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## A NOTE ON STABILIZING AND TIME-OPTIMAL CONTROLLERS FOR DISCRETE-TIME LINEAR SYSTEMS WITH POSITIVE CONTROLS<sup>1</sup>

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Abstract: Design of stabilizing and time-optimal controllers for discrete-time linear systems with positive controls is revisited. Necessary and sufficient conditions for stabilizability of this class of systems are given. A class of stabilizing controllers is designed. We show that a stabilizing time-optimal controller exists if the system is dead-beat controllable. We show by an example that there may also exist time-optimal controllers which are not stabilizing. *Copyright © 1998 IFAC*

Keywords: linear systems, positive controls, stabilizing controllers, time-optimal controllers

### 1. INTRODUCTION

Linear systems with positive controls arise in modelling of biological processes (Zaslavsky, 1990), in economics (Heemels, *et al.*, 1997) and may also be obtained from black-box identification of nonlinear systems, where they are referred to as simple Hammerstein models. Controllability of linear systems with positive controls has been characterized for the continuous-time (CT) and discrete-time (DT) cases in (Saperstone, 1973) and (Evans and Murthy, 1997) respectively. Asymptotic null-controllability of CT linear systems with positive controls has been discussed in (Heemels, *et al.*, 1997). Design of stabilizing controllers for CT linear systems with positive controls has been

addressed in (Zaslavsky, 1990). Several closed loop designs for time-optimal and non-time-optimal dead-beat controllers for DT systems with positive controls have been presented in (Nešić and Mareels). First, we present necessary and sufficient conditions for stabilizability of DT linear systems with positive controls. The proof is constructive: we show how to design stabilizing controllers. Second, we show that for dead-beat controllable DT linear systems with positive controls we can always design time-optimal (dead-beat) controllers, which are *stabilizing*. Third, we show that for dead-beat controllable DT linear systems with positive controls not all time-optimal controllers are stabilizing. The second result contrasts with the general nonlinear case: it is easy to construct examples of general DT nonlinear dead-beat controllable systems for which all time-optimal controllers result in unstable closed-loop systems. On the other hand, the third result contrasts with the linear case since it is known from

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(O'Reilly, 1981) that for DT dead-beat controllable linear systems time-optimal controllers are always stabilizing. We emphasize that the results in (Zaslavsky, 1990), applied to DT systems, do not suffice to design stabilizing *time-optimal* controllers. On the other hand, in (Nešić and Mareels) only time-optimal controllers for DT linear systems with positive controls are considered. However, it has not yet been established if these controllers are stabilizing.

## 2. PRELIMINARIES AND MAIN RESULTS

Let  $\mathbf{N}$ ,  $\mathbf{R}$ ,  $\mathbf{C}$  be the sets of natural, real and complex numbers respectively.  $\mathbf{R}_+ = [0, +\infty[$ , while  $\mathbf{R}_+^n$  is the set of  $n$ -tuples with components belonging to  $\mathbf{R}_+$ .

We consider systems of the form

$$\begin{aligned} x(k+1) &= Ax(k) + bu(k), \\ x(k) &\in \mathbf{R}^n, u(k) \in \mathbf{R}_+ \end{aligned} \quad (1)$$

The spectrum of a square matrix  $A$  is denoted as  $\sigma(A)$ . Denote the set of control sequences of length  $N$  as  $\mathcal{U}_N$ . The state of system (1) at time  $N$ , emanating from the initial state  $x(0)$  under the control sequence  $U_N \in \mathcal{U}_N$  is denoted as  $x(N, x(0), U_N)$ . If  $u(k) = u(x(k))$ , then  $x(N, x(0), u(x(k)))$  is used.

*Definition 1.* System (1) is *positively controllable* if for all  $x_a, x_b \in \mathbf{R}^n$  there exists an  $N \in \mathbf{N}$  and an input sequence  $U_N \in \mathcal{U}_N$  such that  $x(N, x_a, U_N) = x_b$ .

The following result was proved in (Evans and Murthy, 1997).

*Proposition 2.* System (1) is positively controllable *iff*

- (1)  $\text{Rank}[\lambda I - A : b] = n, \forall \lambda \in \mathbf{C}$
- (2)  $\sigma(A) \cap [0, +\infty[ = \emptyset$

A complex number  $\lambda^* \in \sigma(A)$  is called a *positively controllable mode* for system (1) if  $\text{Rank}[\lambda^* I - A : b] = n$  and  $\lambda^* \notin [0, +\infty[$ . If a complex number  $\lambda' \in \sigma(A)$  is not a positively controllable mode for system (1), then it is called a *positively uncontrollable mode*.

