CONTROL OF NONLINEAR SAMPLED-DATA SYSTEMS WITH FIXED SAMPLING PERIODS VIA THEIR APPROXIMATE DISCRETE-TIME MODELS

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ABSTRACT

We present a framework for controller design for nonlinear sampled-data differential inclusions based on their approximate discrete-time models. The results are presented for the case of fixed sampling periods and the considered stability is with respect to arbitrary (not necessarily compact) sets.

1. INTRODUCTION

One of the main stumbling blocks in sampled-data nonlinear control is the absence of a good model for controller design even in cases when a continuous-time plant model is assumed to be known. This has lead to a range of different methods for controller design of nonlinear sampled-data systems (see for instance [1, 2, 3, 4, 9, 12, 14, 17, 19] and references defined therein).

A possible approach is to base the controller design on a discrete-time model of the plant. A large body of nonlinear discrete-time literature *assumes* that the exact discrete-time model of the plant is known, which is typically not true even if the continuous-time plant model is known exactly. A more realistic approach is to base the controller design on an approximate discrete-time model, which can be obtained using a numerical integration method, such as Runge-Kutta. Approximate discrete-time models can be generated in two distinctly different ways:

1. Sampling interval T of the sampled-data system and the integration period h of the underlying numerical integration scheme are equal ($T = h$) and can be arbitrarily assigned. This typically leads to a class of *fast sampling results* that lead to semiglobal stabilization results.
2. Sampling interval T of the sampled-data system is fixed (or there is a fixed lower bound on T) and the integration period h of the underlying numerical scheme satisfies $h \in (0, T]$ and can be arbitrarily assigned. In this case, fast sampling is not possible and stabilization can only be achieved on a bounded region of the state space.

In both of these cases we obtain a family of approximate discrete-time models that is parameterized by the integration period $h > 0$.

The first approach was used in several control applications in [5, 6], in the context of adaptive control in [11] and recently a general framework for stabilization based on this approach was presented in [14]. Results in [14] apply only to the stabilization problem for plants described by differential equations. These results were further generalized in [15] and [16] to deal respectively with stabilization of sampled-data differential inclusions and input-to-state stabilization of systems with disturbances. We have already remarked that these results require *fast sampling* which means that they may not be implementable in practice in cases when the required sampling period is too small to be realized with the available hardware.

The second approach has been used in literature to address stabilization for a range of different examples (see for instance Section V in [9], Example 1 in [1] and Section V in [4]). However, we are not aware of the general conditions that guarantee that this method will work and presenting such conditions is the main contribution of this paper.

We emphasize that whenever the controller design is based on an approximate discrete-time model, it may happen that a family of controllers stabilizes the family of approximate models for all integration periods $h > 0$ but at the same time it destabilizes the exact discrete-time model for all integration pe-

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riods. This fact was illustrated by an example in [14] for the first approach. Hence, a careful investigation is needed when the controller design is based on an approximate discrete-time model irrespective of the way in which the approximate discrete-time model is generated.

Consequently, there is a strong motivation for considering the problem of controller design for sampled-data nonlinear systems based on approximate discrete-time models with a fixed sampling period. In this paper we present a general framework for stabilization of sampled-data nonlinear systems based on the family of approximate discrete-time plant models and for the case of fixed sampling periods. Our results are presented for general differential inclusions and we consider stabilization with respect to arbitrary (not necessarily compact) closed sets. The proof techniques that we use are closely related to the ones exploited in [14] that deal with the stabilization problem of sample-data differential inclusions for the case of fast sampling. Moreover, our results are similar to results in numerical analysis literature on global properties of attractors under discretization (see Chapter 7 in [18] and [8]).

The paper is organized as follows. In Section 3 we present preliminaries. Main results are stated in Section 3.

