Uniting local and global observers for the state estimation of nonlinear continuous-time systems

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***Abstract*— Generic techniques are available for the design of local observers for nonlinear continuous-time systems. By local, we mean that the state estimate is guaranteed to converge to the true state, provided its initial value is nearby the plant initial condition. The main drawback of this approach is the right initialization of the observer, which may not be easy in practice. To overcome this potential issue, we propose to combine two observers: a local one, and an other observer, which provides (approximate) estimates but in a global sense, namely for any initial condition. We explain how to combine these two observers with a hybrid scheme guaranteeing global asymptotic convergence. The hybrid observer, called uniting observer, matches the local observer when the initial conditions are nearby the initial conditions of the observed system. We illustrate the proposed approach by means of a numerical example.**

***Index Terms*— Observers, hybrid systems, local performances.**

I. Introduction

While the systematic design of observers for linear time-invariant dynamical systems is very well understood, see the seminal works of Kalman [12] and Luenberger [17], no general method, that can be always easily applied to observable systems, is available when the system's dynamics are nonlinear. After the first works in the 70's and 80's, many techniques have been proposed in the last decades. Among them, we may identify two classes of asymptotic observers: (semi) global and local. We refer to *(semi) global asymptotic observers* when asymptotic estimation is achieved for any initial condition of the observed plant and any initial condition of the observer. Within this class of observers we recall passivity-based designs, e.g. [2]; techniques based on LMIs conditions, [5], [23], [25]; Luenberger-like observers (also known as Kazantzis-Kravaris Luenberger observers), e.g., [1], [13]; approaches which use normal forms induced by uniform observability conditions such as high-gain observers, e.g., [3], [4], [8], [14]; sliding-mode observers, e.g., [9]. When the system's dynamics are very complex, the design of such observers may be not possible, e.g., because of non-feasibility of the LMIs [2], [5], [25], or too complicated to apply, e.g. because we need to solve partial differential equations [1], or to find analytically the inverse of a nonlinear change of coordinates [3], [9], [14]. On the other hand, we refer to *local asymptotic observers* when asymptotic estimation is guaranteed only when the initial conditions of the observer are close enough to those of the observed plant. Among the various techniques for the design of local observers, the most popular is certainly the extended Kalman filter (EKF), e.g. [7], but other techniques can be found in [11], [22]. The design of local observers, such as EKF, is systematic under mild regularity conditions, but due to their local properties, divergence of the estimation may occur if the initial conditions are not carefully chosen. This wrong initialization problem could be overcome if we were able to build a *global approximate observer*, that is an observer providing, in a finite amount of time, an estimation close enough to the true value of the state. Examples of approximate observers can be found in [18], [24]. Then, intuitively, by waiting long enough, we could use the local observer at hand by picking as initial conditions the ones given by the approximate observer.

In this paper, we follow the ideas of [20], [21], which investigate control problems, but in the context of observers. In particular, our goal is to unite *two given* observers, one possibly local and an other possibly global. In [21], different observers are combined in the context of observer-based output feedback stabilization and not estimation solely. In this work, the observers we consider may have different structures, state dimensions and can be expressed in different coordinates. Under a certain number of sufficient conditions, we show how to design a hybrid observer which guarantees global asymptotic estimation and preserves the behaviour of the local observer when the estimation error is small enough. The proposed hybrid scheme can be used not only for the purpose of improving performances of two continuous-time observers, as done in [6] or [9] in a continuous-time approach with adaptive designs for specific classes of observers, but also to achieve global asymptotic estimation when a continuous-time global asymptotic observer is not (explicitly) known. This novel perspective suggests also to put more efforts in the study of global approximate observers for nonlinear systems.

Note that the idea of exploiting local observers to ensure (semi) global asymptotic stability property for the estimation error was also pursued in [15], [19], where a multi-observer, consisting of a bank of numerous local observers, is proposed. However, the approach pursued in this paper is different as only two observers are required, as well as some auxiliary variables, which ease the implementation in terms of computational cost. Also, here, we do not need to assume that initial condition of the plant lies in a known

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compact set. Finally, it is worth stressing that this work is not simply an extension to the observer case of [20], [21] since a certain number of different technical points need to be addressed. The main difference is that, while, in [20], [21], the equilibrium of the closed-loop system is known *a priori*, in the considered observation framework, the steady-state of the hybrid observer is an unknown time-varying trajectory. As a consequence, in order to decide which observer has to be used, we need to know the norm of the estimation error provided by the global observer and not the norm of the system state.

The paper is organized as follows. In Section II, we state the framework and the uniting observers problem we aim to solve. In Section III, we give a certain number of assumptions under which the uniting problem can be solved. An explicit solution is presented in Section IV. The proof of the main result is omitted for the space reasons. A simple example is given in Section V, where we combine an extended Kalman filter with a high-gain observer in order to improve the performances in presence of measurement noise.

Notation

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{N} = \{1, 2, \ldots\}$. Given a matrix $P \in \mathbb{R}^{n\times m}$, we denote with $\mathrm{vec}(P)$ the vectorization of the matrix P. Given a differential equation $\dot{x} = f(x)$, we denote by $x(t)$ its solution initialized at x° at time t, when it exists. We consider hybrid systems of the form [10]

$$\dot{x} = F(x), \quad x \in \mathcal{C}, \quad x^+ = G(x), \quad x \in \mathcal{D},$$

where $x \in \mathbb{R}^{n_x}$ is the state, $\mathcal{C}$ is the flow set, F is the flow map, $\mathcal{D}$ is the jump set and G is the jump map. We assume that the hybrid model satisfies the basic regularity conditions, see Section 6.2 in [10], which will be the case in our study. We recall some definitions from [10]. Solutions to system (1) are defined on so-called *hybrid time domains*. A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J-1}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \ldots \leq t_J$ and it is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots, J\})$ is a compact hybrid time domain. Given two hybrid times $(t_0, j_0), (t_1, j_1) \in E$, we denote $(t_0, j_0) \leq (t_1, j_1)$ if $t_0 \leq t_1$ and $j_0 \leq j_1$. Given an initial condition $x^\circ \in \mathcal{C} \cup \mathcal{D}$, we denote by $x(t, j)$ a solution to the hybrid system starting at x° at time (t, j) if $(t, j) \in \mathrm{dom}\, x$. Throughout the text, we will refer to solutions as maximal solutions, see Definition 2.7 in [10].

II. Problem Statement

We investigate the estimation problem for autonomous nonlinear systems of the form

$$\dot{x} \;=\; f(x)\,, \qquad y \;=\; h(x) \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state and $y \in \mathbb{R}^{n_y}$ is the measured output, $n_x, n_y \in \mathbb{N}$. Moreover, we suppose that f, h are continuous functions and that there exists a set $\mathcal{X} \subseteq \mathbb{R}^{n_x}$, which is forward invariant for system (1), namely $x(0) \in \mathcal{X}$ implies $x(t) \in \mathcal{X}$ for all $t \in [0, \infty)$. To make precise what we mean by a global approximate observer and a local asymptotic observer, consider the following observer

$$\dot{\zeta} \;=\; \varphi(\zeta, y), \qquad \hat{x} \;=\; \psi(\zeta), \tag{2}$$

where $\zeta \in \mathcal{Z} \subseteq \mathbb{R}^{n_\zeta}$ is the observer state and $\hat{x} \in \mathbb{R}^{n_x}$ is the estimation, $n_\zeta \in \mathbb{N}$. Even if often the dimension of the observer n_ζ coincides with the system dimension n_x, e.g. [2], [4], [14], [25], the notation above allows to consider observers with a state dimension larger than the dimension of the plant, such as in [1], [3], [7]. We use the following definitions.

