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On Modeling Lifetimes, Sharing Risk, and Valuing Promises: Essays in Actuarial Science

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Abstract

This thesis contributes to the modeling, sharing, and valuation of insurance risks in a financial-actuarial setting. Chapters 2 to 4 present the original research contributions of this thesis, except where explicitly stated otherwise in the text.

Chapter 2 introduces time series vine copulas to capture nonlinear and asymmetric serial and cross-sectional dependence in mortality period effects across multiple populations. The general S-vine class, together with its D-vine, M-vine, and stationary COPAR special cases, is considered. This represents one of the first applications of such models in this context. A comparison of time series vine copula models against leading alternatives in the existing literature is provided, including vector autoregressive (VAR), sparse VAR, and ARIMA copula models. Furthermore, a mortality model with multiple period effects is used to showcase the flexible and parsimonious structure of time series vine copulas. The results demonstrate that time series vine copulas consistently outperform other models in both in-sample and out-of-sample settings. The practical implications of model choice are then investigated through a longevity bond pricing exercise based on a KORTIS-bond-style payment structure. The dependence structure between period effects is shown to have a meaningful impact on risk assessment.

Chapter 3 shifts focus to risk-sharing mechanisms in insurance markets. A general framework that extends existing models, which assume Bernoulli-type losses, to the setting of general non-negative loss severities, is developed for compensation-based risk-sharing. These schemes differ from contribution-based risk-sharing, in which participants are fully reimbursed for their losses at time-1 and contributions are determined once the losses are known. In a compensation-based risk-sharing scheme, participants choose their investment into the pool at time-0, and the value of the pooled fund is redistributed at time-1; this redistribution may or may not fully cover participants' individual losses. A key advantage of this structure is the elimination of counterparty risk. Next, the actuarial fairness of such schemes is investigated, and conditions under which they are actuarially fair are established. The role of the administrator is examined through a comparison of active and passive designs, which reveals how administrative structure affects the compensation mechanism and identifies conditions under which participants are indifferent between the two.

Chapter 4 turns to the valuation of insurance liabilities in a dynamic financial-actuarial framework. Three nested information structures are considered: the asset-price filtration, the financial filtration, and the insurer-relevant filtration. Within this framework, the three-step hedge-based valuation approach for hybrid insurance liabilities proposed by Linders (2023) is extended to a multi-period setting. By restricting the insurer's trading strategies to the financial filtration and allowing actuarial valuation within the insurer-relevant filtration, the proposed framework complies with no-insider-trading requirements. It is shown that the class of three-step valuations coincides with the class of fair dynamic valuations. This result generalizes the single-period framework of Linders (2023) to a fully dynamic and filtration-sensitive environment. Next, a discrete multi-step adaptation of the two-step market evaluations of Pelsser and Stadje (2014) is studied. It is shown that this class likewise coincides with fair dynamic valuations, a result that unifies these approaches within a multi-period framework.

Chapter 5 concludes the thesis and outlines directions for future research.

Declaration

I, Atibhav Chaudhry, hereby declare that:

a) this thesis comprises only my original work towards the degree of Doctor of Philosophy, except where indicated in the preface;

b) due acknowledgment has been made in the text to all other material used; and

c) this thesis does not exceed the maximum word limit in length, exclusive of tables, maps, bibliographies, and appendices as approved by the Research Higher Degrees Committee.

Signed: Atibhav Chaudhry

Date: 30 June 2026

Preface

This research was conducted by Atibhav Chaudhry as part of a joint PhD program between the University of Melbourne and KU Leuven, under the supervision of Prof. Dr. Jan Dhaene, Prof. Dr. Rui Zhou, Prof. Dr. Katrien Antonio, and Prof. Dr. Benjamin Avanzi. This thesis is based on the following works.

Chapter 2 of this thesis is joint work with Prof. Dr. Rui Zhou and Prof. Dr. Katrien Antonio. It is based on the following working paper:

Chaudhry, A., Zhou, R., & Antonio, K. (2026). *Modeling mortality dependence with time series vine copulas: A unified framework for serial and cross-sectional dependence.* Working paper.

Chapter 3 of this thesis is joint work with Prof. Dr. Jan Dhaene, Prof. Dr. Ka Chun Cheung, and Mr. Austin Riis-Due. It is based on the following submitted paper:

Dhaene, J., Chaudhry, A., Riis-Due, A., & Cheung, K. C. (2026). *Compensation-based risk-sharing.* Submitted for publication to the *Annals of Actuarial Science* on 24 June 2026.

Chapter 4 of this thesis is joint work with Prof. Dr. Jan Dhaene, Prof. Dr. Michele Vanmaele, and A/Prof. Dr. Daniël Linders. It is based on the following working paper:

Chaudhry, A., Vanmaele, M., Dhaene, J., & Linders, D. (2026). *Information-constrained fair dynamic valuations of hybrid insurance liabilities.* Working paper.

No part of this work has been submitted toward any other qualification, nor was any portion completed prior to the commencement of the PhD candidature.

All co-authorship has taken place in accordance with the Graduate Research Training Policy (MPF1321) and the Authorship Policy (MPF1181) of the University of Melbourne.

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A PhD can look solitary from the outside, but I never experienced it to be so. This thesis carries my name, yet it is the product of the support, generosity, and guidance of many people. Much of what sustained me over the years happened far from the desk, in the encouragement of the people around me. Every one mattered.

First and foremost, I thank my supervisors Professor Jan Dhaene, Professor Rui Zhou, Professor Katrien Antonio, and Professor Benjamin Avanzi. Without any one of you, this PhD would not have been possible. I feel deeply fortunate to have been your student. Thank you for the time and care you invested in me, for your high standards, and for the intellectual openness you brought to our work. Your feedback improved every draft and every detail, and our collaboration has been both professionally formative and personally meaningful.

They say all it takes is one person believing in you. For me, Jan, that person was you. Whether it was our conversations over coffee or the moments when I sat in your office anxious and overwhelmed, you were always there to reassure me. Your faith in me has been a major reason I am writing these words today. You are an exceptional supervisor, a mentor, and most importantly, a kind soul. Thank you for your generosity throughout. You made this journey truly unforgettable.

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A thesis is also shaped by the people who build it with you. I am grateful to my co-authors Professor Daniel Linders, Professor Michele Vanmaele, Professor Ka Chun Cheung, Dr. Rodrigue Kazzi, and Mr. Austin Riis-Due for the opportunity to work together and learn from each one of them. Thank you for your time, patience, and for the discussions that strengthened both the ideas and the writing.

As this work reached its final stage, it found careful readers in those who examined it. To my thesis examiners Professor Emiliano Valdez, Professor Bertrand Tavin, Professor Daniel Linders, and Dr. Enrique Calderin, thank you for the seriousness with which you engaged with this thesis. Your thoughtful feedback and attention to detail improved the manuscript. I am equally grateful for your patience with the administrative complexities that come with a joint-PhD examination. My gratitude also goes to Professor Piet Sercu for agreeing to chair my public defense.

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A PhD thesis alone never tells the whole story of one's journey. Behind every page lies a transformation — you begin knowing very little, and slowly become someone shaped by the people around you. The manuscript is merely a record of what was found. But those people are what made one capable of finding it at all.

Atibhav Chaudhry
Leuven, 2026

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Chapter 1

Introduction

The insurance and broader risk-transfer industries have undergone a profound transformation over the last two decades (Cummins & Weiss, 2009). This evolution has also been documented by international supervisory and policy bodies such as the International Association of Insurance Supervisors (2025) and the Organisation for Economic Co-operation and Development (2025). This transformation was driven not by specific events but by the interaction of several factors, such as demographic changes, innovation in financial systems, technological advances, evolution of regulation, and the development of new approaches for sharing risk. The confluence of these external changes has reshaped how actuaries model risk, design equitable risk-sharing mechanisms, and value future liabilities. Traditional actuarial methodologies have often proved inadequate for various stakeholders given the complexity and dynamic nature of modern risk (Cairns et al., 2008; Eling & Lehmann, 2018). The research carried out in this thesis addresses the need for advanced modeling tools, equitable risk-sharing, and fair valuation methodologies that are aligned with contemporary industry realities.

Across these developments, a common unifying theme can be identified: the central role of *risk transfer* in modern insurance systems. Whether through reinsurance arrangements, longevity-linked securities, collective risk-sharing schemes, or hybrid insurance products, insurance entities increasingly rely on structured mechanisms to transfer, share, or redistribute risk across stakeholders. As a result, actuarial work in this context spans three connected tasks: modeling risk, designing mechanisms through which risk can be transferred or shared, and valuing the resulting obligations in a manner that is economically sound, transparent, and consistent over time. This thesis contributes to each of these stages, providing an integrated actuarial perspective on the transfer of risk.

One of the key forces underlying the industry-wide transformation described above was the changing nature of mortality and longevity risk patterns. Sustained mortality improvements were observed throughout the twentieth century (Oeppen & Vaupel, 2002), but these improvements became increasingly heterogeneous across some populations and increasingly homogeneous across others (Marmot, 2005; Vallin & Meslé, 2004). The dependence between mortality improvements across countries and subpopulations became more pronounced in some cases, driven by increasing cross-border mobility, the rapid spread of epidemics, and global convergence in health behaviors and medical technologies. However, in other cases, dependence became less pronounced due to widening socioeconomic disparities, unequal adoption of medical innovations, and structural differences in healthcare systems that generated increasingly distinct mortality improvements. These changes had major implications for insurer entities and financial institutions (Blake et al., 2014; Coughlan et al., 2011), both of which need to forecast mortality improvements accurately and manage basis risk between experience and reference populations when pricing instruments such as annuities, deferred pensions, and longevity-linked securities. The growing awareness that mortality risk is driven by complex demographic, epidemiological, and socioeconomic factors challenged several assumptions in the existing literature. Many models projected mortality improvements separately across populations and implicitly assumed independence

between populations (Booth & Tickle, 2008; Lee & Carter, 1992), while others relied on linear or symmetric dependence structures that could not accurately capture observed mortality dynamics (Dowd et al., 2011; N. Li & Lee, 2005).

At the same time, innovation in longevity-linked financial instruments allowed capital-market participants to gain exposure to or protection against mortality improvement trends (Y. Lin & Cox, 2005). This development created a pressing need for flexible joint modeling frameworks capable of capturing both typical and extreme patterns of dependence in the period effects — the components of mortality models that govern the rate at which mortality improves over time. The development of products such as the KORTIS longevity bond (Hunt & Blake, 2015), survivor swaps (Dowd et al., 2006), q-forwards (Coughlan et al., 2007), and other longevity-linked securities (Cairns et al., 2008; Dowd, 2003; Tan et al., 2014) marked a notable shift in the way mortality risk was priced, transferred, and hedged. Many of these new instruments require the precise joint modeling of period effects across multiple populations. Accurate pricing and the management of the resulting basis risk have significant implications for the risk management practices of insurance entities and capital-market participants (Cairns et al., 2014). Their structure requires not only accurate forecasts of expected period effects, but also reliable modeling of dependence between the period effects of populations in stress scenarios such as pandemics, where such dependence may increase. A model that underestimated or ignored these features could misprice instruments, understate required capital, or generate inappropriate hedging strategies (Cairns et al., 2014; J. S.-H. Li & Hardy, 2011), with the potential to materially affect insurance entities or investors. Therefore, the development of such instruments created a need for joint modeling of period effects, which captures both serial and cross-sectional dependence between the period effects of populations in a flexible manner. This need is addressed in Chapter 2, where time series vine copulas provide a unified, flexible, and parsimonious framework for modeling joint mortality dynamics, supporting the quantification and transfer of longevity risk through insurance and capital-market instruments.

Risk-sharing arrangements have also evolved substantially in recent times. The increasing prevalence of decentralized risk-sharing arrangements (Feng, 2023) that bypass a central party (e.g., an insurer) has placed greater emphasis on transparency, equity, and sustainability. In such schemes, gains and losses must be allocated among participants according to principles that are easy to explain to those without an actuarial background and that remain consistent with actuarial fairness principles. Technological advancements have increased innovation in this area, giving rise to peer-to-peer insurance networks (Feng et al., 2023), cooperative risk pools (Santos et al., 2021), and decentralized mutual-aid arrangements (Abdikerimova & Feng, 2022). Such arrangements create a context in which participants share risks collectively without the presence of an insurer. This gives rise to a different set of challenges regarding how losses arising from heterogeneous risk should be shared equitably and how adjustments must be made when the scheme is run by an administrator who neither bears losses nor contributes their own risk to the scheme. In Chapter 3, a framework is developed that presents compensation-based principles for sharing losses in a decentralized risk-sharing scheme, providing an alternative to classical contribution-based principles (Denuit et al., 2022). The framework focuses on the transfer and redistribution of primarily idiosyncratic risks among participants, while ensuring actuarial fairness and reducing counterparty risk in modern risk-sharing arrangements.

Another major force that reshaped actuarial practices was the transformation of regulatory and accounting frameworks that govern the valuation of insurance liabilities. The implementation of Solvency II, the Swiss Solvency Test, and the internationally adopted IFRS 17 standard represented a shift towards the principles of market consistency, model consistency, and dynamic valuation of liabilities (European Parliament and Council of the European Union, 2009; International Accounting Standards Board, 2017; Swiss Financial Market Supervisory Authority, 2024). Such frameworks require insurers to value liabilities using a methodology that accurately reflects the underlying nature of each insurance contract. In particular, hedgeable components of a contract payoff must be valued using risk-neutral principles, non-hedgeable components must be valued using a best estimate and a risk margin, and hybrid contracts must combine both principles. Unit-linked products, variable annuities

with embedded guarantees, and participating contracts are examples of hybrid contracts. Regulatory and accounting frameworks also require insurance entities to value their liabilities consistently over multiple periods. Since insurers are prohibited from trading on non-public information to avoid any possibility of insider trading, they must base valuation and hedging decisions strictly on publicly available information. This constraint requires the separation of information into distinct categories: financial-market information, systematic risk information, and policyholder-specific information. The evolution of these information structures must be incorporated into the valuation of liabilities. This motivates the theoretical development in Chapter 4, which extends the concept of fair valuation to a multi-period setting with nested filtrations, providing the mathematical foundations required to value hybrid liabilities arising from risk-transfer arrangements in accordance with modern regulatory and accounting standards.

In addition, advances in technology have allowed insurance entities to create actuarial tools more suited to this evolving landscape. For mortality modeling in particular, the use of time series vine copulas — a computationally intensive method to capture dependence — has now become feasible (Dißmann et al., 2013), as demonstrated in Chapter 2. Technological progress has also allowed for faster and more efficient valuation of liabilities at multiple time points (Klöppel & Schweizer, 2007). This has made it possible for insurers to gather and incorporate multiple sources of information into their valuations in a scalable manner. Valuation operators that are not only theoretically sound at each time point but also dynamically consistent and computationally tractable are now feasible due to these developments, as demonstrated in Chapter 4.

Across shifts in mortality dynamics, the growth of longevity-linked financial innovation, regulatory reform, and the emergence of decentralized insurance arrangements, the unifying challenge for the actuarial profession is the design and evaluation of effective risk-transfer mechanisms. The three research projects in this thesis address this challenge from complementary perspectives. Chapter 2 develops advanced tools for modeling and quantifying longevity risk, enabling its transfer across insurance and capital markets. Chapter 3 proposes principled mechanisms for sharing primarily idiosyncratic risks among participants in collective risk-sharing arrangements. Chapter 4 establishes a multi-period valuation framework for pricing and valuing hybrid insurance liabilities generated by such risk-transfer mechanisms in a fair and dynamically consistent manner.

