Idle speed control using linear time varying model predictive control and discrete time approximations

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***Abstract*— This paper addresses the problem of idle speed control of hydrogen fueled internal combustion engine (H2ICE) using model predictive control (MPC) and sampled data control (SDC) theories. In the first step, results from SDC theory and a version of MPC are collectively employed to obtain a rigorously developed new generic control strategy. Here, a controller, based on a family of approximate discrete time models, is designed within a previously proposed framework to have guaranteed practical asymptotic stability of the exact (unknown) discrete time model. Controller design, accomplished using MPC theory, is facilitated by successive online linearizations of the nonlinear discrete time model at each sampling instant. In the second step, the technique is implemented in the idle speed control of hydrogen internal combustion engine (H2ICE). Various conditions under which this theory can be implemented are presented and their validity for idle speed control problem are discussed. Simulations are presented to illustrate the effectiveness of the control scheme.**

I. INTRODUCTION

Idle speed control (ISC) of internal combustion engines (ICEs) is a crucial component of overall engine control system. An average passenger car spends about one third of its on board fuel in normal city driving during idling [1]. Thus it is important to optimize the vehicle operation at idle to obtain best fuel economy and minimize emissions.

One of the significant developments in the ICE technologies is the ultra-lean burn engine. H2ICEs correspond to this class of engines and are the focus of this paper for the development of ISC scheme. Ultra-lean burn engines, as the name suggests, use a leaner mixture of air to fuel in comparison to stoichiometric proportions and have the potential to simultaneously deliver much improved fuel economy and reduction in emissions [2]. The main aim in ISC is to maintain the engine at a predefined set-point while rejecting any load disturbances. While doing so maximum fuel economy and minimization of emissions should be ensured. Simultaneously, the constraints associated with the control inputs must be taken into the consideration, which are present due the engine hardware and other engine control system considerations. Thus, ISC of a hydrogen engine is a multivariable problem subject to constraints and hence model predictive control theory is an appropriate choice [5], [3], [4].

In order to develop a suitable version of MPC theory for ISC it is necessary to discretize the continuous time control oriented engine model. However, due to the nonlinear nature of the control oriented model, exact discretization is usually not feasible and the usual practice is to obtain an approximate discrete time model and then use this model to design control schemes. Reference [6] shows that it is possible to have a family of controllers which asymptotically stabilizes approximate models but fails to stabilize the exact model. The paper addresses this problem by deriving sufficient checkable conditions guaranteeing the practical asymptotic stabilizability of the exact discrete time model using the controller based on a family of approximate discrete time models. However, the publication is of prescriptive nature and does not cover a systematic way to develop a controller within its framework. In this paper, we build on the existing results on SDC by developing a receding horizon controller within the framework proposed in [6].

In addition, one of the factors that limits the practical usability of MPC schemes to real systems is the computational burden due to nonlinear MPC and should be adequately addressed. This motivates the study of alternative MPC formulations which require solution of simpler and faster optimization algorithms so that real time ISC can be facilitated. In this paper, we counter the computational issues by successive online linearizations of the approximate discrete time plant about the current trajectory to obtain a linear time varying model predictive control (LTV-MPC) strategy. Similar control strategies have previously been proposed where knowledge of the *exact* discrete time model is assumed ([7], [8]). As pointed out earlier, exact discrete time models are generally unobtainable for nonlinear systems. Hence, it is necessary to develop the family of LTV-MPC controllers within a framework which provides appropriate stability guarantees of exact model under the controllers based on approximations models. In [9], the framework of [6] is used to develop finite and infinite horizon control schemes, results of which are extended in [12] where a version of classical MPC formulation is proposed within the setup of [6] but does not address the issues of computational burden.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

A. Engine model

Consider the following description of the general class of continuous time nonlinear systems:

$$\dot{x} = f(x,u), \qquad x(0) = x_0 \tag{1}$$

$$y = h(x) \tag{2}$$

where, $x \in \Re^n$, $u \in \mathcal{U} \subset \Re^m$, $\mathcal{U}$ is compact, $f : \Re^n \times \mathcal{U} \to \Re^n$, with $f(0,0) = 0$.

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Our aim is to develop appropriate MPC control theory to solve the idle speed control problem of H2ICE so as to maintain the engine speed, N_e, to a predefined idle speed set-point. In the context of reduced order control oriented engine model of H2ICE for ISC , the state vector $x(t) = [P_{im}, N_e]^T$ includes intake manifold pressure (P_{im} $[Pa]$) and engine speed (N_e $[RPM]$). The vector of inputs $u = [u_{th}, u_{SA}, u_{\lambda}, T_{load}]^T$ where throttle area (u_{th} $[m^2]$), spark angle in degrees before top dead center (u_{SA}) and air to fuel ratio (u_{λ}) are the control inputs while T_{load} represents the load fluctuations. A systematic procedure for reducing the order of control oriented engine models can be found in [13]. The dynamics of P_{im} are governed by manifold filling and emptying dynamics while those of N_e are based on the engine torque. Accordingly, in the context of engine idle speed control model, the function $f(.,.)$ in the right hand side of equation (1) takes the following form:

$$f(x,u) = \begin{bmatrix} \left(\frac{\gamma R T_{im}}{V_{im}}\right)\left[\dot{m}_{im_{in}}(x,u) - \dot{m}_{im_{out}}(x,u)\right] \\ \frac{1}{J_e}\left(T_e(x,u) - T_{load}\right) \end{bmatrix} \tag{3}$$

where, $\dot{m}_{im_{in}}$, $\dot{m}_{im_{out}}$, η_{vol} and T_e denote the mass-flows in and out of the intake manifold, volumetric efficiency and engine brake torque, respectively and are defined as follows:

$$\dot{m}_{im_{in}} = \frac{P_{amb}}{\sqrt{RT_{amb}}} u_{th} \Psi\left(\frac{P_{im}}{P_{amb}}\right) \tag{4}$$

$$\dot{m}_{im_{out}} = \frac{P_{im} V_d n_{cyl} \eta_{vol} N_e}{4\pi R T_{im}} \tag{5}$$

$$\eta_{vol} = \eta_{vol_1}(P_{im}) \times \eta_{vol_2}(N_e) \tag{6}$$

$$\begin{aligned} T_e = {} & 136.14 - 0.002P_{im} + 111.1u_{\lambda} - 6.2u_{SA} \\ & -2.1N_e - 1.9u_{\lambda}^2 - .16u_{SA}^2 + .86\times 10^{-7}P_{im}^2 \\ & -.0025P_{im}u_{\lambda} - .55\times 10^{-5}P_{im}u_{SA} \\ & +.17\times 10^{-3}P_{im}N_e + 3.5u_{\lambda}u_{SA} - 2.34u_{\lambda}N_e \\ & +.09u_{SA}N_e \end{aligned} \tag{7}$$

where,

$\eta_{vol_1}(P_{im}) = 1 + \frac{1}{r_c}\left[1 - \left(\frac{P_{amb}}{P_{im}}\right)^{\frac{1}{\gamma}}\right]$; $\eta_{vol_2}(N_e) = \gamma_0 + \gamma_1 N_e$;

$$\Psi\left(\frac{P_{im}}{P_{amb}}\right) = \begin{cases} \sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}, & \frac{P_{im}}{P_{amb}} \le 0.5283 \\ \sqrt{\frac{2\gamma}{\gamma-1}\left(\left(\frac{P_{im}}{P_{amb}}\right)^{1.4} - \left(\frac{P_{im}}{P_{amb}}\right)^{1.7}\right)}, & \frac{P_{im}}{P_{amb}} > 0.53. \end{cases}$$

Expressions of $\dot{m}_{im_{in}}$ and $\dot{m}_{im_{out}}$ in (4)-(5) are based on [11] and, volumetric efficiency and torque regression in (6) and (7) are obtained by using idle speed data which was gathered from a six cylinder hydrogen fueled engine. The values of the constants used in expression (3)-(5) are summarized in Table I.

Next, we propose an idle speed control approach for hydrogen engine described by (3)-(7). The controller is to be based on approximate discrete time models while ensuring the stability of the exact (unknown) discrete time model. Simultaneously, the new control algorithm should alleviate the computational complexity so that the real time implementations of this theory may be facilitated.

TABLE I

Inertia, J_e	0.15 kg-m^2
Ratio of specific heats, γ	1.38
Gas constant, R	287.058 Jkg^{-1}K^{-1}
Intake manifold volume, V_{im}	4×10^{-3} m^3
V_d	7.6902×10^{-4} m^3
n_{cyl}	6
Compression ratio, r_c	10.3
γ_0	0.43042
γ_1	0.0018766
P_{amb}, T_{amb}	101325 Pa, 289 K

In order to preserve and emphasize the generality, the new control theory is developed for the general class of nonlinear systems described by (1)-(2). Then, in sections IV and V its implementation to address the important problem of ISC of H2ICE is discussed.

***Assumption** 1: $f(.,.)$ is continuous and for any pair of positive numbers $\left(\Delta', \Delta''\right)$, $\exists\, L_f = L_f\left(\Delta', \Delta''\right)$ such that*

$$\|f(x,u) - f(y,u)\| \le L_f\|x-y\|, \tag{8}$$

$\forall x, y \in \mathcal{B}_{\Delta'}$, $u \in \mathcal{B}_{\Delta''}$.

***Remark** 1:* It is easy to see that the Assumption 1 holds for idle speed control engine model of (1)-(2).

Assumption 1 guarantees the existence of a unique solution for (1). As a result, the exact discrete time nonlinear model for the original system (1) can be expressed as

$$\Sigma_{Exact}: \quad x_{k+1} = F^E_{T,h}(x_k, u_k) \tag{9}$$

where, $F^E_{T,h}(x,u) := \phi^E(T,x,u)$, $T > 0$ denotes the sampling period and ϕ^E represents the solution of (1).