Definition 1 (Local asymptotic observer): System (2) is said to be a *local asymptotic observer* for system (1) if there exists a set-valued map $\mathcal{B}(x) \subseteq \mathcal{Z}$ such that for any $(x(0), \zeta(0)) \in \mathcal{X} \times \mathcal{B}(x(0))$, then the corresponding solutions to (1), (2) verifies $\lim_{t\to\infty} |x(t) - \hat{x}(t)| = 0$. □

Definition 2 (Global approximate observer): System (2) is said to be a *global approximate observer* for (1) if there exists a $\varepsilon > 0$ such that, for any initial condition $(x(0), \zeta(0)) \in \mathcal{X} \times \mathcal{Z}$, the corresponding solution to (1), (2) verifies $\limsup_{t\to\infty} |x(t) - \hat{x}(t)| \leq \varepsilon$. □

In this paper, we assume to know a local asymptotic observer and a global approximate observer for system (1). Any of the techniques in literature (such as [1], [3], [5], [7], [14], [18], [24], [25]) satisfying Definitions 1 and 2 can be used to construct these estimators. Note that any observer guaranteeing global asymptotic convergence is by definition a global approximate observer. The local asymptotic observer is given by

$$\dot{\zeta}_0 \;=\; \varphi_0(\zeta_0, y)\,, \qquad \hat{x}_0 \;=\; \psi_0(\zeta_0)\,, \tag{3}$$

with $\zeta_0 \in \mathcal{Z}_0 \subseteq \mathbb{R}^{n_0}$ and $\hat{x}_0 \in \mathbb{R}^{n_x}$, and the global approximate observer is written as

$$\dot{\zeta}_1 \;=\; \varphi_1(\zeta_1, y)\,, \qquad \hat{x}_1 \;=\; \psi_1(\zeta_1)\,, \tag{4}$$

with $\zeta_1 \in \mathcal{Z}_1 \subseteq \mathbb{R}^{n_1}$, and $\hat{x}_1 \in \mathbb{R}^{n_x}$. The subscript 0 is hence used to denote the local observer, while the subscript 1 is used to denote the global one. For the sake of convenience, we suppose that, given any solution x to (1) in $\mathcal{X}$, the set $\mathcal{Z}_0$, respectively $\mathcal{Z}_1$, is forward invariant for observer (3), respectively (4). This is assumption is stated in order to avoid the finite escape time phenomenon; we will work on relaxing this condition in future work.

By combining observers (3) and (4), we aim at designing a hybrid scheme in order to achieve global asymptotic estimation of system (1). The hybrid observer has the form

$$\dot{\xi} \;=\; \varphi(\xi, y), \quad \xi \in \mathcal{C}, \quad \xi^+ \;=\; w(\xi, y), \quad \xi \in \mathcal{D}, \tag{5}$$

and $\hat{x} = \psi(\xi)$, with $\xi \in \mathbb{R}^{n_\xi}$, $\hat{x} \in \mathbb{R}^{n_x}$, $\mathcal{C} \subseteq \mathbb{R}^{n_\xi}$, $\mathcal{D} \subseteq \mathbb{R}^{n_\xi}$. As a result, systems (1) and (5) lead to the overall system below

$$\begin{aligned}
&\left.\begin{array}{rcl} \dot{x} &=& f(x) \\ \dot{\xi} &=& \varphi(\xi, y) \end{array}\right\} \qquad (x, \xi) \in \mathcal{X} \times \mathcal{C}, \\
&\left.\begin{array}{rcl} x^+ &=& x \\ \xi^+ &=& w(\xi, y) \end{array}\right\} \qquad (x, \xi) \in \mathcal{X} \times \mathcal{D}, \\
&\quad y \;=\; h(x), \quad \hat{x} \;=\; \psi(\xi)\,.
\end{aligned} \tag{6}$$

Our objective is to solve the *uniting problem*, as defined next.

Definition 3 (Uniting problem): The uniting problem is solved if there exists an integer $n_\xi \geq \max\{n_0, n_1\}$, closed sets $\mathcal{C}$, $\mathcal{D}$, continuous functions $\varphi : \mathcal{C} \times h(\mathcal{X}) \to \mathbb{R}^{n_\xi}$, $w : \mathcal{D} \times h(\mathcal{X}) \to \mathbb{R}^{n_\xi}$ and $\psi : (\mathcal{C} \cup \mathcal{D}) \to \mathbb{R}^{n_x}$, $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geq 0}$, $\delta > 0$ and a (non-zero) matrix $M \in \mathbb{R}^{n_x \times n_\xi}$, such that the following holds.

- *(Completeness property)* Any solution (x, ξ) to (6) is complete.
- *(Global asymptotic property)* Any solution (x, ξ) to (6) satisfies

$$\lim_{t+j \to \infty} |x(t,j) - \hat{x}(t,j)| = 0 \,. \tag{7}$$

- *(Local property)* Any solution (x, ξ) to (6) satisfying $\alpha(x(0), \xi(0)) \leq \delta$ has the hybrid time domain $[0, \infty) \times \{0\}$ and $(x(t,0), M\xi(t,0)) = (\bar{x}(t), \bar{\zeta}_0(t))$ for all $t \in [0, \infty)$, where $(\bar{x}, \bar{\zeta}_0)$ is a solution to (1), (3). □

The *global asymptotic property* states that hybrid observer (5) guarantees global asymptotic estimation of x. On the other hand, the *local property* ensures that, when the initial conditions of hybrid observer (5) are nearby to the initial conditions of system (1), then hybrid observer (5) behaves as the local observer (3). In other words, solutions to (1), (5) correspond to solutions to (1), (3). This property guarantees that the performances and the robustness properties of local observer (3) are locally retained by hybrid observer (5).

III. Assumptions

As mentioned above, we will use observers (3) and (4), which we assume to know, to construct hybrid observer (5). The main idea is the following. When the state estimation is large, we need to use the global approximate observer, as the local one is not ensured to convergence in this case. After a sufficiently long time, the estimation error will become smaller than ε in view of Definition 2. Then, we can switch observer and use the local one. In order to implement such a mechanism, the following assumptions are needed.

Assumption 1 (State correspondence): There exists a continuous function $\gamma : \mathbb{R}^{n_1} \to \mathbb{R}^{n_0}$ such that $\gamma(\mathcal{Z}_1) \subseteq \mathcal{Z}_0$ and $\psi_1(\zeta_1) = \psi_0(\gamma(\zeta_1))$ for all $\zeta_1 \in \mathcal{Z}_1$. □

When we will switch from one observer to the other, we will use the function $\gamma(\cdot)$ to maps estimates of global observer (4) into estimates for local observer (3). Basically, if we want to initialize local observer (3) at time T, we select $\zeta_0(T) = \gamma(\zeta_1(T))$.

Assumption 2 (Local observer): There exist a continuous function $V_0 : \mathbb{R}^{n_x} \times \mathbb{R}^{n_0} \to \mathbb{R}_{\geq 0}$ and a real number $\varepsilon_0 > 0$ such that the following holds.

1) System (3) is a *local asymptotic observer* for system (1) with a basin of attraction $\mathcal{B}_0$ containing the set

$$\Omega_0 \;:=\; \{(x, \zeta_0) \in \mathcal{X} \times \mathcal{Z}_0 : V_0(x, \zeta_0) \leq \varepsilon_0\} \,. \tag{8}$$

2) The set Ω_0 is forward invariant for system (1), (3). □

Assumption 2 states that observer (3) is a local asymptotic observer. The function V_0 is usually a Lyapunov function for observer (3), used to prove convergence, see [7], [22], for instance. Forward invariance of the set Ω_0 is an extra assumption which comes for free when V_0 is a Lyapunov function. In the following, it is used to prevent undesired jump.