By developing mortality modeling, risk-sharing, and liability valuation methodologies that are mathematically rigorous, conceptually unified, and aligned with contemporary industry challenges, this thesis contributes to the actuarial profession's capacity to address the evolving needs of insurers, regulators, financial institutions, and other market participants. Together, these contributions support a modern actuarial perspective in which risk is not only quantified, but also shared and valued in a manner that is fair, transparent, and consistent with the economic reality.

Chapter 2

Modeling mortality dependence with time series vine copulas - A unified framework for serial and cross-sectional dependence

2.1 Introduction

Mortality improvements are uncertain, leading to longevity risk—the risk that individuals live longer than expected—posing a significant challenge for insurers due to the financial impact, particularly on annuity-type products that may require extended payouts. Mortality improvements are often correlated between populations, driven by advancements in healthcare, public health, technology, and socio-economic factors. Joint modeling of mortality across multiple populations enables insurers to better quantify longevity risk within their portfolios. For example, insurers may benefit from a natural hedge by offering both annuities and death benefit policies. Joint modeling of mortality across multiple populations also supports effective risk management strategies. Mortality-linked financial instruments, such as longevity bonds or mortality swaps, offer insurers a way to hedge against longevity risk by linking payouts to mortality outcomes. However, because these instruments are typically tied to standardized indices, they may not align precisely with specific insured populations, leading to population basis risk (Hunt & Blake, 2015; Villegas et al., 2017). By accurately modeling mortality dependence across populations, insurers can better understand the basis risk and optimize strategies in the face of uncertain mortality improvements.

A common approach to constructing a multi-population mortality model is to first select a single-population model for each population, such as the Lee-Carter (Lee & Carter, 1992) or Cairns-Blake-Dowd (Cairns et al., 2006b) models and their variants, to decompose mortality into age, period, and cohort components. The dependence between the period and/or cohort components between populations is then further modeled (Cairns, Blake, Dowd, Coughlan, & Khalaf-Allah, 2011; J. S.-H. Li et al., 2015; Zhou, 2019). Two types of dependencies are typically considered: serial dependence, which captures how past values of a component affect its current value, and cross-sectional dependence, which reflects how one component influences another, either within the same population or across populations.

In this chapter, we propose the use of time series vine copulas to simultaneously capture serial and cross-sectional dependence in multi-population mortality data. A vine copula is a flexible hierarchical

structure that decomposes the joint distribution into bivariate copulas, allowing for nonlinear and asymmetric relationships—including tail dependence—while accommodating high-dimensional data. To our knowledge, this is one of the first studies to adopt time series vine copulas to capture mortality dependence. By effectively modeling nonlinear and asymmetric dependence in both serial and cross-sectional dimensions, the time series vine copula approach offers great flexibility. This approach also incorporates efficient parameter estimation processes to achieve parsimony, which is particularly valuable in high-dimensional applications where reducing complexity without overly sacrificing accuracy is essential. In addition, it enables intuitive visualization of dependence structures, aiding interpretation, and providing insights into relationships within and between populations.

Although vine copulas have been widely applied to cross-sectional dependence modeling, their extension to serial dependence is relatively recent. The use of copulas for serial dependence was originally introduced by Smith et al. (2010) for univariate time series and has since been extended to multivariate time series by various researchers. Examples include the copula autoregressive (COPAR) model of Brechmann and Czado (2015), the D-vine model of Smith (2015), and the M-vine model of Beare and Seo (2015). These approaches differ in the specific vine structure employed but share the goal of constructing a model that ensures stationarity, which is critical for consistent and reliable time series analysis. In particular, Nagler et al. (2022) generalized these methods to a new class of stationary vines (S-vines), explicitly designed to guarantee stationarity in time series modeling.

Time series vine copulas have been applied in modeling financial data (Beare & Seo, 2015; Nagler et al., 2022), electricity market data (Smith, 2015; Smith et al., 2010), and macro-economic indices (Brechmann & Czado, 2015). Their potential for modeling mortality dependence remains unexplored. Existing methods for modeling mortality dependence fall primarily into two categories: vector autoregressive (VAR)-based methods and ARIMA-copula approaches.

VAR-based models assume linear dependence for both serial and cross-sectional relationships. They describe each component as a linear combination of its past values and those of other components. These models have been extensively applied in the mortality literature. For example, Zhou et al. (2014) compare time series models for period effects, including a VAR model applied to the Lee-Carter components of two populations. Yang and Wang (2013) extend the VAR model by adding co-integration terms to form a Vector Error Correction Model (VECM). Other studies, such as Cairns, Blake, Dowd, Coughlan, and Khalaf-Allah (2011), Dowd et al. (2011), J. S.-H. Li and Hardy (2011), and N. Li and Lee (2005), use combinations of autoregressive (AR) and random-walk-with-drift models, which can be seen as special cases of VAR models.

One significant limitation of VAR-based models is the large number of parameters required for multiple populations, leading to overfitting and high uncertainty in parameter estimates. Penalized VAR-based models (Chang & Shi, 2021; Guibert et al., 2019; H. Li & Lu, 2017; H. Li & Shi, 2021) address this issue by reducing the parameter space. However, both standard and penalized VAR models assume linear and symmetric dependence and often rely on normality assumptions — an approach that does not account for tail dependence.

Recent studies, such as Wang et al. (2015) and Zhou (2019), highlight the presence and importance of asymmetric tail dependence in mortality modeling. Asymmetric tail dependence indicates that extreme values in one time series may coincide with extreme values in another. This feature has significant implications for quantifying longevity risk, as it sheds light on how extreme improvements in mortality may coincide across populations and thus amplify financial impacts. It is also critical to price financial instruments associated with longevity risk, for example, the KORTIS bond (Hunt & Blake, 2015). The principal repayment of the KORTIS bond is reduced when there is an extreme divergence in mortality improvements between two populations. Accurately capturing tail dependence is essential to avoid mispricing and inadequate risk assessment. The time series vine copula approach overcomes the limitations of VAR-based models by capturing nonlinear and asymmetric dependence in a parsimonious manner.

The autoregressive integrated moving average-copula (ARIMA-copula) approach offers an alter-

native by introducing nonlinearity and asymmetry through the use of copulas in a two-step process. First, an ARIMA model (or an extension like ARIMA-GARCH) is applied separately to each time series to capture serial dependence. In the second step, a copula is used to model cross-sectional dependence between the residuals, allowing for more complex relationships, including tail dependence. Various copula models have been proposed, including static and time-varying multivariate copulas (T. Lin et al., 2015; Wang et al., 2015), hierarchical Archimedean copulas (H. Li & Lu, 2019; J. Li et al., 2021; Zhu et al., 2017), and factor copulas for high-dimensional settings (H. Chen et al., 2015, 2017). Vine copulas, which decompose multivariate dependence into bivariate copulas, offer another promising approach (Zhou, 2019; Zhou & Ji, 2021). A comprehensive review is provided by Grosser and Okhrin (2022).

However, the ARIMA-copula approach has its limitations. Although it introduces nonlinearity and asymmetry for cross-sectional dependence, serial dependence remains constrained by the linear and symmetric assumptions of the ARIMA model. This restricts its ability to capture complex dependency structures. Furthermore, Smith (2015) suggests that a one-step framework, which captures both types of dependence simultaneously, often performs better than the two-step procedure used for the ARIMA-copula approach in various applications. The time series vine copula approach addresses these limitations by jointly modeling serial and cross-sectional dependence in a one-step framework that allows for flexible dependence structures in both dimensions. However, we acknowledge that this approach sacrifices the simplicity of separate modeling in ARIMA-copula in exchange for the ability to capture more complex dependency structures.

In this chapter, we explore various time series vine copulas, including the general class of S-vines and its special cases—D-vines, M-vines, and stationary COPAR models—to model mortality dependence. We conduct a comprehensive comparison of time series vine copula models with VAR, sparse VAR, and ARIMA-copula models, evaluating both fitting and forecasting performance. These models are applied to mortality data from four populations involving eight period components. Our results show that time series vine copulas outperform existing methods in both in-sample and out-of-sample performance. This side-by-side evaluation provides insight into the strengths and trade-offs of each approach.

We further examine the practical implications of model choice by analyzing its impact on longevity bond pricing, using a payment structure similar to the KORTIS bond. Our analysis shows that while the choice of model may not significantly affect the median of the mortality/longevity index, it plays a critical role in capturing distributional properties that are important for risk assessment.

Section 2.2 describes the construction of the multi-population mortality model and introduces the single-population model that serves as its building block. Section 2.3 reviews existing approaches for jointly modeling period components. Section 2.4 introduces the time series vine copula and its application to modeling mortality dependence. Section 2.5 details the mortality data and the fitting of the base mortality structures. Section 2.6 illustrates the fitted models with a two-population example. Section 2.7 compares the fit and forecasting performance of various models to capture the mortality dependence. Section 2.8 demonstrates the application of these models to evaluate key pricing factors for a KORTIS-like longevity bond. Finally, Section 2.9 concludes.

2.2 Single-population mortality models

Multi-population mortality models are often constructed using a two-step approach, as described by J. S.-H. Li et al. (2015). In the first step, the mortality rates of each population are modeled individually using a single-population mortality model, which typically decomposes mortality rates into components such as age, period, and cohort effects. In the second step, these estimated components from different populations are jointly modeled as a multivariate time series to capture mortality dependence. We follow this approach in this chapter.

In this section, we present the generalized structure of single-population mortality models (Hunt & Blake, 2021). Consider the mortality rates of J populations in the age range of $[x_0, x_1]$ in the period of $[1, \ldots, T]$. The mortality model is expressed as follows:

$$\eta_{x,t,j} = \alpha_{x,j} + \sum_{i=1}^{I} \beta_{x,j}^{(i)} \kappa_{t,j}^{(i)} + \beta_{x,j}^{(0)} \gamma_{t-x,j}, \tag{2.1}$$

where $j = 1, \ldots, J$, $x = x_0, \ldots, x_1$, $t = 1, \ldots, T$, and $\eta_{x,t,j}$ is a predictor, which is a transformation of the death probabilities of population j at age x in year t. In the existing literature, $\eta_{x,t,j}$ is often set to the logarithm of central mortality rates (Lee & Carter, 1992) or the logistic transformation of death probabilities (Cairns et al., 2006b).

This model includes a static age function $\alpha_{x,j}$, which represents the age effects for population j; I age/period terms, in which $\kappa_{t,j}^{(i)}$, referred to as the i^{th} period effect for population j, reflects mortality evolution over time and $\beta_{x,j}^{(i)}$, which determines the pattern of mortality change across ages; an age/cohort term consisting of the cohort effect $\gamma_{t-x,j}$ for population j, which remains fixed for each birth year, and an age function $\beta_{x,j}^{(0)}$ that determines the age pattern of the cohort effect for population j. This representation of single-population mortality is a generalization of the Lee-Carter model (Lee & Carter, 1992), the classic Age-Period-Cohort (APC) model (Hobcraft et al., 1985), the Cairns-Blake-Dowd (CBD) model (Cairns et al., 2006b), and various others.

These models can be estimated using either the least squares method or maximum likelihood estimation with distributional assumptions. Identifiability constraints are necessary to ensure the uniqueness of the parameter estimates. In Section 2.5, we will describe the specific model setup, parameter constraints, and estimation methods used in this chapter.

Following the estimation of the model, further time series modeling can be performed on the estimated period and cohort effects to enable mortality forecasting. In this chapter, we do not further model the cohort effect $\gamma_{t-x,j}$, since for forecasting purposes we only consider cohorts present during the fitting period. Forecasting the cohort effect presents some challenges. Cairns et al. (2009) and Dowd et al. (2010) document that in the cohort-extended Lee-Carter model, the cohort component is not robust to the choice of the fitting window: re-estimating the model on a slightly shorter or longer sample can produce quantitatively different fits. Dowd et al. (2010) mention that this occurs due to multiple maxima in the likelihood. So to avoid forecasting a non-robust object, we project mortality only at ages for which the relevant forecast cohort was already estimated in-sample. If cohort effects must be forecast, the same modeling principles outlined later in this chapter may be used, with one important difference. The period effects typically require first-order differencing to achieve stationarity. The cohort effects, on the other hand, are fitted under an identifiability constraint that forces them to fluctuate around zero with no linear trend (Cairns et al., 2009). Typically, the cohort effect is modeled using a low-order ARMA process with mean-reversion to zero (Cairns et al., 2009). As a result, the cohort effects can be modeled directly on their levels, without differencing. Some care must nevertheless be taken when projecting corner cohort effects, which are estimated from very few observations.

2.3 Jointly calibrating period effects for multiple populations

The existing literature has devised several multivariate time series approaches to model mortality dependence, including VAR, SparseVAR, and ARIMA-copula. In this section, we review these approaches and explore how they have been used to jointly model period effects.

Since period effects $\kappa_{t,j}^{(i)}$ often exhibit a time trend due to mortality improvement over time, the first order difference of period effects, $\Delta\kappa_{t,j}^{(i)} = \kappa_{t,j}^{(i)} - \kappa_{t-1,j}^{(i)}$, is typically taken to achieve stationarity

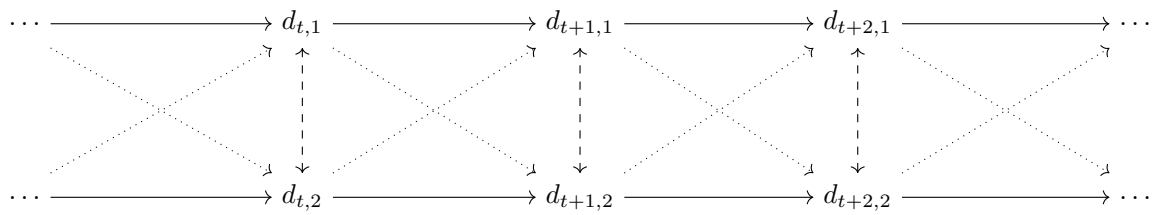


Figure 2.1: An illustration of different types of dependence between two period effects. We consider two populations, each with one period effect. Solid lines indicate serial dependence, dashed lines represent contemporaneous cross-sectional dependence, and dotted lines denote lagged cross-sectional dependence.

(J. Li et al., 2021; Wang et al., 2015; Zhou, 2019; Zhou & Ji, 2021). To simplify the notation, we define $d_{t,k} = \Delta\kappa_{t,j}^{(i)}$, where $k = i \times j$. This simplification allows us to use a single index k running from 1 to $I \times J$ to distinguish differenced period effects in the time series analysis. This simplification is justified because, although the period effects in Equation (2.1) are distinguished by the index i, which represents the ith age/period term, and j, which represents the population, this distinction is not explicitly utilized in the time series modeling. Instead, each series is treated as a distinct time series.

We further denote $\boldsymbol{d}_t$ as the vector of differenced period effects at time t:

$$\boldsymbol{d}_t = [d_{t,1}, d_{t,2}, \ldots, d_{t,I\times J}]',$$

where the prime symbol ($'$) represents the transpose of a vector or matrix.

2.3.1 Two types of dependence between period effects

The differenced period effects are often jointly modeled as multivariate time series to capture two types of dependence: serial dependence and cross-sectional dependence.

- The serial dependence examines the relationship between different time points for a specific period effect, specifically between $d_{t,k}$ and $d_{t-l,k}$, where l represents the time lag.
- The cross-sectional dependence can be divided further into contemporaneous cross-sectional dependence and lagged cross-sectional dependence:
  - Contemporaneous cross-sectional dependence, referred to simply as "cross-sectional dependence" hereon, investigates the relationship between the changes in different period effects at the same time point, represented by the dependence between d_{t,k_1} and d_{t,k_2} where $k_1 \neq k_2$.
  - The lagged cross-sectional dependence explores the relationship between the changes in different period effects at lagged time points, indicated by the dependence between d_{t,k_1} and d_{t-l,k_2}, where $k_1 \neq k_2$.