2. PRELIMINARIES

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given an arbitrary set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the following notation is used

$$|x|_{\mathcal{A}} := \inf_{s \in \mathcal{A}} |x - s|,$$

where $|x|$ denotes the Euclidean norm of the vector x. Given two set valued maps F and H, we often write $F(H(x))$ to denote $\{F(w) : w \in H(x)\}$. We will use the following definition in the sequel: for a closed set $\mathcal{A}$ and non negative real numbers $0 \leq \delta \leq \Delta$ we define

$$\mathcal{H}_{\mathcal{A}}(\delta, \Delta) := \{x \in \mathbb{R}^n : \delta \leq |x|_{\mathcal{A}} \leq \Delta\} \ . \tag{1}$$

We present a framework for understanding the design of sampled-data control laws for differential inclusions

$$\dot{x} \in F(x, u) \ , \tag{2}$$

based on *approximate* discrete-time models for (2), where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are respectively the state and control input of the plant and $F(\cdot, \cdot)$ is a set valued map. We make the following assumption for the set-valued map $F(\cdot, u)$:

Assumption 1 *For each $u \in \mathbb{R}^m$, the set-valued map $F(\cdot, u)$ satisfies the following basic conditions: 1) it is upper semi-continuous, i.e., for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $|\xi - x| \leq \delta$ we have*

$$F(\xi, u) \subseteq F(x, u) + \varepsilon \overline{\mathcal{B}}_n \tag{3}$$

where $\overline{\mathcal{B}}_n$ denotes the closed unit ball in $\mathbb{R}^n$, 2) for each $x \in \mathbb{R}^n$ the set $F(x, u)$ is nonempty, compact and convex. ■

This assumption guarantees that for each fixed u there exists at least one solution to (2). We will use $\mathcal{S}(x, u)$ to denote the set of solutions to (2) starting at x with u held constant.

We consider the case when the sampling period $T > 0$ of the sampled-data systems is fixed. This fact imposes fundamental limitations to the performance of the closed-loop sampled-data system since it typically leads to a compact domain of attraction due to the possibility of finite escape times for large initial states.

For a given $h > 0$ and for each $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ we use the following notation

$$F_h^e(x, u) := \{\xi \in \mathbb{R}^n : \xi = \phi(h, x, u), \ \phi \in \mathcal{S}(x, u)\} \ .$$

We consider plants (2) for which we assume that the control u is held constant between the sampling instants kT, where $T > 0$ is a fixed sampling period, and the exact discrete-time model (when exists) can be written in the form:

$$x_{k+1} \in F_T^e(x_k, u_k) \ . \tag{4}$$

Since T is fixed, we write F^e instead of F_T^e. Note that (4) is in general well defined only for a subset of $\mathbb{R}^n \times \mathbb{R}^m$ and it is in general unknown on the set where it is defined. Hence, we introduce a family of approximate discrete-time models that can be written in the form:

$$x_{k+1} \in F_i^a(x_k, u_k) \ , \tag{5}$$

where $i = 1, 2, 3, \ldots$. There are many possible ways in which the family F_i^a can be obtained. For example, choose an underlying numerical integration scheme, such as Runge-Kutta, with integration period $h \in (0, T]$. We denote this integration scheme as follows:

$$x_{k+1} \in f_h(x, u) \ . \tag{6}$$

For example, if the underlying integration scheme is the Euler method we have $f_h(x, u) := x + hF(x, u)$

Then we can generate a family of numerically integrated approximate models F_i^a by letting $h_i = T/i, i = 1, 2, \dots$ and integrating the continuous-time dynamics (2) over one sampling interval while keeping the control constant. In particular, if we define

$$\begin{aligned} f_h^1(x,u) &:= f_h(x,u) \\ f_h^{i+1}(x,u) &:= f_h(f_h^i(x,u),u)\ ,\quad i = 1,2,\dots\ , \end{aligned} \tag{7}$$

we can introduce the family of approximate discrete-time models (5) where

$$F_i^a(x,u) \quad := \quad f_{h_i}^i(x,u)\ ,\quad i = 1,2,\dots\ , \tag{8}$$

where $h_i = T/i, i \in \mathbb{N}$. We will assume that a family of discrete-time controllers

$$\begin{aligned} z_{k+1} &\in G_i(z_k,x_k) \\ u_k &\in H_i(z_k,x_k) \end{aligned} \tag{9}$$

has been designed based on (5), where $z \in \mathbb{R}^{n_c}$ is the controller state variable and G_i, H_i are set values maps. The controller is designed so that practical stability with respect to a nonempty closed set $\tilde{\mathcal{A}}$ with a certain domain of attraction has been achieved for the closed loop system (5), (9). The question arises: what would this imply for the exact closed-loop system (4), (9). We use the following definitions:

