpha\varphi(w)} \right). \tag{1.5.66}$$

[26]This is also referred to as the conformal momentum in the literature. We defer to [50] here, which refers to this parameter as charge. This will also allow us to follow the notation of [34] later in section 2.3.3, which reserves the term momentum for a related vector which we define in (1.5.73) and (1.5.162).

This calculation then involves the OPE formed by contracting φ and $\partial\varphi$. We can obtain this by integrating the OPE of $\partial\varphi$ with itself and find

$$\partial\varphi(z)\varphi(w) = -\frac{1}{z-w} + O((z-w)^0). \tag{1.5.67}$$

Using this, the singular OPE of $T(z)$ and $e^{\sqrt{2}i\alpha\varphi}$ is

$$T(z)e^{\sqrt{2}i\alpha\varphi(w)} = \frac{2\alpha^2 e^{\sqrt{2}i\alpha\varphi(w)}}{(z-w)^2} + \frac{1}{z-w}\partial e^{\sqrt{2}i\alpha\varphi(w)} + O((z-w)^0). \tag{1.5.68}$$

From this, we see that

$$V_\alpha(z) := e^{\sqrt{2}i\alpha\varphi(z)}, \tag{1.5.69}$$

is a primary field of conformal dimension $h = \alpha^2$. These vertex operators are the fields used for the CFT side of the original AGT correspondence which we will cover in chapter 2.

For later use, we note that the field $V_{-\alpha}$ has the same conformal dimension as the field V_α. Since correlation functions in CFTs depend only on the coordinates and conformal dimensions of the fields, this symmetry $\alpha \mapsto -\alpha$ can be exploited in theories involving these vertex operators to obtain information about correlation functions involving them.

Remark 1.5.6.2. Since their introduction, vertex operators have been subsequently generalized into the concept of *vertex operator algebras* (VOAs). VOAs provide an alternative, axiomatic framework to study CFTs to the one we have presented here.

Remark 1.5.6.3. These vertex operators are the first place we have to be mindful about the chiral nature of our CFTs. In the mode expansion of the free boson (1.5.55), we note the presence of the, not purely holomorphic, zero-mode term φ_0. Due to this, naievely combining two chiral free bosons $\varphi^*(z,\overline{z}) = \varphi(z)+\overline{\varphi}(\overline{z})$, in an attempt to obtain the full CFT, duplicates this zero-mode. We must instead think of $\varphi(z)$ as containing the holomorphic dependence of the free boson, where the full CFT is obtained through the equation $\varphi(z,\overline{z}) = \varphi(z)+\overline{\varphi}(\overline{z})-\varphi_0$. The corresponding vertex operator is commonly notated as $V_\alpha(z,\overline{z})$. This behaviour naturally carries over to our constructed vertex operators. In this case, purely holomorphic or anti-holomorphic vertex operators only make sense when paired their respective other parts in a full vertex operator within a correlation function. With this caveat in mind, we will continue to write our vertex operators as purely holomorphic in the sequel.

1.5.7 Liouville Conformal Field Theory

Liouville field theory is a model which has conformal symmetry in special cases. Of interest to us is the fact that, although Liouville is an *interacting* field theory (that is, not a free field theory), we can study it using free fields. To do so, we will build the theory using copies of the free boson vertex operator , and in doing so will obtain a *free-field realization.* Although the idea of free-field realizations for intereacting theories is an interesting one, we will not delve into it in any detail outside of our need of it here. The interested reader can read, for example, [81].

This material is mostly a brief review of [75, §3]. To define a Liouville field theory, we need two parameters: the *background charge* $Q \in \mathbb{C}$, and *coupling constant* $b \in \mathbb{C}$. The field theory is conformal when

$$Q = b + \frac{1}{b}, \tag{1.5.70}$$

and in this case has Vir central charge

$$c = 1 + 6Q^2. \tag{1.5.71}$$

In this sense, we think of Liouville CFT as a family of CFTs, parameterized by a central charge $c \in \mathbb{C}$.

Liouville theory will form the CFT side of the original AGT correspondence, and we will consider the case where the Liouville theory has a minimal model central charge (1.5.52) and the case where the central charge is generic (non-minimal). In both cases, we will consider functions that involve the three Liouville parameters c, Q, and b and study their analytic properties. Due to this, we think it is valuable to understand the relationship between these parameters for different ranges of c. To collect this information, we reproduce a nice table of these relationships here [75, **eq (2.1.22)**]:

central charge	c	$\mathbb{C}$	≤ 1	$1-6\frac{(p-p')^2}{pp'}$	1	$[1,25]$	25	≥ 25
background charge	Q	$\mathbb{C}$	$i\mathbb{R}$	$i\frac{p-p'}{\sqrt{pp'}}$	0	$[0,2]$	2	≥ 2
coupling constant	b	$\mathbb{C}^*$	$i\mathbb{R}$	$i\sqrt{\frac{p}{p'}}$	i	$e^{i\mathbb{R}}$	1	$\mathbb{R}$

(1.5.72)

The reader should take particular notice of the values when $c = 1-6\frac{(p-p')^2}{pp'}$ which correspond to minimal models. The new results presented within this thesis hinge on analysis of a gauge theory object dependent on a the corresponding Q and b values for miniaml models, which we will introduce in chapter 2.

For a fixed value of c, the spectrum of a Liouville CFT is continuous. In fact, we will take the definition of Liouville as a CFT to be the presence of a continuous spectrum where each representation of Vir occurs with multiplicity zero or one, and one other assumption we will clarify below. This means that in Liouville CFT we can have primary fields with a continuum of conformal dimensions. The primary fields will be the vertex operators we constructed for the free boson above for a continuous charge parameter α. In this setting, the vertex operators are often referred to as *Liouville exponentials*, and form the free-field realization of Liouville theory.

It is also common to parameterize the conformal dimension of a Liouville exponential by a new parameter $P \in \mathbb{C}$, called the *momentum*. In this case we have that

$$\alpha^2 = h(P) := \frac{Q^2}{4} - P^2. \tag{1.5.73}$$

When utilizing this parameterization one has the symmetry $P \mapsto -P$. This terminology is *not uniform* across the literature, some sources refer to the parameter α as momentum, so one must always confirm which convention is being used. Our final axiom to define Liouville CFT is that the correlation functions are also meromorphic functions of the coupling constant b and the momenta.

The spectrum of primary fields is defined by

$$P \in i\mathbb{R} \iff h \in \frac{c-1}{24} + \mathbb{R}_{\geq 0}, \tag{1.5.74}$$

and is one where the holomorphic and anti-holomrphic representations for a primary field are isomorphic. Such a CFT is said to have a *diagonal* spectrum.

We note that when $c \leq 1$ ($Q \in i\mathbb{R}$), the presence of fields with minimal model momenta $h_{r,s}$ (see (1.5.46)) create poles in the s-channel conformal blocks. To fix this, the spectrum is perturbed to contain only fields with

$$P \in i\mathbb{R} + \epsilon, \quad \epsilon \in \mathbb{R} \setminus \{0\}, \tag{1.5.75}$$

and can be shown to be independent of the perturbation parameter ϵ chosen. We must therefore add in the minimal model primary fields as an additional assumption, and restrict the OPEs of the theory, so that the minimal model primary fields do not appear and cause a pole in the calculation of the conformal blocks.

These fields and parameters form the data for the basic form of the CFTs that are AGT dual to generic $SU(2)$ gauge theories, as we shall see in chapter 2. More importantly in section 2.2.4, we will consider gauge theories that are dual to CFTs with central charge $c = 1 - 6\frac{(p-p')^2}{pp'}$ for two coprime integers p, p'. By comparison with (1.5.52), we see that in this case we can have degenerate primary fields that form minimal models. This will be the main focus of this thesis.

1.5.8 The Coulomb-Gas Formalism for Minimal Models

When discussing the minimal model primary fields in this thesis, we will use the *Coulomb-gas formalism*, originally introduced in the series of papers [82, 83, 84]. The utility of the Coulomb-gas formalism is that, as we did for Liouville above, we can realize the minimal models using the free boson. Many of these results and arguments will be repeated when we discuss the Coulomb-Gas for Toda CFTs in section 1.5.15, and again in chapter 3 when deriving the Burge conditions for our generalized AGT conjecture. In this section we will collect the notation, objects, and results from [50, §9] we will require.

The motivation for this framework is derived from physical concerns, and as such, to go into them would be outside the scope of this thesis. Instead we will take as an axiom that the N-point correlation function of Liouville exponentials[27]

$$\langle\prod_{i=1}^{N} V_{\alpha^{(i)}}(z_i)\rangle, \tag{1.5.76}$$

can only be non-zero if we have the *charge neutrality condition*

$$\sum_{i=1}^{N} \alpha^{(i)} = 2\alpha_0, \tag{1.5.77}$$

where $4\alpha_0^2 = -Q^2$. In the Coulomb-gas formalism, α_0 is often referred to as the background charge instead of Q. With this background charge, the vertex operator $V_\alpha = e^{\sqrt{2}i\alpha\varphi}$ has conformal dimension[28]

$$\Delta(\alpha) = \alpha^2 - 2\alpha_0\alpha. \tag{1.5.78}$$

In this case, the conformal dimension of V_α and $V_{2\alpha_0-\alpha}$ are the same. We will exploit this symmetry $\alpha \mapsto 2\alpha_0 - \alpha$ when discussing correlation functions below.

We now investigate what the neutrality condition *seems* to imply for the spectrum of a Liouville CFT, realized using vertex operators. Consider, for instance, the 2-point function $\langle V_\alpha(z)V_\alpha(w)\rangle$ of two vertex operators with the same charge α, located at two distinct points z and w. Under the symmetry $\alpha \mapsto 2\alpha_0 - \alpha$ we expect that this should be equivalent to the 2-point correlation function

$$\langle V_{2\alpha_0-\alpha}(z)V_\alpha(w)\rangle = \frac{1}{(z-w)^{\Delta(\alpha)}}, \tag{1.5.79}$$

[27] While we use the Coulomb-Gas formalism for minimal models *not* Liouville, the two models are limits of each other. The reader unfamiliar with this should consult [75, §1].

[28] We have notated the conformal dimension here using Δ instead of h. Here and in chapter 2, the chapters which constitute a review of standard material, we will endeavour to match our notation to the cited references that we most closely follow, for the ease of the reader who is learning this material using these references. In all subsequent chapters we will notate conformal dimension *only* using Δ.

for any allowed values of α in our CFT. We also note that this correlation function satisfies our charge axiom (1.5.77). On the other hand, the neutrality condition suggests that the only vertex operators who have a non-vanishing 2-point function with themselves $\langle V_\alpha(z)V_\alpha(w)\rangle$ are for $\alpha = \alpha_0$, so that our theory only has two primary fields. This shows an incongruity between the charge neutrality condition, and the form of the 2-point correlation functions (1.5.27), which is determined by conformal symmetry.

Since we wish to consider CFTs with many distinct primary fields, we must find a way to change the charges associated to the fields in the correlation function without changing their conformal dimensions. This leads to the idea of *screening operators* or *screening charges*, operators which have conformal dimension 0 and non-zero charge. The insertion of these operators will allow us to change the charge associated to correlation functions without changing the conformal dimension. In this way, we can construct correlation functions that satisfy both the charge neutrality condition and the form determined by conformal symmetry.

To construct such an operator A, we can use the zero mode of a primary field ϕ of conformal dimension $\Delta_\phi = 1$, with non-zero charge. We can extract this using the contour integral[29]

$$A = \oint_z dz\phi(z). \tag{1.5.80}$$

In our case, the vertex operators

$$V_\pm(z) := V_{\alpha_\pm}(z), \quad \alpha_\pm := \alpha_0 \pm \sqrt{\alpha_0 + 1}, \tag{1.5.81}$$

have the conformal dimension

$$\Delta(\alpha_\pm) = \alpha_0^2 + \alpha_0^2 + 1 \pm 2\sqrt{\alpha_0+1}\alpha_0 - 2\alpha_0^2 \mp 2\sqrt{\alpha_0+1}\alpha_0 = 1, \tag{1.5.82}$$

required. We say that $\alpha_\pm$ are *screening charges*, and we note that they are the *same* as the parameters (1.5.47) that we used in our calculation of the Kac determinant. The screening operators are then defined to be

$$Q_\pm = \oint dz V_\pm(z). \tag{1.5.83}$$

We can now insert the operators $Q_\pm$ to a correlation function any amount of times so that, for instance, the 2-point correlator

$$\langle V_\alpha(z)V_\alpha(w)Q_-^m Q_+^n\rangle, \tag{1.5.84}$$

[29] The specifics on this contour integral can get quite technical (see [85, §5]), for our purposes we will assume this operation is well-defined and well-behaved.

now has the neutrality condition

$$2\alpha + m\alpha_- + n\alpha_+ = 2\alpha_0 = \alpha_+ + \alpha_-. \tag{1.5.85}$$

We note that this equation means that the charge of a highest weight state and a vertex operator insertion differ by $2\alpha_0$ and we will use this fact when considering AGT involving minimal models.

We also interpret this equation as meaning that if we have a theory where α is going to be equivalent to $2\alpha_0 - \alpha$ within the 2-point correlation functions, 2α must be some integer multiple of α_+ and α_-. One can then consider a variety of different correlators to arrive at the conclusion that this condition on 2α is necessary for any primary field in this CFT to have non-trivial correlation functions. Thus we parameterise

$$\alpha_{r,s} := -\frac{1}{2}(r-1)\alpha_+ - \frac{1}{2}(s-1)\alpha_-, \tag{1.5.86}$$

and we will sometimes refer to $\alpha_{r,s}$ as *degenerate charge*. The corresponding vertex operator $V_{\alpha_{r,s}}$ has conformal dimension

$$\Delta(\alpha_{r,s}) = \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 - \alpha_0^2, \tag{1.5.87}$$

which is of the same form as (1.5.51) for the primary fields in the minimal models, although it is not precisely the same as r and s are not restricted. To connect to the minimal models, we consider the case where we have $\alpha_+/\alpha_- \in \mathbb{Q}$, or equivalently

$$p'\alpha_+ + p\alpha_- = 0, \tag{1.5.88}$$

and further restrict to $p > p'$. This second condition has the solution

$$\alpha_+ = \sqrt{\frac{p}{p'}}, \quad \alpha_- = -\sqrt{\frac{p'}{p}}, \tag{1.5.89}$$

which further implies that

$$\alpha_{r,s} = \frac{1}{2\sqrt{pp'}}\left(p(1-r) - p'(1-s)\right), \quad \alpha_0 = \frac{p-p'}{2\sqrt{pp'}}. \tag{1.5.90}$$

We can use these to calculate the central charge and conformal dimensions of primary fields in the Liouville conformal field theory, where all correlators are assumed to have Coulomb-gas screening charges inserted into them to satisfy the neutrality condition (1.5.77), as

$$c_{p,p'} = 1 - \frac{6(p-p')^2}{pp'}, \tag{1.5.91}$$

$$\Delta_{r,s} = \frac{(rp-sp')^2 - (p-p')^2}{4pp'}. \tag{1.5.92}$$

These are the central charge (1.5.53) and conformal dimension (1.5.53) of minimal model primary fields.

Remark 1.5.8.1. Fixing the possible values of the integer parameters to $r = 1, 2, \ldots, p$ and $s = 1, 2, \ldots, p'$, can be done by calculating the 3-point function

$$\langle V_{r_1,s_1} V_{r_2,s_2} V_{r_3,s_3} Q_+^r Q_-^s \rangle. \tag{1.5.93}$$

This process is necessary to realize the minimal models using the Coulomb-Gas formalism, as a priori we could insert more screening charges than this and obtain non-trivial correlation functions with fields that do not have the correct minimal model conformal dimension. To do so would be outside the scope of this thesis, although it can be found in [50, §9.2] The result of this calculation is that $1 \leq r \leq p$ and $1 \leq s \leq p'$.

1.5.9 Wess-Zumino-Witten Models

The exposition in this section and the one below are a review of the material in [50, §15]. A Wess-Zumino-Witten (WZW) model is a theory that exhibits Lie algebra symmetry. For a simple Lie algebra $\mathfrak{g}$ a $\widehat{\mathfrak{g}}_k$-WZW model is a field theory which has a current $J(z)$ of $\mathfrak{g}$-valued modes (a $\mathfrak{g}$-symmetry). To define the theory, we fix the OPE of this current, and this is what gives the WZW model its $\widehat{\mathfrak{g}}_k$-structure.

In this section we will describe the *Sugawara construction*, which shows that $\widehat{\mathfrak{g}}_k$-WZW models contain the Virasoro energy-momentum tensor $T(z)$ and are therefore not just field theories, but also CFTs. Mathematically, this is equivalent to saying that the completion of the universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{g}})$ of any affine Lie algebra $\widehat{\mathfrak{g}}$ contains a Vir subalgebra. As discussed in the previous sections, a WZW model contains two commuting copies of Vir, and all statements below should be read as applying to a pair of commuting holomorphic and anti-holomorphic sector simultaneously.

We begin by making precise the defining properties of an $\widehat{\mathfrak{g}}_k$-WZW model we described above. Let $J(z) = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}$ be a current of $\mathfrak{g}$-valued modes on a Riemann surface Σ (for our purposes, Σ will always be the Riemann sphere). Let $\{t^a\}_{a\in I}$ be a basis for a $\mathfrak{g}$-representation labelled by an index set I. Using this basis we write the current as

$J(z) = \sum_a J^a(z)t^a = \sum_{n,a} J^a_n t^a z^{-n-h}$. We define the *structure constants* f^{ab}_c for $\mathfrak{g}$ by

$$[t^a, t^b] = \sum_c i f^{ab}_c t^c, \tag{1.5.94}$$

and fix the OPE of the mode $J^a(z)$ with itself to be

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \sum_c i f^{ab}_c \frac{J^c(w)}{z-w} + O(1). \tag{1.5.95}$$

We can then use this OPE to obtain an OPE for the current $J(z)$ with itself. Then, remembering our discussion in section 1.5.2, the current $J(z)$ generates an $\widehat{\mathfrak{g}}_k$-representation in the CFT by its action on the space of states, where the modes $\{J^a_n\}_a$ for fixed $n \in \mathbb{Z}$ are identified with the generators for $\widehat{\mathfrak{g}}$ at grade n. Here the constant k in the $O((z-w)^{-2})$ term of the OPE is the level of the representation, and we will always assume that k is a positive integer[30]. Mathematically, we will treat the fields in $\widehat{\mathfrak{g}}_k$-WZW models as $\widehat{\mathfrak{g}}_k$ representations.

We begin by showing that a CFT containing $J(z)$ also contains the Vir energy-momentum tensor. The *Sugawara energy-momentum tensor* $T_{\widehat{\mathfrak{g}}_k}(z)$ is constructed using the modes of $J(z)$ by

$$T_{\widehat{\mathfrak{g}}_k}(z) = \frac{-1}{2(k+g)} \sum_a (J^a J^a)(z), \tag{1.5.96}$$

where k is the level of the $\widehat{\mathfrak{g}}$ representation, g is the dual Coxeter number, $(J^a J^a)$ represents the normal ordered product as defined in 1.5.2, and the modes J^a_m are assumed to be orthonormal with respect to the Killing form. This generates a Vir representation of central charge

$$c(\widehat{\mathfrak{g}}_k) = \frac{k\dim(\mathfrak{g})}{k+g}, \tag{1.5.97}$$

and the Virasoro modes $\{L_n\}_{n\in\mathbb{Z}}$ for $T_{\widehat{\mathfrak{g}}_k}(z)$ are

$$L_n = \frac{1}{2(k+g)} \sum_a \left(\sum_{m \leq -1} J^a_m J^a_{n-m} + \sum_{m \geq 0} J^a_{n-m} J^a_m \right). \tag{1.5.98}$$

Note that J^a_m and J^a_{n-m} commute for $n \neq 0$, and that for $n = 0$ we are simply placing the terms with larger subscript on the right. This is just the definition of normal ordering, so we can write

$$L_n = \frac{1}{2(k+g)} \sum_a \left(\sum_{m\in\mathbb{Z}} : J^a_m J^a_{n-m} : \right). \tag{1.5.99}$$

This construction ensures that a $\widehat{\mathfrak{g}}_k$-WZW model has a Vir energy-momentum tensor and hence is a CFT.

[30]One can see this from a physics perspective, and the reader familiar with classical field theory can find this calculation in [50, §15.1]

We define an $\widehat{\mathfrak{g}}_k$-*WZW primary field* $\phi_{\bar{\Lambda}}$ to be associated to an irreducible finite-dimensional representation of $\mathfrak{g}$. Such representations are the modules $L_{\bar{\Lambda}}$ from section 1.2.1 associated to dominant integral weights $\bar{\Lambda} \in P^+(\mathfrak{g})$. Let $\phi_{\bar{\Lambda}}(z)$ be an $\widehat{\mathfrak{g}}_k$-WZW primary field and denote its corresponding primary state by $|\phi_{\bar{\Lambda}}\rangle$. If we let $\{t^a_{\bar{\Lambda}}\}$ be a basis for $L_{\bar{\Lambda}}$ we have the following defining properties of the WZW primary state $|\phi_{\bar{\Lambda}}\rangle$

$$J^a_0 \, |\phi_{\bar{\Lambda}}\rangle = -t^a_{\bar{\Lambda}} \, |\phi_{\bar{\Lambda}}\rangle \,, \quad J^a_n \, |\phi_{\bar{\Lambda}}\rangle = 0, \quad n > 0. \tag{1.5.100}$$

Such states are also Vir primaries, although the converse is not true (Vir primary states are *not* necessarily WZW primaries). The conformal dimension of $|\phi_{\bar{\Lambda}}\rangle$ is

$$h_{\bar{\Lambda}} = \frac{(\bar{\Lambda}, \bar{\Lambda} + 2\rho)}{2(k+g)}. \tag{1.5.101}$$

The descendant states are then constructed as

$$J^a_{-n_1} J^b_{-n_2} \dots |\phi_{\bar{\Lambda}}\rangle \,, \quad n_1, n_2, \dots > 0. \tag{1.5.102}$$

Note that we do not need to use the Vir modes L_{-n} for $n > 0$ as the Vir energy-momentum tensor (and hence Vir modes) were constructed out of $J(z)$.

We now fix the basis for $\widehat{\mathfrak{g}}_k$ to be the Cartan-Weyl basis from section 1.3, for which the primary field definitions (1.5.100) translate to

$$H^i_0 \, |\bar{\Lambda}\rangle = H^i_0(\bar{\Lambda}) \, |\bar{\Lambda}\rangle \,, \tag{1.5.103}$$

$$E^{\pm\alpha}_n \, |\bar{\Lambda}\rangle = H^i_n \, |\bar{\Lambda}\rangle = E^{\alpha}_0 \, |\bar{\Lambda}\rangle \,, \quad n > 0, \; \alpha \in \Delta_+. \tag{1.5.104}$$

We will assume that all $\widehat{\mathfrak{g}}_k$-WZW model primary fields in any theory we consider will be such that the level k affine weight Λ with finite part $\bar{\Lambda}$ is dominant and integral[31]. Then the equations (1.5.103) and (1.5.102) ensure that *all* states in these theories form the highest weight integrable $\mathfrak{g}_k$-modules L_Λ we discussed in section 1.3. Due to this, we identify the primary states $|\bar{\Lambda}\rangle$ as states defined by the level k affine weight Λ, whose finite part is $\bar{\Lambda}$, and when doing so we notate the primary state as $|\Lambda\rangle$. Thus, $\widehat{\mathfrak{g}}_k$-highest weight modules generate the spectrum for $\widehat{\mathfrak{g}}_k$-WZW models, for which only the finite algebra $\mathfrak{g}$ is a symmetry (as the Vir modes do not commute with all the modes J^a_n).

The OPE of $J^a(z)$ and a WZW primary field $\phi_\Lambda(w)$ associated to a primary state $|\Lambda\rangle$ is

$$J^a(z)\phi_\Lambda(w) = \frac{-t^a_\Lambda \phi_\Lambda(w)}{z-w} + O(1), \tag{1.5.105}$$

[31] In fact, one can show that in a WZW model with at least one such state (which can be taken to be the vacuum), all states that do not correspond to integrable $\widehat{\mathfrak{g}}_k$ modules in this way decouple from the theory. To do so is unnecessary for us, and would take us outside the scope of this thesis.

where we note that t^a_Λ are the basis t^a in the $\mathfrak{g}$-representation whose highest weight $\bar{\Lambda}$ is the finite part of Λ.

Finally, we define the character for the state $|\bar{\Lambda}\rangle$ to be

$$\chi_\Lambda^{\widehat{\mathfrak{g}}_k}(q,\hat{\mathfrak{t}}) = \mathrm{Tr}_{L_\Lambda} q^{L_0 - c/24} \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{H_i}, \tag{1.5.106}$$

and a corresponding $\widehat{\mathfrak{g}}_k$-WZW q-character as

$$\chi_\Lambda^{\widehat{\mathfrak{g}}_k}(q) = \mathrm{Tr}_{L_\Lambda} q^{L_0 - c/24} = q^{h_\Lambda - c/24} \sum_{n=0}^{\infty} \dim(L_\Lambda^{(n)}) q^n, \tag{1.5.107}$$

where we have notated the number of linearly independent states at level n (with regards to the Vir generators) in L_Λ as $\dim(L_\Lambda^{(n)})$. We will focus on the case of $\widehat{\mathfrak{g}}_k = \widehat{\mathfrak{sl}}(n)_N$ and consider the module L_Λ, whose Vir-character we shall need in chapter 4. Here, L_0 acts as $L_0 \mapsto h_\Lambda \mathrm{Id} - \hat{d}$, where Id is the identity operator and we recall that $\hat{d}$ is the derivation on $\widehat{\mathfrak{sl}}(n)$. Then (1.5.106) yields

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q,\hat{\mathfrak{t}}) = q^{h_\Lambda}\, \mathrm{Tr}_{L(\Lambda)}\, q^{-D} \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{H_i}\,. \tag{1.5.108}$$

Now note that (1.5.108) can be written as a sum of terms $\exp(\beta)$ with β of the form

$$\beta = \Lambda - k\delta - \sum_{j=1}^{n} e_j(l_j - l_{j-1}) = \Lambda - k\delta - \sum_{j=1}^{n} l_j \alpha_j \tag{1.5.109}$$

for some $k \in \mathbb{Z}$. Then, because $\beta(D) = -k$ and $\beta(H_i) = d_i + l_{i-1} - 2l_i + l_{i+1}$, (1.5.108) yields:

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q,\hat{\mathfrak{t}}) = q^{h_\Lambda} \sum_{\mathbf{l}\in\mathbb{Z}^{n-1}} \bar{\sigma}_{\mathbf{l}}^{\Lambda}(q) \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{d_i + l_{i-1} - 2l_i + l_{i+1}}\,, \tag{1.5.110}$$

where $\bar{\sigma}_{\mathbf{l}}^{\Lambda}(q)$ are normalized $\widehat{\mathfrak{sl}}(n)$ string functions. Alternatively, this may be expressed using the usual $\widehat{\mathfrak{sl}}(n)$ string functions $\sigma^{\Lambda}_{\gamma(\mathbf{l})}(q)$ as

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q,\hat{\mathfrak{t}}) = q^{h_\Lambda} \sum_{\mathbf{l}\in\mathbb{Z}^{n-1}} \sigma^{\Lambda}_{\gamma(\mathbf{l})}(q) \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{\gamma(\mathbf{l})_i}, \tag{1.5.111}$$

after defining $\gamma(\mathbf{l}) = [\gamma_0, \gamma_1, \ldots, \gamma_{N-1}] \in P_{n,N}$, by setting

$$\gamma_i = d_i + l_{i-1} - 2l_i + l_{i+1} = d_i - \sum_{j=1}^{n-1} A_{ij} l_j, \tag{1.5.112}$$

for each $i \in \mathcal{I}_n$. Then using a combinatorial expression for $\widehat{\mathfrak{sl}}(n)$ string functions

$$\sigma^{\Lambda}_{\gamma(\mathbf{l})}(q) = (q;q)_\infty \sum_{\lambda \in \mathcal{M}^\sigma_{\mathbf{l}}} q^{\,k_0(\lambda)}\,, \tag{1.5.113}$$

we can also express (1.5.110) as

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q,\hat{\mathfrak{t}}) = q^{\,h_\Lambda}\,(q;q)_\infty \sum_{\lambda \in \mathcal{M}^\sigma} q^{\,k_0(\lambda)} \prod_{i=1}^{N-1} \hat{\mathfrak{t}}_i^{\,d_i+\delta k_{i-1}(\lambda)-2\delta k_i(\lambda)+\delta k_{i+1}(\lambda)}, \tag{1.5.114}$$

where we set $\delta k_0(\lambda) = \delta k_N(\lambda) = 0$. This final expression will be used in chapter 4 to check our proposed AGT correspondence.

We have therefore shown that $\widehat{\mathfrak{g}}_k$-WZW models are also CFTs. Moreover we can apply the machinery described for CFTs described throughout this section on $\widehat{\mathfrak{g}}_k$-WZW models. In the subsection below, we will use this line of thinking to derive differential equations satisfied by the correlation functions for WZW models. In our case, we will focus on the 4-point function for the case $\widehat{\mathfrak{g}}_k = \widehat{\mathfrak{sl}}(n)_N$.

1.5.10 The KZ Differential Equation for WZW Models

We can derive a differential equation that must be satisfied by WZW n-point functions, which we will use to test our generalized AGT hypothesis in chapter 4. To do so, we will construct an *affine* null vector which decouples from the theory (analogously to the singular vectors for minimal models in section 1.5.5). This singular vector will have a form involving Vir and $\widehat{\mathfrak{g}}$-modes. We will then substitute the action of the Vir modes as differential operators from section 1.5.2 to a correlation function involving this singular vector to obtain a differential equation.

We begin by using the WZW expression for the Vir modes (1.5.98) to obtain the following action by L_{-1} on a WZW primary field $|\Lambda\rangle$

$$L_{-1}\,|\Lambda\rangle = \frac{1}{k+g}\sum_a (J^a_{-1}J^a_0\,|\Lambda\rangle) = \frac{-1}{k+g}\sum_a (J^a_{-1}t^a_\Lambda)\,|\Lambda\rangle\,. \tag{1.5.115}$$

We can use this to construct the null vector

$$|\chi\rangle = \left(L_{-1} + \frac{1}{k+g}\sum_a (J^a_{-1}t^a_\Lambda)\right)|\Lambda\rangle = 0. \tag{1.5.116}$$

We can obtain the action of J^a_{-1} on a WZW primary field $\phi(z)$ in an n-point correlation function as

$$\begin{aligned}\langle\phi_1(z_1)\dots(J^a_{-1}\phi_k(z_k))\dots\phi_n(z_n)\rangle &= \frac{1}{2\pi i}\oint_{z_k} dz\frac{1}{z-z_k}\langle J^a(z)\phi_1(z_1)\dots\phi_n(z_n)\rangle \\ &= \frac{1}{2\pi i}\sum_{j\neq k}\oint_{z_j} dz\frac{1}{z-z_k}\frac{t^a_{\Lambda_j}}{z-z_j}\langle\phi_1(z_1)\dots\phi_n(z_n)\rangle,\end{aligned}$$

where we have used the compactness of Σ to reverse the contour and encircle all poles *except* the one at z_k. We substitute this expression, together with the differential operator expression for L_{-1} from (1.5.23), to obtain the Knizhnik-Zamolodchikov (KZ) differential equation for correlations functions [36]

$$\left[\partial_{z_i}+\frac{1}{k+g}\sum_{j\neq i}\frac{\sum_a t^a_{\Lambda_i}\otimes t^a_{\Lambda_j}}{z_i-z_j}\right]\langle\phi_1(z_1)\dots\phi_n(z_n)\rangle = 0. \tag{1.5.117}$$

In the case of the $\widehat{\mathfrak{sl}}(n)_k$-WZW 4-point correlation function[32] of primary fields, we follow our discussion from section 1.5.3. First we fix the three points $z_2 = 0$, $z_3 = 1$, $z_4 = \infty$ so that there is only one independent variable $z_1 = q$, which is the cross-ratio. We restrict the four primary fields to be two transforming in the fundamental representation and two transforming in the anti-fundamental.

The Casimir $\sum_a t^a_{\Lambda_i}\otimes t^a_{\Lambda_j}$ acts on the tensor product of a fundamental and anti-fundamental representation. Using the Littlewood-Richardson rules (see section 1.2.3), these tensor products decompose into two factors. In this setting, the KZ equation has solutions built using a basis of two solutions. For a 4-point correlation function composed of two fundamental and two anti-fundamental primary fields, the KZ differential equation reduces to a second order ordinary differential equation[33]

$$\frac{q(1-q)}{n^2}\left(n^2\kappa^2\partial_q^2+A(q)\partial_q+B(q)\right)f_1(q) = 0, \tag{1.5.118}$$

where $\kappa = k+n$, $q^r(1-q)^s f_1$ is one of the linearly independent conformal blocks, and s and r are to be determined. The functions $A(q)$ and $B(q)$ are rational functions involving all the other parameters and their specific form is irrelevant for our purposes. Explicit expressions can be found in [50, **eqs (15.161), (15.162)**].

[32]Note that the n here which defines the rank of the affine Lie algebra, is *not* related to the number of primary fields in the correlation function, which we have confusingly notated as n throughout this section. Here, we are considering a 4-point correlation function for the $\widehat{\mathfrak{sl}}(n)_k$-WZW model of arbitrary rank.

[33]Although this derivation consists of standard algebraic manipulation for differential equations, and using some basic Lie algebra representation theory identities, it is very long and not so enlightening. As such we will omit it in this text, but point the reader to [50, §15.4] to see the full details.

This ODE has 3 singular points so can be transformed into the hypergeometric differential equation

$$\big(q(1-q)\partial_q^2 + [c-(a+b+1)]\partial_q - ab\big)\, f(q) = 0, \tag{1.5.119}$$

where a, b, c are functions of k and n, and q is the cross ratio

$$q = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{1.5.120}$$

Doing so involves fixing the parameters r and s to

$$r = r_+ = \frac{1}{\kappa n}, \quad \text{or } r = r_- = -\frac{n^2-1}{\kappa n}, \tag{1.5.121}$$

$$s = s_+ = \frac{1}{\kappa n}, \quad \text{or } s = s_- = 1 - \frac{n^2-1}{\kappa n}. \tag{1.5.122}$$

Since this is a linear second-order ordinary differential equation, we can construct general solutions to it out of a basis of two linearly independent solutions as expected. Solutions to the hypergeometric differential equation are built out of the hypergeometric function

$${}_2F_1(a, b, c; z) = \sum_{n\in\mathbb{Z}_{\geq 0}} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{1.5.123}$$

where $(x)_n = x(x-1)\ldots(x-n+1)$ is the falling Pochammer symbol. In the case of the KZ equation (1.5.118), the two linearly independent functions are

$$f_1^{(-)}(q) = {}_2F_1\left(\frac{1}{\kappa}, -\frac{1}{\kappa}, 1 - \frac{n}{\kappa}; q\right), \tag{1.5.124}$$

$$f_1^{(+)}(q) = {}_2F_1\left(\frac{n-1}{\kappa}, \frac{n+1}{\kappa}, 1 + \frac{n}{\kappa}; q\right). \tag{1.5.125}$$

This process is generic in that solutions of the KZ equations can be expressed in terms of generalized hypergeometric functions (see [86] and [87]). In chapter 4, we will use the following known series expressions for specific $\widehat{\mathfrak{sl}}(n)_k$-WZW 4-point conformal blocks.

Example 1.5.10.1. The $\widehat{\mathfrak{sl}}(n)_N$-WZW conformal blocks (note that here we have notated the level by N not k to keep in line with our notation in [88]), for the 4-point function on the Riemann sphere of primary fields with fundamental and anti-fundamental representations (which we denote by $\square$ and $\bar{\square}$ respectively), schematically denoted by

$$\langle \bar{\square}(\infty)\, \square(1)\, \square(z)\, \bar{\square}(0)\rangle_{\mathbb{P}^1}^{\widehat{\mathfrak{sl}}(n)_N}, \tag{1.5.126}$$

were obtained in [36], as solutions to the Knizhnik-Zamolodchikov equation (1.5.117), as

$$
\begin{aligned}
\mathcal{F}_1^{(0)}(z) &= z^{-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(-\frac{1}{n+N},\frac{1}{n+N};\frac{N}{n+N};z\right),\\
\mathcal{F}_2^{(0)}(z) &= \frac{1}{N}\,z^{1-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(1-\frac{1}{n+N},1+\frac{1}{n+N};1+\frac{N}{n+N};z\right),\\
\mathcal{F}_1^{(1)}(z) &= z^{h_\theta-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(\frac{n-1}{n+N},\frac{n+1}{n+N};1+\frac{n}{n+N};z\right),\\
\mathcal{F}_2^{(1)}(z) &= -n\,z^{h_\theta-2h_\square}\,(1-z)^{h_\theta-2h_\square}\,{}_2F_1\left(\frac{n-1}{n+N},\frac{n+1}{n+N};\frac{n}{n+N};z\right),
\end{aligned}
\tag{1.5.127}
$$

where $h_\square = \frac{n^2-1}{2n(n+N)}$ is the conformal dimension of the four primary fields, and $h_\theta = \frac{n}{n+N}$ is the conformal dimension of the adjoint field with weight $\theta = [N-1,1,0,\ldots,0,1]$. These four solutions correspond to two choices of the representations of states in the internal channel which follow from the fusion of $\square$ and $\overline{\square}$, and $\mathcal{F}_1^{(0)}(z),\mathcal{F}_2^{(0)}(z)$ (resp. $\mathcal{F}_1^{(1)}(z),\mathcal{F}_2^{(1)}(z)$) corresponds to the identity (resp. adjoint) field conformal block in the s-channel. Under a hypergeometric transformation

$$
z \quad\to\quad \mathfrak{q} := \frac{z}{z-1},
\tag{1.5.128}
$$

the Gauss hypergeometric function transforms as

$$
{}_2F_1\left(\alpha,\beta;\gamma;z\right) = (1-\mathfrak{q})^{\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\mathfrak{q}\right),
\tag{1.5.129}
$$

and the $\widehat{\mathfrak{sl}}(n)_N$-WZW 4-point conformal blocks (1.5.127) are expressed, in the $\mathfrak{q}$-module, as

$$
\begin{aligned}
\widehat{\mathcal{F}}_1^{(0)}(\mathfrak{q}) &:= z^{2h_\square}\mathcal{F}_1^{(0)}(z) = (1-\mathfrak{q})^{2h_\square-\frac{n+1}{n+N}}\,{}_2F_1\left(-\frac{1}{n+N},\frac{N-1}{n+N};\frac{N}{n+N};\mathfrak{q}\right),\\
\widehat{\mathcal{F}}_2^{(0)}(\mathfrak{q}) &:= z^{2h_\square}\mathcal{F}_2^{(0)}(z) = -\frac{\mathfrak{q}}{N}\,(1-\mathfrak{q})^{2h_\square-\frac{n+1}{n+N}}\,{}_2F_1\left(\frac{N-1}{n+N},1-\frac{1}{n+N};1+\frac{N}{n+N};\mathfrak{q}\right),\\
\widehat{\mathcal{F}}_1^{(1)}(\mathfrak{q}) &:= \frac{z^{2h_\square}}{n}\mathcal{F}_1^{(1)}(z) = \frac{(-\mathfrak{q})^{h_\theta}}{n}\,(1-\mathfrak{q})^{2h_\square-\frac{n+1}{n+N}}\,{}_2F_1\left(\frac{n-1}{n+N},1-\frac{1}{n+N};1+\frac{n}{n+N};\mathfrak{q}\right),\\
\widehat{\mathcal{F}}_2^{(1)}(\mathfrak{q}) &:= \frac{z^{2h_\square}}{n}\mathcal{F}_2^{(1)}(z) = -\,(-\mathfrak{q})^{h_\theta}\,(1-\mathfrak{q})^{2h_\square-\frac{n+1}{n+N}}\,{}_2F_1\left(-\frac{1}{n+N},\frac{n-1}{n+N};\frac{n}{n+N};\mathfrak{q}\right).
\end{aligned}
\tag{1.5.130}
$$

1.5.11 Fusion in WZW Models

We will now describe fusion in WZW models and present a combinatorial method to calculate the fusion rules in WZW models [89, 90]. In chapter 4, we will calculate series expansions for simple 4-point conformal blocks of primary fields in $\widehat{\mathfrak{sl}}(n)_N$-models. The primary fields in these correlation functions must respect the $\widehat{\mathfrak{sl}}(n)_N$-WZW fusion rules, and we will use this algorithm to determine the possible combinations primary fields we can consider.

First, we describe the fusion of two $\widehat{\mathfrak{g}}_k$-WZW primary fields. As primary fields in WZW models are labelled by $\widehat{\mathfrak{g}}$ dominant integrable weights, we will schematically represent fusion of conformal families associated to primary fields of highest weight Λ_1 and Λ_2 by

$$[\phi_{\Lambda_1}] \times [\phi_{\Lambda_2}] \mapsto \sum_{\Lambda_3} \mathcal{N}_{\Lambda_1 \Lambda_2}^{\Lambda_3} [\phi_{\Lambda_3}], \tag{1.5.131}$$

where the sum is over dominant integral $\widehat{\mathfrak{g}}$ weights Λ_3. In an $\widehat{\mathfrak{g}}_k$-WZW model the algorithm for fusion is as follows:

1. Add the finite part of Λ_1 to the finite part of each weight Λ_2 in the representation. Denote the finite $\mathfrak{g}$ weight produced by this by $\bar{\Lambda}_3$.
2. Extend the finite weight $\bar{\Lambda}_3$ to an affine weight Λ_3 of level k.
3. Check the action of affine Weyl group W on each Λ_3. If we have $(s_i w)\Lambda_3 = w\Lambda_3$ for some $w \in W$, then we ignore the weight in this algorithm[34]. Otherwise we take the weight $w\Lambda_3$ such that $w\Lambda_3 \in P^+_{k,r}$, to appear in the fusion rules.

This is easier to see combinatorially, where we can describe fusion as an extension to the Littlewood-Richardson rules for tensor products we described in section 1.2. To calculate fusion, we first calculate the tensor product decomposition for the finite parts of Λ_1 and Λ_2 using the Littlewood-Richardson rules (leaving the tensor product decomposition in terms of Young diagrams).

Then, consider a weight $\bar{\Lambda}_3'$ that appears in this decomposition. If $\mathrm{par}(\bar{\Lambda}_3)_1 \leq k$ then $\bar{\Lambda}_3'$ extends to a dominant integrable affine weight Λ_3, otherwise we remove a boundary strip of length

$$t = \mathrm{par}(\bar{\Lambda}_3)_1 - k - 1, \tag{1.5.132}$$

from $\mathrm{par}(\bar{\Lambda}_3)$ beginning at the end of the first row and moving down and to the left and place it at the n-th row moving up and right. We then eliminate any rows of length n. If this process produces a standard Young diagram λ that corresponds to an $\widehat{\mathfrak{g}}_k$ weight Λ_3 (that is if $\lambda = \mathrm{par}(\Lambda_3)$) then the weight Λ_3 appears in the fusion rules for Λ_1 and Λ_2. We will illustrate this process performed after the Littlewood-Richardson rules with an example from $\widehat{\mathfrak{sl}}(4)_5$ fusion, which shows the generic behaviour of this combinatorial algorithm.

Example 1.5.11.1. Consider the candidate weight $\Lambda' = [3, 6, 4]$ from a tensor product decomposition using the LR rules. We have $\mathrm{par}(\Lambda') = (13, 10, 4)$, and the algorithm at each step

[34] We cannot see why this is true from the combinatorial algorithm presented here. This method is a combinatorial representation of the algorithm giving the Kac-Walton formula [91, 92], for which such weights appear with a weight of opposite parity and thus cancel with each other. A pedagogical introduction to the WZW fusion rules, the Kac-Walton formula, and this combinatorial representation is the subject of [50, §16].

gives us

$$(13, 10, 4) \longrightarrow (9, 7, 6, 5) \longrightarrow (4, 2, 1) \tag{1.5.133}$$

In terms of Young diagrams this can be visualised as

$$\begin{ytableau} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \end{ytableau} \longrightarrow \begin{ytableau} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{ytableau}$$
$$\longrightarrow \begin{ytableau} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot \\ \cdot \end{ytableau} \tag{1.5.134}$$

where we have used 0's to visually represent the boundary strip we have moved in between the first and second diagrams. The third diagram is then obtained by deleting the 4 columns of length 4.

1.5.12 The GKO Construction and Cosets

Originally used to prove the unitarity of the $M(p, p+1; 2)$ minimal models [93], the Goddard-Kent-Oliver (GKO) construction [94] is a method to construct new CFTs out of WZW models. Consider an affine Lie algebra $\widehat{\mathfrak{g}}$ of rank r, with generators J^i_n for $n \in \mathbb{Z}$ and $i = 1, \ldots, \dim(\mathfrak{g})$ (note that J^i_0 for $i = 1, \ldots, \dim(\mathfrak{g})$ are the generators of the subalgebra $\mathfrak{g}$ in $\widehat{\mathfrak{g}}$). Let $\widehat{\mathfrak{p}} \subset \widehat{\mathfrak{g}}$ be an affine Lie subalgebra, with generators denoted by $\bar{J}^i_n$. Since $\widehat{\mathfrak{p}} \subset \widehat{\mathfrak{g}}$, each $\bar{J}^i_n$ can be expressed as a linear combination of the J^i_n.

Recall that in an $\widehat{\mathfrak{g}}_k$-WZW mode, we can construct a Vir energy-momentum tensor by the Sugawara construction that we will denote by $T_{\widehat{\mathfrak{g}}}(z)$. We can also define a second Vir energy-momentum tensor $T_{\widehat{\mathfrak{p}}}$ associated to the $\widehat{\mathfrak{p}}$ subalgebra in this way. Now consider the *coset energy-momentum tensor* $T_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}}$ defined by

$$T_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}} = T_{\widehat{\mathfrak{g}}} - T_{\widehat{\mathfrak{p}}}. \tag{1.5.135}$$

We can take the OPE of $T_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}}$ with itself (see (1.5.2.3)) and see that it generates a Vir representation of central charge[35] $c_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}} = c_{\widehat{\mathfrak{g}}} - c_{\widehat{\mathfrak{p}}}$, and we denote its Virasoro generators by

[35] Not strictly true, for general cosets we must be careful about the embedding x_e index of $\widehat{\mathfrak{p}}$ in $\widehat{\mathfrak{g}}$. The embedding index is defined as the ratio of the projection of the norm-square of longest root $\theta_{\widehat{\mathfrak{g}}}$ for $\widehat{\mathfrak{g}}$ to the norm-square of $\theta_{\widehat{\mathfrak{p}}}$ for $\widehat{\mathfrak{p}}$

$$x_e = \frac{(\theta_{\widehat{\mathfrak{g}}}, \theta_{\widehat{\mathfrak{g}}})}{(\theta_{\widehat{\mathfrak{p}}}, \theta_{\widehat{\mathfrak{p}}})}. \tag{1.5.136}$$

We will only consider diagonal coset models for which $x_e = 1$ and we can use this equation for the coset central charge without fear.

$L_n^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}}$ for $n \in \mathbb{Z}$. Informally, we say the *coset model* $\frac{\widehat{\mathfrak{g}}}{\widehat{\mathfrak{p}}}$ is a CFT whose energy-momentum tensor is $T_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}}$. Formally, one defines[36] the *coset algebra* $\frac{\widehat{\mathfrak{g}}}{\widehat{\mathfrak{p}}}$ as the largest Lie algebra A such that $A \otimes \widehat{\mathfrak{p}} \subset \widehat{\mathfrak{g}}$. By an abuse of terminology, we will use the terms coset model and coset algebra interchangeably throughout this thesis. In general, we will refer to coset models and algebras as simply *cosets*. We will sometimes refer to the algebras $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{p}}$ as the numerator and denominator of the coset respectively.

Objects related to the subalgebra $\widehat{\mathfrak{p}}$ decouple from a coset model, lending validity to the name coset. By considering $\frac{\widehat{\mathfrak{g}}}{\widehat{\mathfrak{p}}}$ as a subalgebra of $\widehat{\mathfrak{g}}$, we see that $\frac{\widehat{\mathfrak{g}}}{\widehat{\mathfrak{p}}}$ commutes with $\widehat{\mathfrak{p}}$ since

$$[L_j^{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}}, \bar{J}_m^i] = 0 \quad \forall i, j, m \tag{1.5.137}$$

This commutativity[37] is what decouples $\widehat{\mathfrak{p}}$ in the coset model.

We have not discussed the fields of coset models, nor will we in much depth, as we wil never directly consider coset fields when considering the cosets present in our work in chapters 4 and 5. In chapter 4, we will trivialize the coset factor in our generalized AGT correspondence, leaving only an $\widehat{\mathfrak{sl}}(n)_N$-WZW model. While in chapter 5, we will work with coset models, but only at the level of characters. As such, we will not need to use the fields of a coset theory, although we will briefly touch on them when discussing coset characters below.

General cosets are quite complicated, and in this thesis we will only study *diagonal cosets*, which are cosets of the form

$$\mathfrak{g} = \frac{\widehat{\mathfrak{g}}_{k_1} \times \widehat{\mathfrak{g}}_{k_2}}{\widehat{\mathfrak{g}}_{k_1+k_2}}. \tag{1.5.138}$$

If we let $(J_n^a)^{(1)}$ and $(J_m^b)^{(2)}$ denote the $\widehat{\mathfrak{g}}_{k_1}$ and $\widehat{\mathfrak{g}}_{k_2}$ generators respectively, diagonal cosets correspond to embedding $\widehat{\mathfrak{g}}_{k_1+k_2}$ using the generators $(J_n^a)^{(1)} \times (J_n^a)^{(2)}$.

We will finish this section by discussing the character functions of coset models. In practice, to describe the states in a coset representation, we branch from the integrable representations of the numerator algebra $\widehat{\mathfrak{g}}$ to the denominator algebra $\widehat{\mathfrak{p}}$. We know that if $\widehat{\mathfrak{g}}$ contains a $\widehat{\mathfrak{p}}$ subalgebra, the highest weight $\widehat{\mathfrak{g}}$-module L_Λ decomposes into $\widehat{\mathfrak{p}}$-modules by the branching rules

$$L_\Lambda \mapsto \bigoplus_\mu b_\Lambda^\mu L_\mu, \tag{1.5.139}$$

where μ are dominant integral $\widehat{\mathfrak{p}}$ weights. This suggests that the coset characters $\chi^{\mu/\lambda}$ should be taken to be the branching functions b_λ^μ for $\widehat{\mathfrak{g}}$ to $\widehat{\mathfrak{p}}$. This brief discussion misses some of the subtleties in this argument, and the reader should consult [50, §18.2] for a more in depth

[36] One can also define cosets using the modern framework of vertex operator algebras. To understand the work in this thesis we will only need to understand the GKO construction of $T_{\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}}$.

[37] When considering cosets from a VOA perspective, the coset $\widehat{\mathfrak{g}}/\widehat{\mathfrak{p}}$ is defined to be the commutant of $\widehat{\mathfrak{p}}$ within $\widehat{\mathfrak{g}}$.

discussion. Throughout this thesis we will take the coset characters and branching functions in (1.5.139) to be interchangeable.

We will consider only the case where $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}}(n)_{k_1} \times \widehat{\mathfrak{sl}}(n)_{k_2}$ and $\widehat{\mathfrak{p}} = \widehat{\mathfrak{sl}}(n)_{k_3}$ in chapter 5. This is described by the branching of $\widehat{\mathfrak{sl}}(n)$ representations to other $\widehat{\mathfrak{sl}}(n)$ representations, and we will not need to worry about these subtleties.

We have the following example to illustrate the use of the GKO construction to understand the minimal models in terms of WZW models. This example also presents an important character function expression that we will use in chapters 4 and 5.

Example 1.5.12.1. The GKO construction of the minimal model $M(p+2, p+3; 2)$ is via the coset [94]

$$M(p+2, p+3; 2) \simeq \frac{\widehat{\mathfrak{sl}}(2)_p \times \widehat{\mathfrak{sl}}(2)_1}{\widehat{\mathfrak{sl}}(2)_{p+1}}. \tag{1.5.140}$$

The branching function/character formula for the $M(p+2, p+3; 2)$ minimal model with conformal weight parameterised by $1 \leq r < p$ and $1 \leq s < p+1$ can be calculated by the Rochi-Caridi expression [95]

$$\chi^{p,p+1}_{r,s}(q) = \frac{1}{(q;q)_\infty} \sum_{k\in\mathbb{Z}} q^{k^2 p(p+1)+k((p+1)r-ps)} - q^{(kp+r)(k(p+1)+s)}. \tag{1.5.141}$$

We will generalize this construction in (1.5.143) below, with a corresponding generalized character expression in (1.5.144).

1.5.13 $\mathcal{W}$-Algebra Minimal Models

Here we discuss a generalisation of the Virasoro minimal models, the $\mathcal{W}$-algebra minimal models. This will mostly constitute a brief review of $\mathcal{W}$-algebras, before defining certain principal $\mathcal{W}_r$-algebra minimal models as cosets of diagnoal $\widehat{\mathfrak{sl}}(n)$ cosets in (1.5.143). The presentation of the material in this section will be non-standard as an introduction to $\mathcal{W}$-algebras.

Throughout this thesis, we will only see $\mathcal{W}$-algebras in two settings: Indirectly, through the $\mathcal{W}$-minimal model characters, and directly as a symmetry of $\mathcal{A}_{N-1}$-Toda field theories. As such, we present only enough material to understand both cases. The reader who wishes to understand $\mathcal{W}$-algebras as algebraic objects should consult [96]. The reader interested in a review from the perspective of CFTs should see [97]. An older review on their application in integrable models can be found in [98], whereas the reader who wishes to see a review covering these different approaches should see [99]. We begin by defining what we mean when we say a $\mathcal{W}$-*algebra*.

Definition 1.5.13.1. ([12]) A quantum $\mathcal{W}$-algebra is a mCFT, whose Hilbert space $\mathcal{H}$ contains a finite number of distinguished chiral states $|i\rangle$ including $|T\rangle$, whose corresponding vertex operators $W^{(s_i)}(z) = V(|i\rangle, z)$ (where the Virasoro energy-momentum tensor $T(z) = W^{(2)}(z)$) are quasiprimary fields of integer conformal dimension $h_i = s_i$. Furthermore, the entire space of fields is generated by normal ordered products of $W^{(s_i)}(z)$ and their derivatives.

A quantum *principal* $\mathcal{W}_r$-algebra (or just $\mathcal{W}_r$-algebra) is a quantum $\mathcal{W}$-algebra where $s_i = i+1$ for $i = 1, \dots, r-1$ (in the notation of [12] $\mathcal{W}_r$ is $W(2,3,\dots,r)$). These states, and their OPEs, define an action of an algebra on the spectrum $\mathcal{H}$ of this CFT, generated by the states $|i\rangle$. The algebra of this representation can be defined through relations that can be obtained, in theory, through the OPEs of the distinguished states in this CFT. This algebra is a $\mathcal{W}$-algebra. Throughout this thesis, when we say a $\mathcal{W}$-algebra we will always mean a principal $\mathcal{W}_r$-algebra obtained in this way.

Example 1.5.13.2. The Vir algebra can be obtained in this way through the action of the energy-momentum tensor $T(z) = W^{(2)}(z)$. Therefore Vir is the $\mathcal{W}_2$-algebra.

This is one primary motivating factor for studying $\mathcal{W}_r$ algebras (and more general $\mathcal{W}$-algebras), as generalizations of Vir. The $\mathcal{W}$-algebras were first introduced in [100], where the principal algebra $\mathcal{W}_3$ was studied. In more modern applications, $\mathcal{W}$-algebras are created through quantum Hamiltonian reduction, see [101] and [102].

When considering AGT correspondences involving an $SU(N)$ gauge theory on $\mathbb{C}^2$, the central CFT object will be the correlation functions of *$\mathcal{W}_r$-primary fields.* For CFTs with $\mathcal{W}_r$-symmetry, we define the $\mathcal{W}_r$-primary fields by the properties

$$L_0 |\alpha\rangle = \Delta(\alpha) |\alpha\rangle, \quad W_0^{(k)} |\alpha\rangle = w^{(k)}(\alpha) |\alpha\rangle, \quad L_n |\alpha\rangle = W_n^{(k)} |\alpha\rangle = 0, \quad n > 0, \tag{1.5.142}$$

where $W_j^{(k)}$ is the j-th mode of the current $W^{(k)}$, $\Delta(\alpha)$ is the conformal dimension of $|\alpha\rangle$, and the eigenvalues $w^{(k)}(\alpha)$ are the quantum numbers for the $\mathcal{W}$-algebra currents.

Originally conjectured in [103, 104] and proven in [105], the $\mathcal{W}_r$-minimal models, which we notate by $M(p, p'; r)$, are CFTs parameterized by two coprime integers p and p'. Of note, the so-called *unitary minimal models*[38] $M(n+p, n+p+1; n)$ have a coset construction for $p \in \mathbb{Z}_{\geq 1}$

$$M(n+p, n+p+1; n) \simeq \frac{\widehat{\mathfrak{sl}}(n)_p \times \widehat{\mathfrak{sl}}(n)_1}{\widehat{\mathfrak{sl}}(n)_{p+1}}. \tag{1.5.143}$$

We will only be concerned with the unitary minimal models in this thesis, and in chapter 5 we will use this coset construction when calculating coset characters. From this, we can see that primary fields in the unitary W_r-minimal models are labelled by two dominant integrable $\widehat{\mathfrak{sl}}(n)$ weights whose levels different by 1.

[38]Where we choose to notate the rank of the $\mathcal{W}$-algebra by n, in line with our subsequent AGT notation.

In the sequel, we will use a character expression from [47], for $M(n+p, n+p'; n)$-minimal models. This formula is seen to generalize the Rochi-Caridi expression above (1.5.141). The character for a unitary minimal model labelled by the two $\widehat{\mathfrak{sl}}(n)$ weights $\xi \in P^+_{n,p-n}$ and $\zeta \in P^+_{n,p'-n}$ is

$$\chi^{n,p,p'}_{\xi,\zeta}(q) = \frac{q^{\Delta^r_{\xi,\zeta}}}{(q;q)^{n-1}_\infty} \sum_{k_1+\cdots+k_n=0} q^{p'\sum_{i=1}^n k_i(\frac{1}{2}pk_i - \nu_i + i)} \det_{1\leq s,t\leq n}\left(q^{(\mu_t - t)(pk_s - \nu_s + s + \nu_t - t)}\right), \tag{1.5.144}$$

where $\mu = par(\zeta) = (\mu_1, \ldots, \mu_n)$ and $\nu = par(\xi) = (\nu_1, \ldots, \nu_n)$ as defined in (1.3.20) and $\Delta^r_{\xi,\zeta} = \frac{1}{2pp'}|p'(\xi+\rho) - p(\zeta+\rho)|^2 - \frac{1}{24}(r-1)$.

In our case, we will be interested in using these to calculate other coset characters combinatorially using coloured cylindric partitions. To do so, we connect these minimal model characters to the cylindric generating functions from 1.1. By comparing (1.4.4) to $\chi^{n,p,p'}_{0,\zeta}(q)$ we have the following equality

$$\frac{1}{(q;q)_\infty}\chi^{n,p,p'}_{0,\zeta}(q) = q^{\Delta^r_{0,\zeta}} X_\Lambda(q), \tag{1.5.145}$$

which will form a combinatorial basis for calculating characters and branching functions in chapter 5.

1.5.14 $\mathcal{A}_{N-1}$-Toda CFTs

Toda field theories (which we will refer to as Toda) are generalizations of Liouville field theory that we met in section 1.5.7 and are defined for a simple Lie algebra $\mathfrak{g}$. While sharing many similarities to Liouville theory, analysis of Toda is much more complicated, mostly due to the presence of $\mathcal{W}$-algebra symmetry, while for Liouville CFT there is only a Vir-symmetry. Due to this, this section will be light on explicit calculations and proofs, although we will provide references for each result for the interested reader. We will also only consider the case where $\mathfrak{g} = \mathfrak{sl}(N)$, corresponding to the root system $\mathcal{A}_{N-1}$, although many results are phrased for general $\mathfrak{g}$.

As for Liouville theory, Toda is defined by a coupling constant b and background charge Q. Toda is a CFT when

$$Q = b + \frac{1}{b}. \tag{1.5.146}$$

We construct primary fields out of the $(N-1)$-component field $\varphi = (\varphi_1, \ldots, \varphi_{N-1})$, where each φ_i for $i = 1, \ldots, N-1$ is a free boson (see section 1.5.6).

Additionally, we postulate the existence of $N-1$ holomorphic currents $W^k(z)$ of spins $k = 2, 3, N-1$ where

$$W^2(z) = T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi : +(Q\rho, \partial\varphi), \tag{1.5.147}$$

is the Vir energy-momentum tensor, and $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+(\mathfrak{g})}\alpha$ and $(\ ,\ )$ denote the Weyl vector and the scalar product on $\mathfrak{h}^*$ respectively. This Vir energy-momentum tensor, generates a representation of Vir with central charge [106]

$$c = N - 1 + 12(Q\rho, Q\rho) = (N-1)(1 + N(N-1)(b + \frac{1}{b})^2). \tag{1.5.148}$$

The currents $W^k(z)$ generate a representation of $\mathcal{W}_N$ in Toda.

Remark 1.5.14.1. For the reader familiar with integrability we can provide a construction for the currents. Note that while we can give an explicit expression for $W^2(z)$, the higher spin currents, for $N > 2$, are too complicated to provide closed form expressions. These currents are related to the field φ through the Miura transformation [104]

$$\prod_{i=0}^{N-1}\left((b + \frac{1}{b}) + (h_{N-i}, \partial\varphi)\right) = \sum_{k=0}^{N} W^{N-k}(z)\left((b + \frac{1}{b})\partial\right)^k, \tag{1.5.149}$$

where h_i for $i = 1, \ldots, N$ are the weights of the fundamental representation (which is the highest weight module $L_{\bar{\Lambda}_1}$) of $\mathfrak{sl}(N)$. Thus, we have that $h_1 = \bar{\Lambda}_1$ and $h_k = \bar{\Lambda}_1 - \sum_{i=1}^{k-1}\alpha_i$ This relation illustrates the link between the study of Toda (and more broadly, CFTs in general) and integrability. Historically, this link was one motivating factor to study CFTs.

Again as for Liouville, the primary fields of $\mathcal{A}_{N-1}$-Toda field theory are vertex operators. We will construct these using tensor products between $(N-1)$ copies of the free boson, denoted as φ_i for $i = 1, \ldots, N-1$, and the simple roots α_i.

Let $\varphi = \sum_{i=1}^{N-1}\varphi_i\alpha_i = (\varphi_1, \ldots, \varphi_{N-1})$ be a vector composed of $(N-1)$ copies of the free boson in this way. We similarly define an $(N-1)$-component charge vector $\alpha = \sum_{i=1}^{N-1}\alpha^{(i)}\alpha_i$, where $\alpha^{(i)}$ for $i = 1, \ldots, N-1$ are some scalar coefficients. We define the vertex operator primary fields for Toda by

$$V_\alpha = e^{i(\alpha,\varphi)}, \tag{1.5.150}$$

where here we note that the Killing form acts as $(\alpha, \varphi) = \sum_{i=i}^{N-1}\varphi_i(\alpha, \alpha_i)$. The leading order behaviour of the singular OPE of V_α and the currents $W^k(z)$ is of the form

$$W^k(z)V_\alpha(w) = \frac{w^{(k)}(\alpha)V_\alpha(z)}{(z-w)^k} + \ldots. \tag{1.5.151}$$

The functions $w^{(k)}(\alpha)$ are known as the *quantum numbers*[39] for the $\mathcal{W}$-algebra CFT in physics, and they are known explicitly [104]. In particular

$$w^{(2)}(\alpha) = \Delta(\alpha) = \frac{(\alpha, 2Q\rho - \alpha)}{2}, \tag{1.5.152}$$

is the conformal dimension of the field V_α. One can obtain this OPE (and hence the quantum numbers) using the OPEs for the free boson (from section 1.5.6), and the $\mathfrak{sl}(N)$ Cartan matrix (1.3.40) (to get the $\mathfrak{sl}(N)$ matrix from the $\widehat{\mathfrak{sl}}(N)$, we delete the 0-th row and column). The singular OPE's defined above imply that V_α is a $\mathcal{W}_N$-primary field in Toda.