Figure 2.1 visually represents these dependencies for an example involving two populations, A and B, each with one period effect. The illustration considers serial dependence of lag-1, contemporaneous cross-sectional dependence, and cross-sectional dependence of lag-1. For serial dependence or lagged cross-sectional dependence, a variable at time t can affect itself or another variable in a future period (for example, $d_{t,1}$ influences $d_{t+1,2}$), but not vice versa. Therefore, a one-sided arrow is used to indicate directionality. For contemporaneous cross-sectional dependence, where two variables at the same period can affect each other, a double-sided arrow is used.

2.3.2 Vector Autoregression

Let $\{\boldsymbol{d}_t\}$ represent the set $\{\boldsymbol{d}_t \mid t = 2, \ldots, T\}$. The Vector Autoregression model of order p, VAR(p), for the differenced period effects $\{\boldsymbol{d}_t\}$ can be expressed as follows (Hamilton, 2020):

$$\boldsymbol{d}_t = \boldsymbol{c} + \boldsymbol{\Phi}_1 \boldsymbol{d}_{t-1} + \boldsymbol{\Phi}_2 \boldsymbol{d}_{t-2} + \ldots + \boldsymbol{\Phi}_p \boldsymbol{d}_{t-p} + \boldsymbol{\epsilon}_t, \tag{2.2}$$

where $\boldsymbol{\Phi}_l$, for $l = 1, \ldots, p$, is a $(I \times J)$-by-$(I \times J)$ coefficient matrix and $\boldsymbol{\epsilon}_t$ is the vector of error terms defined as:

$$\boldsymbol{\epsilon}_t = [\epsilon_{t,1}, \ldots, \epsilon_{t,I\times J}]'.$$

The error vector $\boldsymbol{\epsilon}_t$ is assumed to follow a multivariate normal distribution, that is, $\boldsymbol{\epsilon}_t \sim \mathcal{N}(\boldsymbol{0}, \Omega)$, where Ω is the covariance matrix. Additionally, it is assumed that the error terms are serially uncorrelated, that is, $E[\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}'_{t-l}] = 0$ for all $l \neq 0$. VAR models are often estimated by Ordinary Least Squares (OLS) or Maximum Likelihood Estimation (MLE). Under normality, OLS and MLE produce the same coefficient estimates, but differ in their estimation of the covariance matrix Ω. In this chapter, we estimate the VAR models using MLE and select the order p that leads to the lowest Bayesian Information Criterion (BIC). [1]

Example 2.3.1 *To capture the dependence structure depicted in Figure 2.1, we consider a VAR(1) model for two differenced period effects, $\boldsymbol{d}_t = [d_{t,1}, d_{t,2}]'$, representing a scenario with two populations, each with a single period effect. The model is given by:*

$$d_{t,1} = c_1 + \phi_{1,1} d_{t-1,1} + \phi_{1,2} d_{t-1,2} + \epsilon_{t,1}, \tag{2.3}$$
$$d_{t,2} = c_2 + \phi_{2,1} d_{t-1,1} + \phi_{2,2} d_{t-1,2} + \epsilon_{t,2}. \tag{2.4}$$

In Equation (2.3), the relationship between $d_{t,1}$ and $d_{t-1,1}$ represents serial dependence, while the relationship between $d_{t,1}$ and $d_{t-1,2}$ captures cross-sectional dependence at lag one. Both types of dependence are linear in VAR models. Linear relationships also imply symmetric impacts, which means that a one-unit increase in $d_{t-1,k}$ produces an equal and opposite effect on $d_{t,k}$ as a one-unit decrease. The contemporaneous cross-sectional dependence between $d_{t,1}$ and $d_{t,2}$ is captured by allowing correlation between the error terms $\epsilon_{t,1}$ and $\epsilon_{t,2}$ by assuming a multivariate normal distribution for $\boldsymbol{\epsilon}_t$.

The VAR model provides a flexible framework for capturing both serial and cross-sectional dependence in differenced period effects. This approach has been widely used in the mortality modeling literature. For example, Zhou et al. (2014) use a Lee-Carter model as the base mortality structure for two populations and then model the differenced period effects from the two populations with the Vector Autoregressive (VAR) model. Yang and Wang (2013) take a similar approach but instead consider four populations. Cairns, Blake, Dowd, Coughlan, and Khalaf-Allah (2011), Dowd et al. (2011), J. S.-H. Li and Hardy (2011), and N. Li and Lee (2005) assume a dominant population whose mortality is not influenced by other populations. They use a random walk with drift model for the period effect of the dominant population and an AR(1) model for the difference in period effects between secondary and dominant populations. This approach can be viewed as a special case of the VAR(1) model.

The VAR approach and its variants have several limitations. First, the assumed linearity and symmetry in serial and cross-sectional dependence may be too restrictive, as real-world mortality often exhibits nonlinear and asymmetric dependence. Second, assuming a multivariate normal distribution for $\boldsymbol{\epsilon}_t$ implies a lack of tail dependence, which may underestimate the probability of extreme co-movements, an important consideration when modeling systemic risks such as longevity risk. Finally, the number of parameters scales rapidly with the number of populations. The VAR(p) model in Equation (2.2) has $I \times J + p(I \times J)^2$ autoregressive coefficients and $\frac{(I\times J)(I\times J+1)}{2}$ covariance parameters, leading to potential overfitting and reduced interpretability in high-dimensional settings.

[1] BIC is defined as $-2L + k\ln(n)$, where L is the log-likelihood, n is the number of observations in the data and k is the number of parameters in the model (Schwarz, 1978).

2.3.3 Sparse Vector Autoregression

One way to mitigate the overfitting and parameter explosion problem in VAR models is to introduce sparsity. A Sparse Vector Autoregressive (sVAR) model achieves this by applying regularization techniques such as lasso, ridge regression, or elastic net, reducing the number of estimated parameters and improving model interpretability. See Han et al. (2015) and the references therein for an overview of the sVAR regularization techniques. This approach is particularly useful in high-dimensional settings, where standard VAR models struggle due to the large number of parameters relative to the available data.

Let $\hat{\boldsymbol{d}}_t$ denote the estimates of $\boldsymbol{d}_t$. The OLS estimates of the VAR model in Equation (2.2) are obtained by minimizing

$$\sum_{t=2}^{T} \|\boldsymbol{d}_t - \hat{\boldsymbol{d}}_t\|_2^2,$$

where $\|.\|_2$ denotes the L2 norm of a vector or matrix. In contrast, using an sVAR model with lasso penalty (Cavalcante et al., 2017), the estimates $\hat{d}_t$ are obtained by minimizing

$$\sum_{t=2}^{T} \|\boldsymbol{d}_t - \hat{\boldsymbol{d}}_t\|_2^2 + \lambda \|\boldsymbol{\Phi}\|_1, \tag{2.5}$$

where $\boldsymbol{\Phi} = [\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2, \ldots, \boldsymbol{\Phi}_p]$, $\|.\|_1$ denotes the L1 norm of a vector or matrix, and λ is a positive regularization parameter, typically chosen by cross-validation. Higher λ values result in a heavier penalty for parameters that deviate from zero, which leads to a more parsimonious model. The L1 penalty can potentially regularize a parameter exactly to 0.

Given a value for λ, numerical optimization is used to obtain parameter estimates that minimize Equation (2.5). Since the optimization algorithm can be sensitive to the scales of the input data, it is common to standardize the data before performing the optimization. We tune the value of λ using five-fold cross-validation. In this process, the dataset is divided into five parts. The model is trained on four parts and validated on the remaining part, and this process is repeated five times with different parts as the validation set. The average validation error is then calculated for each candidate value of λ, and the one that minimizes this error is selected as the optimal regularization parameter. This ensures that the model balances complexity and predictive accuracy.

The sVAR approach has also been utilized in the mortality literature. Guibert et al. (2019) use a sVAR model with an elastic-net penalty to model differenced log mortality rates. The elastic-net penalty is a combination of the lasso penalty and the ridge penalty, expressed as $\lambda_1\|\boldsymbol{\Phi}\|_1 + \lambda_2\|\boldsymbol{\Phi}\|_2$, where λ_1 controls the L1 penalty and λ_2 controls the L2 penalty. Similarly, H. Li and Shi (2021) employ an sVAR model with a lasso penalty but enforce age coherence by imposing constraints to the coefficients of the sVAR.

Although the sVAR model resolves the overparameterization and overfitting issues of the standard VAR model, it still maintains the restrictive dependence structure of the VAR model, including linear and symmetric serial and lagged cross-sectional dependence, and the common assumption of normally distributed error terms, which does not account for tail behavior in the cross-sectional dependence.

Given these limitations, a more flexible approach is needed, one that allows for nonlinear, asymmetric, and tail-dependent relationships while preserving temporal dynamics. This motivates the use of copula-based methods, which we introduce next.

2.3.4 ARIMA-copula

The copula approach is based on the seminal work of Sklar (1959), which establishes that marginal distributions and dependence structures can be modeled separately, offering greater flexibility in multivariate modeling. According to Sklar's theorem, the joint probability density function of the random

variables $Y_1, Y_2, \ldots, Y_n$ can be expressed as:

$$f(y_1, y_2, \ldots, y_n) = f_1(y_1) \cdot f_2(y_2) \cdot \cdots \cdot f_n(y_n) \cdot c(F_1(y_1), F_2(y_2), \ldots, F_n(y_n)), \tag{2.6}$$

where $f(y_1, y_2, \ldots, y_n)$ represents the joint probability density function of the random variables $Y_1, Y_2, \ldots, Y_n$. Here, $F_k(y_k)$ and $f_k(y_k)$ denote the marginal cumulative distribution function and the marginal probability density function of the random variable Y_k, respectively. The function $c(\cdot)$ is an n-dimensional copula density that captures the dependence structure among the variables.

Although traditional copula models effectively capture cross-sectional dependence, they do not inherently account for temporal dependence in time series data. To address this, the ARIMA-copula approach extends the copula framework by first modeling the marginal time series dynamics with ARIMA models before applying copulas to capture dependence among the residuals.

Building on this, the ARIMA-copula approach models the joint distribution of period effects in two steps. Let $\{d_{t,k}\}$ represent the set $\{d_{t,k} \mid t = 2, \ldots, T; k = 1, \ldots, I \times J\}$. Assuming that $\{d_{t,k}\}$ is stationary, the first step, referred to as the 'marginal model', involves fitting an ARIMA$(p, 0, q)$ model to each time series $\{d_{t,k}\}$ with $k = 1, \ldots, I \times J$, expressed as:

$$\left(1 - \sum_{i=1}^{p} \phi_i L^i\right) d_{t,k} = \delta + \left(1 + \sum_{j=1}^{q} \theta_j L^j\right) \epsilon_{t,k},$$

where L denotes the lag operator (Hyndman & Khandakar, 2008). We have that $Lx_t = x_{t-1}$ and $L^i x_t = x_{t-i}$, for $i \in \mathbb{N}$. The error term $\epsilon_{t,k}$ is commonly assumed to follow a normal distribution $N(0, \sigma_k)$, although other distributions may be used. This model can be estimated using OLS or MLE, and the AR order p and the MA order q are typically selected based on criteria such as AIC or BIC. This step yields $(I \times J)$ residual series, $\{\hat{\epsilon}_{t,1}\}, \ldots, \{\hat{\epsilon}_{t,I\times J}\}$.

In the second step, a copula model (Sklar, 1959) is estimated for the error terms $\epsilon_{t,1}, \ldots, \epsilon_{t,I\times J}$, using the residuals obtained earlier. The residuals $\{\hat{\epsilon}_{t,k} \mid t = 2, \ldots, T\}$ are assumed to be i.i.d. samples of the corresponding error term, $\epsilon_{t,k}$. The dependence structure of the error terms is assumed to remain time-invariant. Likelihood-based methods are commonly used for copula estimation.

Several authors have applied the ARIMA-copula approach to jointly model period effects in mortality literature, utilizing different choices of copulas and variants of ARIMA models. For example, Wang et al. (2015) employed an ARIMA model with error terms following a jump diffusion and generalized hyperbolic distributions, while considering multivariate Clayton, Gaussian, Student's t, and skew-Student's t copulas. H. Li and Lu (2019) used a random walk with drift for the marginal model and proposed the use of hierarchical Archimedean copulas for the dependence structure. More recently, Zhou and Ji (2021) implemented an ARIMA-GARCH model with various error term distributions for the marginal and a vine copula for the dependence.

ARIMA-copula models offer the advantage of allowing nonlinear cross-sectional dependence through the use of copulas. However, these models still retain a linear and symmetric serial dependence due to the ARIMA model used for the marginal. This linearity in serial dependence might not fully capture more complex dynamics. Moreover, the ARIMA-copula model captures serial dependence and cross-sectional dependence in two separate steps. Smith (2015) suggests that a one-step model simultaneously capturing both cross-sectional and serial dependence improves empirical analysis in many applications.

2.3.5 ARIMA-copula with a vine copula

Traditional copulas, such as Gaussian and Student-t copulas, are restrictive in high-dimensional settings with multiple time series because they impose a single dependence structure across all variables. Vine

copulas address this limitation by constructing a hierarchical dependence structure, offering greater flexibility in modeling complex dependence across multiple time series.

In this section, we introduce vine copulas and provide an example of their application in the ARIMA-copula approach. Since vine copulas form the foundation for our proposed time series vine copula models, the key features introduced here will serve as the basis for later sections.

Initially explored by Joe (1996), a vine copula is a multivariate copula constructed from bivariate copulas to allow for a more flexible dependence structure. Bedford and Cooke (2001, 2002) formalized this concept and introduced the graphical tree representation of such a copula. A regular vine (R-vine) $\mathcal{V}$ for n variables, $Y_1, Y_2, \ldots, Y_n$, consists of $(n-1)$ trees, $T_1, \ldots, T_{n-1}$, with the following properties (Bedford & Cooke, 2001, 2002):

1. T_i denotes the ith tree with the node set N_i and the edge set E_i. There are $n-i+1$ nodes in the node set N_i.
2. For $i = 2, \ldots, n-1$, the node set of the ith tree is the edge set of the $(i-1)$th tree, that is $N_i = E_{i-1}$.
3. Proximity condition: each edge in the tree T_i connects two nodes that share a common node in the tree T_{i-1}, for $i = 2, \ldots, n-1$.

These properties are illustrated using an example in Example 2.3.2. Two special cases of R-vine, C-vine and D-vine, are commonly used:

- A C-vine is an R-vine where there is a central node in each tree connected to all other nodes in that tree.
- A D-vine is an R-vine where no node is connected to more than two nodes.

To construct an R-vine copula, we associate each edge in the trees with a bivariate copula that captures the dependence between the connected nodes. These copulas can be selected from various families, such as Gaussian, Clayton, and Gumbel, allowing flexibility in modeling different types of dependence.