To solve our uniting problem, we need the estimation error provided by global observer (4) to be small enough in order to guarantee convergence of local asymptotic observer (3) when initialized with the estimate provided by global observer (4). For this, let $\Upsilon_1 \subset \mathcal{X} \times \mathcal{Z}_1$ be defined as

$$\Upsilon_1 := \left\{(x, \zeta_1) \in \mathcal{X} \times \mathcal{Z}_1 : V_0(x, \gamma(\zeta_1)) \leq \varepsilon_0\right\}, \tag{9}$$

where V_0 and ε_0 are given in Assumption 2. Note that by construction, $\gamma(\Upsilon_1) = \Omega_0$. Loosely speaking, the set Υ_1 is the "projection" of the Ω_0 in the ζ_1-coordinates. As a consequence, we will ask that global observer (4) to converge to a subset of Υ_1, in the next assumption.

Assumption 3 (Global approximate observer): There exist a continuous function $V_1 : \mathbb{R}^{n_x} \times \mathbb{R}^{n_1} \to \mathbb{R}_{\geq 0}$, and a real number $\varepsilon_1 > 0$ such that the following holds.

1) For all solutions x to system (1) evolving in $\mathcal{X}$ and all corresponding solutions ζ_1 to observer (4) starting in $\mathcal{Z}_1$ we have $\limsup_{t \to \infty} V_1(x(t), \zeta_1(t)) \;<\; \varepsilon_1$.
2) The set Ω_1 defined as

$$\Omega_1 \;:=\; \left\{(x, \zeta_1) \in \mathcal{X} \times \mathcal{Z}_1 \;:\; V_1(x, \zeta_1) \leq \varepsilon_1\right\} \tag{10}$$

is forward invariant for system (1), (4). Moreover, the set Ω_1 is strictly contained in the set Υ_1 defined by (9), namely $\Omega_1 \subset \Upsilon_1$. □

Item 1) of Assumption 3 means that observer (4) is a global approximate observer for system (1). When V_1 is a Lypaunov function, Ω_1 is a Lyapunov level set of V_1 and forward invariance comes directly. Finally, item 2) implies that the estimate provided by global observer (4) converges inside the set Υ_1 defined in (9).

Let $\Omega := \Omega_0 \cup \Omega_1$, with Ω_0 and Ω_1 from Assumption 2-3,

$$\begin{aligned}\Omega := \big\{(x, \zeta_0, \zeta_1) \in \mathcal{X} \times \mathcal{Z}_0 \times \mathcal{Z}_1 : V_0(x, \zeta_0) \leq \varepsilon_0, \\ V_1(x, \zeta_1) \leq \varepsilon_1\big\}.\end{aligned} \tag{11}$$

By construction, when both observers are initialized inside the set Ω, convergence of the uniting observer should be guaranteed. In order to satisfy the *local property* of Definition 3, the following assumption is needed.

Assumption 4 (Basin of attractions): There exist a continuous function $\theta : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}_\theta, \bar{\alpha}_\theta$ satisfying

$$\underline{\alpha}_\theta(|\hat{x}_0 - \hat{x}_1|) \;\leq\; \theta(\hat{x}_0, \hat{x}_1) \;\leq\; \bar{\alpha}_\theta(|\hat{x}_0 - \hat{x}_1|) \tag{12}$$

for all $\hat{x}_0, \hat{x}_1 \in \mathbb{R}^{n_x}$, and such that, by letting $\hat{x}_q = \psi_q(\zeta_q)$, $q = 0, 1$, the following holds.

1) By letting Ω defined as in (11), we have

$$c_0 := \sup\left\{\theta(\hat{x}_0, \hat{x}_1) : (x, \zeta_0, \zeta_1) \in \Omega\right\} \;>\; \varepsilon_1 \,. \tag{13}$$

2) The set Υ_0 defined as

$$\Upsilon_0 := \left\{(x, \zeta_0) \in \mathcal{X} \times \mathcal{Z}_0 : \exists\, \zeta_1 \in \Omega_1,\ \theta(\hat{x}_0, \hat{x}_1) < c_0 + 1\right\} \tag{14}$$

is strictly contained in the set $\mathcal{B}_0$ given by Assumption 2, namely $\Upsilon_0 \subset \mathcal{B}_0$. □

The function $\theta(\cdot)$ in Assumption 4 is a distance function, which is needed to detect whether the estimates provided by the local observer and the global one are too far when the state of global observer (4) is inside the set Ω_1 defined in (10). Further comments are postponed to the end of this section and in the forthcoming Section IV where the design of the hybrid observer solving the uniting problem is proposed.

Finally, the last key ingredient we need to implement the hybrid observer solving the uniting problem is the switching rule from the global observer to the local one. We mentioned above that we aim at switching between the observers depending on the state estimation error $\hat{x}_1 - x$ provided by global observer (4). The latter is unknown, we therefore need to estimate it. Hence, we assume below that we can design this estimator. In particular, we assume how to upper-bound the function V_1 which takes place of the estimation errors $\hat{x}_1 - x$, and which is all we need to switch from one observer to the other in view of item 2) of Assumption 3.

Assumption 5 (Error-norm estimator): There exist a continuous function $\rho : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$, a class $\mathcal{KL}$ function β and $c_1 > 0$ satisfying $\varepsilon_1 < c_1 < c_0$, with ε_1 given by Assumption 3 and c_0 given by Assumption 4, such that, the following holds.

1) For any solution (x, ζ_1) to (1), (4) with $(x(0), \zeta_1(0)) \in \mathcal{X} \times \mathcal{Z}_1$,

$$V_1(x(t), \zeta_1(t)) \leq z(t) + \beta(|x(0)| + |\zeta_1(0)| + |z(0)|,\ t)$$

for all $t \in [0, \infty)$, where z is the solution of

$$\dot{z} = -z + \rho(y, h(\hat{x}_1)) \tag{15}$$

with initial condition $z(0) \in \mathbb{R}_{\geq 0}$.

2) For each solution to (1), (4) starting in the set Ω_1, defined in (10), we have $\rho(y(t), h(\hat{x}_1(t))) < c_1$ for all $t \in [0, \infty)$. □

The existence of (15) satisfying the previous assumption is related to the notion of norm-estimator, see [16]. In this case, however, we ask that an upper bound for V_1 can be estimated from $|y - h(\hat{x}_1)|$. Moreover, in view of item 2) of Assumption 5, after a time long enough, we can expect that if the norm of z is small, then the state of global observer (4) is inside the set Ω_1 defined in (10). We can therefore switch observer. The existence of (15) is always guaranteed when V_1 is a Lyapunov function satisfying

$$\langle \nabla V_1(x, \zeta_1), (f(x, u), \varphi_1(x, u, y)) \rangle \ \leq\ -V_1(x, \zeta_1) + \varepsilon_1 \ ,$$

since we can select $\rho(\cdot) := \rho_1(\cdot) + \varepsilon_1$ where $\rho_1 : \mathbb{R}^{n_y} \times \mathbb{R}^{n_y} \to \mathbb{R}_{\geq 0}$ is any continuous function satisfying

$$\max_{(x,\zeta_1)\in\Omega_1} \rho_1(y, h(\hat{x}_1)) < c_1 - \varepsilon_1 \ .$$

Finally, note that the estimation of an upper bound for V_0 is not necessary, since the information whether the state of local observer (3) is inside the "right" set Ω_0, defined in (8), can be indirectly obtained by using the function $\theta(\cdot)$ of Assumption 4.

IV. Main Result

We are now in the position to state our main result concerning the existing of a hybrid observer solving the uniting problem.

Theorem 1: Under Assumptions 1-5, there exists a hybrid observer (5) *solving the uniting problem.*

The proposed hybrid observer is made of four components:

- local observer (3), which provides an estimate $\hat{x}_0$;
- global approximate observer (4), which provides an estimate $\hat{x}_1$;
- the dynamical estimator z given in Assumption 5;
- a discrete variable q which takes value in $\{0, 1\}$ defining which state estimate, $\hat{x}_0$ or $\hat{x}_1$, we need to use.