*Remark 3.* In general, the number of steps needed to control a system from a state  $x_a$  to  $x_b$  depends on the states  $x_a$  and  $x_b$ . However, it can be seen from the proof of Proposition 2 in (Evans and Murthy, 1997), that if system (1) is positively controllable, there exists a uniform bound  $\tilde{N}$  on the number of steps needed to control the system

from  $x_a$  to  $x_b$  (uniform meaning here that  $\tilde{N}$  is not depending on  $x_a$  or  $x_b$ ).

*Definition 4.* System (1) is *positively dead-beat controllable or pdbc* if for all  $x(0) \in \mathbf{R}^n$  there exists an  $N \in \mathbf{N}$  and an input sequence  $U_N \in \mathcal{U}_N$  such that  $x(N, x(0), U_N) = 0$ .

The following result was proved in (Nešić and Mareels).

*Proposition 5.* System (1) is pdbc *iff*

- (1)  $\text{Rank}[\lambda I - A : b] = n, \forall \lambda \in \mathbf{C} \setminus \{0\}$
- (2)  $\sigma(A) \cap ]0, +\infty[ = \emptyset$

A *cone*  $C$  is a subset of  $\mathbf{R}^n$  if for all  $x \in C, \alpha x \in C$  for all  $\alpha \in \mathbf{R}_+$ .

The pair  $(A, b)$  is *stabilizable* if the linear system  $x(k+1) = Ax(k) + bu(k)$ , with unconstrained input  $u \in \mathbf{R}$ , is stabilizable or equivalently if  $\text{Rank}[\lambda I - A : b] = n$ , for all  $\lambda \in \mathbf{C}$  not belonging to the open unit disk.

A *stabilizing positive controller* for system (1) is a map  $u(x) : \mathbf{R}^n \rightarrow \mathbf{R}_+$  such that the origin is an equilibrium point of the closed-loop system formed by (1) and  $u(x)$ , which is globally asymptotically stable.

A *time-optimal positive controller* for system (1) is a map  $u(x) : \mathbf{R}^n \rightarrow \mathbf{R}_+$  such that the origin is a globally attractive equilibrium point of the closed-loop system resulting from (1) and  $u(x)$  and such that every initial state of the closed-loop system is transferred to the origin in minimum time. Notice that the origin of the closed-loop system resulting from (1) and a time-optimal controller is not necessarily stable.

A positive controller for system (1) which is both stabilizing and time-optimal is a *stabilizing time-optimal positive controller*.

System (1) is *positively stabilizable* if there exists a stabilizing positive controller for (1).

System (1) is *positively stabilizable in minimum time* if there exists a stabilizing time-optimal positive controller for (1).

*Definition 6.* A map  $u(x) : \mathbf{R}^n \rightarrow \mathbf{R}_+$  satisfies the small control property (SCP) if

$$\forall \epsilon > 0, \exists \delta > 0 : \|x\| < \delta \Rightarrow |u(x)| < \epsilon \quad (2)$$

**Main results:** The main results of this paper can be summarized as follows.

*Theorem 7.* System (1) is positively stabilizable *iff*

- (1) The pair  $(A, b)$  is stabilizable.
- (2)  $\sigma(A) \cap [1, +\infty[ = \emptyset$ .

In other words, system (1) is positively stabilizable iff all its positively uncontrollable modes belong to the open unit disk.

**Theorem 8.** System (1) is positively stabilizable in minimum time iff system (1) is pdbc.

The main results of this paper can be rephrased using the following sets:

$\mathcal{S}$ : The set of all stabilizing positive controllers for system (1).

$\mathcal{S}_d$ : The set of all time-optimal positive controllers for system (1). It is known from (Nešić and Mareels) that  $\mathcal{S}_d \neq \emptyset$  iff system (1) is pdbc. A design procedure for the family of controllers  $\mathcal{S}_d$  can be found in (Nešić and Mareels).

$\mathcal{S}_p$ : a subset of  $\mathcal{S}$  which we construct in this paper. The construction relies on designs presented in (Nešić and Mareels) and (Zaslavsky, 1990).