Even though existence of Σ_{Exact} can be guaranteed (through Assumption 1), in the case of nonlinear systems establishing Σ_{Exact} is generally a very cumbersome task and is often impossible to accomplish. The usual approach is to compute an approximate discrete time model:

$$\Sigma_{Approx}: \quad x_{k+1} = F^A_{T,h}(x_k, u_k) \tag{10}$$

where, h is the tuning parameter, typically the integration step of the numerical approximation method used to approximate $F^E_{T,h}$. Details on the methods for the numerical solutions of the ordinary differential equations can be found in [10].

This paper considers the case of fixed T while integration step h can take any arbitrary value independent of T.

***Assumption** 2:* Both $F^E_{T,h}$ and $F^A_{T,h}$ are continuous, uniformly in h.

B. Basic definitions and background

Our primary objective is to devise a feedback controller $v^A_{T,h}$ for ISC based on the approximate discrete time model, Σ_{Approx} (given by (10)) which also stabilizes the exact discrete time model, Σ_{Exact}, in an appropriate sense.

To this end, we design the new controller with the framework proposed in [6] which presents sufficient conditions that guarantee the stabilizability of exact discrete time model by the same family of controllers developed on the basis of family of approximate discrete time models, $F^A_{T,h}$. This result is summarized in the form of Theorem 1. Theorem 1 is based on *Properties 1-3* which are presented first.

- *Property 1- Uniformly bounded control*: Given any positive real numbers $(T, \Delta^{'}, \Delta^{''})$, $\exists\ h^* > 0$ such that
$$\underset{\{x\in\mathcal{B}_{\Delta^{'}},\ h\in(0,h^*]\}}{sup} \|v^A_{T,h}\| \le \Delta^{''} \tag{11}$$
- *Property 2- Consistency property*: Given any positive real numbers $(T, \Delta^{'}, \Delta^{''})$, $\exists \gamma \in \mathcal{K}$ and $h^* > 0$ such that
$$\|F^A_{T,h}(x,u) - F^E_T(x,u)\| \le T\gamma(h) \tag{12}$$
$\forall\ (x,u) \in \mathcal{B}_{\Delta'} \times \mathcal{B}_{\Delta''};\ \forall\ h \in (0,h^*]$.
- *Property 3- Stability with continuous Lyapunov function*: Given any positive real numbers $(T, \Delta^{'})$, a family of functions $V_{T,h} : \Re^n \to \Re_{\ge 0}$, functions $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ and a positive definite function $\sigma_3 : \Re_{\ge 0} \to \Re_{\ge 0}$. Suppose that for any pair of strictly positive numbers (δ_1, δ_2) with $\delta_2 < D\ \exists\ h^* > 0$ and $c > 0$ such that $\forall\ x \in \mathcal{B}_D$, $\forall\ h \in (0,h^*]$
$$\sigma_1\left(\|x\|\right) \le V_{T,h}(x) \le \sigma_2\left(\|x\|\right) \tag{13}$$
$$V_{T,h}\left(F^A_{T,h}\left(x, v^A_{T,h}(x)\right)\right) - V_{T,h}(x) \le -T\sigma_3\left(\|x\|\right) \tag{14}$$
and for all $x_1, x_2 \in B_D - B_{\delta_2}$, with $\|x_1 - x_2\| \le c$ we have
$$\|V_{T,h}(x_1) - V_{T,h}(x_2)\| \le \delta_1 \tag{15}$$
Then, we say that the family (10) is (T, D) stable with a continuous Lyapunov function.

Theorem 1: [6] *Suppose there exists a set of positive real numbers* $(T, \Delta^{'}, \Delta^{''})$ *such that*

1) *Family of closed loop systems* $\left(F^A_{T,h}, v^A_{T,h}\right)$ *is* $(T, \Delta^{'})$ *stable with a continuous Lyapunov function.*
2) *the family of controllers* $v^A_{T,h}$ *is* $(T, \Delta^{'}, \Delta^{''})$*-uniformly bounded;*
3) *the family* $F^A_{T,h}$ *is* $(T, \Delta^{'}, \Delta^{''})$ *consistent with* $F^E_{T,h}$.

Then, there exists $\beta \in \mathcal{KL},\ D_1 \in (0, \Delta^{'})$ *and for any* $\delta > 0$ $\exists\ h^*$ *such that for all* $x_0 \in \mathcal{B}_{D_1}$ *and* $h \in (0,h^*]$, *the solution of the family* $\left(F^E_{T,h}, v^A_{T,h}\right)$ *satisfies*
$$\|\phi^E_{T,h}(k, x_0)\| \le \beta(\|x_0\|, kT) + \delta \tag{16}$$
Let $\Gamma \subset \Re^n$ be a given compact set containing the neighborhood of the origin and $\mathcal{U}_h$ be a family of admissible control inputs defined as $\mathcal{U}_h = \{\mathbf{u^h} = \left(u^h_0, u^h_1, \ldots, u^h_k\right)\}$ parameterized by h.