In an R-vine structure, the dependence is hierarchically modeled across multiple levels, where each edge in tree i represents a conditional dependence given a conditioning set. Specifically, consider an edge $e \in E_i$, denoted as $e = \{j(e)|D(e), k(e)|D(e)\}$. Here, $j(e)$ and $k(e)$ are the two nodes connected by the edge, and $D(e)$ represents the conditioning set, which consists of variables defined in earlier trees of the vine. This setup is visually illustrated in Example 2.3.2. The bivariate copula density associated with this edge is given by:

$$c_{j(e),k(e)|D(e)}\left(F_{j(e)|D(e)}\left(y_{j(e)}|\mathbf{y}_{D(e)}\right), F_{k(e)|D(e)}\left(y_{k(e)}|\mathbf{y}_{D(e)}\right)\right),$$

where $F_{j(e)|D(e)}(.)$ and $F_{k(e)|D(e)}(.)$ denote the conditional distribution functions of $Y_{j(e)}$ and $Y_{k(e)}$ given $\boldsymbol{Y}_{D(e)}$.

The joint density function of the n variables can then be expressed using the R-vine decomposition:

$$f(y_1, \ldots, y_n) = \prod_{k=1}^{n} f_k(y_k) \times \prod_{i=1}^{n-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)}\left(F_{j(e)|D(e)}\left(y_{j(e)}|\mathbf{y}_{D(e)}\right), F_{k(e)|D(e)}\left(y_{k(e)}|\mathbf{y}_{D(e)}\right)\right). \tag{2.7}$$

R-vine copulas can be constructed in many different ways, depending on how variables are paired and conditioned across different layers. The vine structure determines the order in which the dependence is modeled, and different structures can lead to different dependency representations. Given the vast number of possible vine structures, selecting an optimal R-vine structure is a key step in the modeling process.

The selection and estimation of an R-vine copula can be performed using a sequential approach proposed by Dißmann et al. (2013). This process begins by constructing the first tree, which captures the strongest pairwise unconditional dependence and selects the best-fitting bivariate copula for each pair based on AIC/BIC. The process then proceeds layer by layer, constructing subsequent trees that capture the strongest conditional dependence and selecting the best-fitting copulas for each pair of conditional variables. This approach ensures that the complex dependence among multiple variables is captured accurately while maintaining computational feasibility.

To use vine copulas in the ARIMA-copula approach for modeling period effects, we first fit an ARIMA model to each period effect and obtain its residuals. We then construct a copula for the error terms $\boldsymbol{\epsilon}_t = [\epsilon_{t,1}, \ldots, \epsilon_{t,I\times J}]'$, using these residuals.

Example 2.3.2 *As an illustration, consider an R-vine structure for the error terms, $\epsilon_{t,1}, \epsilon_{t,2}, \ldots, \epsilon_{t,5}$, from the ARIMA models for five differenced period effects. Hereby we assume that each population in the base mortality structure is modeled using a single period effect, resulting in five residual series in total.*

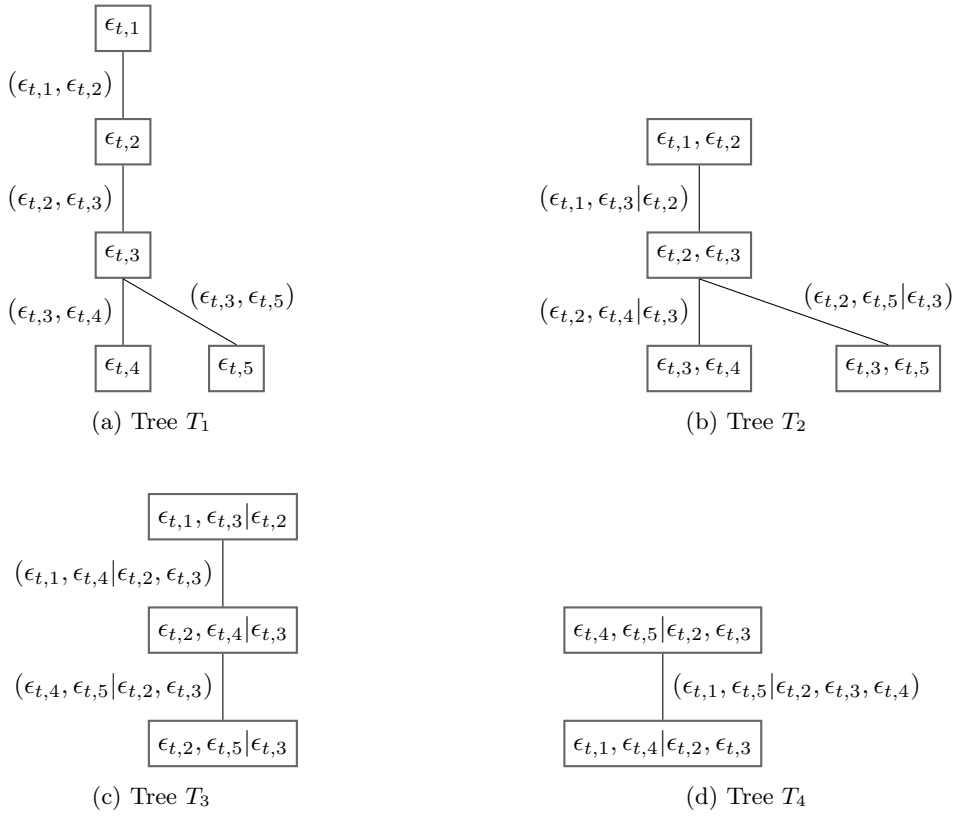
To select an R-vine structure, a sequential approach proposed by Dißmann et al. (2013) is used. A possible R-vine structure is presented in Figure 2.2.

- *The structure consists of four trees.*
- *The tree T_1 is presented in Figure 2.2a. Its node set N_1 contains all the error terms under consideration and $N_1 = \{\epsilon_{t,1}, \epsilon_{t,2}, \epsilon_{t,3}, \epsilon_{t,4}, \epsilon_{t,5}\}$. Its edge set is $E_1 = \{(\epsilon_{t,1}, \epsilon_{t,2}),\ (\epsilon_{t,2}, \epsilon_{t,3}), (\epsilon_{t,3}, \epsilon_{t,4}), (\epsilon_{t,3}, \epsilon_{t,5})\}$. The dependence between the error terms in each element of the edge set is modeled using a bivariate copula.*
- *The tree T_2 is presented in Figure 2.2b. Its node set N_2 is equal to E_1, the edge set of the tree T_1. By the proximity condition, an edge between $(\epsilon_{t,1}, \epsilon_{t,2})$ and $(\epsilon_{t,2}, \epsilon_{t,3})$ is possible because they share a common node $\epsilon_{t,2}$ in tree T_1.*
- *The trees T_3 and T_4 are presented in Figures 2.2c and 2.2d, respectively. It is straightforward to verify that they satisfy the properties of an R-vine.*
- *Consider the edge $\{\{\epsilon_{t,1}, \epsilon_{t,2}, \epsilon_{t,3}\}, \{\epsilon_{t,4}, \epsilon_{t,2}, \epsilon_{t,3}\}\}$ in the tree T_3. Here, we have $e = \{\{\epsilon_{t,1}|\epsilon_{t,2}, \epsilon_{t,3}\}, \{\epsilon_{t,4}|\epsilon_{t,2}, \epsilon_{t,3}\}\}$. We see that the nodes connected by this edge in the tree T_3, $\{\epsilon_{t,1}, \epsilon_{t,3}|\epsilon_{t,2}\}$ and $\{\epsilon_{t,2}, \epsilon_{t,4}|\epsilon_{t,3}\}$, are both edges in the tree T_2. The bivariate copula density associated with this edge in the tree T_3 is given by:*

$$c_{\{\epsilon_{t,1},\epsilon_{t,4}|\epsilon_{t,2},\epsilon_{t,3}\}}\left(F_{\{\epsilon_{t,1}|\epsilon_{t,2},\epsilon_{t,3}\}}\left(y_{\epsilon_{t,1}}|\boldsymbol{y}_{\{\epsilon_{t,2},\epsilon_{t,3}\}}\right), F_{\{\epsilon_{t,4}|\epsilon_{t,2},\epsilon_{t,3}\}}\left(y_{\epsilon_{t,4}}|\boldsymbol{y}_{\{\epsilon_{t,2},\epsilon_{t,3}\}}\right)\right)$$

Although the previous example illustrates one possible R-vine tree structure, alternative structures exist. Figures 2.3a and 2.3b illustrate the first trees of possible C-vine and D-vine structures to model these error terms. In Figure 2.3a, a node $\epsilon_{t,1}$ is connected to all other nodes, forming a star pattern, characteristic of C-vines. In Figure 2.3b, each node is connected to at most two others, resulting in a path-like structure, typical of D-vines. The selected vine structure is driven by the observed dependence patterns in $\boldsymbol{\epsilon}_t$, and other tree structures may also be considered.

Vine copulas provide a suitable framework for modeling period effects, offering flexibility in capturing cross-sectional dependence through the exclusive use of bivariate copulas. This dependence has



(a) Tree T_1 (b) Tree T_2

(c) Tree T_3 (d) Tree T_4

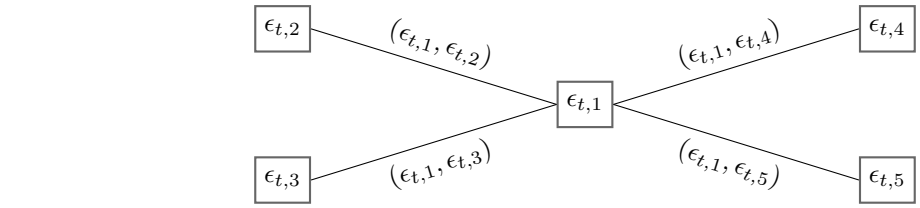
Figure 2.2: A possible R-vine structure for the error terms of ARIMA models fitted to five period effects.

been observed by various authors (T. Lin et al., 2015; Zhu et al., 2017). These structures further accommodate non-linear and asymmetric tail dependence between period effects. Moreover, C-vines and D-vines impose meaningful hierarchical structures, which are particularly useful when prior knowledge about the cross-sectional dependence between period effects is available.

2.4 Time series vine copula and its application to modeling period effects

In this section, we introduce the time series vine copula and discuss its application to modeling period effects. The time series vine copula overcomes several drawbacks of the commonly used approaches reviewed in Section 2.3. Specifically, it allows for fully asymmetric and nonlinear serial and cross-sectional dependence. Additionally, it captures both types of dependence in one step, with a built-in parameter selection procedure in the model fitting process to achieve parsimony. However, the flexibility of time series vine copula models comes at the risk of overfitting. In addition, both estimation and simulation entail computational burden owing to their complex algorithms.

We first describe how a vine copula can model serial dependence in a univariate time series, before extending it to the multivariate case, where it jointly models both serial and cross-sectional dependence.



(a) C-vine structure



(b) D-vine structure

Figure 2.3: First tree of possible C-vine and D-vine structures for five error terms.

2.4.1 Univariate time series vine copula

Smith et al. (2010) introduce the use of a vine copula to construct the joint density of a univariate time series $\{Y_t\}$. Using a D-vine structure, Smith et al. (2010) show that the joint density function of the time series can be written as

$$f(y_1, \ldots, y_T) = \left\{ \prod_{t=1}^{T} f_t(y_t) \right\} \left\{ \prod_{t=2}^{T} \prod_{k=1}^{t-1} c_{t,k|t-1,\ldots,k+1}(u_{t|t-1,\ldots,k+1}, u_{k|t-1,\ldots,k+1}) \right\},$$

where $u_t = F_t(y_t)$ is the distribution function of Y_t evaluated at y_t, $u_{t|D} = F_{t|D}(y_t|\boldsymbol{y}_D)$ is the conditional distribution function of Y_t given $\boldsymbol{Y}_D$ evaluated at y_t given $\boldsymbol{y}_D$, and c is the density of the bivariate copula associated with the edge $\big\{\{t, t-1, \ldots, k+1\}, \{k, t-1, \ldots, k+1\}\big\}$.

Example 2.4.1 *Let us illustrate the time series vine copula with a univariate time series consisting of four time points, denoted $\{Y_1, \ldots, Y_4\}$. A possible D-vine structure for this time series is presented in Figure 2.4. Since $t = 4$, the joint density function based on the D-vine structure can be expressed as a product of marginal densities and bivariate copulas using Equation (2.7):*

$$\begin{aligned} f(y_1, y_2, y_3, y_4) = & f_1(y_1) \cdot f_2(y_2) \cdot f_3(y_3) \cdot f_4(y_4) \\ & \cdot c_{1,2}(u_1, u_2) \cdot c_{2,3}(u_2, u_3) \cdot c_{3,4}(u_3, u_4) \\ & \cdot c_{1,3|2}(u_{1|2}, u_{3|2}) \cdot c_{2,4|3}(u_{2|3}, u_{4|3}) \\ & \cdot c_{1,4|2,3}(u_{1|2,3}, u_{4|2,3}). \end{aligned} \tag{2.8}$$

In Equation (2.8), the joint density function involves six bivariate copulas and four marginal distributions. The model is too complex considering that there are only four observations. Two assumptions could be imposed on the time series to reduce the complexity of the model: strict stationarity and Markov property (Nagler et al., 2022).

A time series is strictly stationary if the joint distribution of any subset of the series remains unchanged when all the time indices are shifted by the same amount. Strict stationarity implies that the joint distribution of the time series remains constant over time. For example, the marginal distribution of Y_t, $F_t(.)$, remains unchanged over time t. Similarly, the joint distribution function $F(y_t, y_k|y_{t-1}, \ldots, y_{k+1})$ remains unchanged when both the time indices t and k are shifted by j, and the same holds for its corresponding bivariate copula $c_{t,k|t-1,\ldots,k+1}$. As a result, we have $c_{t,k|t-1,\ldots,k+1} = c_{t+j,k+j|t+j-1,\ldots,k+j+1}$ under strict stationarity.

The Markov property means that the future state depends on the current state and a finite number of preceding states. The pth-order Markov property can be expressed as follows:

$$\Pr[Y_{t+1} \leq y | Y_t, \ldots, Y_1] = \Pr[Y_{t+1} \leq y | Y_t, \ldots, Y_{t-p+1}]$$

The Markov property of order p is equivalent to imposing an independence constraint on the bivariate copula $c_{t,j|t-1,\ldots,j+1}$ for all $j \leq t-p$, as shown by Smith et al. (2010).

Assuming strict stationarity, the density and distribution functions $f_t(.)$ and $F_t(.)$ in Equation (2.8) remain invariant over time. Therefore, we can denote them as $f(y)$ and $F(y)$, respectively. The bivariate copulas $c_{1,2}$, $c_{2,3}$, and $c_{3,4}$ are identical due to stationarity and can be represented by a single copula c_1. Similarly, the conditional bivariate copulas $c_{1,3|2}$ and $c_{2,4|3}$ are equal and can be denoted as c_2. Furthermore, assuming a Markov order of 2, the copula $c_{1,4|2,3}$ is the independence copula, thus $c_{1,4|2,3}(.) = 1$. Consequently, the joint density function in Equation (2.8) simplifies to:

$$\begin{aligned} f(y_1, y_2, y_3, y_4) = & f(y_1) \cdot f(y_2) \cdot f(y_3) \cdot f(y_4) \\ & \cdot c_1(u_1, u_2) \cdot c_1(u_2, u_3) \cdot c_1(u_3, u_4) \\ & \cdot c_2(u_{1|2}, u_{3|2}) \cdot c_2(u_{2|3}, u_{4|3}), \end{aligned} \tag{2.9}$$

where $u_t = F(y_t)$. Equation (2.9) only involves one marginal distribution and two bivariate copulas. Therefore, the assumptions of stationarity and a Markov order of 2 significantly reduce the complexity of the joint density function. The vine structure copula allows for the selection of specific copulas for c_1 and c_2 in Equation (2.9), so that features, such as asymmetric tail dependence between lagged variables, can be incorporated. When using a Gaussian marginal distribution for Y_t and a Gaussian copula for c_1 and c_2, the joint density function in Equation (2.9) is equivalent to that of a Gaussian AR(2) model.