Definition 1 *Let $\widetilde{\mathcal{A}}$ be a nonempty closed subset of $\mathbb{R}^p$, $T > 0$ and let $\mathcal{F}_{i,\epsilon}(\cdot)$ be a family of set-valued maps from $\mathbb{R}^p$ to subsets of $\mathbb{R}^p$. For the parameterized family of systems*

$$\xi_{k+1} \in \mathcal{F}_{i,\epsilon}(\xi_k)\ ,\quad i = 1,2,\dots \tag{10}$$

the set $\widetilde{\mathcal{A}}$ is said to be (Δ,β)-practically asymptotically ((Δ,β)-PA) stable in i, uniformly in small ϵ, if there exist $\Delta > 0$, $\beta \in \mathcal{KL}$, and, for each $\delta > 0$, there exist $i^* > 0$ and $\epsilon^* > 0$ such that, for all $i \geq i^*$, $\epsilon \in [0,\epsilon^*]$ and all $\xi \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0,\Delta)$, all solutions $\phi_{i,\epsilon}(\cdot,\xi)$ of (10) satisfy

$$|\phi_{i,\epsilon}(k,\xi)|_{\widetilde{\mathcal{A}}} \leq \beta(|\xi|_{\widetilde{\mathcal{A}}}, kT) + \delta\ . \tag{11}$$

If (10) does not depend on a parameter ϵ, and the above property holds, we simply say that the set $\widetilde{\mathcal{A}}$ is (Δ,β)-PA stable, uniformly in i. ■

In the sequel we investigate stability of the system (5), (9) or (4), (9) with respect to a nonempty closed set $\widetilde{\mathcal{A}} \subset \mathbb{R}^{n\,+\,n_c}$. We also make use of the following set:

$$\mathcal{A} := \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^{n_c} \text{ such that } (x,z) \in \tilde{\mathcal{A}}\}\ . \tag{12}$$

Definition 2 *The family F_i^a is (T,Δ_1,Δ_2)-upper semi-consistent with F^e, with respect to $\mathcal{A}$ if for the given (T,Δ_1,Δ_2) and for any $\epsilon > 0$ there exists $i^* > 0$ such that for all $(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta_1) \times \Delta_2\overline{\mathcal{B}}_m$ and all $i \geq i^*$ we have*

$$F^e(x,u) \subseteq F_i^a(x,u) + \epsilon T\overline{\mathcal{B}}_n\ .$$

■

Sufficient checkable conditions for (T,Δ_1,Δ_2)-upper semi-consistency are presented in the next section.

3. MAIN RESULTS

In this section we state and prove the main results of the paper (Theorem 1 and Proposition 1). Theorem 1 is a straightforward consequence of Definition 2 and an appropriate robust stability property of the system (5), (9). In Proposition 1 we present Lyapunov conditions on the family (5), (9) that guarantee the appropriate robust stability property needed in Theorem 1. The proof of Proposition 1 relies on Lemmas 1 and 2 that may be of independent interest. Finally, we present sufficient conditions for the consistency property of Definition 2 that is used in Theorem 1. Due to space limitations most proofs are omitted.

The following stability result follows directly from Definition 2.

Theorem 1 *For a given $T > 0$, suppose that the following conditions hold:*

1. *There exists a nonempty closed set $\tilde{\mathcal{A}} \subset \mathbb{R}^{n\,+\,n_c}$, $\Delta > 0$ and $\beta \in \mathcal{KL}$ such that the parameterized family of systems:*

$$\begin{aligned} x_{k+1} &\in F_i^a(x_k,u_k) + \epsilon T\overline{\mathcal{B}}_n \\ z_{k+1} &\in G_i(z_k,x_k) \\ u_k &\in H_i(z_k,x_k)\ , \end{aligned} \tag{13}$$

   is (Δ,β)-PA stable in i with respect to $\tilde{\mathcal{A}}$, uniformly in small ϵ.