Example 1.5.14.2. Let $h_1 = \bar{\Lambda}_1$, $h_2 = \bar{\Lambda}_2 - \bar{\Lambda}_1$, and $h_3 = -\bar{\Lambda}_2$ be the three weights of the highest weight $\mathfrak{sl}(3)$-representation $L_{\bar{\Lambda}_1}$. We have the following closed form expression for $w^{(3)}$ [107]

$$w^{(3)}(\alpha) = i\sqrt{\frac{48}{22+5c}}(\alpha - Q\rho, h_1)(\alpha - Q\rho, h_2)(\alpha - Q\rho, h_3). \tag{1.5.153}$$

1.5.15 The Coulomb-Gas Formalism for Toda Field Theories

The material covered in this section is only a brief review of the results on Toda correlation functions and minimal models. Most of the material covered can be found in [108] and [109, 110] with more depth, and the interested reader should consult these works to fully understand the derivations required to obtain these results.

As for theories whose primary fields are Liouville exponentials, a naive study of the correlation functions in Toda seems to suggest that there are few primary fields with non-trivial correlation functions. For instance, the n-point correlation function $\langle \prod_{i=1}^{n} V_{\alpha^{(i)}}(z)\rangle$ is non-zero only if the neutrality condition

$$\sum_{i=1}^{n} \alpha^{(i)} = 2Q\rho, \tag{1.5.154}$$

is satisfied. We follow the same arguments as we did for the Liouville exponentials in section 1.5.8, and use the same method to form a theory with a continuum of primary fields. We again introducing screening charges Q_k and $\tilde{Q}_k$, that are the zero modes obtained by integrating the fields $V_{\alpha^{(i)}} = e^{b(\alpha^{(i)},\varphi)}$ and $V_{\alpha^{(i)}} = e^{b^{-1}(\alpha^{(i)},\varphi)}$ of conformal dimension 1.

We then modify the n-point correlation functions by adding in these screening charges to obtain $\langle \prod_{i=1}^{n} V_{\alpha^{(i)}}(z) \prod_{k=1}^{N-1} Q_k^{s_k} \prod_{k=1}^{N-1} \tilde{Q}_k^{r_k}\rangle$, where now the the neutrality condition reads

$$(2Q\rho - \sum_{i=1}^{n} \alpha^{(i)}, \bar{\Lambda}_k) = bs_k + b^{-1}r_k. \tag{1.5.155}$$

[39] In physics, quantum numbers are used to classify states in a system, and are eigenvalues of operators.

We now generalise the Coulomb-gas formalism for Liouville described in section 1.5.8 to theories whose primary fields are the Toda vertex operators (1.5.150). By doing so, we will obtain a description for the $\mathcal{W}_N$-minimal models from the previous section. In this general framework, the Coulomb-gas formalism for Liouville is the special $\mathcal{W}_2 \simeq Vir$ case.

First we define the screening charge parameters

$$\alpha_+ = \sqrt{\frac{p}{p'}}, \quad \alpha_- = -\sqrt{\frac{p'}{p}}, \tag{1.5.156}$$

where $p, p' \in \mathbb{Z}$ are coprime, and the new background charge

$$\alpha_0 = \alpha_+ + \alpha_-. \tag{1.5.157}$$

Note that we only consider the case where $p\alpha_- + p'\alpha_+ = 0$, as we will never use the Coulomb-gas formalism for CFTs that aren't minimal models. Using this, we parameterize the central charge of the Vir subrepresentation in a minimal model manner as

$$c_N^{p,p'} = (N-1)(1 - N(N+1)\alpha_0^2). \tag{1.5.158}$$

In this case, the charges of the vertex operators are parameterized using two $(N-1)$-tuples of integers $\mathbf{r} = (r_1, \ldots, r_{N-1})$ and $\mathbf{s} = (s_1, \ldots, s_{N-1})$ as

$$\alpha_{\mathbf{r},\mathbf{s}} = -\sum_{i=1}^{N-1} \left((r_i - 1)\alpha_+ + (s_i - 1)\alpha_-\right) \bar{\Lambda}_i. \tag{1.5.159}$$

Under this parameterization, the vertex operator[40]

$$V_{\alpha_{\mathbf{r},\mathbf{s}}} = e^{i(\alpha_{\mathbf{r},\mathbf{s}}, \varphi)}, \tag{1.5.160}$$

has conformal dimension

$$\Delta(\alpha_{\mathbf{r},\mathbf{s}}) = \frac{1}{2}\left(P_{\mathbf{r},\mathbf{s}}^2 - \alpha_0^2\rho^2\right), \tag{1.5.161}$$

where $\rho^2 = (\rho, \rho)$ and $\alpha_0^2 = (\alpha_0, \alpha_0)$, that is parameterized using the momentum vector

$$P_{\mathbf{r},\mathbf{s}} = -\sum_{i=1}^{N-1} \left(r_i\alpha_+ + s_i\alpha_-\right) \bar{\Lambda}_i. \tag{1.5.162}$$

We will use this formalism in section 2.3.3, and again in chapter 3.

[40]Note, the additional factor of i in the definition of the vertex operator when considering minimal models with the Coulomb-gas formalism.

1.6 Constructing the ALE Space of $\mathbb{C}^2/\mathbb{Z}_n$ and the Instantons

In chapter 2 we will review minimal model AGT for $SU(N)$ gauge theories on $\mathbb{C}^2$, while the new results in chapters 3 4, and 5 will be based on extending this theory to the ALE spaces. ALE spaces, first introduced to physics in [11] (see [111] for a review), are spaces diffeomorphic to the minimal resolutions of singularities of $\mathbb{C}^2/\mathbb{Z}_n$ [112]. Though not essential to understanding the results presented in the sequel, we will review their construction below.

Then, since the fundamental object on the 4D gauge side of AGT dualities is Nekrasov's instanton partition function, we will briefly review the ADHM construction of instantons on $\mathbb{C}^2$ [113] and a generalization to the ALE space diffeomorphic to $\mathbb{C}^2/\mathbb{Z}_n$ [114] below in sections 1.6.2 and 1.6.3. In principle, one does not need to understand the instantons themselves to understand AGT dualities. Starting from the results of Nekrasov's calculation of the instanton partition function [4] is perfectly reasonable. In our case, we introduce these two constructions to explain the rationale behind adding colours to multipartitions when moving from AGT for gauge theories on $\mathbb{C}^2$ to theories on the ALE space $\mathbb{C}^2/\mathbb{Z}_n$ in chapter 3.

1.6.1 Constructing ALE space associated to $\mathbb{C}^2/\mathbb{Z}_n$

In this section we will follow the approach of [114] and [112] to construct the ALE spaces diffeomorphic to $\mathbb{C}^2/\mathbb{Z}_n$ as hyper-Kähler quotients. By an abuse of notation, we will denote these ALE spaces as $\mathbb{C}^2/\mathbb{Z}_n$. For a more complete story, the interested reader is pointed towards the paper [42]. We will closely follow the exposition in [115], but replace the Kleinian subgroup[41] Γ of $SU(2)$ with $\mathbb{Z}_n$.

Let $\rho_R : \mathbb{Z}_n \longrightarrow \mathrm{End}(R)$ be the regular representation of $\mathbb{Z}_n$. That is, R is the complex vector space whose basis is given by $\{e_h \mid h \in \mathbb{Z}_n\}$. The action of $\mathbb{Z}_n$ on R is given by $g \cdot e_h = e_{gh}$, for any $g, h \in \mathbb{Z}_n$. We use this action to define an action of $\mathbb{Z}_n$ on $\mathrm{End}(R)$ by conjugation. Let $\rho_Q^{SU(2)} : SU(2) \longrightarrow \mathrm{End}(Q)$ be the defining representation of $SU(2)$, where $Q \cong \mathbb{C}^2$ and $SU(2)$ action is by matrix multiplication. Note that we denote $\mathbb{C}^2$ by Q to match the notation in [114]. Consider the representation $\rho_Q : \mathbb{Z}_n \longrightarrow \mathrm{End}(Q)$ induced by the embedding $\mathbb{Z}_n \hookrightarrow SU(2)$ via

$$\mathbb{Z}_n = \left\{ \begin{pmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & e^{-\frac{2\pi i k}{n}} \end{pmatrix} \middle| k = 0, 1, \ldots, n-1 \right\}. \tag{1.6.1}$$

[41] ALE spaces can be constructed for any Kleinian (discrete) subgroup Γ of $SU(2)$ as $\mathbb{C}^2/\Gamma$, although we will only discuss $\Gamma = \mathbb{Z}_n$. References for Kleinian subgroups can be found in [116, 117]. Importantly, the construction presented here makes use of the McKay correspondence between Kleinian subgroups of $SU(2)$ and the ADE classification of Dynkin diagrams [118].

We will construct the ALE space associated to $\mathbb{Z}_n$ as a quotient of the manifold

$$\Xi = (Q \otimes \mathrm{End}(R))^{\mathbb{Z}_n}, \tag{1.6.2}$$

where we use the conjugation action of $\mathbb{Z}_n$ on $\mathrm{End}(R)$ from above, and where the $\mathbb{Z}_n$ superscript means to restrict to the $\mathbb{Z}_n$-invariant elements of $Q \otimes \mathrm{End}(R)$. As a representation of $\mathbb{Z}_n$, we decompose Q as

$$Q \simeq Q_1 \oplus Q_2, \tag{1.6.3}$$

where $Q_i = \mathbb{C}e_i$ is a one-dimensional representation of $\mathbb{Z}_n$, and $\{e_1, e_2\}$ are the standard basis for $Q \cong \mathbb{C}^2$. We now describe the $\mathbb{Z}_n$-invariant elements of $Q \otimes \mathrm{End}(R)$. Let $\zeta = e_1 \otimes \alpha + e_2 \otimes \beta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in Q \otimes \mathrm{End}(R) = (Q_1 \otimes \mathrm{End}(R)) \oplus (Q_2 \otimes \mathrm{End}(R))$, where $\alpha, \beta \in \mathrm{End}(R)$, be an arbitrary element. For any $\gamma \in \mathbb{Z}_n$ we can write the action of γ on $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ as

$$\gamma \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \rho_Q(\gamma) \begin{pmatrix} (\rho_R(\gamma)^{-1} \alpha \rho_R(\gamma)) \\ (\rho_R(\gamma)^{-1} \beta \rho_R(\gamma)) \end{pmatrix} = \rho_Q(\gamma) \rho_R^{-1}(\gamma) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rho_R(\gamma). \tag{1.6.4}$$

Note that $\rho_Q(\gamma)$ acts on the basis elements $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, while leaving α and β invariant. Therefore, $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is $\mathbb{Z}_n$-invariant if any only if

$$\gamma_R^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \gamma_R = \gamma_Q^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \tag{1.6.5}$$

for each element $\gamma \in \mathbb{Z}_n$, where $\gamma_R = \rho_R(\gamma)$ and $\gamma_Q = \rho_Q(\gamma)$ are matrix representations of γ in $\mathrm{End}(R)$ and $\mathrm{End}(Q)$ respectively. This specifies the manifold Ξ. We now make explicit the pairs of endomorphisms (α, β) on the manifold Ξ, following the exposition of [119].

We note that matrix representatives for each basis element in the regular representation of $\mathbb{Z}_n$ can be written as $\rho_R(e_k) = \rho_R(e_1)^k$ for $k = 0, \dots, n-1$ where

$$\rho_R(e_1) = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \tag{1.6.6}$$

Using this, we can solve the invariance condition (1.6.5). First we make a change of basis on R so that each matrix $\rho_R(e_k)$ is diagonal. Let $\omega_n = e^{-2\pi i/n}$ be an n-th root of unity. The matrices $\rho_R(e_k)$ are circulant[42] matrices, so are diagonalized by discrete Fourier transform matrices

$$S = \left(\frac{1}{\sqrt{n}}\omega_n^{mm'}\right)_{m,m'=0,1,\ldots,n-1}, \qquad S^{-1} = \left(\frac{1}{\sqrt{n}}\omega_n^{mm'}\right)_{m,m'=0,1,\ldots,n-1}. \tag{1.6.7}$$

Under this change of basis we have

$$\rho_R(e_k) = S\left(\mathrm{diag}(1, \omega^k, \omega^{2k}, \ldots, \omega^{(n-1)k})\right) S^{-1}. \tag{1.6.8}$$

We let $A = S^{-1}\alpha S$ and $B = S^{-1}\beta S$ be the matrix representations of α and β in this basis respectively. We also denote $D_k = \mathrm{diag}(1, \omega^k, \ldots, \omega^{(n-1)k})$. The invariance condition for $\gamma = e_k\, \mathbb{Z}_n$ in this basis now reads

$$\begin{aligned} \begin{pmatrix} \omega_n^k & 0 \\ 0 & \omega_n^{-k} \end{pmatrix} \begin{pmatrix} SAS^{-1} \\ SBS^{-1} \end{pmatrix} &= \begin{pmatrix} SD_{-k}S^{-1}\alpha SD_kS^{-1} \\ SD_{-k}S^{-1}\beta SD_kS^{-1} \end{pmatrix} \\ \begin{pmatrix} \omega_n^k A \\ \omega_n^{-k} B \end{pmatrix} &= \begin{pmatrix} D_{-k}AD_k \\ D_{-k}BD_k \end{pmatrix}. \end{aligned} \tag{1.6.9}$$

We note that for a matrix $M = (M_{ij})_{0\le i,j\le n-1}$, we can write

$$D_{-k}MD_k = \left(\omega_n^{(j-i)k}M_{ij}\right)_{0\le i,j\le n-1}, \tag{1.6.10}$$

so that $\omega_n^k M = D_{-k}MD_k$ implies that $M_{ij} = 0$ for $j - i \neq 1$. From this, we see that the $\mathbb{Z}_n$-invariance condition in this basis is satisfied by endomorphisms A and B such that the matrix representation of A has only non-zero entries when $j - i = 1, -n + 1$ and that of B has only non-zero entries when $j - i = -1, n - 1$. Therefore these matrices are parameterized by $2n$ complex numbers $u_0, \ldots, u_{n-1}$ and $v_0, \ldots, v_{n-1}$ as

$$A = \begin{pmatrix} 0 & u_0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & u_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & u_{n-2} \\ u_{n-1} & 0 & \ldots & \ldots & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & v_0 \\ v_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & v_2 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & v_{n-1} & 0 \end{pmatrix}. \tag{1.6.11}$$

We now describe the quotient we take of Ξ to obtain the ALE space. We denote the quaternion algebra by $\mathbb{H}$. We also choose an invariant hermitian metric g_R on R. Using this we can define a real structure, an anti-linear involution, on $\mathrm{End}(R)$ by defining a hermitian adjoint

[42] A *circulant matrix* is a matrix where the (i, j) entry is equal to the $(i - 1, j - 1)$ entry.

operator

$$g_R(\alpha X, Y) = g_R(X, \alpha^\dagger Y), \quad X, Y \in T_x R, \quad x \in R, \quad \alpha \in \mathrm{End}(R). \tag{1.6.12}$$

We can give the space $Q \otimes \mathrm{End}(R)$ the structure of an $\mathbb{H}$-module. This quaternion structure can then be seen by writing the pairs of endomorphisms $p = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in Q \otimes \mathrm{End}(R)$ as *quaternions of matrices*. These quaternions of matrices represent a quaternion-valued matrix as a complex valued matrix, where matrix multiplication is equivalent to multiplication of quaternions. We associate to p the following quaternion of a matrix

$$p = \begin{pmatrix} \alpha & \beta \\ -\beta^\dagger & \alpha^\dagger \end{pmatrix}. \tag{1.6.13}$$

Again, we represent α and β as complex matrices. We can then write each in terms of real matrices as $\alpha = \alpha_1 + \alpha_2 i$ and $\beta = \beta_1 + \beta_2 i$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in M_{n\times n}(\mathbb{R})$. We also associate the matrix quaternion[43]

$$h_p = \alpha_1 + \alpha_2 i + (\beta_1 + \beta_2 i)\, j = \alpha_1 + \alpha_2 i + \beta_1 j + \beta_2 k, \tag{1.6.14}$$

to the pair of endomorphisms $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Then, if the two quaternions of matrices p_1 and p_2, associated to pairs of endomorphisms $\begin{pmatrix} \alpha^{(1)} \\ \beta^{(1)} \end{pmatrix}$ and $\begin{pmatrix} \alpha^{(2)} \\ \beta^{(2)} \end{pmatrix}$, have corresponding matrix quaternions h_{p_1} and h_{p_2}, the right matrix multiplication $p_1 p_2$ corresponds to right multiplication of quaternions $h_{p_1} \cdot h_{p_2}$. Note that we have notated quaternion multiplication using a dot to differentiate it from matrix multiplication.

This is a generalization of the quaternion structure on $\mathbb{C}^2$, obtained by representing the quaternion $h = \mathrm{Re}(z) + \mathrm{Im}(z)i + \mathrm{Re}(w)j + \mathrm{Im}(z)k$ by the 2×2 complex matrix

$$\begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}, \quad (z, w) \in \mathbb{C}^2, \tag{1.6.15}$$

with matrix multiplication corresponding to right multiplication of quaternions.

Let g_Ξ be the Riemannian metric tensor on Ξ, and ∇ the Levi-Civita connection. Since Ξ is an $\mathbb{H}$-module, it has 3 complex structures I, J, K that satisfy the quaternion algebra relations and are covariantly constant with respect to ∇

$$I^2 = J^2 = K^2 = -1, \;\; IJ = -JI = K, \;\; \nabla I = \nabla J = \nabla K = 0. \tag{1.6.16}$$

[43] We take *matrix quaternion* to mean a quaternion valued matrix.

Using these complex structures, we can define three symplectic (closed, non-degenerate 2-)forms ω_I, ω_J, and ω_K on $Q \otimes \mathrm{End}(R)$ as

$$\omega_I(v,w) = g_\Xi(v, Iw), \tag{1.6.17}$$
$$\omega_J(v,w) = g_\Xi(v, Jw), \tag{1.6.18}$$
$$\omega_K(v,w) = g_\Xi(v, Kw), \tag{1.6.19}$$

where $v, w \in T_x\Xi$ for $x \in \Xi$ are vectors. A manifold with 3 symplectic forms defined from 3 complex structures in this way is called a *hyper-Kähler manifold*[44].

Let $U(R)$ denote the set of unitary transformations on R with respect to g_R and denote by $F \subset U(R)$ the ones that commute with the action of $\mathbb{Z}_n$. We also denote by $\mathfrak{f} = \mathrm{Lie}(F)$ the corresponding Lie algebra to F. There is a natural action of F on $Q \otimes \mathrm{End}(R)$ by

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \mapsto f^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} f, \quad f \in F, \quad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in Q \otimes \mathrm{End}(R). \tag{1.6.20}$$

As F commutes with $\mathbb{Z}_n$, it defines an action on Ξ that preserves the complex structures I, J, and K, and hence leaves the associated symplectic forms ω_I, ω_J, and ω_K invariant.

To construct the ALE space from Ξ, we will take a symplectic quotient with respect to F. Below we will briefly review symplectic quotients to remind the reader of what this means. For the reader unfamiliar with the material below we recommend the reference text [121] and the lecture notes [122]. The author also used [123] in their first foray into symplectic geometry.

1.6.1.1 Symplectic Quotients

Let X be a symplectic manifold with symplectic form ω_X, and let G be a compact Lie group with corresponding Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$. Assume X carries a Hamiltonian G-action, that is that the action preserves ω. We can associate to each $\xi \in \mathfrak{g}$ a vector field V_ξ of G whose flow lines correspond to the action of G (in this case by mapping $\xi \mapsto \exp(\xi)$) on X. We define the *moment map* $\mu : X \longrightarrow \mathfrak{g}^*$ by

$$d([\mu(x)](\xi)) = \omega_x(V_\xi, \cdot), \tag{1.6.21}$$

where $[\mu(x)](\xi)$ is the action of the map $\mu(x) \in \mathfrak{g}^*$ on ξ. By considering only $\zeta \in Z := (\mathfrak{g}^*)^G \subset \mathfrak{g}^*$, where Z is the G-invariant[45] subalgebra $(\mathfrak{g}^*)^G$ of $\mathfrak{g}^*$, we have that $\mu^{-1}(\zeta)$ is

[44] The reader unfamiliar with hyper-Kähler geometry should first begin with Kähler gemoetry, which is the study of manifolds with one complex structure and one symplectic form, and is pointed to the lecture notes [120].

[45] Here the G-action on $\mathfrak{g}^*$ is the *co-adjoint* action.

invariant under the action of G. The space Z is the dual space to the centre of $\mathfrak{g}$. Assuming the action is free, we can then form the *symplectic quotient* of this action by taking the manifold

$$X_\zeta = \mu^{-1}(\zeta)//G, \tag{1.6.22}$$

on which the symplectic form ω_X descends to another symplectic form ω_ζ (see [121] or [122] for details).

This construction can be generalised to hyper-Kähler manifolds as follows. Let G be a Lie group acting on a hyper-Kähler manifold X which leaves the 3 symplectic forms invariant. Denote again its Lie algebra by $\mathfrak{g} = \mathrm{Lie}(G)$. This action then has 3 associated moment maps μ_I, μ_J, μ_K associated to the symplectic forms $\omega_I, \omega_J, \omega_K$ respectively. We then collect these in one map $\mu : X \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g}^*$, where $\mu(x) = (\mu_I(x), \mu_J(x), \mu_K(x))$ for $x \in X$.

As above we denote the dual to the centre of $\mathfrak{g}$ by Z. For $\zeta \in Z$ the symplectic quotient X_ζ is again a hyper-Kähler manifold [124]. When performing a symplectic quotient on a hyper-Kähler manifold we refer to it as a *hyper-Kähler quotient*.

1.6.1.2 Constructing the ALE Space as a Hyper-Kähler Quotient

Thus to construct the ALE space associated to $\mathbb{Z}_n$ from Ξ, we perform a hyper-Kähler quotient on Ξ using the Lie group F of unitary transformations that commute with the action of $\mathbb{Z}_n$. The associated moment map is $\mu = (\mu_I, \mu_J, \mu_K) : X \longrightarrow \mathfrak{f}^* \times \mathfrak{f}^* \times \mathfrak{f}^*$. Explicitly, the specific moment maps on X are defined purely in terms of the associated endomorphisms α and β:

$$\mu_I(\alpha, \beta) = \frac{1}{2} i \left([\alpha, \alpha^\dagger], [\beta, \beta^\dagger] \right) \tag{1.6.23}$$

$$\mu_J(\alpha, \beta) = \frac{1}{2} \left([\alpha, \beta], [\alpha^\dagger, \beta^\dagger] \right) \tag{1.6.24}$$

$$\mu_K(\alpha, \beta) = \frac{1}{2} i \left(-[\alpha, \beta], [\alpha^\dagger, \beta^\dagger] \right) \tag{1.6.25}$$

Following the notation of [124], we collect these maps into $\mu_\mathbb{R} = \mu_I$ and $\mu_\mathbb{C} = \mu_J + i\mu_K$ (note $\mu_\mathbb{C}(\alpha, \beta) = [\alpha, \beta]$). We use these to perform a symplectic quotient by choosing a regular point of the dual space which can be written as the triple $\zeta = (\zeta_1, \zeta_2, \zeta_3) \in Z \times Z \times Z$, where $Z \subset \mathfrak{f}^*$ is the dual to the centre of $\mathfrak{f}$. This constructs the quotient hyper-Kähler manifold

$$X_\zeta = \mu^{-1}(\zeta)//F. \tag{1.6.26}$$

The space X_ζ is the ALE space and as shown in [112, **Cor 3.12**], is diffeomorphic to the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_n$.

1.6.2 The ADHM Construction on $\mathbb{C}^2$

Central to this thesis is a study of the mathematical structure of the instanton partition function, whose calculation involves integration over a *moduli space* $\mathcal{M}_{k,N}$, $k \in \mathbb{Z}_{>0}$ (the case $k \in \mathbb{Z}_{<0}$ are the so called *anti-instantons*) of linear operators satisfying the *ADHM constraints*. We will call these the k-instantons and say that $\mathcal{M}_{k,N}$ is the *k-instanton moduli space*. The construction described here is for 4D $U(N)$ super Yang-Mills on $\mathbb{C}^2$ (or equivalently $\mathbb{R}^4$). We also define $\mathcal{M}_N = \cup_{k=1}^{\infty}\mathcal{M}_{k,N}$, which we refer to as the *instanton moduli space*.

We begin by considering two complex vector spaces V and W, with complex dimensions k and N respectively. Associated to these we introduce 4 linear operators, $B_i \in \mathrm{End}(V)$ for $i \in \{1,2\}$, $I \in \mathrm{Hom}(W,V)$, and $J \in \mathrm{Hom}(V,W)$. The group $GL(V)$ acts on the vector space

$$\mathrm{End}(V) \oplus \mathrm{End}(V) \oplus \mathrm{Hom}(W,V) \oplus \mathrm{Hom}(V,W), \tag{1.6.27}$$

induced by the natural $GL(V)$ action on V. An *instanton* is a quadruple (B_1, B_2, I, J) such that the ADHM constraints

$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0, \tag{1.6.28}$$
$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0, \tag{1.6.29}$$

are satisfied. Note that the dagger superscript denotes the hermitian conjugate.

Remark 1.6.2.1. The ADHM constraints notation has been chosen suggestively, and they are in fact the moment maps of a hyper-Kähler quotient as described in the previous section.

1.6.3 Constructing Instantons on the ALE Space

The instantons on X_ζ are constructed in a similar fashion to the ADHM construction on $\mathbb{C}^2$. As before we have the data of (B_1, B_2, I, J) such that $B_i \in \mathrm{End}(V)$ for $i \in \{1,2\}$, $I \in \mathrm{Hom}(W,V)$, and $J \in \mathrm{Hom}(V,W)$, with the added property that V and W are $\mathbb{Z}_n$-modules and that B_1 and B_2 are $\mathbb{Z}_n$-equivariant endomorphisms (they commute with the action of $\mathbb{Z}_n$). Since V and W are $\mathbb{Z}_n$-modules, they decompose under the $\mathbb{Z}_n$ action as

$$V = \bigoplus_{i=0}^{n-1} V_i \otimes R_i, \quad W = \bigoplus_{i=0}^{n-1} W_i \otimes R_i, \tag{1.6.30}$$

where R_k is the k-th irreducible representation of $\mathbb{Z}_n$ (that is, generated by $e^{2\pi ik/n}$), and we have

$$k = \dim(V) = \sum_{i=0}^{n-1} v_i, \quad N = \dim(W) = \sum_{i=0}^{n-1} w_i. \tag{1.6.31}$$

We then impose the generalised ADHM constraints on this data

$$\zeta_{\mathbb{R}} = [B_1, B_2] + IJ = 0, \tag{1.6.32}$$

$$\zeta_{\mathbb{C}} = [B_1, B_2^\dagger] + [B_1, B_2^\dagger] - II^\dagger + JJ^\dagger = 0. \tag{1.6.33}$$

where $(\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in Z \times Z \times Z$ as described in the construction of the ALE space (that is, they are the moment maps for $GL(V)$-action).

Chapter 2

AGT on $\mathbb{C}^2$

The original AGT conjecture [2], and the generalizations of AGT that we consider in this thesis (AGT-$\mathcal{W}$ [6] and coset AGT [10]), proposed a duality between 2D conformal field theories (CFTs) and certain 4D $\mathcal{N} = 2$ supersymmetric (SUSY) gauge theories. We have already covered the necessary CFT background material in section 1.5 that is required to understand the 2D side of our results, and will not need such an in-depth discussion on SUSY gauge theories to understand the 4D side. In this chapter, we will first define the objects and tools that we will need for our work from the 4D side, and then discuss the original AGT conjecture and its generalization to AGT-$\mathcal{W}$ for $SU(2)$ and $SU(N)$ gauge theories on $\mathbb{C}^2$ respectively.

Within this thesis, we will use AGT dualities to to calculate 4-point conformal blocks and character functions for CFTs using Nekrasov's instanton partition function (defined for $SU(N)$ theories on $\mathbb{C}^2$ in (2.1.14)) from the dual 4D gauge theory. As such, we will endeavor to only introduce the data required to mathematically define the SUSY gauge theories and the instanton partition function we will use. The physics of AGT dualities is built on the work of Seiberg and Witten [3, 125], and the reader interested in fully appreciating AGT first needs to understand Seiberg-Witten theory and its subsequent developments. The review [126] provides an introduction to Seiberg-Witten theory, and the thesis [127] has an in-depth computation of Nekrasov's partition function. Those unfamiliar with supersymmetry might find use in the lecture notes [128], together with the standard reference [129].

2.1 The Instanton Partition Function for $\mathcal{N} = 2$ $SU(N)$ Gauge Theories

Central to our study of AGT correspondences will be Nekrasov's instanton partition function (which we will refer to as the instanton partition function, or partition function when clear), which is associated to a sector of 4D $\mathcal{N} = 2$ SUSY gauge theories. Specifically, the 4D theories on the gauge side of the duality are special gauge theories referred to as *class S* gauge theories, first introduced in [1].

In this section, we will introduce all the mathematical data needed to define the instanton partition for the specified class S gauge theories we will encounter in this thesis. To do so, we will need to know some of the mathematical data defining these class S gauge theories.

2.1.1 Coulomb Branch Data

Nekrasov's instanton partition function is defined in the *low-energy sector* of $\mathcal{N} = 2$ SUSY gauge theories. Specifically, in the low-energy sector we consider the theory on the *Coulomb branch*. This is in contrast to the *Higgs branch*, or a mixing of the two (both of which we will not consider in this work). Thus, when discussing class S theories with a pair of pants decomposition (defined in 2.1.2) in the sequel, the reader should always note that we are not discussing the full theory, only the Coulomb branch.

We will begin by discussing the SUSY gauge theories. Gauge theories are defined on a pseudo-Riemannian manifold (M, g) which is referred to as *space-time*, If we define $\dim_{\mathbb{R}}(M) = d$, we will say that a gauge theory is a d-dimensional (dD) gauge theory. To specify the gauge theory we require a Lie group G, called the *gauge group*, with its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. In SUSY gauge theories we also need an integer $\mathcal{N} = 1, 2, 4, 8$ which is referred to as the *amount of supersymmetry*. In our case, we will only consider theories where $\mathcal{N} = 2$.

We will only consider gauge theories with *flavours of matter*, which have an additional associated set of groups $G_1, \ldots, G_k$ for some $k > 0$ which we use to define a group referred to as a *flavour symmetry group* $G_1 \times \cdots \times G_k$ (in our case each factor of the flavour symmetry group will be of the form $G_i = U(M)$ or $SU(M)$ for some $M \in \mathbb{Z}_{>0}$) with associated *mass parameters*[1] $m_0, \ldots, m_l \in \mathbb{C}$ where $l + 1$ is the sum of the dimensions of the defining representations for each $G_1, \ldots, G_k$.

[1]Throughout this thesis, we will employ the non-standard convention that vectors of parameters in 4D gauge theories start their labelling with 0. As we will be identifying gauge theory objects with $\widehat{\mathfrak{sl}}(N)$ weights on the CFT side, this convention will make our notation cleaner later.

We specify the Coulomb branch by the parameters $a_0, \ldots, a_{\mathrm{rank}(G)-1} \in \mathbb{C}$, called the *Coulomb parameters*. The Coulomb parameters specify an element ϕ of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in the adjoint representation.

Finally, we need to specify a collection of fields in our gauge theories. In supersymmetric gauge theories, these are grouped into sets called *multiplets*, and we will only need to know the multiplets in our theory to write down the instanton partition function. In this thesis, we will only consider theories composed of a *vector multiplet*[2] and a collection of other *matter multiplets* associated to representations of $\mathfrak{g}$. We will refer to the matter multiplets by the representations they are associated to. Specifically, our theories will be composed of some combination of *fundamental*, *anti-fundamental*, and *bifundamental* multiplets.

The flavours of matter in our theories correspond to the matter multiplets. Each mass parameter is associated to one matter field in the gauge theory, and each matter field is said to be a *flavour of matter*. We will focus on theories with the flavour symmetry group $U(N) \times U(N)$, which has $2N$ mass parameters $\mathbf{m} = (m_0, \ldots, m_{N-1})$ and $\mathbf{m}' = (m'_0, \ldots, m'_{N-1})$. To denote that our theory has $2N$ flavours of matter, it is common to collect the number of flavours of matter in an integer N_f. Then in our case, we have $N_f = 2N$. We will also sometimes associate flavours of matter to their representations, and this is reflected in the flavour symmetry group. In this vein, the matter in our theory can be further classified into N flavours of fundamental and N flavours of anti-fundamental matter.

Class S gauge theories are a subset of $\mathcal{N} = 2$ SUSY gauge theories on a 4D manifold M that follow an ADE classification. They are defined in terms compact Riemann surface $\Sigma_{g,n}$ of genus g with n punctures for $(g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0)$. Below in section 2.1.2, we will describe the construction of compact Riemann surfaces via gluing pairs of pants that shows these pairs (g, n) are not allowed. We note that these theories are obtained from 6D SUSY gauge theories on $M \times \Sigma_{g,n}$. The multiplets of the class S theories we will consider are fully specified by the compact Riemann surface $\Sigma_{g,n}$, although to understand how this works is outside the scope of this thesis. Instead we will just need the results of this, which we will describe below. We will denote the class S theory with gauge group G on $M = \mathbb{C}^2$ associated to $\Sigma_{g,n}$ by $\mathcal{T}^G_{g,n}$.

In this thesis, we will focus on $\mathcal{A}_{N-1}$-type class S theories associated to the Riemann surface $\Sigma_{0,4}$, the Riemann sphere with 4 punctures, so that we are considering $\mathcal{T}^{SU(N)}_{0,4}$. Since ϕ is in the Cartan subalgebra of the adjoint representation of $\mathfrak{su}(n)$ on the Coulomb branch, we

[2] The Cartan element ϕ defines one of the fields in the vector multiplet.

can write it explicitly as a diagonal matrix

$$\phi = \begin{pmatrix} a_0 & 0 & \dots & \dots & 0 \\ 0 & a_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & a_{N-1} \end{pmatrix}, \quad a_0, \dots, a_{N-1} \in \mathbb{C}. \tag{2.1.1}$$

Then the traceless condition for $\mathfrak{su}(n)$ imposes $\sum a_i = 0$. The set $\{a_0, \dots, a_{N-1}\}$ are the Coulomb parameters for the $SU(N)$ gauge theories we will consider. We will often collect the Coulomb parameters into a vector $\mathbf{a} = (a_0, \dots, a_{N-1})$.

Remark 2.1.1.1. In physics, the Coulomb parameters are complex numbers that are referred to as vacuum expectation values (VEVs). In the literature, making a choice of Coulomb parameters is often called fixing the vacuum.

2.1.2 Gluing Pair of Pants and Quivers

The next step is to link the multiplet content of the theory $\mathcal{T}_{0,4}^{SU(N)}$ to a pair of pants decomposition of $\Sigma_{0,4}$. Once we have done this, we will be able to write down the instanton partition function. A *pair of pants* is a surface that is homeomorphic to the sphere with 3 punctures, a typical example of which is depicted in figure 2.1. In our case, we will take each pair of pants to be a Riemann surface (have a complex structure).

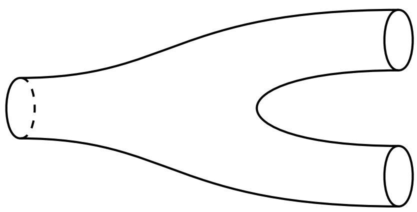


FIGURE 2.1: A standard pair of pants surface.

We will briefly review the gluing (or sewing) procedure for Riemann surfaces here. Consider two compact Riemann surfaces C_1 and C_2 (which in our case will be two copies of a pair of pants), each with punctures (or marked points) $p_1 \in C_1$ and $p_2 \in C_2$. We define closed disks around each puncture $p_1 \in D_1 \subset C_1$ and $p_2 \in D_2 \subset C_2$ with radii $\rho_1 \in \mathbb{R}$ and $\rho_2 \in \mathbb{R}$ respectively.

For a compact Riemann surface C_i, a complex closed disk $D_i \setminus \{p_i\} \subset C_i$ of radius ρ_i with the centre point $p_i \in D_i$ removed is homeomorphic to a semi-infinite cylinder as we will now review. We pick a local coordinate z_i on C_i, such that in this coordinate we have

$$D_i \setminus \{p_i\} = \{z_i = re^{i\theta} + z_i(p_i) \in \mathbb{C} | 0 < r \leq \rho_1,\ 0 \leq \theta < 2\pi\}. \tag{2.1.2}$$

The points in $D_i \setminus \{p_i\}$ are now parameterized by θ and r, and we notate the point $re^{i\theta} + p_i = w \in D_i$ as (r, θ). We then define the homeomorphism between $D_i \setminus \{p_i\}$ and the semi-infinite cylinder $S^1 \times [1/\rho_i, \infty)$ by $(r, \theta) \mapsto (e^{i\theta}, 1/r)$.

Remark 2.1.2.1. The pair of pants pictured in figure 2.1 is a visual representation of $\mathbb{CP}^1 \setminus \{w_1, w_2, w_3\}$, the Riemann sphere with punctures at the points $w_1, w_2, w_3 \in \mathbb{CP}^1$, with this cylinder homeomorphism applied to a disk around each puncture.

We note that as $p_1 \in C_2$ and $p_2 \in C_2$ are punctures, the two disks $D_1 \setminus \{p_1\}$ and $D_2 \setminus \{p_2\}$ are homeomorphic to cylinders. We define two complex parameters z_1 and z_2 around p_1 and p_2 such that $z_i = 0$ at p_i for $i = 1, 2$. We can then *glue* these two cylinders by using a parameter $q \in \mathbb{C} \setminus \{0\}$ to identify neighbourhoods of p_1 and p_2 in D_1 and D_2 through the imposed relationship $z_1 z_2 = q$. Doing so allows us to define a new Riemann surface C, where the two cylinders are glued.

The coordinates z_1 and z_2 are local and therefore we cannot have $z_1, z_2 = \infty$. Thus, we cannot allow $q \longrightarrow \infty$ either. We note that the phase of q tells us how the two ends of the cylinders are rotated in relation to each other before being glued.

Each compact Riemann surface $\Sigma_{g,n}$ where $(g, n) \neq (0, 0), (0, 1), (0, 2), (1, 0)$ can be constructed by gluing some number of pair of pants in this way, and this is how the restriction of Riemann surfaces used to define class S theories is determined. Thus, we can associate a pair of pants decomposition[3] and set of gluing parameters $q_1, \ldots, q_m \in \mathbb{C}$ to any $\Sigma_{g,n}$. In our case, we decompose $\Sigma_{0,4}$ into two glued pairs of pants.

We now apply this gluing procedure to two pairs of pants and glue them along one of each of their punctures to obtain $\Sigma_{0,4}$, the compact Riemann surface of genus 0 with four punctures. The pair of pants decomposition for $\Sigma_{0,4}$ is represented in figure 2.2. In this setting, we have one sewing parameter that we will denote by $q \in \mathbb{C}$.

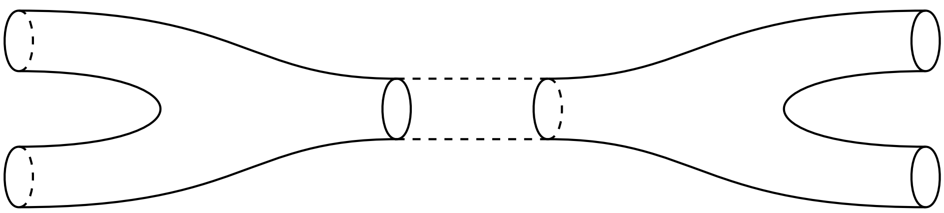


FIGURE 2.2: Two pairs of pants being glued along cylinders around punctures.

[3]There may be more than one pair of pants decomposition for one $\Sigma_{g,n}$. We only consider $\Sigma_{0,4}$ in this thesis, which has only one such decomposition, so that we need not worry about this.

Finally, we will also associate a *generalized quiver* to a pair of pants decomposition of a compact Riemann surface. A generalized quiver is a directed graph for which loops and multiple arrows between nodes is allowed, with extra data associated to the nodes. Quiver gauge theories were first introduced in [114], although we will not need the full formal description of them. The definition we provide here is *not* the conventional definition of a quiver gauge theory, and is instead sufficient to describe the data we require for the AGT calculations within this thesis.

Let Σ be constructed by gluing n pairs of pants which we number $i = 1, \dots, n$. We will let the graph $Q_\Sigma = (E, V)$, with edge set E and vertices V, be the associated quiver. The vertices V will be further split into two types, which we will refer to as *circular* nodes V_c or *rectangular* nodes V_r. The circular nodes will correspond to each gluing operation and the rectangular nodes will correspond to the remaining unglued punctures. In our case, we will only consider quivers with one circular node and four rectangular nodes. We further associate each $v_r \in V_r$ an additional label i corresponding to the pair of pants the puncture initially lay on. Finally, the edges will be between nodes corresponding to one pair of pants, specifically between the circular and rectangular nodes.

When drawing the quiver Q_Σ, we will draw a pair of edges corresponding to a pair of punctures originally on one pair of pants as one edge that splits into two rectangular nodes, shown below in figure 2.3. When using this convention, each edge that is attached to the circular

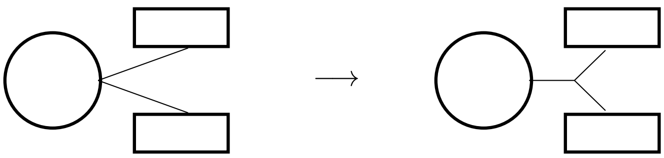


FIGURE 2.3: Drawing a pair of edges associated to unglued punctures on a pair of pants as one edge splitting into two.

node corresponds to one pair of pants, and the number of rectangular nodes it connects to tells one of the number of any punctures. This eliminates the need to track the label i, as it is now clear which punctures correspond to which pair of pants originally. An example of this is shown for $\Sigma_{0,4}$ in figure 2.4. The utility of this will be clear when we describe the AGT correspondence in section 2.2.

Having formalized this process, it is clearer to think about it in an informal way. Informally, we can think of the quiver in the following way from our pair of pants decomposition for $\Sigma_{0,4}$: The round node represents the Riemann sphere, and each rectangular node represents punctures. The split edges then correspond to the individual pairs of pants in the decomposition.

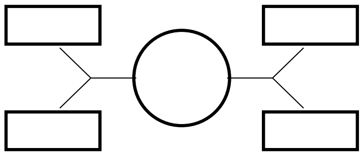


FIGURE 2.4: The quiver diagram associated to $\Sigma_{0,4}$ the Riemann sphere with 4 punctures. Note that the punctures corresponding to each pair of pants are on the left and right side of the circular node respectively, cf: figure 2.2 above.

2.1.3 Class S Quiver Gauge Theories for the Riemann Sphere

We can now describe how to associate a gauge group G and flavour symmetries to the quiver for $\Sigma_{0,4}$. To each circular node we associate a Lie group $G^{(i)} = SU(n_i)$ such that the gauge group is the product of groups in circular nodes $G = G^{(1)} \times \cdots \times G^{(k)}$. Flavour symmetries are then associated to the rectangular nodes. In our case, we place $G = SU(N)$ in the circular node and a flavour symmetry factor of $U(N) \simeq U(1) \rtimes SU(N)$ split between the top and bottom boxes on both the left and right side of the quiver. The vector multiplet is associated to the circular node, while the N flavours of fundamental (anti-fundamental) matter multiplets are associated to the left (right) pairs of rectangular nodes.

In this thesis, we will take the Riemann surface and quiver associated to class S theories as being descriptive of the gauge theory. The reader should be aware that they are actually *prescriptive* instead. To understand these constructions involves a thorough understanding of work of Seiberg and Witten, which is outside the scope of this work. The excellent review of AGT [130], and the references therein, covers this construction for the reader already familiar with Seiberg-Witten theory. With this caveat in mind, we are ready to define what we mean when we say class S gauge theory in this work.

Definition 2.1.3.1. A class S gauge theory is a gauge theory defined by gauge group $G = G^{(1)} \times \cdots \times G^{(k)}$, flavour symmetry group $G_1 \times \cdots \times G_{k'}$, a 4D manifold M, and a compact Riemann surface $\Sigma_{g,n}$ for $(g,n) \neq (0,0),(0,1),(0,2),(1,0)$ together with a pair of pants decomposition of $\Sigma_{g,n}$. We denote this theory by $\mathcal{T}^{G}_{g,n}$ when clear to do so.

In practice, we will describe our theories by filling in the nodes of the quiver diagram associated to $\Sigma_{g,n}$ with the gauge and flavour symmetry group factors. The purpose of this is twofold. Firstly, it allows us to see all the data of our gauge theory. Secondly, as we will describe below in section 2.2, the structure of the quiver is diagrammatically similar to the correlation function that is AGT dual to the gauge theory. The theory $\mathcal{T}^{SU(N)}_{0,4}$ is represented by the quiver in figure 2.5.

In this case, we associate to $G_1 = U(N)$ the fundamental representation $\rho_1 : \mathfrak{su}(N) \to \mathrm{End}(L_{\bar{\Lambda}_1})$ of $\mathfrak{su}(N)$ and to $G_2 = U(N)$ the anti-fundamental representation $\rho_2 : \mathfrak{su}(N) \to$

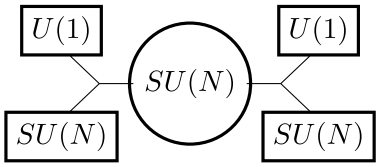


FIGURE 2.5: The quiver diagram for the $SU(N)$ class S theory associated to $\Sigma_{0,4}$ the Riemann sphere with 4 punctures.

$\mathrm{End}(L_{\bar{\Lambda}_{N-1}})$, where we recall that $L_{\bar{\Lambda}}$ refers to the highest weight $\mathfrak{su}(N) \cong \mathfrak{sl}(N)$-module of highest weight $\bar{\Lambda}$ as described in section 1.2.2. This theory has $N_f = 2N$ flavours of matter.

2.1.4 Nekrasov's Instanton Partition Function

In this section we will define Nekrasov's instanton partition function for $\mathcal{N} = 2$ $U(N)$ gauge theories on $\mathbb{C}^2$, which we will use to compute conformal blocks in Liouville and $\mathcal{A}_{N-1}$ Toda CFTs on $\mathbb{C}^2$. The instanton partition function is associated to the instantons in $\mathcal{T}_{0,4}^{SU(N)}$, which we constructed in section 1.6.2. Below in section 2.3, we will use this for the AGT(-$\mathcal{W}$) correspondence.

The instanton partition function has only been calculated directly for $U(N)$ gauge theories, while we are interested in it for the 4D theory $\mathcal{T}_{0,4}^{SU(N)}$. In section 2.2.1, we will discuss how to obtain the instanton partition function we will need from the $U(N)$ one. Until then, we will refer to the 4D $U(N)$ theory with the same multiplets as $\mathcal{T}_{0,4}^{SU(N)}$ as $\mathcal{T}_{0,4}^{U(N)}$ for ease of notation.

To understand the mathematical definition of the instanton partition function we need to understand some geometry of gauge theories. Mathematically, the matter fields are sections of a bundle V over $\mathbb{C}^2$, which carries a $G = U(N)$ action[4]. We refer to V as the *gauge bundle*. The ADHM construction defines a self-dual connection on V, which then defines an instanton. We additionally associate the mass parameters to these fields by taking the tensor product of V with the flavour space $M = \mathbb{C}^{N_f} = \mathbb{C}^N \oplus \mathbb{C}^N$, which contains the complex mass parameters.

We define the instanton partition function for $\mathcal{T}_{0,4}^{U(N)}$ as the integral ([4, **eq (1.7)**])

$$Z_{inst}(\mathbf{a}, \mathbf{m}, \mathbf{m}'; q) := \sum_{k \geq 0} \int_{\mathcal{M}_{k,N}} \mathrm{Eu}(V \otimes M) q^k, \tag{2.1.3}$$

over the k-instanton moduli spaces $\mathcal{M}_{k,N}$ from section 1.6.2, where the parameter q is the gluing parameter for the pair of pants decomposition of $\Sigma_{0,4}$. In this definition, M has an action of the flavour symmetry group $U(N) \times U(N)$, while both M and $\mathcal{M}_{k,N}$ have

[4]Formally one can think of them as sections of associated bundles to a $U(N)$-principal fibre bundle. The bundle V is then the sum of all these individual bundles.

an action of the gauge group $U(N)$. The notation Eu denotes the $U(N) \times U(N) \times U(N)$-equivariant Euler class. The $3N$ parameters contained in the vectors $\mathbf{a} = (a_0, \ldots, a_{N-1})$, $\mathbf{m} = (m_0, \ldots, m_{N-1})$, and $\mathbf{m}' = (m'_0, \ldots, m'_{N-1})$ are the equivariant parameters coming from the $U(N) \times U(N) \times U(N)$-action. The moduli space is non-compact and a calculation of the integral was not possible until the work of Nekrasov [4].

By implementing a so-called *topological twist* (which we will not discuss, but is explained in the original papers [4, 5] and the reviews [130, 131]) and the Ω-*deformation* (see [5, §2.2]), Nekrasov was able to calculate Z_{inst} for a wide variety of $U(N)$ gauge theories. To do so, he used equivariant cohomology or what physicists call *supersymmetric localization*. Localization in this case is very technical (see [132] and [133] for reviews) and an understanding of these technicalities is unnecessary for our purposes. We only need the results of this calculation, which we will describe here.

The Ω-deformation (or Ω-*background*) is parameterised by two numbers $\epsilon_1, \epsilon_2 \in \mathbb{R}$ which are called the *deformation parameters*. They parameterize the action of the torus $T^2 \cong U(1) \times U(1)$ on $\mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (e^{i\epsilon_1} z_1, e^{i\epsilon_2} z_2). \tag{2.1.4}$$

It is a deformation of the metric of the 6D space $\mathbb{C}^2 \times \Sigma_{g,n}$ that the class S theories are obtained from. The instanton partition function of $\mathcal{T}_{0,4}^{U(N)}$ can be calculated for theories with the Ω-deformation applied to them.

Considering the Ω-deformation as the action of the torus T^2 on $\mathbb{C}^2$ (that is on the base space of V), the integrals over $\mathcal{M}_{k,N}$ localize to the fixed points of the combined action of T^2 with the actions of the gauge group[5] and flavour symmetry group, denoted by $T^2 \times U(N) \times U(N) \times U(N)$. This procedure then reduces the integrals over the moduli spaces $\mathcal{M}_{k,N}$ to regular contour integrals, by invoking the Duistermaat-Heckman formula [134].

After transforming (2.1.3) into regular contour integrals, the coefficient of q^k for a fixed k can be found by a residue calculation. These residues are rational functions of the parameters and thus the expression (2.1.3) is a power series in q whose coefficients are rational functions. We can write the results of this calculation in terms of N-tuples of Young diagrams $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$ with k boxes, that is $|\lambda| = k$, where each term in the series defining $Z_{inst}(\mathbf{a}, \mathbf{m}, \mathbf{m}'; q)$ (note that Z_{inst} also depends on the deformation parameters) corresponds to one N-tuple λ. For a fixed N-tuple of Young diagrams λ, the function defining the coefficient of q^k has a factorized form corresponding to the multiplets present in our gauge theory. We will refer to each of these factors as the *contribution of the multiplet*. The residues are sometimes referred to as the instantons that contribute to the partition function.

[5]Note again that although we discuss the Coulomb parameters as being associated to $U(N)$ they are actually parameterizing elements of the Cartan subalgebra of the Lie algebra $\mathfrak{u}(N)$. When discussing the torus action above, we are in a situation where the group $U(N)$ has been broken to $U(1)^N$.

Here we will define the contributions from each multiplet for the $G = U(N)$ theories on $\mathbb{C}^2$ we consider in the sequel (see [126]). All multiplet contributions to Z_{inst} will be built using the *building block function* $E(x, \tilde{\lambda}, \lambda')$, which depends on the deformation parameters ϵ_1 and ϵ_2 and takes in as its arguments: a parameter $x \in \mathbb{C}$, two Young diagrams $\tilde{\lambda}$ and λ' that are not necessarily distinct, and a box $\square = (i, j)$ in either $\tilde{\lambda}$ or λ'. Explicitly, the building block function is defined to be

$$E(x, \tilde{\lambda}, \lambda', \square) := x - \epsilon_1 L_{\lambda'}(\square) + \epsilon_2 A^{+}_{\tilde{\lambda}}(\square). \tag{2.1.5}$$

We recall that $L_{\lambda^{(m)}}(\square)$ and $A_{\lambda^{(l)}}$ are the leg and arm length respectively, defined in (1.1.2), while the superscript $+$ notation is also defined in (1.1.3). In all cases we will consider, x will be the difference of two Coulomb parameters or a mass parameter and the pair of Young diagrams $\tilde{\lambda}$ and λ' will both be from the N-tuple of Young diagrams λ. Due to this second property, we will notate $\tilde{\lambda}$ and λ' as $\lambda^{(l)}$ and $\lambda^{(m)}$ in the sequel, where it is always assumed that $0 \leq l, m \leq N - 1$.

Remark 2.1.4.1. We recall from our discussion in section 1.1, that if $\square \notin \lambda^{(l)}$ we have $A^{+}_{\lambda^{(l)}} \leq 0$, and if $\square \notin \lambda^{(m)}$ we have $L_{\lambda^{(m)}} < 0$.

All results obtained in this thesis will be predicated on a strategy of finding pairs of Young diagrams $(\lambda^{(l)}, \lambda^{(m)})$ for which the building block function vanishes for fixed ϵ_1, ϵ_2, x, and a box $\square$ contained in either $\lambda^{(l)}$ or $\lambda^{(m)}$. We will then restrict the summation in the definition of Z_{inst} to avoid any N-tuples λ containing these pairs. As such, we note that due to $A^{+}_{\lambda^{(l)}}(\square)$, $L_{\lambda^{(m)}}(\square) \in \mathbb{Z}$, the equation

$$0 = E(x, \lambda^{(l)}, \lambda^{(m)}, \square) = x - \epsilon_1 L_{\lambda^{(m)}}(\square) + \epsilon_2 A^{++}_{\lambda^{(l)}}(\square), \tag{2.1.6}$$

implies that

$$x = k^{(m)} \epsilon_1 + k^{(l)} \epsilon_2, \tag{2.1.7}$$

for some integers $k^{(m)}$ and $k^{(l)}$. As we will see, gauge theories where this relation is true are special, and in the sequel we will provide evidence that they are AGT dual to minimal model CFTs.

We will split the discussion of the gauge theories $\mathcal{T}^{SU(N)}_{0,4}$, their instanton partition functions, and AGT relations into two cases: $N = 2$ and $N > 2$. This is due to multiple reasons, one is that for $\mathfrak{su}(2)$ the highest weight fundamental representation L_{Λ_1} is isomorphic to its complex conjugate, the anti-fundamental representation. Furthermore, the AGT correspondence for $SU(2)$ and $SU(N)$ gauge theories have subtle differences that we will discuss in section 2.3. All the subsequent expressions we reproduce in the review below are originally due to Nekrasov [4].

2.1.4.2 $G = U(2)$

We note that in this case, the partition function is defined as a sum over pairs $\lambda = (\lambda^{(0)}, \lambda^{(1)})$ of Young diagrams. Similarly, we have a pair of Coulomb parameters $\mathbf{a} = (a_0, a_1)$, and two pairs of mass parameters $\mathbf{m} = (m_0, m_1)$ and $m' = (m'_0, m'_1)$.

The summands of the partition function for $\mathcal{T}_{0,4}^{U(2)}$ have factors corresponding to three types of multiplets, the vector multiplet (consisting of fields in the adjoint representation of $U(2)$), fundamental multiplets (consisting of fields in the highest weight $\mathfrak{u}(2)$ representation L_{Λ_1}), and anti-fundamental multiplets (consisting of fields in complex conjugate of fundamental representation, note that this is isomorphic to L_{Λ_1}).

We begin by defining the contribution of the multiplet containing the matter in the fundamental representation, or the *fundamental contribution*. We let $m \in \mathbb{R}$ be the mass parameter associated to this multiplet. The contribution can be written as

$$Z_{fun}(\mathbf{a}, \lambda; m) := \prod_{i=0}^{1} \prod_{\square \in \lambda^{(i)}} \left(a_i + \epsilon_1 A^{+}_{\lambda^{(i)}}(\square) + \epsilon_2 L^{+}_{\lambda^{(i)}}(\square) - m \right). \tag{2.1.8}$$

Similarly the contribution from the multiplet containing the matter in the anti-fundamental representation, called the *anti-fundamental contribution*, can be written as

$$Z_{afun}(\mathbf{a}, \lambda; m) := Z_{fun}(\mathbf{a}, \lambda, \epsilon_1 + \epsilon_2 - m) = \prod_{j=0}^{1} \prod_{\square \in \lambda^{(j)}} \left(a_i + \epsilon_1 A_{\lambda^{(j)}}(\square) + \epsilon_2 L_{\lambda^{(j)}}(\square) - m \right). \tag{2.1.9}$$

In sections 2.2 and 2.3, and later in chapter 3, we will often work with the function $Z_{vec}(\mathbf{a}, \lambda)$, which is the *inverse* of contribution of the vector multiplet, when looking at non-physical poles in Z_{inst} for certain gauge theories. For ease of notation and terminology, we will refer then refer to Z_{vec} iteslf as *the contribution of the vector multiplet*. This can be written as

$$\begin{aligned} Z_{vec}(\mathbf{a}, \lambda) := \prod_{i,j=0}^{1} \prod_{\square \in \lambda^{(i)}} & E(a_i - a_j, \lambda^{(i)}, \lambda^{(j)}, \square) \\ & \prod_{\blacksquare \in \lambda^{(j)}} \left(\epsilon_1 + \epsilon_2 - E(a_j - a_i, \lambda^{(j)}, \lambda^{(i)}, \blacksquare) \right), \end{aligned} \tag{2.1.10}$$

where the products run over pairs of (not necessarily distinct) boxes $\square$ and $\blacksquare$ in pairs of (not necessarily distinct) Young diagrams. Note that the boxes $\square$ and $\blacksquare$ are *not* related to any colouring of the Young diagrams, which are all uncoloured, they are just used to notate different boxes[6]. Using these, we can write the instanton partition function for $\mathcal{T}_{0,4}^{U(2)}$ gauge

[6] We will also use this notation to differentiate boxes in products in chapter 3, where we *will* be considering coloured Young diagrams. Within this thesis, all coloured boxes will be labelled by numbers and not shadings..

theories on $\mathbb{C}^2$ as

$$Z_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}';q)=\sum_{\lambda}\frac{Z_{fun}(\mathbf{a},\lambda;m_1)Z_{fun}(\mathbf{a},\lambda;m_2)Z_{afun}(\mathbf{a},\lambda;m_1')Z_{afun}(\mathbf{a},\lambda;m_2')}{Z_{vec}(\mathbf{a},\lambda)}q^{|\lambda|}, \tag{2.1.11}$$

where the summation runs over all pairs of Young diagrams $\lambda=(\lambda^{(0)},\lambda^{(1)})$. By comparing this expression to (2.1.3), we see that the residues of the poles contained in $\mathcal{M}_{k,N}$ can be described by pairs of Young diagrams λ with k boxes as claimed.

2.1.4.3 $G=SU(N)$ for $N>2$

In this case, the partition function is written as a sum over N-tuples of Young diagrams $\lambda=(\lambda^{(0)},\dots,\lambda^{(N-1)})$ of Young diagrams. Similarly, we have N Coulomb parameters $\mathbf{a}=(a_0,\dots,a_{N-1})$, and two N-tuples of mass parameters $\mathbf{m}=(m_0,\dots,m_{N-1})$ and $\mathbf{m}'=(m_0',\dots,m_{N-1}')$, corresponding to $N_f=2N$.

The form of the partition function for $\mathcal{T}_{0,4}^{U(N)}$ has a factorized form corresponding to two types of multiplets, a vector multiplet (again consisting of fields in the adjoint representation of $U(N)$) and two bifundamental[7] multiplets (consisting of fields in the bifundamental representation of $U(N)$). In our case, the bifundamental multiplets will have either a trivial fundamental or anti-fundamental factor, so that the fields belonging to these multiplets will only be in an anti-fundamental or fundamental representation of $U(N)$ respectively.

We begin by defining a function Z_{bif}, that depends on two vectors of N complex numbers $\mathbf{a}=(a_0,a_1,\dots,a_{N-1})$, $\mathbf{b}=(b_0,b_1,\dots,b_{N-1})\in\mathbb{C}^{N-1}$, and two N-tuples of Young diagrams[8] $\lambda_{(1)}=(\lambda_{(1)}^{(0)},\dots,\lambda_{(1)}^{(N-1)})$ and $\lambda_{(2)}=(\lambda_{(2)}^{(0)},\dots,\lambda_{(2)}^{(N-1)})$

$$\begin{aligned}Z_{bif}(\mathbf{a},\lambda_{(1)};\mathbf{b},\lambda_{(2)}):=\prod_{i=0}^{N-1}\prod_{j=0}^{N-1}&\prod_{\square\in\lambda_{(1)}^{(i)}}\left(E\left(a_i-b_j,\lambda_{(1)}^{(i)},\lambda_{(2)}^{(j)},\square\right)\right)\\&\prod_{\blacksquare\in\lambda_{(2)}^{(j)}}\left(\epsilon_1+\epsilon_2-E\left(b_j-a_i,\lambda_{(2)}^{(j)},\lambda_{(1)}^{(i)},\blacksquare\right)\right).\end{aligned} \tag{2.1.12}$$

We will use Z_{bif} to define the contribution of the N fundamental and N anti-fundamental multiplets in one concise form. The contribution of the vector multiplet Z_{vec} depends on the Coulomb parameters $\mathbf{a}=(a_0,a_1,\dots,a_{N-1})$, and one multipartition $\lambda=(\lambda^{(0)},\dots,\lambda^{(N-1)})$,

[7] A *bifundamental representation* of a Lie algebra is the tensor product of a fundamental representation and an anti-fundamental representation.

[8] Note that we have notated these multipartitions using subscripts with brackets to differentiate this notation from that corresponding to their rows.

and can be written as

$$Z_{vec}(\mathbf{a},\lambda) := Z_{bif}(\mathbf{a},\lambda;\mathbf{a},\lambda;\mathbf{0}) = \prod_{i,j=0}^{N-1} \prod_{\square\in\lambda^{(i)}} E\Big(a_i - a_j, \lambda^{(i)}, \lambda^{(j)}, \square\Big) \prod_{\blacksquare\in\lambda^{(j)}} \Big(\epsilon_1 + \epsilon_2 - E(a_j - a_i, \lambda^{(j)}, \lambda^{(i)}, \blacksquare)\Big) \tag{2.1.13}$$

Using these we can write the instanton partition function for $U(N)$ gauge theories on $\mathbb{C}^2$ as

$$Z_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}';q) = \sum_{\lambda} \frac{Z_{bif}(\mathbf{a},\lambda;-\mathbf{m},\varnothing)Z_{bif}(\mathbf{m}',\varnothing;\mathbf{a},\lambda;m)}{Z_{vec}(\mathbf{a},\lambda)} q^{|\lambda|}, \tag{2.1.14}$$

where the summation runs over all N-tuples of Young diagrams $\lambda = (\lambda^{(0)},\ldots,\lambda^{(N-1)})$.

We again remind the reader that these instanton partition functions we have just defined are for $U(N)$ gauge theories. Whereas the AGT correspondences we will meet in the subsequent sections and chapters are between 2D CFTs and $SU(N)$ gauge theories. An important part of performing AGT style computations is extracting the $SU(N)$ partition function from the $U(N)$ one, and in chapter 3 we will perform our own generalization of this for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$.

Finally, we define a generating function for the instantons[9], which we take to be generating function for the N-tuples of Young diagrams defining Z_{inst}. Thus for generic $U(N)$ and $SU(N)$ gauge theories on $\mathbb{C}^2$ we define

$$X^{U(N)}(q) = X^{SU(N)}(q) = \sum_{\lambda\in\mathrm{Par}^N} q^{|\lambda|}. \tag{2.1.15}$$

In subsequent chapters, we will generalise this definition to gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$.