To construct and estimate a time series vine copula, we begin by identifying an appropriate marginal distribution $F(y)$ for the data. Applying the probability integral transform to each observation y_t yields pseudo-observations $u_t = F(y_t)$. These pseudo-observations, which are uniformly distributed over the interval [0,1], are then used to sequentially select and estimate the bivariate copulas, following the methodology used in the modeling of R-vine copulas (Dißmann et al., 2013).

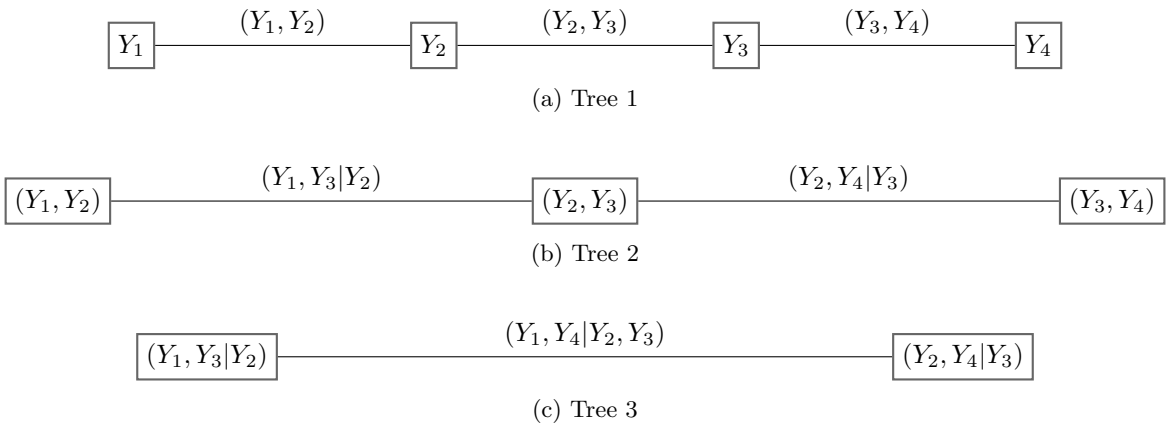
To visualize the dependence structure of a longer time series, it is not necessary to plot all time periods in the vine structure. It is sufficient to present the vine copula trees for $Y_t, Y_{t+1}, \ldots, Y_{t+p}$, where p is the Markov order. This is due to strict stationarity, which ensures that the statistical properties and dependence relationships remain constant over time, and the Markov property of order p, which ensures that each state depends only on the previous p states, making this segment representative of the entire time series.

2.4.2 Multivariate time series vine copulas

The vine copula can also be utilized to simultaneously capture serial and cross-sectional dependence in multivariate time series. Unlike the ARIMA-copula approach, the time series vine copula for multivariate time series is constructed directly on the time series, rather than the residuals from fitting the marginal ARIMA models. Consider a K-dimensional time series denoted by $\boldsymbol{Y}_t = [Y_{t,1}, \ldots, Y_{t,K}]'$, with corresponding realizations $\boldsymbol{y}_t = [y_{t,1}, \ldots, y_{t,K}]'$. Let $\boldsymbol{y} = [\mathbf{y}_1', \mathbf{y}_2', \ldots, \mathbf{y}_T']'$ represent the vectorized form of the entire time series observation over T time periods. According to Sklar's theorem, the joint density function of the multivariate time series can be expressed as

$$f(\boldsymbol{y}) = \left\{ \prod_{t=1}^{T} \prod_{k=1}^{K} f_{t,k}(y_{t,k}) \right\} c(u_{1,1}, \ldots, u_{1,K}, u_{2,1}, \ldots, u_{2,K}, \ldots, u_{T,1}, \ldots, u_{T,K})$$

where $f_{t,k}(.)$ denotes the marginal density function of $Y_{t,k}$, and $u_{t,k} = F_{t,k}(y_{t,k})$ represents the corresponding marginal cumulative distribution function evaluated at $y_{t,k}$.



(a) Tree 1

(b) Tree 2

(c) Tree 3

Figure 2.4: A univariate time series vine structure for $\{Y_t\}_{t=1,\dots,4}$.

The copula density function $c(.)$ encapsulates the dependence structure, including both serial and cross-sectional dependence, among all variables at all time points. Various researchers, including Brechmann and Czado (2015), Beare and Seo (2015), Smith (2015), and Nagler et al. (2022), have demonstrated that vine copulas are well suited to effectively capture these dependencies in multivariate time series. These approaches often achieve better performance than the VAR or ARIMA-copula approaches. The methodology involves:

- Cross-sectional dependence modeling: Constructing a vine structure that captures the cross-sectional dependence among $Y_{t,1}, \dots, Y_{t,K}$ at the same time point t. The cross-sectional structure, which remains the same for all $t = 1, \dots, T$, is a component of the time series vine.
- Temporal linking: In the first tree of the time series vine, link the cross-sectional structures at time t and $t+1$ by connecting a node of the cross-sectional structure at t to a node from that at $t+1$. This "temporal linking" allows for modeling temporal dependence, encompassing both serial dependence and cross-sectional lagged dependence.

Imposing strict stationarity and a Markov order on multivariate time series can significantly simplify the dependence structure, resulting in a more parsimonious model. In this context, these concepts are defined similarly to their univariate counterparts. For example, strict stationarity implies that the joint distribution of the multivariate series remains unchanged when all the time indices are shifted by the same amount. These assumptions impose specific conditions on the structure of the vine copula. Nagler et al. (2022) introduce a generalized structure for such vines, named stationary vine (S-vine), which adheres to strict stationarity.

In this section, we present both the S-vine and its special cases. Due to the limited space, we do not provide technical details of the S-vine and refer interested readers to Nagler et al. (2022) for a detailed discussion.

2.4.2.1 D-vine and M-vine

Before selecting or estimating a time series vine copula, we first obtain pseudo-observations that follow a uniform (0,1) distribution for each time series, as follows:

1. A parametric or non-parametric marginal distribution F_j is estimated separately for each time series, $\{d_{t,j}\}_{j=1,\dots,T}$, as if the observations are independent.

2. The pseudo-observations of each time series are extracted using a probability integral transform (or via the empirical cdf) and denoted as $u_{t,j} = F_j(d_{t,j})$. Let us denote $\boldsymbol{u}_t = (u_{t,1}, \ldots, u_{t,J})'$.

Smith (2015) and Beare and Seo (2015) propose the D-vine and M-vine structures to model multivariate time series, respectively. The selection and estimation procedure is as follows:

1. A cross-sectional D-vine copula is estimated from $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_T\}$, using the procedure described by Aas et al., 2009, as if they are independent observations of a J-dimensional random vector. This estimated cross-sectional D-vine copula applies to all time periods, $t = 1, \ldots, T$.

2. For the temporal link in the first tree of the time series vine copula, both D-vine and M-vine structures have two connection options:

   - In the time series D-vine copula, the opposite ends of the cross-sectional trees must be connected. The two options are: connecting the first variable of the current time point to the last variable of the next time point, or connecting the last variable of the current time point to the first variable of the next time point.
   - In the time series M-vine copula, the same ends of the cross-sectional trees must be connected. The two options are: connecting the first variable of the current time point to the first variable of the next time point, or connecting the last variable of the current time point to the last variable of the next time point.

   The connection with the higher Kendall's tau value is chosen to capture the strongest temporal dependence. A bivariate copula for a temporal link is selected based on BIC and estimated subsequently.

3. The higher tree structures are automatically determined.

   - For the time series D-vine, the higher trees are uniquely determined by the proximity condition.
   - For the time series M-vine, the higher trees are uniquely determined by the "M" shape and the proximity condition.

   The selection and the estimation of the bivariate copulas are performed sequentially, tree by tree.

The D-vine structure proposed by Smith (2015) for multivariate time series has the following characteristics:

- The cross-sectional tree that captures the cross-sectional dependence among $Y_{t,1}, \ldots, Y_{t,k}$ is a D-vine. This D-vine remains unchanged with time because of strict stationarity.
- In the first tree of the time series vine, two cross-sectional structures are connected at the time points t and $t+1$ using an edge which links the opposite ends of the cross-sectional trees.
- Once the first tree is fixed, the remaining trees are determined automatically due to the proximity condition.

The M-vine of Beare and Seo (2015) also uses a D-vine for the cross-sectional tree at each time point. However, in the first tree of the time series vine, two cross-sectional structures at time points t and $t+1$ are connected by nodes that lie at the same end of the cross-sectional trees. Similar to the D-vine, once the first tree is specified, the remaining trees of the M-vine are determined automatically based on the proximity condition.

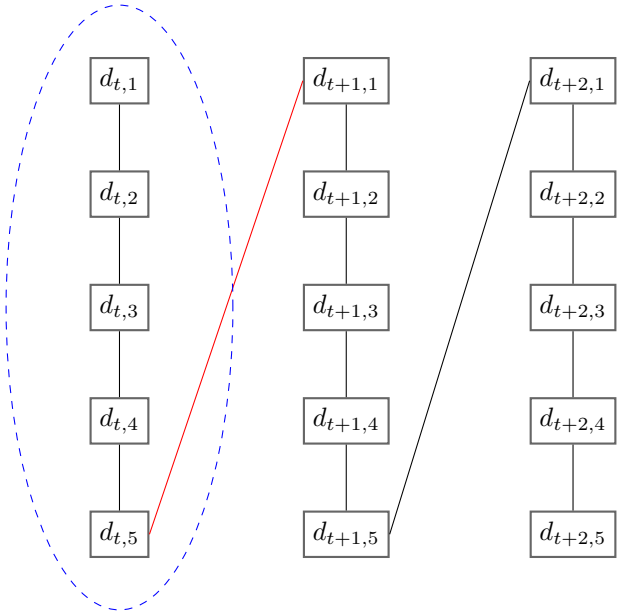


Figure 2.5: First tree of a D-vine for five series of differenced period effects at three time points.

Example 2.4.2 *Consider a five-dimensional time series, $\{\boldsymbol{d}_t\}_{t=1,\ldots,T}$ with each dimension representing a differenced period effect, thus $\boldsymbol{d}_t = [d_{t,1}, d_{t,2}, d_{t,3}, d_{t,4}, d_{t,5}]'$. Figures 2.5 and 2.6 present the first tree of a D-vine and a M-vine for this five-dimensional time series at three time points, respectively.*

The cross-sectional tree for $\boldsymbol{d}_t$ is circled with a blue dashed line, while the temporal link between t and $t+1$ is highlighted as a red line.

Figures 2.5 and 2.6 illustrate how a time series vine copula models different types of dependence in a single step. The cross-sectional tree, shown in the blue dashed circle, follows a path pattern due to the use of a D-vine for cross-sectional dependence. The structure remains the same over time $t, t+1, \ldots$.

The temporal linking between the two figures is different. In Figure 2.5, it is established by connecting $d_{t,5}$ and $d_{t+1,1}$, which are at opposite ends of their respective cross-sectional trees. In Figure 2.6, the connection is between $d_{t,1}$ and $d_{t+1,1}$, which lie at the same end of their respective cross-sectional trees.

While the cross-sectional tree captures contemporaneous cross-sectional dependence, the temporal links represent lagged cross-sectional dependence. The serial dependence, such as between $d_{t,j}$ and $d_{t+1,j}$, is implicitly implied by the vine structure. This is in contrast to the ARIMA-copula based models, which capture serial dependence and cross-sectional dependence in two separate steps.

Once the first tree of the D-vine or M-vine is determined, the remaining trees are determined automatically due to the proximity condition. In Figure 2.5, the first tree of the D-vine stretches to a path, as are the remaining trees. Therefore, these trees together form a D-vine for the multivariate time series. This is a special feature of the D-vine copula for multivariate time series.

Assuming a Markov order of p for the time series vine copula, $d_{t_1,k_1}|\boldsymbol{d}_D$ and $d_{t_2,k_2}|\boldsymbol{d}_D$ are independent when $t_1 > t_2 + p$. This results in a large number of independence copulas at higher trees of the time series vine copula, considerably simplifying the joint density function (see Brechmann et al. (2012) for more details). The strict stationarity condition further simplifies the model, assuming that the joint distribution of $d_{t_1,k_1}|\boldsymbol{d}_D$ and $d_{t_2,k_2}|\boldsymbol{d}_D$ remains the same when all time indices, including t_1, t_2, and those in D, are shifted by the same amount.

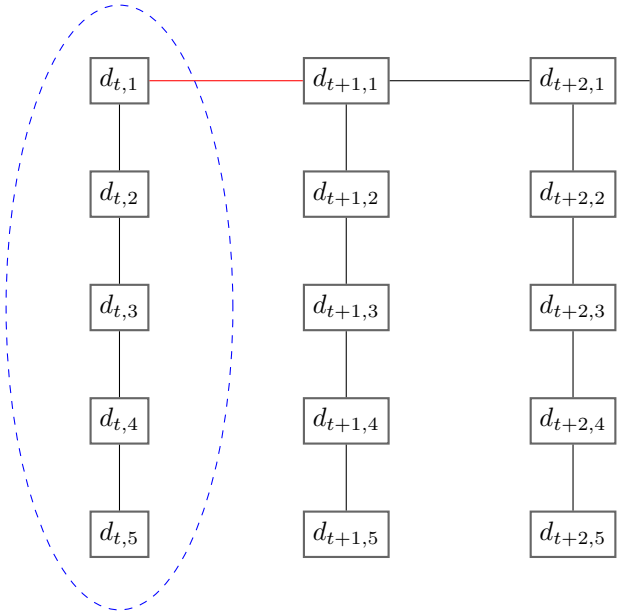


Figure 2.6: First tree of an M-vine for five series of differenced period effects at three time points.

2.4.2.2 Stationary COPAR

The stationary COPAR (S-COPAR) model is proposed by Nagler et al. (2022), by modifying the COPAR model proposed by Brechmann and Czado (2015) to achieve stationarity. We propose a methodology for its use in mortality modeling. First, obtain the pseudo-observations $\boldsymbol{u}_t = (u_{t,1}, \ldots, u_{t,J})$ as described in section 2.4.2.1.

In what follows, "serial tau" of a time series refers to the Kendall's tau computed between the observation of the underlying random variable at time t and its observation at time $t-1$. For instance, the serial tau for $\{d_{t,j}\}$ represents the Kendall's tau between $d_{t,j}$ and $d_{t-1,j}$. The selection and estimation procedures for S-COPAR proceed as follows:

1. Select and estimate the cross-sectional C-vine trees.
   (a) For each $j = 1, \ldots, J$, compute the empirical serial tau for the time series $\{d_{t,j}\}$. Among the J time series, select as the center node of the first tree the one with the maximum absolute serial tau.
   (b) Use BIC as the selection criterion to select and then estimate the bivariate copulas in the first tree. Compute observations of conditional variables, which are conditional distributions based on the estimated first tree. These observations, such as $F(d_{t,1}|d_{t,2})$, due to the time index, can also be viewed as observations of time series. Therefore, we refer to them as conditional time series.
   (c) Compute the empirical serial tau for each conditional time series obtained from the previous step. Set the center node of the second tree to be the conditional time series with the largest absolute serial tau.
   (d) Use BIC as the selection criterion to select and then estimate the bivariate copulas in the second tree. Compute observations of conditional time series for the third tree.
   (e) For the third tree onward, proceed the same way as with the second tree.
2. The center nodes of the cross-sectional trees are used for the temporal links. Use BIC to select the bivariate copula for the temporal link and then estimate the selected copula.