2. *There exist $M > 0$, $\nu > 0$ and $i^* > 0$ such that with $\tilde{\Delta} := \beta(\Delta,0) + \nu$ we have*

$$\sup_{\left\{(x,z)\in\mathcal{H}_{\tilde{\mathcal{A}}}(0,\tilde{\Delta})\ ,\ w\in H_i(z,x)\ ,\ i\geq i^*\right\}} |w| \leq M\ . \tag{14}$$

3. *The family F_i^a is $(T,\widetilde{\Delta},M)$-upper semi-consistent with F^e with respect to $\mathcal{A}$ defined in (12).*

Under these conditions the system (4), (9) is (Δ,β)-PA stable in i with respect to $\tilde{\mathcal{A}}$. ■

Proof of Theorem 1: Let $\phi^a_{i,\epsilon}(\cdot,\xi)$ represent a solution of (13) starting at ξ and let $\phi^e_i(\cdot,\xi)$ represent a solution of (4), (9) starting at ξ. Let $\delta > 0$ be given and let ν come from item 2. Define $\delta_1 := \min\{\delta,\nu\}$. Since the family (13) is (Δ,β)-PA stable in i with respect to $\widetilde{\mathcal{A}}$, uniformly in ϵ, there exists $\beta \in \mathcal{KL}$ and, for the given δ_1, there exist $i_1^* > 0$ and $\epsilon^* > 0$ such that, for all $\xi = (x,z) \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0,\Delta)$, $i \geq i^*$ and $\epsilon \in [0,\epsilon^*]$, the solutions of (13) satisfy (11). Let $\widetilde{\Delta} := \beta(\Delta,0) + \nu$ and with this $\widetilde{\Delta}$, let M and $i_2^* > 0$ come from item 2. Let item 3 with ϵ^* generate i_3^* from $(T,\widetilde{\Delta},M)$-upper semi-consistency. Choose $i^* := \max\{i_1^*, i_2^*, i_3^*\}$. Consider arbitrary $i \geq i^*$. Note that $\xi \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0,\widetilde{\Delta})$ implies $x \in \mathcal{H}_{\mathcal{A}}(0,\widetilde{\Delta})$. Hence, for all $i \geq i^*$ and all $\xi \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0,\Delta)$, we have $|\phi^e_i(k,\xi)|_{\widetilde{\mathcal{A}}} \leq \widetilde{\Delta}, \forall k$. This, in turn, implies that the solutions of (4), (9) satisfy (11) for all $(x,z) \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0,\Delta)$ and $i \geq i^*$. Hence, the system (4), (9) is (Δ,β)-PA stable in i with respect to $\widetilde{\mathcal{A}}$. ■

Proposition 1 *Let $\tilde{\mathcal{A}} \subset \mathbb{R}^{n+n_c}$ be a nonempty closed set and $T > 0$. Suppose that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ and a number $D > 0$ such that for any pair of strictly positive real numbers (δ_1,δ_2) with $\delta_2 \leq D$ there exist $i^* > 0$ and $c > 0$ and for all $i \geq i^*$ there exists $V_i : \mathbb{R}^{n+n_c} \to \mathbb{R}_+$ such that for all $(x,z) \in \mathcal{H}_{\tilde{\mathcal{A}}}(0,D)$, we have*

$$\alpha_1(|(x,z)|_{\tilde{\mathcal{A}}}) \leq V_i(x,z) \leq \alpha_2(|(x,z)|_{\tilde{\mathcal{A}}}) \qquad (15)$$

for all $w_1 \in F^a_i(x, H_i(x,z))$, $w_2 \in G_i(x,z)$ we have

$$V_i(w_1,w_2) - V_i(x,z) \leq -T\alpha_3(|(x,z)|_{\tilde{\mathcal{A}}}) + T\delta_1 \;, \qquad (16)$$

and, for all $(x_1,z),(x_2,z) \in \mathcal{H}_{\tilde{\mathcal{A}}}(\delta_2,D)$, with $|x_1 - x_2| \leq c$ we have

$$|V_i(x_1,z) - V_i(x_2,z)| \leq \delta_1 \;. \qquad (17)$$

Then there exists $\beta \in \mathcal{KL}$ such that for any Δ, with $\alpha_1^{-1} \circ \alpha_2(\Delta) < D$, we have for the system (13) that the set $\tilde{\mathcal{A}}$ is (Δ,β)-PA stable in i, uniformly in small ϵ. ■

Remark 1 *We emphasize that no regularity assumptions on H_i and G_i are needed in Proposition 1. In particular, H_i and G_i may be discontinuous in x and z. Moreover, we do not need a continuity assumption on the z dependence of V_i.* ■

The following two results are used in the proof of Proposition 1 and they may be of independent interest in stability investigations of difference inclusions.