2.2 AGT for $SU(2)$ gauge theories with $N_f = 4$ on $\mathbb{C}^2$

We are now ready to understand the original AGT correspondence. In 2009 Alday, Gaiotto, and Tachikawa suggested a link between the class S theories $\mathcal{T}_{0,4}^{SU(2)}$ and $\mathcal{T}_{1,1}^{SU(2)}$ on $\mathbb{C}^2$ and CFTs on the Riemann sphere and torus respectively [2]. They found that under a suitable parameter identification, the instanton partition function we discussed above agrees with the conformal blocks (see section 1.5.3) of Liouville CFT (section 1.5.7) on the compact Riemann surfaces $\Sigma = \mathbb{CP}^1$, T^2 defining class S gauge theories.

Remark 2.2.0.1. In fact they found a stronger result than this. They conjectured that one could obtain the full Liouville correlation functions from the gauge theory. As part of their

[9]Formally, the residues resulting from supersymmetric localization.

conjecture they obtained he DOZZ factors (named for Dorn and Otto, and A. and Al. Zamolodchikov, who first found them independently in [135] and [136]) of Liouville CFT (see [75, §3]) in the gauge theory.

In this section, we will review their results, with most of this material taken from the original work [2]. The reader unfamiliar with AGT but familiar with CFT and SUSY is recommended the excellent (physics-slanted) review [130] and references therein. While the for reader unfamiliar with SUSY gauge theories, CFTs, and AGT we recommend the thesis [131] and references within.

We will focus here on the conformal blocks of Liouville theory on the Riemann sphere, which corresponds to $\mathcal{T}_{0,4}^{SU(2)}$. All subsequent material will also be for CFTs on the sphere, and our results will be for this case. We will develop the dictionary of parameters necessary to identify Nekrasov's instanton partition function with the conformal blocks of Liouville theory.

2.2.1 Gauge Theory Parameters and Stripping the $U(1)$ Factor

We begin by motivating a specific reparameterization for the mass parameters of the gauge theory. This will also begin the process of stripping the $U(1)$ factor for the instanton partition function. We note that this parameterization is for a 4D $U(N)$ gauge theory, and stripping the $U(1)$ factor from its instanton partition function allows us to obtain the instanton partition function for a 4D $SU(N)$ gauge theory.

We begin by showing this parameterization explicitly for $\mathcal{T}_{0,4}^{SU(2)}$. This was done in the original paper [2], which we will follow when discussing AGT involving $\mathcal{T}_{0,4}^{SU(N)}$. We choose to parameterize the mass parameters m_0, m_1, m_0', and m_1' using new mass parameters n_0, n_1, n_0', and n_1' as

$$m_0 = n_0 + n_0', \;\; m_1 = n_0 - n_0', \;\; m_0' = n_1 + n_1', \;\; m_1' = n_1 - n_1'. \tag{2.2.1}$$

These n and n' parameters will be identified with the conformal momentum for fields in the Liouville theory. Importantly, we have the relations

$$\frac{1}{2}(m_0 - m_1) = n_0', \quad \frac{1}{2}(m_0 + m_1) = n_0, \quad \frac{1}{2}(m_0' - m_1') = n_1', \quad \frac{1}{2}(m_0' + m_1') = n_1. \tag{2.2.2}$$

We will use generalizations of these relations when discussing AGT-$\mathcal{W}$, which involves 4D $SU(N)$ gauge theories, without making this explicit reparameterization.

We now explain the general idea of this reparameterization fo $\mathcal{T}_{0,4}^{SU(N)}$, which has flavour symmetry $U(N) \times U(N)$. Note that we begin here simply to obtain the correct multiplet arrangement for our $U(N)$ theory. In this setting, one $U(N)$ flavour symmetry factor acts on N complex valued fields as a matrix. We claim that it acts on the fields as if they are in the adjoint representation of $U(N)$ on its Lie algebra $\mathfrak{u}(N)$. We then treat the matrix of mass parameters as being from the adjoint representation of $U(N)$ and split it into two terms corresponding to the factorization $U(N) \simeq U(1) \rtimes SU(N)$. To do so, we recall the Lie algebra $\mathfrak{su}(N)$ is the subpace of the complex-valued traceless matrices in $\mathfrak{u}(N)$.

Following the arguments of the paragraph above, we can now understand this reparameterization from a representation theory perspective. To illustrate this, we review the example from [131, §7.3]. We consider the pair of matter multiplets with mass parameters m_0 and m_1, which we write in the matrix $M = \mathrm{diag}(m_0, m_1)$. We then decompose as

$$\begin{pmatrix} m_0 & 0 \\ 0 & m_1 \end{pmatrix} = \begin{pmatrix} n_0 + n_0' & 0 \\ 0 & n_0 - n_0' \end{pmatrix} = n_0 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U(1)} + n_0' \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{SU(2)} \tag{2.2.3}$$

where we view n_0 term as coming from the $U(1)$ Cartan subalgebra (where we have taken the adjoint action of $U(1)$ to act as diagonal 2×2 matrices on the space of diagonal matrices), and the n_0' term as coming from the Cartan subalgebra of $SU(2)$ (represented on its Lie algebra $\mathfrak{su}(2)$). Note that the parameters m_0 and m_1 correspond to one factor of $U(2)$ flavour symmetry and m_0' and m_1' to the other.

Note that the parameters n_0, n_1 corresponding to the $U(1)$ factor still appear in the $SU(2)$ partition function. As we will see below, they also appear in the $U(1)$ factor. This reasoning generalizes to $\mathcal{T}_{0,4}^{SU(N)}$ as we will see in section 2.3.

We recall that the Coulomb parameters in a $U(2)$ gauge theory parameterize an element ϕ in the Cartan subalgebra of $\mathfrak{u}(2)$. The element ϕ is then a complex valued diagonal matrix, and we have two independent complex parameters which we collect in the vector $\mathbf{a} = (a_0, a_1) \in \mathbb{C}^2$. To move from the $U(2)$ gauge theory to the $SU(2)$ gauge theory, we must restrict to the case where ϕ is in the Cartan subalgebra of $\mathfrak{su}(2)$. Then ϕ will be a complex valued traceless diagonal matrix so that $\sum_{i=0}^{1} a_i = 0$. Due to this, we set the Coulomb parameters (a_1, a_2) on the gauge side to $(a, -a)$.

Once we have parameterized the gauge theory in the manner described above we can strip the $U(1)$ factor in the $U(2)$ instanton partition function to obtain the $SU(2)$ instanton partition function, which is a process we will use this in every AGT correspondence and conjecture we consider in this thesis. In general AGT relations involving gauge groups $SU(N)$, we are looking for an equation of the form

$$Z_{inst}^{U(N)} = (U(1) \text{ factor}) \times Z_{inst}^{SU(N)}. \tag{2.2.4}$$

In each case we consider in this work, and in most cases for AGT[10], the $U(1)$ factor is a geometric series, raised to some power involving the deformation parameters and mass parameters corresponding to the $U(1)$ factor of the flavour symmetry group.

For $U(2)$ gauge theories, Alday, Gaiotto, and Tachikawa found that the correct factorization was [2, **eq (3.9)**]:

$$Z_{inst}^{U(2),N_f=4}(a,n_0,n_0',n_1,n_1';q) = (1-q)^{2n_0(\epsilon_1+\epsilon_2-n_1)} \times Z_{inst}^{SU(2)}(a,n_0,n_0',n_1,n_1';q), \quad (2.2.5)$$

where we note that the $U(1)$ factor is a geometric series raised to a power involving only the deformation parameters and the mass parameters corresponding to the $U(1)$ factor.

Remark 2.2.1.1. In the original paper [2], the mass parameters are denoted by μ_i for $i = 1,\ldots,4$ and the parameters we have denoted by n_i and n_i' for $i = 0,1$ are denoted by m_i and m_i'. In this case, we have instead deferred to the notation of [33] and [34], which the work in this thesis directly builds on.

2.2.2 Liouville Conformal Blocks from $SU(2)$ Gauge theories

In this section, we will provide the AGT dictionary that identifies the $SU(2)$ instanton partition function with the Liouville 4-point conformal block of primary fields.

We begin by looking at the parameters needed to define the Liouville conformal block. We recall our visual representation of conformal blocks in figure 1.3, which we have reproduced here in figure 2.6. Note that we have represented the insertion of primary fields by their conformal dimensions, here notated by Δ's. With Δ_i corresponding to vertex operators $\phi_i(z) = V_{\alpha^{(i)}}(z)$ for $i = 1,2,3,4$ and Δ being the conformal weight of primary field defining the conformal family flowing in the channel.

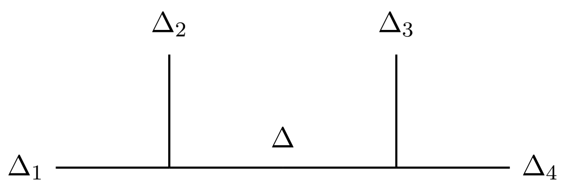


FIGURE 2.6: A 4-point Liouville conformal block, with primary fields labelled by their conformal dimensions.

[10] The author is not aware of any case where this is not true. When considering the dual expression involving conformal blocks for 2D CFTS, this $U(1)$ factor can be considered as corresponding to the free field correlation function of chiral vertex operators (see [19, §4]). This correlation function has the form of a geometric series, which suggests that the dual $U(1)$ factor in the 4D gauge theory should also have this form.

We recall that the central charge c of Liouville is parameterised using the background charge $Q \in \mathbb{C}$ and coupling constant $b \in \mathbb{C}$ as

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}. \tag{2.2.6}$$

The background charge and coupling constants are the Liouville objects that are identified with the Ω-deformation parameters of the gauge theory in the AGT correspondence. Explicitly, we identify

$$\epsilon_1 = b, \;\; \epsilon_2 = \frac{1}{b}, \implies Q = \epsilon_1 + \epsilon_2. \tag{2.2.7}$$

Next, we specify the Liouville exponentials in the 4-point block using gauge theory data. The four legs have conformal dimensions determined from the mass parameters of the gauge theory. The primary field for the internal conformal family that flows in the channel, has conformal dimension determined from the Coulomb parameters. In this case, we identify the gauge theory parameters with the conformal charge of the CFT (see sections 1.5.6 and 1.5.7). The identification is

$$\Delta_1 = \alpha_0(Q-\alpha_0), \;\; \Delta_2 = n_0(Q-n_0), \;\; \Delta_3 = n_1(Q-n_1), \;\; \Delta_4 = \alpha_1(Q-\alpha_1), \;\; \Delta = \alpha(Q-\alpha), \tag{2.2.8}$$

where the Liouville charges α, α_0, and α_1 are defined in terms of the mass and Coulomb parameters as

$$\alpha_0 = \frac{Q}{2} + n_0', \;\; \alpha_1 = \frac{Q}{2} + n_1', \;\; \alpha = \frac{Q}{2} + a. \tag{2.2.9}$$

We note that our reparameterization makes clear the asymmetry between the mass parameters n_0 and n_1 corresponding to the $U(1)$ in the flavour symmetry group and the parameters n_0' and n_1' corresponding to the $SU(2)$ factor on the CFT side of the correspondence.

Identifying the Coulomb parameters with the conformal family flowing in the channel and the mass parameters with the primary fields of the block is a general feature of AGT style correspondences. Comparing figure 2.5 with 2.6 we can see this diagrammatically. We associate the mass parameters to the flavour symmetries (legs of the quiver) and the Coulomb parameters to the gauge group (the internal node of the quiver), this structure is similar to the conformal block.

We have made this correspondence explicit in figure 2.7, where we have placed a quiver and conformal block side-by-side with the parameters associated to each object labelled. Note that we have drawn the conformal block in a slightly modified form here to make this identification clearer. The external legs are still labelled by Δ_1 and Δ_4, while the internal are labelled by Δ_2 and Δ_3. Furthermore, the Riemann surface $\Sigma_{0,4}$ can be thought of as the Riemann sphere, which contains the 2D Liouville CFT, with 4 punctures corresponding to the insertion of the 4 primary fields.

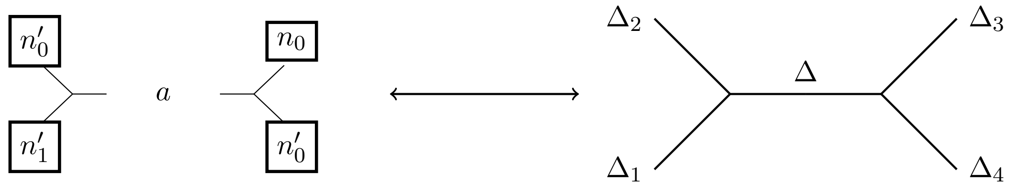


FIGURE 2.7: A comparison between the diagrammatic structure and parameters associated to a quiver diagram for a 4D SUSY gauge theory and a 2D Liouville conformal block.

2.2.3 AGT for $U(2)$ Gauge Theories

In this section we review [19], which proved the AGT correspondence for $SU(2)$ gauge theories and a special set of primary fields on the CFT side from first principles in the CFT. We will also review their exposition on the AGT interpretation of the full $U(2)$ instanton partition function in CFT terms. This explicit proof, for the AGT correspondence in one context provides both tools to attempt more general proofs and validity to conjecture.

Until now, we have spoken only of a connection between $SU(2)$ gauge theories and CFTs with symmetry algebra Vir. We now discuss the connection between the full $U(2)$ theory and a 2D CFT. To do so, we interpret the $U(1)$ factor in the $U(2)$ instanton partition function from a CFT perspective. Following [19], on the CFT side we take the $U(1)$ factor to correspond to a copy of the Heisenberg algebra $\mathcal{H}$ with generators $\{a_n\}_{n\in\mathbb{Z}}$ and relations

$$[a_n, a_m] = \frac{n}{2}\delta_{n+m}, \tag{2.2.10}$$

in the symmetry algebra. Under this identification, the CFT that is AGT dual to $U(2)$ gauge theories on $\mathbb{C}^2$ has the combined symmetry algebra $\mathcal{A} = \mathcal{H} \otimes Vir$, with the usual generators $a_n \in \mathcal{H}, L_m \in Vir$ for $n, m \in \mathbb{Z}$, and additional relation

$$[a_n, L_m] = 0. \tag{2.2.11}$$

We now construct such a theory and its primary fields, using the Liouville vertex operators. As before, we parameterise the central charge in Liouville form

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b} = \epsilon_1 + \epsilon_2. \tag{2.2.12}$$

We now define *new* free fields using the Heisenberg modes (note that these are free fields built out of the positive or negative modes of the free boson from section 1.5.6)

$$\varphi_\pm := i \sum_{\pm n>0} \frac{a_n}{n} z^{-n}, \tag{2.2.13}$$

which we use to define the *free exponential*

$$\mathcal{V}_\alpha = e^{2(\alpha-Q)\varphi_-} e^{2\alpha\varphi_+}. \tag{2.2.14}$$

Then, the $\mathcal{A}$-primary fields V_α of conformal charge $\alpha \in \mathbb{C}$ are constructed as the product of a free and Liouville exponential as

$$V_\alpha := \mathcal{V}_\alpha \cdot V_\alpha^L, \tag{2.2.15}$$

where V_α^L are Vir primary fields of conformal dimension $\Delta = \alpha(Q-\alpha)$.

We now use these primary fields to build highest weight modules. We build an $\mathcal{A}$-highest weight Verma module with highest weight vector $|P\rangle$, where we recall that P is the momentum vector described in (1.5.73). The conformal dimension of this state is the eigenvalue of L_0, constructed using the Heisenberg modes as for the free boson, defined by $L_0 |P\rangle = (Q^2/4 - P)|P\rangle$. As usual we define the highest weight state by the relations $a_n |P\rangle = L_n |P\rangle = 0$ for $n > 0$, and the inner product $\langle P|P\rangle = 1$. The highest weight module is then built using the basis of descendant states

$$a_{-l_m} \dots a_{-l_1} L_{-k_n} \dots L_{-k_1} |P\rangle, \quad k_1 \geq \dots \geq k_n > 0, \;\; l_1 \geq \dots \geq l_m > 0. \tag{2.2.16}$$

As the labels $\{l_i\}_{i\in\mathbb{Z}_{>0}}$ and $\{k_j\}_{j\in\mathbb{Z}_{>0}}$ are sequences of weakly decreasing integers, it is natural to define this basis in terms of Young diagrams of length m and n. We define the pair of Young diagrams $\lambda^{(0)} = (l_1, l_2, \dots, l_m)$ and $\lambda^{(1)} = (k_1, k_2, \dots, k_n)$ and define the new notation involving Young diagrams for the descendant state defined above

$$\hat{a}_{(-\lambda^{(0)})} \hat{L}_{(-\lambda^{(1)})} |P\rangle := a_{-l_m} \dots a_{-l_1} L_{-k_n} \dots L_{-k_1} |P\rangle. \tag{2.2.17}$$

In this module one can construct a unique basis that reproduces the form of the instanton partition function for the conformal blocks, leading us to the following proposition.

Proposition 2.2.3.1. *([19, **Prop 2.1**]) For $\lambda = (\lambda^{(0)}, \lambda^{(1)})$, there exists a unique orthogonal basis $|P\rangle_\lambda$ for the Verma module of the form*

$$|P\rangle_\lambda = \sum_{|\mu|=|\lambda|} C_\lambda^{\mu^{(0)}\mu^{(1)}}(P) \hat{a}_{(-\mu^{(0)})} \hat{L}_{(-\mu^{(1)})} |P\rangle, \tag{2.2.18}$$

with Hermitian conjugate

$$_\lambda\langle P| = \sum_{|\mu|=|\lambda|} C_\lambda^{\mu^{(0)}\mu^{(1)}}(P) \langle P| \left(\hat{a}_{(-\mu^{(0)})}\right)^\dagger \left(\hat{L}_{(-\mu^{(1)})}\right)^\dagger, \tag{2.2.19}$$

such that the matrix elements for this basis have the following form

$$\frac{{}_{\mu}\langle P'|\, V_\alpha\, |P\rangle_\lambda}{{}_{\varnothing}\langle P|\, V_\alpha\, |P\rangle_\varnothing} = Z_{bif}(P', \mu; P, \lambda; \alpha). \tag{2.2.20}$$

The coefficients $C_\lambda^{\mu^{(0)}\mu^{(1)}}(P)$ in (2.2.18) and (2.2.19) are determined by the equation (2.2.20). This proposition was proved by finding the coefficients for $\lambda = (\varnothing, \lambda^{(1)})$ and $(\lambda^{(0)}, \varnothing)$, where $\varnothing$ is the empty Young diagram. In this case, it was found that the coefficients $C_\lambda^{\mu^{(0)}\mu^{(1)}}(P)$ involve special linear combinations of the Jack polynomials $J_\lambda^{(-1/b^2)}$. Then the well known properties of the Jack polynomials (see for example [46]) were used to describe an algorithm to generate $|P\rangle_\lambda$ for all λ.

This proposition provides an interpretation for the combinatorial form of the conformal blocks purely through CFT considerations in a special case. That is, it proves the AGT conjecture for this case. In doing so, we have more evidence for the validity of more complicated and generalized AGT correspondences.

Before finishing this section, we will provide the dictionary for the $SU(2)$ AGT conjecture on $\mathbb{C}^2$, from [2], which will be useful to refer back to throughout this thesis.

Gauge Theory	Conformal Field Theory
Deformation Parameters ϵ_1, ϵ_2	Liouville Parameters b, Q and Central Charge c $(\epsilon_1, \epsilon_2) = (b, 1/b)$ $Q = b + 1/b, c = 1 + 6Q^2$
Flavour symmetry $U(2) \times U(2)$	A 4-point correlation function on the sphere
Mass Parameter m associated to a flavour symmetry	Insertion of a Liouville Exponential $e^{2m\varphi}$
One $SU(2)$ gauge group with associated sewing parameter $q = e^{2\pi i\tau}$	A thin neck (or channel) with sewing parameter
Coulomb parameters a for the complex scalar field for $SU(2)$ gauge group	Primary field $e^{2\alpha\varphi}$ flowing in the channel $\alpha = Q/2 + a$
Z_{inst}	Conformal blocks

(2.2.21)

2.2.4 $SU(2)$ Gauge Theories and Minimal Models

In this section we will review an AGT correspondence between $SU(2)$ gauge theories and minimal model CFTs, first considered in [33]. All new results in chapters 3, 4, and 5 are obtained utilizing a similar flow of logic and arguments as here. As such, this section should be well understood by the reader before attempting to understand the results presented in this thesis.

This process will involve working on *both* sides of an AGT correspondence simultaneously. The flow of logic goes as follows: First, we make use of the AGT dictionary above to choose an Ω-deformation on the gauge side such that the dual CFT is a minimal model. Then, we fix the gauge theory through its mass and Coulomb parameters so that dual object to Z_{inst} is a 4-point conformal block of minimal model primary fields. To do so, we make use of the Coulomb gas formalism, reviewed in section 1.5.8, which describes the minimal model CFT with screening charges. After using the Coulomb gas screening charges to parameterize the conformal momentum of all primary fields in the conformal block, we return to the gauge theory where the corresponding mass and Coulomb parameters are now parameterized in terms of screening charges. By considering the usual definition of the instanton partition function for these gauge theories, we obtain an expression for the instanton partition function that contains non-physical poles and is ill-defined. Finally, we restrict the summation of the partition function so that we remove these poles, by imposing the *Burge conditions* on pairs (N-tuples in subsequent sections on $SU(N)$ gauge theories) of Young diagrams.

To assist with understanding of this flow of logic, we have created a flow chart depicting this process in figure 2.8. Note that in the flowchart, the calculation starts with an AGT identificiation, then moves *across* the duality and back again. We have also colour coded boxes depending on which side of the correspondence they occur on.

As described in section 1.5.5, the central charge c and conformal dimensions $\Delta_{r,s}$ of primary fields are constrained in the minimal models. When considering the dictionary above, we see that there should be special values of the deformation parameters ϵ_1 and ϵ_2 (corresponding to a minimal model central charge), Coulomb parameters a (corresponding to the conformal dimension of the primary field flowing in the channel), and mass parameters m_i, m_j' (corresponding to the conformal dimensions of the legs of the conformal block) of the gauge theory, for which the gauge theory has behaviour analogous to removing the null states in a minimal model.

This line of thinking was considered in [33], and led to the idea that the summations defining instanton partition functions of guage theories that are AGT dual to minimal models should be restricted to special sets of Young diagrams. In the case of AGT for $SU(2)$ gauge theories, these are *Burge pairs*, as we shall now review.

We match the notation in [33], and choose to write the central charge $c_{p,p'}$ of a minimal model using the parameter $a_{p,p'}$ as

$$c_{p,p'} = 1 - 6\left(a_{p,p'} - \frac{1}{a_{p,p'}}\right)^2, \quad a_{p,p'} = \sqrt{\frac{p'}{p}}. \tag{2.2.22}$$

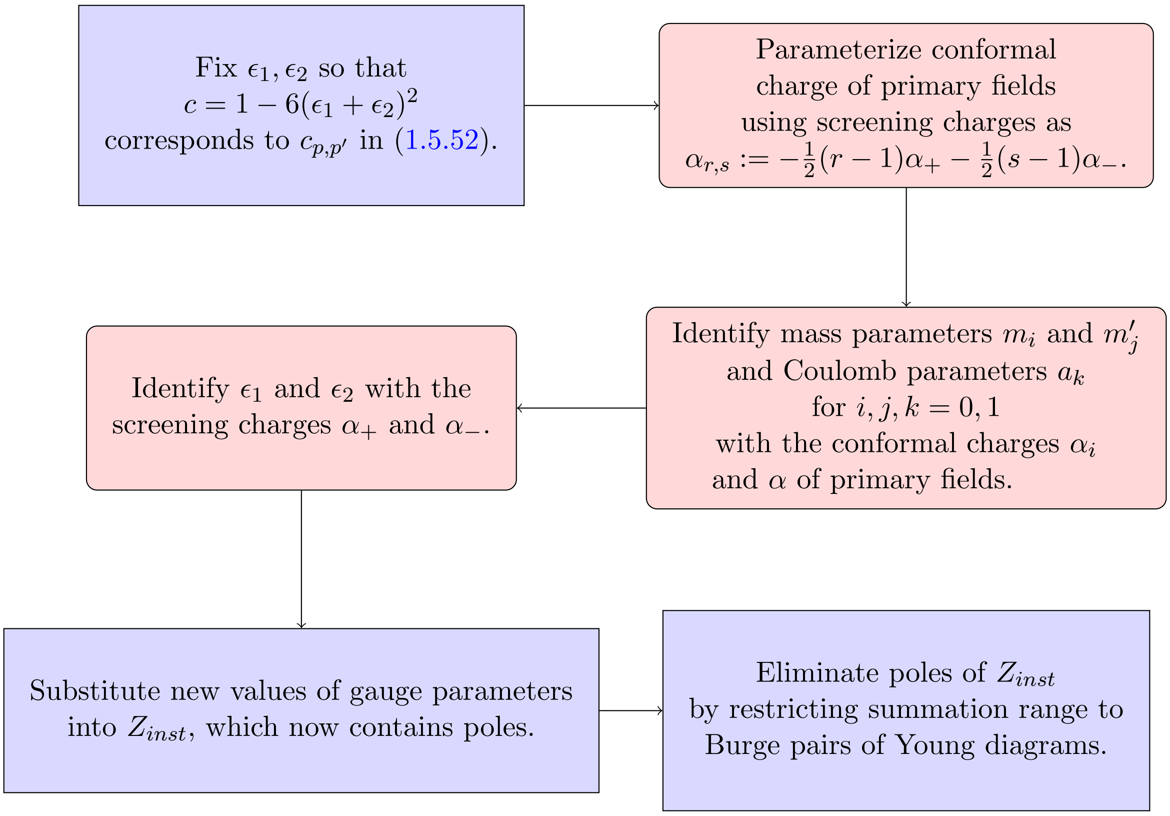


FIGURE 2.8: A flowchart representing the flow of logic in our subsequent minimal model calculations from AGT.

We then identify the parameter $a_{p,p'}$ with the screening charges $\{\alpha_+, \alpha_-\}$ of the Coulomb-gas formalism, from section 1.5.8, as

$$\alpha_+ = a_{p,p'}, \quad \alpha_- = -\frac{1}{a_{p,p'}}. \tag{2.2.23}$$

When considering AGT involving minimal models, we make a slight modification to the AGT identification for the deformation parameters, and identify them with the screening charges

$$\epsilon_1 < 0 < \epsilon_2, \ \ \epsilon_1 = \alpha_-, \ \ \epsilon_2 = \alpha_+. \tag{2.2.24}$$

Remark 2.2.4.1. Note that the identification of the deformation parameters here differs by a factor of i from the AGT dictionary described above. This factor comes from the form of the coupling constant b (see (1.5.72)).

We then move to the CFT side of the correspondence and parameterize *all* conformal charges parameterizing the vertex operator primary fields using the screening charges as in (1.5.86). Then we identify these minimal model charges with gauge theory parameters using the dictionary (2.2.21). In this case, we have two distinct objects that are parameterized in terms of the Coulomb-gas formalism, the charge of a highest weight $M(p, p'; 2)$ state and the charge of the vertex operator insertions. Remembering our discussion in section 1.5.8 we are led to identify the gauge theories that are dual to CFT Vir-minimal models as ones where the mass parameters m_i and m'_j and Coulomb parameters $\mathbf{a} = (a, -a)$ have the following expressions

$$m_i^{r,s}, m'^{r,s}_j = -\frac{r-1}{2}\alpha_+ - \frac{s-1}{2}\alpha_-, \quad a^{r,s} = -\frac{r}{2}\alpha_+ - \frac{s}{2}\alpha_-, \tag{2.2.25}$$

for $0 < r < p$ and $0 < s < p'$. The superscripts for these parameters denotes their parameterization in terms of screening charges and represents the fact that these are identified with minimal model CFT parameters. The instanton partition function is then expected to be AGT dual to the 4-point conformal block of 4 minimal model primary fields. Gauge theories whose parameters satisfy the above relations are central to this thesis, and show unique behaviour (as we will see in the sequel), leading us to the following definition.

Definition 2.2.4.2. An $\mathcal{N} = 2$ class S gauge theory with the Ω-deformation, whose deformation parameters ϵ_1 and ϵ_2 satisfy (2.2.24), and mass and Coulomb parameters satisfy (2.2.25), such that the theory is AGT dual to a minimal model CFT, is called a gauge theory under a *minimal model identification.*

The expressions above both correspond to the conformal charges of fields, so are expected to be symmetric, but we only consider the case of a correlation function where a field flows in the channel. Due to this we must respect the fusion rules for Vir minimal models, so that the the legs are arbitrary minimal model primary fields and the family that flows in the channel is the product of the fusion of minimal model primary fields.

We now consider the instanton partition function Z_{inst} of the gauge theory under these parameter identifications. As we will see, allowing Z_{inst} to be defined by an unrestricted summation over pairs of Young diagrams leads to terms whose denominator vanishes. Moreover, each term where this would be true has a factor in the numerator that also vanishes leading to indeterminate expressions of the form $\frac{0}{0}$.

To resolve this, we redefine the definition of Z_{inst} for gauge theories under a minimal model identification to be a *restricted* summation of pairs of Young diagrams, where the restriction is to pairs of Young diagrams for which there is no zero in the denominator. We will make this restriction explicit below in proposition 2.2.4.3, where it is written as conditions on the pairs of Young diagrams.

We recall the definition (2.1.14) of Z_{inst} for $U(2)$ gauge theories with $N_f = 4$ flavours of matter (2 fundamental and 2 anti-fundamental matter multiplets). Consider a summand in its series definition, corresponding to a pair of Young diagrams $\lambda = (\lambda^{(0)}, \lambda^{(1)})$, whose denominator term Z_{den} is the contribution of the vector multiplet (2.1.10) which we reproduce here

$$Z_{den}(a,m,m',\lambda) = Z_{vec}(a,\lambda) = \prod_{i,j=0,1}\left(\prod_{\square\in\lambda^{(i)}}\left(E(a_i-a_j,\lambda^{(i)},\lambda^{(j)},\square\right)\right.$$
$$\left.\prod_{\blacksquare\in\lambda^{(j)}}\left(\epsilon_1+\epsilon_2-E(a_j-a_i,\lambda^{(j)},\lambda^{(i)},\blacksquare)\right)\right). \quad (2.2.26)$$

We specialise to the case of gauge theories under a minimal model identification, so that the gauge theory parameters satisfy (2.2.25) and (2.2.24). Investigating the zeroes of Z_{den} leads one to the following proposition.

Proposition 2.2.4.3. *([33, **Prop 4.1**]) For fixed r, s, such that $0 < r < p$ and $0 < s < p'$, the function Z_{den} $(a^{r,s},\lambda) \neq 0$ if and only if*

$$\lambda_i^{(1)} \geq \lambda_{i+s-1}^{(0)} - r + 1, \quad \lambda_i^{(0)} \geq \lambda_{i+p'-s-1}^{(1)} - (p-r) + 1, \quad (2.2.27)$$

for any $i \leq \theta$ where $\theta = \min(l(\lambda^{(1)}), l(\lambda^{(0)} - s + 1))$.

In section 2.3.3 we will review a variation of this proposition, for $SU(N)$ gauge theories, which we will prove using similar methods. Then, in chapter 3, we will present a new (although superficially similar) result for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, again using a variation on this proof. Due to this, we will reproduce the full proof, edited to align better with our own proof for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ of proposition 3.4.2.1, for this proposition here, which will be beneficial to refer back to later.

Proof. We begin by considering the various factors present in $Z_{den}(a^{r,s},\lambda)$. Each factor corresponds to a choice of an ordered pair of Young diagrams $(\lambda^{(i)},\lambda^{(j)})$ for $i,j = 0,1$ and a box $\square \in \lambda^{(i)}$ or $\blacksquare \in \lambda^{(j)}$. We introduce new notation for these factors, which is a white or black box with superscript (i,j). Under this notation we have, for instance, the factor corresponding to the square $\blacksquare \in \lambda^{(1)}$ and the ordered pair $(\lambda^{(0)},\lambda^{(1)})$ notated as $\blacksquare^{(0,1)}$. Note that this superscript is *not* related to the lattice coordinates of the box in question.

We also introduce new terminology for boxes that cause a factor of Z_{den} to vanish. If we find a an condition equivalent to $\square^{(i,j)} = 0$ or $\blacksquare^{(i,j)} = 0$ for some i,j respectively, we say that condition is a *vanishing* or *zero condition.* On the other hand, if we can find some condition that guarantees $\square^{(i,j)} \neq 0$ or $\blacksquare^{(i,j)} \neq 0$, we say that we have found a *non-vanishing* or *non-zero* condition. In our case, we will find zero conditions in the form of inequalities on pairs of Young diagrams, and the corresponding non-zero conditions will involve reversing these inequalities.

Using this notation, we begin by considering the case where $i = j$, so that the factor $\square^{(i,i)}$ can be written as

$$E(a_i - a_i, \lambda^{(i)}, \lambda^{(i)}, \square) = E(0, \lambda^{(i)}, \lambda^{(i)}, \square) \tag{2.2.28}$$

$$= A^+_{\lambda^{(i)}}(\square)\alpha_+ - L_{\lambda^{(i)}}(\square)\alpha_-. \tag{2.2.29}$$

Since $\square \in \lambda^{(i)}$ and $\alpha_+ > 0$ and $\alpha_- < 0$, we have $A^+_{\lambda^{(i)}}(\square) > 0$ and $L_{\lambda^{(i)}}(\square) \geq 0$ and hence

$$\square^{(i,i)} = E(a_i - a_i, \lambda^{(i)}, \lambda^{(i)}, \square) > 0. \tag{2.2.30}$$

This means that the factor $\square^{(i,i)}$ can never vanish, and hence will never cause Z_{den} to vanish. Similarly, $\blacksquare^{(i,i)}$ gives us

$$\blacksquare^{(i,i)} = -A_{\lambda^{(i)}}(\blacksquare)\alpha_+ + L^+_{\lambda^{(i)}}(\blacksquare)\alpha_- < 0, \tag{2.2.31}$$

by the same argument as above, so that $\blacksquare^{(i,i)}$ also cannot vanish and hence cause Z_{den} to vanish.

We now check the remaining factors of Z_{den} for zeroes. Consider, for instance, the equation

$$\square^{(0,1)} = A^+_{\lambda^{(0)}}(\square)\alpha_+ - L_{\lambda^{(1)}}(\square)\alpha_- = 0, \tag{2.2.32}$$

which is of the form

$$C_+\alpha_+ + C_-\alpha_- = 0, \tag{2.2.33}$$

for $C_\pm \in \mathbb{Z}$. Equations of this form, where the screening charges have the form (2.2.23), have solutions of the following form

$$C_+ = cp, \quad C_- = cp', \tag{2.2.34}$$

where $c \in \mathbb{Z}$ is a constant that is to be determined for the specific equation in question[11]. We will encounter equations of this form frequently within this thesis, and the subsequent method of finding its solution will be invoked each time.

Substituting our minimal model parameterisation for a, and using $a_0 - a_1 = a - (-a) = 2a$, the vanishing $\square^{(0,1)} = 0$ is equivalent to the following two equations

$$A^+_{\lambda^{(0)}}(\square) - r = cp, \quad -L_{\lambda^{(1)}}(\square) - s = cp', \tag{2.2.35}$$

for some $\square \in \lambda^{(0)}$. Assuming this is true, we can then find the implications of the existence of such a box $\square = (i, j)$. We know that there is a box $\square'$ at the end of the i-th row of $\lambda^{(0)}$ with coordinates $\square' = (i, j + A_{\lambda^{(0)}}(\square))$ so that the $(j + A_{\lambda^{(0)}}(\square))$-th column has length at

[11] This notation may be confusing, as this c is *not* the central charge of the Vir representation. It is simply some integer constant which we will fix.

least i, giving us the following inequality

$$(\lambda^{(0)})^T_{j+A_{\lambda^{(0)}}(\square)} \geq i. \tag{2.2.36}$$

We then solve the condition $cp' + L_{\lambda^{(1)}}(\square) + s = cp' + (\lambda^{(1)})^T_j - i + s = 0$, to obtain

$$i = cp' + (\lambda^{(1)})^T_j + s, \tag{2.2.37}$$

and hence

$$(\lambda^{(0)})^T_{j+A_{\lambda^{(0)}}(\square)} \geq cp' + (\lambda^{(1)})^T_j + s. \tag{2.2.38}$$

Similarly, we can substitute $A_{\lambda^{(0)}}(\square) = r + cp - 1$ into the inequality above to obtain

$$(\lambda^{(0)})^T_{j+r+cp-1} \geq cp' + (\lambda^{(1)})^T_j + s. \tag{2.2.39}$$

This inequality is a property for the *Young diagram* $\lambda^{(0)}$ which is equivalent to the vanishing of the factor $\square^{(0,1)}$ for some $\square \in \lambda^{(0)}$. Thus, we can reverse this inequality to guarantee that no such $\square \in \lambda^{(0)}$ exists

$$(\lambda^{(0)})^T_{j+r+cp-1} < cp' + (\lambda^{(1)})^T_j + s \tag{2.2.40}$$

$$\Longleftrightarrow (\lambda^{(0)})^T_{j+r+cp-1} \leq cp' + (\lambda^{(1)})^T_j + s - 1, \tag{2.2.41}$$

which we write as

$$(\lambda^{(1)})^T_j \geq (\lambda^{(0)})^T_{j+r+cp-1} - cp' - s + 1. \tag{2.2.42}$$

We now determine the value of c that will eliminate all possible boxes. We do this in two steps, first we determine the largest set of possible values for c, then we choose the value in this set corresponding to the strongest inequality. We can fix the set of all possible c by noting that, as $\square \in \lambda^{(0)}$, we have the bound $A^+_{\lambda^{(0)}}(\square) = cp + r > 0$, and since $0 < r < p$ we have that $c \geq 0$. Furthermore, we already know that $c \in \mathbb{Z}$, so that the largest set of possible values for c is $c \in \mathbb{Z}_{\geq 0}$.

The weakly decreasing property of Young diagrams states that $(\lambda^{(0)})^T_{j+r-1} \geq (\lambda^{(0)})^T_{j+r+cp-1}$, since $-s + 1 \geq -cp' - s + 1$ for $c \geq 0$. Thus, the strongest bound is obtained for $c = 0$, allowing us to write our first non-zero condition as the inequality

$$(\lambda^{(1)})^T_j \geq (\lambda^{(0)})^T_{j+r-1} - s + 1. \tag{2.2.43}$$

Finally, we translate this condition into an equivalent one involving the original Young diagrams, not their transposes. To do this, we note that this inequality is equivalent to saying that the last box in the j-th row of $(\lambda^{(1)})^T$ is to the right of the box $(s-2)$ boxes from the end of the $(j+r-1)$-th row of $(\lambda^{(0)})^T$.

We can visualise this condition by first shifting the Young diagram $(\lambda^{(0)})^T$ up and to the left by $(r-1)$ and $(s-1)$ slots respectively on its lattice, so that the top left box now has coordinates $\square = (-r+1, -s+1)$. We then define a new diagram λ', which is composed of all the boxes from the shifted diagram $\square = (i,j) \in (\lambda^{(0)})^T$ that still lay in the positive x and y-quadrant of the lattice. These are defined by $i - r + 1 > 0$ and $j - s + 1 > 0$. Then the boxes in the i-th row for $i = 1, \dots, l(\lambda')$ in the new shifted diagram are to left of the end of the i-th row of $(\lambda^{(1)})^T$.

Said in a more geometric way, we can informally say that the outline $\mathcal{L}_{\lambda'}$ (from section 1.1) of λ' never lies to the right of the outline $\mathcal{L}_{\lambda^{(1)})^T}$ of $(\lambda^{(1)})^T$. More formally, we can say that if we start $\mathcal{L}_{\lambda^{(1)})^T}$ at a point $(a,b) \in \mathbb{Z}^2$ and $\mathcal{L}_{\lambda'}$ at $(a-1, b-1)$, the two paths are non-intersecting lattice paths[12].

We now take the transpose of this relationship of shifts between diagrams, and consider how these shifts act on the the non-transposed diagrams $\lambda^{(0)}$ and $\lambda^{(1)}$. The shifts are now $(r-1)$ slots to the left and $(s-1)$ slots up, but the relationship between outlines is still the same. Then by implementing this logic in reverse, we see that these inequalities on transposed diagrams are equivalent to the following inequalities for the original diagrams

$$\lambda_j^{(1)} \geq \lambda_{j+s-1}^{(0)} - r + 1, \tag{2.2.44}$$

where we note, that this process interchanges the roles of s and r in the inequalities. This is the first inequality claimed to be true in the proposition.

We now repeat the same argument as above for $\square^{(1,0)}$ to obtain

$$A^+_{\lambda^{(1)}}(\square) + r = dp, \quad -L_{\lambda^{(0)}}(\square) + s = dp', \tag{2.2.45}$$

where we define a new constant $d \in \mathbb{Z}$ to be determined in the role of c from before. By following our previous arguments, a square $\square \in \lambda^{(1)}$ satisfying these zero conditions is equivalent to the following inequality

$$(\lambda^{(0)})_j^T \geq (\lambda^{(1)})^T_{j+dp-1-r} 1 + s - dp'. \tag{2.2.46}$$

This time we have $A_{\lambda^{(1)}}(\square) = dp - r - 1 \geq 0$ so that $d \in \mathbb{Z}_{>0}$, and as before we must choose the smallest possible d to obtain the strongest bound, which gives the following *non-zero*

[12] As Burge multipartitions (defined by the Burge conditions) are a subset of cylindric partitions (see section 1.1), we note that this idea of non-intersecting lattice paths forms the basis of Gessel and Krattenthalers computation of the generating function of cylindric partitions [35].

condition where $d = 1$

$$(\lambda^{(0)})^T_j \geq (\lambda^{(1)})^T_{j+p-1-r} + 1 + s - p'. \tag{2.2.47}$$

We again follow our previous arguments to obtain the inequality

$$\lambda^{(0)}_j \geq \lambda^{(1)}_{j+p'-1-s} + 1 + r - p, \tag{2.2.48}$$

involving the non-transposed diagrams. This is the second inequality of the proposition.

The two inequalities obtained by considering the $\blacksquare^{(i,j)}$ factors contributing zeros are weaker than the bounds obtained here. For example $\blacksquare^{(0,1)} = 0$ is equivalent to the following equations for some $e \in \mathbb{Z}$

$$-A_{\lambda^{(1)}}(\blacksquare) - r = ep, \quad L^{++}_{\lambda^{(0)}}(\blacksquare) - s = ep', \tag{2.2.49}$$

from which it is apparent that we obtain a similar inequality to the second non-zero condition above where we replace s and r by $s+1$ and $r-1$ respectively. Thus this bound is weaker than the second non-zero condition. The case $\blacksquare^{(1,0)}$ follows an analogous argument with the first non-zero condition. $\square$

This proposition tells us that to remove the poles that appear in instanton partition function for $SU(2)$ gauge theories under a minimal model identification, we must restrict our sum to *Burge pairs*, and here lies our interest in Burge multipartitions. In this simple case, where we are considering models that are AGT dual to the well understood Vir minimal models, we can use the AGT correspondence to find interesting behaviour on the gauge side. In chapters 3, 4, and 5 we will work in the reverse direction, we will aim to use the gauge theory to study the less well understood $\widehat{\mathfrak{sl}}(n)$-WZW models and, more generally, CFTs with symmetry algebra $\mathcal{A}(N, n; p)$.

As a consequence of proposition 2.2.4.3, we define the *Burge reduced instanton partition function* $\mathcal{Z}$ for $U(2)$ gauge theories with two multiplets of each fundamental and anti-fundamental matter under a minimal model identification as follows

$$\begin{aligned}\mathcal{Z}(\mathbf{a},\mathbf{m},\mathbf{m}';q) &:= Z_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}';q)\\ &= \sum_{\lambda\in\mathcal{C}_{r-1,s-1}} \frac{Z_{fun}(\mathbf{a},\lambda;m_1)Z_{fun}(\mathbf{a},\lambda;m_2)Z_{afun}(\mathbf{a},\lambda;m_1')Z_{afun}(\mathbf{a},\lambda;m_2')}{Z_{vec}(\mathbf{a},\lambda)} q^{|\lambda|},\end{aligned} \tag{2.2.50}$$

which is identical to (2.1.14), except we are restricting the sum to $\mathcal{C}_{r-1,s-1}$ which we define to be the set of all Burge pairs (equivalently cylindric partitions). This process, where we redefine the definition of Z_{inst} to be a sum of Burge multipartitions instead of *all* pairs of Young diagrams, is interpreted as a gauge theoretic equivalent to removing null states with Vir minimal models.

2.2.5 The Instanton Generating Function and Minimal Model Characters

For gauge theories under a minimal model identification, we note that our redefinition of Z_{inst} implies we should similarly define the *Burge-reduced instanton generating function* $\widehat{X}^{SU(2)}$ using the usual instanton generating function (2.1.15). In this case, we go one step further than for the partition function and factorize the $U(1)$ (free boson) factor to obtain the $SU(2)$ Burge-reduced instanton generating function should be taken to be

$$\widehat{X}^{SU(2)}_{r,s}(q) = (q;q)_\infty \times X^{U(2)}_{r,s}(q) = (q;q)_\infty \times \sum_{\lambda \in \mathcal{C}_{r-1,s-1}} q^{|\lambda|}. \tag{2.2.51}$$

Since it is known that $X^{U(2)}_{r,s}(q)$ is equal to the character function $\chi^{p,p+1}_{r,s}(q)$ (1.5.141) for Vir-minimal models, up to a factor of $(q;q)^{-1}_\infty$ [40, 20], we see that the Burge-reduced instanton generating function is equal to $\chi^{p,p+1}_{r,s}(q)$. Thus, we see the AGT dual object to the instanton generating function is the character function of the dual 2D CFT. We will utilize this correspondence when testing our proposed generalization in chapter 4, and in chapter 5 this will form the basis for our new combinatorial identities for $\widehat{\mathfrak{sl}}(n)$-string functions and coset characters.

2.3 $SU(N)$ AGT-$\mathcal{W}$ with $N_f = 2N$ on $\mathbb{C}^2$

We now discuss a generalisation of AGT to a correspondence between 4D $\mathcal{N} = 2$ SUSY gauge theories on $\mathbb{C}^2$ with gauge group $G = SU(N)$ and 2D $\mathcal{A}_{N-1}$-Toda field theory, first suggested in [6]. This generalized correspondence is commonly referred to as AGT-$\mathcal{W}$ due to the $\mathcal{W}$-algebra symmetry of Toda (see section 1.5.14). The AGT correspondence discussed in the previous sections, between $\mathcal{N} = 2$ $SU(2)$ gauge theories on $\mathbb{C}^2$ and Liouville CFT, is the $N = 2$ case of this more general framework. In our case, we will restrict our focus to the class S theory $\mathcal{T}^{SU(N)}_{0,4}$ and the 4-point conformal block for $\mathcal{A}_{N-1}$-Toda field theory. We recall, that the theory $\mathcal{T}^{SU(N)}_{0,4}$ has a $U(N) \times U(N) \simeq U(1)^2 \times SU(N)^2$ flavour symmetry and has $N_f = 2N$ flavours of matter, composed of N flavours of fundamental and anti-fundamental matter.

2.3.1 $\mathcal{W}_N$-Chiral Blocks

The $\mathcal{A}_{N-1}$-Toda conformal field theory on the 2D side of the correspondence, has a $\mathcal{W}_N$ symmetry algebra, as explained in section 1.5.14. As such, we will begin by briefly discussing $\mathcal{W}_N$-chiral blocks, which are the conformal blocks of this theory.

The calculation of conformal (or chiral) blocks (and by extension, correlation functions) in $\mathcal{W}_N$ CFTs for $N > 2$ is more complicated than for Vir CFTs. The most important difference is that the general n-point correlation functions of $\mathcal{W}_N$ primary fields can *not* be determined in terms of only the 3-point functions of $\mathcal{W}_N$-primaries (see (1.5.142) for the defining property of $\mathcal{W}_N$-primary fields).

This means that a full computation of *all* correlation functions in $\mathcal{W}_N$ CFTs using the generic $\mathcal{W}_N$ chiral blocks (this is the equivalent of Vir conformal blocks for $\mathcal{W}_N$ algebras) is so far unknown. Thus, the $SU(N)$ AGT-$\mathcal{W}$ relation follows from the results of [108, 109] and restricts to a special class of n-point correlation functions which are determined in terms of the 3-point functions of primary fields. In our case we will only consider $n = 4$.

We recall our discussion from section 1.5.14, where we constructed primary fields in Toda as vertex operators. These fields are parameterized by a $\mathfrak{sl}(N)$ weight, which is their conformal charge. We will calculate the same conformal block in this generalized correspondence as in figure 2.6 for these primary fields. For Toda blocks, we will label the legs by the conformal charges parameterizing the conformal dimensions of the primary fields. We depict this in figure 2.9. In this case, the conformal dimension Δ of a Toda primary field is now calculated

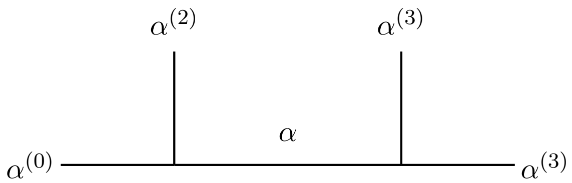


FIGURE 2.9: A 4-point $\mathcal{A}_{N-1}$-Toda chiral block, with primary fields labelled by their conformal charges.

by (1.5.152), which we reproduce here

$$\Delta(\alpha) = \frac{(\alpha, 2Q\rho - \alpha)}{2}, \tag{2.3.1}$$

where we recall that $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ is the Weyl vector. We allow the conformal charge of the *external* legs, labelled by $\alpha^{(0)}$ and $\alpha^{(3)}$, to be any $\mathfrak{sl}(N)$ weights, while the *internal* legs $\alpha^{(1)}, \alpha^{(2)}$ are restricted to have conformal charge of the form

$$\alpha^{(i)} \in \{k\bar{\Lambda}_1,\, k'\bar{\Lambda}_{N-1} | k, k' \in \mathbb{R}\}, \quad i = 1, 2, \tag{2.3.2}$$

where we recall that $\bar{\Lambda}_i$ denotes the i-th fundamental weight of $\mathfrak{sl}(N)$. In the language of the dual 4D gauge theory, we will take this special restriction to correspond to the factorisation of the flavour symmetry group. We also recall that the central charge of Toda is parameterised as

$$c = (N-1)(1 + N(N+1)(Q\rho, Q\rho)), \quad Q = b + \frac{1}{b}. \tag{2.3.3}$$

As for $SU(2)$ gauge theories, we make the AGT identification $\epsilon_1 = b$ and $\epsilon_2 = b^{-1}$ and $Q = \epsilon_1 + \epsilon_2$. We form the following conformal charges from the mass parameters of the gauge theory [6]

$$\alpha^{(0)} = \frac{Q\rho}{2} + \sum_{i=1}^{N-1} (m_{i+1} - m_i)\,\bar{\Lambda}_i, \quad \alpha^{(1)} = \sum_{i=1}^{N} m_i \bar{\Lambda}_1, \tag{2.3.4}$$

$$\alpha^{(3)} = \frac{Q\rho}{2} + \sum_{i=1}^{N-1} \left(m'_i - m'_{i+1}\right)\bar{\Lambda}_i, \quad \alpha^{(2)} = \sum_{i=1}^{N} m'_i \bar{\Lambda}_{N-1}, \tag{2.3.5}$$

and we can see here that $\alpha^{(1)}$ and $\alpha^{(2)}$ have been restricted to the special values described above in (2.3.2). We also note that form of $\alpha^{(0)}$ and $\alpha^{(3)}$ generalizes the formulas (2.2.2).

Remark 2.3.1.1. One can take the point of view of 4 dimensional gauge theory to explain this lack of symmetry in the vertex operator momenta. Each puncture on the Riemann sphere with four punctures in $\mathcal{T}^G_{0,4}$ corresponds to the insertion of a vertex operator in a correlation function via AGT. In the case of the class S theories we are considering, there is extra information associated to each puncture. In the case of $\mathcal{T}^G_{0,4}$, we have been considering theories with two types of punctures, one on each pair of pants, which correspond to the two different types of legs in the 4-point block.

Remark 2.3.1.2. We can also make a parameter matching argument. On the $SU(N)$ gauge side we have $2N$ mass parameters and $N-1$ Coulomb parameters for a total of $3N-1$ parameters defining our theory, whereas on the CFT side for an unrestricted 4-point correlation function, we have 5 weight labels with $N-1$ components. We can then see that in the case of $N=2$ we have $3N-1 = 5(N-1) = 5$ but that this is the only such case for $N \geq 2$. This suggests that not every parameter on the CFT side that is AGT dual to $SU(N)$ gauge theories for $N > 2$ can be free.

We then identify the Coulomb parameters $\mathbf{a} = (a_0, \dots, a_{N-1})$ with the charge of the internal channel as follows. We recall our discussion in section 1.3.3, and embed the dual $\mathfrak{h}^*$ of the Cartan algebra for $\mathfrak{su}(N)$ into $\mathbb{C}^N$ by fixing the basis $\{e_i\}$ and defining[13] $\alpha_i = e_i - e_{i+1}$. We define $e_0 = \frac{1}{N-1}\sum_{i=1}^{N} e_i$ and $\varepsilon_i = e_i - e_0$ and in this notation the internal momentum α is related to the Coulomb parameters by

$$2\alpha = Q\rho + \sum_{i=1}^{N} a_{i-1}\varepsilon_i. \tag{2.3.6}$$

To confirm that this is a natural generalization of $SU(2)$ AGT, we take the $N=2$ case of what this setup, and we see that $\alpha^{(0)}$ and $\alpha^{(3)}$ reduce to $\alpha_1 + \alpha_2$ and $\alpha_1 - \alpha_2$ (in the language of [2]) as expected, while $\alpha^{(1)}$, $\alpha^{(2)}$, and α also obviously reduce to their $SU(2)$ counterparts.

[13] Note that here α_i refers to the i-th root of $\mathfrak{su}(N)$ *not* the conformal charge of a vertex operator.

2.3.2 Gauge Theory Parameters and Stripping the $U(1)$ Factor

Recall that our $U(N)$ theory has $2N$ mass parameters and N Coulomb parameters collected in the vectors $\mathbf{m} = (m_0, \ldots, m_{N-1})$, $\mathbf{m}' = (m'_0, \ldots, m'_{N-1})$, and $\mathbf{a} = (a_0, \ldots, a_{N-1})$. In the $SU(N)$ gauge theory we additionally have the relation $\sum_{i=0}^{N-1} a_i = 0$.

As before, we strip the $U(N)$ instanton partition function of the $U(1)$ factor to obtain the $SU(N)$ partition function. In this case, the factorization is conjectured to be [6]

$$Z_{inst}^{U(N)} = (1-q)^{\frac{\left(\sum_{i=0}^{N-1} m_i\right)\left(\epsilon_1+\epsilon_2-\frac{1}{N}\sum_{i=0}^{N-1} m'_i\right)}{\epsilon_1\epsilon_2}} Z_{inst}^{SU(N)}. \tag{2.3.7}$$

Here we have written the $U(1)$ factor in terms of the gauge theory parameters, whereas in the AGT literature this factor is often written in terms of the Q and α parameters of the CFT side. We do so to match with our later notation and to emphasize this factor as naturally occurring in the gauge theory.

Remark 2.3.2.1. Having stripped the $U(1)$ factor and formed an AGT dictionary between $SU(N)$ theories on $\mathbb{C}^2$ and $\mathcal{A}_{N-1}$-Toda CFTs, it is natural to attempt to extend the approach of the proof for AGT involving $SU(2)$ gauge theories on $\mathbb{C}^2$ reviewed in section 2.2.3, to the case of $SU(N)$ gauge theories. This was done in [137], where a special basis for the highest weight modules of the corresponding Toda CFT symmetry algebra $\mathcal{H}\otimes\mathcal{W}_N$ was constructed using Jack polynomials. The matrix elements of this basis reproduced the bifundamental multiplet contribution for the corresponding $SU(N)$ gauge theory. As noted in section 2.1.4, all other multiplet contributions in the gauge theory can be obtained as special cases of the bifundamental multiplet contribution, so that the conformal blocks in the CFT can be identified with the instanton partition function in the gauge theory.

2.3.3 $SU(N)$ AGT and $\mathcal{W}_N$-Minimal Models

As we did for the AGT correspondence between $SU(2)$ gauge theories and CFTs with a Vir symmetry algebra, we now consider the case of gauge theories that are AGT dual to $\mathcal{W}_N$-minimal models, reviewing the results of [34]. We recall that the $\mathcal{W}_N$ minimal models are labelled by two coprime integers $p < p'$, and we denote the $\mathcal{W}_N$ minimal models by $M(p,p';N)$. We shall describe $M(p,p';N)$ using the Coulomb-gas formalism of section 1.5.15, analogously to our discussion of Liouville minimal models in section 2.2.4.

In the Coulomb-gas formalism, primary fields in $M(p,p';N)$ are labelled by their conformal charges

$$\alpha_{\mathbf{r},\mathbf{s}} = -\sum_{i=1}^{N-1}\left((r_i-1)\alpha_+ + (s_i-1)\alpha_-\right)\bar{\Lambda}_i, \tag{2.3.8}$$

where we recall that $\alpha_\pm$ are the screening charges (1.5.156). It is convenient to supplement our $\mathbf{r}$ and $\mathbf{s}$ parameters with two strictly positive integers r_0 and s_0, defined by

$$\sum_{i=0}^{N-1} r_i = p, \quad \sum_{i=0}^{N-1} s_i = p'. \tag{2.3.9}$$

Under this parameterization, the conformal charges of the internal legs are restricted to be of the following two forms

$$\alpha_{r_1,s_1} = ((r_1-1)\alpha_+ + (s_1-1)\alpha_-)\,\bar{\Lambda}_1, \;\; ((r_1-1)\alpha_+ + (s_1-1)\alpha_-)\,\bar{\Lambda}_{N-1}. \tag{2.3.10}$$

The momentum vector $P_{\mathbf{r},\mathbf{s}}$ (see (1.5.162)) for the field with charge $\alpha_{\mathbf{r},\mathbf{s}}$ in this theory is

$$P_{\mathbf{r},\mathbf{s}} = -\sum_{i=1}^{N-1} (r_i\alpha_+ + s_i\alpha_-)\,\bar{\Lambda}_i. \tag{2.3.11}$$

Using this we can write the conformal dimension $\Delta_{\mathbf{r},\mathbf{s}}$ of a primary field labelled by $\mathbf{r}$ and $\mathbf{s}$ as

$$\Delta_{\mathbf{r},\mathbf{s}} = \frac{1}{2}(P_{\mathbf{r},\mathbf{s}} + \alpha_0\rho)\cdot(P_{\mathbf{r},\mathbf{s}} - \alpha_0\rho) = \frac{1}{2}\left(P_{\mathbf{r},\mathbf{s}}^2 - \alpha_0^2\rho^2\right). \tag{2.3.12}$$

Remark 2.3.3.1. In this equation α_0 is denoting the background charge for the Coulomb-gas formalism, not the affine root of $\widehat{\mathfrak{sl}}(N)$. We have denoted both of these objects in this confusing way throughout this thesis, as both of these notations are uniform across the literature. When using the $SU(N)$ AGT dictionary for correspondences involving minimal models this notation will always reference the background charge. Importantly, the charge vectors for $\mathcal{A}_{N-1}$-Toda only involve $\mathfrak{sl}(N)$ weights, never affine roots.

To identify the Coulomb parameters $\mathbf{a} = (a_0, \ldots, a_{N-1})$ with minimal model conformal momenta, we parameterize

$$a_i = a_i^+\alpha_+ + a_i^-\alpha_-, \tag{2.3.13}$$

and identify

$$a_i^+ = \sum_{j=1}^{N-1}(\Lambda_j, h_{i+1})r_j, \quad a_i^- = \sum_{j=1}^{N-1}(\Lambda_j, h_{i+1})s_j, \tag{2.3.14}$$

where h_j for $j = 1, \ldots, N-1$ are the weight vectors of the fundamental representation of $\mathfrak{sl}(N)$ (defined in remark 1.5.14.1). Taking the fundamental representation to have highest weight $\bar{\Lambda}_{N-1}$, the weight vectors are $h_1 = \bar{\Lambda}_{N-1}$ and $h_j = \bar{\Lambda}_{N-1} - \sum_{k=j}^{N-1}\alpha_k$ for $j = 2, \ldots, N-1$, where we recall that the set $\{\alpha_i\}_{i=1,\ldots,N-1}$ is the set of simple roots for $\mathfrak{sl}(N)$. We then have $h_i - h_{i+1} = \alpha_i$ which fixes $(N-1)$ of the Coulomb parameters.

Remark 2.3.3.2. Note that the weight vectors h_i for the fundamental representation are the embedded weight vectors we defined for (2.3.6) above.

The final Coulomb parameter follows from the condition $\sum_{i=0}^{N-1} a_i = 0$, where we substitute (2.3.14), and the definition of the screening charges which we reproduce here

$$\alpha_+ = \sqrt{\frac{p'}{p}}, \quad \alpha_- = -\sqrt{\frac{p}{p'}}. \tag{2.3.15}$$

Solving these conditions, shows us that the difference of successive coulomb parameters is of the form

$$a_i^+ - a_{i+1}^+ = -r_i, \quad a_i^- - a_{i+1}^- = -s_i, \quad i = 1, \dots, N-1, \tag{2.3.16}$$

while the sum of *all* the differences must add to $p\alpha_+ + p'\alpha_-$, so that the final difference involves the strictly positive integers r_0 and s_0, defined in (2.3.9), and is of the form

$$a_{N-1}^+ - a_0^+ = -r_0, \quad a_{N-1}^- - a_0^- = -s_0. \tag{2.3.17}$$

This finishes our parameterization of $SU(N)$ gauge theories under a minimal model identification. We now use this identification and write the denominator of a term in $Z_{inst}^{U(N)}$ as

$$\begin{aligned} Z_{den}(\mathbf{a}, \mathbf{m}, \mathbf{m}') = \prod_{i,j=0}^{N-1} \prod_{\square \in \lambda^{(i)}} &\left(E(-\sum_{k=i}^{j-1}(r_i\alpha_+ + s_i\alpha_-), \lambda^{(i)}, \lambda^{(j)}, \square) \right) \\ &\prod_{\blacksquare \in \lambda^{(j)}} \left(\epsilon_1 + \epsilon_2 - E(\sum_{k=i}^{j-1}(r_i\alpha_+ + s_i\alpha_-), \lambda^{(j)}, \lambda^{(i)}, \blacksquare) \right). \end{aligned} \tag{2.3.18}$$

In this formula we let the i and j labels be defined modulo N so that if $j-1 < i$ we have

$$\sum_{k=i}^{j-1} = \sum_{k=i}^{N-1} + \sum_{k=0}^{j-1} \tag{2.3.19}$$

Remark 2.3.3.3. We note that for $(i,j) = (i, i+1)$ the corresponding factors in $Z_{den}^{U(N)}$ look analogous to the factors present in $Z_{den}^{U(2)}$.

We can now see that, as for $SU(2)$ gauge theories, using the usual definition for the instanton partition function of $SU(N)$ gauge theories under a minimal model identification will not work. Under this parameterization of the gauge theory parameters, Z_{den} vanishes for some N-tuples of Young diagrams. As before, we can create a well-defined partition function by modifying the definition of Z_{inst} to eliminate these poles. To do so, we define it as a sum over a restricted set of N-tuples of Young diagrams. This leads to the following proposition.

Proposition 2.3.3.4. *([34, **Prop 4.1**]) The denominator function Z_{den} in $Z_{inst}^{SU(N)}$ for gauge* *theories under a minimal model identification does not vanish for an N-tuple of Young diagrams $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$ if and only if λ satisfies the following Burge conditions*

$$\lambda_j^{(i)} \geq \lambda_{j+s_i-1}^{(i+1)} - r_i + 1, \quad i = 0, \ldots, N-1, \quad 1 \leq j \leq \min(l(\lambda^{(i)}), l(\lambda^{(i+1)}) - s_i + 1), \tag{2.3.20}$$

where the Young diagram superscript labels are defined modulo N, and r_i, s_i for $i = 0, \ldots, N-1$ are strictly positive integers that parameterize the conformal momentum of the conformal family that flow in the channel.

The proof of this proposition is analogous to the proof of proposition 2.2.4.3. In this case, the Burge conditions follow from eliminating the zeros in the products corresponding to the pairs of Young diagrams $(\lambda^{(i)}, \lambda^{(i+1)})$, and the proof for these pairs is exactly the same as for $SU(2)$.