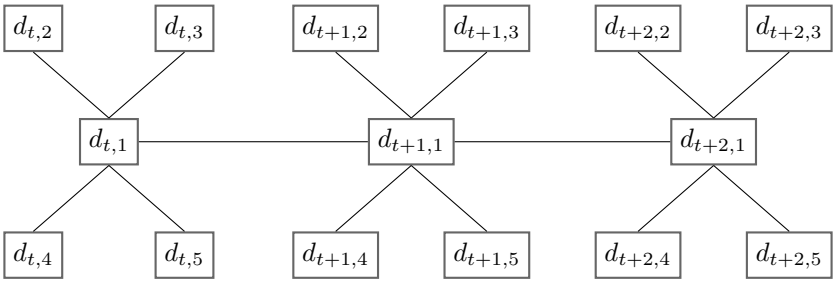


Figure 2.7: First tree of an S-COPAR for five series of differenced period effects at three time points.

This S-COPAR has several features:

- A C-vine is used for the cross-sectional dependence.
- In the first tree of the time series vine, a primary time series is assumed and set as the center node of the C-vine. The center nodes of the cross-sectional structures at time t and $t+1$ are then connected to capture the serial dependence.
- Unlike M-vine and D-vine, the structures of higher trees in S-COPAR are not unique, thus requiring selection.

Example 2.4.3 *Consider a five-dimensional time series, $\{\boldsymbol{d}_t\}_{t=1,\dots,T}$, with each dimension representing a differenced period effect, thus $\boldsymbol{d}_t = [d_{t,1}, d_{t,2}, d_{t,3}, d_{t,4}, d_{t,5}]'$. Figure 2.7 presents the first tree of an S-COPAR for this five-dimensional time series at three time points, respectively.*

In Figure 2.7, $\{d_{t,1}\}$ serves as the "primary" time series and the center node of the cross-sectional trees. The center nodes of cross-sectional trees $d_{t,1}, d_{t+1,1}, \dots, d_{t+2,1}, \dots$ are connected to capture the temporal dependence.

Assuming a primary time series is reasonable in the context of multi-population mortality modeling, especially when considering a large population and its sub-populations. For instance, an insurer operating in a single country with multiple product lines might have each insured population as a sub-population of the national population. Joint modeling of the national population and its sub-populations provides several benefits. The national population data captures macro-level trends and risks that might not be apparent in subpopulation data alone. National data is often of high quality and completeness, which can improve the prediction of subpopulation mortality.

However, there are cases where this approach may be less suitable or less desirable. If no single population clearly dominates, selecting a primary time series may introduce bias or fail to capture important heterogeneity among populations. In such cases, other criteria may be more appropriate for determining the center node of the C-vine. For example, instead of defaulting to the largest population, we might choose the time series with the highest serial dependence as the primary series to ensure better temporal consistency in the model. Alternatively, in highly heterogeneous populations, a more flexible structure—such as S-vine—might be preferable to allow for greater adaptability in capturing dependence.

Since the tree structure of an S-COPAR is not unique, we propose a selection method to capture the most serial dependence in lower trees.

2.4.2.3 S-vine

Nagler et al. (2022) propose a generalization of the models presented by Beare and Seo (2015) and Smith (2015), known as stationary time series vine copula models, or "stationary vine copula" (S-vine)

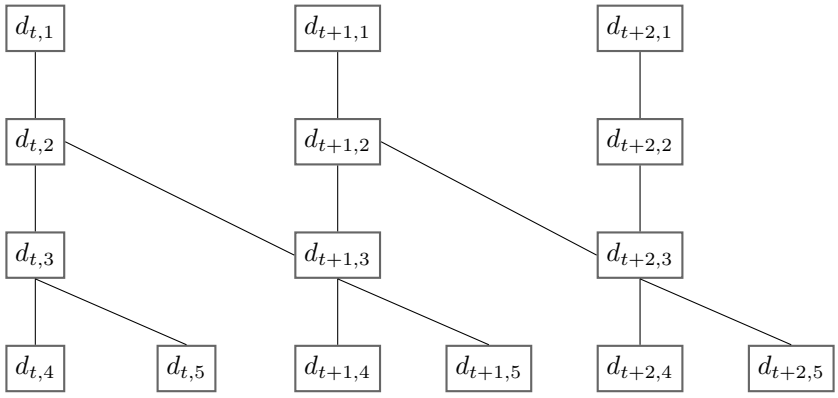


Figure 2.8: Tree one of an S-vine for five series of differenced period effects at three time points.

for short. The S-vine model achieves high versatility by allowing flexible cross-sectional structures and temporal links while maintaining strict stationarity.

The sequential procedures for selecting and estimating the S-vine time series copula are provided by Nagler et al., 2022. We provide a brief summary below:

1. A cross-sectional R-vine structure is selected and estimated for $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_T\}$, following the sequential method suggested by Dißmann et al., 2013.

2. The temporal link is chosen to connect the two nodes with the highest absolute value of Kendall's tau. For instance, in the first tree, the temporal link should connect the pair $u_{t,i}$ and $u_{t+1,j}$ that has the highest absolute Kendall's tau for all $i, j \in \{1, \ldots, J\}$.

The S-vine has the following features:

- S-vine allows for the use of a more flexible R-vine for the cross-sectional structure.
- S-vine allows for arbitrary nodes to be connected over time, provided that the proximity condition is satisfied.
- The M-vine, D-vine, and S-COPAR models are all special cases of the S-vine model.

Example 2.4.4 *Consider a five-dimensional time series, $\{\boldsymbol{d}_t\}_{t=1,\ldots,T}$, with each dimension representing a differenced period effect, thus $\boldsymbol{d}_t = [d_{t,1}, d_{t,2}, d_{t,3}, d_{t,4}, d_{t,5}]'$. Figure 2.8 presents the first tree of an S-vine for this five-dimensional time series at three time points, respectively.*

For Example 2.4.4, we intentionally present an R-vine structure for the cross-sectional tree in Figure 2.8. Since D-vine and C-vine are special cases of R-vine, adopting an R-vine provides greater flexibility in modeling cross-sectional dependence. The temporal link in Figure 2.8 is established by connecting $d_{t,2}$ and $d_{t+1,3}$. Unlike the time series D-vine, M-vine, and S-COPAR, there are no constraints on which two nodes from adjacent time points can be linked. This further improves the flexibility of the S-vine.

The flexibility of the S-vine tree structure allows us to tailor the model to the underlying data while maintaining parsimony. Nagler et al. (2022) propose a sequential method for selecting and estimating the S-vine model, aiming to maximize dependence capture in the lower trees. As a result, more independent copulas are used in the upper trees.

2.5 Data and estimated period effects

In this section, we describe the mortality data used in our analysis. We outline the setup of the single-population mortality model introduced in Section 2.2 and fit it to each population. In addition, we conduct an exploratory analysis of the estimated period effects from the fitted mortality structures. For the following sections, all analyses are implemented in R [version 4.2.0]. We use the `svines` package (Nagler, 2025) to fit and simulate time series vine copula models. For illustrations, we use the `ggplot2` package (Wickham, 2016).

2.5.1 Data

We obtained the number of deaths and exposure-at-risk for the years 1950-2019 and ages 50-90 for four populations, Great Britain (GBR), Denmark (DNK), the United States (USA), and Australia (AUS) — from the Human Mortality Database (Human Mortality Database, 2022). Our analysis uses aggregated gender data that sum male and female deaths and exposures for the death counts and exposure measures, respectively.

The data set is divided into two periods: 1950-2000 for training and 2001-2019 for testing. Two sets of analyses are performed. The first set, involving Great Britain and Australia, illustrates how time series vine copulas can be applied in mortality modeling and how their fitting and forecasting performance compare to existing approaches. The second set, involving all four populations, further demonstrates the usefulness of time series vine copulas in higher dimensions.

2.5.2 Base mortality structure

We adopt a modified version of the M6 model (Cairns et al., 2009), referred to as "log-M6" model hereafter, as the base mortality structure. This model is expressed as follows:

$$\ln \mu_{x,t,j} = \kappa_{t,j}^{(1)} + (x - \overline{x})\, \kappa_{t,j}^{(2)} + \gamma_{t-x,j}, \tag{2.10}$$

where $\overline{x}$ denotes the average of the age range, $x_0, \ldots, x_1$.

The log-M6 model differs from the original M6 model in that it uses a log-link for mortality rates rather than a logit-link for death rates. The M6 model has been shown to have a reasonably good fit to mortality data by various researchers (Cairns, Blake, Dowd, Coughlan, Epstein, & Khalaf-Allah, 2011; J. S.-H. Li et al., 2015; Wen et al., 2021). However, we find that the first order difference of the period effects from the original M6 model can still show a significant degree of non-stationarity. The use of log-link with mortality rates alleviates the problem.

Denote the number of deaths and exposure-at-risk at age x in year t from population j by $D_{x,t,j}$ and $E_{x,t,j}$, respectively. Assuming that $D_{x,t,j} \sim \text{Poisson}(E_{x,t,j}\mu_{x,t,j})$, the log-likelihood function for the log-M6 model can be written as follows:

$$\sum_{x,t,j} \left[D_{x,t,j} \log\left(E_{x,t,j}\mu_{x,t,j}\right) - E_{x,t,j}\mu_{x,t,j} - \log\left(D_{x,t,j}!\right) \right].$$

MLE is applied to estimate the model. To ensure model identifiability, the following parameter constraints are imposed: $\sum_{c\in\mathcal{C}} \gamma_{c,j} = 0$, and $\sum_{c\in\mathcal{C}} c\gamma_{c,j} = 0$, where $\mathcal{C}$ is the set of years of birth. Note that the identifiability constraints are model-specific and vary depending on the model chosen. See Cairns et al. (2009) for a detailed discussion. For instance, the M7 model requires an additional constraint, $\sum_{c\in\mathcal{C}} c^2\gamma_{c,j} = 0$, to ensure identifiability in the presence of an extra period effect term $\kappa_{t,j}^{(3)}$.

Alternative setups within the general framework of Hunt and Blake (2021) could also be used. The further modeling of period effects follows a similar approach.

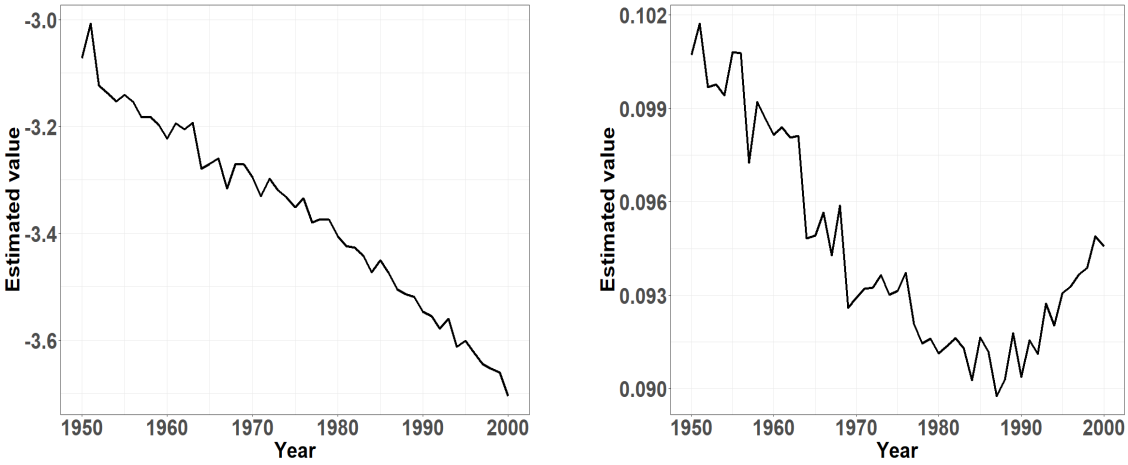
2.5.3 Estimated period effects

In this section, we explore the statistical properties of the estimated values for $\kappa_{t,j}^{(i)}$. In particular, we are interested in their stationarity, distribution, and dependence structure. This initial exploration underpins our model choices.

2.5.3.1 Stationarity

Stationarity in time series is crucial, as it is a key assumption underlying most time series models, including VAR and time series vine copula models. Figure 2.9 presents the estimates for $\kappa_{t,\text{GBR}}^{(1)}$ and $\kappa_{t,\text{GBR}}^{(2)}$, the two period effects for Great Britain. It is evident that both period effects are non-stationary, thus requiring differencing.

To ensure that a first-order difference is sufficient for stationarity, we apply the Augmented Dickey-Fuller (ADF) test to the differenced period effects. Given that there are two period effects for each of the four populations, the test is conducted on all eight period effects. Table 2.1 summarizes the p-values of the ADF tests. At a significance level of 5%, we reject the null hypothesis that there exists a unit root for six of the eight tests. For the two period effects where the null hypothesis cannot be rejected at the 5% level, the p-values are 0.053 and 0.132, indicating a small degree of non-stationarity. However, taking a second-order difference to achieve stationarity may lead to unreasonable mortality forecasts. Therefore, we proceed with the first-order differenced series, which is also consistent with common practice in the existing literature.



(a) $\kappa_{t,\text{GBR}}^{(1)}$ (b) $\kappa_{t,\text{GBR}}^{(2)}$

Figure 2.9: Estimated period effects for Great Britain.

Population j	USA	GBR	DNK	AUS
$\Delta\kappa_{t,j}^{(1)}$	0.046	<0.01	0.047	0.053
$\Delta\kappa_{t,j}^{(2)}$	0.132	<0.01	0.045	<0.01

Table 2.1: p-values of ADF tests performed on differenced period effects.

2.5.3.2 Normality

When using VAR and ARIMA-copula models (and their extensions) for period effects, normally distributed error terms are commonly assumed for mortality forecasting and simulation (Guibert et al., 2019; Zhou, 2019; Zhou et al., 2014). However, as shown in Table 2.2, there is significant skewness and excess kurtosis in the period effects for some populations. This deviation from normality suggests that assuming normality may lead to an inaccurate understanding of mortality and longevity risks.

Skewness and kurtosis are particularly pronounced for Great Britain. We also performed the Shapiro-Wilk normality test (Royston, 1982) on $\kappa^{(1)}_{t,\text{GBR}}$ and $\kappa^{(2)}_{t,\text{GBR}}$. The normality tests result in p-values of 0.0113 and 0.0029, respectively. Thus, the null hypothesis of normal distribution is rejected at the 5% significance level for the period effects of Great Britain.

Population j		USA	GBR	DNK	AUS
$\Delta\kappa^{(1)}_{t,j}$	Skewness	0.29	−0.76	−0.32	0.07
	Excess kurtosis	−0.52	2.48	0.16	−0.90
$\Delta\kappa^{(2)}_{t,j}$	Skewness	0.06	−0.99	−0.62	−0.22
	Excess kurtosis	−0.20	1.22	0.63	−0.87

Table 2.2: Skewness and excess kurtosis of $\Delta\kappa^{(i)}_{t,j}$.

2.5.3.3 Asymmetric tail dependence

The use of VAR and SparseVAR models assumes a linear and symmetric dependence structure for period effect time series. However, several authors (Chuliá et al., 2016; H. Li & Lu, 2019; Wang et al., 2015; Zhou, 2019) have noted the presence of nonlinear cross-sectional dependence, while Wang et al. (2015) identified the existence of asymmetric tail dependence.