Lemma 1 *Let $V : \mathbb{R}^n \to \mathbb{R}_+$, let $F : \mathbb{R}^n \to$ subsets of $\mathbb{R}^n$. Suppose $0 \leq \ell_1 \leq \ell_2$, $\alpha \in \mathcal{K}_\infty$ is locally Lipschitz*[1]*, $\ell_3 \geq 0$ and $T > 0$ satisfy*

$$V(x^+) - V(x) \leq -T\alpha(V(x)) \qquad (18)$$

for all $x \in \mathcal{V}(\ell_1,\ell_2)$, $x^+ \in F(x) \bigcap \mathcal{V}(\ell_1, \ell_2 + \ell_3)$, where α is locally Lipschitz, and

$$V(x^+) - V(x) \leq \ell_3 \qquad \forall x \in \mathcal{V}(0,\ell_2) \;, x^+ \in F(x) \;.$$

Define $\beta(y_\circ,\cdot)$ to be the maximal solution of the differential equation

$$\dot{y} = -\alpha(y) \qquad y(0) = y_\circ \geq 0 \;. \qquad (19)$$

(Note that $\beta \in \mathcal{KL}$.)

If $\ell_1 + \ell_3 \leq \ell_2$ then, for all $x_\circ \in \mathcal{V}(0,\ell_2)$, the solutions $\phi(\cdot,x_\circ)$ of the difference inclusion

$$x^+ \in F(x) \qquad (20)$$

satisfy for all $k \in \{0,1,2,\dots\}$

$$V(\phi(k,x_\circ)) \leq \max\{\beta(V(x_\circ),kT), \ell_1 + \ell_3\} \;. \qquad (21)$$

Lemma 2 *Let $V : \mathbb{R}^n \to \mathbb{R}_+$, let $F_1 : \mathbb{R}^n \to$ subsets of $\mathbb{R}^n$ and $F_2 : \mathbb{R}^n \to$ subsets of $\mathbb{R}^n$. Let $0 \leq \ell_1 \leq \ell_2$, $\alpha \in \mathcal{K}_\infty$ and $T > 0$. Suppose that the following conditions hold:*

1. *for all $x \in \mathcal{V}(0,\ell_2)$ and $x_1^+ \in F_1(x)$, we have*
$$V(x_1^+) - V(x) \leq -T[2\alpha(V(x)) - \frac{1}{2}\alpha(\ell_1)] \;,$$
2. *for all $x \in \mathcal{V}(0,\ell_2)$ and any $x_2^+ \in F_2(x)$ with $V(x_2^+) \geq \ell_1$ there exists $x_1^+ \in F_1(x)$ such that*
$$\left|V(x_2^+) - V(x_1^+)\right| \leq \frac{T}{2}\alpha(\ell_1) \;, \qquad (22)$$
3. $\ell_1 + \max\{\ell_1 + \frac{T}{2}\alpha(\ell_1), T\alpha(\ell_1)\} \leq \ell_2$

Then, we have that:

$$V(x_2^+) - V(x) \leq -T\alpha(V(x))$$

for all $x \in \mathcal{V}(\ell_1,\ell_2)$, $x_2^+ \in F_2(x) \bigcap \mathcal{V}(\ell_1, \ell_2 + \ell_3)$ and

$$V(x_2^+) - V(x) \leq \ell_3 \;,$$

for all $x \in \mathcal{V}(0,\ell_2)$, $x_2^+ \in F_2(x)$, where $\ell_3 := \max\{\ell_1 + \frac{T}{2}\alpha(\ell_1), T\alpha(\ell_1)\}$. ■

Now we present checkable sufficient conditions for the consistency property of Definition 2. We do this by introducing two appropriate consistency properties between the numerical integration scheme (6), defined by f_h, and the exact discrete-time model defined