After showing this, we are left to show that the zeros occurring in the factors corresponding to the other possible pairs of Young diagrams $(\lambda^{(i)}, \lambda^{(i+k)})$ for $k \neq \pm 1$ are also eliminated if the Burge conditions are satisfied. This amounts to showing that the $(\lambda^{(i)}, \lambda^{(i+k)})$ non-zero conditions are weaker bounds than the Burge conditions.

Proof. Consider the pair of Young diagrams $(\lambda^{(i)}, \lambda^{(i+k)})$, for which there is a corresponding factor in Z_{vec}

$$\begin{aligned}\square^{(i,i+k)} &:= \prod_{\square \in \lambda^{(i)}} \left(E(-\sum_{j=i}^{i+k-1} (r_j \alpha_+ + s_j \alpha_-), \lambda^{(i)}, \lambda^{(i+k)}, \square) \right) \\ &= \prod_{\square \in \lambda^{(i)}} \left(E(-\sum_{j=i}^{i+k-1} (r_j \alpha_+ + s_j \alpha_-), \lambda^{(i)}, \lambda^{(i+k)}, \square) \right).\end{aligned} \tag{2.3.21}$$

We note that this factor is of a similar form as the factor in the $SU(2)$ which we notated as $\square^{(0,1)}$. The differences being that there is a different parameter x in the building block function $E(x, \lambda^{(l)}, \lambda^{(m)}, \square)$ and the labels of the Young diagrams. Thus, we can follow the arguments from our previous proof of proposition 2.2.4.3 with different Young diagram labels and we replace our previous parameter with this new one, which amounts to replacing

$$r_i \mapsto \sum_{j=i}^{i+k-1} r_i, \quad s_i \mapsto \sum_{j=i}^{i+k-1} s_i. \tag{2.3.22}$$

After doing so, we see that to eliminate the zeros in this factor we find the same inequalities as before except with the parameters replaced as above. Thus, we find the non-zero conditions

$$\lambda_j^{(i)} \geq \lambda_{j+(\sum_{l=i}^{i+k-1} s_i)-1}^{(i+k)} - \left(\sum_{l=i}^{i+k-1} r_i \right) + 1. \tag{2.3.23}$$

We claim that the strongest bound one finds in this way is obtained using the non-zero conditions found for the pairs of Young diagrams of the form $(\lambda^{(i)}, \lambda^{(i+1)})$, which we will refer to as *sequential conditions*. To show this, we will compare the bounds obtained for pairs of diagrams of the form $(\lambda^{(i)}, \lambda^{(i+k)})$ for $k > 1$ using the sequential conditions against non-zero conditions on $(\lambda^{(i)}, \lambda^{(i+k)})$ obtained using the arguments from the proof of proposition 2.2.4.3.

We begin by considering what the sequential conditions obtained for the pairs of diagrams of the form $(\lambda^{(i)}, \lambda^{(i+1)})$ and $(\lambda^{(i+1)}, \lambda^{(i+2)})$ imply as a restriction on pairs of diagrams of the form $(\lambda^{(i)}, \lambda^{(i+1)})$. Explicitly, we have the two sequential conditions

$$\lambda_j^{(i)} \geq \lambda_{j+s_i-1}^{(i+1)} - r_i + 1, \tag{2.3.24}$$

and

$$\lambda_j^{(i+1)} \geq \lambda_{j+s_{i+1}-1}^{(i+2)} - r_{i+1} + 1, \tag{2.3.25}$$

which together imply that

$$\lambda_j^{(i)} \geq \lambda_{j+s_i-1+s_{i+1}-1}^{(i+2)} - r_i + 1 - r_{i+1} + 1 = \lambda_{j+s_i+s_{i+1}-2}^{(i+2)} - r_i - r_{i+1} + 2. \tag{2.3.26}$$

We repeat this argument, using induction, and see that the sequential conditions imply the following inequalities between pairs of diagrams of the form $(\lambda^{(i)}, \lambda^{(i+k)})$

$$\lambda_j^{(i)} \geq \lambda_{j+\sum_{l=i}^{i+k-1}(s_l-1)}^{(i+k)} - \sum_{l=i}^{i+k-1} (r_l - 1). \tag{2.3.27}$$

If we now compare (2.3.23) to (2.3.27), we see that the second is stronger for $k > 1$, so that the sequential conditions also imply all other non-zero conditions for this factor. Thus, if we restrict to N-tuples of Young diagrams that satisfy (2.3.27) we remove *all* zeros from this factor. We now repeat this process for the pair $(\lambda^{(i)}, \lambda^{(i+k)})$ and the corresponding factor

$$\blacksquare^{(i,i+k)} := \prod_{\blacksquare \in \lambda^{(i+k)}} \left(\epsilon_1 + \epsilon_2 - E\Big(\sum_{k=i}^{i+k-1} (r_i\alpha_+ + s_i\alpha_-), \lambda^{(i+k)}, \lambda^{(i)}, \blacksquare\Big) \right). \tag{2.3.28}$$

As before, this corresponds to a factor $\blacksquare^{(0,1)}$ in the $SU(2)$ proof with a changed parameter and a relabelling of the Young diagrams, so that, as above, we obtain the same inequalities between this pair of diagrams to remove all zeros in this factor. As explained in the $SU(2)$ proof, these inequalities are actually *weaker* than those to eliminate the zeroes coming from the other factor, so do not contribute any new information.

This is only half of the possible pairs of diagrams, and hence half the possible factors in Z_{den}. To repeat this argument for the other half, we substitute $i + k \mapsto i - k$ and consider the case of $k > 0$ under this substitution. We have that

$$a_i - a_{i-k} = \sum_{j=i-k}^{i-1} (r_j\alpha_+ + s_j\alpha_-) = p\alpha_+ + p'\alpha_- - \left(\sum_{j=0}^{i-k-1} + \sum_{j=i}^{N-1}\right)(r_j\alpha_+ + s_j\alpha_-), \quad (2.3.29)$$

where in the second equality we have used the definitions of r_0 and s_0

$$r_0 = p - \sum_{j=1}^{N-1} r_j, \quad s_0 = p - \sum_{j=1}^{N-1} s_j. \quad (2.3.30)$$

We can now repeat the same arguments as for the pairs of diagrams of the form $(\lambda^{(i)}, \lambda^{(i+k)})$ except we now compare the $SU(N)$ factors $\square^{(i,i-k)}$ and $\blacksquare^{(i,i-k)}$ with the $SU(2)$ factors we notated as $\square^{(1,0)}$ and $\blacksquare^{(1,0)}$ respectively. Thus, we obtain the non-zero conditions

$$\lambda_j^{(i)} \geq \lambda^{(i-k)}_{j+p'-(\sum_{j=0}^{i-k-1}+\sum_{j=i}^{N-1})(s_i)-1} - p + \left(\sum_{j=0}^{i-k-1} + \sum_{j=i}^{N-1}\right)(r_i) + 1, \quad (2.3.31)$$

which compare with the non-zero conditions implied on the pair $(\lambda^{(i)}, \lambda^{(i-k)})$ by the sequential conditions. In this case, we repeat the process described above. When using the inequality

$$\lambda_j^{(N-1)} \geq \lambda^{(0)}_{j+s_0-1} - r_0 + 1, \quad (2.3.32)$$

we substitute the definition of r_0 and s_0 to obtain

$$\lambda_j^{(N-1)} \geq \lambda^{(0)}_{j+p-(\sum_{j=1}^{N-1} s_j)-1} - p + \sum_{j=1}^{N-1} r_j + 1. \quad (2.3.33)$$

Then, using induction as above, we see that the sequential conditions imply the following inequalities between the pair $(\lambda^{(i)}, \lambda^{(i-k)})$

$$\lambda_j^{(i)} \geq \lambda^{(i-k)}_{j+p'-(\sum_{j=0}^{i-k-1}+\sum_{j=i}^{N-1})(s_i-1)} - p + \left(\sum_{j=0}^{i-k-1} + \sum_{j=i}^{N-1}\right)(r_i + 1). \quad (2.3.34)$$

As before we see that (2.3.34) is a set of stronger bounds than (2.3.31). This shows that we need only to restrict our summation to the Burge conditions implied by the adjacent factors to eliminate all zeros in the denominator of Z_{inst} and completes our proof of the proposition. □

Remark 2.3.3.5. In essence, relying on the $N = 2$ case and using the sequential conditions works as each sequential condition compares the i-th and $(i + s_i - 1)$-th row of sequential diagrams. When repeatedly applied, we can think of this as moving down the rows of diagrams repeatedly, where we move by s_i down and then one back up the diagram. This is in contrast to the bounds on non-adjacent pairs, which move by s_i down for each row but then back up one only once. Similar arguments follow for the r_i parameters, where we recall our discussion in section 2.2.4 and interpret these as shifting along columns. We will use this argument later when considering the case of AGT for gauge theories on the ALE space $\mathbb{C}^2/\mathbb{Z}_n$.

These Burge multipartitions, which are defined by the labels $\mathbf{r}$ and $\mathbf{s}$ of primary field defining the conformal family that flows in the channel, are weight (ζ, ξ)-Burge multipartitions where $\zeta = [s_0 - 1, \ldots, s_{N-1} - 1]$ and $\xi = [r_0 - 1, \ldots, r_{N-1} - 1]$ in the notation of (1.1.11). When discussing them in the context of physics, we will prefer the CFT notation involving $\mathbf{r}$ and $\mathbf{s}$ and notate as $\mathcal{C}^{\mathbf{r},\mathbf{s}}$ the set of Burge multipartitions of weight (ζ, ξ), when it is clear to do so. When discussing these inequalities in the context of representation theory and algebraic combinatorics, we will prefer to notate them using ζ and ξ. Here we make a distinction between the set of strictly positive integers $\mathbf{r}$ and $\mathbf{s}$, and Dynkin labels ξ and ζ. We will sometimes refer to this set as the set of $(\mathbf{r}, \mathbf{s})$-Burge partitions.

We now define the Burge-reduced instanton partition function $\mathcal{Z}^{U(N)}$ which is conjectured to be in AGT correspondence with the 4-point comformal blocks of $\mathcal{H} \oplus \mathcal{W}_N$ minimal models on the sphere

$$\mathcal{Z}^{U(N)}(\mathbf{a}, \mathbf{m}, \mathbf{m}'; q) := Z_{inst}^{U(N)}(\mathbf{a}, \mathbf{m}, \mathbf{m}'; q) = \sum_{\lambda \in \mathcal{C}^{\mathbf{r},\mathbf{s}}} q^{|\lambda|} \frac{Z_{bif}(\mathbf{a}, \mathbf{m}, \lambda) Z_{bif}(\mathbf{a}, \mathbf{m}', \lambda)}{Z_{vec}(\mathbf{a}, \lambda)}. \quad (2.3.35)$$

In [34], this was checked against the $\mathcal{W}_3$ minimal model $M(8, 9; 3)$. To do this, they fixed one of the Toda vertex operator insertions (in our notation, α_1) to be a $\mathcal{W}_3$ null state, which determines the labels of this vertex operator. A null-state is a descendant field of a degenerate primary field, and by representing the action of the algebra generators of the $\mathcal{W}_3$-symmetry as differential operators, one can show that the correlation function satisfies a third-order ordinary differential equation of Pochammer type, whose solutions are constructed using the ${}_3F_2$ hypergeometric function.

By expanding the instanton partition function up to order $|\lambda| = 4$ the authors confirmed that $\mathcal{Z}^{U(N)}$ agreed with ${}_3F_2$ term-by-term up to some overall q factor, after stripping the $U(1)$ factor from $\mathcal{Z}^{U(N)}$. This allowed a tangible prediction to be tested and provides validity to this process which we shall repeat for AGT on $\mathbb{C}^2/\mathbb{Z}_n$ in chapter 4. In our case, we will utilize the KZ differential equation from (1.5.117), and check the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ space against it.

We finish by again checking the gauge theory instanton generating function against the CFT character function. Having defined a Burge-reduced partition function we again define the

Burge reduced $SU(2)$ instanton generating function as

$$\widehat{X}^{U(N)}_{\mathbf{r},\mathbf{s}}(q) := (q;q)_\infty \times X^{U(N)}_{\mathbf{r},\mathbf{s}}(q) = (q;q)_\infty \times \sum_{\lambda \in \mathcal{C}^{\mathbf{r},\mathbf{s}}} q^{|\lambda|}, \tag{2.3.36}$$

which is known to be equal to the character function (1.5.144) for $\mathcal{W}_N$-minimal models [41].

Chapter 3

AGT for $\mathcal{N}=2$ $SU(N)$ Gauge Theories on $\mathbb{C}^2/\mathbb{Z}_n$

In this chapter we discuss a conjectured extension of the AGT correspondence to $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ with CFTs with symmetry algebra

$$\mathcal{A}(N,n;p) = \frac{\widehat{\mathfrak{gl}}(p-N)_N}{\widehat{\mathfrak{gl}}(p-N-n)_N} \cong \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{p+n-N}}, \tag{3.0.1}$$

first suggested in [10], which we call *coset AGT*. We begin by reviewing the material of [42], which calculated the instanton partition function for $U(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ as a sum of N-tuples of coloured Young diagrams. We then propose how to strip the $U(1)$ factor and obtain the $SU(N)$ instanton partition function.

We then propose an explicit dictionary between the mass, and deformation parameters of the gauge theory and the conformal charge of primary fields in the CFT. We use this dictionary to identify gauge theories that we conjecture are AGT dual to $\mathcal{A}(N,n;p)$-CFTs that involve minimal models (in a sense that we will make precise in section 3.4.1). We will then show that in these gauge theories, the usual definition of the partition function contains non-physical poles and is ill-defined. Finally, we eliminate these poles by imposing the Burge conditions, calculated in proposition 3.4.2.1, on our N-tuples of *coloured* Young diagrams to obtain a well-defined partition function for these theories. The material in this section is based on the content from [43] co-authored by the author.

3.1 Instanton Partition Function on $\mathbb{C}^2/\mathbb{Z}_n$

In this section, we discuss how the instanton partition function for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ is defined using the building blocks of the instanton partition function on $\mathbb{C}^2$ (2.1.12) and the action of $\mathbb{Z}_n$. We begin by discussing the action of $\mathbb{Z}_n$ on both $\mathbb{C}^2$ and the Coulomb parameters, and how this action affects the instantons. We will then see that the residues that contribute to instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ are now characterized by n-coloured N-tuples of Young diagrams.

3.1.1 Gauge Theories and Instantons on $\mathbb{C}^2/\mathbb{Z}_n$

We begin by considering the action of $\mathbb{Z}_n$ on the space $\mathbb{C}^2$, where we have an $\mathcal{N}=2$ $SU(N)$ class S gauge theory with $N_f=2N$ flavours of matter (with N flavours of both fundamental and anti-fundamental matter) as described in chapter 2. We again introduce the Ω-deformation parameters $\epsilon_1,\epsilon_2\in\mathbb{R}$ by the $U(1)^2$-torus action (see (2.1.4))

$$\mathbb{C}^2\to\mathbb{C}^2,\ \ (z_1,z_2)\mapsto(e^{i\epsilon_1}z_1,e^{i\epsilon_2}z_2), \tag{3.1.1}$$

and define the action of $\mathbb{Z}_n$ on $\mathbb{C}^2$ to be

$$\begin{aligned}\rho:\ \ &\mathbb{Z}_n\times\mathbb{C}^2\longrightarrow\mathbb{C}^2\\ &(\sigma,(z_1,z_2))\mapsto(e^{\frac{2\pi i}{n}\sigma}z_1,e^{\frac{-2\pi i}{n}\sigma}z_2),\quad \sigma\in\mathbb{Z}_n.\end{aligned} \tag{3.1.2}$$

Let $a_i\in\mathbb{C}$, $i=0,1,\dots,N-1$ denote the Coulomb parameters of the $U(N)$ gauge theory. These Coulomb parameters also transform under this $\mathbb{Z}_n$-action as

$$\mathbb{Z}_n:\ \ e^{ia_i}\mapsto e^{\frac{2\pi i}{n}\sigma_i}e^{ia_i}, \tag{3.1.3}$$

where each parameter a_i has an associated $\mathbb{Z}_n$-charge $\sigma_i\in\mathbb{Z}_n$ for $i=0,\dots,N-1$. Note that $\sigma=0$ is the trivial transformation. Below in section 3.1.2, we will use the invariance of the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ under the action of $\mathbb{Z}_n$ to restrict which possible representations can appear in this way, as conditions on the charges σ_i for $i=0,\dots,N-1$.

Remark 3.1.1.1. Mathematically, this involves associating irreducible $\mathbb{Z}_n$-representations to the a_i as follows. As the Coulomb parameters are complex numbers, we can consider the circles

$$C_j=\{e^{ir}e^{ia_j}|r\in\mathbb{R}\},\quad j=0,1,\dots,N-1, \tag{3.1.4}$$

where $\mathbb{Z}_n$ acts on C_j by the rotation $(\sigma,e^{ir}e^{ia_j})\mapsto e^{\frac{2\pi i\sigma}{n}}e^{ir}e^{ia_j}$. The action of $\mathbb{Z}_n$ on C_j extends to a one dimensional complex representation of $\mathbb{Z}_n$. Note that there are n-distinct one dimensional complex irreducible representations of the $\mathbb{Z}_n$ labelled by the elements

$0, 1, \ldots, n-1 \in \mathbb{Z}_n$. The charge $\sigma_i \in \mathbb{Z}_n$ then determines which irreducible representation of the $\mathbb{Z}_n$-action we associate to each a_i.

In [42], it was shown that the instanton partition function for these theories can again be computed using localization. The new information for this calculation on $\mathbb{C}^2/\mathbb{Z}_n$ is carried in the $\mathbb{Z}_n$-charges σ_i of the Coulomb parameters a_i for $i = -0, \ldots, N-1$, which appears as the charges used to *colour* the N-tuples of Young diagrams $\lambda^\sigma = (\lambda^{(0,\sigma_0)}, \ldots, \lambda^{(N-1,\sigma_{N-1})})$ describing the residues of the contour integrals. When discussing AGT for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ in the sequel, each Young diagram will be coloured in n-colours in this way. As such we will drop the charge superscript σ_i for Young diagrams so that when discussing N-tuples of Young diagrams $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$ we have

$$\lambda^{(i)} := \lambda^{(i,\sigma_i)}, \tag{3.1.5}$$

except when we explicitly state that we are considering an uncoloured Young diagram.

We will assume (without loss of generality) that the charges satisfy

$$\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{N-1}, \tag{3.1.6}$$

as this can always be achieved by rearranging the labels of the Coulomb parameters a_i.

We recall (see sections 1.6.2 and 1.6.3) that we can use the ADHM construction, and its generalization to $\mathbb{C}^2/\mathbb{Z}_n$, to construct the instantons on $\mathbb{C}^2/\mathbb{Z}_n$. These solutions describe a self-dual or anti-self dual connection on a $U(N)$-principal bundle, this is the *gauge bundle* (cf: section 2.1.4), over $\mathbb{C}^2/\mathbb{Z}_n$. Mathematically, these self-dual and anti-self dual connections for this bundle *are* the instantons [113, 112]. The instanton solutions are then classified by both the first and second Chern classes, c_1, c_2 of the gauge bundle with these connections, which is called the *instanton bundle*, over the space $\mathbb{C}^2/\mathbb{Z}_n$.

Following [112, 42], we can write down the Chern classes of the instanton bundle using the Chern classes of a tautological $\mathbb{Z}_n$-bundle $\mathcal{T} = \mathbb{C}^2 \times_{\mathbb{Z}_n} \mathbb{C}[\mathbb{Z}_n]$ on $\mathbb{C}^2/\mathbb{Z}_n$. The fibres of this bundle $\mathcal{T}$ are isomorphic to the regular representation $\mathbb{C}[\mathbb{Z}_n]$ of $\mathbb{Z}_n$. We then decompose $\mathbb{C}[\mathbb{Z}_n] = \oplus_{i=0,\ldots,n-1} R_i$, where $\{R_i\}_{i=0,\ldots,n-1}$ are the 1-dimensional irreducible representations of $\mathbb{Z}_n$, and the instanton bundles as follows

$$\mathcal{T} = \bigoplus_{i=0,\cdots,n-1} \mathcal{T}_i, \ \text{ where } \mathcal{T}_i := \mathbb{C}^2 \times_{\mathbb{Z}_n} R_i. \tag{3.1.7}$$

Recall from section 1.6.3 that when constructing the instantons on $\mathbb{C}^2/\mathbb{Z}_n$ we have two vector spaces V and W, that are also $\mathbb{Z}_n$-modules. These vector spaces decompose under the $\mathbb{Z}_n$-action as

$$V=\bigoplus_{i=0}^{n-1} V_i\otimes R_i, \quad W=\bigoplus_{i=0}^{n-1} W_i\otimes R_i, \tag{3.1.8}$$

with dimensions $k_i=\dim(V_i)$ and $N_i=\dim(W_i)$ for $i=0,\dots,n-1$.

The first and second Chern classes c_1 and c_2 of the gauge bundle are defined in terms of the Chern classes of the line bundles $\mathcal{T}_i$, the dimensions of the vector spaces V_i and W_i under the $\mathbb{Z}_n$-decomposition, and the instanton data on $\mathbb{C}^2/\mathbb{Z}_n$ as

$$c_1=\sum_{i=0}^{n-1}\mathfrak{c}_i c_1(\mathcal{T}_i), \tag{3.1.9}$$

$$c_2=\sum_{i=0}^{n-1}\mathfrak{c}_i c_2(\mathcal{T}_i)+\frac{k}{|\mathbb{Z}_n|}, \tag{3.1.10}$$

where $k=\sum_{i=0}^{n-1}k_i$, and $c_1(\mathcal{T}_i)$ and $c_2(\mathcal{T}_i)$ are the first and second Chern classes of $\mathcal{T}_i$ for $i=0,\dots,n-1$ respectively. The coefficients $\mathfrak{c}_i$ are functions of k_i and N_i, and are given by

$$\mathfrak{c}_i=N_i-2k_i+k_{i+1}+k_{i-1}, \quad \text{where } k_{i+n}=k_i, \quad i=0,\dots,n-1. \tag{3.1.11}$$

Note that the 1-dimensional irreducible representation corresponding to $i=0$ is the trivial representation, as such the Chern class vanishes, $c_1(\mathcal{T}_0)=0$.

Remark 3.1.1.2. Recall the $\widehat{\mathfrak{sl}}(n)$ Cartan matrix A, defined by (1.3.40), which we reproduce here (remembering that we label our rows and columns by $i,j=0,1,\dots,n-1$)

$$A_{ij}=\begin{cases}2, & i=j,\\ -1, & i\equiv j\pm 1 \mod n,\\ 0, & \text{else.}\end{cases} \tag{3.1.12}$$

By forming the vectors $\mathbf{k}=(k_0,\dots,k_{n-1})$, $\mathbf{N}=(N_0,\dots,N_{n-1})$, $\mathfrak{c}=(\mathfrak{c}_0,\dots,\mathfrak{c}_{n-1})$, we can rephrase the n equations (3.1.11) into one matrix equation as

$$A\cdot\mathbf{k}=(\mathbf{N}-\mathfrak{c}). \tag{3.1.13}$$

Then by defining $\delta k_i=k_i-k_0$ we have the following $(n-1)$ equations

$$\mathfrak{c}_i=N_i-2\delta k_i+\delta k_{i+1}+\delta k_{i-1}, \quad i=2,\dots,n-2, \tag{3.1.14}$$

together with

$$\mathfrak{c}_1 = N_1 - 2\delta k_1 + \delta k_2, \tag{3.1.15}$$

$$\mathfrak{c}_{n-1} = N_{n-1} - 2\delta k_{n-1} + \delta k_{n-2}. \tag{3.1.16}$$

In matrix form we can write this system of equations as

$$\bar{A} \cdot \delta \mathbf{k} = (\bar{\mathbf{N}} - \bar{\mathfrak{c}}), \tag{3.1.17}$$

where $\bar{A}$ is the $\mathfrak{sl}(n)$ Cartan matrix, $\bar{\mathbf{N}} = (N_1, \ldots, N_{n-1})$, and $\bar{\mathfrak{c}} = (\mathfrak{c}_1, \ldots, \mathfrak{c}_{n-1})$. The utility of this second form for this system of equations lies in the fact that the finite Cartan matrix $\bar{A}$ is invertible.

By rephrasing the equations in this manner using A we see that classification of the instanton solutions through the Chern classes uses the structure of $\widehat{\mathfrak{sl}}(n)$. As we will show in chapter 4, we can finetune the Ω-deformation used to calculate the instanton partition function for these theories on $\mathbb{C}^2/\mathbb{Z}_n$ to obtain the characters and conformal blocks of $\widehat{\mathfrak{sl}}(n)$-WZW models. It then seems to be no coincidence that the Chern classes are of this form, when considering that the primary fields of $\widehat{\mathfrak{sl}}(n)$-WZW models form integrable highest weight $\widehat{\mathfrak{sl}}(n)$-modules.

We will take the Chern classes c_1 and c_2 of the gauge bundle to be fixed parameters defining the instantons. Through equations (3.1.11) this defines a set of possible solutions for $\mathbf{N}$ and $\mathbf{k}$, which we take to define the possible instanton solutions for our gauge theory. In this sense, we will ignore the Chern classes and parameterize our theories using two sets of integers $\mathbf{N}$ and $\mathbf{k}$.

Example 3.1.1.3. ([32]) Let $N = 2$ be fixed and $c_1 = 0$. In this case we have the matrix equation

$$\bar{A} \cdot \delta \mathbf{k} = \bar{\mathbf{N}}, \tag{3.1.18}$$

which we can invert using the inverse $\mathfrak{sl}(n)$ Cartan matrix

$$\bar{A}^{-1}_{ij} = \min(i, j) - \frac{ij}{n}, \tag{3.1.19}$$

to solve for the differences δk_i for $i=1,\ldots,n-1$. For $n=2,3,4$ we have the following table of integer solutions to this matrix equation:

$n=2$	$\{\mathbf{N}\}$	$\{\delta\mathbf{k}\}$
	$(2,0)$	$(0,0)$
	$(0,2)$	$(1,1)$
$n=3$	$\{\mathbf{N}\}$	$\{\delta\mathbf{k}\}$
	$(2,0,0)$	$(0,0,0)$
	$(0,1,1)$	$(0,1,1)$
$n=4$	$\{\mathbf{N}\}$	$\{\delta\mathbf{k}\}$
	$(2,0,0,0)$	$(0,0,0,0)$
	$(0,1,0,1)$	$(0,1,1,1)$
	$(0,0,2,0)$	$(0,1,2,1)$

(3.1.20)

3.1.2 Defining the Partition Function

Here we will recall the form of $SU(N)$ instanton partition function for $\mathbb{C}^2/\mathbb{Z}_n$ first derived in [42]. The instanton partition function is defined as a series over the set of N-tuples of coloured Young diagrams with fixed charges $\sigma=(\sigma_0,\ldots,\sigma_{N-1})\in(\mathbb{Z}_n)^N$ and colour data $\delta\mathbf{k}=(\delta k_0,\ldots,\delta k_{n-1})\in\mathbb{Z}^{n-1}$, denoted by $\mathcal{P}^{\sigma}_{\delta\mathbf{k}}$. The charges in the vector σ are the $\mathbb{Z}_n$-charges associated to the Coulomb parameters $\mathbf{a}=(a_0,\ldots,a_{N-1})$ and the differences $\delta\mathbf{k}=(\delta k_1,\ldots,\delta k_{n-1})$ are calculated using the dimensions of the vector spaces V_i used when constructing the instantons as

$$\delta k_i = k_i - k_0, \quad k_i=\dim(V_i). \tag{3.1.21}$$

To calculate the terms in this series, we will start with the terms from the partition function on $\mathbb{C}^2$ and project out all factors that are invariant under the action of $\mathbb{Z}_n$ (we will make this process explicit in example 3.1.2.2). Due to this, the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ is constructed using the building block $E(x,\lambda^{(l)},\lambda^{(m)},\square)$ function used to construct the partition function on $\mathbb{C}^2$ (2.1.5), which we reproduce below. We recall that E depends on a complex parameter x, a pair of Young diagrams $(\lambda^{(l)},\lambda^{(m)})$, and a box $\square=(i,j)$ in either $\lambda^{(l)}$ or $\lambda^{(m)}$, and is defined as

$$E(x,\lambda^{(l)},\lambda^{(m)},\square)=x-\epsilon_1 L_{\lambda^{(m)}}(\square)+\epsilon_2 A^{+}_{\lambda^{(l)}}(\square), \tag{3.1.22}$$

where $L_\lambda(\square)$ and $A_\lambda(\square)$ are the leg and arm length of the box $\square$ in the Young diagram λ respectively (see (1.1.2) and (1.1.3)). Using this, we can define Z^*_{bif}, the contribution of the bifundamental multiplet on $\mathbb{C}^2/\mathbb{Z}_n$, that depends on two vectors of N complex numbers

$\mathbf{a} = (a_0, a_1, \ldots, a_{N-1})$ and $\mathbf{b} = (b_0, b_1, \ldots, b_{N-1}) \in \mathbb{C}^{N-1}$, and two N-tuples of Young diagrams $\lambda_{(1)} = (\lambda_{(1)}^{(0)}, \ldots, \lambda_{(1)}^{(N-1)})$ and $\lambda_{(2)} = (\lambda_{(2)}^{(0)}, \ldots, \lambda_{(2)}^{(N-1)})$

$$Z^*_{bif}(\mathbf{a}, \lambda_{(1)}; \mathbf{b}, \lambda_{(2)}) = \prod_{i,j=0}^{N-1} \prod^{*}_{\square \in \lambda_{(1)}^{(i)}} E\left(a_i - b_j, \lambda_{(1)}^{(i)}, \lambda_{(2)}^{(j)}, \square\right)$$
$$\prod^{*}_{\blacksquare \in \lambda_{(2)}^{(j)}} \left(\epsilon_1 + \epsilon_2 - E\left(b_j - a_i, \lambda_{(2)}^{(j)}, \lambda_{(1)}^{(i)}, \blacksquare\right)\right), \tag{3.1.23}$$

where the asterisks mean to take a product over *only* the $\mathbb{Z}_n$-invariant factors modulo[1] 2π. Below we will explicitly describe what this means by calculating which boxes $\square \in \lambda_{(1)}^{(i)}$ and $\blacksquare \in \lambda_{(2)}^{(j)}$ for $i, j = 0, \ldots, N-1$ contribute as factors for Z^*_{bif}. Note that Z^*_{bif} is composed of the $\mathbb{Z}_n$-invariant factors of Z_{bif}, the contribution of the bifundamental multiplet on $\mathbb{C}^2$, defined in (2.1.12).

We remind the reader that the colours of the two boxes $\square \in \lambda_{(1)}^{(i)}$ and $\blacksquare \in \lambda_{(2)}^{(j)}$ are *not* related to the colourings associated to $\lambda_{(1)}^{(i)}$ and $\lambda_{(2)}^{(j)}$, which we still denote by the integers $0, 1, \ldots, n-1$. In this case, we are denoting the boxes in these products to distinguish the boxes $\square$ which are from diagrams $\lambda_{(1)}^{(i)}$ in the first N-tuple of Young diagrams and the boxes $\blacksquare$ which are from the diagrams $\lambda_{(2)}^{(j)}$ in the second.

Remark 3.1.2.1. In [42], the form of (3.1.23) was again calculated using supersymmetric localization. The process to do so is similar to the calculation for $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2$, as the fixed points of the instanton moduli space for $\mathbb{C}^2/\mathbb{Z}_n$ are also the fixed points for $\mathbb{C}^2$. The difference between these two calculations is that the contribution of these fixed points for the gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ only comes from boxes whose contribution is invariant under the action of $\mathbb{Z}_n$ on this gauge theory.

We will now calculate which boxes in the N-tuples of Young diagrams $\lambda_{(1)}$ and $\lambda_{(2)}$ correspond to $\mathbb{Z}_n$-invariant factors. Under a $\mathbb{Z}_n$-transformation, the gauge theory parameter transformations are generated by

$$\epsilon_1 \mapsto \epsilon_1 + \frac{2\pi}{n}, \quad \epsilon_2 \mapsto \epsilon_2 - \frac{2\pi}{n}, \quad a_i \mapsto a_i + \frac{2\pi}{n}\sigma_i, \quad b_j \mapsto b_j + \frac{2\pi}{n}\sigma'_j. \tag{3.1.24}$$

We recall our notation from the proof of proposition 2.2.4.3, where for fixed $i, j = 0, 1, \ldots, N-1$ and a box $\square$ in either $\lambda_{(1)}^{(i)}$ or $\lambda_{(2)}^{(j)}$, we denote the factor

$$E\left(a_i - b_j, \lambda_{(1)}^{(i)}, \lambda_{(2)}^{(j)}, \square\right), \tag{3.1.25}$$

[1] Here and in the sequel, when we say modulo 2π we mean that two numbers are equal up to the addition of a term of the form $2\pi m$ for some $m \in \mathbb{Z}$.

in (3.1.23) by $\square^{(i,j)}$. Similarly, we denote

$$\epsilon_1+\epsilon_2-E\left(b_j-a_i,\lambda^{(j)}_{(2)},\lambda^{(i)}_{(1)},\blacksquare\right), \tag{3.1.26}$$

by $\blacksquare^{(i,j)}$. To show $\square^{(i,j)}$ is invariant modulo 2π under a $\mathbb{Z}_n$-transformation it is sufficient to show that it is invariant under (3.1.24). To do so, we apply (3.1.24) to the parameters and then substitute these transformed parameters into (3.1.23). We denote by $\square^{(i,j)}_{\mathbb{Z}_n}$ and $\blacksquare^{(i,j)}_{\mathbb{Z}_n}$ these factors after a $\mathbb{Z}_n$-transformation. In the following example, we follow these steps explicitly for one such factor and derive the necessary conditions on a box $\square=(i,j)$, in terms of the Coulomb parameter charges σ_0 and σ_0', to correspond to a factor invariant under the action of $\mathbb{Z}_n$ modulo 2π.

Example 3.1.2.2. Consider the $\square^{(0,0)}$ factor

$$\square^{(0,0)}=E\left(b_0-a_0,\lambda^{(0)}_{(1)},\lambda^{(0)}_{(2)},\square\right)=b_0-a_0-\epsilon_1L_{\lambda^{(0)}_{(2)}}(\square)+\epsilon_2A^+_{\lambda^{(0)}_{(1)}}(\square),\quad \square\in\lambda^{(0)}_{(1)}. \tag{3.1.27}$$

After applying the $\mathbb{Z}_n$-transformation and substituting in the transformed gauge parameters we obtain

$$\square^{(0,0)}_{\mathbb{Z}_n}=b_0+\frac{2\pi}{n}\sigma_0'-a_i-\frac{2\pi}{n}\sigma_0-(\epsilon_1+\frac{2\pi}{n})L_{\lambda^{(0)}_{(2)}}(\square)+(\epsilon_2-\frac{2\pi}{n})A^+_{\lambda^{(0)}_{(1)}}(\square),\quad \square\in\lambda^{(0)}_{(1)}. \tag{3.1.28}$$

The asterisk product in (3.1.23) means to take only terms satisfying

$$\square^{(0,0)}\equiv\square^{(0,0)}_{\mathbb{Z}_n}\mod 2\pi, \tag{3.1.29}$$

which we have shown in (3.1.28) is equivalent to the condition that

$$\sigma_0'-\sigma_0-L_{\lambda^{(0)}_{(2)}}(\square)-A^+_{\lambda^{(0)}_{(1)}}(\square)\equiv 0\mod n. \tag{3.1.30}$$

The computation in example 3.1.2.2 can be generalized to all $i,j=0,\ldots,N-1$ and boxes $\square\in\lambda^{(i)}_{(1)}$ and $\blacksquare\in\lambda^{(j)}_{(2)}$. After doing so, we obtain the following equations, which we refer to as the *orbifold conditions*, that determine which boxes correspond to $\mathbb{Z}_n$-invariant factors:

$$\sigma_j'-\sigma_i-L_{\lambda^{(j)}_{(2)}}(\square)-A^+_{\lambda^{(i)}_{(1)}}(\square)\equiv 0\mod n,\quad \square\in\lambda^{(i)}_{(1)}, \tag{3.1.31}$$

$$\sigma_i-\sigma_j'-L_{\lambda^{(i)}_{(1)}}(\blacksquare)-A^+_{\lambda^{(j)}_{(2)}}(\blacksquare)\equiv 0\mod n,\quad \blacksquare\in\lambda^{(j)}_{(2)}. \tag{3.1.32}$$

We use Z^*_{bif} to define Z^*_{vec}, the inverse of the vector multiplet contribution on $\mathbb{C}^2/\mathbb{Z}_n$, which depends on a single N-tuple of Young diagrams $\lambda=(\lambda^{(0)},\ldots,\lambda^{(N-1)})$

$$Z^*_{vec}(\mathbf{a},\lambda)=Z^*_{bif}(\mathbf{a},\lambda;\mathbf{a},\lambda). \tag{3.1.33}$$

Using this we can define the instanton partition function $Z_{inst}^{U(N)}$ for $\mathcal{N}=2$ $U(N)$ class S gauge theories with N flavours of fundamental and anti-fundamental matter on $\mathbb{C}^2/\mathbb{Z}_n$. The partition function is defined using the mass parameters $\mathbf{m}=(m_0,\dots,m_{N-1})\in\mathbb{C}^N$ and $\mathbf{m}'=(m'_0,\dots,m'_{N-1})\in\mathbb{C}^N$ (defined in section 2.1.1) and Coulomb parameters $\mathbf{a}=(a_0,\dots,\ a_{N-1})\in\mathbb{C}^N$. We remind again that the $SU(N)$ factor of this $U(N)$ partition function is conjectured to be equal to the conformal block of a CFT with symmetry algebra $\mathcal{A}(N,n;p)$ under a suitable identification of parameters. The definition of $Z_{inst}^{U(N)}$ on $\mathbb{C}^2/\mathbb{Z}_n$ will involve *empty* coloured N-tuples of Young diagrams $\varnothing^{\mathbf{b}}$, where we will consider N-tuples of Young diagrams which have no boxes. As they are coloured they will still have charges $\mathbf{b}=(b_0,\dots,b_{N-1})\in(\mathbb{Z}_n)^N$, which are referred to as the $\mathbb{Z}_n$-*boundary charges.*

Theorem 3.1.2.3. ([32]) Using equivariant localization, the $U(N)$ instanton partition function $Z_{inst}^{U(N)}$ for the class S theory with N fundamental and N anti-fundamental matter multiplets on the ALE space $\mathbb{C}^2/\mathbb{Z}_n$ can be written as the following sum over N-tuples of coloured Young diagrams λ with charge vector $\sigma=(\sigma_0,\dots,\sigma_{N-1})$ and colour data $\delta\mathbf{k}=(\delta k_1,\dots,\delta k_{n-1})$

$$Z_{\sigma;\delta\mathbf{k}}^{U(N)}(\mathbf{a},\mathbf{m},\mathbf{m}',\mathbf{b},\mathbf{b}';q)=\sum_{\lambda\in\mathcal{P}^{\sigma}_{\delta\mathbf{k}}}\frac{Z^*_{bif}(\mathbf{m},\varnothing^{\mathbf{b}};\mathbf{a},\lambda)Z^*_{bif}(\mathbf{a},\lambda;-\mathbf{m}',\varnothing^{\mathbf{b}'})}{Z^*_{vec}(\mathbf{a},\lambda)}q^{\frac{1}{n}|\lambda|},\tag{3.1.34}$$

where $\mathcal{P}^{\sigma}_{\delta\mathbf{k}}$ is the set of all N-tuples of Young diagrams with charges $\sigma=(\sigma_0,\dots,\sigma_{N-1})$ and colour differences $\delta\mathbf{k}=(\delta k_1,\dots,\delta k_{n-1})$, $\mathbf{a}=(a_0,\dots,a_{N-1})\in\mathbb{C}^N$ are the Coulomb parameters, $\mathbf{m}=(m_0,\dots,m_{N-1})\in\mathbb{C}^N$ and $\mathbf{m}'=(m'_0,\dots,m'_{N-1})\in\mathbb{C}^N$ are the mass parameters for the fundamental and anti-fundamental multiplets associated to a $U(N)\times U(N)$ flavour symmetry, and $\mathbf{b}=(b_0,\dots,b_{N-1})\in(\mathbb{Z}_n)^N$ and $\mathbf{b}'=(b'_0,\dots,b'_{N-1})\in(\mathbb{Z}_n)^N$ are the $\mathbb{Z}_n$-boundary charges, which are assigned to empty coloured Young diagrams $\varnothing^{\mathbf{b}}$ and $\varnothing^{\mathbf{b}'}$.

We assume the $\mathbb{Z}_n$-boundary charges are ordered as

$$b_0\geq b_1\geq\cdots\geq b_{N-1},\quad b'_0\geq b'_1\geq\cdots\geq b'_{N-1}.\tag{3.1.35}$$

Remark 3.1.2.4. Note that the instanton partition function for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ has explicit dependence on the $\mathbb{Z}_n$-boundary charges $\mathbf{b}$ and $\mathbf{b}'$. In the sequel, we will employ the shorthand

$$Z_{\mathbf{b},\mathbf{b}'}^{U(N)}(\mathbf{a},\mathbf{m},\mathbf{m}';q):=Z_{\sigma;\delta\mathbf{k}}^{U(N)}(\mathbf{a},\mathbf{m},\mathbf{m}',\mathbf{b},\mathbf{b}';q),\tag{3.1.36}$$

which is *not* $Z_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}';q)$, the instanton partition function on $\mathbb{C}^2$ (2.1.14) (which has no dependence on $\mathbb{Z}_n$-boundary charges). We will also always notate the multiplet contributions with an asterisk superscript to differentiate them from the multiplet contributions on $\mathbb{C}^2$.

3.1.3 Stripping the $U(1)$ Factor in $Z_{inst}^{U(N)}$

The coset AGT correspondence is between the instanton partition function for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ and the 4-point conformal blocks for a 2D CFT with symmetry algebra $\mathcal{A}(N,n;p)$. As in the case of AGT for gauge theories on $\mathbb{C}^2$, only the $U(N)$ instanton partition function (3.1.34) has been calculated explicitly [42]. As explained in chapter 2, we therefore must find a $U(1)$ factor in $Z_{\mathbf{b},\mathbf{b}'}^{U(N)}$ to obtain $Z_{\mathbf{b},\mathbf{b}'}^{SU(N)}$, the $SU(N)$ partition function on $\mathbb{C}^2/\mathbb{Z}_n$.

To do so, we begin by recalling that we must impose the traceless condition

$$\sum_{i=0}^{N-1} a_i = 0, \tag{3.1.37}$$

on the Coulomb parameters a_i for $i=0,\dots,N-1$, so that they parameterize an element ϕ of the Cartan subalgebra of $\mathfrak{sl}(n)$. Then following [2, 6, 26, 138] we propose a $U(1)$ factor for the instanton partition function of the following form

$$Z^{U(1)} := (1-q)^{\frac{\left(\sum_{i=0}^{N-1} m_i\right)\left(\epsilon_1+\epsilon_2-\frac{1}{N}\sum_{i=0}^{N-1} m_i'\right)}{n\epsilon_1\epsilon_2}}. \tag{3.1.38}$$

We regard this as a generalization of the form for the $U(1)$ factor for $SU(N)$ gauge theories on $\mathbb{C}^2$ (cf: (2.2.5) for the $SU(2)$ case and (2.3.7) for the general $SU(N)$ case, which we recall was obtained in [6]), and of the form for $SU(2)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$. We will check this proposal against the $U(1)$ factors known for $N=3$, $n=1$ and $N=2$, $n=2$ below. In chapter 4, we will factorize the partition function as

$$Z_{\mathbf{b},\mathbf{b}'}^{U(N)}(\mathbf{a},\mathbf{m},\mathbf{m}';q) = Z^{U(1)} \times Z_{\mathbf{b},\mathbf{b}'}^{SU(N)}(\mathbf{a},\mathbf{m},\mathbf{m}';q), \tag{3.1.39}$$

and identify the conformal blocks of the 4-point $\widehat{\mathfrak{sl}}(n)$-WZW function with $Z_{\mathbf{b},\mathbf{b}'}^{SU(N)}(\mathbf{a},\mathbf{m},\mathbf{m}';q)$. By doing so we will see that our proposed fator $Z^{U(1)}$ naturally corresponds to the Heisenberg algebra factor $\mathcal{H}$ for our CFT. Then by the arguments in chapter 2, we can identify $Z^{U(1)}$ as the correct $U(1)$ factor of the $U(N)$ partition function.

Example 3.1.3.1. In the case of $SU(3)$ for gauge theories $\mathbb{C}^2$ (that is $N=3$, $n=1$), the instanton partition function $Z_{inst}^{U(3)}$ was found to factorize in the form [139, 25]

$$Z_{inst}^{U(3)}(\mathbf{a},\mathbf{m},\mathbf{m}';q) = Z^{U(1)} Z_{inst}^{SU(3)}(\mathbf{a},\mathbf{m},\mathbf{m}';q) \tag{3.1.40}$$

$$= (1-q)^{\frac{\left(\sum_{i=0}^{2} m_i\right)\left(\epsilon_1+\epsilon_2-\frac{1}{3}\sum_{i=0}^{2} m_i'\right)}{\epsilon_1\epsilon_2}} Z_{inst}^{SU(3)}(\mathbf{a},\mathbf{m},\mathbf{m}';q). \tag{3.1.41}$$

For $SU(2)$ on $\mathbb{C}^2/\mathbb{Z}_2$ (that is $N=2$, $n=2$) the instanton partition function $Z^{U(2)}_{inst}$ was found to factorize in the form [26, 28, 140]

$$Z^{U(2)}_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}',\mathbf{b},\mathbf{b}';q) = Z^{U(1)} Z^{SU(2)}_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}';q) \tag{3.1.42}$$

$$= (1-q)^{\frac{\left(\sum_{i=0}^{1} m_i\right)\left(\epsilon_1+\epsilon_2-\frac{1}{2}\sum_{i=0}^{1} m'_i\right)}{2\epsilon_1\epsilon_2}} Z^{SU(2)}_{inst}(\mathbf{a},\mathbf{m},\mathbf{m}',\mathbf{b},\mathbf{b}';q). \tag{3.1.43}$$

As we can clearly see by the form we have written the $U(1)$ factors for both cases above, our proposed $U(1)$ factor (3.1.38) reduces to these for the specified choices of parameters.

3.2 The Algebra $\mathcal{A}(N,n;p)$

In this section we consider the 2D CFTs with the symmetry algebra $\mathcal{A}(N,n;p)$, which have been conjectured to be dual to our gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$. The algebra $\mathcal{A}(N,n;p)$ is of the following form [10, 32, 138, 34]

$$\mathcal{A}(N,n;p) = \frac{\widehat{\mathfrak{gl}}(p-N)_N}{\widehat{\mathfrak{gl}}(p-N-n)_N} \cong \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{p+n-N}}, \tag{3.2.1}$$

where $n,N,p \in \mathbb{Z}_{>0}$, $n \leq p-N$, and $p \geq N$, $\mathcal{H}$ is the Heisenberg algebra. We will only consider the second presentation of the algebra $\mathcal{A}(N,n;p)$, which itself has 3 distinct components, in the subsequent material of this chapter and chapter 4. We will then explore the first presentation in chapter 5. On the CFT side, the parameter p is fixed in terms of the Ω-deformation parameters for the gauge theory by

$$\frac{\epsilon_1}{\epsilon_2} = -\frac{n+p}{p}. \tag{3.2.2}$$

When the deformation parameters satisfy this for $n,p \in \mathbb{Z}$, they define what is referred to as the *rational* Ω*-background.*

A CFT with the symmetry algebra $\mathcal{A}(N,n;p)$ represents a *combined* system of three 2D CFTs: a free boson (corresponding to $\mathcal{H}$, see section 1.5.6), an $\widehat{\mathfrak{sl}}(n)_N$-WZW model, and a diagonal coset model which is referred to as a n-th $\mathcal{W}_N$-parafermion system that we notate as $\mathcal{W}^{para}_{n,N}$ (see [141] and references therein). As explained in chapter 2, the Heisenberg factor on the CFT side corresponds to the $U(1)$ factor in the gauge group $U(N)$ on the gauge side of the correspondence. The $U(1)$ factor appearing in the $U(N)$ instanton partition function then appears as the correlator of a free boson on the CFT side.

There are two important special cases of this algebra $\mathcal{A}(N,n;p)$ to highlight, the case of $n=1$ with $p>N$, and the case $p=N$ with $n>1$.

3.2.0.1 $n=1$ and $p>N$

For the first case, the symmetry algebra is

$$\mathcal{A}(1,N;p)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(1)_N\oplus\frac{\widehat{\mathfrak{sl}}(N)_1\oplus\widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{p-N+1}}. \tag{3.2.3}$$

As $\widehat{\mathfrak{sl}}(1)_N$-WZW models are trivial (see [50]), for these choices of parameters the symmetry algebra is of the form

$$\mathcal{A}(1,N;p)=\mathcal{H}\oplus\frac{\widehat{\mathfrak{sl}}(N)_1\oplus\widehat{\mathfrak{sl}}(N)_{p-N}}{\widehat{\mathfrak{sl}}(N)_{p-N+1}}. \tag{3.2.4}$$

Since $p>N$ we can redefine $(p-N)\mapsto p$ and see that (3.2.4) is the same as (1.5.143) with one additional Heisenberg factor. Thus we obtain a combined CFT which describes one free boson and $M(N+p,N+p+1;N)$ (a $(p,p+1)$-$\mathcal{W}_N$ unitary minimal model). On the gauge side, we see that for $n=1$ we have that $\mathbb{C}^2/\mathbb{Z}_n\overset{n=1}{\mapsto}\mathbb{C}^2$ and this structure reduces to the AGT-$\mathcal{W}$ correspondence for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2$ described in chapter 2.

Remark 3.2.0.2. We will base our proposed generalization around the form of the AGT-$\mathcal{W}$ conjecture discussed in section 2.3. Thus, our identifications between the gauge theory and CFT parameters will be of the same form as (2.3.4). When we further restrict to gauge theories that are AGT dual to minimal model CFTs, the gauge theory parameters will be in the form (2.3.8).

3.2.0.3 $p=N$ and $n>1$

For the second case, we trivialize the coset factor in the second presentation of $\mathcal{A}(N,n;p)$ in (3.2.1) (as now $p-N=0$). This leaves us with the symmetry algebra

$$\mathcal{A}(N,n;N)=\mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_N. \tag{3.2.5}$$

In this case, we are considering a CFT composed of a free boson and an $\widehat{\mathfrak{sl}}(n)$-WZW model, and this will be the topic of chapter 4. There we will show that one can calculate the characters and conformal blocks of $\widehat{\mathfrak{sl}}(n)$-WZW models using the instanton generating function and instanton partition function of $\mathcal{N}=2$ $SU(N)$ gauge theories with specific mass and deformation parameters.

For later reference, we state here the central charge for a CFT with symmetry algebra $\mathcal{A}(N,n;p)$, by adding the individual central charges of its factors. For the first two factors we have [50]

$$c(\mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_N)=1+\frac{N(n^2-1)}{n+N}, \tag{3.2.6}$$

whereas the $\mathcal{W}^{para}_{n,N}$ central charge can be calculated string theoretically as [10]

$$c(\mathcal{W}^{para}_{n,N}) = \frac{n(N^2-1)}{n+N} - \frac{nN(N^2-1)}{p(p+n)} \tag{3.2.7}$$

$$= \frac{n(N^2-1)}{n+N} + \frac{N(N^2-1)}{n}\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}. \tag{3.2.8}$$

3.3 AGT Dictionary for $\mathcal{N} = 2$ $SU(N)$ Gauge Theories on $\mathbb{C}^2/\mathbb{Z}_n$

Here we provide a proposal [43] which identifies the parameters in (3.1.34) with those defining the conformal blocks of the 4-point $\mathcal{W}^{para}_{n,N}$ correlation function on $\mathbb{P}^1$ between primary fields $\psi_{\alpha_r}(z_r)$, $r = 0,1,2,3$, which are labelled by the charge $\alpha_r \in P^+(\mathfrak{sl}(N))$ and where the primary fields are inserted at $z_r \in \mathbb{P}^1$. After a $PSL(2,\mathbb{C})$ transformation we can fix 3 of the z_r coordinates to $0,1,\infty$, so that the 4-point function

$$\langle \psi_{\alpha_0}(\infty)\psi_{\alpha_1}(1)\psi_{\alpha_2}(q)\psi_{\alpha_3}(0)\rangle^{\mathcal{W}^{para}_{n,N}}_{\mathbb{P}^1}, \tag{3.3.1}$$

depends on one variable q, which is the cross-ratio (1.5.31). We identify the cross ratio q with the variable q used to define the series expansion of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ in (3.1.34).

Following the discussion of section 2.3 we again only consider special values of α_1, α_2 in the 4-point functions (3.3.1) on the CFT side of the conjectured correspondence below. As in the case for CFTs whose symmetry algebras are the $\mathcal{W}_N$ algebras, we will restrict to the case where the charge of the two internal legs are taken to be scalar multiples of $\bar{\Lambda}_1$ and $\bar{\Lambda}_{N-1}$. This leads to the following conjecture.

Conjecture 3.3.0.1. *([43]) The charges α_r for $r = 0,1,2,3$, which define the primary fields for an $\mathcal{A}(N,n;p)$ 4-point function on $\mathbb{C}$ (3.3.1), are related to the mass parameters m_i, m'_i and the deformation parameters ϵ_1 and ϵ_2 for an $\mathcal{N}=2$ $SU(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ as follows:*

$$2\alpha_0 = Q\rho + \sum_{i=0}^{N-2}(m_{i+1}-m_i)\,\bar{\Lambda}_{i+1}, \quad 2\alpha_1 = \sum_{i=0}^{N-1} m_i\bar{\Lambda}_1 \tag{3.3.2}$$

$$2\alpha_3 = Q\rho + \sum_{i=0}^{N-2}(m'_i-m'_{i+1})\,\bar{\Lambda}_{i+1}, \quad 2\alpha_2 = \sum_{i=0}^{N-1} m'_i\bar{\Lambda}_{N-1}. \tag{3.3.3}$$

Here $Q = \epsilon_1+\epsilon_2$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ is the Weyl vector, and $\{\bar{\Lambda}_i\}_{i=1,\dots,N-1}$ are the $\mathfrak{sl}(N)$ fundamental weights. Furthermore, the internal charge α^s (see section 1.5.3) for the 4-point *function is related to Coulomb parameters a_i as*

$$2\alpha^s = Q\rho + \sum_{i=0}^{N-1} a_i\varepsilon_i, \tag{3.3.4}$$

where we have used the notation $\varepsilon_i := e_{i+1} - e_0$ we introduced for AGT-$\mathcal{W}$ involving $SU(N)$ gauge theories on $\mathbb{C}^2$ in section 2.3.

Remark 3.3.0.2. Comparing these identifications to the $\mathbb{C}^2$ identifications (2.3.4) and (2.3.5), we see that this is a generalization of AGT-$\mathcal{W}$ for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2$. In fact this is the same identification, as might be expected. The only difference is that the levels of the associated Dynkin labels defining our minimal model representations have a larger upper bound of $(p+n)$ compared to $(p+1)$ for AGT for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2$.

We rewrite the conformal charge explicitly $\alpha_r = \sum_{i=1}^{N-1} \alpha_r^{(i)} \bar{\Lambda}_{i+1}$ for $r = 0,1,2,3$ (note that for the restricted charge α_1 and α_2, the only non-zero labels are $\alpha_1^{(N-1)}$ and $\alpha_2^{(1)}$), and invert these relations to express the mass parameters in terms of the conformal charge as

$$m_i = \left(i - \frac{N+1}{2}\right) Q + \frac{2}{N}\left(-\sum_{j=1}^{i-1} j\alpha_0^{(j)} + \sum_{j=i}^{N-1}(N-j)\alpha_1^{(j)} + \alpha_1^{(N-1)}\right), \tag{3.3.5}$$

$$m_i' = -\left(i - \frac{N+1}{2}\right) Q + \frac{2}{N}\left(\sum_{j=1}^{i-1} j\alpha_3^{(j)} + \sum_{j=i}^{N-1}(N-j)\alpha_0^{(j)} - \alpha_2^{(0)}\right). \tag{3.3.6}$$

As for the mass parameters we can invert the equation defining the internal charge (3.3.4) above, to express the Coulomb parameters in terms of the conformal charge of defining the conformal dimension of the family that flows in the channel by using the pairing on the root lattice $P(\mathfrak{sl}(N))$

$$a_i = \frac{1}{N}\sum_{i=0}^{N-1}\langle 2\alpha^s - Q\rho, e_i\rangle. \tag{3.3.7}$$

3.4 Burge Conditions

We wish to consider AGT correspondences involving minimal models on the CFT side. Following section 2.3.3, we can achieve this by restricting the Ω-deformation parameters so that the conjectured corresponding central charge on the CFT side is that of a minimal model. We then employ the Coulomb-gas formalism for the conformal charges (from section 1.5.15) of the minimal model primary fields for a CFT with symmetry algebra $\mathcal{W}_{n,N}^{para}$ and identify these CFT charge with the mass and Coulomb parameters of the gauge theory[2]. For ease of

[2]The $\mathcal{W}_{n,N}^{para}$ models have not been proven to exist. We simply assume their existence and show through calculation that our methods are consistent.

notation, we will sometimes refer to the combined CFT involving these minimal models as a $\mathcal{A}(N,n;p)$-minimal model, although only the $\mathcal{W}^{para}_{n,N}$ factor is a minimal model in the usual sense. In this case, we will see that the usual definition for the instanton partition will have poles. In proposition 3.4.2.1, we will eliminate these poles by restricting the summation to N-tuples of n-coloured Young diagrams with specified colour content that satisfy the Burge inequalities. This is a new generalization of the results we reviewed in sections 2.2.4 and 2.3.3, and contains both as subcases.

3.4.1 Minimal Model Identification and $\mathbb{Z}_n$-Charge Conditions

By analogy with the known $\mathcal{W}_N$-minimal model CFTs (see sections 1.5.13 and 1.5.15) and our discussion on other AGT correspondences involving minimal models in chapter 2, we propose that the charge for primary fields in $\mathcal{W}^{para}_{n,N}$ models should take the values

$$2\alpha^{\mathbf{r},\mathbf{s}} := \sum_{i=1}^{N-1} \left((r_i-1)\epsilon_1 + (s_i-1)\epsilon_2\right)\bar{\Lambda}_i, \tag{3.4.1}$$

where $r_i > 0$ and $s_i > 0$, and $\sum_{i=1}^{N-1} r_i \leq p$ and $\sum_{i=1}^{N-1} s_i \leq n+p = p'$, which we call *degenerate charge*. We note that our proposal for the degenerate charge is identical (up to a scalar) to the known case of $\mathcal{W}_N$-minimal models under the Coulomb-gas formalism of sections 1.5.15 and 2.3.3. We also note that the proposed degenerate charge also reproduces the singular vectors for CFTs with the symmetry algebra $\mathcal{A}(2,2;p)$ (the Neveu-Schwarz-Ramond algebra) [28].

These values of the charge are conjectured to parameterize the conformal dimensions of minimal model primary fields with associated Vir-highest weight modules that have null states. We define the additional parameters r_0 and s_0 such that $\sum_{i=0}^{N-1} r_i = p$ and $\sum_{i=0}^{N-1} s_i = p'$ and we can collect these into the strictly positive $\widehat{\mathfrak{sl}}(N)$ Dynkin labels $r = [r_0, r_1, \ldots, r_{N-1}] \in P^{++}_{N,p}$ and $s = [s_0, s_1, \ldots, s_{N-1}] \in P^{++}_{N,n+p}$.

For $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, we consider the so-called *rational* Ω*-background*[3]

$$p\epsilon_1 + p'\epsilon_2 = 0, \tag{3.4.2}$$

where $p \geq N$, p and p' are coprime, and we take $p' = p+n$. In our case, if $\gcd(p,p) = d \neq 1$ we see that in (3.2.2) we have that

$$-\frac{n+p}{p} = -\frac{k'd}{kd} = -\frac{k'}{k}, \quad \gcd(k,k') = 1, \tag{3.4.3}$$

[3]Note that this is the same Ω-background we considered in chapter 2. The reader should also note the similarity of this formula to (1.5.88), which connected the Coulomb-gas formalism to minimal models.

which corresponds to the rational Ω-background $k\epsilon_1+k'\epsilon_2=0$. This background is equivalent to the minimal model identification we made for the deformation parameters in 2.2.4 and 2.3.3.

For the rational Ω-background, the inverse of the vector contribution Z^*_{vec} (3.1.33), which appears as the denominator in the summand of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ (3.1.34), can vanish and cause a pole in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$. This occurs for the special values of the Coulomb parameters corresponding to the degenerate charge

$$a_i^{\mathbf{r},\mathbf{s}} := -\sum_{j=1}^{N-1}\langle\bar{\Lambda}_j,\mathbf{e}_i\rangle\,(r_j\epsilon_1+s_j\epsilon_2) = -\sum_{j=i}^{N-1}(r_j\epsilon_1+s_j\epsilon_2)+\frac{1}{N}\sum_{j=1}^{N-1} j\,(r_j\epsilon_1+s_j\epsilon_2)\,. \tag{3.4.4}$$

Remark 3.4.1.1. The second summation in (3.4.4) enforces the traceless property for the Coulomb parameters, ensuring that the set $\{a_i^{\mathbf{r},\mathbf{s}}\}_{i=0,\ldots,N-1}$ parameterize an element of the Cartan subalgebra $\widehat{\mathfrak{h}}_{\widehat{\mathfrak{sl}}(N)}$. Since the building block functions only involve the difference $(a_i-a'_j)$, adding the same term to all Coulomb parameters does not change its value.

By restricting on the CFT side of the conjectured correspondence to minimal models, we have been able to consider their AGT dual gauge theories to propose a specific parameterization of the gauge theory parameters for a special subset of gauge theories. As we will show in the sequel, the usual definition of the instanton partition function for these theories is ill-defined, and must be altered to obtain a well-defined one. By parameterizing the mass and Coulomb parameters, of an $\mathcal{N}=2$ $SU(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ with the rational Ω-background, in this way by (3.3.5) and (3.4.1), and (3.4.4) respectively we have a gauge theory under a *minimal model identification*, as it identifies this gauge theory as AGT dual to a CFT minimal model.

Remark 3.4.1.2. By letting $p'=p+n$ we see that under our conjectured AGT correspondence the rational Ω-background corresponds to the central charge of the algebra $\mathcal{A}(N,n;p)$. Thus our conjectured AGT dual CFT to $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ should have a process dual to removing these non-physical poles. Remembering our discussion in sections 2.2.4 and 2.3.3, this suggests that the AGT dual CFT to $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ with mass parameters defined by (3.3.5) and (3.4.1) and Coulomb parameters by (3.4.4) should be a minimal model. In this CFT, the process that is dual to removing poles should be the removal of null states. This supports our conjecture that the dual CFTs should have the symmetry algebra $\mathcal{A}(N,n;p)$.

Under this minimal model identification, we obtain restrictions on the $\mathbb{Z}_n$-charges of the Coulomb parameters gauge theory parameters by considering the $\mathbb{Z}_n$-invariance of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$. Consider a $\mathbb{Z}_n$-transformation (3.1.24) on the minimal model Coulomb parameters (3.4.4). We see that the r and s parameters are related to the $\mathbb{Z}_n$-charges σ_i associated to the Coulomb parameters a_i for $i = 0, \dots, N-1$ by

$$\sigma_i - \sigma_{i+1} \equiv s_i - r_i \mod n, \tag{3.4.5}$$

where the relative between r_j and s_j is due to the differing transformation properties of ϵ_1 and ϵ_2. We refer to (3.4.5) as the $\mathbb{Z}_n$*-charge conditions*.

3.4.2 Eliminating the Poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$

Below in the proof of proposition 3.4.2.1, we will show that when considering gauge theories under a minimal model identification there are unphysical poles in the unrestricted instanton partition function that we must eliminate. To eliminate them, we restrict the summation range of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ analogously to the case of $Z^{U(N)}_{inst}$ for gauge theories on $\mathbb{C}^2$, as discussed in section 2.3.3. This process is then an AGT dual process to removing null states in the minimal model CFT, and must be performed when comparing $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ to conformal blocks, and when calculating CFT characters.