According to the discrete variable q, the hybrid observer has two different operating modes.

a) *(Global mode $q = 1$)* The global observer is "running" and we the local observer is "turned off", namely we keep its state constant. Consequently, we set $\hat{x} = \hat{x}_1$. The z-dynamics is used to detect whether the error estimate $|\hat{x}_1 - x|$ is small enough. In particular, when $|z| \leq c_1$, with c_1 given in Assumption 5, we allow for a jump and we change the operating mode.

b) *(Local mode $q = 0$)* The estimate $\hat{x}$ is provided by the local observer, namely $\hat{x} = \hat{x}_0$. To avoid wrong initializations, we must detect if the local observer is outside its basin of attraction, see Assumption 2. Since we do not have a direct knowledge of the behaviour of the local observer when $(x, \zeta_0) \notin \mathcal{B}_0$, we let run both observers and we compare the estimates $\hat{x}_0$ and $\hat{x}_1$. These two estimates should be respectively close enough if both observers are working in the right region, namely $(x, \hat{x}_0) \in \mathcal{B}_0$ and $(x, \hat{x}_1) \in \Omega_1$, defined in (10). As a consequence, if the difference between the two estimates $|\hat{x}_0 - \hat{x}_1|$ is too large, which is detected thanks to the function $\theta(\cdot)$ in Assumption 4, then we impose a jump and change the operating mode. Moreover, for robustness purposes, if $|\hat{x}_1 - x|$ becomes too large, we also impose a jump and we change operating mode. This can be observed by using the function $\rho(\cdot)$ in Assumption 5.

According to the previous operating modes, by letting $n_\xi = n_0 + n_1 + 2$ and by taking $\xi = (\zeta_0, \zeta_1, z, q) \in \mathbb{R}^{n_\xi} = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \mathbb{R} \times \mathbb{R}$, the proposed hybrid observer (5) solving the uniting problem is given by

$$\left.\begin{array}{rcl} \dot{\zeta}_0 &=& (1-q)\varphi_0(\zeta_0, y) \\ \dot{\zeta}_1 &=& \varphi_1(\zeta_1, y) \\ \dot{z} &=& -z + \max\left\{\rho(y, h(\hat{x}_1)),\right. \\ && \qquad\qquad \left.(1-q)\theta(\hat{x}_0, \hat{x}_1)\right\} \\ \dot{q} &=& 0 \end{array}\right\} \quad \xi \in \mathcal{C} \tag{16}$$

$$\left.\begin{array}{rcl} \zeta_0^+ &=& \gamma(\zeta_1) \\ \zeta_1^+ &=& \zeta_1 \\ z^+ &=& z \\ q^+ &=& 1-q \end{array}\right\} \quad \xi \in \mathcal{D} \tag{17}$$

$$\hat{x}_0 \,=\, \psi_0(\zeta_0)\,, \quad \hat{x}_1 \,=\, \psi_1(\zeta_1)\,, \quad \hat{x} \,=\, (1-q)\hat{x}_0 + q\hat{x}_1\,, \tag{18}$$

where $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$, $\mathcal{D} = \mathcal{D}_0 \cup \mathcal{D}_1$

$$\begin{array}{rcl} \mathcal{C}_0 &:=& \left\{\xi \in \Xi : 0 \le z \le c_0,\ q = 0\right\}, \\ \mathcal{C}_1 &:=& \left\{\xi \in \Xi : c_1 \le z,\ q = 1\right\}, \\ \mathcal{D}_0 &:=& \left\{\xi \in \Xi : c_0 \le z,\ q = 0\right\}, \\ \mathcal{D}_1 &:=& \left\{\xi \in \Xi : 0 \le z \le c_1,\ q = 1\right\}, \end{array} \tag{19}$$

$\Xi := \mathcal{Z}_0 \times \mathcal{Z}_1 \times \mathbb{R}_{\geq 0} \times \{0,1\}$ and $c_1 < c_0$ are given in Assumptions 4 and 5. Moreover, the local property in Definition 3 is verified with $M := (I_{n_0}\, 0)$, where I_{n_0} is the identity matrix in $\mathbb{R}^{n_0 \times n_0}$,

$$\alpha(x,\xi) := \max\left\{\frac{1}{\varepsilon_0}V_0(x,\zeta_0),\ \frac{1}{\varepsilon_1}V_1(x,\zeta_1),\ \frac{1}{c_0}|z|,\ q\right\}$$

and for some value $0 < \delta < 1$, with V_0, V_1, ε_0 ε_1, defined by Assumptions 2-3 and c_0 given by (13).

The main idea of the proof of Theorem 1 is to show that the set $\mathcal{A} \subset \mathcal{X} \times (\mathcal{C} \cup \mathcal{D})$, defined as

$$\mathcal{A} := \left\{(x,\xi) \in \mathcal{X} \times \Xi : (x,\zeta_0,\zeta_1) \in \Omega,\ z \le c_0,\ q = 0\right\},$$

is globally attractive for (6) and that any solution lying inside a compact subset of $\mathcal{A}$ for a long enough time satisfies (7).

V. Illustrative Example

In the following example we want to exploit the robustness properties of extended Kalman filters (EKF), e.g. [7], [22], with respect to measurement noise, as opposed to high-gain observers (HGO), [14]. We consider, in particular, the perturbed Duffing oscillator

$$\dot{x} = Sx + B\phi(x) + Q_c u, \qquad y = Cx + R_c v, \tag{20}$$

where $x = (x_a, x_b)^T \in \mathbb{R}^2$ is the state, $y \in \mathbb{R}$ is the output, $\phi(x) = x_a - x_a^3$, and (S,B,C) is a triplet in prime form, namely $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = (0,1)^T$, $C = (1,0)$. In (20), $u \in \mathbb{R}^2$ is some unmeasured disturbance affecting the plant and $v \in \mathbb{R}$ is the measurement noise. Moreover, $Q_c = Q_c^T > 0$ and $R_c > 0$ are supposed to be known. Note that for initial condition $x(0)$ ranging in some given compact set $\mathcal{X}^\circ \subset \mathbb{R}^2$, and for any (small) bounded input $u(t)$, there exists a set $\mathcal{X} \supseteq \mathcal{X}^\circ$, which is forward invariant for (20). It can be computed numerically.

By following [7], [22], the local asymptotic observer in (3) is an EKF with state $\zeta_0 = (\hat{x}_0^T, \mathrm{vec}(P))^T$ and dynamics given by

$$\begin{array}{rcl} \dot{\hat{x}}_0 &=& S\hat{x}_0 + B\phi(\hat{x}_0) + K(y - C\hat{x}_0) \\ \dot{P} &=& A(\hat{x}_0)P + PA(\hat{x}_0)^T - PC^T R_c^{-1} CP + Q_c \\ K &=& PC^T R_c^{-1} \end{array} \tag{21}$$

where $\hat{x}_0 = (\hat{x}_{0a}, \hat{x}_{0b}) \in \mathbb{R}^2$ is the estimate, $P \in \mathbb{R}^{2\times 2}$, Q_c, R_c are chosen according to (20), and $A(\hat{x}_0) := S + B\partial\phi(\hat{x}_0)/\partial x$. Since system (20) is uniformly observable, the matrix $P(t)$ is guaranteed to be bounded for all $t \geq 0$ and convergence of observer (21) can be established with the Lyapunov function $V_0 = (\hat{x}_0 - x)^T P^{-1}(\hat{x}_0 - x)$ for $|\hat{x}(0) - x(0)|$ small enough, see [7], [22]. The global observer in (4) is given by a HGO designed with state $\zeta_1 = \hat{x}_1$ and dynamics given by

$$\dot{\hat{x}}_1 \,=\, S\hat{x}_1 + B\phi_s(\hat{x}_1) + D_\kappa L(y - C\hat{x}_1) \tag{22}$$

where $\hat{x}_1 = (\hat{x}_{1a}, \hat{x}_{1b}) \in \mathbb{R}^2$ is the observer state, $L \in \mathbb{R}^{2\times 1}$ is any matrix chosen such that $S - LC$ is Hurwitz, $D_\kappa = \mathrm{diag}(\kappa, \kappa^2)$, where $\kappa \geq 1$ is the high-gain parameter, and the function $\phi_s(\cdot) := \mathrm{sat}_r(\phi(\cdot))$ where sat_r is any (continuous) saturation function with saturation level $r \geq \max_{x\in\mathcal{X}} |\phi(x)|$. Observer (22) is a semi-global asymptotic observer for (20) when $\kappa \geq 1$ is large enough. This can be established with the Lyapunov function $V_1 = \hat{e}_1^T H \hat{e}_1$, where H is solution of $H(S-LC) + (S-LC)^T H = -I$, and $\hat{e}_1 = \kappa D_k^{-1}(\hat{x}_1 - x)$, see [14].