[1] There is no loss of generality in assuming that α is locally Lipschitz since if it is not we can always find another function $\alpha_1 \in \mathcal{K}_\infty$ that is locally Lipschitz and such that $\alpha(s) \geq \alpha_1(s), \forall s \geq 0$ (see [7, pg. 139]).

by F_h^e. We use respectively notation $\mathcal{S}_h^a(x,u)$ and $\mathcal{S}^e(x,u)$ to denote sets of solutions of the difference inclusion (6) and differential inclusion (2). The solutions of the difference inclusion (6) and differential inclusion (2) are respectively denoted as $\phi_h^a(k,x,u)$ and $\phi^e(t,x,u)$, that is $\phi_h^a \in \mathcal{S}_h^a(x,u)$ and $\phi^e \in \mathcal{S}^e(x,u)$.

Definition 3 *The family f_h is said to be* one-step upper semi-consistent with F_h^e uniformly with respect to $\mathcal{A}$ *if, for each pair of positive real numbers (Δ, M) there exist $\rho \in \mathcal{K}_\infty$ and $h^* > 0$ such that, for all $(x,u) \in \mathcal{H}_\mathcal{A}(0,\Delta) \times M\overline{\mathcal{B}}_m$ and all $h \in (0,h^*)$, we have*

$$F_h^e(x,u) \subseteq f_h\left(x,u\right) + h\rho(h)\overline{\mathcal{B}}_n \tag{23}$$

■

A sufficient condition for one-step upper semi-consistency is stated below.

Lemma 3 *[15] If*

1. *for each $\widetilde{\Delta} \geq 0$ there exists $M > 0$ such that*

$$\sup_{\left\{(x,u)\in\mathcal{H}_\mathcal{A}(0,\widetilde{\Delta})\times\widetilde{\Delta}\overline{\mathcal{B}}_m\ ,\ w\in F(x,u)\right\}} |w| \leq M\ , \tag{24}$$

2. *there exists a set-valued map $\widetilde{F}(\cdot,\cdot)$ such that*
   (a) *for each $u \in \mathbb{R}^m$, the set-valued map $\widetilde{F}(\cdot,u)$ satisfies the basic conditions of Assumption 1,*
   (b) *f_h is one-step upper semi-consistent with*
   $$\widetilde{f}_h^{Euler}(x,u) := x + h\widetilde{F}(x,u)$$
   uniformly with respect to $\mathcal{A}$,
   (c) *for each $\widetilde{\Delta} \geq 0$ there exists $\widetilde{\rho} \in \mathcal{K}_\infty$ such that $(x,\xi,u) \in \mathcal{H}_\mathcal{A}(0,\widetilde{\Delta}) \times \mathcal{H}_\mathcal{A}(0,\widetilde{\Delta}) \times \widetilde{\Delta}\overline{\mathcal{B}}_m$ implies*
   $$F(\xi,u) \subseteq \widetilde{F}(x,u) + \widetilde{\rho}(|\xi - x|)\overline{\mathcal{B}}_n$$

then f_h is one-step upper semi-consistent with F_h^e uniformly with respect to $\mathcal{A}$. ■

Definition 4 *The family f_h is said to be multi-step upper semi-consistent with F_h^e if, for each 4-tuple of strictly positive real numbers $(T, \eta, \Delta_1, \Delta_2)$ there exist a function $\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ and $h^* > 0$ such that, for all $h \in (0,h^*)$, $(x,u),(y,u) \in \mathcal{H}_\mathcal{A}(0,\Delta_1) \times \Delta_2\overline{\mathcal{B}}_m$ and $|x-y| \leq \delta$ we have*

$$F_h^e(x,u) \subseteq f_h(y,u) + \alpha(\delta,h)\overline{\mathcal{B}}_n$$

and $i \leq T/h$ implies

$$\alpha^i(0,h) := \overbrace{\alpha\left(\cdots\alpha\left(\alpha\right.\right.}^{i}\left.\left.\left(0,h\right),h\right)\cdots,h\right) \leq \eta\ .$$

A sufficient condition for multi-step upper semi-consistency is given in the following:

Lemma 4 *If, for each pair of strictly positive real numbers (Δ_1, Δ_2), there exist $K > 0$, $\rho \in \mathcal{K}_\infty$ and $h^* > 0$ such that for all $h \in (0,h^*)$ and all $(x,u),(y,u) \in \mathcal{H}_\mathcal{A}(0,\Delta_1) \times \Delta_2\overline{\mathcal{B}}_m$ we have*

$$F_h^e(x,u) \subseteq f_h(y,u) + \left[(1+Kh)\left|x-y\right| + h\rho(h)\right]\overline{\mathcal{B}}_n$$

then f_h is multi-step consistent with F_h^e.