In terms of these sets, our main results are:

**Corollary 9.**  $\mathcal{S} \neq \emptyset$  iff

- (1) The pair  $(A, b)$  is stabilizable.
- (2)  $\sigma(A) \cap [1, +\infty[ = \emptyset$ .

**Corollary 10.**  $\mathcal{S}_d \cap \mathcal{S} \neq \emptyset$  iff system (1) is pdbc.

Another interesting relationship is illustrated in the paper.

**Proposition 11.** There exist systems of the form (1) such that  $\mathcal{S}_d \not\subset \mathcal{S}$ .

In other words, there exist systems which admit a time-optimal positive controller that is not stabilizing.

### 3. CONSTRUCTION OF STABILIZING POSITIVE CONTROLLERS

In this section we prove the main results of the paper.

*Claim:* We assume in the sufficiency parts of the proofs of Theorem 7 and 8 that system (1) is positively controllable (see Proposition 2).

The proof of this claim is omitted.

#### Proof of Theorem 7

*Necessity:* Necessity of condition 1 is trivial. It follows from (Evans and Murthy, 1997) that condition 2 is also necessary since if  $A$  has an eigenvalue  $\lambda \in [1, +\infty[$ , then there exists an initial state  $x_0$  such that for all  $k \in \mathbf{N}$  and for all control sequences  $U_k \in \mathcal{U}_k$  with positive controls,  $\|x(k, x_0, U_k)\| \geq \delta > 0$  with  $\delta$  some strictly positive real number. This implies that the origin of system (1), using an arbitrary positive control sequence is not attractive. But then the origin of

the closed-loop system, formed by system (1) and an arbitrary feedback  $u(x)$  taking values in  $\mathbf{R}_+$ , is not attractive.

*Sufficiency:* Assume that system (1) is positively controllable.

Consider a positively controllable system (1) in its controllability canonical form

$$x(k+1) = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & & & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u(k), u(k) \in \mathbf{R}_+ \quad (3)$$

By the classical pole-placement theorem, the poles of the closed-loop system resulting from (3) with unconstrained control  $u(k) \in \mathbf{R}$  and a linear feedback  $u(x) = f^T x$  may be arbitrarily assigned. Place the poles in the open unit disk. The closed-loop system is then described by

$$x(k+1) = (A + bf^T)x(k) := A_{cl}x(k) \quad (4)$$

where  $A_{cl}$  is a Schur matrix. However, the constraint  $u(x) \in \mathbf{R}_+$  will not be satisfied for all  $x \in \mathbf{R}^n$ . We show that particular choices of  $f^T$  yield an invariant cone  $C_0$  for the closed-loop system (meaning that if  $x(0) \in C_0$ , then  $x(k, x(0), u(x(k))) \in C_0$ , for all  $k \in \mathbf{N}$ ) such that  $u(x) \geq 0$ , for all  $x \in C_0$ .

Choose  $f^T$  such that the poles of the closed-loop system are  $\{\lambda_1, \dots, \lambda_p\} \subset [0, 1[$  where  $p \leq n$ . Consider the closed half space defined by  $F := \{x \in \mathbf{R}^n | f^T x \geq 0\}$ .

*Claim:* There exists a basis of  $\mathbf{R}^n$  of generalized eigenvectors  $\{e_1^1, \dots, e_{m_1}^1, \dots, e_1^p, \dots, e_{m_p}^p\}$  (Kailath, 1980) of  $A_{cl}$  such that for all  $i = 1, \dots, p$

$$\begin{cases} f^T e_j^i \geq 0, \forall j = 1, \dots, m_i \\ A_{cl} e_1^i = \lambda_i e_1^i \\ A_{cl} e_j^i = \lambda_i e_j^i + \sum_{k < j} \beta_k^i(j) e_k^i, \beta_k^i(j) \geq 0, \\ \forall j = 2, \dots, m_i \end{cases} \quad (5)$$

where the  $\beta_k^i(j)$  are well-chosen, but not necessarily unique positive real numbers.

The proof of this claim is omitted.