Definition 1: [12] The exact discrete time model (9) is practically asymptotically controllable (PAC) from Γ to the origin with the parameterized family of control input $\mathcal{U}_h$, if there exists a $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\sigma(\cdot) \in \mathcal{K}$ which are independent of parameters T and h and such that for any r there exists an h^* such that $\forall\ x \in \Gamma$ and $\forall\ h \in (\ 0, h^*\ ]$ there exists a control sequence $\mathbf{u}^h(x) \in \mathcal{U}_h$, such that $u^h_k \le \sigma(\|x\|)$ and the corresponding solution ϕ of (9)
$$\|\phi\left(x, \mathbf{u}^h\right)\| \le max\{\beta\left(\|x\|, kT\right), r\} \tag{17}$$

***Assumption** 3:* $\exists T^*$ *such that the exact discrete time model,* Σ_{Exact}*, is PAC from* Γ *to the origin for all* $T \in (\ 0, T^*\ ]$, *where* $\Gamma \subset \mathcal{B}_{\Delta'}$.

Our aim is to stabilize the exact discrete time model, Σ_{Exact}, using a family of controllers developed via approximate discrete time model. This is only feasible if the solutions of the exact and approximate models are sufficiently close in an appropriate sense (as per *Property 2*). Hence we need a closeness property of the following form.

***Assumption** 4: The family* $F^A_{T,h}(x,u)$ *is* $(T, \Delta^{'}, \Delta^{''})$*-consistent with* $F^E_{T,h}$.

***Remark** 2:* Provided Assumption 1 holds true, then $\exists h^*$ such that $\forall h \in (0,h^*]$ family of approximate models $F^A_{T,h}$ obtained by various consistent Runge-Kutta integration methods satisfies consistency property (details can be found in [10], Lemma 222A, page 131) and therefore truncation error asymptotically decays as the integration step $h \to 0^+$. Therefore, when one step methods are used, Assumption 4 holds without loss of the generality.

Next we formulate a controller design strategy based on Σ_{Approx} while ensuring that the *Properties* 1-3 are satisfied. The work fits well with the existing literature since [6] only provides a framework for controller design while does not elaborate on an appropriate procedure to develop such a controller. A version of receding horizon control from [7] and [8] is considered and conditions are laid to ensure that the *Properties* 1-3 are satisfied to achieve *practical asymptotic stability* of exact discrete time model.

III. Receding horizon control problem

In this section, a version of receding horizon control is proposed which is based on approximate discrete time models and stabilizes the exact discrete time model. The receding horizon control strategy is based on the successive online linearization of the approximate discrete time nonlinear model about the current trajectory to obtain a LTV-MPC strategy. The existing LTV-MPC schemes are based on the premise that an exact discrete time model is available. This is often not true, especially, while working with nonlinear systems. Here we present conditions on LTV-MPC which guarantee the satisfaction of *Properties* 1-3 and discuss the validity of these conditions.

The following cost function, $J_{T,h}\left(N, x_k, \mathbf{u}_k\right)$ is defined:
$$J_{T,h}\left(N, x_k, \mathbf{u}_k\right) = \sum_{i=k}^{k+N-1} Tl_h\left(x_i, u_i\right) + \mathcal{P}\left(x_{k+N}\right) \tag{18}$$
where, $l_h\left(x_i, u_i\right)$ denotes the stage cost with $l_h\left(0, 0\right) = 0$ and $\mathcal{P}\left(x_{k+N}\right)$ represents the terminal cost. Let $\mathbf{u}_k = [u_k, \ldots, u_{k+N-1}]$ be a sequence of inputs over a horizon of length N.

The optimization problem is defined as follows:
$$\underset{\mathbf{u}_k}{min}\ J_{T,h}\left(N, x_k, \mathbf{u}_k\right) \tag{19}$$

subject to

$$x_{i+1,k} = F^A_{T,h}(x_i, u_i); \quad i = k, \ldots, k+N-1 \quad (20)$$
$$x_{i,k} \in \Gamma; \quad i = k+1, \ldots, k+N \quad (21)$$
$$u_{i,k} \in \mathcal{U}; \quad i = k, \ldots, k+N-1 \quad (22)$$
$$x_{k,k} = x(k) \quad (23)$$

where, $x_{i,k}$ for $i = k+1, \ldots, k+N$ denote predicted states given state $x_{k,k}$.