We begin by discussing the poles themselves. The partition function $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ (3.1.34) is defined as a sum over N-tuples of coloured Young diagrams $\lambda = (\lambda^{(0)}, \dots, \lambda^{(N-1)})$, where each term is a rational function of ϵ_1 and ϵ_2. Furthermore, the numerator of each term is finite for any λ, so that all the poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ are when Z^*_{vec} vanishes for some λ. Using (3.1.23) and (3.1.33), we can write the inverse vector multiplet contribution as

$$\begin{aligned} Z^*_{vec}(\mathbf{a}^{\mathbf{r},\mathbf{s}}, \lambda) = \prod_{i,j=0}^{N-1} \prod^{*}_{\square \in \lambda^{(i)}} & E\left(a^{\mathbf{r},\mathbf{s}}_i - a^{\mathbf{r},\mathbf{s}}_j, \lambda^{(i)}, \lambda^{(j)}, \square\right) \\ & \prod^{*}_{\blacksquare \in \lambda^{(j)}} \left(\epsilon_1 + \epsilon_2 - E\left(a^{\mathbf{r},\mathbf{s}}_j - a^{\mathbf{r},\mathbf{s}}_i, \lambda^{(j)}, \lambda^{(i)}, \blacksquare\right)\right). \end{aligned} \tag{3.4.6}$$

Due to the similarity of Z^*_{vec} to Z_{vec} (the vector multiplet contribution for gauge theories on $\mathbb{C}^2$), we will use the same logic and terminology as we did in chapter 2, which we recall here. The poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ correspond to the zeroes of $Z^*_{vec}(\mathbf{a}^{\mathbf{r},\mathbf{s}}, \lambda)$. We associate these zeroes to an N-tuple of Young diagrams $\lambda = (\lambda^{(0)}, \dots, \lambda^{(N-1)})$ and a box in one of the $\lambda^{(i)}$ for some $i = 0, \dots, N-1$ as follows. Each term in the series (3.1.34) defining $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ corresponds to one N-tuple of Young diagrams, therefore we can associate the pole of one term to an N-tuple λ in this way. We also note that due to the factorized form of $Z^*_{vec}(\mathbf{a}^{\mathbf{r},\mathbf{s}}, \lambda)$, these poles are caused by the vanishing of a factor $\square^{(i,j)}$ or $\blacksquare^{(i,j)}$ (notation from section 2.2.4), and we use this to further associate the pole to a box in one of the diagrams of the N-tuple λ.

We will therefore think of individual boxes as *causing* zeros in Z^*_{vec} and hence poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$. Thus a box $\square \in \lambda^{(i)}$ such that $E\left(a^{\mathbf{r},\mathbf{s}}_i - a^{\mathbf{r},\mathbf{s}}_j, \lambda^{(i)}, \lambda^{(j)}, \square\right) = 0$ or such that $\big(\epsilon_1 + \epsilon_2 - E\big(a^{\mathbf{r},\mathbf{s}}_j - a^{\mathbf{r},\mathbf{s}}_i, \lambda^{(j)}, \lambda^{(i)}, \square\big)\big) = 0$ is said to *cause a pole* in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$.

We will then reduce our search to poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ to searching for $\square \in \lambda^{(i)}$ which cause poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$. For gauge theories with a minimal model identification we must then restrict the summation of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ to N-tuples of coloured Young diagrams which have no boxes that cause a pole.

To do so, we use the same argument that we used in chapter 2. In proposition 3.4.2.1 below, we will see that to eliminate all poles we must again restrict the summation of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ to N-tuples of Young diagrams that satisfy the Burge inequalities.

The proof will proceed analogously as in section 2.3.3, except we have a new condition on which boxes we sum over defined by (3.1.31) and (3.1.32) and the new $\mathbb{Z}_n$-charge conditions (3.4.5). We will also remove the restriction that p and $p' = n + p$ are coprime integers, that we used for the $SU(N)$ minimal model identification, and craft our proof to work for *all* $p' \in \mathbb{Z}$ such that $p' > p$. We can do this without changing the rational Ω-background of the gauge theories we are considering since, as seen in (3.4.3), when p and $(p+n)$ are not coprime they still correspond to *some* rational Ω-background with k and k' coprime. While this does not effect the computation of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ on the gauge side, it allows us to use this AGT conjecture to compute conformal blocks in a broader spectrum of CFTs.

Proposition 3.4.2.1. *([43, **Prop 4.3**]) To eliminate the poles of the instanton partition function $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ for an $\mathcal{N}=2$ $SU(N)$ gauge theory on $\mathbb{C}^2/\mathbb{Z}_n$ under a minimal model identification, the summation in (3.1.34) must be restricted to N-tuples of coloured charged Young diagrams that satisfy the Burge conditions*

$$\lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+r_i-1} - s_i + 1, \tag{3.4.7}$$

where r_i and s_i for $i = 0, \ldots, N-1$ parameterize the Coulomb parameters, $a^{r,s}_j$ for $j = 0, \ldots, N-1$, corresponding to the degenerate charge (3.4.4).

We assume that the Coulomb parameters take the degenerate values $a^{\mathbf{r},\mathbf{s}}_i$. Again our logic will be as follows: We will first assume that some box $\square$ in one of our coloured Young diagrams $\lambda^{(i)}$ causes a pole in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$, so that the denominator of a term in the instanton partition vanishes. We will then show that the existence of $\square$ is equivalent inequalities on the N-tuple of coloured Young diagrams λ that we will calculate.

Recall that the conditions on an N-tuple of coloured Young diagrams λ such that λ contains a box $\square \in \lambda^{(i)}$ for some $i \in \{0, \ldots, N-1\}$ which causes Z_{vec} to vanish (equivalently, a pole in

$Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$) are called *zero-conditions* or *vanishing-conditions*. Similarly, conditions that ensure no such box exists are referred to as *nonzero-conditions*. We also refer to an equation which is equivalent to $Z_{vec} = 0$ as a *vanishing equation*.

We will then restrict to the definition of $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ to be a sum over N-tuples of Young diagrams that do not satisfy these zero-conditions. This will be Burge conditions, and by applying them to $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ we will have therefore eliminated the non-physical poles.

Proof. For a gauge theory under a minimal model identification, $Z_{vec} = 0$ if and only if there exists $\square = (x, y) \in \lambda^{(i)}$ such that one of the following equations

$$E(a^{\mathbf{r},\mathbf{s}}_i - a^{\mathbf{r},\mathbf{s}}_{i+k}, \lambda^{(i)}, \lambda^{(i+k)}, \square) = 0, \tag{3.4.8}$$

$$\epsilon_1 + \epsilon_2 - E(a^{\mathbf{r},\mathbf{s}}_{i+k} - a^{r,s}_i, \lambda^{(i+k)}, \lambda^{(i)}, \square) = 0, \tag{3.4.9}$$

is true for some $i \in \{0, 1, \ldots, N-1\}$ and $k \in \mathbb{Z}$. Note that here we have substituted $j = i+k$ into (3.4.6). We can combine both of these equations into one for the rational Ω-background parameterized by $p\epsilon_1 = -p'\epsilon_2$ with $p' - p = n$, as

$$E^{r,s}_{i,i+k}(\square) + \eta = 0, \quad \eta = 0, n, \tag{3.4.10}$$

where $E^{r,s}_{i,i+k}(\square) = \frac{p}{\epsilon_2} E(a^{r,s}_i - a^{r,s}_{i+k}, \lambda^{(i)}, \lambda^{(i+k)}, \square)$.

We begin again by considering the case $N - i \geq k > 0$, and later we will consider the case where $k > N - i$ which is equivalent (as we define the labels of Young diagrams modulo N) to $-N - i \leq k < 0$. Since the second term in our Coulomb parameter parameterization is a constant for all i, we have that the difference of two Coloumb parameters under a minimal model identification can be written as

$$a^{\mathbf{r},\mathbf{s}}_i - a^{\mathbf{r},\mathbf{s}}_{i+k} = \sum_{j=i}^{i+k-1} (r_j\epsilon_1 + s_j\epsilon_2). \tag{3.4.11}$$

Similarly by substituting k with $-k$ we have

$$a^{\mathbf{r},\mathbf{s}}_i - a^{\mathbf{r},\mathbf{s}}_{i-k} = -\sum_{j=i-k}^{i-1} (r_j\epsilon_1 + s_j\epsilon_2) \tag{3.4.12}$$

$$= -p\epsilon_1 - p'\epsilon_2 + \left(\sum_{j=0}^{i-k-1} + \sum_{j=i}^{N-1}\right)(r_j\epsilon_1 + s_j\epsilon_2). \tag{3.4.13}$$

After substituting these values for the differences of Coulomb parameters into $E^{\mathbf{r},\mathbf{s}}_{i,i\pm k}$ above, we see that we have the same vanishing equations as we did for $SU(N)$ on $\mathbb{C}^2$ in 2.3.3. In this case, we must additionally constrain any solutions to this vanishing equation by the orbifold conditions (3.1.31) and (3.1.32).

As before, we can immediately discount the case where $k=0$ from introducing a pole as this corresponds to the vanishing equation

$$E^{\mathbf{r},\mathbf{s}}_{i,i}(\square)+\eta=0 \tag{3.4.14}$$

$$\implies \frac{p}{\epsilon_2}\left(-\epsilon_1 L_{\lambda^{(i)}}(\square)+\epsilon_2 A^+_{\lambda^{(i)}}(\square)\right)=-\eta \tag{3.4.15}$$

$$\implies p' L_{\lambda^{(i)}}(\square)+pA^+_{\lambda^{(i)}}(\square)=-\eta, \tag{3.4.16}$$

for some $\square\in\lambda^{(i)}$. As for the case of gauge theories on $\mathbb{C}^2$ we have that $L_{\lambda^{(i)}}(\square)>0$ and $A^+_{\lambda^{(i)}}(\square)>0$ for $\square\in\lambda^{(i)}$. Therefore this equation can never be satisfied, and the zero conditions can only come from factors where $k\neq 0$.

Case 1 $k>0$

For this case the zero condition becomes $E^{r,s}_{i+k,i}(\square)+\eta=0$, where $\square\in\lambda^{(i+k)}$, $0\le i\le N-2$, and $1\le k\le N-1$. Substituting the values (3.4.4) for $a^{r,s}_i$ and $a^{r,s}_{i+k}$, we have this zero condition is explicitly

$$\sum_{j=i}^{i+k-1}(r_j p'-s_j p)+p' L_{\lambda^{(i+k)}}(\square)+pA^+_{\lambda^{(i)}}(\square)+\eta=0. \tag{3.4.17}$$

We now introduce a new element to the proof we described in section 2.3.3, and define $d=\gcd(p',p)$ so that $p'=dp'_d$ and $p=dp_d$, where $p_d,p'_d\in\mathbb{Z}_{>0}$ and $\gcd(p_d,p'_d)=1$. We then factor d out of the zero condition, leaving us with the same equations for the leg length and arm length as before[4]

$$L_{\lambda^{(i+k)}}(\square)=-\left(\sum_{j=i}^{i+k-1} r_j+cp_d+\delta_{\eta n}\right), \tag{3.4.18}$$

$$A_{\lambda^{(i)}}(\square)=\sum_{j=i}^{i+k-1} s_j+cp'_d+\delta_{\eta n}-1, \tag{3.4.19}$$

where $c\in\mathbb{Z}$ is an constant to be determined, and we have replaced p and p' with p_d and p'_d respectively.

We know that for any Young diagram λ and $\square=(x,y)\in\lambda$ the definition $L_\lambda(\square)=\lambda^T_y-x$ implies that

$$\lambda^T_y\ge x. \tag{3.4.20}$$

[4]Note the $\delta_{\eta n}$ term comes from writing $n=p'-p$.

In our situation, we apply this to the box at the *end* of the x-th row (the one that contains $\square = (x, y)$ that causes a pole) which has coordinates $(x, y + A_{\lambda^{(i)}}(\square))$ and obtain

$$(\lambda^{(i)})^T_{y+A_{\lambda^{(i)}}} \geq x. \tag{3.4.21}$$

We then substitute (3.4.19) into this inequality and solve equation (3.4.18) for x to obtain the inequality

$$(\lambda^{(i+k)})^T_{y+\sum_{j=i}^{i+k-1} s_j + cp'_d + \delta_{\eta n} - 1} \geq (\lambda^{(i)})^T_y + \sum_{j=i}^{i+k-1} r_j + cp_d + \delta_{\eta n}, \tag{3.4.22}$$

which is a zero-condition for Z_{vec}.

We now consider how the $\mathbb{Z}_n$-charge conditions (3.4.5), and the two orbifold conditions, (3.1.31) and (3.1.32), for $\square$ restrict the possible values for the indeterminate constant c in (3.4.18) and (3.4.19). We telescope the $\mathbb{Z}_n$-charges of the Coulomb parameters by

$$\sigma_i - \sigma_{i+l} = \sigma_i - \sigma_{i+1} - \sigma_{i+2} + \cdots + \sigma_{i+l-1} - \sigma_{i+l}, \tag{3.4.23}$$

and note that

$$\begin{aligned} \sigma_i - \sigma_{i+k} &= \sigma_i - \sigma_{i+1} - \sigma_{i+2} + \cdots + \sigma_{i+k-1} - \sigma_{i+k} \\ &\equiv s_i - r_i + s_{i+1} - r_{i+1} + \cdots + s_{i+k-1} - r_{i+k-1} \bmod n, \end{aligned} \tag{3.4.24}$$

through (3.4.5). We now substitute (3.4.18) and (3.4.19) into the orbifold condition (3.1.31) together with $p' - p = n$ to obtain the following condition satisfied by c

$$-c(p'_d - p_d) = -\frac{cn}{d} \equiv 0 \mod n. \tag{3.4.25}$$

As this is an equation between integers, we see that c must be of the form $c = dc_d$ for some $c_d \in \mathbb{Z}$, analogously to p and p'. This new parameter c_d then satisfies

$$-c_d n \equiv 0 \mod n. \tag{3.4.26}$$

Now we wish to eliminate the poles caused by any $\square \in \lambda^{(i)}$. As the existence of such a pole is equivalent to the zero-condition (3.4.22) defined by the inequalities, we can eliminate these poles by restricting to Young diagrams that *do not* satisfy these inequalities. Thus, we consider Young diagrams that satisfy the nonzero-conditions

$$(\lambda^{(i+k)})^T_{y+\sum_{j=i}^{i+k-1} s_j + cp'_d + \delta_{\eta n} - 1} < (\lambda^{(i)})^T_y + \sum_{j=i}^{i+k-1} r_j + cp_d + \delta_{\eta n}. \tag{3.4.27}$$

We then substitute the new parameter c_d, which allows us to obtain a zero condition using our original p and p' parameters, and move from a strict bound a nonstrict one to obtain (note the additional -1 term on the right-hand side)

$$(\lambda^{(i+k)})^T_{y+\sum_{j=i}^{i+k-1} s_j + c_d p' + \delta_{\eta n} - 1} \leq (\lambda^{(i)})^T_y + \sum_{j=i}^{i+k-1} r_j + c_d p + \delta_{\eta n} - 1. \tag{3.4.28}$$

The final step to obtaining the zero condition is to find the values of c_d and η that will give us the strongest such bound. Since $A^+_{\lambda^{(i)}}(\square) > 0$ and $\sum s_i \leq p' + n$ we can see from (3.4.19) that we must also have $c_d \geq 0$.

As for the case for theories on $\mathbb{C}^2$, we can use the weakly decreasing property of Young diagrams to see that any inequality satisfied by the *highest* Young diagram row on the left-hand side (that is, the one with the smallest row index) will imply all lower rows satisfy the same inequality. Whereas the smallest number on the right-hand side of the bound implies all larger numbers will also satisfy the bound. Using the equation satisfied by c_d (3.4.28) and both of these facts, we see that $\gamma_d = 0$ and $\eta = 0$ give us the strongest bound.

We also note that once we have found the strongest zero condition that satisfies our $\mathbb{Z}_n$-charge and orbifold conditions, we can safely follow the arguments we used for the $\mathbb{C}^2$ to see that the $k = 1$ inequality implies all other inequalities. Finally, we translate this back into a condition for the original Young diagrams, instead of the transposed ones, to obtain the necessary condition to eliminate all the zeros from Z_{vec}, and hence poles from $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$, as

$$\lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+r_i-1} - s_i + 1. \tag{3.4.29}$$

Case 2: $k < 0$

In this case we will substitute $k \mapsto -k$ so that we are considering vanishing equations of the form

$$E(a^{\mathbf{r},\mathbf{s}}_i - a^{\mathbf{r},\mathbf{s}}_{i-k}, \lambda^{(i)}, \lambda^{(i-k)}, \square) = 0, \tag{3.4.30}$$

$$\epsilon_1 + \epsilon_2 - E(a^{\mathbf{r},\mathbf{s}}_{i-k} - a^{r,s}_i, \lambda^{(i-k)}, \lambda^{(i)}, \square) = 0, \tag{3.4.31}$$

for some $\square \in \lambda^{(i)}$.

We repeat the proof above, where we now have the vanishing condition:

$$-\sum_{j=i-k}^{i-1} \left(r_j p' - s_j p\right) + p' L_{\lambda^{(i-k)}}(\square) + p A^+_{\lambda^{(i)}}(\square) + \eta = 0. \tag{3.4.32}$$

To eliminate zeroes in Z_{vec} the N-tuples of Young diagrams must then satisfy

$$(\lambda^{(i-k)})^T_{x+\sum_{j=i-k}^{i-1} s_j+c_d p'+\delta_{\eta n}-1} \leq (\lambda^{(i)})^T_x - \sum_{j=i-k}^{i-1} r_j + c_d p + \delta_{\eta n} - 1. \tag{3.4.33}$$

In this case we now have that $c_d \geq 1$, where this bound is obtained from analogous equations to (3.4.18) and (3.4.19). The equation satisfied by c_d is again (3.4.28), so that we see that $c_d = 1$ and $\eta = 0$ gives the strongest bounds. As before the $k = 1$ bound implies all subsequent inequalities, so that we can eliminate all poles occurring in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ by imposing

$$\lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+r_i-1} - s_i + 1. \tag{3.4.34}$$

Finally we can repeat these arguments for $k = N - i$ and $i = 0$ to obtain

$$\lambda^{(N-1)}_j \geq \lambda^{(0)}_{j+r_0-1} - s_0 + 1, \tag{3.4.35}$$

which completes the cyclic set of inequalities. □

We see that even though we only consider boxes with $\mathbb{Z}_n$-invariant contributions to the partitions function, we can still eliminate the poles in $Z^{U(N)}_{\mathbf{b},\mathbf{b}'}$ by imposing the Burge conditions on the N-tuples of coloured Young diagrams we sum over. It is important to note that for gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ we are considering *coloured* Burge multipartitions, whereas for gauge theories on $\mathbb{C}^2$ we consider *uncoloured* ones.

As a corollary to this proposition, we see that the generating function for the instantons for $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ under a minimal model identification is the generating function of Burge multipartitions (1.4.3) up to some q-factor. This fact is central to the work in chapters 4 and 5.

Chapter 4

Instanton Counting on $\mathbb{C}^2/\mathbb{Z}_n$ and $\widehat{\mathfrak{sl}}(n)_N$-WZW Models

In this chapter, we will test the specialization coset AGT conjecture to gauge theories under a minimal model identification, discussed in chapter 3, against $\widehat{\mathfrak{sl}}(n)_N$-WZW models. We will prove that a generalization of Burge generating functions, to a generating function of coloured Burge multipartitions, can be used to calculate the $\widehat{\mathfrak{sl}}(n)_N$-WZW characters, using the crystal graph techniques described in section 1.3.5. We will then compare simple cases of the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ with solutions to the KZ differential equation, which will reduce to linear combinations of hypergeometric functions. Using the discussion in 1.5.9, special cases of $\widehat{\mathfrak{sl}}(n)_N$-WZW 4-point conformal blocks satisfy this differential equation and this will test our proposed AGT identification from chapter 3.

4.1 Burge Conditions for $\mathcal{N}=2$ $SU(N)$ Gauge Theories when $p=N$

Here we consider the $\mathcal{N}=2$ $SU(N)$ gauge theories under a minimal model identification from chapter 3, and specialise to the case $p=N$ (cf: section 3.2). We will then see that the condition $p=N$ specializes the Burge conditions, which we use have a well-defined instanton partition function in these theories (see proposition 3.4.2.1), to a special case (4.1.8), which we will refer to as cylindric Burge conditions.

As discussed in chapter 3, when the Ω-deformation parameters ϵ_1 and ϵ_2 for the rational Ω-background are parameterized such that $p=N$ by

$$\frac{\epsilon_1}{\epsilon_2} = -\frac{n+p}{p} = -\frac{n+N}{N}, \quad n, N \in \mathbb{Z}_{>0}, \tag{4.1.1}$$

the $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ are conjectured, see 3.3.0.1, to be in AGT correspondence with 2D CFTs that have the symmetry algebra (see section 3.2)

$$\mathcal{A}(N, n; N) = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N \oplus \frac{\widehat{\mathfrak{sl}}(N)_n \oplus \widehat{\mathfrak{sl}}(N)_{N-N}}{\widehat{\mathfrak{sl}}(N)_{N+n-N}} = \mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N. \tag{4.1.2}$$

This symmetry algebra describes a CFT which is the combined system of an $\widehat{\mathfrak{sl}}(n)_N$-WZW model with a free boson.

Recall that for $\mathbf{r} = [r_0, r_1, \ldots, r_{N-1}] \in P^{++}_{N,p}$, and $\mathbf{s} = [s_0, s_1, \ldots, s_{N-1}] \in P^{++}_{N,p+n}$, we denote the Coulomb parameters of theories under a minimal model identification by $\mathbf{a}^{\mathbf{r},\mathbf{s}}$. We also note that the parameters $\mathbf{a}^{\mathbf{r},\mathbf{s}}$ correspond to the degenerate charge (3.4.4) of the conformal family that flows in the channel. Since $\sum_{i=0}^{N-1} r_i \leq p = N$ *and* $r_i > 0$ for all $i = 0, \ldots, N-1$, the only choice for these parameters when $p = N$ is

$$r_i = 1, \quad i = 0, 1, \ldots, N-1. \tag{4.1.3}$$

In this case, $\mathbf{r} = \mathbf{1} = [1, 1, \ldots, 1] \in P^{+}_{N,N}$ and we denote the Coulomb parameters by $a_i^{\mathbf{1},\mathbf{s}}$ or $a_i^{\mathbf{s}}$ for $i = 0, \ldots, N-1$ when it is clear to do so.

As discussed in section 3.4.1, the labels $\mathbf{r}$ and $\mathbf{s}$ are linked to the $\mathbb{Z}_n$-charges assigned to the Coulomb parameters, and equivalently, the charges of N-tuples of coloured Young diagrams, through the $\mathbb{Z}_n$-charge conditions (3.4.5). When $\mathbf{r} = \mathbf{1}$ the $\mathbb{Z}_n$-charge conditions become

$$\sigma_i - \sigma_{i+1} \equiv s_i - 1 \mod n. \tag{4.1.4}$$

Remembering that we order the charges $\{\sigma_i\}_{i=0,\ldots,N-1}$ by size, so that $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_{N-1}$, we have $|\sigma_i - \sigma_{i+1}| \leq n$ and we are free to define

$$s_i = \sigma_i - \sigma_{i+1} + 1, \quad i = 1, \ldots, N-1, \tag{4.1.5}$$

where $\sigma_N = \sigma_0$ and

$$s_0 = \sigma_0 - \sigma_1 + n + 1. \tag{4.1.6}$$

Remark 4.1.0.1. When $p = N = 1$ so that $\mathbf{r} = [1]$, the contributing instantons for the partition function are described by one coloured Young diagram $\lambda = (\lambda^{(0)})$. The singular Burge condition (3.4.7) for this case, where we take $s = s_0$, then reads

$$\lambda_j^{(0)} \geq \lambda_j^{(0)} - s + 1, \quad j = 1, \ldots, l(\lambda^{(0)}), \tag{4.1.7}$$

for $n + 1 \geq s \geq 1$. This is true for any Young diagram since $1 - s \leq 0$. $Z^{U(N)}_{inst}$ is then defined as a summation over all Young diagrams. On the CFT side, this is equivalent to the fact that there are no null states to remove for the highest weight $\mathcal{A}(1, n; 1)$-modules.

In the sequel we will specialise to $p = N > 1$. We substitute $\mathbf{r} = \mathbf{1}$ into the Burge inequalities (3.4.7) to obtain

$$\lambda_j^{(i)} \geq \lambda_j^{(i+1)} - s_i + 1, \quad i = 0, \dots, N-1, \quad j = 1, \dots, l(\lambda_j^{(i)}), \tag{4.1.8}$$

where we note that if $j > l(\lambda_j^{(i+1)})$, we take $\lambda_j^{(i+1)} = 0$.

Definition 4.1.0.2. A *coloured cylindric Burge multipartition* $\lambda = (\lambda^{(0)}, \dots, \lambda^{(N-1)})$ is an N-tuple of n-coloured Young diagrams that satisfy the inequalities (4.1.8) for some $\mathbf{s} \in P^{++}_{N,N+n}$.

This definition uses a dominant integral $\widehat{\mathfrak{sl}}(N)$ weight $\mathbf{s}$ with strictly positive Dynkin labels. We can instead express coloured cylindric Burge multipartitions using the notation of section 1.1, defining cylindric Burge multipartitions using a dominant integral $\widehat{\mathfrak{sl}}(N)$ weight ζ, which does not necessarily have strictly positive Dynkin labels.

We define the following $\widehat{\mathfrak{sl}}(n)$ weights, which are *not* strictly positive, $\zeta = \mathbf{s} - \mathbf{1} = [s_0 - 1, \dots, s_{N-1} - 1]$ and $\xi = \mathbf{0} = [0, \dots, 0]$, and $\Lambda = \sum_{i=0}^{N-1} \Lambda_{\sigma_i}$. Then, a coloured cylindric Burge multipartition for $\mathbf{s} \in P^{++}_{N,N+n}$ is a coloured $(\mathbf{0}, \zeta)$-Burge multipartition.

We denote by $\mathcal{C}_\Lambda^{(n;\zeta,\mathbf{0})}$ the set of N-tuples of coloured cylindric Burge multipartitions. Below in proposition 4.2.2.2, we will show that this set is equal to the set of cylindric partitions $\mathcal{M}_\sigma$ with their natural colouring. Note that this set is equal to $\mathcal{C}_\Lambda^n$ defined in section 1.1, and is the coloured version of the set $\mathcal{C}^{\mathbf{r},\mathbf{s}}$ defined in chapter 2. We have chosen to now emphasize the weights $(\mathbf{0}, \zeta)$ defining the Burge inequalities with this new notation. We will also equivalently refer to this set as $\mathcal{C}_\sigma^{\mathbf{r},\mathbf{s}}$ when emphasizing the charges $\sigma = (\sigma_0, \dots, \sigma_{N+1})$ and the CFT labels $\mathbf{r}$ and $\mathbf{s}$.

When it is clear if we are considering N-tuples of coloured Young diagrams that satisfy (4.1.8), we will refer to elements of $\mathcal{C}_\Lambda^{(n;\zeta,\mathbf{0})}$ as cylindric Burge multipartitions. Following the proof of 2.3.3.4, the specialised Burge inequalities (4.1.8) are equivalent to cylindric inequalities (1.1.11) on the transposed diagrams $\lambda^T = ((\lambda^{(0)})^T, \dots, (\lambda^{(N-1)})^T)$, hence the name cylindric Burge multipartitions.

4.2 Burge Generating Functions and $\widehat{\mathfrak{sl}}(n)_N$-WZW Characters

As usual, it is expected that the generating functions for instantons will agree with the characters of CFTs (1.5.10), up to some overall factors. When considering minimal model CFTs, this will mean that the Burge generating functions are expected to correspond to $\mathcal{A}(N, n; p)$-minimal model characters (see discussion in section 3.2), and, in the case of $p = N$, to a product of Heisenberg (the generating function of partitions (1.4.7)) and $\widehat{\mathfrak{sl}}(n)_N$-WZW characters (1.5.106). Since $\widehat{\mathfrak{sl}}(n)_N$-WZW primary fields form irreducible $\widehat{\mathfrak{sl}}(n)_N$-modules, we will use the work in sections 1.1 and 1.5.9 (based on [60, 142]) to write the WZW characters using generating functions for highest-lift cylindric partitions.

4.2.1 Defining New Generating Functions

In this section, we introduce a *refined Burge generating function* of coloured Burge multipartitions which carries the information of the Chern classes on the instanton bundle (3.1.11). Recall the Chern classes c_1, c_2 of the instanton bundle are given by

$$c_1 = \sum_{i=0}^{n-1} \mathfrak{c}_i c_1(\mathcal{T}_i), \quad c_2 = \sum_{i=0}^{n-1} \mathfrak{c}_i c_2(\mathcal{T}_i) + \frac{k}{|\mathbb{Z}_n|}, \tag{4.2.1}$$

where each line bundle $\mathcal{T}_i \to \mathbb{C}^2/\mathbb{Z}_n$ is associated to one of the 1-dimensional irreducible representations of $\mathbb{Z}_n$ (see section 3.1.1), and

$$\begin{aligned} \mathfrak{c}_i &= N_i - 2k_i + k_{i+1} + k_{i-1} \\ &= N_i - 2\delta k_i + \delta k_{i+1} + \delta k_{i-1}, \text{ where } \delta k_i := k_i - k_0. \end{aligned} \tag{4.2.2}$$

We introduce new formal parameters $\{t_i\}_{i=1,\dots,n-1}$ which we collect in a vector $\mathfrak{t} = (t_1, \dots, t_{n-1})$. We then refine the Burge generating function (1.4.3) using these parameters so that the exponents of $\{t_i\}_{i=1,\dots,n-1}$ in the refined generating function correspond to the values $\mathfrak{c}_i$ classifying the instanton solution through their Chern classes. Note, that for $i = 0$ the corresponding 1-dimensional irreducible representation is trivial so we do not introduce a parameter, say t_0, which corresponds to $\mathfrak{c}_0$.

Recall that the contribution of a residue, corresponding to an instanton, to the partition function is associated to an N-tuple of Young diagrams λ. We therefore also associate the Chern classes classifying this instanton solution to the N-tuple λ, and denote them by $\mathfrak{c}_i(\lambda)$.

Definition 4.2.1.1. The $\mathfrak{t}$-*refined Burge generating function* $X_\sigma^{\mathbf{r},\mathbf{s}}(q;\mathfrak{t})$ of coloured Burge multipartitions with colour content defined by $\delta\mathbf{k} = (\delta k_1, \dots, \delta k_{n-1})$ (cf: (1.4.3)) for $\mathbf{r} \in P_{N,p}^{++}$, $\mathbf{s} \in P_{N,p+n}^{++}$, and a partition $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_{N-1})$ with $\sigma_0 \le n$ is defined by

$$X_\sigma^{\mathbf{r},\mathbf{s}}(q;\mathfrak{t}) = \sum_{\lambda \in \mathcal{C}_\sigma^{\mathbf{r},\mathbf{s}}} q^{\frac{1}{n}|\lambda|} \prod_{i=1}^{n-1} \mathfrak{t}_i^{\mathfrak{c}_i(\lambda)}, \tag{4.2.3}$$

where $\mathfrak{c}_i$ is defined by (4.2.2).

Note that $X^{\mathbf{r},\mathbf{s}}_{\sigma}(q;\mathfrak{t})$ is the generating function of instantons for $SU(N)$ theories under a minimal model identification on $\mathbb{C}^2/\mathbb{Z}_n$, where the $\mathbb{Z}_n$-charges $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$ associated to the Coulomb parameters are fixed.

We can use the $\mathfrak{t}$-refined Burge generating function to count the contributing instantons of specific Chern classes. For fixed Chern classes, defined by fixed $\mathfrak{c}_i$ for $i = 1, \ldots, n-1$, the coefficient of $\prod_{i=1}^{n-1} t_i^{\mathfrak{c}_i(\lambda)}$ in $X^{\mathbf{r},\mathbf{s}}_{\sigma}(q;\mathfrak{t})$ (which is a series in q) counts the instantons classified by these Chern classes (which correspond to many different N-tuples of coloured Young diagrams $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})$).

This leads us to define a second new generating function $X^{\mathbf{r},\mathbf{s}}_{\sigma;\mathbf{l}}(q)$, which counts coloured cylindric Burge multipartitions with prescribed colour data defined by the vector of differences $\delta\mathbf{k}$. For a vector of integers $\mathbf{l} = (l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$ we define a generating function of Burge multipartitions with fixed colour data $\delta k_i = l_i$ for $i = 1, \ldots, n$ (note that this is a series *only* in q)

$$X^{\mathbf{r},\mathbf{s}}_{\sigma;\mathbf{l}}(q) = \sum_{\lambda \in \mathcal{C}^{\mathbf{r},\mathbf{s}}_{\sigma;\mathbf{l}}} q^{\frac{1}{n}|\lambda|}. \tag{4.2.4}$$

The $\mathfrak{c}_i$ defining the Chern classes depend on two vectors of integers: $\mathbf{N} = (N_0, \ldots, N_{n-1})$ where N_i is the number of diagrams with charge $i \in \mathbb{Z}_n$, and the colour differences $\delta\mathbf{k} = (\delta k_1, \ldots, \delta k_{n-1})$. From this, we can see that the Chern classes are fixed across sets of instantons corresponding to Young diagrams with fixed charges $\sigma = (\sigma_0, \ldots, \sigma_{N-1})$ and colour data defined by $\delta\mathbf{k}$. Thus, we can rewrite the $\mathfrak{t}$-refined Burge generating function (4.2.3) as a sum over the Burge generating functions with prescribed colour data $X^{\mathbf{r},\mathbf{s}}_{\sigma;\mathbf{l}}(q)$ as (note that we have substituted the definition (4.2.2), with fixed $\delta k_i = l_i$ for $i = 1, \ldots, n-1$, of $\mathfrak{c}_i$ in the exponent of $\mathfrak{t}_i$ here)

$$X^{\mathbf{r},\mathbf{s}}_{\sigma}(q;\mathfrak{t}) = \sum_{\mathbf{l} \in \mathbb{Z}^{n-1}} X^{\mathbf{r},\mathbf{s}}_{\sigma;\mathbf{l}}(q) \prod_{i=1}^{n-1} \mathfrak{t}_i^{N_i - 2l_i + l_{i+1} + l_{i-1}}. \tag{4.2.5}$$

4.2.2 Calculating $\widehat{\mathfrak{sl}}(n)_N$-WZW Characters Using the Instanton Generating Function

In this section we will prove that the generating function of instantons for $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ with a minimal model identification can be identified with the Vir character function $\chi^{\widehat{\mathfrak{sl}}(n)_N}_{\Lambda}$ (1.5.114) for integrable $\widehat{\mathfrak{sl}}(n)_N$-modules when $p = N$.

We begin by recalling some facts about integrable $\widehat{\mathfrak{sl}}(n)_N$-modules. Let $\Lambda = [d_0, \ldots, d_{n-1}] \in P^{+}_{n,N}$ and L_Λ be the irreducible highest weight $\widehat{\mathfrak{sl}}(n)_N$-module with highest weight Λ. As explained in section 1.5.9, L_Λ is a Vir-module and we can write its conformal dimension h_Λ

(as the eigenvalue of $L_0 \in \text{Vir}$ (1.5.96)) and central charge c (1.5.97) as

$$h_\Lambda = \frac{\langle \Lambda, \Lambda + 2\rho \rangle}{2(N+n)}, \qquad c = \frac{N(n^2-1)}{N+n}, \tag{4.2.6}$$

where $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ is the Weyl vector. Using the parameter $\hat{\mathfrak{t}} = (\hat{t}_1, \ldots, \hat{t}_{n-1})$, the (graded-) character for L_Λ as a Vir-module is defined to be

$$\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q, \hat{\mathfrak{t}}) = \mathrm{Tr}_{L_\Lambda} q^{L_0} \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{H_i}, \tag{4.2.7}$$

where $H_i \in \widehat{\mathfrak{h}} \subset \widehat{\mathfrak{sl}}(N)$ for $i = 1, \ldots, n-1$ are the Chevalley basis elements for the Cartan subalgebra. We will use this form of the character function to identify the $\mathfrak{c}_i$, defining the Chern classes classifying the instanton solutions on the gauge side of the conjecture 3.3.0.1, for $p = N$ with elements of the Cartan subalgebra of $\widehat{\mathfrak{sl}}(N)$ for the CFT side, below in proposition 4.2.2.2.

Having discussed the form of the character function on the CFT side, we now discuss the form of the instanton generating function we will use on the gauge side. We begin by noting that the Burge conditions for $p = N$ (4.1.8) have the form $\lambda_j^{(i)} \geq \lambda_j^{(i+1)} - s_i + 1$, which compares the size of the j-th row of i and $(i+1)$-th Young diagrams with a shift defined by the i-th element of the vector $\mathbf{s}$. We shall refer to pairs such as this as *sequential Young diagrams* (cf: the proof of 2.3.3.4). On the other hand, the set $\mathcal{M}^\sigma$ of cylindric multipartitions in the formula of $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}$ satisfies the inequalities $\lambda_j^{(i)} \geq \lambda_{j+\sigma_j-\sigma_{j+1}}^{(i+1)}$, for $j \geq 1$, $0 \leq i \leq N-2$ and $\lambda_j^{(N-1)} \geq \lambda_{j+\sigma_{N-1}-\sigma_0+n}^{(0)}$, for $j \geq 1$ (which are equations (1.4.10) and (1.4.11)), which compare *different* parts of sequential Young diagrams with *no* shift. To compare the generating function of instantons with the character function $\chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}$, we will use following the following lemma.

Lemma 4.2.2.1. ([43]) The map of Young diagrams $\lambda \mapsto \lambda^T$ gives the following equality of sets of Burge multipartitions

$$\mathcal{C}^{\mathbf{r},\mathbf{s}} = \mathcal{C}^{\mathbf{s},\mathbf{r}}. \tag{4.2.8}$$

Proof. This proof uses an idea from the proofs of the propositions 2.2.4.3, 2.3.3.4, 3.4.2.1. Consider an N-tuple of Young diagrams $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(N-1)})) \in \mathcal{C}^{\mathbf{r},\mathbf{s}}$ satisfies the Burge inequalities

$$\lambda_j^{(i)} \geq \lambda_{j+r_i-1}^{(i+1)} - s_i + 1. \tag{4.2.9}$$

The N-tuple of its transposes $\lambda^T = ((\lambda^{(0)})^T, \ldots, (\lambda^{(N-1)})^T)$ then satisfy different Burge inequalities given by

$$(\lambda^{(i)})_j^T \geq (\lambda^{(i+1)})_{j+s_i-1}^T - r_i + 1, \tag{4.2.10}$$

so that $\lambda^T \in \mathcal{C}^{\mathbf{s},\mathbf{r}}$. Thus, the map $\lambda \mapsto \lambda^T$ provides a bijection between $\mathcal{C}^{\mathbf{r},\mathbf{s}}$ and $\mathcal{C}^{\mathbf{s},\mathbf{r}}$ as required. $\square$

Putting together both sides of this discussion leads us to the following proposition.

Proposition 4.2.2.2. *([43]) For a partition $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$ for which $\sigma_0 < n$, define $\boldsymbol{s} = [s_0, s_1, \ldots, s_{N-1}]$ by (4.1.5) and (4.1.6), and set $\Lambda = \sum_{i=0}^{N-1} \Lambda_{\sigma_i}$. Then*

$$X_\sigma^{\boldsymbol{1},\boldsymbol{s}}(q, \mathfrak{t}) = \frac{q^{w_\Lambda - h_\Lambda}}{(q,q)_\infty} \chi_\Lambda^{\widehat{\mathfrak{sl}}(n)_N}(q, \hat{\mathfrak{t}}), \tag{4.2.11}$$

where $\hat{\mathfrak{t}} = (\hat{\mathfrak{t}}_1, \ldots, \hat{\mathfrak{t}}_{n-1})$ is related to $\mathfrak{t} = (\mathfrak{t}_1, \ldots, \mathfrak{t}_{n-1})$ by

$$\hat{\mathfrak{t}}_i = q^{-\frac{1}{2n} i(n-i)}\, \mathfrak{t}_i \tag{4.2.12}$$

for $1 \le i < n$, and

$$w_\Lambda = \frac{1}{2n} \sum_{i=1}^{n-1} i(n-i) N_i, \tag{4.2.13}$$

where $\Lambda = [N_0, N_1, \ldots, N_{n-1}]$.

Remark 4.2.2.3. The partition σ defines the charges of the component partitions of the Burge multipartitions.

Proof of Proposition 4.2.2.2. Comparison of the conditions (1.4.10) and (1.4.11) with (3.4.7) for $\mathbf{r} = \mathbf{1}$ shows that there is a bijection $\mathcal{M}^\sigma \to \mathcal{C}^{\mathbf{s},\mathbf{1}}$ (note that in this case, we have *reversed* the usual order of $\mathbf{r}$ and $\mathbf{s}$), with the map $\lambda^\sigma \mapsto \lambda$ from the former to the latter being the forgetful map, which maps a coloured Young diagram λ^σ to the uncoloured Young diagram obtained by forgetting its colouring.

We have chosen to notate the charges for λ to emphasize the contrast between the uncoloured $\lambda \in \mathcal{C}^{\mathbf{s},\mathbf{1}}$ and coloured nature of $\lambda^\sigma \in \mathcal{M}^\sigma$. Combining this with the bijection described by (4.2.8) then yields a bijection $\mathcal{M}^\sigma \to \mathcal{C}^{\mathbf{1},\mathbf{s}}$ described by $\lambda^\sigma \mapsto \lambda \mapsto \lambda^T$.

Because of the differing ways in which the colours are ordered in $\mathcal{M}^\sigma$ and $\mathcal{C}_\sigma^{\mathbf{1},\mathbf{s}}$, colouring λ^T to give an element of $\mathcal{C}_\sigma^{\mathbf{1},\mathbf{s}}$, results in the coloured Young diagram $(\lambda^T)^\sigma$. As explained in lemma 4.2.2.1 we are free to describe Burge, and therefore cylindric, multipartitions using diagrams λ or their transposes λ^T. Thus, in the expression (4.2.3), $\mathcal{C}_\sigma^{\mathbf{s},\mathbf{1}} \equiv \mathcal{C}_\sigma^{\mathbf{1},\mathbf{s}}$ can be replaced by $\mathcal{M}^\sigma$. Noting that $|\lambda| = \sum_{i=0}^{n-1} k_i(\lambda) = nk_0(\lambda) + \sum_{i=1}^{n-1} \delta k_i(\lambda)$, and using (4.2.2), then gives

$$\begin{aligned} X_\sigma^{\mathbf{1},\mathbf{s}}(q, \mathfrak{t}) &= \sum_{\lambda \in \mathcal{M}^\sigma} q^{k_0(\lambda)} \prod_{i=1}^{n-1} q^{\frac{1}{n}\delta k_i(\lambda)} \mathfrak{t}_i^{N_i + \delta k_{i-1}(\lambda) - 2\delta k_i(\lambda) + \delta k_{i+1}(\lambda)} \\ &= \sum_{\lambda \in \mathcal{M}^\sigma} q^{k_0(\lambda)} \prod_{i=1}^{n-1} \mathfrak{t}_i^{N_i} \left(q^{\frac{1}{n}} \frac{\mathfrak{t}_{i-1}\mathfrak{t}_{i+1}}{\mathfrak{t}_i^2} \right)^{\delta k_i(\lambda)}, \end{aligned} \tag{4.2.14}$$

where we set $\mathfrak{t}_0 = \mathfrak{t}_n = 1$. Substituting for each $\mathfrak{t}_i$ using (4.2.11) then shows that

$$X_\sigma^{\mathbf{1},\mathbf{s}}(q,\mathfrak{t}) = q^{w_\Lambda} \sum_{\lambda\in\mathcal{M}^\sigma} q^{k_0(\lambda)} \prod_{i=1}^{n-1} \hat{\mathfrak{t}}_i^{N_i} \left(\frac{\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_{i+1}}{\hat{\mathfrak{t}}_i^2}\right)^{\delta k_i(\lambda)} \tag{4.2.15}$$

which yields (4.2.11) using (1.5.114). □

We now recall that the primary fields of $\widehat{\mathfrak{sl}}(N)$-WZW models form integrable highest weight representations of $\widehat{\mathfrak{sl}}(N)$, so that the WZW characters are also $\widehat{\mathfrak{sl}}(N)$ characters for a highest weight module L_Λ. We can use this fact to re-express the cylindric Burge generating function for fixed colours $X_{\sigma;\mathbf{l}}^{\mathbf{1},\mathbf{s}}(q)$ in (4.2.4) in terms of the $\widehat{\mathfrak{sl}}(n)$ string functions $\sigma_\mu^\Lambda(q)$ in (1.3.32) through the identification (4.2.11). To do so, we define a different notation for the $\widehat{\mathfrak{sl}}(n)_N$ string functions, using a vector of integers $\mathbf{l} = (l_1,\dots,l_{n-1}) \in \mathbb{Z}^{n-1}$, which parameterizes the weight μ of the descendent state in L_Λ while ignoring the grade as,

$$\sigma_{\mathbf{l}}^\Lambda(q) = \sigma^\Lambda_{\Lambda-\sum_{i=1}^{n-1} l_i\alpha_i}(q). \tag{4.2.16}$$

The descendant state $\Lambda - \sum_{i=1}^{n-1} l_i\alpha_i$ is obtained by a sequence composed of l_i applications of the lowering operators f_i for $i = 1,\dots,n-1$. Using this, we have the following corollary to proposition 4.2.2.2:

Corollary 4.2.2.4. *For a partition $\sigma = (\sigma_0,\sigma_1,\dots,\sigma_{N-1})$ for which $\sigma_0 < n$, define $\boldsymbol{s} = [s_0, s_1, \dots, s_{N-1}]$ by (4.1.5) and (4.1.6), and set $\Lambda = \sum_{i=1}^N \Lambda_{\sigma_i}$. Then for each $\boldsymbol{l} = (l_1,\dots, l_{n-1}) \in \mathbb{Z}^{n-1}$,*

$$X_{\sigma;\boldsymbol{l}}^{\boldsymbol{1},\boldsymbol{s}}(q) = \frac{q^{\frac{1}{n}|\boldsymbol{l}|}}{(q,q)_\infty}\,\sigma_{\boldsymbol{l}}^\Lambda(q), \tag{4.2.17}$$

where we set $|\boldsymbol{l}| = \sum_{i=1}^{n-1} l_i$.

Proof. This results from reexpressing the left and right sides of (4.2.11) using (1.5.110) and (4.2.12), and then using the fact that the finite Cartan matrix $\bar{A}$ is invertible. □

Until now, we have considered the Burge generating function (that is, the generating function for instantons) $X_\sigma^{\mathbf{1},\mathbf{s}}$ to be defined for a fixed vector of charges for an N-tuple of Young diagrams $\sigma = (\sigma_0,\dots,\sigma_{N-1})$, while we have considered the $\widehat{\mathfrak{sl}}(n)_N$ characters to be defined for a dominant integral weight $\Lambda \in P^+(\widehat{\mathfrak{sl}}(n))$. Thus, to connect both sides of (4.2.12) (or equivalently (4.2.17)), we will instead define $X_\sigma^{\mathbf{1},\mathbf{s}}$ for a dominant integral $\widehat{\mathfrak{sl}}(n)$ weight.

This will give us a uniform way of notating character and generating functions on both sides of our generalized conjecture, allowing one to see that our proposed identifications are natural. In the sequel, we will be notating both the generating function of instantons and the $\widehat{\mathfrak{sl}}(n)_N$ character functions using *one* dominant integral weight.

To do this, we use the vector of integers $\mathbf{N} = (N_0, \ldots, N_{n-1})$ used to define the Chern classes of the instanton bundle through (3.1.11). In terms of the N-tuples of coloured Young diagrams labelling an instanton, each N_i corresponds to the number of Young diagrams of charge i. Then, as each $N_i \geq 0$, we can instead consider $\mathbf{N}$ to be a dominant integral $\widehat{\mathfrak{sl}}(n)_N$ weight.

Since we have ordered the Young diagram charges by size, specifying the vector $\mathbf{N}$ is equivalent to specifying the vector of charges $\sigma = (\sigma_0, \ldots, \sigma_{N-1})$. So, we let $\mathbf{N} = [N_0, N_1, \ldots, N_{n-1}] \in P^+_{n,N}$ be such that each $N_i \geq 0$ with $\sum_{i=0}^{n-1} N_i = N$. If we regard $\mathbf{N}$ as an $\widehat{\mathfrak{sl}}(n)$ weight, then $\mathbf{N} = \sum_{i=0}^{n-1} N_i \Lambda_i \in P^+_{n,N}$. We rewrite $\mathbf{N} = \sum_{j=0}^{N-1} \Lambda_{\sigma_j}$ and define $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{N-1})$. Then $\sigma = \lambda^T$, the partition conjugate to $\lambda = \mathrm{par}(\mathbf{N})$ defined by (1.3.20).

Remark 4.2.2.5. Note that in this case, $\mathbf{N}$ is a cyclic permutation of the dual weight $(\mathbf{s}-\mathbf{1})^\dagger \in P^+_{n,N}$ (see (1.3.42)). In chapter 5, we will extend this process for identifying the generating function of cylindric Burge partitions with a CFT character to general Burge partitions.

Now define $\mathbf{s} = [s_0, s_1, \ldots, s_{N-1}]$ by (4.1.5) and (4.1.6), and define the $SU(N)$ $\mathfrak{t}$-refined Burge-reduced generating function of coloured Young diagrams, by factoring out the Heisenberg factor $\mathcal{H}$, whose character is $\chi_{\mathcal{H}}(q) = (q;q)_\infty^{-1}$, by

$$\widehat{X}^{\mathrm{red}}_{\mathbf{N}}(q, \mathfrak{t}) := (q,q)_\infty \times X^{\mathbf{1},\mathbf{s}}_{\sigma}(q, \mathfrak{t}). \tag{4.2.18}$$

We choose to factorize the Burge generating function in this way to match the form of $\mathcal{A}(N, n; N)$ (4.1.2). This factorization leads us to propose that $\widehat{X}^{\mathrm{red}}_{\mathbf{N}}$ is the natural gauge theoretic object to be identified with the $\widehat{\mathfrak{sl}}(n)_N$-WZW characters (that is, not with the character of the combined system $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N$) through the coset AGT conjecture 3.3.0.1 for gauge theories under a minimal model identification when $p = N$. This is shown through the following corollary to proposition 4.2.2.2.

Corollary 4.2.2.6. *If* $\boldsymbol{N} \in P^+_{n,N}$*, then*

$$\widehat{X}^{\mathrm{red}}_{\boldsymbol{N}}(q, \mathfrak{t}) = q^{w_{\boldsymbol{N}} - h_{\boldsymbol{N}}} \chi^{\widehat{\mathfrak{sl}}(n)_N}_{\boldsymbol{N}}(q, \hat{\mathfrak{t}}), \tag{4.2.19}$$

where $\hat{\mathfrak{t}}$ *is related to* $\mathfrak{t}$ *by* (4.2.12)*, and* $h_{\boldsymbol{N}}$ *and* $w_{\boldsymbol{N}}$ *are given by* (4.2.6) *and* (4.2.13)*.*

From this corollary we see that $\widehat{X}^{\mathrm{red}}_{\mathbf{N}}$ *is* the natural object to use in our generalized AGT conjecture, as we predicted, since it is equal term-by-term, up to a factor of $q^{w_{\mathbf{N}} - h_{\mathbf{N}}}$, with an $\widehat{\mathfrak{sl}}(n)_N$-WZW character.

This corollary implies that the Chern classes (3.1.11) on the gauge side are AGT dual objects to the eigenvalues of Cartan elements H_i of $\widehat{\mathfrak{sl}}(n)$ on the CFT side, by identifying the $\mathfrak{c}_i$ with the H_i. To see this, we expand both sides of (4.2.19) term-by-term and note that in the factorization (4.2.18) we have not factorized out any function of the parameters $\{\mathfrak{t}_i\}_{i=1,\ldots,n-1}$. Thus in the q-series expansion of $\widehat{X}^{\text{red}}_{\mathbf{N}}$, we retain the form of the $\mathfrak{t}$ factors of $X^{\mathbf{r},\mathbf{s}}_{\sigma}$. By using the $\mathfrak{t}$ factors in (4.2.3), which carry the information of the Chern classes, and comparing them to the $\mathfrak{t}$ factors in (4.2.7), we see that we can identify the Chern classes with the Cartan eigenvalues. This natural identification between the Chern classes on the gauge side and the Cartan eigenvalues on the CFT side of the generalized AGT conjecture, provides strong evidence for its validity. We note that this identification supports our observations in remark 3.1.1.2.

Example 4.2.2.7. In the case of $N=1$, (4.2.19) is particularly simple, because then $h_{\mathbf{N}} = w_{\mathbf{N}}$. For instance, for $(N,n)=(1,2)$ we have

$$\begin{aligned}
\widehat{X}^{red}_{[1,0]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X_{(0);(-l)}(q)\,\mathfrak{t}^{2l} = \frac{1}{(q;q)_\infty}\sum_{j\in\mathbb{Z}} q^{j^2}\,\hat{\mathfrak{t}}^{\,2j} = \chi^{\widehat{\mathfrak{sl}}(n)_N}_{[1,0]}(q,\hat{\mathfrak{t}}),\\
\widehat{X}^{red}_{[0,1]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X_{(1);(-l)}(q)\,\mathfrak{t}^{2l+1} = \frac{1}{(q;q)_\infty}\sum_{j\in\mathbb{Z}+\frac{1}{2}} q^{j^2}\,\hat{\mathfrak{t}}^{\,2j} = \chi^{\widehat{\mathfrak{sl}}(n)_N}_{[0,1]}(q,\hat{\mathfrak{t}}),
\end{aligned} \tag{4.2.20}$$

where $\hat{\mathfrak{t}} = q^{-\frac{1}{4}}\,\mathfrak{t}$.

In section 4.4, we will give explicit examples of corollary 4.2.2.6 for $(N,n)=(2,2)$, $(2,3)$ and $(3,2)$ by comparing with the $\widehat{\mathfrak{sl}}(n)_N$-WZW characters computed using the Weyl-Kac character formula (1.3.36).

Note that when $\mathfrak{t}=(1,\ldots,1)$, we have that

$$\widehat{X}^{\text{red}}_{\mathbf{N}}(q,(1,\ldots,1)) = (q,q)_\infty \times X^{\mathbf{1},\mathbf{s}}_{\sigma}(q), \tag{4.2.21}$$

gives the $\widehat{\mathfrak{sl}}(n)$ principally specialised character (1.3.38). Therefore we say $\mathfrak{t}=(1,\ldots,1)$ is the principally specialised case of the $\mathfrak{t}$-refined Burge reduced generating function.

4.3 Burge-reduced instanton partition functions and $\widehat{\mathfrak{sl}}(n)_N$-WZW conformal blocks

In this section, we use the generalized AGT conjecture 3.3.0.1 for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ under a minimal model matching for $p=N$ to extract integrable $\widehat{\mathfrak{sl}}(n)_N$-WZW conformal blocks from the well-defined instanton partition function $\widehat{\mathcal{Z}} := Z^{SU(N)}$. We then make explicit conjectures of this nature for specific choices of parameters, such that on the CFT side we extract conformal blocks that satisfy the KZ differential equation (1.5.117) when it reduces to the hypergeometric differential equation (specifically, those of example 1.5.10.1). Finally, we show that the series expansions of the instanton partition function for these cases matches the of solutions to the KZ equation, and hence integrable $\widehat{\mathfrak{sl}}(n)_N$ conformal blocks, up to a given order.

4.3.1 $U(1)$ instanton partition function

We begin by considering $U(1)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$, which are conjectured to be AGT dual with CFTs whose symmetry algebra is the $N=1$ algebra $\mathcal{A}(1,n;p) = \mathcal{H}\oplus\widehat{\mathfrak{sl}}(n)_1$. Note that since $N=1$, the instantons are labelled by *one* coloured Young diagram. Equivalently, we consider cases where $\mathbf{N}$ is a fundamental $\widehat{\mathfrak{sl}}(n)$ weight. We choose to notate this level 1 weight $\mathbf{N}=\mathbf{N}_\sigma$, where σ is the charge of Young diagram.

We now consider the instanton partition function for this gauge theory. Following corollary 4.2.2.6, the corresponding module in $\widehat{\mathfrak{sl}}(n)_1$ is the highest-weight module with $\Lambda=\Lambda_\sigma$. For $b,b'\in\mathbb{Z}_n$, $m,m'\in\mathbb{C}$, and $N_\sigma\in P^+_{n,1}$ we define

$$Z^{b,b'}_{\mathbf{N}_\sigma}(m,m';q) =: Z^{U(N)}_{\sigma;0}(0,b,b',m,,m';q), \tag{4.3.1}$$

and make the following conjecture.

Conjecture 4.3.1.1. *The $U(1)$ instanton partition function (4.3.1) on $\mathbb{C}^2/\mathbb{Z}_n$ with $b'=b$ and $\boldsymbol{N}_0=[1,0,\dots,0]$ is*

$$Z^{b,b}_{\boldsymbol{N}_0}(m,m';q) = (1-q)^{\frac{m(\epsilon_1+\epsilon_2-m')}{n\,\epsilon_1\,\epsilon_2}}\,(1-q)^{-2\,h_b}\,, \tag{4.3.2}$$

where $h_b = h_{\boldsymbol{N}_b} = \frac{b(n-b)}{2n}$ is the conformal dimension of the highest-weight state $|N_b\rangle$ in the $\widehat{\mathfrak{sl}}(n)_1$-WZW model. The first factor is the $U(1)$ factor $Z_{\mathcal{H}}(m,m';q)$ in (3.1.38) for $N=1$, and the second factor is the 2-point function (see (1.5.27)) of $\widehat{\mathfrak{sl}}(n)_1$-WZW primary fields with highest-weights Λ_b and Λ_{n-b}

Note that in this case the Burge conditions are vacuous. We also note that we have a closed form expression for the CFT (right-hand) side. In general, only the 2-point function is sufficiently restricted by conformal invariance which gives one simple closed form expressions such as these on the CFT side, while on the gauge side only when $N=1$ will the instanton partition function involve an unrestricted sum over Young diagrams.

4.3.2 $SU(N)$ Burge-reduced instanton partition functions

For $N\geq 2$, in the same way that we defined the Burge-reduced generating function (4.2.18) of coloured Young diagrams, we now introduce a reduced version of the instanton partition function (3.1.34) by imposing the specialized Burge conditions (3.4.7) with $\mathbf{r} = \mathbf{1} \in P^{++}_{N,N}$ and $\mathbf{s} \in P^{++}_{N,N+n}$,

$$\mathcal{Z}^{\mathbf{s};\mathbf{b},\mathbf{b}'}_{\sigma;\mathbf{l}}\left(\mathbf{a},\mathbf{m},\mathbf{m}';q\right) = \sum_{\lambda^\sigma \in \mathcal{C}^{\mathbf{s}}_{\sigma;\mathbf{l}}} \frac{Z_{\text{bif}}\left(\mathbf{m},\boldsymbol{\emptyset}^{\mathbf{b}};\mathbf{a},\lambda^\sigma\right) Z_{\text{bif}}\left(\mathbf{a},\lambda^\sigma;-\mathbf{m}',\boldsymbol{\emptyset}^{\mathbf{b}'}\right)}{Z_{\text{vec}}\left(\mathbf{a},\lambda^\sigma\right)} q^{\frac{1}{n}|\lambda^\sigma|}, \tag{4.3.3}$$

where $\sum_{i=0}^{N-1} a_i = 0$ is assumed. The Coulomb parameters $\mathbf{a} = (a_0,\ldots,a_{N-1})$, and the mass parameters $\mathbf{m} = (m_0,\ldots,m_{N-1})$, $\mathbf{m}' = (m'_0,\ldots,m'_{N-1})$, are related to the internal charge α^s, and the external charge $\alpha_{r=0,1,2,3}$, of a 4-point conformal block in a $\mathcal{W}^{para}_{n,N}$ CFT, by the relations (3.3.7) and (3.3.5) respectively. The gauge theory in the rational Ω-background (3.2.2) for $p = N$,

$$\frac{\epsilon_1}{\epsilon_2} = -1 - \frac{n}{N}, \tag{4.3.4}$$

is expected, via the proposed AGT correspondence, to describe a minimal model CFT whose charge take the degenerate values (3.4.1) when $r_i = 1$ for $i = 0,\ldots,N-1$,

$$\begin{aligned} 2\,\alpha^s &= -\sum_{i=1}^{N-1}(s_i-1)\,\epsilon_2\,\overline{\Lambda}_i, \\ \overset{(3.3.7)}{\Longrightarrow} \quad a_{i-1} = a^{\mathbf{s}}_{i-1} &:= -\sum_{j=i}^{N-1}\left(s_j - 1 - \frac{n}{N}\right)\epsilon_2 + \frac{1}{N}\sum_{j=1}^{N-1} j\left(s_j - 1 - \frac{n}{N}\right)\epsilon_2, \end{aligned} \tag{4.3.5}$$

parametrized by $\mathbf{s} = [s_0, s_1, \ldots, s_{N-1}] \in P^{++}_{N,N+n}$. Using these degenerate values, we write the charge of the 4 primary fields on the CFT side using the new parameters $\mathbf{s}^{(0)} = [s^{(0)}_0, s^{(0)}_1, \ldots, s^{(0)}_{N-1}] \in P^{++}_{N,N+n}$, $\mathbf{s}^{(3)} = [s^{(3)}_0, s^{(3)}_1, \ldots, s^{(3)}_{N-1}] \in P^{++}_{N,N+n}$, and

$$\begin{aligned} \mathbf{s}^{(1)} &= [s^{(1)}_0, s^{(1)}_1, \ldots, s^{(1)}_{N-1}] = [s^{(1)}_0, 1, \ldots, 1, s^{(1)}_{N-1}] \in P^{++}_{N,N+n}, \\ \mathbf{s}^{(2)} &= [s^{(2)}_0, s^{(2)}_1, \ldots, s^{(2)}_{N-1}] = [s^{(2)}_0, s^{(2)}_1, 1, \ldots, 1] \in P^{++}_{N,N+n}, \end{aligned} \tag{4.3.6}$$

which parameterize the mass parameters $\mathbf{m}$ and $\mathbf{m}'$ for a gauge theory under a minimal model identification

$$\begin{aligned} 2\,\alpha_0 &= -\sum_{i=1}^{N-1}\left(s^{(0)}_i - 1\right)\epsilon_2\,\overline{\Lambda}_i, \quad 2\,\alpha_1 = -\left(s^{(1)}_{N-1} - 1\right)\epsilon_2\,\overline{\Lambda}_{N-1}, \\ 2\,\alpha_3 &= -\sum_{i=1}^{N-1}\left(s^{(3)}_i - 1\right)\epsilon_2\,\overline{\Lambda}_i, \quad 2\,\alpha_2 = -\left(s^{(2)}_1 - 1\right)\epsilon_2\,\overline{\Lambda}_1. \end{aligned} \tag{4.3.7}$$

Note that we have $\alpha_1 \propto \overline{\Lambda}_{N-1}$ and $\alpha_1 \propto \overline{\Lambda}_1$. This allows us express the dual gauge theory mass parameters using (3.3.2) and (3.3.5) as

$$\begin{aligned} m_{i+1} = m_{i+1}^{\mathbf{s}^{(0)},\mathbf{s}^{(1)}} :=& -\left(i - \frac{N+1}{2}\right)\frac{n}{N}\epsilon_2 \\ &+ \frac{1}{N}\left(\sum_{j=1}^{i-1} j\left(s_j^{(0)}-1\right) - \sum_{j=i}^{N-1}(N-j)\left(s_j^{(0)}-1\right) - \left(s_{N-1}^{(1)}-1\right)\right)\epsilon_2, \end{aligned} \tag{4.3.8}$$

$$\begin{aligned} m'_{i+1} = m'^{\mathbf{s}^{(2)},\mathbf{s}^{(3)}}_{i+1} :=& \left(i - \frac{N+1}{2}\right)\frac{n}{N}\epsilon_2 \\ &+ \frac{1}{N}\left(-\sum_{j=1}^{i-1} j\left(s_j^{(3)}-1\right) + \sum_{j=i}^{N-1}(N-j)\left(s_j^{(3)}-1\right) - \left(s_1^{(2)}-1\right)\right)\epsilon_2. \end{aligned} \tag{4.3.9}$$

By using (4.1.5), we can fix the $\mathbf{s}$ parameters defining the Coulomb parameters for gauge theories under a minimal model identification as $s_i = \sigma_{i-1} - \sigma_i + 1$, from the ordered $\mathbb{Z}_n$-charges $\sigma_0 \geq \ldots \geq \sigma_{N-1}$. Similarly, we now fix the $\mathbf{s}$ parameters $\mathbf{s}^{(0)}$ and $\mathbf{s}^{(3)}$ defining the mass parameters using the $\mathbb{Z}_n$-boundary charges as follows. Taking a shift by the central $U(1)$ factor in the $U(N)$ flavor symmetry (from (3.3.5)) into account, one obtains the $\mathbb{Z}_n$-boundary charge conditions (note the similarity to the $\mathbb{Z}_n$-charge conditions obtained for the Coulomb parameters in (3.4.5))

$$s_i^{(0)} - 1 \equiv b_{i-1} - b_i \mod n, \quad s_i^{(3)} - 1 \equiv b'_{i-1} - b'_i \mod n, \quad i = 1, \ldots, N-1. \tag{4.3.10}$$

We can then determine the independent parameters in $\mathbf{s}^{(0)}$ and $\mathbf{s}^{(3)}$ as

$$s_i^{(0)} = b_{i-1} - b_i + 1, \quad s_i^{(3)} = b'_{i-1} - b'_i + 1, \quad i = 1, \ldots, N-1. \tag{4.3.11}$$

The remaining independent parameters $s_{N-1}^{(1)}$ and $s_1^{(2)}$ in (4.3.6) will be determined below in (4.3.15), by imposing that these parameters satisfy the $\widehat{\mathfrak{sl}}(n)$-WZW fusion rules on the CFT side in the conformal blocks, when applying the conjecture.