Remark 2 Relative to the one-step consistency condition, the condition of Lemma 4 is guaranteed by one-step consistency plus the following type of Lipschitz condition on either the family F_h^e or the family f_h:
for each (Δ_1, Δ_2) there exist $K > 0$ and $i^* > 0$ such that for all $(x,u),(y,u) \in \mathcal{H}_\mathcal{A}(0,\Delta_1) \times \Delta_2\overline{\mathcal{B}}_m$ and all $i \geq i^*$,

$$f_h(x,u) \subseteq f_h(y,u) + (1+Kh)|x-y|\overline{\mathcal{B}}_n\ . \tag{25}$$

This condition is guaranteed for F_h^e when $F(x,u)$ is locally Lipschitz. The condition given in Lemma 4 for multi-step consistency is similar to conditions used in the numerical analysis literature (e.g., see conditions (i) and (iii) of Assumption 6.1.2 in [18, pg.429]). ■

In terms of trajectory error over "continuous-time" intervals with length of order one, multi-step consistency gives the following:

Lemma 5 *If f_h is multi-step consistent with F_h^e then for each 4-tuple of strictly positive real numbers $(T, \eta, \Delta_1, \Delta_2)$ there exists $h^* > 0$ such that, if h, i, u and ξ such that for all $h \in (0,h^*)$, $|u| \leq \Delta_2$, $ih \in [0,T]$, we have*

$$\phi_h^a(i,\xi,u) \in \mathcal{H}_\mathcal{A}(0,\Delta_1) \cap \mathcal{S}_h^a(x,u)\ , \tag{26}$$

then for any $\phi_h^a(i,\xi,u) \in \mathcal{S}_h^a(\xi,u)$ and $\phi^e(ih,\xi,u) \in \mathcal{S}^e(\xi,u)$ we have

$$|\phi^e(ih,\xi,u) - \phi_h^a(i,\xi,u)| \leq \eta \qquad \forall i : ih \in [0,T]\ .$$

■

A simple consequence of the above lemma is a sufficient condition for consistency presented in Definition 2. Note that the family F_i^a is generated by the numerical integration scheme using the formulas (7) and (8) and the surrounding discussion.

Corollary 1 *Let f_h be multiple-step upper semi-consistent with F_h^e, let the family F_i^a be defined using (7) and (8). Given any $(T, \epsilon, \Delta_1, \Delta_2)$ there exists $h^* > 0$ and $i^* \geq T/h^*$ such that if the condition (26) holds then for all $(x,u) \in \mathcal{H}_\mathcal{A}(0,\Delta_1) \times \Delta_2\overline{\mathcal{B}}_m$ and all $i \geq i^*$, we have*

$$F^e(x,u) \subseteq F_i^a\left(x,u\right) + \epsilon T\overline{\mathcal{B}}_n \tag{27}$$

■

Remark 3 *We note that in order to verify that consistency in Definition 2 holds, one needs to show that the condition (26) holds. This may be hard to do in general with a given triple* (T, Δ_1, Δ_2). *However, if the family* f_h *satisfies a Lipschitz condition, uniform in* h, *then given any* (T, Δ_1, Δ_2) *there exist* $(\widetilde{\Delta}_1, \widetilde{\Delta}_2)$ *and* $h^* > 0$ *such that if* $(\xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \widetilde{\Delta}_1) \times \widetilde{\Delta}_2 \overline{\mathcal{B}}_m$ *and* $h \in (0, h^*)$, *then* $\phi_h^a(k, \xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta_1)$ *for all* k *such that* $kh \in [0, T]$.

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