To explore whether asymmetric tail dependence is present in our dataset, we select bivariate copulas that best fit various time series pairs, including $\left(\Delta\kappa^{(i_1)}_{t,j_1}, \Delta\kappa^{(i_2)}_{t,j_2}\right)$ and $\left(\Delta\kappa^{(i_1)}_{t,j_1}, \Delta\kappa^{(i_2)}_{t-1,j_2}\right)$, where $j_1, j_2 \in \{\text{AUS, GBR, DNK, USA}\}$ and $i_1, i_2 \in \{1, 2\}$. We first obtain pseudo-observations of each time series by ranking the data and scaling these ranks to the unit interval $[0, 1]$ using the empirical distribution function. We then select the best-fitting bivariate copula for each pair from the list of copulas, including independence, Gaussian, Student's t, Clayton, Gumbel, Frank, and Joe.

The functional form of each copula is denoted as $C(u, v)$, where $u, v \sim \text{Uniform}(0, 1)$. $\Phi^{-1}(\cdot)$ represents the inverse CDF of the standard normal distribution. $\Phi_\rho(\cdot, \cdot)$ is the joint CDF of a bivariate normal distribution with a mean vector of zero and correlation parameter ρ. $t^{-1}_\nu(\cdot)$ represents the inverse CDF of a Student's t-distribution with ν degrees of freedom. $t_{\nu,\rho}(\cdot, \cdot)$ represents the joint CDF of a bivariate Student's t-distribution with a mean vector of zero, the correlation parameter ρ, and ν degrees of freedom.

Table 2.4 lists three pairs that demonstrate asymmetric tail dependence for serial, contemporaneous cross-sectional, and lagged cross-sectional dependence, respectively. The choice of a rotated Clayton 90° copula for $\left(\Delta\kappa^{(2)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t-1,\text{AUS}}\right)$ with an estimated Kendall's tau of −0.22 represents that when the variable at time $t-1$ is at a low value, its value at time t tends to be high. Although Kendall's tau is relatively small, the fact that an independence copula is not selected indicates the presence of mild dependence. A rotated Gumbel 180° copula with an estimated Kendall's tau of 0.32 is selected for $(\Delta\kappa^{(2)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t-1,\text{USA}})$. This suggests that lower extremes in one variable are moderately associated with lower extremes in the other. A rotated Joe 270° copula with an estimated Kendall's tau of −0.29

Family	$C(u,v)$	Parameter
Independence	uv	N/A
Gaussian	$\Phi_\rho\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$	$\rho \in [-1,1]$
Student's t	$t_{\nu,\rho}\left(t_\nu^{-1}(u), t_\nu^{-1}(u)\right)$	$\rho \in [-1,1]$, $\nu > 0$
Clayton	$\left[\max\left(u^{-\theta}+v^{-\theta}-n+1, 0\right)\right]^{-1/\theta}$	$\theta \in [-1,\infty)\backslash\{0\}$
Gumbel	$\exp\left(-\left(-\log(u)^{\theta}-\log(v)^{\theta}\right)^{\frac{1}{\theta}}\right)$	$\theta \in [1,\infty)$
Frank	$-\frac{1}{\theta}\log\left[1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right]$	$\theta \in (-\infty,\infty)\backslash\{0\}$
Joe	$1-\left[1-\left(1-(1-u)^{\theta}\right)\left(1-(1-v)^{\theta}\right)\right]^{1/\theta}$	$\theta \in [1,\infty)$

Table 2.3: List of the copula families used in our analysis and their corresponding functional forms and parameter domains (Nelsen, 2007).

is selected for $\left(\Delta\kappa^{(1)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t-1,\text{AUS}}\right)$, indicating that a low value at time $t-1$ observed for $\Delta\kappa^{(2)}_{t-1,\text{AUS}}$ is associated with a high value at time t for $\Delta\kappa^{(1)}_{t,\text{AUS}}$.

Therefore, there is evidence of asymmetric tail dependence in each dependence type—serial, contemporaneous cross-sectional, and lagged cross-sectional. Among the models introduced in Sections 2.3 and 2.4, the time series vine copula can effectively capture the asymmetric dependence in all three types of dependence.

Time series pair	Dependence Type	Selected copula	Kendall's Tau
$\left(\Delta\kappa^{(2)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t-1,\text{AUS}}\right)$	Serial	Rotated Clayton 90°	-0.22
$\left(\Delta\kappa^{(2)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{USA}}\right)$	Cross-sectional	Rotated Gumbel 180°	0.32
$\left(\Delta\kappa^{(1)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t-1,\text{AUS}}\right)$	Lagged cross-sectional	Rotated Joe 270°	-0.29

Table 2.4: Selected bivariate copulas and estimated Kendall's tau for three pairs of time series.

2.6 Estimated models for period effects with a two-population example

In this section, we model the estimated period effects using VAR, sVAR, ARIMA-copula, and time series vine copulas. To facilitate easier interpretation, we focus on the models estimated for two populations, Great Britain and Australia.

2.6.1 Estimated VAR model for GBR and AUS

Table 2.5 presents the estimated parameters for the VAR model applied to the differenced period effects of GBR and AUS. The lag order of 1 is selected for the VAR models, as it resulted in the lowest BIC. The p-values of the estimated coefficients indicate that, although there is significant serial and

cross-sectional dependence at a 5% significance level, not all coefficients are significant. The presence of several insignificant coefficients implies that the model could potentially be simplified.

The estimated covariance matrix with the corresponding correlations for the residuals of the VAR model is presented in Table 2.6. Pearson's correlation tests were conducted to assess the significance of these cross-sectional correlations. Significant correlations, which indicate contemporaneous cross-sectional dependence, were observed at the 5% significance level in several pairs of variables, including $(\Delta\kappa^{(1)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{GBR}})$, $(\Delta\kappa^{(1)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t,\text{AUS}})$, and $(\Delta\kappa^{(2)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{AUS}})$. The presence of several insignificant correlations in the residuals suggests that the covariance structure may be simplified.

Dependent variable	Intercept	Coefficient for $\Delta\kappa^{(1)}_{t-1,\text{GBR}}$	Coefficient for $\Delta\kappa^{(1)}_{t-1,\text{AUS}}$	Coefficient for $\Delta\kappa^{(2)}_{t-1,\text{GBR}}$	Coefficient for $\Delta\kappa^{(2)}_{t-1,\text{AUS}}$
$\Delta\kappa^{(1)}_{t,\text{GBR}}$	−0.0251***	−0.3720**	−0.3184**	−4.4843	7.1140*
$\Delta\kappa^{(1)}_{t,\text{AUS}}$	−0.0174***	0.0224	−0.2662*	6.0381	−11.2621***
$\Delta\kappa^{(2)}_{t,\text{GBR}}$	−0.0007***	−0.0064	−0.0131**	−0.3312*	0.1535
$\Delta\kappa^{(2)}_{t,\text{AUS}}$	−0.0001	−0.0130*	−0.0185***	0.4403**	−0.2290

Table 2.5: Estimated parameters of the VAR(1) model fitted to the differenced period effects of GBR and AUS. Significance levels: '***' for p-values $p \leq 0.01$, '**' for $0.01 < p \leq 0.05$, '*' for $0.05 < p \leq 0.10$. Blank indicates p-value $p \geq 0.1$.

Variables	$\Delta\kappa^{(1)}_{t,\text{GBR}}$	$\Delta\kappa^{(1)}_{t,\text{AUS}}$	$\Delta\kappa^{(2)}_{t,\text{GBR}}$	$\Delta\kappa^{(2)}_{t,\text{AUS}}$
$\Delta\kappa^{(1)}_{t,\text{GBR}}$	551.70 (1***)	86.19 (0.152)	11.24 (0.479***)	1.53 (0.06)
$\Delta\kappa^{(1)}_{t,\text{AUS}}$	86.19 (0.152)	584.50 (1***)	−0.21 (−0.009)	8.45 (0.338**)
$\Delta\kappa^{(2)}_{t,\text{GBR}}$	11.24 (0.479***)	−0.21 (−0.009)	1.00 (1***)	0.50 (0.481***)
$\Delta\kappa^{(2)}_{t,\text{AUS}}$	1.53 (0.063)	8.45 (0.338**)	0.50 (0.481***)	1.07 (1***)

Table 2.6: Variance-covariance matrix with correlations in brackets for the residuals of the VAR model fitted to the differenced period effects of GBR and AUS (scaled by 10^6). Significance levels for Pearson's correlation tests between variables are indicated as follows: '***' for p-values $p \leq 0.01$, '**' for $0.01 < p \leq 0.05$, '*' for $0.05 < p \leq 0.10$. Blank indicates p-value $p \geq 0.1$.

2.6.2 Estimated sVAR model for GBR and AUS

Table 2.7 presents the estimated coefficients for the sVAR model. Since these coefficients are derived from standardized data, their magnitudes are not directly comparable to those in Table 2.5. Standardizing the data enhances the interpretability of the coefficients, as their magnitudes now reflect the impact of the corresponding variables. We observe that three coefficients have been reduced to zero, suggesting that these variables contribute minimally to the prediction power. These same coefficients are also insignificant at the 10% significance level in the VAR model, as shown in Table 2.5, further supporting their lack of predictive power. This demonstrates a key advantage of sVAR over VAR. By enforcing sparsity, sVAR achieves greater model parsimony while retaining essential predictive relationships.

Among the four coefficients for the dependent variable $\Delta\kappa^{(1)}_{t,\text{GBR}}$, the largest comes from $\Delta\kappa^{(1)}_{t-1,\text{GBR}}$, indicating that its own lagged value has the most significant impact. The largest coefficient for $\Delta\kappa^{(2)}_{t,\text{GBR}}$ comes from $\Delta\kappa^{(1)}_{t-1,\text{AUS}}$, suggesting that the lagged value of a different variable plays the most significant

role. Therefore, the sVAR model also suggests that there exists significant serial and lagged cross-sectional dependence. The estimated covariance matrix for the sVAR model is similar to that of the VAR model presented in Table 2.6, so we do not provide it here.

Dependent variable	Intercept	Coefficient for $\Delta\kappa^{(1)}_{t-1,\text{GBR}}$	Coefficient for $\Delta\kappa^{(1)}_{t-1,\text{AUS}}$	Coefficient for$\Delta\kappa^{(2)}_{t-1,\text{GBR}}$	Coefficient for $\Delta\kappa^{(2)}_{t-1,\text{AUS}}$
$\Delta\kappa^{(1)}_{t,\text{GBR}}$	-0.040	-0.387	-0.132	0	0.060
$\Delta\kappa^{(1)}_{t,\text{AUS}}$	-0.021	0	-0.235	0.112	-0.368
$\Delta\kappa^{(2)}_{t,\text{GBR}}$	-0.009	-0.170	-0.194	-0.187	0
$\Delta\kappa^{(2)}_{t,\text{AUS}}$	0.023	-0.063	-0.458	0.087	-0.035

Table 2.7: Estimated parameters of the sVAR model fitted to the standardised differenced period effects of GBR and AUS.

2.6.3 Estimated ARIMA-copula model for GBR and AUS

The ARIMA orders selected for $\Delta\kappa^{(i)}_{t,j}$ based on BIC are presented in Table 2.8. These ARIMA models are estimated using MLE, assuming that the errors follow a normal distribution. The residuals from these ARIMA models are then extracted and further modeled using an R-vine copula, as described in Section 2.3.4. These residuals are assumed to be marginally normally distributed with variance parameters obtained from MLE estimation, consistent with most of the existing literature (H. Chen et al., 2015; T. Lin et al., 2015; Zhu et al., 2017). We consider the bivariate copula families listed in Table 2.3 and their rotations for the construction of the R-vine copula.

Series	ARIMA order
$\Delta\kappa^{(1)}_{t,\text{GBR}}$	(0,0,1)
$\Delta\kappa^{(1)}_{t,\text{AUS}}$	(1,0,0)
$\Delta\kappa^{(2)}_{t,\text{GBR}}$	(0,0,1)
$\Delta\kappa^{(2)}_{t,\text{AUS}}$	(0,0,0)

Table 2.8: Selected ARIMA orders for $\Delta\kappa^{(i)}_{t,j}$ based on BIC.

The first tree of the selected R-vine copula for the four residual series is shown in Figure 2.10. The two higher trees are not presented since they select an independent copula for all edges. In the first tree, the dependence between $\Delta\kappa^{(1)}_{t,\text{GBR}}$ and $\Delta\kappa^{(2)}_{t,\text{GBR}}$ is best explained by a Student's t copula with the parameter estimate of 0.5 and degrees of freedom of 11.3. While the selection of a Student's t copula suggests the presence of symmetric upper- and lower-tail dependence between the two time series, the high degrees of freedom imply that this tail dependence is rather mild. The copula selected between $\Delta\kappa^{(2)}_{t,\text{GBR}}$ and $\Delta\kappa^{(2)}_{t,\text{AUS}}$ is a Survival Gumbel copula (Gumbel rotated 180°) with a parameter estimate of 1.36, indicating lower tail dependence between the two variables.

Interestingly, the copula connecting $\Delta\kappa^{(2)}_{t,\text{AUS}}$ and $\Delta\kappa^{(1)}_{t,\text{AUS}}$ is chosen to be independent. Given that the higher trees consistently select independence copulas, the independence between $\Delta\kappa^{(2)}_{t,\text{AUS}}$ and $\Delta\kappa^{(1)}_{t,\text{AUS}}$ implies that $\Delta\kappa^{(1)}_{t,\text{AUS}}$ is independent from the other three time series, both contemporaneously and with respect to any lagged dependence. However, this finding contrasts with the results from the VAR and sVAR models, which indicate strong contemporaneous and lagged cross-sectional dependence.

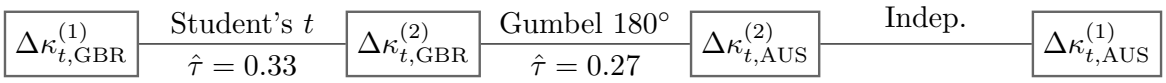


Figure 2.10: First tree of the R-vine in the estimated ARIMA-copula model.

This discrepancy may arise from the limitation of the ARIMA-copula approach, which only explicitly models contemporaneous cross-sectional dependence, even though it effectively captures serial dependence within each time series. As a result, the ARIMA-copula might overlook lagged dependence between the time series, potentially leading to an incomplete understanding of the relationships and less accurate inferences about the overall dependence structure.

2.6.4 Estimated time series vine copula models

To estimate time series vine copulas, we begin with selecting a marginal distribution for each time series based on BIC. Given that $\Delta\kappa^{(i)}_{t,j}$ can take both positive and negative values, we consider the following marginal distributions: Normal, skew-Normal, Laplace, Gumbel, Cauchy, Student's t, skew-Student's t, Generalized error, skew-Generalized error, and log-Gamma. This selection includes distributions with different skewness and kurtosis properties. The parameter estimates of these distributions are obtained using MLE. The selected marginal distributions are presented in Table 2.9.

The Laplace distribution is selected for the two GBR period effects. As shown in Table 2.2, both $\Delta\kappa^{(1)}_{t,\text{GBR}}$ and $\Delta\kappa^{(2)}_{t,\text{GBR}}$ exhibit large excess kurtosis, suggesting heavy tails in mortality improvement. As a result, the Laplace distribution, with an excess kurtosis of 3, is preferred over the Normal distribution which has no excess kurtosis.