By factorizing out the $U(1)$ factor (3.1.38), as in the case of the $\mathfrak{t}$-refined Burge-reduced generating function (4.2.18), we define a Burge-reduced instanton partition function labelled by $\mathbf{N} = [N_0, \ldots, N_{n-1}] \in P^+_{n,N}$, $\mathbf{l} = (l_1, \ldots, l_{n-1}) \in \mathbb{Z}^{n-1}$, and $\mathbb{Z}_n$-boundary charges $\mathbf{b} = (b_0, \ldots, b_{N-1})$ and $\mathbf{b}' = (b'_0, \ldots, b'_{N-1})$ as follows.

Definition 4.3.2.1. The $SU(N)$ Burge-reduced instanton partition function is defined by [43]

$$\widehat{\mathcal{Z}}^{\mathbf{b},\mathbf{b}'}_{\mathbf{N};\mathbf{l}}(q) = Z_{\mathcal{H}}\left(\mathbf{m}^{\mathbf{s}^{(1)},\mathbf{s}^{(2)}}, \mathbf{m'}^{\mathbf{s}^{(3)},\mathbf{s}^{(4)}}; q\right)^{-1} \times \mathcal{Z}^{\mathbf{s};\mathbf{b},\mathbf{b}'}_{\sigma;\mathbf{l}}\left(\mathbf{a}^{\mathbf{s}}, \mathbf{m}^{\mathbf{s}^{(1)},\mathbf{s}^{(2)}}, \mathbf{m'}^{\mathbf{s}^{(3)},\mathbf{s}^{(4)}}; q\right). \quad (4.3.12)$$

Here the Coulomb parameters $\mathbf{a}^{\mathbf{s}} = (a^{\mathbf{s}}_0, \ldots, a^{\mathbf{s}}_{N-1})$ are given by (4.3.5) with $s_i = \sigma_{i-1} - \sigma_i + 1$ in (4.1.5), and the mass parameters $\mathbf{m}^{\mathbf{s}^{(0)},\mathbf{s}^{(1)}} = (m_0^{\mathbf{s}^{(0)},\mathbf{s}^{(1)}}, \ldots, m_{N-1}^{\mathbf{s}^{(0)},\mathbf{s}^{(1)}})$ and $\mathbf{m'}^{\mathbf{s}^{(2)},\mathbf{s}^{(3)}} = (m'^{\mathbf{s}^{(2)},\mathbf{s}^{(3)}}_0, \ldots, m'^{\mathbf{s}^{(2)},\mathbf{s}^{(3)}}_{N-1})$ are given by (4.3.8) and (4.3.9) with $s_i^{(0)}$, $s_i^{(3)}$ in (4.3.10) and $s^{(1)}_{N-1}$, $s^{(2)}_1$ determined below in (4.3.15) with the fusion rules.

By corollary 4.2.2.6, the set $\{N_i\}_{i=0,\ldots,n-1}$, determined from the $\mathbb{Z}_n$-charges σ, is identified with the eigenvalues of the action of the $\widehat{\mathfrak{sl}}(n)_N$ Cartan algebra. Therefore we take the vector $\mathbf{N}$ to define a level N highest weight for the $\widehat{\mathfrak{sl}}(n)_N$-WZW model.

We now use the preceding discussion to conjecture which $\widehat{\mathfrak{sl}}(n)_N$-WZW 4-point conformal blocks are dual to $\widehat{\mathcal{Z}}^{\mathbf{b},\mathbf{b}'}_{\mathbf{N};\mathbf{l}}(q)$. We begin by forming two new $\widehat{\mathfrak{sl}}(n)_N$ weights associated to the $\mathbb{Z}_n$-boundary charges by (note that this defines two sets of Dynkin labels as well)

$$\mathbf{B} = [B_0, B_1, \ldots, B_{n-1}] := \sum_{i=1}^{N} \Lambda_{b_i}, \qquad \mathbf{B}' = [B'_0, B'_1, \ldots, B'_{n-1}] := \sum_{i=1}^{N} \Lambda_{b'_i}. \quad (4.3.13)$$

We recall our discussion of 4-point conformal blocks in section 1.5.3. We consider the conformal blocks from the 4-point correlation function between primary fields

$$\langle \psi_{\alpha_0}(\infty)\psi_{\alpha_1}(1)\psi_{\alpha_2}(q)\psi_{\alpha_3}(0)\rangle^{\widehat{\mathfrak{sl}}(n)}_{\mathbb{P}^1}, \quad (4.3.14)$$

which we calculate by taking an OPE between $\psi_{\alpha_1}(1)\psi_{\alpha_2}(q)$, so that our calculation is for the s-channel. We also recall that this 4-point function satisfies the KZ differential equation, and that we can reduce this KZ differential equation to the hypergeometric differential equation for special choices of primary fields (see section 1.5.10).

We propose that for the $\widehat{\mathfrak{sl}}(n)_N$-WZW 4-point conformal blocks, the integrable representations of the primary fields for the two external legs are of highest weight, with the highest weights being $\mathbf{B}$ and $\mathbf{B}'$, the $\widehat{\mathfrak{sl}}(n)_N$ weights corresponding to the $\mathbb{Z}_n$-boundary charges $\mathbf{b}$ and $\mathbf{b}'$ defined above. We then notate the highest weights of the primary fields corresponding to the internal legs as $\mathbf{B}_c, \mathbf{B}'_c \in P(\widehat{\mathfrak{sl}}(n))$ (note that as of now, we have *not* fixed these) and now represent the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{\mathbf{b},\mathbf{b}'}_{\mathbf{N};\mathbf{l}}(q)$ (4.3.12) pictorially in figure 4.1 (cf: figure 1.3).

In this diagram, we have labelled the primary fields by their associated dominant integrable $\widehat{\mathfrak{sl}}(n)_N$ highest weights. In terms of the labelling of primary fields used for figure 1.3, we have

$$\mathbf{B} \sim \phi_1, \quad \mathbf{B}_c \sim \phi_2, \quad \mathbf{B}'_c \sim \phi_3, \quad \mathbf{B}' \sim \phi_4, \quad \mathbf{N} \sim \phi_s.$$

We have additionally included the associated vectors of $\mathbb{Z}_n$-charges on the gauge side, which are used in the definition of the instanton partition function, directly below their associated $\widehat{\mathfrak{sl}}(n)$ weights (through (4.3.13)).

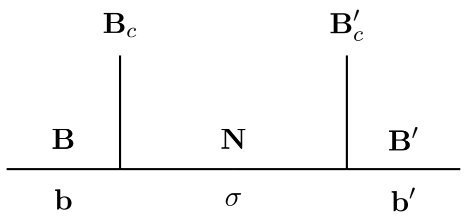


FIGURE 4.1: Pictorial representation of a 4-point conformal block in $\widehat{\mathfrak{sl}}(n)_N$-WZW models obtained from the instanton partition function for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ under a minimal model identification when $p=N$.

We also represent figure 4.1 schematically by $\mathbf{b} - \mathbf{b}_c - (\mathbf{N}) - \mathbf{b}'_c - \mathbf{b}'$, where the vectors $\mathbf{b}_c, \mathbf{b}'_c \in (\mathbb{Z}_n)^N$ are associated to the weights $\mathbf{B}_c$ and $\mathbf{B}'_c$ through equations analogous to (4.3.13). The representations with highest weights $\mathbf{B}_c$ and $\mathbf{B}'_c$ of the remaining two of the four external primary fields need to be taken so that the highest weights $\mathbf{B}_c$ and $\mathbf{B}'_c$ respect the WZW fusion rules from section 1.5.11, which apply from right to left in this diagram, of the $\widehat{\mathfrak{sl}}(n)_N$-WZW model when $\mathbf{N}$, $\mathbf{b}$ and $\mathbf{b}'$ are fixed. The choice of the integers $\mathbf{l}$ for $\widehat{\mathcal{Z}}^{\mathbf{b},\mathbf{b}'}_{\mathbf{N};\mathbf{l}}(q)$, which indicate the states of internal channel following corollary 4.2.2.6, is also restricted by the fusion rules of $\mathbf{b}'$ and $\mathbf{b}'_c$.

In (4.3.10) and (4.3.11), the parameters in $\mathbf{s}^{(0)}$ and $\mathbf{s}^{(3)}$ were fixed using the $\mathbb{Z}_n$-boundary charge conditions. We now fix the remaining parameters $s^{(1)}_{N-1}, s^{(2)}_1$ in (4.3.6) using the fusion rules. Let $\mathbf{b}_c = (b_{c,0}, \ldots, b_{c,N-1})$ and $\mathbf{b}'_c = (b'_{c,0}, \ldots, b'_{c,N-1})$ be boundary charges associated with $\mathbf{B}_c$ and $\mathbf{B}'_c$, respectively.[1] We propose that they satisfy the same type of boundary charge conditions with (4.3.8) as $s^{(1)}_i - 1 \equiv b_{c,i} - b_{c,i-1} \pmod n$ and $s^{(2)}_i - 1 \equiv b'_{c,i-1} - b'_{c,i} \pmod n$ for the parameters in (4.3.6). As a result, these boundary charges are

$$\begin{aligned} \mathbf{b}_c &\equiv (b_c, b_c, \ldots, b_c, b_c + s^{(1)}_{N-1} - 1) \pmod n, \\ \mathbf{b}'_c &\equiv (b'_c + s^{(2)}_1 - 1, b'_c, b'_c, \ldots, b'_c) \pmod n, \end{aligned} \tag{4.3.15}$$

where $b_c, b'_c \in \{0, 1, \ldots, n-1\}$, and $s^{(1)}_{N-1}$, $s^{(2)}_1$ should be determined by the fusion rules. For definiteness, we restrict $s^{(2)}_{N-1}, s^{(3)}_1 \in \{1, \ldots, n\}$, and if $N=2$ we take $b_c + s^{(2)}_1 \le n$, $b'_c + s^{(3)}_1 \le n$ so that the boundary charges are $\mathbf{b}_c = (b_c, b_c + s^{(2)}_1 - 1)$ and $\mathbf{b}'_c = (b'_c + s^{(3)}_1 - 1, b'_c)$.

[1] We will not assume the ordering of the boundary charges $\mathbf{b}_c$ and $\mathbf{b}'_c$.

4.3.3 Conjectured $\widehat{\mathfrak{sl}}(n)$-WZW Conformal Blocks from Instanton Partition Functions

We propose the following conjectures on the relation between the $SU(N)$ Burge-reduced instanton partition functions (4.3.12) on $\mathbb{C}^2/\mathbb{Z}_n$ and the $\widehat{\mathfrak{sl}}(n)_N$-WZW conformal blocks. To describe our conjectures, we represent $\Lambda = [d_0, d_1, \dots, d_{n-1}] \in P^+_{n,N}$ as a Young diagram by a partition $\lambda = \mathrm{par}(\Lambda)$ using (1.3.20).

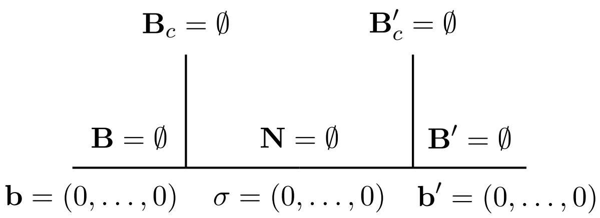
Conjecture 4.3.3.1 (The case corresponding to the diagram $\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$)**.** *The trivial $\widehat{\mathfrak{sl}}(n)_N$-WZW correlation function of the type*

$$\langle \emptyset(1)\, \emptyset(q)\rangle^{\widehat{\mathfrak{sl}}(n)_N}_{\mathbb{P}^1}\,, \tag{4.3.16}$$

agrees with the following Burge-reduced instanton partition function

$$\widehat{\mathcal{Z}}^{\mathbf{0},\mathbf{0}}_{[N,0,\dots,0];\mathbf{0}}(q) \quad = \quad (1-q)^{-2h_\emptyset} = 1. \tag{4.3.17}$$

Here $\boldsymbol{s} = \boldsymbol{s}_1 = \boldsymbol{s}_2 = \boldsymbol{s}^{(3)} = \boldsymbol{s}^{(4)} = [n+1, 1, \dots, 1]$ are fixed by (4.1.5), (4.3.11) and (4.3.15), and $h_\emptyset = 0$ is the conformal dimension for the representation $\emptyset = [N, 0, \dots, 0]$.
Visually this is represented as



Remark 4.3.3.2. The 4-point correlation function in this case reduces to the the 2-point function (4.3.16) as it is a pairing between the states $\langle \emptyset(1)|$ and $|\emptyset(q)\rangle$ together with the insertion of two empty vertex operators. In this case, the vertex operators are trivial so we are left with the trivial 2-point function.

Conjecture 4.3.3.3 (The case corresponding to $\emptyset - [N-1, 0, \dots, 0, 1] - (\square) - \square - \emptyset$)**.** *The $\widehat{\mathfrak{sl}}(n)_N$-WZW 2-point conformal block of the type $\langle \overline{\square}(1)\, \square(q)\rangle^{\widehat{\mathfrak{sl}}(n)_N}_{\mathbb{P}^1} = (1-q)^{-2h_\square}$ agrees with the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{\mathbf{0},\mathbf{0}}_{[N-1,1,0\dots,0];\mathbf{0}}(q)$. That is*

$$\widehat{\mathcal{Z}}^{\mathbf{0},\mathbf{0}}_{[N-1,1,0\dots,0];\mathbf{0}}(q) \quad = \quad (1-q)^{-2h_\square}\,. \tag{4.3.18}$$

Here $\boldsymbol{s} = \boldsymbol{s}^{(3)} = [n, 2, 1, \dots, 1]$, $\boldsymbol{s}_1 = \boldsymbol{s}^{(4)} = [n+1, 1, \dots, 1]$ and $s^{(2)}_{N-1} = n$ are fixed by (4.1.5), (4.3.11) and (4.3.15), and $h_\square = \frac{n^2-1}{2n(n+N)}$ is the conformal dimension for the representation $\square = [N-1, 1, 0\dots, 0]$.

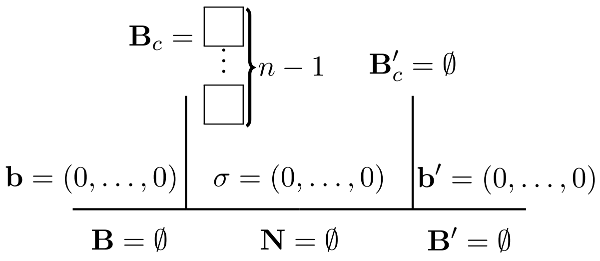


FIGURE 4.2: The conformal block for conjecture 4.3.3.3.

Visually this is represented in figure 4.2.

Conjecture 4.3.3.4 (The case corresponding to $\square - \square - (\emptyset$ or $[N-2,1,0,\dots,0,1]) - \square -$ $[N-1,0,\dots,0,1]$)**.** *The $\widehat{\mathfrak{sl}}(n)_N$-WZW 4-point conformal blocks of the type*

$$\langle \overline{\square}(\infty)\, \square(1)\, \square(q)\, \overline{\square}(0) \rangle_{\mathbb{P}^1}^{\widehat{\mathfrak{sl}}(n)_N}\,,$$

agree with, up to certain overall factors, the following Burge-reduced instanton partition functions,[2]

$$\begin{aligned}
&\widehat{\mathcal{Z}}^{(1,0,\dots,0),(n-1,0,\dots,0)}_{[N,0,\dots,0];\boldsymbol{l}}(q) \\
&= \begin{cases} (1-q)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(-\frac{1}{n+N}, \frac{N-1}{n+N}; \frac{N}{n+N}; q\right), & \textit{for } \ \boldsymbol{l} = \mathbf{0}, \\ \frac{1}{N}\, q^{\frac{1}{n}}\, (1-q)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(\frac{N-1}{n+N}, 1 - \frac{1}{n+N}; 1 + \frac{N}{n+N}; q\right), & \textit{for } \ \boldsymbol{l} = (-1,\dots,-1), \end{cases}
\end{aligned} \tag{4.3.19}$$

and

$$\begin{aligned}
&\widehat{\mathcal{Z}}^{(1,0,\dots,0),(n-1,0,\dots,0)}_{[N-2,1,0,\dots,0,1];\boldsymbol{l}}(q) \\
&= \begin{cases} (1-q)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(-\frac{1}{n+N}, \frac{n-1}{n+N}; \frac{n}{n+N}; q\right), & \textit{for } \ \boldsymbol{l} = \mathbf{0}, \\ \frac{1}{n}\, q^{1-\frac{1}{n}}\, (1-q)^{2h_\square - \frac{n+1}{n+N}}\, {}_2F_1\left(\frac{n-1}{n+N}, 1 - \frac{1}{n+N}; 1 + \frac{n}{n+N}; q\right), & \textit{for } \ \boldsymbol{l} = (1,\dots,1). \end{cases}
\end{aligned} \tag{4.3.20}$$

Here, by (4.1.5), (4.3.11) and (4.3.15), for (4.3.19) $\boldsymbol{s} = [n+1,1,\dots,1]$, $\boldsymbol{s}_1 = \boldsymbol{s}^{(3)} = [n,2,1,\dots,1]$, $\boldsymbol{s}^{(4)} = [2,n,1,\dots,1]$ and $s^{(2)}_{N-1} = 2$ are fixed, and for (4.3.20) $\boldsymbol{s} = [2,n-1,2,1,\dots,1]$, $\boldsymbol{s}_1 = \boldsymbol{s}^{(3)} = [n,2,1,\dots,1]$, $\boldsymbol{s}^{(4)} = [2,n,1,\dots,1]$ and $s^{(2)}_{N-1} = 2$ are fixed, where when $N=2$, $[2,n-1,2,1,\dots,1]$ means $[3,n-1]$. The integers $\boldsymbol{l} = \delta\boldsymbol{k}$ are taken so that the corresponding modules on the CFT side, following corollary 4.2.2.6, are in the fundamental chamber under the action of affine Weyl group of $\widehat{\mathfrak{sl}}(n)$, and the second ones in (4.3.19) and

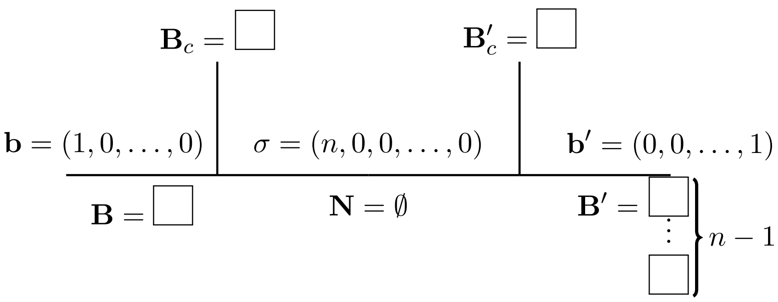
[2] (4.3.19) and (4.3.20) correspond to, respectively, the 4-point WZW conformal blocks $\widehat{\mathcal{F}}^{(0)}_{i=1,2}(q)$ and $\widehat{\mathcal{F}}^{(1)}_{i=2,1}(q)$ in (1.5.130)

(4.3.20) *respect the fusion rules by*

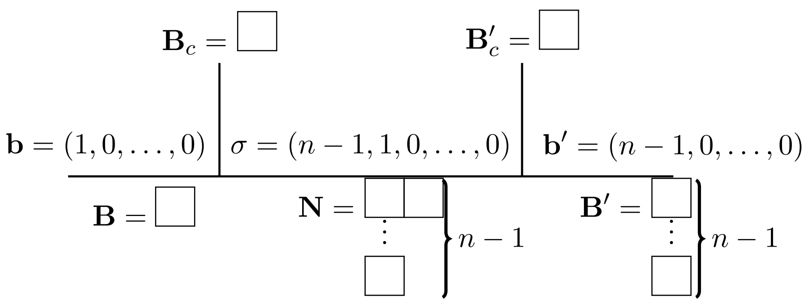
$$\begin{aligned} \boldsymbol{N} = [N,0,\dots,0] = \emptyset \quad &\xrightarrow{\delta \boldsymbol{k} = (-1,\dots,-1)} \quad \mathfrak{c} = [N-2,1,0,\dots,0,1]\,, \\ \boldsymbol{N} = [N-2,1,0,\dots,0,1] \quad &\xrightarrow{\delta \boldsymbol{k} = (1,\dots,1)} \quad \mathfrak{c} = [N,0,\dots,0] = \emptyset, \end{aligned} \tag{4.3.21}$$

where $\mathfrak{c} = [\mathfrak{c}_0, \mathfrak{c}_1, \dots, \mathfrak{c}_{n-1}]$ *are defined by the Chern classes* (3.1.11)*. When* $n = 2$*,* $[N-2,1,0,\dots,0,1]$ *means* $[N-2,2] = \square\square$ *and then* $\sigma = (1,1,0,\dots,0)$*.*

Visually we represent $\widehat{\mathcal{Z}}^{(1,0,\dots,0),(n-1,0,\dots,0)}_{[N,0,\dots,0];\boldsymbol{l}}(q)$ *as*



and $\widehat{\mathcal{Z}}^{(1,0,\dots,0),(n-1,0,\dots,0)}_{[N-2,1,0,\dots,0,1];\boldsymbol{l}}(q)$ *by*



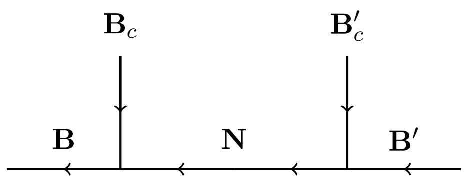


FIGURE 4.3: Pictorial representation of the fusion rules respected for the 4-point conformal block in $\widehat{\mathfrak{sl}}(n)_N$-WZW models obtained from the instanton partition function for $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ under a minimal model identification when $p = N$.

Here we include an example of the fusion calculations required to determine which arrangements of highest weights (in terms of $\widehat{\mathfrak{sl}}(n)$ weights) are allowed in these conformal blocks. We begin by noting that the fusion rules needing to hold from right to left is represented in figure 4.3. There are two vertices in the diagram, and both represent fusion between primary fields. The arrows pointing towards a vertex are the two fields fusing and the arrow pointing away from the vertex represents the product of this fusion. Our example will focus on the right hand vertex of the block.

Example 4.3.3.5. Consider the fusion in conjecture 4.3.3.4, involving the primary fields of weights $\Lambda^{(2)} = [1,0,\dots,0]$ and $\Lambda^{(3)} = [0,\dots,1]$ with corresponding Young diagrams $\mathrm{par}(\Lambda^{(2)}) = (1)$ and $\mathrm{par}(\Lambda^{(3)}) = (\underbrace{1,\dots,1}_{n-1})$. We first use the Littlewood-Richardson rules for $\Lambda^{(3)} \otimes \Lambda^{(2)}$ which gives the two Young diagrams $\lambda = (\underbrace{2,\dots,1}_{n-1})$ and $\mu = (\underbrace{1,\dots,1}_{n})$. We now use the fusion rules for $\widehat{\mathfrak{sl}}(n)_N$ and remove all columns of length n. This leaves the two diagrams $\lambda = (\underbrace{2,\dots,1}_{n-1})$ and $\mu = (\emptyset)$ corresponding to the weights $[1,0,\dots,1]$ and $[0,\dots,0]$. This correctly gives the two cases in the conjecture.

4.4 Examples of $SU(N)$ Burge-reduced instanton counting on $\mathbb{C}^2/\mathbb{Z}_n$

We illustrate the statement of corollary 4.2.2.6 and check conjectures 4.3.3.1, 4.3.3.3 and 4.3.3.4 for $(N,n) = (2,2), (2,3)$ and $(3,2)$. In particular we demonstrate how one can extract their $\widehat{\mathfrak{sl}}(n)_N$-WZW conformal blocks from the Burge-reduced instanton partition functions.[3]

4.4.1 $(N,n) = (2,2)$ and $\widehat{\mathfrak{sl}}(2)_2$-WZW model

For $(N,n) = (2,2)$, there are three highest-weight representations

$$\emptyset = [2,0], \quad \square = [1,1], \quad \square\square = [0,2], \tag{4.4.1}$$

with conformal dimensions

$$h_{[k_0,k_1]} = \frac{k_1\,(k_1+2)}{16} : \quad h_\emptyset = 0, \quad h_\square = \frac{3}{16}, \quad h_{\square\square} = \frac{1}{2}. \tag{4.4.2}$$

[3] The computations in this section heavily rely on Mathematica. We have also checked conjectures 4.3.3.1, 4.3.3.3 and 4.3.3.4 for $(N,n) = (2,4)$ up to $O(q^5)$.

4.4.1.1 Burge-reduced generating functions of coloured Young diagrams

The $\mathfrak{t}$-refined Burge-reduced generating functions (4.2.18) for $(N,n)=(2,2)$ are obtained as

$$\begin{aligned}
\widehat{X}^{\rm red}_{[2,0]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[3,1]}_{(0,0);(-l)}(q)\,\mathfrak{t}^{2l} = X^{[2,0]}_{[2,0]}(q)\,f_0(q,\hat{\mathfrak{t}}) + X^{[2,0]}_{[0,2]}(q)\,f_1(q,\hat{\mathfrak{t}}),\\
\widehat{X}^{\rm red}_{[0,2]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[3,1]}_{(1,1);(-l)}(q)\,\mathfrak{t}^{2l+2} = X^{[0,2]}_{[0,2]}(q)\,f_1(q,\hat{\mathfrak{t}}) + X^{[0,2]}_{[2,0]}(q)\,f_0(q,\hat{\mathfrak{t}}),\\
\widehat{X}^{\rm red}_{[1,1]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[2,2]}_{(1,0);(-l)}(q)\,\mathfrak{t}^{2l+1} = X^{[1,1]}_{[1,1]}(q)\,g(q,\hat{\mathfrak{t}}),
\end{aligned} \tag{4.4.3}$$

where $\hat{\mathfrak{t}} = q^{-\frac14}\,\mathfrak{t}$,

$$\begin{aligned}
&X^{[2,0]}_{[2,0]}(q) = 1+q+3q^2+5q^3+10q^4+16q^5+28q^6+43q^7+70q^8+105q^9+161q^{10}+\cdots\ ,\\
&X^{[2,0]}_{[0,2]}(q) = q^{\frac12}+2q^{\frac32}+4q^{\frac52}+7q^{\frac72}+13q^{\frac92}+21q^{\frac{11}{2}}+35q^{\frac{13}{2}}+55q^{\frac{15}{2}}+86q^{\frac{17}{2}}+130q^{\frac{19}{2}}+\cdots\ ,\\
&X^{[1,1]}_{[1,1]}(q) = 1+2q+4q^2+8q^3+14q^4+24q^5+40q^6+64q^7+100q^8+\cdots\ ,\\
&X^{[0,2]}_{[0,2]}(q) = X^{[2,0]}_{[2,0]}(q), \quad X^{[0,2]}_{[2,0]}(q) = X^{[2,0]}_{[0,2]}(q),
\end{aligned} \tag{4.4.4}$$

and

$$f_\sigma(q,\hat{\mathfrak{t}}) = \sum_{j\in 4\,\mathbb{Z}+2\,\sigma} q^{\frac18\,j^2}\,\hat{\mathfrak{t}}^{\,j}, \quad \sigma=0,1, \quad g(q,\hat{\mathfrak{t}}) = \sum_{j\in 2\,\mathbb{Z}+1} q^{\frac18\,j^2+\frac18}\,\hat{\mathfrak{t}}^{\,j}. \tag{4.4.5}$$

The Burge-reduced generating functions (4.4.3) agree with the $\widehat{\mathfrak{sl}}(2)_2$-WZW characters computed by (1.3.36),

$$\widehat{X}^{\rm red}_{[2,0]}(q,\mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_2}_{[2,0]}(q,\hat{\mathfrak{t}}), \quad \widehat{X}^{\rm red}_{[0,2]}(q,\mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_2}_{[0,2]}(q,\hat{\mathfrak{t}}), \quad \widehat{X}^{\rm red}_{[1,1]}(q,\mathfrak{t}) = q^{\frac{1}{16}}\,\chi^{\widehat{\mathfrak{sl}}(2)_2}_{[1,1]}(q,\hat{\mathfrak{t}}), \tag{4.4.6}$$

and corollary 4.2.2.6 is confirmed. Up to an overall factor, the functions (4.4.4) are the $\widehat{\mathfrak{sl}}(2)$ string functions of level-2 in [57] and given by (*cf.* Corollary 4.2.2.4),

$$X^{[2,0]}_{[2,0]}(q) + X^{[2,0]}_{[0,2]}(q) = \frac{\left(-q^{\frac12};q\right)_\infty}{(q;q)_\infty}, \quad X^{[2,0]}_{[2,0]}(q) - X^{[2,0]}_{[0,2]}(q) = \frac{\left(q^{\frac12};q^{\frac12}\right)_\infty}{(q;q)^2_\infty}, \quad X^{[1,1]}_{[1,1]}(q) = \frac{(q^2;q^2)_\infty}{(q;q)^2_\infty}. \tag{4.4.7}$$

Using the Jacobi triple product identity

$$\sum_{l\in\mathbb{Z}} x^l\,y^{\frac12 l(l-1)} = (-x;y)_\infty \left(-\frac{y}{x};y\right)_\infty (y;y)_\infty\,, \tag{4.4.8}$$

one can easily obtain (4.2.21) for the principal characters of $\widehat{\mathfrak{sl}}(2)$,

$$\begin{aligned} \widehat{X}^{\mathrm{red}}_{[2,0]}(q,1) &= \widehat{X}^{\mathrm{red}}_{[0,2]}(q,1) = \mathrm{Pr}\,\chi^{\widehat{\mathfrak{sl}}(2)}_{[2,0]}(q) = \left(-q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty (-q;q)_\infty\,, \\ \widehat{X}^{\mathrm{red}}_{[1,1]}(q,1) &= \mathrm{Pr}\,\chi^{\widehat{\mathfrak{sl}}(2)}_{[1,1]}(q) = \left(-q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty \left(-q^{\frac{1}{2}};q\right)_\infty . \end{aligned} \tag{4.4.9}$$

4.4.1.2 Burge-reduced instanton partition functions

For $N = 2$ with general n, the Burge-reduced instanton partition functions (4.3.12) are determined by the parameters in $\mathbf{s} = [s_0, s_1] \in P^{++}_{2,n+2}$ and $\mathbf{s}_r = [s_{r,0}, s_{r,1}] \in P^{++}_{2,n+2}$, $r = 1,2,3,4$, fixed by the relations (4.1.5), (4.3.11):

$$s_1 = \sigma_0 - \sigma_1 + 1, \quad s_1^{(1)} = b_0 - b_1 + 1, \quad s_1^{(4)} = b_0' - b_1' + 1, \tag{4.4.10}$$

and (4.3.15) from the ordered charges $\sigma_0 \geq \sigma_1$, $b_0 \geq b_1$ and $b_0' \geq b_1'$. The Coulomb parameters are then determined from the parameter $s := s_1$ by (4.3.5):

$$a_0 = -\frac{1}{2}\left(s - 1 - \frac{n}{2}\right)\epsilon_2, \quad a_1 = \frac{1}{2}\left(s - 1 - \frac{n}{2}\right)\epsilon_2, \tag{4.4.11}$$

and the mass parameters $\mathbf{m} = (m_0, m_1)$ and $\mathbf{m}' = (m_0', m_1')$ are determined from the parameters in $\mathbf{s}_1, \mathbf{s}_2$ and $\mathbf{s}^{(3)}, \mathbf{s}^{(4)}$, respectively, by (4.3.7).

Let us consider the case of $(N,n) = (2,2)$ with the rational Ω-background $\epsilon_1/\epsilon_2 = -2$ in (4.3.4).[4]

Example 4.4.1.3 ($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0];(l)}(q)$ and take $l = 0$ in the fundamental chamber, which respects the fusion rules, as in conjecture 4.3.3.1. Here $\mathbf{s} = \mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [3,1]$ are fixed. Then, the Burge-reduced instanton partition function is obtained as

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0];(0)}(q) = (1-q)^{-2h_\emptyset} = 1, \qquad h_\emptyset = 0, \tag{4.4.12}$$

which agrees with conjecture 4.3.3.1.

Example 4.4.1.4 ($\emptyset - \square - (\square) - \square - \emptyset$). Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1];(l)}(q)$ and take $l = 0$ in the fundamental chamber as in conjecture 4.3.3.3. Here $\mathbf{s} = \mathbf{s}_2 = \mathbf{s}^{(3)} = [2,2]$ and $\mathbf{s}_1 = \mathbf{s}^{(4)} = [3,1]$ are fixed. Then we see that the Burge-reduced instanton partition function is

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1];(0)}(q) = (1-q)^{-2h_\square} = 1 + \frac{3q}{8} + \frac{33q^2}{128} + \frac{209q^3}{1024} + \frac{5643q^4}{32768} + \frac{39501q^5}{262144} + \cdots\,, \tag{4.4.13}$$

[4] Examples 4.4.1.3, 4.4.1.4 and 4.4.1.5 are confirmed up to $O(q^6)$.

where $h_{\square} = 3/16$, which agrees with conjecture 4.3.3.3.

Example 4.4.1.5 ($\square - \square - (\emptyset) - \square - \square$ and $\square - \square - (\square\square) - \square - \square$). For conjecture 4.3.3.4, first consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];(l)}(q)$, where $\mathbf{s} = [3,1]$ and $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [2,2]$ are fixed. Then we find that the Burge-reduced instanton partition functions for $l = 0, -1$ in the fundamental chamber are

$$
\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];(0)}(q) &= (1-q)^{2h_{\square}-\frac{3}{4}}\,{}_2F_1\left(-\frac{1}{4},\frac{1}{4};\frac{1}{2};q\right) \\
&= 1 + \frac{q}{4} + \frac{11q^2}{64} + \frac{35q^3}{256} + \frac{949q^4}{8192} + \frac{3333q^5}{32768} + \frac{47909q^6}{524288} + \cdots , \\
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];(-1)}(q) &= \frac{q^{\frac{1}{2}}}{2}(1-q)^{2h_{\square}-\frac{3}{4}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};\frac{3}{2};q\right) \\
&= \frac{q^{\frac{1}{2}}}{2} + \frac{q^{\frac{3}{2}}}{4} + \frac{23q^{\frac{5}{2}}}{128} + \frac{37q^{\frac{7}{2}}}{256} + \frac{2013q^{\frac{9}{2}}}{16384} + \frac{3537q^{\frac{11}{2}}}{32768} + \cdots ,
\end{aligned}
\tag{4.4.14}
$$

where $h_{\square} = 3/16$, and the second one respects the fusion rules by (4.3.21). Next consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[0,2];(l)}(q)$, where $\mathbf{s} = [3,1]$ and $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [2,2]$ are fixed. Then we obtain the Burge-reduced instanton partition functions for $l = 0, 1$ in the fundamental chamber as

$$
\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[0,2];(0)}(q) &= (1-q)^{2h_{\square}-\frac{3}{4}}\,{}_2F_1\left(-\frac{1}{4},\frac{1}{4};\frac{1}{2};q\right) = \widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];0}(q), \\
\widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[0,2];(1)}(q) &= \frac{q^{\frac{1}{2}}}{2}(1-q)^{2h_{\square}-\frac{3}{4}}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};\frac{3}{2};q\right) = \widehat{\mathcal{Z}}^{(1,0),(1,0)}_{[2,0];-1}(q),
\end{aligned}
\tag{4.4.15}
$$

where the second one respects the fusion rules by (4.3.21). The above results (4.4.14) and (4.4.15) support conjecture 4.3.3.4. By

$$
{}_2F_1\left(-\frac{1}{4},\frac{1}{4};\frac{1}{2};q\right) = \left(\frac{1+\sqrt{1-q}}{2}\right)^{\frac{1}{2}}, \quad \frac{q^{\frac{1}{2}}}{2}\,{}_2F_1\left(\frac{1}{4},\frac{3}{4};\frac{3}{2};q\right) = \left(\frac{1-\sqrt{1-q}}{2}\right)^{\frac{1}{2}}, \tag{4.4.16}
$$

they are also consistent with the results in [27].[5]

4.4.2 $(N,n) = (2,3)$ and $\widehat{\mathfrak{sl}}(3)_2$-WZW model

For $(N,n) = (2,3)$, there are six highest-weight representations

$$
\emptyset = [2,0,0], \quad \square = [1,1,0], \quad \square\square = [0,2,0], \quad \begin{smallmatrix}\square\\\square\end{smallmatrix} = [1,0,1], \quad \begin{smallmatrix}\square\square\\\square\end{smallmatrix} = [0,1,1], \quad \begin{smallmatrix}\square\square\\\square\square\end{smallmatrix} = [0,0,2], \tag{4.4.17}
$$

[5] More precisely, in [27], the generic Ω-background, without the Burge conditions, was discussed. Then the first one of (4.4.14) and the second one of (4.4.15), with $\mathfrak{c} = 0$, were obtained as prefactors combined with the $\mathcal{N} = 1$ super-Virasoro Ramond conformal blocks $H_{\pm}(q)$, $F_{\pm}(q)$, $\widetilde{H}_{\pm}(q)$ and $\widetilde{F}_{\pm}(q)$. What we found is that, when we impose the specific Burge conditions, the conformal blocks are trivialized as $H_{\pm}(q), F_{\pm}(q) \to 1$ and $\widetilde{H}_{\pm}(q), \widetilde{F}_{\pm}(q) \to 0$, and only the prefactors are obtained.

with conformal dimensions

$$
\begin{aligned}
h_{[k_0,k_1,k_2]} &= \frac{k_1^2+k_2^2+k_1k_2+3k_1+3k_2}{15}\ : \\
h_{\emptyset} = 0, \quad h_{\ydiagram{1}} = h_{\ydiagram{1,1}} &= \frac{4}{15}, \quad h_{\ydiagram{2}} = h_{\ydiagram{2,2}} = \frac{2}{3}, \quad h_{\ydiagram{2,1}} = \frac{3}{5}.
\end{aligned}
\tag{4.4.18}
$$

4.4.2.1 Burge-reduced generating functions of coloured Young diagrams

The $\mathfrak{t}$-refined Burge-reduced generating functions (4.2.18) for $(N,n)=(2,3)$ are obtained as

$$
\begin{aligned}
\widehat{X}^{\mathrm{red}}_{[2,0,0]}(q,(\mathfrak{t}_1,\mathfrak{t}_2)) &= (q;q)_\infty \sum_{(l_1,l_2)\in\mathbb{Z}^2} X^{[4,1]}_{(0,0);(-l_1,-l_2)}(q)\, \mathfrak{t}_1^{2l_1-l_2}\, \mathfrak{t}_2^{-l_1+2l_2} \\
&= X^{[2,0,0]}_{[2,0,0]}(q)\, f_{00}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[2,0,0]}_{[0,1,1]}\, g_{00}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[0,2,0]}(q,(\mathfrak{t}_1,\mathfrak{t}_2)) &= (q;q)_\infty \sum_{(l_1,l_2)\in\mathbb{Z}^2} X^{[4,1]}_{(1,1);(-l_1,-l_2)}(q)\, \mathfrak{t}_1^{2+2l_1-l_2}\, \mathfrak{t}_2^{-l_1+2l_2} \\
&= X^{[0,2,0]}_{[0,2,0]}(q)\, f_{10}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[0,2,0]}_{[1,0,1]}(q)\, g_{10}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[0,0,2]}(q,(\mathfrak{t}_1,\mathfrak{t}_2)) &= (q;q)_\infty \sum_{(l_1,l_2)\in\mathbb{Z}^2} X^{[4,1]}_{(2,2);(-l_1,-l_2)}(q)\, \mathfrak{t}_1^{2l_1-l_2}\, \mathfrak{t}_2^{2-l_1+2l_2} \\
&= X^{[0,0,2]}_{[0,0,2]}(q)\, f_{01}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[0,0,2]}_{[1,1,0]}(q)\, g_{01}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[1,1,0]}(q,(\mathfrak{t}_1,\mathfrak{t}_2)) &= (q;q)_\infty \sum_{(l_1,l_2)\in\mathbb{Z}^2} X^{[3,2]}_{(1,0);(-l_1,-l_2)}(q)\, \mathfrak{t}_1^{1+2l_1-l_2}\, \mathfrak{t}_2^{-l_1+2l_2} \\
&= X^{[1,1,0]}_{[1,1,0]}\, g_{01}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[1,1,0]}_{[0,0,2]}(q)\, f_{01}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[0,1,1]}(q,(\mathfrak{t}_1,\mathfrak{t}_2)) &= (q;q)_\infty \sum_{(l_1,l_2)\in\mathbb{Z}^2} X^{[3,2]}_{(2,1);(-l_1,-l_2)}(q)\, \mathfrak{t}_1^{1+2l_1-l_2}\, \mathfrak{t}_2^{1-l_1+2l_2} \\
&= X^{[0,1,1]}_{[0,1,1]}(q)\, g_{00}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[0,1,1]}_{[2,0,0]}(q)\, f_{00}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2), \\
\widehat{X}^{\mathrm{red}}_{[1,0,1]}(q,(\mathfrak{t}_1,\mathfrak{t}_2)) &= (q;q)_\infty \sum_{(l_1,l_2)\in\mathbb{Z}^2} X^{[2,3]}_{(2,0);(-l_1,-l_2)}(q)\, \mathfrak{t}_1^{2l_1-l_2}\, \mathfrak{t}_2^{1-l_1+2l_2} \\
&= X^{[1,0,1]}_{[1,0,1]}(q)\, g_{10}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) + X^{[1,0,1]}_{[0,2,0]}(q)\, f_{10}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2),
\end{aligned}
\tag{4.4.19}
$$

where $\hat{\mathfrak{t}}_1 = q^{-\frac{1}{3}}\,\mathfrak{t}_1$, $\hat{\mathfrak{t}}_2 = q^{-\frac{1}{3}}\,\mathfrak{t}_2$,

$$
\begin{aligned}
&X^{[2,0,0]}_{[2,0,0]}(q) = 1+2q+8q^2+20q^3+52q^4+116q^5+256q^6+522q^7+\cdots\ , \\
&X^{[2,0,0]}_{[0,1,1]}(q) = q^{\frac{1}{3}}+4q^{\frac{4}{3}}+12q^{\frac{7}{3}}+32q^{\frac{10}{3}}+77q^{\frac{13}{3}}+172q^{\frac{16}{3}}+365q^{\frac{19}{3}}+740q^{\frac{22}{3}}+\cdots\ , \\
&X^{[0,1,1]}_{[0,1,1]}(q) = 1+4q+13q^2+36q^3+89q^4+204q^5+441q^6+908q^7+\cdots\ , \\
&X^{[0,1,1]}_{[2,0,0]}(q) = 2q^{\frac{2}{3}}+7q^{\frac{5}{3}}+22q^{\frac{8}{3}}+56q^{\frac{11}{3}}+136q^{\frac{14}{3}}+300q^{\frac{17}{3}}+636q^{\frac{20}{3}}+1280q^{\frac{23}{3}}+\cdots\ , \\
&X^{[0,2,0]}_{[0,2,0]}(q) = X^{[0,0,2]}_{[0,0,2]}(q) = X^{[2,0,0]}_{[2,0,0]}(q), \quad X^{[0,2,0]}_{[1,0,1]}(q) = X^{[0,0,2]}_{[1,1,0]}(q) = X^{[2,0,0]}_{[0,1,1]}(q), \\
&X^{[1,1,0]}_{[1,1,0]}(q) = X^{[1,0,1]}_{[1,0,1]}(q) = X^{[0,1,1]}_{[0,1,1]}(q), \quad X^{[1,1,0]}_{[0,0,2]}(q) = X^{[1,0,1]}_{[0,2,0]}(q) = X^{[0,1,1]}_{[2,0,0]}(q),
\end{aligned}
\tag{4.4.20}
$$

and f_{00}, f_{10}, f_{01}, g_{00}, g_{10}, g_{01} are

$$\begin{aligned} f_{\sigma_1\sigma_2}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) &= \sum_{\substack{(j_1,j_2)\in\{2\,\mathbb{Z}\}^2\\ j_1-j_2\in 6\,\mathbb{Z}+2(\sigma_1-\sigma_2)}} q^{\frac{1}{6}\left(j_1^2+j_2^2+j_1\,j_2\right)}\,\hat{\mathfrak{t}}_1^{j_1}\,\hat{\mathfrak{t}}_2^{j_2},\\ g_{\sigma_1\sigma_2}(q,\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2) &= \sum_{\substack{(j_1,j_2)\in\{2\,\mathbb{Z}+1\}^2\\ j_1-j_2\in 6\,\mathbb{Z}+2(\sigma_1-\sigma_2)}} q^{\frac{1}{6}\left(j_1^2+j_2^2+j_1\,j_2\right)+\frac{1}{6}}\,\hat{\mathfrak{t}}_1^{j_1}\,\hat{\mathfrak{t}}_2^{j_2}\\ &\quad+\sum_{\substack{(j_1,j_2)\in\{2\,\mathbb{Z}\}\times\{2\,\mathbb{Z}+1\}\\ j_1-j_2\in 6\,\mathbb{Z}+1+2(\sigma_1-\sigma_2)}} q^{\frac{1}{6}\left(j_1^2+j_2^2+j_1\,j_2\right)+\frac{1}{6}}\left(\hat{\mathfrak{t}}_1^{j_1}\,\hat{\mathfrak{t}}_2^{j_2}+\hat{\mathfrak{t}}_1^{j_2}\,\hat{\mathfrak{t}}_2^{j_1}\right). \end{aligned} \tag{4.4.21}$$

The Burge-reduced generating functions (4.4.19) agree with the $\widehat{\mathfrak{sl}}(3)_2$-WZW characters computed by (1.5.114), using the change of parameters

$$\frac{x_i}{x_{i+1}}=\frac{\hat{\mathfrak{t}}_{i-1}\hat{\mathfrak{t}}_{i+1}}{\hat{\mathfrak{t}}_i^2}\qquad\Longleftrightarrow\qquad x_i=\frac{\hat{\mathfrak{t}}_{i-1}}{\hat{\mathfrak{t}}_i}\,\frac{\hat{\mathfrak{t}}_M}{\hat{\mathfrak{t}}_{M-1}}\,x_M, \tag{4.4.22}$$

as

$$\begin{aligned} &\widehat{X}^{\rm red}_{[2,0,0]}(q,(\mathfrak{t}_1,\mathfrak{t}_2))=\chi^{\widehat{\mathfrak{sl}}(3)_2}_{[2,0,0]}(q,(\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2)),\quad \widehat{X}^{\rm red}_{[0,2,0]}(q,(\mathfrak{t}_1,\mathfrak{t}_2))=\chi^{\widehat{\mathfrak{sl}}(3)_2}_{[0,2,0]}(q,(\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2)),\\ &\widehat{X}^{\rm red}_{[0,0,2]}(q,(\mathfrak{t}_1,\mathfrak{t}_2))=\chi^{\widehat{\mathfrak{sl}}(3)_2}_{[0,0,2]}(q,(\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2)),\quad \widehat{X}^{\rm red}_{[1,1,0]}(q,(\mathfrak{t}_1,\mathfrak{t}_2))=q^{\frac{1}{15}}\,\chi^{\widehat{\mathfrak{sl}}(3)_2}_{[1,1,0]}(q,(\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2)),\\ &\widehat{X}^{\rm red}_{[0,1,1]}(q,(\mathfrak{t}_1,\mathfrak{t}_2))=q^{\frac{1}{15}}\,\chi^{\widehat{\mathfrak{sl}}(3)_2}_{[0,1,1]}(q,(\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2)),\quad \widehat{X}^{\rm red}_{[1,0,1]}(q,(\mathfrak{t}_1,\mathfrak{t}_2))=q^{\frac{1}{15}}\,\chi^{\widehat{\mathfrak{sl}}(3)_2}_{[1,0,1]}(q,(\hat{\mathfrak{t}}_1,\hat{\mathfrak{t}}_2)), \end{aligned} \tag{4.4.23}$$

which agrees with corollary 4.2.2.6. Up to an overall factor, the functions (4.4.20) are the $\widehat{\mathfrak{sl}}(3)$ string functions of level-2 in [57] and given by (*cf.* Corollary 4.2.2.4),

$$\begin{aligned} X^{[2,0,0]}_{[2,0,0]}(q)-q^{\frac{1}{6}}\,X^{[2,0,0]}_{[0,1,1]}(q) &= \frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty\left(q;q^{\frac{5}{2}}\right)_\infty\left(q^{\frac{3}{2}};q^{\frac{5}{2}}\right)_\infty\left(q^{\frac{5}{2}};q^{\frac{5}{2}}\right)_\infty}{(q;q)_\infty^4},\\ X^{[2,0,0]}_{[0,1,1]}(q) &= q^{\frac{1}{3}}\,\frac{\left(q^2;q^2\right)_\infty\left(q^2;q^{10}\right)_\infty\left(q^8;q^{10}\right)_\infty\left(q^{10};q^{10}\right)_\infty}{(q;q)_\infty^4},\\ X^{[0,1,1]}_{[0,1,1]}(q) &= \frac{\left(q^2;q^2\right)_\infty\left(q^4;q^{10}\right)_\infty\left(q^6;q^{10}\right)_\infty\left(q^{10};q^{10}\right)_\infty}{(q;q)_\infty^4},\\ q^{\frac{1}{6}}\,X^{[0,1,1]}_{[0,1,1]}(q)-X^{[0,1,1]}_{[2,0,0]}(q) &= q^{\frac{1}{6}}\,\frac{\left(q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty\left(q^{\frac{1}{2}};q^{\frac{5}{2}}\right)_\infty\left(q^2;q^{\frac{5}{2}}\right)_\infty\left(q^{\frac{5}{2}};q^{\frac{5}{2}}\right)_\infty}{(q;q)_\infty^4}. \end{aligned} \tag{4.4.24}$$

By taking $\mathfrak{t}_1 = \mathfrak{t}_2 = 1$, the principal characters of $\widehat{\mathfrak{sl}}(3)$ are obtained as in (4.2.21):

$$\begin{aligned}
\widehat{X}^{\mathrm{red}}_{[2,0,0]}(q,(1,1)) = \widehat{X}^{\mathrm{red}}_{[0,2,0]}(q,(1,1)) = \widehat{X}^{\mathrm{red}}_{[0,0,2]}(q,(1,1)) &= \mathrm{Pr}\,\chi^{\widehat{\mathfrak{sl}}(3)}_{[2,0,0]}(q) \\
&= \frac{(q;q)_\infty}{\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_\infty \left(q^{\frac{2}{3}};q^{\frac{5}{3}}\right)_\infty \left(q;q^{\frac{5}{3}}\right)_\infty}, \\
\widehat{X}^{\mathrm{red}}_{[1,1,0]}(q,(1,1)) = \widehat{X}^{\mathrm{red}}_{[0,1,1]}(q,(1,1)) = \widehat{X}^{\mathrm{red}}_{[1,0,1]}(q,(1,1)) &= \mathrm{Pr}\,\chi^{\widehat{\mathfrak{sl}}(3)}_{[1,1,0]}(q) \\
&= \frac{(q;q)_\infty}{\left(q^{\frac{1}{3}};q^{\frac{1}{3}}\right)_\infty \left(q^{\frac{1}{3}};q^{\frac{5}{3}}\right)_\infty \left(q^{\frac{4}{3}};q^{\frac{5}{3}}\right)_\infty}.
\end{aligned} \tag{4.4.25}$$

4.4.2.2 Burge-reduced instanton partition functions

For $(N,n) = (2,3)$, the rational Ω-background (4.3.4) yields $\epsilon_1/\epsilon_2 = -5/2$. The parameters in $\mathbf{s} = [s_0, s_1] \in P^{++}_{2,5}$ and $\mathbf{s}_r = [s_{r,0}, s_{r,1}] \in P^{++}_{2,5}$, $r = 1,2,3,4$, which determine the Burge-reduced instanton partition functions, are fixed as in (4.4.10).[6]

Example 4.4.2.3 ($\emptyset-\emptyset-(\emptyset)-\emptyset-\emptyset$). Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0,0];(l_1,l_2)}(q)$ and take $(l_1,l_2) = (0,0)$ in the fundamental chamber as in conjecture 4.3.3.1. Here $\mathbf{s} = \mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [4,1]$ are fixed. Then we see that the Burge-reduced instanton partition function is

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[2,0,0];(0,0)}(q) = (1-q)^{-2h_\emptyset} = 1, \qquad h_\emptyset = 0, \tag{4.4.26}$$

which agrees with conjecture 4.3.3.1.

Example 4.4.2.4 ($\emptyset-\substack{\square\\\square}-(\square)-\square-\emptyset$). Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1,0];(l_1,l_2)}(q)$ and take $(l_1,l_2) = (0,0)$ in the fundamental chamber as in conjecture 4.3.3.3. Here $\mathbf{s} = \mathbf{s}^{(3)} = [3,2]$, $\mathbf{s}_1 = \mathbf{s}^{(4)} = [4,1]$ and $\mathbf{s}_2 = [2,3]$ are fixed. Then the Burge-reduced instanton partition function is

$$\widehat{\mathcal{Z}}^{(0,0),(0,0)}_{[1,1,0];(0,0)}(q) = (1-q)^{-2h_\square} = 1 + \frac{8q}{15} + \frac{92q^2}{225} + \frac{3496q^3}{10125} + \frac{46322q^4}{151875} + \frac{3149896q^5}{11390625} + \cdots, \tag{4.4.27}$$

where $h_\square = 4/15$. This expansion agrees with conjecture 4.3.3.3.

Example 4.4.2.5 ($\square-\square-(\emptyset)-\square-\substack{\square\\\square}$ and $\square-\square-(\boxplus)-\square-\substack{\square\\\square}$). For conjecture 4.3.3.4, consider, first, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0),(2,0)}_{[2,0,0];(l_1,l_2)}(q)$, where $\mathbf{s} = [4,1]$, $\mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = [3,2]$ and $\mathbf{s}^{(4)} = [2,3]$ are fixed. Then we find that the Burge-reduced instanton

[6] Examples 4.4.2.3, 4.4.2.4 and 4.4.2.5 are confirmed up to $O(q^5)$.

partition functions for $(l_1, l_2) = (0,0)$ and $(-1,-1)$ in the fundamental chamber are

$$\begin{aligned}\widehat{\mathscr{Z}}^{(1,0),(2,0)}_{[2,0,0];(0,0)}(q) &= (1-q)^{2h_\square - \frac{4}{5}}\, {}_2F_1\left(-\frac{1}{5}, \frac{1}{5}; \frac{2}{5}; q\right) \\ &= 1 + \frac{q}{6} + \frac{34q^2}{315} + \frac{67q^3}{810} + \frac{49309q^4}{722925} + \frac{254267q^5}{4337550} + \cdots\ , \\ \widehat{\mathscr{Z}}^{(1,0),(2,0)}_{[2,0,0];(-1,-1)}(q) &= \frac{q^{\frac{1}{3}}}{2}(1-q)^{2h_\square - \frac{4}{5}}\, {}_2F_1\left(\frac{1}{5}, \frac{4}{5}; \frac{7}{5}; q\right) \\ &= \frac{q^{\frac{1}{3}}}{2} + \frac{4q^{\frac{4}{3}}}{21} + \frac{79q^{\frac{7}{3}}}{630} + \frac{4619q^{\frac{10}{3}}}{48195} + \frac{16237q^{\frac{13}{3}}}{206550} + \cdots\ ,\end{aligned} \tag{4.4.28}$$

where $h_\square = 4/15$, and the second one respects the fusion rules by (4.3.21). Consider next, the Burge-reduced instanton partition function $\widehat{\mathscr{Z}}^{(1,0),(2,0)}_{[0,1,1];(l_1,l_2)}(q)$, where $\mathbf{s} = \mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = [3,2]$ and $\mathbf{s}^{(4)} = [2,3]$ are fixed. Then we see that the Burge-reduced instanton partition functions for $(l_1, l_2) = (0,0)$ and $(1,1)$ in the fundamental chamber are

$$\begin{aligned}\widehat{\mathscr{Z}}^{(1,0),(2,0)}_{[0,1,1];(0,0)}(q) &= (1-q)^{2h_\square - \frac{4}{5}}\, {}_2F_1\left(-\frac{1}{5}, \frac{2}{5}; \frac{3}{5}; q\right) \\ &= 1 + \frac{2q}{15} + \frac{13q^2}{150} + \frac{8792q^3}{131625} + \frac{218507q^4}{3948750} + \frac{54190157q^5}{1135265625} + \cdots\ , \\ \widehat{\mathscr{Z}}^{(1,0),(2,0)}_{[0,1,1];(1,1)}(q) &= \frac{q^{\frac{2}{3}}}{3}(1-q)^{2h_\square - \frac{4}{5}}\, {}_2F_1\left(\frac{2}{5}, \frac{4}{5}; \frac{8}{5}; q\right) \\ &= \frac{q^{\frac{2}{3}}}{3} + \frac{7q^{\frac{5}{3}}}{45} + \frac{1867q^{\frac{8}{3}}}{17550} + \frac{32582q^{\frac{11}{3}}}{394875} + \frac{18575621q^{\frac{14}{3}}}{272463750} + \cdots\ ,\end{aligned} \tag{4.4.29}$$

where the second one respects the fusion rules by (4.3.21). The above results (4.4.28) and (4.4.29) support conjecture 4.3.3.4.