Define the nonsingular matrix  $E := [e_1^1 \dots e_{m_1}^1 \dots e_1^p \dots e_{m_p}^p]$  and consider the cone  $C_0$  generated by  $\{E\alpha | \alpha \in \mathbf{R}_+^n\}$ . It is clear that  $C_0 \subset F$  and that  $C_0$  has nonempty interior since  $\{e_1^1, \dots, e_{m_1}^1, \dots, e_1^p, \dots, e_{m_p}^p\}$  is a basis of  $\mathbf{R}^n$ . We show that  $C_0$  is an invariant cone for the closed-loop system, resulting from (1) and  $u(x) = f^T x$ . Decompose an arbitrary  $x_0 \in C_0$  along

$e_1^1, \dots, e_{m_p}^p$ :  
 $x_0 = \sum_{i=1}^p \sum_{j=1}^{m_i} \alpha_j^i e_j^i$ ,  $\alpha_j^i \geq 0$ , for all  $j = 1, \dots, m_i$  and  $i = 1, \dots, p$ . It follows from (5) that  $A_{cl}x_0 = \sum_{i=1}^p \sum_{j=1}^{m_i} \alpha_j^i (\lambda_i e_j^i + \sum_{k < j} \beta_k^i(j) e_k^i)$ . Since  $\{\lambda_1, \dots, \lambda_p\} \subset [0, 1]$ ,  $\alpha_j^i \geq 0$  and  $\beta_k^i(j) \geq 0$  for all  $j, k = 1, \dots, m_i$  and for all  $i = 1, \dots, p$ , it follows that  $A_{cl}x_0 \in C_0$ . This proves that  $C_0$  is an invariant cone of (4). Notice that the feedback  $u(x) = f^T x$  satisfies the constraint  $u(x) \geq 0$ , for all  $x \in C_0$  since  $C_0 \subset F$ .

To complete the design of a stabilizing controller,  $u(x)$  still needs to be defined for  $x \in \mathbf{R}^n \setminus C_0$ . We show that it is possible to construct a class of sets  $\{C_1, \dots, C_{N^*}\}$  with  $N^* \leq \tilde{N}$ ,  $C_0 \subset C_1 \subset \dots \subset C_{N^*}$  and  $C_{N^*} = \mathbf{R}^n$ . These sets are defined recursively: for all  $j = 1, \dots, N^*$ ,  $C_j := \{y \in \mathbf{R}^n | \exists U_1 \in \mathcal{U}_1 : x(1, y, U_1) \in C_{j-1}\}$ . Thus  $C_j$  is the *maximal* set of states which can be steered to the cone  $C_0$  in at most  $j$  steps, with an appropriate positive control sequence. We also show that this control can be implemented as a feedback law. These ideas are borrowed from (Nešić and Mareels).

First calculate the set  $C_1$  where  $C_1$  is the maximal set of states which can be mapped to  $C_0$  in at most one step by a suitable positive control for system (3). Rewrite  $C_0$  in the form  $C_0 = \{x \in \mathbf{R}^n | Mx \in \mathbf{R}_+^n\}$ , where  $M = E^{-1}$ . Denote the rows of  $M$  as  $m_1^T, \dots, m_n^T$ , i.e.  $M := [m_1, \dots, m_n]^T$ . Consider the following decompositions resulting from equation (3) and the definition of  $C_0$

$$m_i^T Ax + m_i^T bu \geq 0, \forall i = 1, \dots, n \quad (6)$$

and split these inequalities into three groups depending on the sign of  $m_i^T b$ . The set of indices for which  $m_i^T b = 0$ ,  $m_i^T b > 0$ ,  $m_i^T b < 0$  are relabeled as  $\mathcal{Z}_1 = \{z_1, \dots, z_{n_0}\}$ ,  $\mathcal{P}_1 = \{p_1, \dots, p_{n_+}\}$  and  $\mathcal{N}_1 = \{n_1, \dots, n_{n_-}\}$  respectively. There exists a positive control  $u$  which steers a state  $x \in C_1$  to  $C_0$  only if the following inequalities are satisfied

$$\begin{cases} u \geq 0 \\ \min_{i \in \mathcal{N}_1} \frac{-m_i^T Ax}{m_i^T b} \geq u \geq \max_{j \in \mathcal{P}_1} \frac{-m_j^T Ax}{m_j^T b} \end{cases} \quad (7)$$