A. Linear Time Varying (LTV) model approximation

Let, $x_0 \in \Gamma$ and $u_0 \in \mathcal{U}$ be an operating point and $\hat{x}_0(k)$ be the state trajectory obtained by applying the input sequence $u(k) = u_0$ to the approximate discrete time model (10) for $k \geq 0$ with $\hat{x}_0(0) = x_0$:

$$\hat{x}_0(k+1) = F^A_{T,h}(\hat{x}_0(k), u(k)) \quad (24)$$
$$u(k) = u_0 \quad (25)$$
$$\hat{x}_0(0) = x_0 \quad (26)$$

Using (24)-(26), a linearized model of the approximate discrete time nonlinear system (10) can be written as follows:

$$\delta x(k+1) = A^{T,h}_{k,0}\delta x(k) + B^{T,h}_{k,0}\delta u(k) \quad (27)$$

where, $\delta x(k) = x(k) - \hat{x}_0(k)$, $\delta u(k) = u(k) - u_0$. The matrices $A^{T,h}_{k,0} \in \Re^{2\times 2}$ and $B^{T,h}_{k,0} \in \Re^{2\times 3}$ are obtained upon the linearization of system (10) and are defined as follows:

$$A^{T,h}_{k,0} = \left.\frac{\partial F^A_{T,h}}{\partial x}\right|_{\hat{x}_0(k),u_0}, \qquad B^{T,h}_{k,0} = \left.\frac{\partial F^A_{T,h}}{\partial u}\right|_{\hat{x}_0(k),u_0}. \quad (28)$$

The linear time varying system (27)-(28) depicts the deviations of the system from the state trajectory $\hat{x}_0(k)$ under the control input u_0. Equation (27) can be rewritten as

$$x(k+1) = A^{T,h}_{k,0}x(k) + B^{T,h}_{k,0}u(k) + d^{T,h}_{k,0}(k) \quad (29)$$

where, $d^{T,h}_{k,0} = \hat{x}_0(k+1) - A^{T,h}_{k,0}\hat{x}_0(x) - B^{T,h}_{k,0}u_0$.

Next, we reformulate the MPC problem (19)-(23) using the system description (29):

$$\min_{\mathbf{u}_k} J_{T,h}(N, x_k, \mathbf{u}_k) \quad (30)$$

subject to

$$x_{i+1,k} = A^{T,h}_{i,k}x_{i,k} + B^{T,h}_{i,k}u_{i,k} + d^{T,h}_{i,k}; \quad i = k, \ldots, k+N-1 \quad (31)$$
$$x_{i,k} \in \Gamma; \; i = k+1, \ldots, k+N \quad (32)$$
$$u_{i,k} \in \mathcal{U}; \; i = k, \ldots, k+N-1 \quad (33)$$
$$x_{k+N,k} \in \Gamma_f \quad (34)$$

where, $0 \in \Gamma_f = \{x : \mathcal{P}(x) \leq \eta\} \subset \Gamma$ with $\eta > 0$. The solution of the optimization problem (30)-(34) at a k^{th} instant is denoted by $\mathbf{u}^*_k$ and the control law is given by $v^A_{T,h} = u^*_k$.

Our objective is to establish conditions under which $\mathbf{u}_k$ obtained by solving (30)-(34) will stabilize Σ_{Exact} in an appropriate sense. In other words, we formulate some properties of LTV-MPC that ascertain satisfaction of *Properties* 1-3.

***Assumption** 5: All elements of two Jacobian matrices $A^{T,h}_{i,k}$ and $B^{T,h}_{i,k}$, defined by (28), are bounded, uniformly in $h \in (\,0, h^*\,]$.*

***Remark** 3:* The satisfaction of Assumption 5 is governed by the boundedness of nominal state trajectory $\hat{x}_0(k)$ which is true for H2ICE due to its dissipative nature.

***Assumption** 6:* We assume that $A^{T,h}_{i,k} = A^{T,h}_{k,k}$ and $B^{T,h}_{i,k} = B^{T,h}_{k,k}$, $\forall\ i = k, \ldots, k+N-1$.

***Assumption** 7: The following hold:*

1) *$\mathcal{P}$ is continuous, there exists a $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that $\gamma_1(\|x\|) \leq \mathcal{P}(x) \leq \gamma_2(\|x\|)$, and for any $\Delta' > 0$ there exists a $L_{\mathcal{P}} > 0$ such that $\|\mathcal{P}(x) - \mathcal{P}(y)\| \leq L_{\mathcal{P}}\|x - y\|$ for all $x, y \in \mathcal{B}_{\Delta'}$.*
2) *l_h is continuous with respect to x and u, uniformly in small h, and for any $\left(\Delta', \Delta''\right) > 0$ there exist $h^* > 0$ and $L_l > 0$ such that $\|l_h(x,u) - l_h(y,u)\| \leq L_l\|x-y\|$ for all $h \in (0, h^*]$, $x, y \in \mathcal{B}_{\Delta'}$ and $u \in \mathcal{B}_{\Delta''}$.*
3) *There exists a $h^* > 0$ and $\varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that the inequality $\varphi_1(\|x\| + \|u\|) \leq l_h(x,u) \leq \varphi_2(\|x\| + \|u\|)$ for all $h \in (0, h^*]$, $x, y \in \Re^n$ and $u \in \mathcal{U}$.*

Stage cost and terminal cost are chosen as quadratic functions of state and control inputs:

$$l_h(x_i, u_i) = \left[x_i^TQx_i + u_i^TRu_i\right] \quad (35)$$
$$\mathcal{P}(x_N) = x_N^TPx_N \quad (36)$$

where, Q, R and P are positive definite matrices. Consequently, satisfaction of Assumption 7 is ensured.