We construct the hybrid observer (16)-(18) with $\gamma(\hat{x}_1) := (\hat{x}_1, \mathrm{vec}(I_{2\times 2}))^T$, $\rho(y, C\hat{x}_1) := |y - C\hat{x}_1|$, and $\theta(\hat{x}_0, \hat{x}_1) := |\hat{x}_0 - \hat{x}_1|$, for $\hat{x}_1, \hat{x}_0 \in \mathbb{R}^2$. In the simulations, we suppose that u and v are generated by

$$\begin{array}{rclcl} \dot{w}_i &=& \Omega_i w_i, && i \in \{1,2,3\}, \\ u &=& Q_1 w_1 + Q_2 w_2, && v = Q_3 w_3 \end{array}$$

where $\Omega_i = (0\ \omega_i, -\omega_i\ 0)$ and the matrices $Q_1, Q_2 \in \mathbb{R}^{2\times 2}$ and $Q_3 \in \mathbb{R}^{1\times 2}$ have unitary norms. We selected $\omega_1 = 5$, $\omega_2 = 27$, $\omega_3 = 32$, $Q_1 = (0.99\ \ 0.12, -0.12\ \ 0.99)$, $Q_2 = (0.85\ 0.53, -0.53\ 0.85)$, $Q_3 = (0.65\ 0.76)$, $Q_c = \mathrm{diag}(0.1, 0.1)$, $R_c = 0.5$. The initial conditions of (20) range in the compact set $\mathcal{X}^\circ := \{(x_a, x_b) \in \mathbb{R}^2 : x_a^2 + x_b^2 \leq 25\}$ and we have $\mathcal{X} \subset \{(x_a, x_b) \in \mathbb{R}^2 : |x_a| \leq 6, |x_b| \leq 20\}$. This can be computed numerically. According to this bound we selected $L = (1,1)$, $r = 210$ and $\kappa = 10$. Since observer (22) is asymptotic, ε_1 in Assumption 3 is equal to 0 and therefore c_1 in Assumption 5 can be chosen arbitrarily small. We selected $c_0 = 1.4$, $c_1 = 0.7$. Simulations have been made to verify convergence of EKF (21) with such values.

In Figures 1, 2, we show the behaviours of system (20) and hybrid observer (16)-(18) with initial conditions $x(0,0) = (2,3)$, $\hat{x}_0(0,0) = (-3,-5)$, $P(0,0) = I_{2\times 2}$, $\hat{x}_1(0,0) = (2.5, 3.5)$, $z(0,0) = 0$, $q(0,0) = 0$ to show the case of a "wrong initialization". With such initial conditions, 2 jumps occur at $t_1 = 0.25$ and $t_2 = 1.31$ during the simulation. Figure 1 shows the plots of $x(t,j)$, $\hat{x}_0(t,j)$ and $\hat{x}_1(t,j)$, for $(t,j) \in [0,3] \times \{0,1,2\}$, while Figure 2 shows the plot of the error estimate $e(t,j)$ defined as $e := x - \hat{x}$ and the value of the error norm estimator $z(t,j)$, for $(t,j) \in [0,3] \times \{0,1,2\}$. It is possible to appreciate the better performances in steady state $x_0 = (x_{0a}, x_{0b})^T$ of EKF (21) when compared to the estimate $x_1 = (x_{1a}, x_{1b})^T$ of HGO (22), in particular for $\hat{x}_{0b}$ and $\hat{x}_{1b}$. At $(0,0)$, local EKF (21) is far from the system state (20) while global HGO (22) is nearby (20). However, $q(0,0) = 0$ and $z(0,0) = 0$ impose the use of the local observer for $(t,j) \in [0, t_1) \times \{0\}$. After some time the large error between $\hat{x}_0$ and $\hat{x}_1$ is detected by z through the function

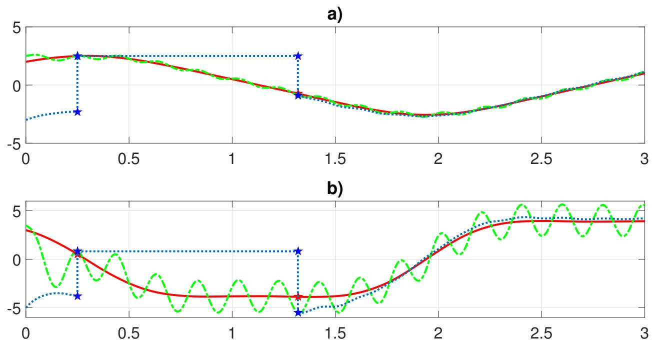

Fig. 1: State $x = (x_a, x_b)$ of Duffing oscillator (20) and components of uniting observer (16)-(18) constructed by combining EKF observer (21), with state $\hat{x}_0 = (\hat{x}_{0a}, \hat{x}_{0b})$, and HGO observer (22), with state $\hat{x}_1 = (\hat{x}_{1a}, \hat{x}_{1b})$. Plot a): x_a, red line; $\hat{x}_{0a}$, blue dotted line; $\hat{x}_{1a}$, dashed green line. Plot b): x_b, red line; $\hat{x}_{0b}$, blue dotted line; $\hat{x}_{1b}$, dashed green line.

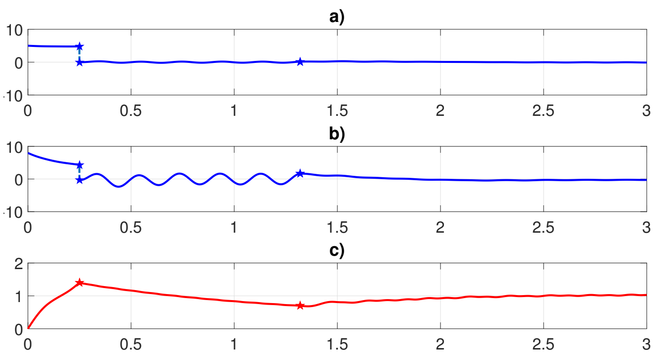

Fig. 2: Estimation error $e = (e_a, e_b) = (x_a - \hat{x}_a, x_b - \hat{x}_b)$ and norm estimator of uniting observer (16)-(18). Plot a): component e_a, blue line. Plot b): component e_b, blue line. Plot c): error-state-norm estimator z, red line.

$\theta(\hat{x}_0, \hat{x}_1)$ and a jump is imposed when $z = c_0$. The local EKF observer is turned off for $(t, j) \in [t_1, t_2) \times \{1\}$. As a consequence, the value of z starts decreasing since the global HGO observer is close to the system state and $\rho(y, C\hat{x}_1)$ is small. Another jump occurs when $z = c_1$. Finally, for $(t, j) \in [t_3, \infty) \times \{2\}$, the local observer is used. The steady-state of the error norm estimator z is different from zero due to the presence of measurement noise which enter through the function $\rho(y, C\hat{x}_1)$.