From these inequalities it is seen that the defining set of inequalities for  $C_1$  are

$$\begin{cases} (\frac{-m_i^T A}{m_i^T b} + \frac{m_j^T A}{m_j^T b})x \geq 0, \forall i \in \mathcal{N}_1, \forall j \in \mathcal{P}_1 \\ \frac{-m_i^T Ax}{m_i^T b} \geq 0, \forall i \in \mathcal{N}_1 \\ m_k^T Ax \geq 0, \forall k \in \mathcal{Z}_1 \end{cases} \quad (8)$$

and  $C_1$  can be written in the form  $C_1 = \{x \in \mathbf{R}^n | M_1 x \in \mathbf{R}_+^q\}$  for some  $q \in \mathbf{N}$ . It is clear that any map  $u_1(x) : C_1 \setminus C_0 \rightarrow \mathbf{R}_+$  taking values in the interval

$$\mathcal{I}_{u_1(x)} := [\max(0, \max_{j \in \mathcal{P}_1} \frac{-m_j^T Ax}{m_j^T b}), \min_{i \in \mathcal{N}_1} \frac{-m_i^T Ax}{m_i^T b}] \quad (9)$$

transfers any initial state in  $C_1 \setminus C_0$  to  $C_0$  in one step. In addition  $u_1(x)$  can be chosen to satisfy the SCP since the lower bound of  $\mathcal{I}_{u_1(x)}$  is a continuous function of  $x$  that converges to zero if  $x \rightarrow 0$ . Thus for instance  $u_1(x) = u_{1,\min}(x) := \min\{y | y \in \mathcal{I}_{u_1(x)}\}$  is a possible choice. Notice however that other choices for  $u_1(x)$ , possessing the SCP, are possible.

The same procedure can be applied to calculate the sets  $C_2, \dots, C_{N^*}$ , the intervals  $\mathcal{I}_{u_2(x)}, \dots, \mathcal{I}_{u_{N^*}(x)}$  and the mappings  $u_{2,\min}(x), \dots, u_{N^*,\min}(x)$ .

It follows from Remark 3 that  $N^* \leq \tilde{N} < +\infty$ . This implies that  $C_{N^*} = \mathbf{R}^n$ .

Define the following map

$$u(x) := \begin{cases} f^T x, & x \in C_0 \\ u_{j,\min}(x), & x \in C_j \setminus C_{j-1}, j = 1, \dots, N^* \end{cases} \quad (10)$$

which also satisfies the SCP.

The origin of the closed-loop system, resulting from (3) and (10) is globally attractive since  $C_0$  is reached in finite time and since the behaviour of the closed-loop system on  $C_0$  is determined by (4) where  $A_{cl}$  is a Schur matrix.

To conclude the proof we need to show that the closed-loop system, resulting from (3) and (10) is stable. This is possible, precisely because  $u(x)$  is chosen such that it satisfies the SCP. We omit the proof.

**QED.**

*Remark 12.* In the proof of Theorem 7 we have constructed a stabilizing positive controller (10) for (3). However, it is clear from the proof that this controller is not the only stabilizing positive controller. A number of design parameters can be changed and still a stabilizing positive controller results. Among these parameters are the linear gain  $f^T$ , the basis of generalized eigenvectors for  $A_{cl}$  (and thus the cone  $C_0$ ) and the mappings  $u_j(x)$ . Adjusting these parameters generates a class of stabilizing controllers for (3), which we denote as  $S_p$ . Since this class of controllers is still fairly wide, we can ask ourselves if there is enough freedom left to design stabilizing *time-optimal* positive controllers, thus improving performance of the closed-loop system. This question will be answered affirmatively in the proof of Theorem 8.

At this point it is useful to refer to the class of *all* time-optimal controllers  $S_d$  for a dbc system (1), designed in (Nešić and Mareels). However, the question if these time-optimal controllers are

also *stabilizing* was not addressed in (Nešić and Mareels). Our aim is to show in the proof of Theorem 8 that if a system is dbc, there always exists a *stabilizing* controller in the class  $\mathcal{S}_d$ .

**Proof of Theorem 8**

*Necessity:* This is trivial.

*Sufficiency:* It has been shown in (Nešić and Mareels) that if system (1) is pdbc, there exists a class  $\mathcal{S}_d$  of time-optimal positive controllers. We show that for a pdbc system (1) there always exist a *stabilizing* controller in  $\mathcal{S}_d$  by proving that there always exists a controller in  $\mathcal{S}_p$  which also belongs to  $\mathcal{S}_d$ .