4.4.3 $(N, n) = (3, 2)$ and $\widehat{\mathfrak{sl}}(2)_3$-WZW model

For $(N, n) = (3, 2)$, there are four highest-weight representations

$$\emptyset = [3,0], \quad \square = [2,1], \quad \square\square = [1,2], \quad \square\square\square = [0,3], \tag{4.4.30}$$

with conformal dimensions

$$h_{[k_0,k_1]} = \frac{k_1(k_1+2)}{20}: \quad h_\emptyset = 0, \quad h_\square = \frac{3}{20}, \quad h_{\square\square} = \frac{2}{5}, \quad h_{\square\square\square} = \frac{3}{4}. \tag{4.4.31}$$

4.4.3.1 Burge-reduced generating functions of coloured Young diagrams

The $\mathfrak{t}$-refined Burge-reduced generating functions (4.2.18) for $(N,n)=(3,2)$ are obtained as

$$\begin{aligned}
\widehat{X}^{\rm red}_{[3,0]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[3,1,1]}_{(0,0,0);(-l)}(q)\,\mathfrak{t}^{2l} = X^{[3,0]}_{[3,0]}(q)\,f_0(q,\hat{\mathfrak{t}}) + X^{[3,0]}_{[1,2]}(q)\,g_0(q,\hat{\mathfrak{t}}),\\
\widehat{X}^{\rm red}_{[0,3]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[3,1,1]}_{(1,1,1);(-l)}(q)\,\mathfrak{t}^{2l+3} = X^{[0,3]}_{[0,3]}(q)\,f_1(q,\hat{\mathfrak{t}}) + X^{[0,3]}_{[2,1]}(q)\,g_1(q,\hat{\mathfrak{t}}),\\
\widehat{X}^{\rm red}_{[2,1]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[2,2,1]}_{(1,0,0);(-l)}(q)\,\mathfrak{t}^{2l+1} = X^{[2,1]}_{[2,1]}(q)\,g_1(q,\hat{\mathfrak{t}}) + X^{[2,1]}_{[0,3]}(q)\,f_1(q,\hat{\mathfrak{t}}),\\
\widehat{X}^{\rm red}_{[1,2]}(q,\mathfrak{t}) &= (q;q)_\infty \sum_{l\in\mathbb{Z}} X^{[2,1,2]}_{(1,1,0);(-l)}(q)\,\mathfrak{t}^{2l+2} = X^{[1,2]}_{[1,2]}(q)\,g_0(q,\hat{\mathfrak{t}}) + X^{[1,2]}_{[3,0]}(q)\,f_0(q,\hat{\mathfrak{t}}),
\end{aligned} \tag{4.4.32}$$

where $\hat{\mathfrak{t}} = q^{-\frac14}\,\mathfrak{t}$,

$$\begin{aligned}
&X^{[3,0]}_{[3,0]}(q) = 1+q+3q^2+6q^3+12q^4+21q^5+39q^6+64q^7+108q^8+\cdots\ ,\\
&X^{[3,0]}_{[1,2]}(q) = q^{\frac12}+2q^{\frac32}+5q^{\frac52}+9q^{\frac72}+18q^{\frac92}+31q^{\frac{11}{2}}+55q^{\frac{13}{2}}+90q^{\frac{15}{2}}+149q^{\frac{17}{2}}+\cdots\ ,\\
&X^{[1,2]}_{[1,2]}(q) = 1+2q+5q^2+10q^3+20q^4+36q^5+64q^6+108q^7+180q^8+\cdots\ ,\\
&X^{[1,2]}_{[3,0]}(q) = q^{\frac12}+3q^{\frac32}+6q^{\frac52}+13q^{\frac72}+24q^{\frac92}+44q^{\frac{11}{2}}+76q^{\frac{13}{2}}+129q^{\frac{15}{2}}+210q^{\frac{17}{2}}+\cdots\ ,\\
&X^{[0,3]}_{[0,3]}(q) = X^{[3,0]}_{[3,0]}(q),\quad X^{[0,3]}_{[2,1]}(q) = X^{[3,0]}_{[1,2]}(q),\quad X^{[2,1]}_{[2,1]}(q) = X^{[1,2]}_{[1,2]}(q),\quad X^{[2,1]}_{[0,3]}(q) = X^{[1,2]}_{[3,0]}(q),
\end{aligned} \tag{4.4.33}$$

and

$$f_\sigma(q,\hat{\mathfrak{t}}) = \sum_{j\in 6\,\mathbb{Z}+3\,\sigma} q^{\frac{1}{12}j^2}\,\hat{\mathfrak{t}}^{\,j},\quad g_\sigma(q,\hat{\mathfrak{t}}) = \sum_{j\in 6\,\mathbb{Z}\pm(2-\sigma)} q^{\frac{1}{12}j^2+\frac16}\,\hat{\mathfrak{t}}^{\,j},\quad \sigma=0,1. \tag{4.4.34}$$

The Burge-reduced generating functions (4.4.32) agree with the $\widehat{\mathfrak{sl}}(2)_3$-WZW characters computed by (1.3.36),

$$\begin{aligned}
&\widehat{X}^{\rm red}_{[3,0]}(q,\mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_3}_{[3,0]}(q,\hat{\mathfrak{t}}),\quad \widehat{X}^{\rm red}_{[0,3]}(q,\mathfrak{t}) = \chi^{\widehat{\mathfrak{sl}}(2)_3}_{[0,3]}(q,\hat{\mathfrak{t}}),\\
&\widehat{X}^{\rm red}_{[2,1]}(q,\mathfrak{t}) = q^{\frac{1}{10}}\,\chi^{\widehat{\mathfrak{sl}}(2)_3}_{[2,1]}(q,\hat{\mathfrak{t}}),\quad \widehat{X}^{\rm red}_{[1,2]}(q,\mathfrak{t}) = q^{\frac{1}{10}}\,\chi^{\widehat{\mathfrak{sl}}(2)_3}_{[1,2]}(q,\hat{\mathfrak{t}}),
\end{aligned} \tag{4.4.35}$$

and corollary 4.2.2.6 is confirmed. Up to an overall factor, the functions (4.4.33) are the $\widehat{\mathfrak{sl}}(2)$ string functions of level-3 in [57] and given by (*cf.* Corollary 4.2.2.4),

$$\begin{aligned}
X^{[3,0]}_{[3,0]}(q) - q^{\frac{1}{6}}\, X^{[3,0]}_{[1,2]}(q) &= \frac{\left(q^{\frac{2}{3}};q^{\frac{5}{3}}\right)_\infty \left(q;q^{\frac{5}{3}}\right)_\infty \left(q^{\frac{5}{3}};q^{\frac{5}{3}}\right)_\infty}{(q;q)^2_\infty}, \\
X^{[3,0]}_{[1,2]}(q) &= q^{\frac{1}{2}}\, \frac{\left(q^3;q^{15}\right)_\infty \left(q^{12};q^{15}\right)_\infty \left(q^{15};q^{15}\right)_\infty}{(q;q)^2_\infty}, \\
X^{[1,2]}_{[1,2]}(q) &= \frac{\left(q^6;q^{15}\right)_\infty \left(q^9;q^{15}\right)_\infty \left(q^{15};q^{15}\right)_\infty}{(q;q)^2_\infty}, \\
q^{\frac{1}{6}}\, X^{[1,2]}_{[1,2]}(q) - X^{[1,2]}_{[3,0]}(q) &= q^{\frac{1}{6}}\, \frac{\left(q^{\frac{1}{3}};q^{\frac{5}{3}}\right)_\infty \left(q^{\frac{4}{3}};q^{\frac{5}{3}}\right)_\infty \left(q^{\frac{5}{3}};q^{\frac{5}{3}}\right)_\infty}{(q;q)^2_\infty}.
\end{aligned} \tag{4.4.36}$$

By taking $\mathfrak{t} = 1$, the principal characters of $\widehat{\mathfrak{sl}}(2)$ are obtained as in (4.2.21):

$$\begin{aligned}
\widehat{X}^{\mathrm{red}}_{[3,0]}(q,1) = \widehat{X}^{\mathrm{red}}_{[0,3]}(q,1) = \Pr \chi^{\widehat{\mathfrak{sl}}(2)}_{[3,0]}(q) &= \frac{\left(-q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty}{\left(q;q^{\frac{5}{2}}\right)_\infty \left(q^{\frac{3}{2}};q^{\frac{5}{2}}\right)_\infty}, \\
\widehat{X}^{\mathrm{red}}_{[2,1]}(q,1) = \widehat{X}^{\mathrm{red}}_{[1,2]}(q,1) = \Pr \chi^{\widehat{\mathfrak{sl}}(2)}_{[2,1]}(q) &= \frac{\left(-q^{\frac{1}{2}};q^{\frac{1}{2}}\right)_\infty}{\left(q^{\frac{1}{2}};q^{\frac{5}{2}}\right)_\infty \left(q^{2};q^{\frac{5}{2}}\right)_\infty}.
\end{aligned} \tag{4.4.37}$$

Note that, these principal characters are related to the principal characters of $\widehat{\mathfrak{sl}}(3)$ in (4.4.25) by

$$\frac{\Pr \chi^{\widehat{\mathfrak{sl}}(2)}_{[3,0]}(q^2)}{(q^2;q^2)_\infty} = \frac{\Pr \chi^{\widehat{\mathfrak{sl}}(3)}_{[2,0,0]}(q^3)}{(q^3;q^3)_\infty}, \qquad \frac{\Pr \chi^{\widehat{\mathfrak{sl}}(2)}_{[2,1]}(q^2)}{(q^2;q^2)_\infty} = \frac{\Pr \chi^{\widehat{\mathfrak{sl}}(3)}_{[1,1,0]}(q^3)}{(q^3;q^3)_\infty}. \tag{4.4.38}$$

4.4.3.2 Burge-reduced instanton partition functions

For $N = 3$ with general n, the Burge-reduced instanton partition functions (4.3.12) are determined from the parameters in $\mathbf{s} = [s_0, s_1, s_2]$, $\mathbf{s}_1 = [s^{(1)}_0, s^{(1)}_1, s^{(1)}_2]$, $\mathbf{s}_2 = [s^{(2)}_0, 1, s^{(2)}_2]$, $\mathbf{s}^{(3)} = [s^{(3)}_0, s^{(3)}_1, 1]$ and $\mathbf{s}^{(4)} = [s^{(4)}_0, s^{(4)}_1, s^{(4)}_2]$ in $P^{++}_{3,n+3}$ that are fixed by the relations (4.1.5), (4.3.11):

$$s_{i+1} = \sigma_{i-1} - \sigma_i + 1, \quad s^{(1)}_{i+1} = b_{i-1} - b_i + 1, \quad s^{(4)}_i = b'_{i-1} - b'_i + 1, \quad i = 1, 2, \tag{4.4.39}$$

and (4.3.15) from the ordered charges $\sigma_0 \geq \sigma_1 \geq \sigma_2$, $b_0 \geq b_1 \geq b_2$, $b'_0 \geq b'_1 \geq b'_2$. The Coulomb parameters are then determined from $\mathbf{s}$ by (4.3.5):

$$\begin{aligned} a_0 &= \frac{1}{3} \sum_{I=1,2} (I-3) \left(s_I - 1 - \frac{n}{3}\right) \epsilon_2, \\ a_1 &= \frac{1}{3} \sum_{I=1,2} (3-2I) \left(s_I - 1 - \frac{n}{3}\right) \epsilon_2, \\ a_2 &= \frac{1}{3} \sum_{I=1,2} I \left(s_I - 1 - \frac{n}{3}\right) \epsilon_2, \end{aligned} \tag{4.4.40}$$

and the mass parameters $\mathbf{m} = (m_0, \ldots, m_{N-1})$ and $\mathbf{m}' = (m'_0, \ldots, m'_{N-1})$ are determined from the parameters in $\mathbf{s}_1, \mathbf{s}_2$ and $\mathbf{s}^{(3)}, \mathbf{s}^{(4)}$, respectively, by (4.3.7).

We now consider the case of $(N, n) = (3, 2)$ with the rational Ω-background $\epsilon_1/\epsilon_2 = -5/3$ in (4.3.4).[7]

Example 4.4.3.3 ($\emptyset - \emptyset - (\emptyset) - \emptyset - \emptyset$). Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[3,0];(l)}(q)$ and take $l = 0$ in the fundamental chamber, which respects the fusion rules, as in conjecture 4.3.3.1. Here $\mathbf{s} = \mathbf{s}_1 = \mathbf{s}_2 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [3, 1, 1]$ are fixed. Then we see that the Burge-reduced instanton partition function is

$$\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[3,0];(0)}(q) = (1-q)^{-2h_\emptyset} = 1, \qquad h_\emptyset = 0, \tag{4.4.41}$$

which agrees with conjecture 4.3.3.1.

Example 4.4.3.4 ($\emptyset - \square - (\square) - \square - \emptyset$). Consider the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[2,1];(\mathbf{l})}(q)$ and take $\mathbf{l} = 0$ in the fundamental chamber as in conjecture 4.3.3.3, where $\mathbf{s} = \mathbf{s}^{(3)} = [2, 2, 1]$, $\mathbf{s}_1 = \mathbf{s}^{(4)} = [3, 1, 1]$ and $\mathbf{s}_2 = [2, 1, 2]$ are fixed. Then the Burge-reduced instanton partition function is obtained as

$$\widehat{\mathcal{Z}}^{(0,0,0),(0,0,0)}_{[2,1];(0)}(q) = (1-q)^{-2h_\square} = 1 + \frac{3q}{10} + \frac{39q^2}{200} + \frac{299q^3}{2000} + \frac{9867q^4}{80000} + \frac{424281q^5}{4000000} + \cdots, \tag{4.4.42}$$

where $h_\square = 3/20$. This series expansion agrees with conjecture 4.3.3.3.

Example 4.4.3.5 ($\square - \square - (\emptyset) - \square - \square$ and $\square - \square - (\square\square) - \square - \square$). For conjecture 4.3.3.4, consider, first, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[3,0];(\mathbf{1})}(q)$, where $\mathbf{s} = [3, 1, 1]$, $\mathbf{s}_1 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [2, 2, 1]$ and $\mathbf{s}_2 = [2, 1, 2]$ are fixed. Then, we find that the Burge-reduced

[7] Examples 4.4.3.3, 4.4.3.4 and 4.4.3.5 are confirmed up to $O(q^{\frac{11}{2}})$.

instanton partition functions for $\mathbf{l} = 0, -1$ in the fundamental chamber are

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[3,0];(0)}(q) &= (1-q)^{2h_{\square}-\frac{3}{5}}\,{}_2F_1\left(-\frac{1}{5},\frac{2}{5};\frac{3}{5};q\right) \\
&= 1+\frac{q}{6}+\frac{13q^2}{120}+\frac{87q^3}{1040}+\frac{8669q^4}{124800}+\frac{344797q^5}{5740800}+\cdots\,, \\
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[3,0];(-1)}(q) &= \frac{q^{\frac{1}{2}}}{3}\,(1-q)^{2h_{\square}-\frac{3}{5}}\,{}_2F_1\left(\frac{2}{5},\frac{4}{5};\frac{8}{5};q\right) \\
&= \frac{q^{\frac{1}{2}}}{3}+\frac{q^{\frac{3}{2}}}{6}+\frac{61q^{\frac{5}{2}}}{520}+\frac{289q^{\frac{7}{2}}}{3120}+\frac{222529q^{\frac{9}{2}}}{2870400}+\frac{25723q^{\frac{11}{2}}}{382720}+\cdots\,,
\end{aligned} \tag{4.4.43}$$

where $h_{\square} = 3/20$, and the second one respects the fusion rules by (4.3.21). Consider, next, the Burge-reduced instanton partition function $\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[1,2];(\mathbf{l})}(q)$, where $\mathbf{s} = \mathbf{s}_2 = [2,1,2]$ and $\mathbf{s}_1 = \mathbf{s}^{(3)} = \mathbf{s}^{(4)} = [2,2,1]$ are fixed. Then we find that the Burge-reduced instanton partition functions for $\mathbf{l} = 0, 1$ in the fundamental chamber are

$$\begin{aligned}
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[1,2];(0)}(q) &= (1-q)^{2h_{\square}-\frac{3}{5}}\,{}_2F_1\left(-\frac{1}{5},\frac{1}{5};\frac{2}{5};q\right) \\
&= 1+\frac{q}{5}+\frac{183q^2}{1400}+\frac{353q^3}{3500}+\frac{796073q^4}{9520000}+\frac{17182143q^5}{238000000}+\cdots\,, \\
\widehat{\mathcal{Z}}^{(1,0,0),(1,0,0)}_{[1,2];(1)}(q) &= \frac{q^{\frac{1}{2}}}{2}\,(1-q)^{2h_{\square}-\frac{3}{5}}\,{}_2F_1\left(\frac{1}{5},\frac{4}{5};\frac{7}{5};q\right) \\
&= \frac{q^{\frac{1}{2}}}{2}+\frac{29q^{\frac{3}{2}}}{140}+\frac{393q^{\frac{5}{2}}}{2800}+\frac{51949q^{\frac{7}{2}}}{476000}+\frac{1725293q^{\frac{9}{2}}}{19040000}+\frac{74432711\mathfrak{q}^{\frac{11}{2}}}{952000000}+\cdots\,,
\end{aligned} \tag{4.4.44}$$

where the second one respects the fusion rules by (4.3.21). The above results (4.4.43) and (4.4.44) support conjecture 4.3.3.4.

Chapter 5

The Full Algebra $\mathcal{A}(N, n; p)$

We now consider the coset AGT correspondence conjectured in the previous two chapters between gauge theories under a minimal model identification and CFTs with the symmetry algebra $\mathcal{A}(N, n; p)$, and let $p \neq N$ with $p > n$. In this case, we choose to consider $\mathcal{A}(N, n; p)$ using the first form

$$\mathcal{A}(N, n; p) = \frac{\widehat{\mathfrak{gl}}(p)_N}{\widehat{\mathfrak{gl}}(p-n)_N}. \tag{5.0.1}$$

In the following, we reindex and use the convention

$$\mathcal{A}(N, n; n+p) = \frac{\widehat{\mathfrak{gl}}(n+p)_N}{\widehat{\mathfrak{gl}}(p)_N}, \tag{5.0.2}$$

instead, where $p \neq N - n$.

Following the ideas presented in 4.2, we will use our conjectured AGT correspondence between 2D CFTs and 4D $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ to calculate the character functions for representations of $\mathcal{A}(N, n; n+p)$ using the generating functions for coloured Burge multipartitions. We propose a non-trivial identification between the minimal model parameters on the CFT side and the parameters defining the Burge inequalities and their colourings for the instantons on the gauge theory side.

We will then provide evidence of this identification in examples where N, n and p are small. We will then discuss new $\widehat{\mathfrak{sl}}(n)$ string function identities that arise as a consequence of this conjecture, and provide evidence of their existence. These string functions identities will come in two forms: as expressions involving $\mathcal{W}_N$-minimal model characters and as q-generating functions of Burge multipartitions, the latter of which we define in (5.3.4).

5.1 Some Coset Character Identities

We begin by recalling that we denote by $P_{N,a}$ the $\widehat{\mathfrak{sl}}(N)$-weight lattice of level $a \in \mathbb{Z}_{>0}$, and by $P^+_{N,a}$ the level a dominant integral weights. For $\xi_a \in P^+_{N,a}$, $\xi_b \in P^+_{N,b}$, and $\xi_{a+b} \in P^+_{N,a+b}$, we also recall, from section 1.3, that we denote the branching functions, describing $L_{\xi_{a+b}}$ as a submodule in the decomposition of the tensor product $L_{\xi_a} \otimes L_{\xi_b}$, by $b^{\xi_a \times \xi_b}_{\xi_{a+b}}$.

We also have the following equality of the branching functions [143]

$$\sum_{\xi_{a+b} \in P^+_{N,a+b}} b^{\xi_a \times \xi_b}_{\xi_{a+b}} b^{\xi_{a+b} \times \xi_c}_{\xi_{a+b+c}} = b^{\xi_a \times \xi_b \times \xi_c}_{\xi_{a+b+c}}, \tag{5.1.1}$$

where $\xi_a \in P^+_{N,a}$, $\xi_b \in P^+_{N,b}$, and $\xi_{a+b+c} \in P^+_{N,a+b+c}$.

As diagonal branching functions of the form $b^{\xi_a \times \xi_b}_{\xi_{a+b}}$ are closely associated with diagonal coset characters (see section 1.5.12), we choose to represent the equation (5.1.1) with a formal product of coset algebras

$$\frac{\widehat{\mathfrak{sl}}(N)_a \times \widehat{\mathfrak{sl}}(N)_b}{\widehat{\mathfrak{sl}}(N)_{a+b}} \times \frac{\widehat{\mathfrak{sl}}(N)_{a+b} \times \widehat{\mathfrak{sl}}(N)_c}{\widehat{\mathfrak{sl}}(N)_{a+b+c}} = \frac{\widehat{\mathfrak{sl}}(N)_a \times \widehat{\mathfrak{sl}}(N)_b \times \widehat{\mathfrak{sl}}(N)_c}{\widehat{\mathfrak{sl}}(N)_{a+b+c}}. \tag{5.1.2}$$

We find this is an efficient way to manipulate branching functions, as this formal product of cosets resembles the usual rules for multiplying fractions. This formal product of diagonal cosets only exists if the level of the denominator of one factor is equal to the level of one of the numerator factors of the other. In this case we say the two coset factors are *compatible*. In (5.1.2), the presence of $\widehat{\mathfrak{sl}}(N)_{a+b}$ in the denominator of the first factor and numerator of the second factor ensures that these two cosets are compatible.

Remark 5.1.0.1. This notation is inspired to be of similar style to the coset arguments used in [143, 27]. In particular in [27], the non-minimal model characters of $\mathcal{A}(n,2;p)$ were calculated and found to agree with the non-Burge reduced generating functions of instantons, which are generating functions of pairs of coloured Young diagrams where the Burge inequalities are not enforced.

As the authors of [27] did not consider minimal model CFTs they had to first enlarge the symmetry generating currents of the product of cosets on the left-hand side of (5.1.2) to obtain a realization of the algebra $\mathcal{A}(n,2;p)$. As we will see in the subsequent text, the minimal model characters for $\mathcal{A}(N,n;n+p)$ are equal to the branching functions of left-hand side without consideration of additional holomorphic currents. We also note that in the context of minimal models, these coset manipulations have been shown to correspond to constructions of CFTs and some primary fields [144]. Due to this, we motivate our forms for the branching functions through these formal coset arguments.

We have the following level-rank duality[1] identity

$$\frac{\widehat{\mathfrak{gl}}(p+1)_N}{\widehat{\mathfrak{gl}}(1)_N \times \widehat{\mathfrak{gl}}(p)_N} = \frac{\widehat{\mathfrak{sl}}(N)_1 \times \widehat{\mathfrak{sl}}(N)_p}{\widehat{\mathfrak{sl}}(N)_{p+1}}, \tag{5.1.3}$$

which is an equality between algebras and is used in similar coset manipulations in [138, 28]. We can then use (5.1.2) and (5.1.3) to obtain a formal expression for the branching functions of the $\mathcal{A}(N, n; n+p)$-minimal models. This is a conjectured equality, which will imply our conjecture 5.4.1. The examples we have computed in section 5.6.1 will provide evidence for its validity.

We begin with an n-fold product of compatible cosets

$$\mathcal{A}(N, n; n+p) = \frac{\widehat{\mathfrak{gl}}(n+p)_N}{\widehat{\mathfrak{gl}}(p)_N} = \frac{\widehat{\mathfrak{gl}}(n+p)_N}{\widehat{\mathfrak{gl}}(n+p-1)_N} \times \frac{\widehat{\mathfrak{gl}}(n+p-1)_N}{\widehat{\mathfrak{gl}}(n+p-2)_N} \times \cdots \times \frac{\widehat{\mathfrak{gl}}(p+1)_N}{\widehat{\mathfrak{gl}}(p)_N}. \tag{5.1.4}$$

For each factor, we formally multiply and divide by $\widehat{\mathfrak{gl}}(1)_N$ which gives us

$$\frac{\widehat{\mathfrak{gl}}(n+p-i)_N}{\widehat{\mathfrak{gl}}(n+p-i-1)_N} = \widehat{\mathfrak{gl}}(1)_N \times \frac{\widehat{\mathfrak{gl}}(n+p-i)_N}{\widehat{\mathfrak{gl}}(1)_N \times \widehat{\mathfrak{gl}}(n+p-i-1)_N}, \quad 0 \le i \le n-1. \tag{5.1.5}$$

We now use the level-rank identity (5.1.3) on the right-hand side of this equality and substitute $\widehat{\mathfrak{gl}}(1)$ as the Heisenberg algebra $\mathcal{H}$ to obtain

$$\widehat{\mathfrak{gl}}(1)_N \times \frac{\widehat{\mathfrak{gl}}(n+p-i)_N}{\widehat{\mathfrak{gl}}(1)_N \times \widehat{\mathfrak{gl}}(n+p-i-1)_N} = \mathcal{H} \times \frac{\widehat{\mathfrak{sl}}(N)_1 \times \widehat{\mathfrak{sl}}(N)_{n+p-i-1}}{\widehat{\mathfrak{sl}}(N)_{n+p-i}}. \tag{5.1.6}$$

We substitute this expression into our compatible coset product expression for $\mathcal{A}(N, n; n+p)$ to obtain

$$\mathcal{A}(N, n; n+p) = \left(\mathcal{H} \times \frac{\widehat{\mathfrak{sl}}(N)_1 \times \widehat{\mathfrak{sl}}(N)_{n+p-1}}{\widehat{\mathfrak{sl}}(N)_{n+p}}\right) \times \cdots \times \left(\mathcal{H} \times \frac{\widehat{\mathfrak{sl}}(N)_1 \times \widehat{\mathfrak{sl}}(N)_p}{\widehat{\mathfrak{sl}}(N)_{p+1}}\right) \tag{5.1.7}$$

We now note that the diagonal cosets in each factor represent the GKO construction for the minimal models as described in section 1.5.12. Thus, we can express the i-th factor in terms of minimal models as

$$\mathcal{H} \times \frac{\widehat{\mathfrak{sl}}(N)_1 \times \widehat{\mathfrak{sl}}(N)_{n-i+p-1}}{\widehat{\mathfrak{sl}}(N)_{n-i+p}} = M(N+n+p-i, N+n+p-i+1; N). \tag{5.1.8}$$

[1] As the name implies, level-rank duality refers to dualities between the conformal blocks of $\widehat{\mathfrak{sl}}(n)_N$ and $\widehat{\mathfrak{sl}}(N)_n$-WZW models, so that the representations $\widehat{\mathfrak{sl}}(n)_N$ and $\widehat{\mathfrak{sl}}(N)_n$ are in a sense dual to each other. The reader interested in learning about level-rank and its origins from conformal embeddings is pointed to [145, 146]. For an example of the application of level-rank duality to cosets see [147].

We substitute this in above to obtain

$$\mathcal{A}(N, n; n+p) = \mathcal{H}^n \times M(N+n+p-1, N+n+p; N) \\ \times M(N+n+p-2, N+n+p-1; N) \times \cdots \times M(N+p-1, N+p; N). \tag{5.1.9}$$

Now the expression (5.1.1) for the branching function of a product of diagonal cosets is in terms of minimal model factors. It is then natural to expect that there will be an expression for the minimal model characters of $\mathcal{A}(N, n; n+p)$ in terms of a sum of products of $\mathcal{W}_N$-minimal model characters $\chi_{\xi\zeta}^{N+k,N+k+1}$ for $k \in \{p-1, n, \ldots, n+p-1\}$. In the following section, we use the explicit series expansion (1.5.144) for $\chi_{\xi\zeta}^{N+k,N+k+1}$ to obtain series expansions for $\mathcal{A}(N, n; n+p)$-minimal model characters.

5.2 Calculating Coset Branching Functions

In this section, we discuss how we will calculate the coset branching functions as sums of products of minimal model characters. To do this, we will calculate the branching functions of the tensor product

$$\underbrace{\widehat{\mathfrak{sl}}(N)_1 \otimes \cdots \otimes \widehat{\mathfrak{sl}}(N)_1}_{n} \otimes \widehat{\mathfrak{sl}}(N)_p \tag{5.2.1}$$

to irreducible $\widehat{\mathfrak{sl}}(N)_{n+p}$-modules.

To calculate this combinatorially, we will use the i-signatures of crystal graphs described in section 1.3.5. In this case, as we are only taking tensor products of level 1 irreducible representations, this greatly simplifies computation. As we will see below, this process is straightforward but tedious and long. To find these branching functions efficiently we introduce new objects called B-matrices in section 5.2.3.

5.2.1 Rules for $\widehat{\mathfrak{sl}}(N)$ Branching

We begin by describing which highest weights appear in the branching of the tensor product of two $\widehat{\mathfrak{sl}}(N)$-modules, calculated using the i-signatures of crystal graphs. Let $\xi = [\xi_0, \ldots, \xi_{N-1}] \in P^+_{N,p}$ and $\zeta = [\zeta_0, \ldots, \zeta_{N-1}] \in P^+_{N,n+p}$ and consider the branching of the tensor product $L_\xi \otimes L_\zeta$ to irreducible $\widehat{\mathfrak{sl}}(N)_{2n+p}$-modules.

Proposition 5.2.1.1. *Let μ be a dominant integral weight such that L_μ is an irreducible $\widehat{\mathfrak{sl}}(N)_{2n+p}$-submodule of $L_\xi \otimes L_\zeta$, then*

$$cls(\mu) = cls(\xi + \zeta). \tag{5.2.2}$$

Although this result is standard in the usual theory of affine Lie algebras, we believe the proof will be instructive to the reader. We will prove this using crystal graph techniques, and the methodology presented will be used to calculate the branching rules below.

Proof. Let $\mathcal{B}_\xi$ and $\mathcal{B}_\zeta$ be the crystal graphs with respective node sets B_ξ and B_ζ of the irreducible highest weight modules L_ξ and L_ζ respectively. We recall that Ω_{L_Λ} denotes the weight space of the module L_Λ. Let $\xi' \in \Omega_{L_\xi}$ and $\zeta' \in \Omega_{L_\zeta}$ be two weights with Dynkin labels $\xi' = [\xi'_0, \ldots, \xi'_{N-1}]$ and $\zeta' = [\zeta'_0, \ldots, \zeta'_{N-1}]$.

The i-signatures of the nodes $b_{\xi'} \in B_\xi$ and $b_{\zeta'} \in B_\zeta$ corresponding to vectors $v_{\xi'} \in L_\xi$ and $v_{\zeta'} \in L_\zeta$ with weights ξ' and ζ' respectively are

$$\hat{\omega}^i_{\xi'} = \underbrace{- \cdots -}_{(\xi')^-_i} \overbrace{+ \cdots +}^{(\xi')^+_i}, \qquad \hat{\omega}^i_{\zeta'} = \underbrace{- \cdots -}_{(\zeta')^-_i} \overbrace{+ \cdots +}^{(\zeta')^+_i}, \tag{5.2.3}$$

where the i-th Dynkin labels of the weights ξ and ζ' are $\xi'_i = \xi'^-_i - \xi'^+_i$ and $\zeta'_i = \zeta'^-_i - \zeta'^+_i$ respectively. We can associate these parameters $\zeta'^\pm_i$ and $\xi'^\pm_i$ to the vectors $v_{\xi'} \in L_\xi$ and $v_{\zeta'} \in L_\zeta$, which will be assumed in the sequel.

Now we consider the node $b = (b_{\xi'} \times b_{\zeta'})$ in the product crystal graph $\mathcal{B}_{\xi\times\zeta}$. If $b \in \mathcal{B}_{\xi\times\zeta}$ corresponds to a highest weight vector $v_b \in L_\xi \otimes L_\zeta$ then

$$e_i v_b = 0, \quad \text{for } i = 0, 1, \ldots, N-1. \tag{5.2.4}$$

In terms of i-signatures, this means that the signatures $\hat{\omega}^i_b$ for each $i = 0, \ldots, N-1$ must not contain a plus sign. To calculate the signature $\hat{\omega}^i_b$, we concatenate the signatures $\hat{\omega}^i_{\xi'}$ and $\hat{\omega}^i_{\zeta'}$ and cancel ordered pairs of $(+-)$. From this we can see that for $\hat{\omega}^i_b$ to correspond to a highest weight vector $\hat{\omega}^i_{\zeta'}$ must also contain no plus signs so that

$$\hat{\omega}^i_{\zeta'} = \hat{\omega}^i_\zeta = \underbrace{- \cdots -}_{\zeta_i}, \tag{5.2.5}$$

and therefore the node $b_{\zeta'}$ corresponds to the highest weight vector of L_ζ.

Having constrained $b_{\zeta'}$ we can determine the different forms of $(b_{\xi'} \times b_{\zeta'})$ by considering the different forms of $b_{\xi'}$. Each weight $\xi' \in \Omega_{L_\xi}$ is such that $cls(\xi') = cls(\xi)$, and you can obtain the node $b_{\xi'}$ corresponding to ξ' by a sequence of applications of the Kashiwara raising $\{e_i\}_i$ and lowering $\{f_i\}_i$ operators on the highest weight node b_ξ. Thus if we can show that the action of Kashiwara operators on the nodes B_ξ is equivalent to the action of Kashiwara operators on $B_{\xi\times\zeta}$ then we have shown that each highest weight vector $(b_{\xi'} \times b_{\zeta'})$ has weights μ that are in the same class as $(\xi + \zeta)$.

The action of a Kashiwara operator can do three things to an i-signature: flip the right most plus sign to a negative sign, add a negative sign (if it flips a plus sign to a negative sign in the $(i-1)$-th or $(i+1)$-th signatures), or remove a negative sign (flipping a negative sign to a plus sign in the $(i-1)$-th or $(i+1)$-th signatures). Let us consider how these three options effect the Dynkin labels associated to a node.

Let μ be the weight associated to b, assuming b is a highest weight vector $\mu_i = \xi_i'^- + \zeta - (\xi')_i^+$. Flipping a plus to a minus sign in $\hat{\omega}^i_{\xi'}$ changes $\xi_i'^- \mapsto \xi_i'^- - 1$ and $\xi_i'^+ \mapsto \xi_i'^+ + 1$ so that $\xi_i' \mapsto \xi_i' + 2$. When concatenating i-signatures this additional plus is paired off so that $\mu_i \mapsto (\xi')_i^- - 1 + (\zeta - 1) - (\xi')_i^+ = \mu_i - 2$. When adding or removing a negative sign we change $(\xi')_i^- \mapsto (\xi')_i^+ \pm 1$ and $\xi_i' \mapsto \xi_i' \pm 1$. From this we have that $\mu_i \mapsto \mu_i \pm 1$ and we see that the action of a raising or lowering operator on $b_{\xi'}$ is equivalent to the action raising or lowering operator on b. Since weights that are obtained from a sequence of raising and lowering operator actions are in the same class, we can conclude that

$$cls(\mu) = cls(\xi + \zeta), \tag{5.2.6}$$

as desired. $\square$

Using this we can write down the branching rules (see (1.3.44)) $L_{\xi\times\zeta}$ as

$$L_{\xi\times\zeta}\Big|_{\widehat{\mathfrak{sl}}(N)_{2n+p}} = \bigoplus_{\substack{cls(\mu)=cls(\xi+\zeta)\\ k\in\mathbb{Z}_{\geq 0}}} b_{(\xi\times\zeta),(\mu-k\delta)} L_{\mu-k\delta}, \quad b_{\xi\times\zeta,(\mu-k\delta)} \in \mathbb{Z}_{\geq 0}. \tag{5.2.7}$$

5.2.2 Determining the $\widehat{\mathfrak{sl}}(N)$ Branching Functions

In this section, we calculate the branching functions of

$$\underbrace{\widehat{\mathfrak{sl}}(N)_1 \otimes \cdots \otimes \widehat{\mathfrak{sl}}(N)_1}_{n} \otimes \widehat{\mathfrak{sl}}(N)_p. \tag{5.2.8}$$

We begin by considering the character of

$$\widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_p, \tag{5.2.9}$$

which is the $n = 1$ case of (5.2.8). As this product will branch to $\widehat{\mathfrak{sl}}(N)_{p+1}$-representations, we have the following identity

$$\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_i}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\xi}(q) = \sum_{\zeta\in P^+_{N,p+1}} b^{\Lambda_i\times\xi}_{\zeta}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\zeta}(q), \tag{5.2.10}$$

where $\Lambda_i \in P^+_{N,1}$ is the i-th $\widehat{\mathfrak{sl}}(N)$ fundamental weight and $\xi \in P^+_{N,p}$. To determine the branching function $b^{\Lambda_i\times\xi}_{\zeta}$ we represent the branching from $L_{\Lambda_i\times\xi}$ to L_ζ by the coset

$$\frac{\widehat{\mathfrak{sl}}(N)_1 \times \widehat{\mathfrak{sl}}(N)_p}{\widehat{\mathfrak{sl}}(N)_{p+1}}, \tag{5.2.11}$$

whose character is the branching function $b^{\Lambda_i\times\xi}_{\zeta}$. As per our discussion in section 1.5.12, this represents the minimal model $M(N+p, N+p+1; N)$ so that the branching function is the minimal model character $\chi^{N,N+p,N+p+1}_{\xi,\zeta}$ when $cls(\zeta) = cls(\Lambda_i + \xi)$ and vanishes otherwise. We therefore have the equality of characters

$$\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_i}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\xi}(q) = \sum_{\substack{\zeta\in P^+_{N,p+1}\\ cls(\zeta)=cls(\xi+\Lambda_i)}} q^{\frac{1}{2}\left(|\zeta-\xi|^2-|\Lambda_i|^2\right)}\chi^{N,N+p,N+p+1}_{\xi,\zeta}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\zeta}(q). \tag{5.2.12}$$

The powers of q in an $\widehat{\mathfrak{sl}}(N)$ character count the grade at which states occur as discussed in section 1.3. To obtain the exponent of the extra q-factor in this expression we consider the occurrence of the weight ζ in the representation $L_\xi \otimes L_{\Lambda_i}$, where $cls(\zeta) = cls(\xi + \Lambda_i)$. This can be found first at grade

$$\frac{1}{2}\left(|\zeta-\xi|^2 - |\Lambda_i|^2\right), \tag{5.2.13}$$

and this gives the q-exponent.

We now repeat this process and consider characters of the associative 3-fold tensor product of $\widehat{\mathfrak{sl}}(N)$ representations

$$\widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_p = \widehat{\mathfrak{sl}}(N)_1 \otimes \left(\widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_p\right), \tag{5.2.14}$$

which we perform in two steps indicated by the brackets. To compute the character of this, we first branch the bracketed tensor product to $\widehat{\mathfrak{sl}}(N)_{p+1}$ and then tensor the result of this with the final $\widehat{\mathfrak{sl}}(N)_1$ representation. As described above, this gives us the equality of characters

$$\begin{aligned}\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_j}(q)\left(\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_i}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\xi}(q)\right) &= \chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_j}(q)\\ &\times\left(\sum_{\substack{\zeta\in P^+_{N,p+1}\\ cls(\zeta)=cls(\xi+\Lambda_i)}} q^{\frac{1}{2}\left(|\zeta-\xi|^2-|\Lambda_i|^2\right)}\chi^{N,N+p,N+p+1}_{\xi,\zeta}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\zeta}(q)\right).\end{aligned} \tag{5.2.15}$$

We then do this branching procedure again on the right-hand side to obtain

$$\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_j}(q)\left(\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_i}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\xi}(q)\right) = \sum_{\substack{\zeta \in P^+_{N,p+1} \\ cls(\zeta)=cls(\xi+\Lambda_i)}} q^{\frac{1}{2}\left(|\zeta-\xi|^2-|\Lambda_i|^2\right)}\chi^{N,N+p,N+p+1}_{\xi,\zeta}(q)$$
$$\times\left(\sum_{\substack{\zeta' \in P^+_{N,p+2} \\ cls(\zeta')=cls(\zeta+\Lambda_j)}} q^{\frac{1}{2}\left(|\zeta'-\zeta|^2-|\Lambda_j|^2\right)}\chi^{N,N+p+1,N+p+2}_{\zeta,\zeta'}(q)\right)\chi^{\widehat{\mathfrak{sl}}(N)}_{\zeta'}(q). \quad (5.2.16)$$

By induction on the 3-fold branching function process we have described, we can algorithmically obtain the branching functions of the n-fold tensor product (5.2.8). It is clear that the resulting expressions will be products of minimal model characters that are contracted over their subscripts, which will be $\widehat{\mathfrak{sl}}(N)$ weights of levels $l = p+1, p+2, \ldots, n+p-1$. This allow us to calculate the branching functions of

$$\underbrace{\widehat{\mathfrak{sl}}(N)_1 \otimes \cdots \otimes \widehat{\mathfrak{sl}}(N)_1}_{n} \otimes \widehat{\mathfrak{sl}}(N)_p, \quad (5.2.17)$$

using matrix multiplication.

Example 5.2.2.1. When $p = 0$, we calculate the characters similarly of an n-fold product of $\widehat{\mathfrak{sl}}(N)_1$ representations

$$\underbrace{\widehat{\mathfrak{sl}}(N)_1 \otimes \cdots \otimes \widehat{\mathfrak{sl}}(N)_1}_{n}, \quad (5.2.18)$$

which we represent by

$$\prod_{i=0}^{n-1}\chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_{\sigma_i}} = \sum_{\substack{\zeta \in P^+_{N,n} \\ cls(\zeta)=cls(\eta)}} b^{\prod_{i=0}^{p-1}\Lambda_{\sigma_i}}_{\zeta}(q)\chi^{\widehat{\mathfrak{sl}}(N)}_{\zeta}(q), \quad (5.2.19)$$

where $\eta = \sum_{i=0}^{n-1}\Lambda_{\sigma_i}$. As we will see in section 5.5, this will give us the $\widehat{\mathfrak{sl}}(N)$ string functions.

5.2.3 B-Matrices

In this subsection, we introduce the *B-Matrices* to compute the branching functions of n-fold products of $\widehat{\mathfrak{sl}}(N)_1$ representations. An individual B-matrix will represent one step of the recursive process described in (5.2.16).

We will refer to each step step of this process as the *level*, as after the l-th step we are considering branching to level l representations. The B-matrices of level l will be a set of matrices $\{B_{\Lambda_i} \mid i = 0, 1, \ldots, N-1\}$ labelled by the $\widehat{\mathfrak{sl}}(N)$ fundamental weights. We define the matrix $B_{\Lambda_i} = B^{(N,l)}_{\Lambda_i}(q)$ of level $l \in \mathbb{Z}_{\geq 1}$, for $N \in \mathbb{Z}_{\geq 2}$, $i = 0, 1, \dots, N-1$, with rows labelled by weights $\xi \in P^+_{N,l}$ and columns by $\zeta \in P^+_{N,l+1}$ as

$$\left(B^{(N,l)}_{\Lambda_i}(q)\right)_{\xi\zeta} = \begin{cases} q^{\frac{1}{2}\left(|\zeta-\xi|^2-|\Lambda_i|^2\right)}\chi^{N,N+l,N+l+1}_{\xi,\zeta}(q) & \text{if } cls(\zeta-\xi-\Lambda_i)=0, \\ 0 & \text{else,} \end{cases} \tag{5.2.20}$$

and supplement these with the $1 \times N$ level 0 matrices

$$\left(B^{(N,0)}_{\Lambda_i}\right)_{1j} = \delta_{ji}. \tag{5.2.21}$$

We see that non-zero B-matrix entries are minimal model characters $\chi^{N,N+l,N+l+1}_{\xi,\zeta}$ with a q prefactor whose exponent is equal to the grade at which ζ first appears in the module $\zeta \times \Lambda_i$.

Example 5.2.3.1. The $N = 3$ matrices for $l = 1$ are

$$B^{(3,1)}_{\Lambda_0} = \begin{pmatrix} \chi^{4,5}_{[100],[200]} & 0 & 0 & 0 & q\chi^{4,5}_{[100],[011]} & 0 \\ 0 & \chi^{4,5}_{[010],[110]} & 0 & 0 & 0 & q\chi^{4,5}_{[010],[002]} \\ 0 & 0 & q\chi^{4,5}_{[001],[020]} & \chi^{4,5}_{[001],[101]} & 0 & 0 \end{pmatrix} \tag{5.2.22}$$

$$B^{(3,1)}_{\Lambda_1} = \begin{pmatrix} 0 & \chi^{4,5}_{[100],[110]} & 0 & 0 & 0 & q\chi^{4,5}_{[100],[002]} \\ 0 & 0 & \chi^{4,5}_{[010],[020]} & \chi^{4,5}_{[010],[101]} & 0 & 0 \\ \chi^{4,5}_{[001],[200]} & 0 & 0 & 0 & \chi^{4,5}_{[001],[011]} & 0 \end{pmatrix} \tag{5.2.23}$$

$$B^{(3,1)}_{\Lambda_2} = \begin{pmatrix} 0 & 0 & q\chi^{4,5}_{[100],[020]} & \chi^{4,5}_{[100],[101]} & 0 & 0 \\ \chi^{4,5}_{[010],[200]} & 0 & 0 & 0 & \chi^{4,5}_{[010],[011]} & 0 \\ 0 & \chi^{4,5}_{[001],[110]} & 0 & 0 & 0 & \chi^{4,5}_{[001],[002]} \end{pmatrix}, \tag{5.2.24}$$

Note here that we have labelled the rows and columns by $\widehat{\mathfrak{sl}}(3)$ dominant integral weights. When writing down an explicit B-matrix, one must make a choice of ordering the row and column weights, in this case we ordered by size of Dynkin labels in numerical order (that is, from the 0th to 2nd).

Let $\eta \in P^+_{N,n}$ be a dominant integral weight and decompose η into a sum of the fundamental weights as $\eta = \sum_{i=p}^{n+p-1} \Lambda_{\sigma_i}$, where we order $\sigma_p \leq \sigma_{p+1} \leq \cdots \leq \sigma_{n+p-1}$, and let p be as in (5.2.17). We define a n-fold product of B-matrices to be

$$B^{(N,n+p;p)}_{\eta}(q) = \prod_{l=p}^{n+p-1} B^{(N,l)}_{\Lambda_{\sigma_l}}(q), \tag{5.2.25}$$

where the order of the matrix multiplication is according to $\sigma_p \leq \sigma_{p+1} \leq \cdots \leq \sigma_{n+p-1}$. The weight η carries the information of the highest weights of each $\widehat{\mathfrak{sl}}(N)_1$ factor in the tensor product (5.2.18) for the fixed ordering of the weight labels $\{\sigma_i\}_{i=p,\dots,n+p-1}$. The matrix elements of $B_\eta^{(N,n+p;p)}$ then give the branching functions $b_\zeta^{\Lambda_{\sigma_p}\times\ldots\Lambda_{\sigma_{n+p-1}}}$, where $\zeta\in P^+_{N,n+p}$, for this branching.

For $p=0$ we can associate the product B-matrix

$$B_\eta^{(N,n;0)}(q)=B^{(N,0)}_{\Lambda_{\sigma_0}}(q)B^{(N,1)}_{\Lambda_{\sigma_1}}(q)\ldots B^{(N,n-1)}_{\Lambda_{\sigma_{n-1}}}(q), \tag{5.2.26}$$

to an n-fold tensor product of $\widehat{\mathfrak{sl}}(N)_1$ representations (5.2.18), which we represent using their highest weights as

$$\Lambda_{\sigma_0}\otimes\cdots\otimes\Lambda_{\sigma_{n-1}}, \tag{5.2.27}$$

where again $\eta=\sum_{i=0}^{n-1}\Lambda_{\sigma_i}$. In this case, the entries of $B_\eta^{(N,n;0)}(q)$ are the branching functions for $\widehat{\mathfrak{sl}}(N)_n$. Explicitly, the entry labelled by $(\Lambda_{\sigma_0},\zeta)$ will give the branching function for the weight ζ, that is

$$\left(B_\eta^{(N,n,0)}(q)\right)_{\Lambda_{\sigma_0}\zeta}=b_\zeta^{\prod_{i=0}^{n-1}\Lambda_{\sigma_i}}(q). \tag{5.2.28}$$

This $p=0$ case reproduces the character expressions in chapter 4.

Up until this point we have just introduced formalism, but we can use this to derive identities between minimal model characters. As mentioned above, the product B-matrix (5.2.25) for fixed $\eta\in P^+_{N,n}$ represents the tensor product of $\widehat{\mathfrak{sl}}(N)$ representations

$$\Lambda_{\sigma_p}\otimes\Lambda_{\sigma_{p+1}}\otimes\ldots\Lambda_{\sigma_{n+p-1}}\otimes\xi,\quad \xi\in P^+_{N,p}, \tag{5.2.29}$$

where we have ordered $\sigma_p\le\cdots\le\sigma_{n+p-1}$. In this case, we can use the well known commutativity isomorphism between tensor products of irreducible $\widehat{\mathfrak{sl}}(N)$-modules to obtain identities between B-matrices.

By commuting two or more factors in the tensor product above, we obtain two branching functions for each tensor product that are identical by the commutativity isomorphism. We can then calculate these equivalent branching functions using *inequivalent* products of B-matrices. The matrix elements of these matrix products are inequivalent sums of products of minimal model characters, which must be equal. We illustrate this below with a simple example.

Example 5.2.3.2. We consider the isomorphism of tensor products of $\widehat{\mathfrak{sl}}(2)_1$ representations

$$\Lambda_i\otimes\Lambda_0\otimes\Lambda_1\cong\Lambda_i\otimes\Lambda_1\otimes\Lambda_0,\quad i=0,1. \tag{5.2.30}$$

We have the following relevant B-matrices

$$B^{(2,1)}_{\Lambda_0}(q)=\begin{pmatrix}\chi^{3,4}_{1,1} & 0 & q\chi^{3,4}_{1,3}\\ 0 & \chi^{3,4}_{2,2} & 0\end{pmatrix},\quad B^{(2,1)}_{\Lambda_1}(q)=\begin{pmatrix}0 & \chi^{3,4}_{1,2} & 0\\ \chi^{3,4}_{2,1} & 0 & \chi^{3,4}_{2,3}\end{pmatrix} \tag{5.2.31}$$

$$B^{(2,2)}_{\Lambda_0}(q) = \begin{pmatrix} \chi^{4,5}_{1,1} & 0 & q\chi^{4,5}_{1,3} & 0 \\ 0 & \chi^{4,5}_{2,2} & 0 & q\chi^{4,5}_{2,4} \\ q\chi^{4,5}_{3,1} & 0 & \chi^{4,5}_{3,3} & 0 \end{pmatrix}, \quad B^{(2,2)}_{\Lambda_1}(q) = \begin{pmatrix} 0 & \chi^{4,5}_{1,2} & 0 & q^2\chi^{4,5}_{1,4} \\ \chi^{4,5}_{2,1} & 0 & q\chi^{4,5}_{2,3} & 0 \\ 0 & \chi^{4,5}_{3,2} & 0 & \chi^{4,5}_{3,4} \end{pmatrix}. \tag{5.2.32}$$

The branching functions of each tensor product are obtained by the two inequivalent products $B^{(2,0)}_{\Lambda_i}(q)B^{(2,1)}_{\Lambda_0}(q)B^{(2,2)}_{\Lambda_1}(q)$ and $B^{(2,0)}_{\Lambda_i}(q)B^{(2,1)}_{\Lambda_1}(q)B^{(2,2)}_{\Lambda_0}(q)$ respectively. By the commutativity isomorphism we must then have

$$B^{(2,0)}_{\Lambda_i}(q)B^{(2,1)}_{\Lambda_0}(q)B^{(2,2)}_{\Lambda_1}(q) = B^{(2,0)}_{\Lambda_i}(q)B^{(2,1)}_{\Lambda_1}(q)B^{(2,2)}_{\Lambda_0}(q), \tag{5.2.33}$$

which is true if $B^{(2,1)}_{\Lambda_1}(q)B^{(2,2)}_{\Lambda_0}(q) = B^{(2,1)}_{\Lambda_1}(q)B^{(2,2)}_{\Lambda_0}(q)$. By considering the non-zero matrix elements we obtain the following identities between minimal model characters

$$\chi^{4,5}_{1,1}(q)\chi^{4,5}_{1,2}(q) + q\chi^{4,5}_{1,3}(q)\chi^{4,5}_{3,2}(q) = \chi^{4,5}_{1,2}(q)\chi^{4,5}_{2,2}(q) \tag{5.2.34}$$

$$q\chi^{4,5}_{1,1}(q)\chi^{4,5}_{1,4}(q) + \chi^{4,5}_{1,3}(q)\chi^{4,5}_{3,4}(q) = \chi^{4,5}_{1,2}(q)\chi^{4,5}_{2,4}(q) \tag{5.2.35}$$

By generalizing this process we have the following identity between B-matrices for $\Lambda, \Lambda' \in P^+_{N,1}$:

$$B^{(N,l)}_{\Lambda}(q)B^{(N,l+1)}_{\Lambda'}(q) = B^{(N,l)}_{\Lambda'}(q)B^{(N,l+1)}_{\Lambda}(q), \tag{5.2.36}$$

which correspond to the following isomorphism of tensor products

$$\cdots \otimes \Lambda \otimes \Lambda' \otimes \ldots \cong \cdots \otimes \Lambda' \otimes \Lambda \otimes \ldots \tag{5.2.37}$$

These identities appear to be generalizations of those in [148] to $\mathcal{W}_N$-minimal models for $N > 2$, and our generalization contains these previously known identities as the $N = 2$ case.

5.2.4 Using the B-Matrices to Calculate $\mathcal{A}(N, n; n+p)$ Minimal Model Characters

We now consider the branchings for an n-fold product of $\widehat{\mathfrak{sl}}(N)_1$ representations with a $\widehat{\mathfrak{sl}}(N)_p$ representation by the formal coset product

$$\frac{\widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_{n+p-1}}{\widehat{\mathfrak{sl}}(N)_{n+p}} \times \frac{\widehat{\mathfrak{sl}}(N)_1 \otimes \widehat{\mathfrak{sl}}(N)_{n+p-2}}{\widehat{\mathfrak{sl}}(N)_{n+p-1}} \times \cdots \times \frac{\widehat{\mathfrak{sl}}(N)_{p+1} \otimes \widehat{\mathfrak{sl}}(N)_1}{\widehat{\mathfrak{sl}}(N)_{p+2}} \times \frac{\widehat{\mathfrak{sl}}(N)_p \otimes \widehat{\mathfrak{sl}}(N)_1}{\widehat{\mathfrak{sl}}(N)_{p+1}}. \tag{5.2.38}$$

Following our discussion in section 5.2.3 above, we can calculate the branching function represented by this coset product using the B-matrix product $B^{(N,n+p;p)}_{\eta}$. Note that here we begin the $\widehat{\mathfrak{sl}}(N)$ product with a level p representation, so that we must begin our matrix product with a B-matrix of level p.

For this calculation, e will need to understand the generic structure of B-matrices, which we now discuss. We have the following properties of the B-matrix product $B_\eta^{(N,n+p;p)}$:

- Its rows are labelled by $\xi \in P^+_{N,p}$ and columns by $\zeta \in P^+_{N,n+p}$.
- Its matrix elements vanish when $cls(\zeta - \xi) \neq cls(\eta)$.
- Its non-zero elements are sums of products of minimal model characters with some additional q-factors.

Let $(\Lambda_{\sigma_p}, \ldots, \Lambda_{\sigma_{n+p-1}})$ be a sequence of $\widehat{\mathfrak{sl}}(N)$ fundamental weights such that $\sigma_j \leq \sigma_{j+1}$ for $p \leq j \leq n+p-1$ and $\eta = \sum_{i=p}^{n+p-1} \Lambda_{\sigma_i}$ a corresponding $\widehat{\mathfrak{sl}}(N)_n$ dominant integral weight. The products of minimal model characters in the non-zero matrix elements $\left(B_\eta^{(N,n+p;p)}\right)_{\xi\zeta}$ can be defined by sequences $\mu = (\mu_{p+1}, \ldots, \mu_{n+p-1})$ of $\widehat{\mathfrak{sl}}(N)$ weights such that $\mu_j \in P^+_{N,j}$ and $cls(\mu_j) = cls(\xi + \sum_{i=p}^{j} \Lambda_{\sigma_i})$ for $p \leq j \leq n+p-1$. The non-zero elements are then formed as a summation over each possible sequence of weights satisfying those conditions, with suitable q factors. Therefore, the branching functions for (5.2.17), which we can represent by the formal coset product (5.2.38), are given by matrix elements of $B_\eta^{(N,n+p;p)}$.

To obtain the minimal model characters of $\mathcal{A}(N, n; n+p)$ from these branching functions, we must include another factor for the n-copies of the Heisenberg algebra $\mathcal{H}$. Since the character of $\mathcal{H}$ is the generator of partitions, $\frac{1}{(q;q)_\infty}$, we define the hatted B-matrix product

$$\hat{B}_\eta^{(n+p;p)}(q) = \frac{1}{(q;q)_\infty^n} B_\eta^{(N,n+p,p)}(q), \tag{5.2.39}$$

so that the matrix elements of $\hat{B}_\eta^{(n+p;p)}(q)$ are coset characters for $\mathcal{A}(N, n; n+p)$. These elements are labelled by $\xi \in P^+_{N,n}$ and $\zeta \in P^+_{N,n+p}$, and combined with η these weights fully determine the coset branching, as there are no additional parameters coming from $\mathcal{H}^n$. In the hatted notation $\hat{B}_\eta^{(n+p;n)}(q)$, we drop the superscript N since the rank of the algebras $\widehat{\mathfrak{sl}}(N)$ used to calculate the branching is clear from the length of the weight η.

5.3 Burge Generating Functions and Dual Dynkin Rings

The AGT conjecture in chapter 3 predicts a relationship between the generating function for the instantons in $\mathcal{N} = 2$ $SU(N)$ supersymmetric gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ and the character functions for $\mathcal{A}(N, n; p)$. The latter are identified as the matrix entries of $\hat{B}_\eta^{(n+p;n)}(q)$ in section 5.2.4 above. We will view this identification as a relationship between two combinatorial objects, the (ξ, ζ)-Burge multipartitions and the sums of products of minimal model characters.

5.3.1 Coloured Burge Generating Functions

We only consider coloured (ξ,ζ)-Burge multipartitions that satisfy the $\mathbb{Z}_n$-charge conditions (3.4.5), which are satisfied by imposing a colouring defined by the charges $\sigma = (\sigma_0,\dots,\sigma_{N-1}) \in (\mathbb{Z}_n)^N$, where

$$\sigma_j = \sigma_0 + \sum_{i=0}^{j}(\zeta_i - \xi_i), \quad j = 0,\dots,N-1. \tag{5.3.1}$$

Remark 5.3.1.1. These coloured Burge multipartitions can be thought of as skew 3D plane partitions built from Young diagrams stacked besides each other, with an additional cyclic condition. In this view point, the box $\square = (1,1)$ in the j-th diagram is lined up with the box $\blacksquare = (1+\zeta_j, 1+\xi_j)$ in the $(j+1)$-th diagram. The charge σ_j ensures this skew plane partition is coloured so that the colour of adjacent boxes in distinct diagrams are the same. Skew plane partitions are outside the scope of this thesis, the interested reader should refer to [149].

We recall some of the notation from section 1.1. We denote the set of coloured (ξ,ζ)-multipartitions with charge $\sigma = (\sigma_0,\dots,\sigma_{N-1})$ by $C^{\sigma}_{\xi\zeta}$. Given a multipartition $\lambda = (\lambda^{(0)},\dots,\lambda^{(N-1)})$ coloured with n colours, we define the total number of i-coloured boxes to be $k_i(\lambda)$ and collect them in the vector $\mathbf{k} = (k_0,\dots,k_{n-1})$. We then define the generating function for (ξ,ζ)-Burge multipartitions with the colouring σ to be

$$X_{\xi,\zeta;n}(\mathfrak{q};\mathbf{z}) = \sum_{\lambda\in\mathcal{C}^{\sigma}_{\xi\zeta}}^{k_0(\lambda)} \prod_{i=1}^{n-1} z_i^{k_i(\lambda)}. \tag{5.3.2}$$

As this generating function counts the number of total coloured boxes, we also define a second generating function using the variables $\mathfrak{q} = \mathfrak{q}/(x_1\dots x_{n-1})$ and $z_i = x_i$ for $i = 1,\dots,n-1$ by

$$X_{\xi,\zeta;n}(\mathfrak{q}/(x_1\dots x_{n-1});\mathbf{x}) = \sum_{\lambda\in\mathcal{C}^{\sigma}_{\xi\zeta}} \mathfrak{q}^{k_0(\lambda)} \prod_{i=1}^{n-1} x_i^{\delta k_i(\lambda)}, \tag{5.3.3}$$

where we recall that $\delta k_i(\lambda) = k_i - k_0$ is the difference of coloured boxes. Finally, for a fixed $\delta\mathbf{k} = (\delta k_1,\dots,\delta k_{n-1}) \in \mathbb{Z}^{n-1}$ we define a third generating function

$$X^{\delta\mathbf{k}}_{\xi,\zeta;n}(q) = [\mathbf{x}^{\delta\mathbf{k}}]X_{\xi,\zeta;n}(\mathfrak{q}/(x_1\dots x_{n-1});\mathbf{x})\Big|_{\substack{x_i\mapsto q\\ \mathfrak{q}\mapsto q^n}}, \tag{5.3.4}$$

where here the prefix $[\mathbf{x}^{\delta\mathbf{k}}]$ means to take only the terms where the exponent of x_i is δk_i for each $i = 1,\dots,n-1$. We will refer to this last generating function as the coloured (ξ,ζ)-Burge q-generating function. The function $X^{\delta\mathbf{k}}_{\xi,\zeta;n}$ counts n-coloured (ξ,ζ)-Burge multipartitions with colouring data specified by $\delta\mathbf{k} = (\delta k_1,\dots,\delta k_{n-1})$ where the exponent of q counts the number of boxes in the multipartition λ. We conjecture that $X^{\delta\mathbf{k}}_{\xi,\zeta;n}$ equals to the matrix elements of $\hat{B}_\eta^{(n+p;p)}(q)$ under some identification of parameters. In the sequel, we will make this identification explicit.

Our notation so far has been used suggestively intentionally, and this equivalence will involve identifying the weight labels ξ and ζ used to define the Burge inequalities with the labels of matrix elements for $\hat{B}_\eta^{(n+p;p)}(q)$. Identifications for η and $\delta\mathbf{k}$ are more involved and will require the dual weights and Dynkin rings from section 1.3.

We will split this discussion into two distinct cases where possible, one where the level and rank are coprime integers and one where they are not coprime. The reason we do this is due to the rotation invariance of Dynkin rings, which means that Dynkin rings represent multiple different $\widehat{\mathfrak{sl}}(n)$ weights whose Dynkin labels are cyclic permutations of each other. When taking a dual of a weight, this rotation invariance will lead to an ambiguity related to classes of weights that needs to be fixed as we will see below.

This combinatorial identification will involve finding the sequence of raising and lowering operators to obtain a descendant state from a highest weight in an $\widehat{\mathfrak{sl}}(N)$-module. Since all weights in a highest weight module have the same class, this will necessitate that we choose a cyclic permutation of a dual weight with a prescribed class.

5.3.2 Dual Dynkin Rings and Fixing Classes

Proposition 5.3.2.2 below shows that we can fix which weight a Dynkin ring should represent, when the level and rank of the weight are coprime, by fixing its class. We first define the automorphism $\tau : P(\widehat{\mathfrak{sl}}(N)) \longrightarrow P(\widehat{\mathfrak{sl}}(N))$ that acts on the weight $\Lambda = [m_0, \ldots, m_{N-1}]$ as

$$\tau(\Lambda) = \tau([m_0, \ldots, m_{N-1}]) = [m_{N-1}, m_0, \ldots, m_{N-2}]. \tag{5.3.5}$$

Clearly, τ has order N. This proposition is then a consequence of the following lemma.

Lemma 5.3.2.1. Let $\Lambda = [m_0, \ldots, m_{N-1}]$ be an $\widehat{\mathfrak{sl}}(N)_n$ weight. For any $j \in \mathbb{Z}$, the class of the weight $\tau^j(\Lambda)$ is related to the class of Λ by

$$cls(\tau^j(\Lambda)) = cls(\Lambda) + jn \bmod N. \tag{5.3.6}$$

Proof. The class of Λ is:

$$cls(\Lambda) = \sum_{i=0}^{N-1} i m_i \bmod N. \tag{5.3.7}$$

We begin with the $j = 1$ case. As above we have that

$$\tau(\Lambda) = \tau([m_0, \ldots, m_{N-1}]) = [m_{N-1}, m_0, \ldots, m_{N-2}]. \tag{5.3.8}$$

So that we can calculate the class of $\tau(\Lambda)$ as

$$\begin{aligned}
cls(\tau(\Lambda)) &= \sum_{i=0}^{N-2}(i+1)m_i && \text{mod } N && (5.3.9)\\
&= \sum_{i=0}^{N-1} im_i + \sum_{i=0}^{N-2} m_i - (n-1)m_{N-1} && \text{mod } N && (5.3.10)\\
&= \sum_{i=0}^{N-1} im_i + \sum_{i=0}^{N-2} m_i + m_{N-1} - m_{N-1} - (N-1)m_{N-1} && \text{mod } N && (5.3.11)\\
&= \sum_{i=0}^{N-1} im_i + \sum_{i=0}^{N-1} m_i - Nm_{N-1} && \text{mod } N && (5.3.12)\\
&= cls(\Lambda) + n && \text{mod } N. && (5.3.13)
\end{aligned}$$

We now apply this result recursively j number of times to obtain

$$cls(\tau^j(\Lambda)) = cls(\Lambda) + jn \text{ mod } N. \tag{5.3.14}$$

□

Proposition 5.3.2.2. *Let D be a Dynkin ring that represents a set W of $\widehat{\mathfrak{sl}}(N)_n$ weights. The class of each weight $\xi \in W$ is unique if N and n are coprime.*

Proof. Let $\Lambda = [m_0, \ldots, m_{N-1}]$ be an $\widehat{\mathfrak{sl}}(N)_n$ weight in W, and fix the numbering of the slots of D so that Λ is the canonical choice of weight for D (that is that there are m_0 empty slots starting with the 1st slot before the 1st bead, m_1 empty slots after that until the 2nd bead etc). The last slot of a Dynkin ring corresponding to an $\widehat{\mathfrak{sl}}(N)$ weight must be occupied, so there are N different weights associated to D. We will define each weight in turn by rotating D clockwise so that the $(N-1)$-th bead is now the N-th bead, and represent this operation on Λ by the automorphism τ which cyclically permutes Dynkin labels defined above.

Two $\widehat{\mathfrak{sl}}(N)_n$ are in the same class if the difference of their classes vanishes modulo N. We rotate the ring $j < N$ times which corresponds to the weight $\tau^j(\Lambda)$. Using lemma 5.3.2.1 above we have that

$$cls(\Lambda) - cls(\tau^j(\Lambda)) = -jn \text{ mod } N. \tag{5.3.15}$$

There are only up to $(N-1)$ unique rotations of D possible, so that the classes of these differ by $n, 2n, \ldots, (N-1)n$. If $\gcd(N, n) = 1$ then jn does not vanish modulo N for $j = 1, \ldots, N-1$ and the classes of these rotations are all unique. □

From this we see that in the case where the level and rank for an $\widehat{\mathfrak{sl}}(N)_n$ weight are coprime, we can uniquely fix j in $\tau^j(\Lambda^\dagger)$ by specifying the value of $cls(\tau^j(\Lambda^\dagger))$.

5.4 The Coset-Burge Character Conjecture

We can now formulate the coset-Burge character conjecture which identifies the matrix elements of the truncated product of B-matrices with the generating function of coloured Burge multipartitions.

Conjecture 5.4.0.1. *The matrix elements of the truncated B-matrix $\hat{B}_\eta^{(n+p;p)}(q)$, defined in (5.2.39), are equal to the q-generating functions $X^{\delta\boldsymbol{k}}_{(\xi,\zeta;n)}(q)$ of (ξ,ζ)-Burge multipartitions with prescribed colour data $\delta\boldsymbol{k} = (\delta k_1, \ldots, \delta k_{n-1})$, where $\xi \in P^+_{N,p}$, $\zeta \in P^+_{N,n+p}$, $\eta \in P^+_{N,n}$. That is, we have*

$$\left(\hat{B}_\eta^{(n;d)}(q)\right)_{\xi,\zeta} = X^{\delta\boldsymbol{k}}_{(\xi,\zeta;n)}(q)q^{\Delta}, \tag{5.4.1}$$

where Δ is some constant. The parameters on both sides are identified using

$$\delta\boldsymbol{k} = A^{-1}\left((\zeta-\xi)^\dagger - \tau^{-j}(\eta^\dagger)\right), \tag{5.4.2}$$

where A is the $\widehat{\mathfrak{sl}}(n)$ Cartan matrix, $\dagger$ means to take the dual weight, τ cyclic permutes Dynkin labels, and[2] *$j = \frac{1}{N}\sum_{i=0}^{N-1} i(\zeta_i - \xi_i - \eta_i)$.*

Remark 5.4.0.2. In section 5.6.1, we will discuss our evidence for the coset-Burge character conjecture. We will also provide 3 fully worked examples of these calculations that include 2 examples for the simpler non-coprime case and one of the coprime case.

Below we motivate the form of the coset-Burge character conjecture and then move on to simple checks that reproduce previous known results in [49]. The specific form of the conjecture was derived through experimentation on Mathematica using the motivation below.

5.4.1 Motivation

We will refer to the combinatorial objects in the coset-Burge character conjecture by their positions in (5.4.1). We will therefore refer to the branching functions of tensor products of $\widehat{\mathfrak{sl}}(N)$ representations, calculated as hatted B-matrix elements, as the left-hand side and q-generating functions for coloured (ξ,ζ)-Burge multipartitions as the right-hand side.