***Remark** 4:* Provided Assumptions 3 and 4 hold, then $\exists\ h^* > 0$ and a feasible control sequence $\mathbf{u}_k$ for all $h \in (0, h^*]$ which solves optimization problem (30)-(34) and hence implies PAC of Σ_{Approx}.

Definition 2: Value of the cost function $J_{T,h}(N, x_k, \mathbf{u}_k)$ at $\mathbf{u}_k = \mathbf{u}^*_k$ where $\mathbf{u}^*_k$ is the solution of the optimization problem (30)-(34) is termed as the value function (denoted by $V^{T,h}_N(x_k)$).

Next we require a stability condition which ensures the stability of the Σ_{Approx} under LTV-MPC approach. Following [5] and [7], we derive the following sufficient condition to ensure the asymptotic stability of Σ_{Approx} under the control law (30)-(34).

Lemma 1: Consider the approximate discrete time model Σ_{Approx} and the MPC controller (30)-(34). Denote $\Delta V(x_k)$ as the difference $V^{T,h}_N(x_k) - V^{T,h}_N(x_{k-1})$ of the value function $V^0_N(x_k)$. Then, $\forall\ h \in (\,0, h^*\,]$

$$V^{T,h}_N(x_k) - V^{T,h}_N(x_{k-1}) \leq -Tl_h\left(x^*_{k-1,k-1}, u^*_{k-1,k-1}\right) \quad (37)$$

if

$$\mathcal{P}\left(F^A_{T,h}\left(x^*_{k+N-1,k-1}, u_{k+N-1,k}\right)\right) + Tl_h\left(\tilde{x}_{k+N-1,k}, u_{k+N-1,k}\right) \leq \mathcal{P}\left(x^*_{k+N-1,k-1}\right) - \Omega\left(\tilde{x}_k, x^*_{k-1}\right) \quad (38)$$

where,

$$\Omega\left(\tilde{x}_k, x^*_{k-1}\right) = T\Sigma_{i=0}^{N-2}\left(\lambda_{max}(Q)\|\tilde{x}_{k+i,k}\|^2 - \lambda_{min}(Q)\|x^*_{k+i,k-1}\|^2\right) \quad (39)$$

with λ_{max} and λ_{min} being the maximum and minimum eigenvalues, respectively.and $\tilde{x}_{(.)}$ is the solution of (31) under a feasible control sequence $\tilde{\mathbf{u}}_{(.)}$.

Proof: From Definition 2, we have

$$V_N^{T,h}(x_{k-1}) = T\Sigma_{i=0}^{N-1}\left(x^{*^T}_{k+i-1,k-1}Qx^*_{k+i-1,k-1} + u^{*^T}_{k+i-1,k-1}Ru^*_{k+i-1,k-1}\right) + \mathcal{P}(x^*_{k+N-1,k-1}) \quad (40)$$

where, $u^*_{k+i-1,k-1}$ for $i = 0, \dots, N-1$ is the optimal solution of (30)-(34) at time $k-1$ and $x^*_{k+i-1,k-1}$ for $i = 1, \dots, N$, is the corresponding state trajectory of system (31). Let us represent the optimal control sequence at time $k-1$ as $\mathbf{u}^*_{k-1} = \left[u^*_{k-1,k-1}, \dots, u^*_{k+N-2,k-1}\right]$. We wish to obtain a feasible control sequence for $x(k)$ and hence an upper bound on $V_N^{T,h}(x_k)$. So, we modify $\mathbf{u}^*_{k-1}$ to obtain a feasible control sequence at time instant k as $\tilde{\mathbf{u}}_k = \left[u^*_{k,k-1}, \dots, u^*_{k+N-2,k-1}, u_{k+N-1,k}\right]$ where $u_{k+N-1,k}$ behaves as a local stabilizing controller within the terminal set Γ_f and $\tilde{x}_{k+i-1}$ for $i = 1, \dots, N$ is the solution of (31). Based on this choice of control sequences, after some mathematical manipulations, we obtain

$$\begin{aligned}\Delta V(x_k) \quad \leq \quad & Tl_h\left(\tilde{x}_{k+N-1,k}, u_{k+N-1,k}\right) \\ & +\Omega\left(\tilde{x}_k, x^*_{k-1}\right) - Tl_h\left(x^*_{k-1,k-1}, u^*_{k-1,k-1}\right) \\ & +\mathcal{P}\left(F^A_{T,h}\left(x^*_{k+N-1,k-1}, u_{k+N-1,k}\right)\right) \\ & -\mathcal{P}(x^*_{k+N-1,k-1}) \end{aligned} \quad (41)$$

where, $\Omega\left(\tilde{x}_k, x^*_{k-1}\right)$ is defined as per (39).