VI. Conclusion

We investigated the uniting problem [20], [21] in the context of estimation. The main idea is to combine a local asymptotic observer, such as an extended Kalman filter, together with a global (possibly approximate) observer in order to design a hybrid observer guaranteeing global asymptotic estimation while inheriting the good properties of the local estimator when the estimation error is small. The uniting observer design is achieved under a set of prescriptive conditions. In future work, we will propose constructive conditions for relevant classes of nonlinear observers and we will analyse the robustness properties of the proposed uniting observer with respect to measurement noise. We also aim at enlarging the considered class of systems by considering systems with inputs, and relaxing some of the assumptions made on the observers used to derive the hybrid-estimator.

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APPENDIX

Proof of Theorem 1. Throughout the proofs, when we refer to system (6), we mean that observer (5) is given by (16)-(19). First, we prove that every solution to (6) is complete. Then, we will prove the local property and finally the global asymptotic property.

(Completeness of solutions) We apply Proposition 6.10 in [10] for this purpose. First of all, in view of the definition of the sets $\mathcal{C}$, $\mathcal{D}$, and the maps $f(\cdot), \varphi(\cdot), w(\cdot)$ of (6), hybrid system (6) satisfies Assumption 6.5 in [10]. Let $(x,\xi) \in \mathcal{C}\backslash\mathcal{D}$. When (x,ξ) is in the interior[1] of $\mathcal{C}$, $T_{\mathcal{C}}(x,\xi) = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \mathbb{R}_{\geq 0} \times \{0\}$ and $(f(x), \phi(\xi, y)) \in \mathcal{X} \times \mathcal{C} \times \mathbb{R}_{\geq 0} \times \{0,1\}$. When (x,ξ) is not in the interior of $\mathcal{C}$, then necessarily, since $(x,\xi) \notin \mathcal{D}$, $z = 0$ and $q = 0$ in view of (19). In this case, $T_{\mathcal{C}}(x,\xi) = \mathbb{R}^{n_0} \times \mathbb{R}^{n_1} \times \mathbb{R}_{\geq 0} \times \{0\}$. Since the flow map of the z-variable takes non-negative values when $z = 0$ and $q = 0$ in view of (16) and (19), $(f(x), \varphi(\xi, y)) \in T_{\mathcal{C}}(x,\xi)$. As a result, the viability condition (VC) in Proposition 6.10 in [10] holds for any $(x,\xi) \in \mathcal{C}\backslash\mathcal{D}$. On the other hand, $w(\mathcal{D}_0, h(\mathcal{X})) \subseteq \mathcal{C}_1$ and $w(\mathcal{D}_1, h(\mathcal{X})) \subseteq \mathcal{C}_0$, hence $w(\mathcal{D}) \subseteq \mathcal{C}$. As a consequence (x,ξ) is either complete or explode in finite time. When $\xi \in \mathcal{C}_0$, the dynamics of ζ_0 and ζ_1 correspond respectively to those of (3) and (4). On the other hand, when $\xi \in \mathcal{C}_1$, we have that $\dot{\zeta}_0 = 0$ and ζ_1 evolves according to the dynamics of (4). As a consequence, since $\mathcal{C} \subseteq \mathcal{Z}_0 \times \mathcal{Z}_1 \times \mathbb{R}_{\geq 0} \times \{0\}$, by definition of $\varphi_0(\cdot), \varphi_1(\cdot)$, no finite escape time phenomenon may occur. We conclude that all solutions are complete by application of Proposition 6.1 in [10].

(Local property) Given $\delta \in (0,1)$, let $\mathcal{A}_\delta \subset \mathcal{X} \times (\mathcal{C} \cup \mathcal{D})$ be defined as

$$\mathcal{A}_\delta := \{(x,\xi) \in \mathcal{X} \times \Xi : \alpha(x,\xi) \leq \delta\}. \tag{23}$$

By construction, we have $\mathcal{A}_\delta \subset \mathcal{A} \subseteq \mathcal{X} \times \mathcal{C}_0$ since $\delta < 1$. The local property follows then by applying the forthcoming claim.

Claim 1: Any solution (x,ξ) to (6) starting in $\mathcal{A}_\delta$ is such that: (i) $\mathrm{dom}(x,\xi) = [0,\infty) \times \{0\}$; (ii) $(x(t,0), \xi(t,0)) \in \mathcal{A}$ for all $t \in [0,\infty)$. □

In view of item (ii) of Claim 1, $(x(t,j), \xi(t,j))$ lies in $\mathcal{A}$ for all $(t,j) \in [0,\infty) \times \{0\}$, and hence in $\mathcal{C}_0$ as long as $(x(0,0), \xi(0,0)) \in \mathcal{A}_\delta$. As a result, since the vector field of ζ_0 in $\mathcal{C}_0$ corresponds to that of the local observer (3), $(x(t,0), M\xi(t,0)) = (\bar{x}(t), \bar{\zeta}_0(t))$ for all $t \in [0,\infty)$ where $(\bar{x}, \bar{\zeta}_0)$ is a solution to (1), (3).

(Global asymptotic property) The asymptotic property follows by combining the next three claims. The first claim shows that the discrete variable q has to take the value 0 at some (hybrid) time. In other words, any (x,ξ) solution to (6) enters the set $\mathcal{C}_0$ at some (hybrid) time. The second claim states that, if the discrete variable q remains equal to 0 after (a sufficiently long) time, i.e. (x,ξ) remains in $\mathcal{C}_0$ for all sufficiently large time, then (7) holds. Finally, the third claim states that the discrete variable cannot switch back and forth between $q = 0$ and $q = 1$, namely a solution cannot switch back and forth between $\mathcal{C}_0$ and $\mathcal{C}_1$ forever.

Claim 2: For any solution (x,ξ) to (6), there exists $(t,j) \in \mathrm{dom}(x,\xi)$ such that $q(t,j) = 0$. □

Claim 3: Let (x,ξ) be a solution to (6). If there exists a hybrid time $(\bar{t},\bar{j}) \in \mathrm{dom}(x,\xi)$, such that $q(t,j) = 0$ for all $(t,j) \in \mathrm{dom}(x,\xi), (t,j) \geq (\bar{t},\bar{j})$, then (7) holds. □

Claim 4: Let (x,ξ) be a solution to (6). There does not exist a nondecreasing sequence of hybrid times $((t_n, j_n)_{n\in\mathbb{N}}) \in \mathrm{dom}(x,\xi)$, such that we have

$$q(t_{2n}, j_{2n}) = 0 \qquad q(t_{2n+1}, j_{2n+1}) = 1 \tag{24}$$

for all $n \in \mathbb{N}$. □

We are in the position to combine the previous claims to prove the global asymptotic property. Pick any solution (x,ξ) to (6). In view of Claim 4, there exists a hybrid time $(t_1, j_1) \in \mathrm{dom}(x,\xi)$ such that either $q(t,j) = 0$ or $q(t,j) = 1$ for all $(t,j) \in \mathrm{dom}(x,\xi)$ with $(t,j) \geq (t_1,j_1)$. By applying Claim 2, we know that it cannot exist a hybrid time $(t_2,j_2) \in \mathrm{dom}(x,\xi)$, $(t_2,j_2) \geq (t_1,j_1)$ such that, $q(t,j) = 1$, for all $(t,j) \in \mathrm{dom}(x,\xi)$ such that $(t,j) \geq (t_2,j_2)$. As a result, the solution (x,ξ) is such that $q(t,j) = 0$ for all $(t,j) \in \mathrm{dom}(x,\xi), (t,j) \geq (t_1,j_1)$. As a consequence, by applying Claim 3 we conclude that (7) holds. This concludes the asymptotic property. ■