Again consider a positively controllable system (1) and consider its controllability canonical form (3). Suppose we place all the poles of  $A_{cl}$  in zero by choosing the row vector  $f^T$  in  $u(x) = f^T x$ . We have to find a cone  $C_0 \subset F$  such that  $C_0$  is invariant for the system (4). Our aim is to show that there exists a particular basis of  $\mathbf{R}^n$  of generalized eigenvectors  $\{e_1, \dots, e_n\}$  of  $A_{cl}$ , such that the cone  $C_0 = \{[e_1 \dots e_n] \alpha | \alpha \in \mathbf{R}_+^n\} \equiv C^*$ , where

$$C^* := \{x \in \mathbf{R}^n | f^T x \geq 0, f^T A_{cl} x \geq 0, \dots, f^T A_{cl}^{n-1} x \geq 0\} \quad (11)$$

It was proved in (Nešić and Mareels) that  $C^*$  is the *maximal* set of states of (3) which can be transferred to the origin by means of  $u(x) = f^T x$ , for all  $x \in C^*$  in at most  $n$  steps and that the behaviour of the system on  $C^*$  is time-optimal (O'Reilly, 1981). Furthermore, it was shown that  $C^*$  is an invariant cone for (4). Then clearly  $C_0$  satisfies the imposed conditions.

A straightforward calculation shows that  $\text{Ker}(A_{cl} - 0I)^n = \text{span}\{z_1, \dots, z_n\}$ , where  $z_i \in \mathbf{R}^n$  for which the  $i^{\text{th}}$  entry equals 1 and all other entries are zero.

On the other hand  $C^*$  can be expressed as

$$C^* = \{M^* \alpha | \alpha \in \mathbf{R}_+^n\} \quad (12)$$

The columns of  $M^*$  are the  $n$ , 1-dimensional faces of  $C^*$  which can be easily calculated from (11). In addition  $M^*$  has the following form

$$M^* = \begin{bmatrix} f_1 & g_2(f_1, f_2) & \dots & g_n(f_1, \dots, f_n) \\ 0 & f_1 & \ddots & \vdots \\ \vdots & & \ddots & g_2(f_1, f_2) \\ 0 & 0 & \dots & f_1 \end{bmatrix} \quad (13)$$

with  $g_i(f_1, \dots, f_i) := g_i \in \mathbf{R}, \forall i = 2, \dots, n$ .

Since  $f_1 = a_0 \neq 0$  ( $A$  is nonsingular since (3) is positively controllable) we can choose  $e_1 = f_1 z_1$ ,  $e_2 = g_2 z_1 + f_1 z_2, \dots, e_n = g_n z_1 + g_{n-1} z_2 + \dots + f_1 z_n$ . This particular choice of  $\{e_1, \dots, e_n\}$  yields a cone  $C_0 = \{[e_1 \dots e_n] \alpha | \alpha \in \mathbf{R}_+^n\}$ , which equals the cone  $C^*$ . Once  $f^T$  and  $C_0$  are known the

rest of the design of Theorem 7 can be carried out. The sets  $C_1, \dots, C_{N^*}$  are obtained and the mappings  $u_j(x), j = 1, \dots, N^*$  can be chosen, such that they satisfy the SCP as shown in the proof of Theorem 1. Finally we obtain a *stabilizing* time-optimal positive controller  $u^*(x)$

$$u^*(x) = \begin{cases} f^T x, & x \in C_0 = C^* \\ u_j(x), & x \in C_j \setminus C_{j-1}, j = 1, \dots, N^* \end{cases} \quad (14)$$

**QED.**

*Remark 13.* The controller (14) is stabilizing because the mappings  $u_j(x)$  can be chosen to satisfy the SCP. This has been shown in the proof of Theorem 1. However, if in (14) we would have chosen different mappings  $u_j(x)$ , satisfying the constraint that  $u_j(x) \in \mathcal{L}_{u_j(x)}$ , but not satisfying the SCP, then the resulting controller would still have been time-optimal, but not stabilizing. This will be illustrated in the example in the next section.

*Remark 14.* We emphasize that the controller  $u^*(x)$  is time-optimal because we construct the cone  $C_0 \equiv C^*$ , which is the maximal set of states of (3) which can be transferred to the origin by means of  $u(x) = f^T x$ , for all  $x \in C^*$ . If instead, we would have applied the procedure from Theorem 7 to find an invariant cone  $C_0 \subset F$  for system (3), then  $C_0$  would in general have been a strict subset of  $C^*$ :  $C_0 \subset C^*$  and  $C_0 \neq C^*$ . Using such a cone  $C_0$ , the controller  $u(x)$  that would have resulted by carrying out the rest of the design of Theorem 7, would not have been time-optimal.