Thus, we have

$$V_N^{T,h}(x_k) - V_N^{T,h}(x_{k-1}) \leq -Tl_h\left(x^*_{k-1,k-1}, u^*_{k-1,k-1}\right) \quad (42)$$

if inequality (38) holds. ■

Next, satisfaction of *Properties 1-3* for the family $\left(F^A_{T,h}, v^A_{T,h}\right)$ under the proposed LTV-MPC scheme is proven to ensure local practical stability of the family $\left(F^E_{T,h}, v^A_{T,h}\right)$. The main result is a consequence of Theorem 1 and is summarized in the form of Corollary 1.

***Corollary** 1:* Suppose Assumptions 1-7 and constraints (38)-(39) of Lemma 1 hold true. Then there exists $\beta \in \mathcal{KL}$ and $D_1 \in (0, \Delta]$ and for any $\delta > 0$ there exists $h^* > 0$ such that for all $x_0 \in \mathcal{B}_{D_1}$ and $h \in (0, h^*]$, the solutions of the family $\left(F^E_{T,h}, v^A_{T,h}\right)$ satisfy the inequality

$$\|\phi^E_k(x_0, k)\| \leq \beta(\|x_0\|, kT) + \delta \quad (43)$$

Proof: To prove Corollary 1 our main task is to demonstrate that Assumptions 1-7 and constraints (38)-(39) of Lemma 1 imply the satisfaction of *Properties* 1-3. Then, (43) directly follows from *Theorem 1*.

- *Property 1* is immediately obvious from the construction of control inputs as they are confined to a compact set for a feasible solution. As per the construction of LTV-MPC control (30)-(34), $x \in \Gamma \subset B_{\Delta'}$. So, for any feasible solution, $x < \Delta'$. Moreover, from the satisfaction of Assumption 3, we obtain $v^A_{T,h}(x) \leq \sigma(\|x\|) \leq \sigma(\Delta') =: \Delta''$.
- *Property 2* is satisfied due to Assumption 4.
- To prove *Property 3,* we consider value function, $V_N^{T,h}(x)$, as a Lyapunov function candidate. Uniform continuity of $V_N^{T,h}(x)$ in h is guaranteed due to Assumptions 2 and 7 (uniform Lipschitz property implies uniform continuity) and thus only the conditions (13) and (14) need to be verified for the family $\left(F^A_{T,h}, v^A_{T,h}\right)$. First we prove (13) holds. To obtain a $\mathcal{K}_\infty$ upper bound on $V_N^{T,h}$ (independent of h), Assumption 7 is used as follows:

$$V_N^{T,h}(x) \leq \mathcal{P}(\|x\|) \leq \gamma_2(\|x\|) =: \sigma_2(\|x\|)$$

A lower bound can be obtained as

$$V_N^{T,h}(x) \geq Tl_h(x, u) \geq T\varphi_1(\|x\|) =: \sigma_1(\|x\|) \quad (44)$$

Finally, the inequality (14) of *Property 3* follows from Lemma 1 upon the satisfaction of constraints (38)-(39). ■

IV. Application to Idle speed control

This section illustrates the application of the approach presented in this paper to the idle speed control of H2ICE described by (3)-(7).

Due to the nonlinear nature of the engine model, (3)-(7), exact discretization is not a viable option and the MPC strategy is devised using an approximately obtained discrete time model. Validity of the Assumptions 1-7 can be easily ascertained which in turn guarantees that the same controller also stabilize the exact discrete time system. The effectiveness of the controller is demonstrated on the original continuous time engine model.

An element of the family of approximate discrete time models, obtained by second order *Runge-Kutta (Heun's method)* (with $T = 2h$), is given as

$$F^A_{T,h}(x_k, u_k) = x_k + \frac{h}{2}\left[f_e(x_k, u_k) + f_e(\alpha, u_k)\right] \quad (45)$$

where, $\alpha = x_k + hf_e(x_k)$.

Let k be the current time instant, x_k be the current state and u_{k-1} be the previous input. We define the following cost function to penalize deviations in engine states and rapid changes in control inputs:

$$\begin{aligned}J_{T,h}(x_k, \Delta\mathbf{u}_k) = \sum_{i=0}^{N} \Big[& (x_{k+i,k} - x_{ref})^T Q\,(x_{k+i,k} - x_{ref}) \\ & +\Delta u^T_{k+i,k} R\Delta u_{k+i,k} \\ & + (u_{SA} - u_{MBT})^T R_{MBT}\,(u_{SA} - u_{MBT})\Big] \end{aligned} \quad (46)$$

where, $x_{ref} = [P_{im,ref}, N_{e,ref}]^T$ is the reference vector, $\Delta u_{k+i,k}$ with $i = 0, \dots, N$ denotes the change in control inputs and u_{MBT} represents the spark angle for MBT (maximum brake torque). A regression for u_{MBT} obtained from the hydrogen engine calibration data is given as follows:

$$u_{MBT} = 9.6281u_\lambda - 0.00011P_{im} + 0.19469N_e - 2.2704$$

Weighting matrices Q and R are chosen to be positive definite diagonal matrices with diagonal entries as $[Q_{P_{im}}, Q_{N_e},]$

and $[R_{th}, R_{SA}, R_\lambda]$, respectively and $R_{MBT} > 0$. Constraints on the states and control inputs for ISC are: $x_{min} = \left[0.52 \times 10^5, 525\right]^T$; $x_{max} = \left[1.01 \times 10^5, 4000\right]^T$; $u_{min} = [0, 10, 2.0]^T$; $u_{max} = [9.9e-05, 50, 5.5]^T$; $\Delta u_{min} = \left[-10^{-5}, -10^{-1}, -10^6\right]^T$; $\Delta u_{min} = \left[10^{-5}, 10^{-1}, 10^{-6}\right]^T$. Horizon length, N, is chosen to be 4.