Proof of Claim 1. Let $\delta \in (0,1)$ and $(x,\xi) \in \mathcal{A}_\delta$ defined as in (23). By definition of $\mathcal{A}_\delta$, $q = 0$ and $|z| < c_0$. Hence $\mathcal{A}_\delta \subseteq (\mathcal{C}_0\backslash\mathcal{D})$. Let (x,ξ) be now a solution to (6) initialized in $\mathcal{A}_\delta$. In order to prove the claim, we first need to prove (x,ξ) experiences no jump on its hybrid time domain. We proceed by contradiction and we assume that there exists $\bar{t} > 0$ such that $(\bar{t},0), (\bar{t},1) \in \mathrm{dom}(x,\xi)$. Hence $q(\bar{t},1) = 1$ in view of (17) and (19). Note that the components of the flow map corresponding to ζ_0 and ζ_1 are equal φ_0 and φ_1, respectively, in $\mathcal{C}_0$. As a consequence, the properties of Assumptions 2, 3 and 5 hold for the ζ_0 and the ζ_1 component of (x,ξ) on $[0,\bar{t}) \times \{0\}$. On the other hand, since $(x(0,0), \xi(0,0)) \in \mathcal{A}_\delta$, then $(x(0,0), \xi(0,0)) \in \Omega$ in view of (11) and (23). As a consequence, by using item 2) of Assumption 2 and item 1) of Assumption 3, we have $(x(t,0), \zeta_0(t,0), \zeta_1(t,0)) \in \Omega$ for all $t \in [0,\bar{t})$. Moreover, by using (13) and item 2) of Assumption 5, we also derive that $\theta(\hat{x}_0(t,0), \hat{x}_1(t,0)) \leq c_0$ and $\rho(h(x(t,0)), h(\psi_1(\zeta_1(t,0)))) < c_1$ for all $t \in [0,\bar{t})$. Now recall that $\mathcal{A}_\delta \subset \mathcal{C}_0$. Then the dynamics of z is given by, in view of (16),

$$\dot{z} = -z + \max\{\rho(y, h(\hat{x}_1)), \theta(\hat{x}_0, \hat{x}_1)\} \tag{25}$$

on $[0,\bar{t}) \times \{0\}$. We introduce the following notation for the sake of convenience

$$\begin{array}{rcl} \rho(t,j) & = & \rho(y(t,j), h(\hat{x}_1(t,j))), \\ \theta(t,j) & = & \theta(\hat{x}_0(t,j), \hat{x}_1(t,j)). \end{array} \tag{26}$$

In view of the definition $\mathcal{A}_\delta$ in (23), we have $0 \leq z(0,0) \leq \delta c_0$. Note that since $\theta(\cdot), \rho(\cdot)$ are functions taking values in $\mathbb{R}_{\geq 0}$, we have $z(t,0) \geq 0$ for all $t \in [0,\bar{t})$. Moreover, by

[1]where $T_{\mathcal{C}}$ denote the tangent cone of $\mathcal{C}$, Definition 5.12 in [10]

recalling Assumptions 4, 5 and that $c_1 < c_0$, we compute from (25), (26)

$$\begin{aligned} z(t,0) =&\ e^{-t}z(0,0) + \int_0^t e^{-(t-s)}\max\{\rho(s,0),\theta(s,0)\}ds \\ \leq&\ e^{-t}\delta c_0 + \left[1-e^{-t}\right]c_0 \\ =&\ c_0 + e^{-t}(\delta-1)c_0 \end{aligned}$$

for all $t\in[0,\bar{t})$. Since $0<\delta<1$, the last inequality implies $z(\bar{t},0) < c_0$ for any $\bar{t}<\infty$. Therefore $z(t,0)\in\mathcal{C}_0\backslash\mathcal{D}_0$ for all $t\in[0,\bar{t}]$. As a result, z cannot jump at $(\bar{t},0)$, we have attained a contradiction. Therefore $\bar{t}=\infty$ (recall that any solution is complete, as already proved). This proves the item (i) of the claim.

To prove the item (ii), note that in view of Assumptions 2 and 3, the sets Ω_0 and Ω_1 are forward invariant for systems (1), (3) and (1), (4), respectively. Since $\mathcal{A}_\delta\subset\Omega\times\mathbb{R}_{\geq 0}\times\{0\}$, we conclude that $(x(t,0),\zeta_0(t,0),\zeta_1(t,0))\in\Omega$ for all $t\in[0,\infty)$. Since $z(t,0)\leq c_0$ for all $t\in[0,\infty)$, and $\mathcal{A}_\delta\subset\mathcal{A}$ we finally conclude that $(x(t,0),\xi(t,0))$ lies in the set $\mathcal{A}$ for all $t\in[0,\infty)$. ■

Proof of Claim 2. We proceed by contradiction. Assume that there exists a solution (x,ξ) to (6)such that that

$$q(t,j)=1 \qquad \forall\,(t,j)\in\mathrm{dom}(x,\xi)\,. \tag{27}$$

We will show that the solution (x,ξ) flowing in $\mathcal{C}_1$ must enter $\mathcal{D}_1$ in finite time violating (27) since $q^+=1$.

In this case $\mathrm{dom}(x,\xi)=[0,\infty)\times\{0\}$. Recall that when ξ flows in $\mathcal{C}_1$, its dynamics are given by

$$\dot{\zeta}_0=0,\quad \dot{\zeta}_1=\varphi_1(\zeta_1,u,y),\quad \dot{z}=-z+\rho(y,h(\hat{x}_1)).$$

According to item 1) of Assumption 3, there exists a $t_1>0$ such that $V_1(x(t,0),\zeta_1(t,0))\leq\varepsilon_1$ for all $t\in[t_1,\infty)$. Therefore, by using item 2) of Assumption 5, we have $\rho(h(x(t,0)),h(\hat{x}_1(t,0)))<c_1$ for all $t\in[t_1,\infty)$. By recalling that z evolves according to (15) in this case, we obtain, for $t\in[t_1,\infty)$

$$|z(t,0)|\leq|z(t_1,0)|e^{-t+t_1}+\max_{s\in[t_1,t]}\rho(y(s,0),h(\hat{x}_1(s,0)))\,.$$

As a consequence, there exists a $t_2 in[t_1,\infty)$ such that $z(t,0)<c_1$ for all $t\in[t_2,\infty)$. In view of the definitions of $\mathcal{C}_1$ and $\mathcal{D}_1$, we conclude that $\xi(t,0)$ leaves the set $\mathcal{C}_1$ and enters $\mathcal{D}_1$ at $(t_2,0)$. After the jump, q becomes equal to 0. This contradicts (27) and concludes the proof of the claim. ■

Proof of Claim 3. Let (x,ξ) be a solution to (6) such that there exists $(\bar{t},\bar{j})\in\mathrm{dom}(x,\xi)$ such that $q(t,j)=0$ for all $(t,j)\geq(\bar{t},\bar{j})$, namely $(x(t,j),\xi(t,j))\in\mathcal{X}\times\mathcal{C}_0$ for all $(t,j)\geq(\bar{t},\bar{j})$. Suppose then, without loss of generality, that $(\bar{t},\bar{j})=(0,0)$. In view of the expression of the flow map in $\mathcal{C}_0$, we can apply Assumptions 2-5 to (x,ξ). In particular, by using Assumption 3, there exists a time $T<\infty$ such that $V_1(x(t,0),\zeta_1(t,0))\leq\varepsilon_1$ for all $t\in[T,\infty)$. Without loss of generality, we assume that $T=0$, namely suppose that the solution (x,ξ) to (6) starts in the set $\{(x,\xi)\in\mathcal{X}\times\mathcal{C}_0 : (x,\zeta_1)\in\Omega_1\}$ with Ω_1 defined as in (10), and satisfies $(x(t,0),\xi(t,0))\in\mathcal{X}\times\mathcal{C}_0$ for all $t\in[0,\infty)$.