*Remark 15.* The stabilizing time-optimal positive controller (14) differs from the controllers from (Zaslavsky, 1990), adapted to DT systems, since the latter controllers are stabilizing but not time-optimal. The reason why the controllers from (Zaslavsky, 1990), adapted to DT systems, are not time-optimal is that on the cone  $C_0$ , where the closed-loop system (4) is linear, the matrix  $A_{cl}$  is not allowed to have eigenvalues in zero. Also, we emphasize that although (14) belongs to the class  $\mathcal{S}_d$  (Nešić and Mareels) of all time-optimal positive controllers, we have proved that (14) is also stabilizing, while stabilizing properties of the class of controllers  $\mathcal{S}_d$  have not been investigated in (Nešić and Mareels). Moreover, it will be illustrated in the example in the next section that there exist a system such that not all time-optimal positive controllers  $\mathcal{S}_d$  from (Nešić and Mareels) are stabilizing.

**4. EXAMPLE**

Consider the following system:

$$x(k+1) = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k),$$

$$u(k) \in \mathbf{R}_+. \quad (15)$$

This system is positively controllable, since the pair  $(A, b)$  is controllable and  $\sigma(A) \cap [0, +\infty[ = \emptyset$ , so we can design controllers belonging to  $\mathcal{S}_p$  and  $\mathcal{S}_d$ .

**Construction of  $u^1(x) \in \mathcal{S}_p$ .**

Place the poles of  $A_{cl}$  in  $\{+\frac{1}{2}, +\frac{1}{4}\}$ . Then  $f^T = (\frac{47}{8} \quad -\frac{17}{4})$ . The eigenvectors associated to  $+\frac{1}{2}$  and  $+\frac{1}{4}$  are  $e_1^T \in \text{span}\{(2 \ 1)\}$  and  $e_2^T \in \text{span}\{(4 \ 1)\}$  respectively. Then an invariant cone  $C_0$  can be defined:  $C_0 := \left\{ \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \alpha \mid \alpha \in \mathbf{R}_+^n \right\}$ . Rewrite  $C_0$

as  $C_0 = \{x \in \mathbf{R}^n \mid m_1^T x \geq 0, m_2^T x \geq 0\}$ , where  $m_1^T = (1 \quad -2)$  and  $m_2^T = (-1 \ 4)$ . It follows that  $u^1(x) = f^T x$  on  $C_0$ . Next, calculate  $C_1$ . It can be shown that  $\mathcal{N}_1 = \{1\}$ ,  $\mathcal{P}_1 = \{2\}$  and  $\mathcal{Z}_1 = \emptyset$  and thus that  $C_1 = \{x \in \mathbf{R}^n \mid \tilde{m}_1^T x \geq 0, \tilde{m}_2^T x \geq 0\}$  where  $\tilde{m}_1^T = (0 \ 1)$  and  $\tilde{m}_2^T = (12 \ 11)$ . It follows that any  $u_1^1(x) \in \mathcal{I}_{u_1^1(x)} = [\max(0, \frac{1}{4}(24 \ 21)x), \frac{1}{2}(12 \ 11)x]$  satisfies the SCP on  $C_1 \setminus C_0$ . Now we calculate  $C_2$ . Notice that  $\tilde{m}_1^T b = 1$  and  $\tilde{m}_2^T b = 11$ , implying that  $\mathcal{N}_2 = \mathcal{Z}_2 = \emptyset$  and that  $\mathcal{P}_2 = \{1, 2\}$ . Thus  $C_2 = \mathbf{R}^n$  and  $u_2^1(x) \in \mathcal{I}_{u_2^1(x)} = [\max(0, \max_{j \in \mathcal{P}_2} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}), +\infty[$  can be chosen such that  $u_2^1(x)$  satisfies the SCP on  $C_2 \setminus C_1$ . A possible choice for a stabilizing positive controller  $u^1(x)$  for system (15) is for instance

$$u^1(x) = \begin{cases} f^T x, & x \in C_0 \\ \max(0, \frac{1}{4}(24 \ 21)x), & x \in C_1 \setminus C_0 \\ \max(0, \max_{j \in \mathcal{P}_2} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}), & x \in C_2 \setminus C_0 \end{cases}$$

Notice that  $u^1(x)$  is not a time-optimal controller, since trajectories of the closed-loop system converge only asymptotically to the origin because  $A_{cl}$  is not nilpotent.