We now describe the parameters on both sides of (5.4.1). On the left-hand side we have matrix elements labelled by two $\widehat{\mathfrak{sl}}(N)$ weights $\xi \in P^+_{N,p}$ and $\zeta \in P^+_{N,n+p}$, which defines two integer parameters $n \in \mathbb{Z}_{>0}$ and $p \in \mathbb{Z}_{\geq 0}$, and a product of a sequence of B-matrices defined by a weight $\eta \in P^+_{N,n}$. The weights ξ and ζ define the irreducible $\widehat{\mathfrak{sl}}(N)_{n+p}$-module we are branching from and the $\widehat{\mathfrak{sl}}(N)_p$-module we are branching to. Whereas η tells one the sequence of level 1 modules we take successive tensor products of.

[2] This is the cls^* operator which we will define later in (5.4.13).

On the right-hand side the coloured Burge multipartitions are defined by the Burge inequalities and prescribed colour data. We define the Burge inequalities by two $\widehat{\mathfrak{sl}}(N)$ weights, $\xi' \in P^+_{N,p}$ and $\zeta' \in P^+_{N,n+p}$ and the colour data by the vector of differences $\delta\mathbf{k} = (\delta k_1, \dots, \delta k_{n-1})$.

Based on level-rank considerations the identification of weights

$$\xi' = \xi, \quad \zeta' = \zeta, \tag{5.4.3}$$

is obvious.

We motivate the identification of colours by analogy to the work in [60] (see also [142] for more details), which identifies cylindric multipartitions with states in irreducible $\widehat{\mathfrak{sl}}(N)$-modules. Under this identification we colour a cylindric multipartition $\lambda' = (\lambda'^{(0)}, \dots, \lambda'^{(N-1)})$ associated to a descendant state v' in an irreducible $\widehat{\mathfrak{sl}}(N)$-module L_Λ using the natural colouring described in 1.1. Then each i-coloured box represents the application of the lowering operator f_i for $i = 0, \dots, N-1$ on the highest weight state. The descendant state associated with λ' then has weight

$$\Lambda' = \Lambda - \sum_{i=0}^{n-1} k_i(\lambda')\alpha_i, \tag{5.4.4}$$

where Λ is the highest weight. Note that as $\sum_{i=0}^{n-1} \alpha_i = \delta$, the Dynkin labels are then completely determined by the differences $\delta\mathbf{k}$.

We take the colour data on right-hand side of (5.4.1) to represent the simple roots in this way. Thus, one can consider the vector of differences $\delta\mathbf{k}$ to represent a sum of simple roots. This represents a level 0 weight of class 0. To naturally form such a weight from the parameters on the left-hand side we therefore must consider the difference of the two dual weights $(\zeta - \xi)^\dagger \in P^+_{n,N}$ and $\eta^\dagger \in P^+_{n,N}$. There is a problem with simply considering the difference

$$(\zeta - \xi)^\dagger - \eta^\dagger = \phi, \tag{5.4.5}$$

as we are taking these to represent dual Dynkin rings, which have rotation invariance, it is possible that ϕ may *not* be of class 0. Therefore, we must consider some cyclic permutation of one of these dual weights so that both dual weights are of the same class.

5.4.2 The Case Where N and n are Coprime

As only the difference in class between these dual weights is important, we fix our convention by fixing the orientation of the ring corresponding to $(\zeta - \xi)$, and rotate the ring corresponding to η. Thus we are trying to find j such that

$$cls\left((\zeta - \xi)^\dagger\right) = cls\left(\tau^j(\eta^\dagger)\right). \tag{5.4.6}$$

In the case of coprime N and n we can use proposition 5.3.2.2 to fix j by choosing the unique solution $k = 0, \ldots, n-1$ such that $cls(\tau^j(\eta^\dagger)) = cls((\zeta - \xi)^\dagger)$. We begin by discussing this case. Explicitly, we then consider the level 0 weight

$$\psi = (\zeta - \xi)^\dagger - \tau^j(\eta^\dagger), \tag{5.4.7}$$

and expand it using the basis of simple roots so

$$\psi = \sum_{i=0}^{n-1} k_i \alpha_i. \tag{5.4.8}$$

This defines the colour data of the Burge multipartition to be $\mathbf{k} = (k_0, k_1, \ldots, k_{n-1})$ and using this we can calculate the $\delta\mathbf{k}$'s for the right-hand side of our correspondence. Note that as described in section 1.3, we can perform this process by multiplying the column vector of $((\zeta-\xi)^\dagger - \tau^j(\eta^\dagger))$ by the inverse Cartan matrix A^{-1}. When N and n are coprime we cannot fix j using this argument, as we describe below.

5.4.3 The Non-Coprime Case

As explained in proposition 5.3.2.2, in the case of $\gcd(N,n) > 1$ we have more than one such choice j that satisfies (5.4.6), so that there exists multiple solutions $j, j' = 0, \ldots, n-1$ such that

$$cls\left(\tau^j(\eta^\dagger)\right) = cls\big(\tau^{j'}(\eta^\dagger)\big), \quad j \neq j'. \tag{5.4.9}$$

If we let $\tau^j(\eta^\dagger) = [d_0, \ldots, d_{n-1}]$ this means that for some $k \in \mathbb{Z}$

$$\sum_{i=0}^{n-1} i d_i = kn + \sum_{i=0}^{n-1} i d_{j'-j+i}, \tag{5.4.10}$$

where we have defined $d_{n+i} = d_i$. This ambiguity is solved by fixing the ambiguity of the equation

$$cls(\zeta - \xi) = cls(\eta). \tag{5.4.11}$$

If we ignore the modular behaviour in this equation for classes we have

$$\sum_{i=0}^{N-1} i(\zeta_i - \xi_i) = kN + \sum_{i=0}^{N-1} i\eta_i, \tag{5.4.12}$$

where k is the last natural integer parameter on the left-hand side. To formalise this we introduce a generalized notion of class, which ignores the modular behaviour. We define the *class star* of a weight $\eta = [\eta_0, \eta_1, \ldots, \eta_{N-1}]$ to be

$$cls^*(\eta) = \sum_{i=0}^{N-1} i\eta_i, \tag{5.4.13}$$

so that the class star is the usual class of a weight defined with no modular arithmetic. Then the correct choice for the weight is $\tau^{-j}(\eta^\dagger)$, where

$$j = \frac{1}{N} cls^*(\zeta - \xi - \eta). \tag{5.4.14}$$

In fact this relationship does not apply only to the weights with non-coprime level and rank, it describes the coprime case as well, as we will show in examples below. It is not immediately clear that $(\zeta - \xi)^\dagger$ and $\tau^{-j}(\eta^\dagger)$ are in the same class, which leads us to prove the following proposition.

Proposition 5.4.3.1. *Let $\zeta \in P^+_{N,n+p}$, $\xi \in P^+_{N,p}$, and $\eta \in P^+_{N,n}$ be three dominant integral $\widehat{\mathfrak{sl}}(n)$ weights. If $cls(\zeta - \xi) = cls(\eta)$ then for*

$$j = \frac{1}{N} cls^*(\zeta - \xi - \eta), \tag{5.4.15}$$

the dual weights $(\zeta - \xi)^\dagger, \tau^{-j}(\eta^\dagger) \in P^+_{n,N}$ satisfy

$$cls\left((\zeta - \xi)^\dagger\right) = cls\left(\tau^{-j}(\eta^\dagger)\right). \tag{5.4.16}$$

Proof. For this we can treat the weight $(\zeta - \xi)$ as one weight $\mu = \zeta - \xi$ since only the difference of ζ and ξ is relevant. We therefore have two weights $\mu, \eta \in P^+_{N,n}$ such that

$$cls(\mu) = cls(\eta). \tag{5.4.17}$$

We wish to show that for

$$j = \frac{1}{N} cls^*(\mu - \eta), \tag{5.4.18}$$

we have that

$$cls(\mu^\dagger) = cls(\tau^{-j}(\eta^\dagger)) \iff cls(\mu^\dagger) - cls(\tau^{-j}(\eta^\dagger)) \equiv 0 \bmod n. \tag{5.4.19}$$

We note that (5.4.17) is equivalent to

$$\sum_{i=0}^{N-1} i(\mu_i - \eta_i) = jN. \tag{5.4.20}$$

We can calculate the dual weights $\mu^\dagger$ and $\eta^\dagger$ explicitly and give their Dynkin labels by

$$\mu^\dagger = [\underbrace{1,0,\ldots,0}_{\mu_0},\underbrace{1,0,\ldots,0}_{\mu_1},\ldots,\underbrace{1,0,\ldots,0}_{\mu_{N-2}},\underbrace{1,0,\ldots,0}_{\mu_{N-1}}], \tag{5.4.21}$$

$$\eta^\dagger = [\underbrace{1,0,\ldots,0}_{\eta_0},\underbrace{1,0,\ldots,0}_{\eta_1},\ldots,\underbrace{1,0,\ldots,0}_{\eta_{N-2}},\underbrace{1,0,\ldots,0}_{\eta_{N-1}}]. \tag{5.4.22}$$

Alternatively, we can characterize these by sums of $\widehat{\mathfrak{sl}}(N)$ fundamental weights by

$$\mu^\dagger = \sum_{i=0}^{N-1} \Lambda_{(\sum_{j=0}^{i} \mu_j)} \tag{5.4.23}$$

$$\eta^\dagger = \sum_{i=0}^{N-1} \Lambda_{(\sum_{j=0}^{i} \eta_j)}. \tag{5.4.24}$$

We can calculate the classes of the dual weights by

$$cls(\mu^\dagger) = \sum_{i=1}^{n-1} i\mu_i^\dagger \bmod n \tag{5.4.25}$$

$$= \mu_0 \cdot 1 + (\mu_0 + \mu_1) \cdot 1 + \cdots + (\mu_0 + \cdots + \mu_{N-2}) \cdot 1 \bmod n \tag{5.4.26}$$

$$\equiv \sum_{i=1}^{N} (N-i)\mu_{i-1} \bmod n, \tag{5.4.27}$$

and similarly

$$cls(\eta^\dagger) = \sum_{i=1}^{N} (N-i)\eta_{i-1} \bmod n. \tag{5.4.28}$$

We now apply lemma 5.3.2.1 to calculate the class of $\tau^{-j}(\eta^\dagger)$ as

$$cls(\tau^{-j}(\eta^\dagger)) = \sum_{i=1}^{N} (N-i)\eta_{i-1} - jN \bmod n. \tag{5.4.29}$$

We can now calculate

$$cls(\mu) - cls(\tau^{-j}(\eta^\dagger)) = \sum_{i=1}^{N} (N-i)(\mu_i - \eta_i) + jN \bmod n \tag{5.4.30}$$

$$= \sum_{i=1}^{N} (N-i)(\mu_i - \eta_i) + \sum_{i=0}^{N-1} i(\mu_i - \eta_i) \bmod n \tag{5.4.31}$$

$$= N \sum_{i=0}^{N-1} (\mu_i - \eta_i) \bmod n \tag{5.4.32}$$

$$= N(n-n) \bmod \text{n}, \tag{5.4.33}$$

which vanishes. Thus $\mu^\dagger$ and $\tau^{-j}(\eta^\dagger)$ are in the same class. $\square$

5.4.4 Checks of the Coset Character Conjecture

In this subsection, we check the Coset Character Conjecture in the two cases where $n = 1$ with $p \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}$ with $p = 0$. In both cases, we verify the conjecture by reducing it to the known results in [47] and [43].

5.4.4.1 The case: $n = 1$ with $p \in \mathbb{Z}_{>0}$

The simplest such check is to let p be arbitrary and non-zero and let $n = 1$, so that we are considering AGT-$\mathcal{W}$ for gauge theories on $\mathbb{C}^2$ as in chapter 2. In this case, we show below that we obtain the $\mathcal{W}_N$-minimal model characters $\chi^{N,N+p,N+p+1}_{\xi,\zeta}$.

Under the assumption $n = 1$, we see that $(\zeta - \xi)^\dagger - \tau^j(\eta^\dagger)$ is a one dimensional vector and $\delta\mathbf{k}$ is therefore trivial. We are therefore considering uncoloured Young diagrams.

For $n = 1$, we consider the hatted product $\hat{B}^{(p+1;p)}_{\Lambda_i} = (q;q)^{-1}_\infty B^{(N,p)}_{\Lambda_i}$ for $i = 0, \ldots, N-1$. We recall the definition of B-matrices (5.2.20) to see that

$$\left(\hat{B}^{(p+1;p)}_{\Lambda_i}(q)\right)_{\xi\zeta} = \frac{1}{(q;q)_\infty}\left(B^{(N,p)}_{\Lambda_i}(q)\right)_{\xi\zeta} \tag{5.4.34}$$

$$= \begin{cases} \frac{q^{\frac{1}{2}\left(|\zeta-\xi|^2-|\Lambda_i|^2\right)}}{(q;q)_\infty}\chi^{N,N+p,N+p+1}_{\xi,\zeta}(q) & \text{if } cls(\zeta-\xi-\Lambda_i) = 0, \\ 0 & \text{else,} \end{cases} \tag{5.4.35}$$

where $\xi \in P^+_{N,p}$ and $\zeta \in P^+_{N,p+1}$. The right-hand side is the q-generating function of uncoloured (ξ,ζ)-Burge multipartitions $X_{\xi,\zeta}(q)$. In [47], it was shown that

$$q^{-\Delta}(q;q)_\infty X_{\xi,\zeta}(q) = \chi^{N,N+p,N+p'}_{\xi,\zeta}, \tag{5.4.36}$$

for the $\widehat{\mathfrak{sl}}(N)$ weights $\xi \in P^+_{N,p}$ and $\zeta \in P^+_{N,p'}$ and Δ is some constant (cf. [47, **3.4**]). By choosing $p' = p+1$ in (5.4.36) we can see that hatted B-matrix elements are equal to the Burge q-generating functions up to an overall factor of q, which supports our conjecture.

5.4.4.2 The case: $n \in \mathbb{Z}_{>0}$ with $p = 0$

The next simplest case is $p = 0$ with arbitrary $n \in \mathbb{Z}_{>0}$, so that we are considering a full (non-truncated) product of B-matrices on the left-hand side. In this case, the coset algebra is isomorphic to $\mathcal{A}(N,n;n)$ and we obtain $\widehat{\mathfrak{sl}}(n)_N$ characters as described in chapter 4. We will now go a step further and show that the coloured $(0,\zeta)$-Burge multipartitions as explained in this chapter are in bijection with cylindric partitions, which provide a model for $\widehat{\mathfrak{sl}}(n)_N$ characters.

Consider a coloured $(0,\zeta)$-Burge multipartition $\lambda = (\lambda^{(0)},\ldots,\lambda^{(N-1)})$ such that the level of ζ is the number of colours n. In this case we see that (5.3.1) simplifies to

$$\sigma_j = \sum_{i=0}^{j-1} \zeta_i, \tag{5.4.37}$$

and that $\sigma_{j+1} - \sigma_j = \zeta_j$ for $0 \le j < N$, which we use this to define the weight

$$\Lambda = \sum_{j=0}^{N-1} \Lambda_{\sigma_j}. \tag{5.4.38}$$

Recall from section 1.4.1, an element of $\mathcal{C}_\Lambda$ consists of an N-tuple of Young diagrams $\bar\lambda = (\bar\lambda^{(0)},\ldots,\bar\lambda^{(N-1)})$ and a vector $\bar\sigma = (\bar\sigma_0,\ldots,\bar\sigma_{N-1})$ ordered by $\bar\sigma_0 \le \cdots \le \bar\sigma_{N-1}$ with $\bar\sigma_{N-1} = n$, where all subscripts are defined modulo N, which defines the Burge inequalities on $\bar\lambda$. We now construct a bijection between the n-coloured $(0,\zeta)$-Burge multipartitions and $\mathcal{C}_\Lambda$ via $(\lambda,\sigma) \mapsto (\bar\lambda,\bar\sigma)$.

We begin by defining the vector $\bar\sigma$, which is very similar to the vector of charges $\sigma = (\sigma_1,\ldots,\sigma_N)$ defined by (5.4.37). By definition, a cylindric partition must have $\bar\sigma_{N-1} = n$, whereas we have $\sigma_j < n$ for $0 < j < N$. Thus, to obtain a vector $\bar\sigma$ defining a cylindric partition $\bar\lambda$ using the vector of charges σ, we fix $j' = \max\{j = 1,\ldots,N | \sigma_j < n\}$ and then define

$$\bar\sigma_j = \begin{cases} 0, & 0 \le j \le j', \\ \sigma_{j-j'+1}, & j' < j \le N. \end{cases} \tag{5.4.39}$$

Here we have taken advantage of the invariance of the Burge and cylindric inequalities under cyclic permutation of their labels. We then use this idea to define $\bar\lambda$ in the same fashion, so that

$$\bar\lambda^{(j)} = \begin{cases} \lambda^{(N-j'+j+1)}, & 0 \le j < j', \\ \lambda^{(j-j'+1)}, & j' \le j \le N-1. \end{cases} \tag{5.4.40}$$

One can easily check that the Burge inequalities on λ imply the cylindric inequalities on $\bar\lambda$. This process uniquely identifies every cylindric partition with a coloured $(0,\zeta)$-Burge multipartition. This shows that the instanton generating function when $p = 0$ and $n \in \mathbb{Z}_{>0}$ is equal to the characters of $\widehat{\mathfrak{sl}}(n)_N$.

5.5 New $\widehat{\mathfrak{sl}}(n)$ String Function Identities

In this section, using the $p = 0$ case of conjecture 5.4.0.1, we obtain the following new conjectural forms for $\widehat{\mathfrak{sl}}(n)_N$ string functions involving minimal model characters using the matrix elements of $\hat{B}^{(0;n)}_\eta(q)$ for $\eta \in P^+_{N,n}$.

Conjecture 5.5.0.1. *For $\zeta \in P^+_{N,n}$ and $\eta \in P^+_{N,n}$, we have the following equality, up to a factor of a power of q, of matrix elements $\hat{B}^{(0;n)}_\eta$ and $\widehat{\mathfrak{sl}}(n)_N$ string functions*

$$\left(\hat{B}^{(0;n)}_\eta\right)_{0\zeta} = \begin{cases} \frac{1}{(q;q)_\infty}\sigma^{\zeta^\dagger}_{\tau^{-j}(\eta^\dagger)}, & cls(\zeta) = cls(\eta), \\ 0, & else \end{cases} \tag{5.5.1}$$

for

$$j = \frac{1}{N} cls^*(\zeta - \eta). \tag{5.5.2}$$

Example 5.5.0.2. When $N = 1$, we can only have an empty weight label η and ζ. In this case, the left-hand side is simply $(q;q)^{-1}_\infty$ multiplied by an empty product. On the right-hand side the string function is trivial, which leaves only the factor of $(q;q)^{-1}_\infty$. In this trivial case, the conjecture 5.5.0.1 is clearly true.

We now explain how to obtain the new identities in conjecture 5.5.0.1 from conjecture 5.4.0.1. Since we are taking $p = 0$, the matrix $\hat{B}^{(0;n)}_\eta(q)$ has only one row and has matrix elements labelled by $\xi = \mathbf{0}$ and $\zeta \in P^+_{N,n}$. In terms of q-characters, this situation corresponds to the following product formula of $\widehat{\mathfrak{sl}}(n)$ characters for $\eta = \sum_{i=1}^N \Lambda_{\rho(i)} \in P^+_{N,n}$:

$$\prod_{i=1}^n \chi^{\widehat{\mathfrak{sl}}(N)}_{\Lambda_{\rho(i)}}(q) = \sum_{\substack{\zeta\in P^+_{N,n} \\ cls(\zeta)=cls(\eta)}} b^{\otimes_i \Lambda_{\rho(i)}}_{\tau^{-j}(\zeta^\dagger)}(q)\chi^{\widehat{\mathfrak{sl}}(n)}_{\tau^{-j}(\zeta^\dagger)}(q), \tag{5.5.3}$$

where $j = \frac{1}{N}cls^*(\zeta - \eta) \in \mathbb{Z}$ is as in conjecture 5.4.0.1, $0 \leq \rho(i) \leq N - 1$ is a sequence of integers defining the order of tensor products of level 1 irreducible representations, and $b^{\otimes_i \Lambda_{\rho(i)}}_\zeta(q)$ are branching functions from $L_{\rho(1)} \otimes \cdots \otimes L_{\rho(N)}$ to L_ζ. We notate the branching of this tensor product in terms of the dual weight $\zeta^\dagger$ to match the form of the coset-Burge conjecture (5.4.1).

From the right-hand side of the conjecture 5.4.0.1, we see that the matrix element labelled $(\mathbf{0}, \zeta)$ of $\left(\hat{B}^{(0;n)}_\eta(q)\right)$ is equal to the branching functions $b^{\otimes_i \Lambda_{\rho(i)}}_{\tau^{-j}(\zeta^\dagger)}(q)$ and the generating functions $X^{\delta\mathbf{k}}_{\mathbf{0},\zeta;n}(q)$ of n-coloured cylindric partitions. To see that we should be comparing this element to generating functions of n-coloured Burge multipartitions, we note that the level of ζ and η is n so that their dual weights $\zeta^\dagger$ and $\eta^\dagger$ are both of rank n.

As explained in section 1.4, the $(\mathbf{0}, \zeta)$-Burge multipartitions are cylindric multipartitions and provide a model for states in $\widehat{\mathfrak{sl}}(n)_N$ irreducible modules. We can now go one step further and use the identification for the generating functions of cylindric partitions and $\widehat{\mathfrak{sl}}(n)_N$ characters (1.4.9), to identify the matrix elements of $\hat{B}^{(0;n)}_\eta$ with the string functions of $\widehat{\mathfrak{sl}}(n)$ characters. When performing this we will need to account for an additional factor of $(q;q)_\infty$, which comes from the relationship between the generating function of FLOTW multipartitions and cylindric partitions (equation (1.4.8), that we reproduce here)

$$X^{\mathbf{k}}_{\Lambda}(\mathfrak{q};\mathbf{z}) = \frac{1}{(\mathfrak{q};\mathfrak{q})_\infty}(X^{\mathbf{k}}_{\Lambda})^*(\mathfrak{q};\mathbf{z}). \tag{5.5.4}$$

Note that the matching between a specific string function $\sigma^{\zeta^\dagger}_{\tau^{-j}(\eta^\dagger)}$ and cylindric generating function X for $\delta\mathbf{k} = A^{-1}\left(\zeta^\dagger - \eta^\dagger\right)$ is from the identification of coloured boxes as corresponding to the action of lowering operators on f_i. We recall the discussion of section 5.4.1 and equation (5.4.4) which we reproduce here in form of

$$\tau^{-j}(\eta^\dagger) = \zeta^\dagger - \sum_{i=0}^{n-1} k_i(\lambda)\alpha_i, \tag{5.5.5}$$

where λ is a $(\mathbf{0},\zeta)$-Burge multipartition (equivalently, cylindric plane partition). We are then free to subtract $\sum_{i=0}^{N-1} k_0(\lambda)\alpha = k_0(\lambda)\delta$ from both sides to obtain

$$\tau^{-j}(\eta^\dagger) = \zeta^\dagger + k_0(\lambda)\delta - \sum_{i=1}^{n-1} \delta k_i(\lambda)\alpha_i. \tag{5.5.6}$$

All cylindric partitions with prescribed differences $\delta\mathbf{k}$ are obtained from the n-core defined by $\delta\mathbf{k}$ by adding lots of n blocks coloured 0 through $n-1$ at a time and keeping only multipartitions that again satisfy the cylindric inequalities. As we can see from (5.5.6) this process is equivalent to counting weights with Dynkin labels $\tau^{-j}(\eta^\dagger)$ at each grade, and by (1.4.8) this gives us precisely $\frac{1}{(q;q)_\infty}\sigma^{\zeta^\dagger}_{\tau^{-j}(\eta^\dagger)}$. We thus have the identity (5.4.1) obtained from conjecture 5.5.0.1.

Below, we will compute both sides of the identities in conjecture 5.5.0.1 in some examples for $n=2$ and $N=3$ and show that their q-expansion coincide up to[3] $O(q^8)$. This gives a strong evidence of conjecture 5.5.0.1. In these, we are comparing series expansions of string functions $\frac{1}{(q;q)_\infty}\sigma^{\zeta^\dagger}_{\tau^{-j}(\eta^\dagger)}$ which can be found in [48] with the series expansions of $\hat{B}^{(0;n)}_{\eta}(q)$ matrix elements obtained using (1.5.144).

5.5.1 Examples for $\widehat{\mathfrak{sl}}(2)_3$

As $n=2$, there are only two distinct classes of weights. We will only show computation here for one $\eta \in P^+_{2,3}$ of each class, namely $\eta = 3\Lambda_0$ and $\eta = 2\Lambda_0 + \Lambda_1$. This corresponds to the product of 3 B-matrices

$$\hat{B}^{(0;3)}_{\eta}(q) = \frac{1}{(q;q)^3_\infty}\prod_{l=0}^{2} B^{(2,l)}_{\Lambda_{\sigma_l}}(q), \tag{5.5.7}$$

[3]Note that we have checked that these examples agree up to $O(q^{10})$, although for the sake of formatting we have chosen to show only up to $O(q^8)$.

where $\eta = \sum_{l=0}^{2} \Lambda_{\sigma_l}$. When matching with string functions we note that we can eliminate the extra $(q;q)_\infty$ factor by considering the matrix elements of $\hat{B}^{(1;3)}_{2\Lambda_0}$ instead of $\hat{B}^{(0;3)}_{\Lambda}$, where $\Lambda = 3\Lambda_0, 2\Lambda_0 + \Lambda_1$, as these matrices have the same matrix elements up to this factor.

By taking the first B-matrix to correspond to Λ_0 or Λ_1, which translates to considering either the first or second row of the matrix $\hat{B}^{(1;3)}_{2\Lambda_0} = \frac{1}{(q;q)^2_\infty} B^{(2,0)}_{\Lambda_{\sigma_0}}(q) B^{(3,1)}_{\Lambda_{\sigma_0}}(q)$, we only need consider the matrix elements of one $\hat{B}$-matrix, namely

$$\hat{B}^{(1;3)}_{2\Lambda_0}(q) = \frac{1}{(q;q)^2_\infty} \begin{pmatrix} q^2\chi^{3,4}_{1,3}\chi^{4,5}_{3,1} + \chi^{3,4}_{1,1}\chi^{4,5}_{1,1} & 0 & q\chi^{3,4}_{1,1}\chi^{4,5}_{1,3} + q\chi^{3,4}_{1,3}\chi^{4,5}_{3,3} & 0 \\ 0 & \chi^{3,4}_{2,2}\chi^{4,5}_{2,2} & 0 & q\chi^{3,4}_{2,2}\chi^{4,5}_{2,4} \end{pmatrix}. \tag{5.5.8}$$

Note that in this notation, a subscript integer label r on a minimal model character $\chi^{*,*}_{r,*}$ or $\chi^{*,*}_{*,r}$ corresponds to a weight $[k-r+1, r-1]$, where k is the level of the weight[4]. This is standard notation from 2D CFT, and using it ensures that the subscripts of minimal model characters coincide with the usual labels for matrix elements. Thus, we have that $\chi^{n,k_1+n,k_2+n}_{r_1,r_2} = \chi_{[k_1-r_1+1,r_1-1],[k_2-r_2+1,r_2-1]}$. For example, the top left entry (usually labelled as (1,1) in a matrix) is labelled by $\chi_{\xi,\zeta}$ in our notation, where $\xi = [2,0]$ and $\zeta = [4,0]$.

To compute the string function for $\widehat{\mathfrak{sl}}(3)_2$, we note that for $\mu, \Lambda \in P^+(\widehat{\mathfrak{sl}}(3)_2)$ with $cls(\Lambda) = cls(\mu)$ we have that

$$\sigma^{\tau^d(\Lambda)}_{\tau^d(\mu)}(q) = \sigma^{\Lambda}_{\mu}(q), \quad d \in \mathbb{Z}. \tag{5.5.9}$$

We can then check the tables in [48, §31], which we will refer to by their root system, class, and level. As discussed in chapter 4, we can also compute these string functions using the expressions from [57].

5.5.1.1 $\eta = [3,0]$

In this case we consider the first row of $\hat{B}^{(1;3)}_{2\Lambda_0}$. The $(1,1)$ entry corresponds to $\zeta - \Lambda_0 = [3,0] = \eta$. Expanding the q-series (note the appearance of the $(q;q)^{-2}_\infty$ factor instead of a $(q;q)^{-3}_\infty$ in line with our discussion above) we obtain

$$\frac{1}{(q;q)^2_\infty}\left(q^2\chi^{3,4}_{1,3}\chi^{4,5}_{3,1} + \chi^{3,4}_{1,1}\chi^{4,5}_{1,1}\right) = 1+2q+8q^2+20q^3+52q^4+116q^5+256q^6+522q^7+1045q^8+\ldots \tag{5.5.10}$$

Taking $[3,0]^\dagger = [2,0,0]$ we can see that this matches the series expansion of $\sigma^{[200]}_{[200]}$ in the table for $A2$, level 3, class 0 as expected.

[4]This connects the r and s parameters for Vir-minimal models from section 1.5.5 with that of the more general description as cosets and $\mathcal{W}$-algebras in sections 1.5.12 and 1.5.13.

The $(3,1)$ entry corresponds to $\zeta - \Lambda_0 = [1,2]$. Expanding the q-series gives

$$\frac{1}{(q;q)_\infty^2}\left(q\chi_{1,1}^{3,4}\chi_{1,3}^{4,5} + q\chi_{1,3}^{3,4}\chi_{3,3}^{4,5}\right) = 2q + 7q^2 + 22q^3 + 56q^4 + 136q^5 + 300q^6 + 636q^7 + 1280q^8 + \dots \tag{5.5.11}$$

In this case we have that $\tau([1,2]^\dagger) = [0,1,1]$, and the series expansion of the hatted B-matrix elements matches $\tau^{-1}(\sigma_{[200]}^{[011]})$ in the table for $A2$, level 2, class 0 as expected.

5.5.1.2 $\eta = [1,2]$

In this case we are considering the second row of $\hat{B}_{2\Lambda_0}^{(0;3)}$. The $(2,2)$ entry corresponds to $\zeta - \Lambda_1 = [3,0]$. We expand the q-series to obtain

$$\frac{1}{(q;q)_\infty^2}\left(\chi_{2,2}^{3,4}\chi_{2,2}^{4,5}\right) = 1 + 4q + 13q^2 + 36q^3 + 89q^4 + 204q^5 + 441q^6 + 908q^7 + 1798q^8 + \dots \tag{5.5.12}$$

As expected this matches the string function $\sigma_{[011]}^{[200]}$ in the table for $A2$, level 2, class 0. Here we note that although $[3,0]^\dagger = [0,0,2]$ and $[1,2]^\dagger = [1,1,0]$, we are free to use $\sigma_{[011]}^{[200]}$, as by (5.5.9) the string function $\sigma_{[110]}^{[002]}$ is equivalent to $\sigma_{[011]}^{[200]}$.

Finally the $(2,4)$ entry corresponds to $\zeta - \Lambda = [1,2]$, and we obtain the q-series

$$\frac{1}{(q;q)_\infty^2} q\chi_{2,2}^{3,4}\chi_{2,4}^{4,5} = 1 + 4q + 12q^2 + 32q^3 + 77q^4 + 172q^5 + 365q^6 + 740q^7 + 1445q^8 + \dots \tag{5.5.13}$$

This series matches $\sigma_{[011]}^{[011]}$ also found in the table for $A2$, level 2, class 0. Note that $\eta^\dagger = \zeta^\dagger = [1,1,0]$, and $\sigma_{[110]}^{[110]} = \sigma_{[011]}^{[011]}$.

5.5.2 New Combinatorial $\widehat{\mathfrak{sl}}(n)$ String Function Identities

In the previous section, we used a truncated product $\hat{B}_\eta^{(3;1)}$ to calculate the matrix elements of the full product of $\hat{B}_{\eta+\Lambda_i}^{(3;0)}$. This approach generalizes for arbitrary n. If we consider the right-hand side of coset-Burge conjecture for $\hat{B}_\eta^{(n;1)}$, we see that we can calculate the $\widehat{\mathfrak{sl}}(n)$ string functions using a (Λ_i, ζ)-Burge multipartition generating function $X_{\Lambda_i,\zeta}^{\delta\mathbf{k}}$ instead of a $(\mathbf{0}, \zeta)$-Burge multipartition with an additional Heisenberg factor. We formalize this idea with the following corollary to 5.4.0.1.

Conjecture 5.5.2.1. *For $\zeta \in P_{N,n}^+$ and $\eta \in P_{N,n-1}^+$, we have the following equality, up to a factor of q, between the generating functions of (Λ_i, ζ)-Burge multipartitions, matrix elements*

$\hat{B}^{(n;1)}_{\eta+\Lambda_i}$, *and* $\widehat{\mathfrak{sl}}(n)$ *string functions*

$$X^{\delta\boldsymbol{k}}_{\Lambda_i,\zeta}(q) = \left(\hat{B}^{(n;1)}_{\eta}\right)_{\Lambda_i\zeta} = \begin{cases} \sigma^{(\zeta-\Lambda_i)^\dagger}_{\tau^{-j}(\eta^\dagger)}, & cls(\zeta-\Lambda_i) = cls(\eta), \\ 0, & else \end{cases} \tag{5.5.14}$$

for

$$j = \frac{1}{N} cls^*(\zeta - \Lambda_i - \eta). \tag{5.5.15}$$

The above identity 5.5.14 gives a new combinatorial interpretation of $\widehat{\mathfrak{sl}}(n)_N$ string functions using $(n+1)$ coloured Burge generating functions This is in contrast to the previously known expressions for $\widehat{\mathfrak{sl}}(n)_N$ string functions, which are in terms of n coloured Burge generating functions as in conjecture 5.5.0.1.

Importantly, we have found a combinatorial model for $\widehat{\mathfrak{sl}}(n)$ characters defined only in terms of the Burge inequalities and colourings of Young diagrams, without having to introduce the FLOTW conditions. Viewed another way, we can use the coloured Burge multipartition framework to calculate $\widehat{\mathfrak{sl}}(n)$ characters without having to put in either an additional factor of $(q;q)^{-1}_\infty$ as seen in [47] or enforcing the highest-lift condition (1.4.2.3).

Interestingly, if we equate the two possible Burge generating functions for the matrix elements of $\hat{B}^{(0;n)}_{\eta+\Lambda_i}$ and $\hat{B}^{(1;n)}_{\eta}$ we find

$$\frac{1}{(q;q)_\infty} X_{\mathbf{0}\zeta}(q) = X^*_\Lambda(q) = X^{\delta\mathbf{k}}_{\Lambda_i\zeta}(q), \tag{5.5.16}$$

which suggests there may be a map between FLOTW multipartitions, which are naturally coloured in n colours, and $(n-1)$ coloured (Λ_i,ζ)-Burge multipartitions. As of now, we have not been able to find such a map.

The FLOTW multipartitions are cylindric partitions which only satisfy inequalities along one axis. They are therefore analogous to plane partitions. The (Λ_i,ζ)-Burge multipartitions have inequalities along two axes and are analogous to skew plane partitions. We find it interesting that there may be some equivalence between these two types of objects.

5.6 Some Coset Character Examples

Here we discuss the evidence we have found for the validity of the coset-Burge character conjecture 5.4.0.1. We will also discuss a selection of simple worked examples explicitly showing the details involved in calculating (5.4.1). The examples are of increasing order of complexity, beginning with the simplest case $(N,n) = (2,3)$ before moving onto $(N,n) = (3,4)$ and finally having a non-coprime case for $(N,n) = (3,6)$.

5.6.1 Evidence For the Coset Burge Character Conjecture

To develop the parameter matching for the coset Burge character conjecture (5.4.1), we only used a selection of weights $\eta \in P^+_{N,n}$ for the level-rank pairs $(N, n) = (2, 3), (2, 4), (3, 2), (3, 4), (3, 6), (3, 7), (4, 3), (4, 6)$. Of these pairs we used all possible weights for $(N, n) = (2, 3), (2, 4), (3, 2)$, where series expansions of $\hat{B}$-matrix elements and coloured Burge q-generating functions were found to agree up to $O(q^{10})$. As seen in the examples below, this involved counting up to thousands of Burge multipartitions.

Having then developed the precise form (5.4.1) of the coset Burge character conjecture we tested our proposed correspondence against *all* possible weights $\eta \in P^+_{N,n}$ for $(N, n) = (3, 4), (3, 6), (3, 7), (4, 3), (2, 7), (2, 8)$ in Mathematica and Maple, where the series expansions of $\hat{B}$-matrix elements and coloured Burge q-generating functions were seen to agree in every case. We were able to check the cases[5] of $(N, n) = (2, 3), (3, 2)$ up to $O(q^{12})$, $(N, n) = (2, 4)$ to $O(q^{10})$, $(N, n) = (3, 4), (4, 3)$ to $O(q^6)$, $(N, n) = (3, 6), (3, 7)$ to $O(q^5)$, $(N, n) = (2, 7)$ to $O(q^4)$, and $(N, n) = (2, 8)$ to $O(q^3)$.

These checks provide strong evidence for the validity of (5.4.1). Below we will show a selection of the examples, that are computed using the machinery developed in this chapter, that demonstrate some of these checks we have described explicitly.

5.6.2 $\widehat{\mathfrak{sl}}(2)_3$

We will check a case for the values $\eta = [3, 0]$ and $[2, 1]$ with $p = 1$, where these matrix entries are equal to $\widehat{\mathfrak{sl}}(3)_2$ string functions. This expands on the examples presented in section 5.5.1 In this case we explicitly show that conjectures 5.5.1, and 5.5.14 hold, and their corresponding identities between string functions and Burge generating functions. As previously described, we obtain the string function q-series from [49].

First, let $\eta = [3, 0]$ which gives $\Omega(\eta) = \{4, 5\}$. The Dynkin rings of η and $\eta^\dagger$ are shown in figure 5.1.

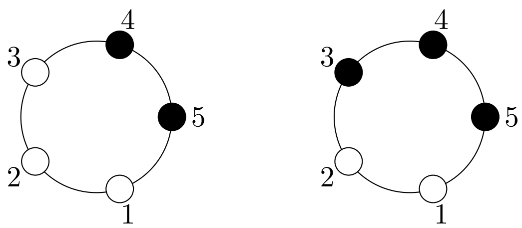


FIGURE 5.1: The Dynkin ring for $\eta = [3, 0]$ its dual $\eta^\dagger$ on the left and right respectively.

[5]Note, we state here the order of the series expansions when factorizing out the additional q factor, undetermined in the conjecture.

There are 3 possible choices for $\tau^{-j}(\eta^\dagger)$, and since $\gcd(2,3)=1$ there is one of each rank 3 class, we have:

$$\tau^{-j}(\eta^\dagger)=\begin{cases}[2,0,0] & \text{class}=0, \quad j=0,\\ [0,0,2] & \text{class}=1, \quad j=1,\\ [0,2,0] & \text{class}=2, \quad j=2.\end{cases} \tag{5.6.1}$$

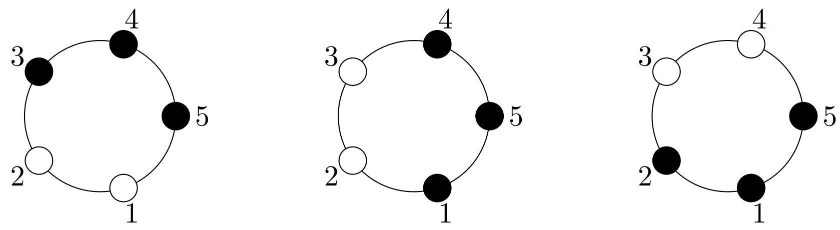


FIGURE 5.2: For $\eta=[3,0]$, the Dynkin rings of each $\tau^{-j}(\eta^\dagger)$ for $j=0,1,2$ respectively.

Next we consider specific matrix elements for the hatted B-matrix, which satisfy the equation

$$\left(\hat{B}^{(3;1)}_{[3,0]}\right)_{\xi,\zeta}=X^{\delta\mathbf{k}}_{(\xi,\zeta;3)}(q). \tag{5.6.2}$$

Consider the entry labelled by $(\xi,\zeta)=([1,0],[4,0])$. Following (5.3.1), we obtain

$$\sigma_0=0 \tag{5.6.3}$$

$$\sigma_1=0+4-1=3\equiv 0 \bmod 3 \tag{5.6.4}$$

$$\sigma_2=\sigma_0, \tag{5.6.5}$$

and our weight vector of Burge charges is $\sigma=[2,0,0]$. We have $cls(\sigma)=0$, so we choose $\tau^0(\eta^\dagger)=[2,0,0]$ and have

$$\sigma-\eta^\dagger=[0,0,0]=\sum_{i=0}^{2}0\cdot\alpha_i, \tag{5.6.6}$$

thus $\mathbf{k}=(0,0,0)$ and $\delta\mathbf{k}=(0,0)$. On the left-hand side of (5.4.1) we have

$$\left(\hat{B}^{(3;1)}_{[3,0]}\right)_{[1,0],[4,0]}=\frac{1}{(q;q)^2_\infty}\Big(\chi^{(3;5,6)}_{[3,0],[4,0]}\left(q^2\chi^{(3;3,4)}_{[1,0],[0,2]}\chi^{(3;4,5)}_{[0,2],[3,0]}+\chi^{(3;3,4)}_{[1,0],[2,0]}\chi^{(3;4,5)}_{[2,0],[3,0]}\right) \tag{5.6.7}$$

$$+q\chi^{(3;5,6)}_{[1,2],[4,0]}\left(q\chi^{(3;3,4)}_{[1,0],[0,2]}\chi^{(3;4,5)}_{[0,2],[1,2]}+q\chi^{(3;3,4)}_{[1,0],[2,0]}\chi^{(3;4,5)}_{[2,0],[1,2]}\right)\Big) \tag{5.6.8}$$

$$=1+3q+15q^2+50q^3+162q^4+457q^5+\dots, \tag{5.6.9}$$

and the right-hand side we have

$$X^{(0,0)}_{([1,0],[4,0];3)}(q)=1+3q+15q^2+50q^3+162q^4+457q^5+\dots, \tag{5.6.10}$$

and our conjectured correspondence is satisfied up to $O\left(q^6\right)$.

Our next example is the entry labelled by $(\xi,\zeta) = ([0,1],[1,3])$. We have

$$\sigma_0 = 0 \tag{5.6.11}$$

$$\sigma_1 = 0 + 1 - 0 = 1 \bmod 3, \tag{5.6.12}$$

and $\sigma = [1,1,0]$. We have $cls(\sigma) = 1$ so we choose $\tau^{-1}(\eta^\dagger) = [0,0,2]$. We have

$$\sigma - \tau^{-1}(\eta^\dagger) = [1,1,-2] = -\alpha_2, \tag{5.6.13}$$

so that $\mathbf{k} = (0,0,-1)$ and $\delta\mathbf{k} = (0,-1)$ and

$$\left(\hat{B}^{(3;1)}_{[3,0]}\right)_{[0,1],[1,3]} = \frac{1}{(q;q)_\infty^2}\Big(q\chi^{(2;3,4)}_{[0,1],[1,1]}\chi^{(2;4,5)}_{[1,1],[0,3]}\chi^{(2;5,6)}_{[0,3],[1,3]} + q\chi^{(2;3,4)}_{[0,1],[1,1]}\chi^{(2;4,5)}_{[1,1],[2,1]}\chi^{(2;5,6)}_{[2,1],[1,3]}\Big) \tag{5.6.14}$$

$$= 2q + 12q^2 + 50q^3 + 172q^4 + 522q^5 + \dots, \tag{5.6.15}$$

and

$$X^{(0,-1)}_{[0,1],[1,3];3}(q) = 2q^{2/3} + 12q^{5/3} + 50q^{8/3} + 172q^{11/3} + 522q^{14/3} + \dots \tag{5.6.16}$$

$$= q^{-1/3}\Big(2q + 12q^2 + 50q^3 + 172q^4 + 522q^5 + \dots\Big), \tag{5.6.17}$$

and the conjecture (5.4.0.1) is satisfied up to a factor of $q^{-1/3}$ (this comes from the number of boxes in the 3-cores divided by the number of colours).

Remark 5.6.2.1. We can also easily see that this fits the generic class star structure. In this case, for $\eta = [3,0]$ we have $\eta^\dagger = [2,0,0]$ and taking $(\xi,\zeta) = ([1,0],[4,0])$ gives

$$cls^*(\zeta - \xi - \eta) = 0, \tag{5.6.18}$$

so that $j = 0$ as above. For $(\xi,\zeta) = ([0,1],[1,3])$ we have

$$cls^*(\zeta - \xi - \eta) = 2, \tag{5.6.19}$$

and $j = 2/2 = 1$ as above.

5.6.3 $\widehat{\mathfrak{sl}}(3)_4$

We begin by noting that $\gcd(3,4) = 1$, so that we can use the coprime structure. We will first consider the $\eta = [4,0,0]$ case. We have $\Omega(\eta) = \{5,6,7\}$, where the relevant Dynkin rings are depicted figure 5.4

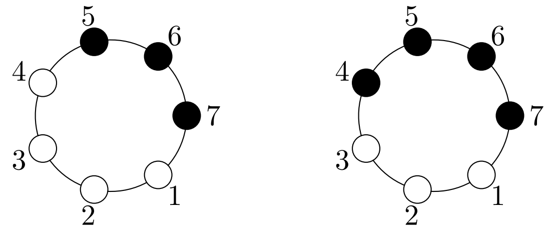


FIGURE 5.3: The Dynkin rings for $\eta = [3,0,0,0]$ and its dual $\eta^\dagger$ on the left and right respectively.

We have the 4 possible choices for $\tau^{-j}(\eta^\dagger)$

$$\tau^{-j}(\eta^\dagger) = \begin{cases} [3,0,0,0] & \text{class} = 0, \quad j = 0, \\ [0,0,0,3] & \text{class} = 1, \quad j = 1, \\ [0,0,3,0] & \text{class} = 2, \quad j = 2, \\ [0,3,0,0] & \text{class} = 3, \quad j = 3. \end{cases} \tag{5.6.20}$$

Similarly to above, the entry labelled by $(\xi, \zeta) = ([5,0,0],[1,0,0])$ has $\sigma = [3,0,0,0]$ with $cls(\sigma) = 0$ so that we choose $\tau^0(\eta^\dagger = [3,0,0,0])$. This gives $\delta\mathbf{k} = (0,0,0)$ and we have

$$\left(\hat{B}^{(4;1)}_{[4,0,0]}\right)_{[1,0,0],[5,0,0]} = 1 + 4q + 24q^2 + 120q^3 + 545q^4 + \ldots, \tag{5.6.21}$$

and

$$X^{(0,0,0)}_{[1,0,0],[5,0,0];4} = 1 + 4q + 24q^2 + 120q^3 + 545q^4 + \ldots, \tag{5.6.22}$$

as expected. We can then easily see that

$$cls^*(\zeta - \xi - \eta) = cls^*([0,0,0]), \tag{5.6.23}$$

giving $j = 0$.

Next, we consider the entry $(\xi, \zeta) = ([0,1,0],[2,2,1])$. We have $\zeta - \xi = [2,1,1]$ so that $\sigma = [1,0,1,1]$ and we choose $\eta^\dagger = [0,0,0,3]$. Then $\sigma - \eta^\dagger = -\alpha_3$ and $\delta\mathbf{k} = (0,0,-1)$[6]. We have

$$\left(\hat{B}^{(4;1)}_{[4,0,0]}\right)_{[0,1,0],[2,2,1]} = 3q + 36q^2 + 264q^3 + 1485q^4 + O\left(q^5\right), \tag{5.6.24}$$

and

$$X^{(0,0,-1)}_{[0,1,0],[2,2,1];4} = q^{-1/4}\left(3q + 36q^2 + 264q^3 + 1485q^4 + \ldots\right), \tag{5.6.25}$$

The class star in this case is

$$cls^*(\zeta - \xi - \eta) = cls^*([-2,1,1]) = 3, \tag{5.6.26}$$

[6]As one moves to higher rank and levels, you can easily calculate the necessary $\delta\mathbf{k}$ values by using the inverse finite Cartan matrix $\left(A^{-1}\right)_{ij} = \min\{i,j\} - \frac{ij}{n}$. This will directly give you the $\delta\mathbf{k}$ values.

and we see that $j = 1$ as expected.

5.6.4 $\widehat{\mathfrak{sl}}(3)_6$

We have $\gcd(6,3) = 3$, so we are in the non-coprime case and we have an ambiguity due to the rotation invariance of the Dynkin ring for the dual weight $(\eta^\dagger)$. This is the only example we provide that we cannot determine $\tau^{-j}(\eta^\dagger)$ purely from class considerations.

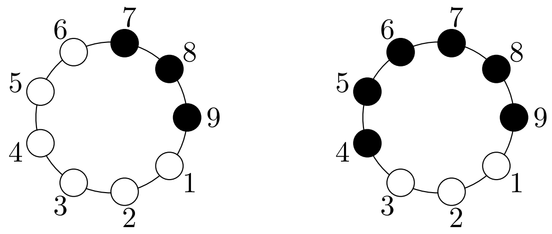


FIGURE 5.4: The Dynkin ring for $\eta = [6, 0, 0, 0]$ and its dual $\eta^\dagger$ on the left and right respectively.

We will only consider the case where $\eta = [6, 0, 0]$. For this case, we fix $\xi = [1, 0, 0]$ and choose three values for ζ to obtain different values of j and hence $\delta\mathbf{k}$ in the conjecture. We have $\Omega(\eta) = \{7, 8, 9\}$ so that $\Omega(\eta^\dagger) = \{1, 2, 3, 4, 5, 6\}$ which gives

$$\tau^{-j}(\eta^\dagger) = \begin{cases} [3,0,0,0,0,0] & \text{class} = 0, \quad j = 0, \\ [0,0,3,0,0,0] & \text{class} = 0, \quad j = 4, \\ [0,0,0,0,3,0] & \text{class} = 0, \quad j = 2, \\ [0,3,0,0,0,0] & \text{class} = 3, \quad j = 5, \\ [0,0,0,3,0,0] & \text{class} = 3, \quad j = 3, \\ [0,0,0,0,0,3] & \text{class} = 3, \quad j = 1. \end{cases} \tag{5.6.27}$$

Consider the matrix element $\left(\hat{B}^{(7;1)}_{[6,0,0]}\right)_{[1,0,0],[7,0,0]}$. We note that $cls^*(\eta - \xi - \zeta) = cls^*([0, 0, 0]) = 0$, so that $j = 0$. Then $(\zeta - \xi)^\dagger - \eta^\dagger = [0, 0, 0, 0, 0, 0]$ so that $\delta\mathbf{k} = (0, 0, 0, 0, 0)$. We now check the matrix element against the generating function of coloured Burge multipartitions. We have

$$\left(\hat{B}^{(7;1)}_{[6,0,0]}\right)_{[1,0,0],[7,0,0]} = 1 + 6q + 48q^2 + 336q^3 + 2142q^4 + \ldots, \tag{5.6.28}$$

and the coloured Burge generating function

$$X^{(0,0,0,0,0)}_{[1,0,0],[7,0,0];6} = 1 + 6q + 48q^2 + 336q^3 + 2142q^4 + \ldots, \tag{5.6.29}$$

which match as expected.

Next, we consider the entry corresponding to $(\xi, \zeta) = ([4,3,0],[1,0,0])$. In this case, we have that $cls^*(\eta - \xi - \zeta) = cls^*([3,3,0]) = 3$ so that $j = 1$. Now, we also have that

$$([4,3,0] - [1,0,0])^\dagger - \tau^{-1}(\eta^\dagger) = [2,0,0,1,0,-3] = -\alpha_4 - 2\alpha_5 \tag{5.6.30}$$

and we see that $\delta\mathbf{k} = (0,0,0,-1,-2)$. The matrix element

$$\left(\hat{B}^{(7;1)}_{[6,0,0]}\right)_{[1,0,0],[4,3,0]} = 15q^2 + 230q^3 + 2190q^4 + \ldots, \tag{5.6.31}$$

and corresponding generating function

$$X^{(0,0,0,-1,-2)}_{[1,0,0],[4,3,0];6} = 15q^{3/2} + 230q^{5/2} + 2190q^{7/2} + \ldots \tag{5.6.32}$$

$$= q^{1/2}\left(15q^2 + 230q^3 + 2190q^4 + \ldots\right), \tag{5.6.33}$$

agree, as expected, up to a factor of $q^{1/2}$.

Finally, we consider the entry $([1,6,0],[1,0,0])$. In this case, we have that $cls^*(\zeta - \xi - \eta) = 6$ so that $j = 2$. Then

$$([1,6,0] - [1,0,0])^\dagger - \tau^{-2}(\eta^\dagger) = [3,0,0,-3,0,] = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 2\alpha_5, \tag{5.6.34}$$

so that $\delta\mathbf{k} = (-1,-2,-3,-4,-2)$. Then the matrix element

$$\left(\hat{B}^{(7;1)}_{[6,0,0]}\right)_{[1,0,0],[1,6,0]} = 21q^4 + 378q^5 + 3767q^6 + \ldots, \tag{5.6.35}$$

and corresponding generating function

$$X^{(-1,-2,-3,-4,-2)}_{[1,0,0],[1,6,0];6} = 21q^2 + 378q^3 + 3767q^4 + \ldots \tag{5.6.36}$$

$$= q^{-2}\left(21q^4 + 378q^5 + 3767q^6 + \ldots\right), \tag{5.6.37}$$

agree, as expected, up to a factor of q^{-2}.

Chapter 6

Conclusion and Outlook

In this thesis, we have provided two distinct approaches to studying coset AGT: First, by providing new evidence for its existence using its connection to $\widehat{\mathfrak{sl}}(n)_N$-WZW models, and second, by calculating branching functions for coset CFTs with symmetry algebra $\mathcal{A}(N,n;p)$ from the instanton generating functions for $\mathcal{N}=2$ $SU(N)$ theories on $\mathbb{C}^2/\mathbb{Z}_n$. In both approaches, we have utilized the combinatorics suggested by coset AGT for gauge theories under a minimal model identification.

To obtain the necessary combinatorics for our study, we have shown that $\mathcal{N}=2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$ under a minimal model identification have ill-defined instanaton partition functions due to the presence of non-physical poles. We then obtained the Burge conditions in these theories, which we used to obtain a well-defined instanton partition function as a sum over coloured Burge multipartitions. In doing so, we introduced the combinatorics of coloured cylindric partitions to the AGT correspondences we considered, which was essential for all results obtained thereafter.

Then, by fine-tuning the Ω-deformation parameters ϵ_1 and ϵ_2 used to calculate the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_n$ and fixing the gauge theory parameters in a minimal model fashion, we were able to conjecture an AGT duality involving $\widehat{\mathfrak{sl}}(n)_N$-WZW models. We have then provided evidence for this correspondence in two ways.

First, we have proved that it is possible to obtain WZW characters corresponding to integrable highest weight $\widehat{\mathfrak{sl}}(n)_N$-modules by applying the combinatorics implied by this correspondence to the instanton generating function. To do this, we compared with expressions for WZW characters using the concept of highest lift cylindric partitions, as developed by the Kyoto school [60, 150, 39]. The evidence obtained in this manner applies to all $\widehat{\mathfrak{sl}}(n)_N$-WZW primary fields corresponding to arbitrary dominant integral $\widehat{\mathfrak{sl}}(n)_N$-weights.

The second set of evidence is from considering the correlation functions of WZW models involving primary fields corresponding to the fundamental and anti-fundamental highest weight irreducible $\widehat{\mathfrak{sl}}(n)_N$-representations. Here we compared series expansions of Z_{inst} to known expressions for conformal blocks in $\widehat{\mathfrak{sl}}(n)_N$-WZW theories involving hypergeometric functions term-by-term, by way of the KZ differential equation [36], and showed they agreed to small order for $(N, n) = (2, 2)$, $(2, 3)$, and $(3, 2)$.

Further work could be done in this direction by extending this program to conformal blocks involving other primary fields. This could then be used to obtain solutions, defined in terms of coloured Burge multipartitions, to the KZ differential equation. Furthermore, any proof of this subcase of AGT would link the partition function for this theory, and by extension coloured Burge multipartitions, to hypergeometric functions.

Previous proofs of other AGT subcases from a CFT perspective make use of special bases of states in the representation theory of CFT symmetry algebras. When considering Liouville CFT and $\mathcal{N} = 2$ $SU(2)$ gauge theories on $\mathbb{C}^2$, bifundamental multiplet contributions can be derived in the CFT using a basis of states corresponding to the Jack polynomials (which themselves are associated to Young diagrams) [19]. In the case of the cosets $\mathcal{A}(1, 2; 1)$ and $\mathcal{A}(2, 2; 2)$ and $\mathcal{N} = 2$ $SU(2)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_2$, a similar basis of states, defined in terms of pairs of checkerboard (2 coloured) Young diagrams involving Ulgov polynomials (first introduced in [151, 152]) has been obtained [27]. We can anticipate that to generalize these results a similar basis of states involving n-coloured N-tuples of Young diagrams exists for $\mathcal{H} \oplus \widehat{\mathfrak{sl}}(n)_N$-modules whose matrix elements reproduce the bifundamental multiplets of $\mathcal{N} = 2$ $SU(N)$ gauge theories on $\mathbb{C}^2/\mathbb{Z}_n$. We expect that in this basis, the singular vectors will be described by N-tuples that do *not* satisfy the Burge conditions.

We then utilized our conjectured coset AGT correspondence involving minimal models to calculate branching functions of the coset $\mathcal{A}(N, n; p)$ for generic values of the parameter p. To do so we first introduced B-matrices, which allowed computation of the branching functions efficiently from a representation theoretic viewpoint by utilizing the crystal graph tools of Kyoto school and Littlemann [66, 61, 62]. We then related Burge multipartitions coloured in n-colours to affine weights of level n (as apposed to the usual case of relating them to weights of rank n) by introducing the new concept of dual weights and their corresponding Dynkin rings.

Having done so, we were able to identify the coloured boxes in Burge multipartitions with affine simple weights (or equivalently, the action of lowering operators) and formed a dictionary between the weights defining the branching functions and the colour data of coloured Burge generating functions. Within this dictionary, we fixed the ambiguity implied by the rotation symmetry of dual Dynkin rings (or equivalently, the existence of multiple candidates for a natural dual weight) by introducing a new property of Dynkin weights, called the class-star. The class-star of a Dynkin weight is related to its usual class and fixes the cyclic invariance inherent to dual weights, allowing us to explicitly write down the coset-Burge conjecture, which was an identification between coset branching functions and coloured Burge generating functions.

By fixing $p = N$ in this conjecture, which corresponds to coset AGT involving *only* the WZW models, we then showed that, due to the form of the B-matrix formulation of branching functions, we obtained new conjectural combinatorial expressions for $\widehat{\mathfrak{sl}}(n)_N$-string functions as a corollary to the coset-Burge conjecture. Previous expressions for these (again using the crystal graph approach of the Kyoto school) involved generating functions of cylindric plane partitions, or in our language $(0, \zeta)$-Burge multipartitions, which are coloured in n colours, with an additional factor of $(q;q)_\infty^{-1}$ (or in the language of AGT, a free boson or Heisenberg factor). Our new conjectural expressions are instead a equality between generating functions of (Λ_i, ζ)-Burge multipartitions coloured with $(n-1)$ colours and $\widehat{\mathfrak{sl}}(n)_N$-string functions.

Since our expressions have no Heisenberg factor, it is natural to conjecture that the (Λ_i, ζ)-Burge multipartitions coloured with $(n-1)$ colours may form a natural labelling of states in highest weight $\widehat{\mathfrak{sl}}(n)_N$-modules. Furthermore, as a corollary to our corollary, we obtain a new conjectural combinatorial expression that states that the generating function of (Λ_i, ζ)-Burge multipartitions coloured with $(n-1)$ colours is equal to a product of a Heisenberg character and the generating function of $(0, \zeta + \Lambda_i)$-Burge multipartitions coloured in n-colours.

To provide evidence for the coset-Burge conjecture and our two corollaries to it, we again took two approaches. The first was to check that it is a genuine generalization of the results in both AGT-$\mathcal{W}$ and coset AGT for $p = N$. By fixing $n = 1$, we showed that the coset-Burge conjecture reduced to AGT-$\mathcal{W}$ results involving the uncoloured Burge generating function (associated to $SU(N)$ theories on $\mathbb{C}^2$) and characters of $\mathcal{W}_N$-algebra minimal models (associated to $\mathcal{A}_{N-1}$-Toda theories). Similarly, by taking $p = N$ we reproduced our own results identifying generating functions of cylindric Burge multipartitions coloured with n colours with WZW characters.

We then provided further evidence by comparing explicit series expansions of coloured Burge generating functions to known series expressions for $\widehat{\mathfrak{sl}}(N)_n$-string functions (derived in [57], but more easily obtained in the tables of [49]) and our own expressions for branching functions using B-matrices to small order. We performed this on a computer, which allowed us to check the coset-Burge conjecture for all pairs $(N, n) = (2,3), (2,4), (3,2), (3,4), (3,6), (3,7), (4,3), (2,7), (2,8)$ of level and rank up to a given order. Furthermore, this data set was also checked against all string functions and found to agree. This evidence is quite strong and gives us strong belief that we have found the correct dictionary for the coset-Burge conjecture.

Currently we are optimistic that a proof of the conjecture is possible, and we are working towards one. The natural obstruction is in a lack of closed form expression for the coloured Burge generating functions, where a combination of both specified colouring *and* Burge style conditions has not been obtained in the literature. Closed form expressions for cylindric Burge multipartitions and coloured tuples of Young diagrams typically involve novel counting methods which often take the form of yokes (in the case of cylindric Burge multiparitions [41]) and abaci (in the case of coloured Young diagrams [45]). Work is ongoing in generalizing these procedures to obtain a counting method suitable for coloured Burge multipartitions. Once obtained, once could then in principle compare this closed form expression against known forms for minimal model characters, such as (1.5.144).

Smaller steps towards a conclusive proof of the coset-Burge conjecture could involve a proof of the $(n-1)$ to n coloured Burge relation purely from a combinatorial viewpoint. Similar expressions have been obtained in the concept of FLOTW partitions [40], which also involved novel counting operations to construct certain multipartitions from a smaller set. This process also introduced the presence of a factor of $(q;q)_\infty^{-1}$ in their expressions. Finding such a procedure may allow one to prove the coset-Burge conjecture without a need for a closed for expression for the coloured Burge generating function.

The author is interested in linking this work to the broader literature in two main ways. In a CFT setting, linking this computational structure to a study of non-integer level $\widehat{\mathfrak{sl}}(N)$-modules is one (see [153, 154, 155] and the introductions [156, 157]). This study of AGT would provide new computational tools for their characters, and by extension their characters modular transformation properties, as well as facilitating conjectural expressions for their conformal blocks. Along this line of reasoning, it would be interesting to compare the results of chapter 4 with that of [158], where $\widehat{\mathfrak{sl}}(2)$-WZW conformal blocks were obtained for $SU(2)$ gauge theories on $\mathbb{C}^2$ (not the ALE space $\mathbb{C}^2/\mathbb{Z}_n$), and a conjecture was made that one could similarly obtain $\widehat{\mathfrak{sl}}(N)$-WZW conformal blocks at level $(-N-\epsilon_1/\epsilon_2)$ from $SU(N)$ theories on $\mathbb{C}^2$. We also note that when $-N-\epsilon_1/\epsilon_2 \in \mathbb{Z}_{>0}$, this conjecture would then obtain the same $\widehat{\mathfrak{sl}}(N)$-WZW conformal blocks we computed in chapter 4. This method then suggests an equality between partition functions of certain $SU(N)$ gauge theories on $\mathbb{C}^2$ and $\mathbb{C}^2/\mathbb{Z}_n$, while also allowing one to obtain conformal blocks which are not able to be computed using our method on the orbifold.

In an algebraic combinatorial setting, the expressions for coloured Burge multipartitions are similar to those obtained in the study of quantum toroidal algebras [30, 31], and their so-called resonance modules. These algebras are known to relate (in a conformal limit) to AGT, and hopefully our expressions make elucidating this link, and the role of these algebras in AGT, clearer. In this second setting, utilizing these larger algebraic structures and their relationship to the results presented in this thesis should illuminate the connection between both sides of coset AGT.

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