When flowing in $\mathcal{C}_0$, the z-dynamics is given by

$$z(t,0)\ =\ e^{-t}z(0,0)+\int_0^t e^{-(s-t)}\max\{\rho(s,0),\theta(s,0)\}ds$$

where we use the notation in (26). Since $z(t,0)\in\mathcal{C}_0$ for all for all $t\in[0,\infty)$, it must satisfy $z(t,0)\leq c_0$ for all $t\in[0,\infty)$, which implies

$$\int_0^t e^{-(s-t)}\rho(s,0)ds\ \leq\ c_0\,,\quad \int_0^t e^{-(s-t)}\theta(s,0)ds\ \leq\ c_0\,. \tag{28}$$

In view of Assumption 5, since $(x(t),\zeta_1(t))\in\Omega_1$ for all $t\in[0,\infty)$, we have $\rho(y(t),h(\hat{x}_1(t,0)))\leq c_1<c_0$ for all $t\in[0,\infty)$. As a consequence the first inequality in (28) is trivially satisfied.

Suppose, that there exists $t_1>0$ such that $\theta(\hat{x}_0(t_1,0),\hat{x}_1(t_1,0))\leq c_0+\frac{1}{2}$. Then, by definition of c_0 and by applying item 2) of Assumption 4, $(x(t_1,0),\zeta_0(t_1,0))$ is in the set $\mathcal{B}_0$, introduced in Assumption 2. As a consequence, in view of item 1) of Assumption 2, the limit (7) holds. Consider now the opposite case in which

$$\theta(\hat{x}_0(t,0),\hat{x}_1(t,0))\geq c_0+\tfrac{1}{2} \tag{29}$$

for all $t\in[0,\infty)$. The inequality (29) implies, in view of (28),

$$\begin{aligned} c_0\geq&\quad\int_0^t e^{-(s-t)}\theta(s,0)ds\geq\int_0^t e^{-(s-t)}(c_0+\tfrac{1}{2})ds \\ \geq&\quad(c_0+\tfrac{1}{2})(1-e^{-t}) \end{aligned}$$

implying $e^{-t}\geq\frac{1}{1+2c_0}$ for all $t\in[0,\infty)$, which cannot hold since $\lim_{t\to\infty}e^{-t}=0$. We deduce that the solution (x,ξ) may not satisfy at the same time (28) and (29) for all $t\in[0,\infty)$. In particular, either a jump occur (contradicting our assumptions), either there must exists a $t_1\in[0,\infty)$ such that $\theta(\hat{x}_0(t_1,0),\hat{x}_1(t_1,0))\leq c_0+\frac{1}{2}$, which implies that (x,ζ_0) is in $\mathcal{B}_0$ in view of Assumption 4. Hence the solution (x,ξ) satisfies (7) in view of Assumption 2. This conclude the proofs. ■

Proof of Claim 4. Let (x,ξ) be a solution to (6). We proceed again by contradiction. Assume that there exists a non-decreasing sequence of hybrid times $(t_n,j_n)_{n\in\mathbb{N}}\in\mathrm{dom}(x,\xi)$, with $(t_n,j_n)\in\mathrm{dom}(x,\xi)$ for all $n\in\mathbb{N}$, such that (24) holds for all $n\in\mathbb{N}$. Without loss of generality and to simplify the notation, we assume that there is no jump between two elements of this sequence and that $j_n=n$. Due to the expression of the function $w(\cdot)$ and the definitions of the sets $\mathcal{C}$ and $\mathcal{D}$, for all $n\in\mathbb{N}$ we have that (x,ξ) to (6) flows in $\mathcal{X}\times\mathcal{C}_0$ between $(t_{2n},2n)$ and $(t_{2n+1},2n)$, and flows in $\mathcal{X}\times\mathcal{C}_1$ between $(t_{2n+1},2n+1)$ and $(t_{2n+2},2n+1)$. We want to show that this behaviour cannot occur for n large since, after a time large enough, the components (x,ζ_0,ζ_1) of (x,ξ) will enter the set Ω, which is forward invariant.

In view of (17), after a jump we have $\zeta_1(t_n,n+1)=\zeta_1(t_n,n)$ for any $n\in\mathbb{N}$, namely the value of ζ_1 changes only during flows but not at jumps. Therefore, in view of Assumption 3, there exists a $N\in\mathbb{N}$ sufficiently large, such that, for any $n\geq N$, we have $V_1(x(t,j),\zeta_1(t,j))\leq\varepsilon_1$ for all

$(t,j) \in [t_{2n}, t_{2n+1}) \times \{2n\}$ and all $(t,j) \in [t_{2n+1}, t_{2n+2}) \times \{2n+1\}$.

Consider a solution $(\bar{x}, \bar{\xi})$ to (6) starting at the time $(t_{2N+1}, 2N+1)$ with initial conditions equal to the value of (x, ξ) at $(t_{2N+1}, 2N+1)$. In view of previous arguments, between $(t_{2N+1}, 2N+1)$ and $(t_{2N+2}, 2N+1)$, the solution $(\bar{x}, \bar{\xi})$ flows in $\mathcal{X} \times \mathcal{C}_1$. By definition of the flow map, ζ_0 is constant during this interval. Then at time $(t_{2N+2}, 2N+1)$ we have a jump. By definition of the jump map and by using Assumption 1,

$$\psi_0(\zeta_0(t_{2N+2}, 2N+2)) = \psi_1(\zeta_1(t_{2N+2}, 2N+2))$$

and therefore, by using Assumption 4,

$$\theta(\hat{x}_0(t_{2N+2}, 2N+2), \hat{x}_1(t_{2N+2}, 2N+2)) = 0 \,,$$

with $\hat{x}_0 = \psi_0(\zeta_0)$ and $\hat{x}_1 = \psi_1(\zeta_1)$. As a consequence, by using item 2) of Assumption 3,

$$(x(t_{2N+2}, 2N+2), \zeta_0(t_{2N+2}, 2N+2)) \in \Omega_0$$

with Ω_0 defined in (8). Consequently, by applying Assumptions 2 and 3

$$(x(t, 2N+2), \zeta_0(t, 2N+2), \zeta_1(t, 2N+2)) \in \Omega$$

for all for all $t \in [t_{2N+2}, t_{2N+3})$. The latter implies, by using Assumption 4,

$$\theta(\hat{x}_0(t, 2N+2), \hat{x}_1(t, 2N+2)) \leq c_0$$

for all $t \in [t_{2N+2}, t_{2N+3})$. In view of Assumption 3

$$\rho(y(t_{2N+2}, 2N+2), h(\hat{x}_1(t_{2N+2}, 2N+2))) \leq c_1 \,.$$

which implies, since $c_1 < c_0$

$$\rho(y(t, 2N+2), h(\hat{x}_1(t, 2N+2))) \leq c_1 \,.$$

for all $t \in [t_{2N+2}, t_{2N+3})$. At time $(t_{2N+2}, 2N+1)$, the solution (x, ξ) is in $\mathcal{D}_1$, implying $z(t_{2N+2}, 2N+2) = z(t_{2N+2}, 2N+1) \leq c_1$. As a consequence, by evaluating the z-dynamics according to (25) between $(t_{2N+2}, 2N+2)$ and $(t_{2N+3}, 2N+2)$ we have

$$\begin{aligned} |z(t, 2N+2)| = \;& e^{-t}|z(t_{2N+2}, 2N+2)| \\ & + \int_0^t e^{-(t-s)} \max\{\rho(s,0), \theta(s,0)\} ds \\ \leq \;& e^{-t} c_1 + \left[1 - e^{-t}\right] c_0 \; < \; c_0 \,, \end{aligned}$$

for $t \in [t_{2N+2}, t_{2N+3}]$, where we used the notation in (26). As a consequence, $z(t_{2N+3}, t_{2N+2}) < c_0$ contradicting the fact that at time $(t_{2N+3}, 2N+2)$ the solution (x, ξ) enters $\mathcal{D}_0$. This concludes the proof of the claim. ■