**Construction of  $u^2(x) \in \mathcal{S}_d$ .**

Place the poles of  $A_{cl}$  in  $\{0, 0\}$  by choosing  $\tilde{f}^T = (6 \ 5)$ . Then  $C^* = C_0 = \{x \in \mathbf{R}^n \mid \tilde{m}_1^T x \geq 0, \tilde{m}_2^T x \geq 0\}$ , where  $\tilde{m}_1^T = (6 \ 5)$  and  $\tilde{m}_2^T = (0 \ 1)$  and  $u^2(x) = \tilde{f}^T x$  on  $C^*$ . Next, calculate  $C_1$ . We obtain that  $\tilde{m}_1^T b = 5$  and  $\tilde{m}_2^T b = 1$ , implying that  $\mathcal{N}_1 = \mathcal{Z}_1 = \emptyset$  and  $\mathcal{P}_1 = \{1, 2\}$  and thus that  $C_1 = \mathbf{R}^n$ . Any  $u_1^2(x) \in \mathcal{I}_{u_1^2(x)} = [\max(0, \max_{j \in \mathcal{P}_1} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}), +\infty[$  can be chosen on  $C_1 \setminus C_0$ .

A possible choice for a time-optimal positive controller

$$u^2(x) = \begin{cases} \tilde{f}^T x, & x \in C_0 \\ \max(0, \max_{j \in \mathcal{P}_1} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}) + 1, & x \in C_1 \setminus C_0 \end{cases}$$

Notice that  $u^2(x)$  does not satisfy the SCP. This implies that  $u^2(x)$  results in an unstable closed-loop system. To see this fix an  $\epsilon^*$  such that  $0 < \epsilon^* < 1$  and consider the closed ball  $B_{\epsilon^*}$  around the origin with radius  $\epsilon^*$ . We will show that for all  $\delta > 0$ , there exists an initial state  $x_0 \in C_1 \setminus C_0$  with  $\|x_0\| < \delta$  such that  $\|x(1, x_0, u^2(x))\| > \epsilon^*$ . Let  $Az = b$ . Then  $z = (-\frac{1}{6}, 0)^T$  and  $\beta z \in C_1 \setminus C_0$ , for all  $\beta > 0$ . For all  $\delta > 0$ , choose  $\beta_\delta > 0$  such that  $\|\beta_\delta z\| < \delta$ . Then for all  $\delta > 0$ , define  $x_0 := \beta_\delta z$ . Thus  $\|x_0\| < \delta$  and  $x_0 \in C_1 \setminus C_0$ . Now  $\|x(1, x_0, u^2(x))\| = \|Ax_0 + bu^2(x_0)\| = |\beta_\delta + u^2(x_0)| \|b\| = |\beta_\delta + u^2(x_0)| > 1 > \epsilon^*$  since  $\beta_\delta > 0$  and  $u^2(x_0) > 1$ .

This example serves as a proof for Proposition 11.

**Construction of  $u^*(x) \in \mathcal{S}_p \cap \mathcal{S}_d$ .**

Consider the previous time-optimal positive controller  $u^2(x)$ , but replace the control law

$$u_1^2(x) = \max(0, \max_{j \in \mathcal{P}_1} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}) + 1, \quad x \in C_1 \setminus C_0$$

$$\text{by } u_1^2(x) = \max(0, \max_{j \in \mathcal{P}_1} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}), \quad x \in C_1 \setminus C_0.$$

Notice that  $u_1^2(x) \in \mathcal{I}_{u_1^2(x)}$ . Thus the following time-optimal positive controller is obtained

$$u^*(x) = \begin{cases} \tilde{f}^T x, & x \in C_0 \\ \max(0, \max_{j \in \mathcal{P}_1} \frac{-\tilde{m}_j^T Ax}{\tilde{m}_j^T b}), & x \in C_1 \setminus C_0 \end{cases}$$

But then  $u^*(x)$  clearly satisfies the SCP. This implies that  $u^*(x)$  is a stabilizing time-optimal positive controller for system (15), as opposed to  $u^2(x)$  which was a destabilizing time-optimal positive controller.

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