Based on section III, the linear time varying model predictive controller solves the following optimization problem each sampling instant:

$$\underset{\Delta \mathbf{u}_k}{min}\ J_{T,h}\left(x_k, \Delta \mathbf{u}_k\right) \tag{47}$$

subject to

$$x_{i+1,k} = A_{i,k}x_{i,k} + B_{i,k}u_{i,k} + d_{i,k}; i = k, \ldots, k+N-1 \tag{48}$$
$$x_{min} \leq x_{i,k} \leq x_{max}; i = k+1, \ldots, k+N \tag{49}$$
$$u_{min} \leq u_{i,k} \leq u_{max}; i = k, \ldots, k+N-1 \tag{50}$$
$$\Delta u_{min} \leq u_{i,k} \leq \Delta u_{max}; i = k, \ldots, k+N-1 \tag{51}$$
$$x_{k+N,k} = x_{ref} \tag{52}$$
$$-\Omega\left(\tilde{x}_k - x_{ref}, x^*_{k-1} - x_{ref}\right) \leq 0 \tag{53}$$

where, $\Delta \mathbf{u}_k^T = \left[\Delta u_{k,k}, \ldots, \Delta u_{k+N-1,k}\right]^T$ and $\Omega\left(\tilde{x}_k - x_{ref}, x^*_{k-1} - x_{ref}\right)$ is based on the stability condition (39).

V. SIMULATION RESULTS

For simulation, we assume a step change in the load demand at time $t = 4$ seconds followed by change in the engine speed set-point at $t = 11$ seconds. Figures 1 and 2 demonstrate the responses of engine states and control inputs. It is clear from figure 1 that the engine speed follows the desired engine speed under changing operating conditions. Under a step change in the T_{load} at 4 seconds from 15 Nm to 35 Nm the engine speed undergoes a momentary dip from 631 rpm to 610 rpm. But under the rapid corrective action due to u_λ the speed recovers to the desired value during the next 2 seconds. Furthermore, the engine speed tracks the change in $N_{e,ref}$ at 11 seconds from 631 rpm to 726 rpm while u_{SA} value tracks the desired u_{MBT} value to ensure maximum possible torque is generated for a given value of fuel.

VI. ACKNOWLEDGMENTS

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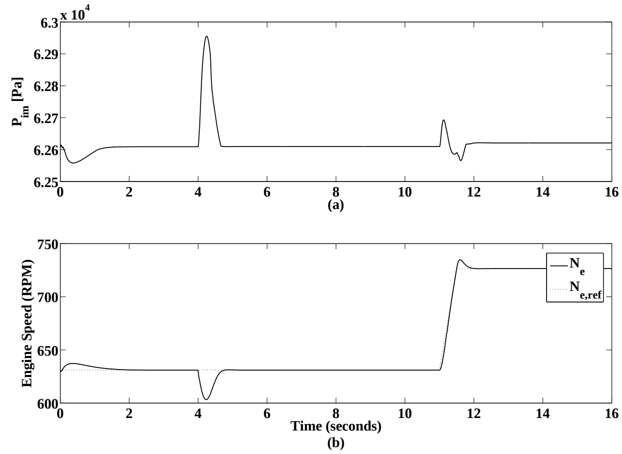

Fig. 1. Responses of intake manifold pressure and engine speed

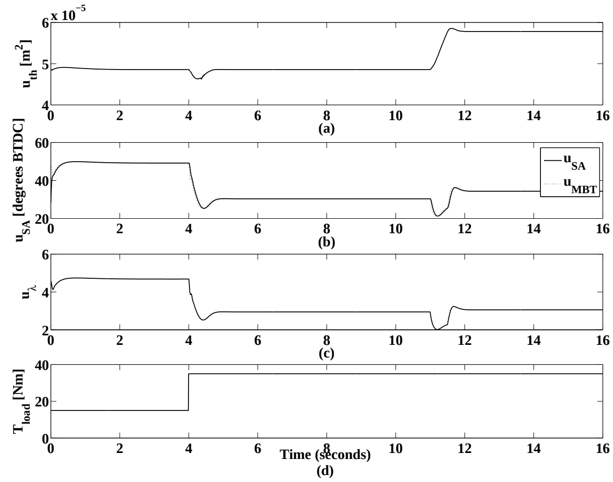

Fig. 2. Responses of Engine control variables and load disturbance input

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