Series	Marginal distribution
$\Delta\kappa^{(1)}_{t,\text{GBR}}$	Laplace $(\mu = -0.0146, \sigma = 0.0204)$
$\Delta\kappa^{(1)}_{t,\text{AUS}}$	Normal $(\mu = -0.0159, \sigma = 0.0292)$
$\Delta\kappa^{(2)}_{t,\text{GBR}}$	Laplace $(\mu = -0.0001, \sigma = 0.0008)$
$\Delta\kappa^{(2)}_{t,\text{AUS}}$	Normal $(\mu = 0.0001, \sigma = 0.0012)$

Table 2.9: Selected and estimated marginal distributions for $\Delta\kappa^{(i)}_{t,j}$, $i = 1, 2$ and $j = \text{GBR}, \text{AUS}$

After determining the marginal distributions, we transform the data into uniform (0,1) values using their estimated marginal CDFs. The transformed data is then used to estimate the time series vine copula models. Since we selected a lag order of 1 for the VAR model, it is reasonable to use a Markov order of 1 for the time series vine copula to ensure consistency and improve comparability. The bivariate copulas considered in the construction of the time series vine copula include those provided in Table 2.3 and their rotations.

Figures 2.11a, 2.11b, 2.11c and 2.11d present the first trees of the estimated time series vine copula models using D-vine, M-vine, S-vine, and S-COPAR respectively. The cross-sectional trees for D-vine, M-vine and S-vine models are identical and connect $\Delta\kappa^{(1)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{AUS}}$, and $\Delta\kappa^{(1)}_{t,\text{AUS}}$ with a path structure. The D-vine and M-vine models require a D-vine (path) structure for the cross-sectional trees by design. The S-vine model, despite having the flexibility to use any R-vine structures, also selects a D-vine structure for the cross-sectional trees because it captures the most cross-sectional dependence among all possible R-vine tree structures. On the other hand, the S-COPAR cross-sectional tree connects $\Delta\kappa^{(1)}_{t,\text{GBR}}$ as the center node of the C-vine.

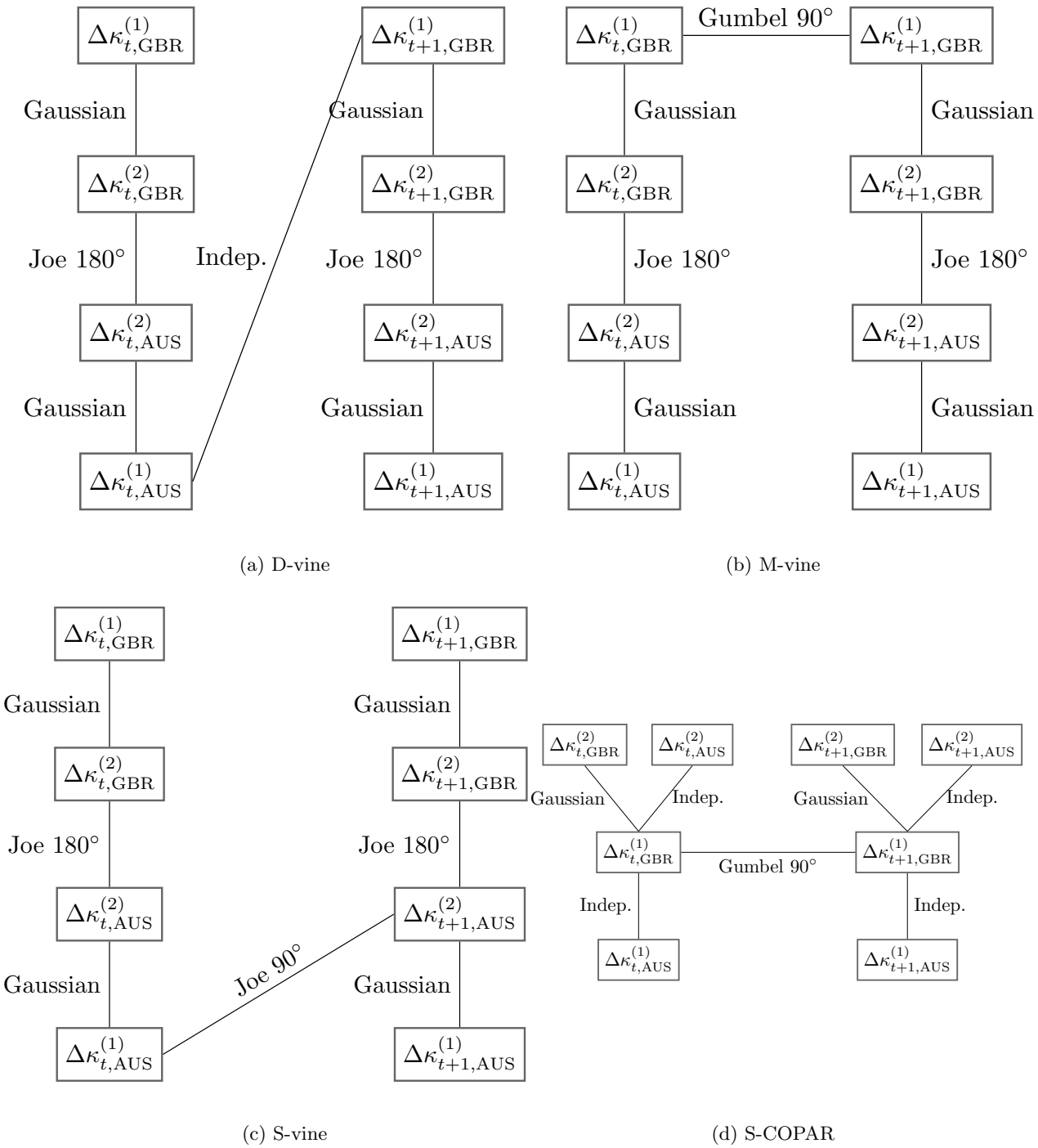


Figure 2.11: First trees of the estimated time series vine copula models using D-vine (a), M-vine (b), S-vine (c) and S-COPAR (d) for the differenced period effects of GBR and AUS.

Cross-sectional dependencies for the D-vine, M-vine and S-vine models in the first tree are significant, since no independence copula is selected for any edge in the cross-sectional structure. While the copulas selected for $(\Delta\kappa^{(1)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{GBR}})$ and $(\Delta\kappa^{(1)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t,\text{AUS}})$ are Gaussian, which has no tail

dependence, the copula chosen for $(\Delta\kappa^{(2)}_{t,\text{GBR}}, \Delta\kappa^{(2)}_{t,\text{AUS}})$ is the Survival Joe copula (Joe copula rotated by 180°) which exhibits lower tail dependence. This indicates that assuming a multivariate normal distribution, as in VAR and sVAR, one may overlook the tail behavior of the cross-sectional dependence. Furthermore, independence copulas are selected to connect the center node with $\Delta\kappa^{(2)}_{t,\text{j}}$, $j = GBR, AUS$, indicating weak dependence between the differenced period effects associated with the first and second period effects of the mortality structure.

Furthermore, for $(\Delta\kappa^{(1)}_{t,\text{AUS}}, \Delta\kappa^{(2)}_{t,\text{AUS}})$, a Gaussian copula is chosen in the M-vine, D-vine and S-vine models, whereas an independence copula is selected in the estimated ARIMA-copula model. This suggests that the ARIMA-copula approach may overlook certain types of dependence present in the data.

The first trees shown in Figures 2.11 differ in their temporal links. For the D-vine model, the temporal link connects $\Delta\kappa^{(1)}_{t,\text{AUS}}$ and $\Delta\kappa^{(1)}_{t+1,\text{GBR}}$, with an independence copula selected for this edge. It may seem counterintuitive to connect two nodes with no significant dependence. However, this arises from the restrictive construction of the time series D-vine copula, which only allows the temporal link to connect the opposite ends of the cross-sectional trees. Although the dependence in both pairs, $(\Delta\kappa^{(1)}_{t,\text{AUS}}, \Delta\kappa^{(1)}_{t+1,\text{GBR}})$ and $(\Delta\kappa^{(1)}_{t,\text{GBR}}, \Delta\kappa^{(1)}_{t+1,\text{AUS}})$, is low, the former shows a relatively higher dependence, leading to its selection. It is important to note that the selection of an independence copula for the temporal link in the first tree does not necessarily indicate a lack of temporal dependence. In the higher trees, non-independent copulas are selected in the temporal links for conditional variables. M-vine models require that the temporal link connect the same ends of the cross-sectional trees. Figure 2.11b connects $\Delta\kappa^{(1)}_{t,\text{GBR}}$ and $\Delta\kappa^{(1)}_{t+1,\text{GBR}}$ since this pair has a greater Kendall's tau than the pair at the other ends, $\Delta\kappa^{(1)}_{t,\text{AUS}}$ and $\Delta\kappa^{(1)}_{t+1,\text{AUS}}$.

Unlike the D-vine and M-vine models, which limit the choice of temporal links, the S-vine allows for the selection of any link that exhibits the highest dependence. Consequently, in Figure 2.11c, the temporal link is chosen to connect $\Delta\kappa^{(1)}_{t,\text{AUS}}$ and $\Delta\kappa^{(2)}_{t+1,\text{AUS}}$. This choice aligns with the estimated sVAR model, where the coefficient with the highest absolute value associates $\Delta\kappa^{(2)}_{t,\text{AUS}}$ with $\Delta\kappa^{(1)}_{t-1,\text{AUS}}$. Furthermore, the copula assigned to the edge connecting $\Delta\kappa^{(1)}_{t,\text{AUS}}$ and $\Delta\kappa^{(2)}_{t+1,\text{AUS}}$ is the Joe copula rotated by 90°, which exhibits negative lower tail dependence. This implies that a high $\Delta\kappa^{(2)}_{t+1,\text{AUS}}$ is likely to occur when $\Delta\kappa^{(1)}_{t,\text{AUS}}$ is a large negative value.

The S-COPAR model in Figure 2.11d, which connects the center node of the cross-sectional trees, selects a Gumbel copula rotated by 90°. This is the same copula as the one selected for temporal linking in the M-vine tree.

2.7 Model performance

In this section, we compare the fitting and forecasting performance of the models described in Sections 2.3 and 2.4. We consider two datasets: the first is the two-population example examined in Sections 2.5 and 2.6, and the second includes all four populations. Since a larger number of populations typically exhibits a more complex dependence structure and imposes greater demands on model parsimony, using four populations (eight period effects in total) allows us to more effectively contrast the performance of these models.

We first fit the models to data from 1950 to 2000 for ages 50 to 90 and use the estimated models to forecast mortality for 2001 to 2019. The out-of-sample forecasting performance is then evaluated. To provide a more robust assessment of forecasting accuracy, we extend the analysis by considering an expanding window approach.

2.7.1 Goodness-of-fit

We use BIC to evaluate the goodness-of-fit of various models for the period effects described in Sections 2.3 and 2.4. For VAR and sVAR, the log-likelihood is calculated assuming that the error terms follow a multivariate normal distribution. The number of parameters includes intercepts, coefficients, and the covariance matrix. For ARIMA-copula and time series vine copulas, the log-likelihood is calculated as the sum of the log-likelihood of the marginals and the log-likelihood of copulas. The total number of parameters is the sum of the parameters from the marginals and the copulas.

Table 2.10 presents the log-likelihoods, the number of parameters estimated, and the BIC values for each model fitted to the data over the period 1950-2000 and the ages 50-90. It is apparent that the time series vine copula models consistently achieve lower BIC values compared to the VAR, sVAR, and ARIMA-copula models, since they attain higher likelihood values with a relatively small number of parameters. Among these models, the S-vine achieves the lowest BIC, highlighting its flexibility in capturing complex dependence in the data.

In contrast, the VAR model yields the highest BIC, primarily due to its larger number of parameters. The ARIMA-vine and sVAR models employ mechanisms to reduce the number of parameters, thereby achieving greater parsimony and, consequently, lower BIC values compared to the VAR model.

Model	GBR, AUS			GBR, AUS, USA, DNK		
	LL	Params	BIC	LL	Params	BIC
VAR	794.3	30	−1471.2	1682.9	108	−2943.4
ARIMA-copula	779.5	13	−1508.2	1654.3	30	−3191.2
sVAR	788.4	18	−1506.3	1632.2	28	−3154.9
D-vine	802.2	18	−1534.0	1706.6	46	−3233.3
M-vine	805.9	16	−1549.2	1703.5	42	−3242.7
S-vine	813.9	17	**−1561.3**	1707.0	42	**−3249.6**
S-COPAR	813.8	17	−1561.1	1701.2	47	−3218.6

Table 2.10: Maximized log-likelihood (LL), number of parameters (Params), and BIC values for models estimated using period effects of two and four populations, based on mortality data from 1950–2000 for ages 50–90.

2.7.2 Forecasting performance

Simulations are used to generate 10,000 future period effect paths based on the estimated models. This choice is consistent with the literature on vine copula models for mortality forecasting. For instance, Zhou (2019) and Zhou and Ji (2021) simulate 5,000 future period effect paths using vine copulas. The corresponding mortality paths are then calculated using Equation (2.10) and estimated cohort effects. We perform jump-off corrections[2] to all simulated mortality rates, as described by Lee and Miller

[2]The jump-off correction adjusts each simulated future mortality rate for individuals aged x from population j by multiplying it by the ratio $m_{x,T,j}/\hat{m}_{x,T,j}$, where $\hat{m}_{x,T,j}$ and $m_{x,T,j}$ are the fitted and observed mortality rates for age x in population j in the final year of the fitting period. This correction mitigates a potentially large and counterintuitive jump between the initial projected mortality rates (the jump-off point) and the observed mortality rates from the last available year of historical data, potentially improving the accuracy of mortality forecasts.

(2001). Using these simulated future mortality paths, we produce the sample distribution and mean forecasts of future mortality.

The simulation procedures for the VAR, sVAR, and ARIMA-copula models are well documented, and we refer interested readers to Brechmann and Czado (2013), Kock et al. (2020), and Lütkepohl (2005) for further details. For time series vine copula models, simulations are performed using a modified version of the Rosenblatt transform (Nagler et al., 2022).

In our single-window study, cohort effects are estimated for the birth years 1860–1950, based on data from 1950 to 2000 and for the ages 50 to 90. For mortality forecasting, we predict only individuals whose cohort effects have already been estimated, since this study does not further model cohort effects. As a result, mortality forecasts are limited to ages 69 to 90 for the years 2001 to 2019.

We use two criteria—root mean squared error (RMSE) and continuous ranked probability score (CRPS)—to compare the out-of-sample forecasting performance across these models. RMSE is defined as follows:

$$\text{RMSE} = \sqrt{\frac{1}{J \times H \times (41-H)} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{x=50+H}^{90} (\tilde{m}_{x,T+h,j} - m_{x,T+h,j})^2}, \tag{2.11}$$

where h is the forecasting horizon ranging from 1 to H, T is the final year of the fitting period, and $\tilde{m}_{x,T+h,j}$ and $m_{x,T+h,j}$ represent the h-year-ahead mean mortality forecast and the observed mortality rate, respectively, for individuals aged x in year $T+h$ of population j. In this section, we set $H = 19$, since the forecasts are made for years 2001 to 2019.

Although RMSE measures the deviation of the observation from the mean forecast, CRPS accounts for the entire distribution of the probabilistic forecast—in our case, the density implied by the simulated future mortality rates. CRPS evaluates the difference between the forecast distribution and the true distribution of the observed value. The empirical CRPS, as defined by Laio and Tamea (2007) for each individual mortality forecast, can be calculated as:

$$\text{CRPS}_{x,T+h,j} = \frac{2}{10000} \sum_{i=1}^{10000} \left(\mathbb{1}_{\left\{ m_{x,T+h,j} < \tilde{m}^{(i)}_{x,t_n+h,j} \right\}} - \frac{i-0.5}{10000} \right) \left(\tilde{m}^{(i)}_{x,T+h,j} - m_{x,T+h,j} \right),$$

where $\{\tilde{m}^{(i)}_{x,T+h,j}\}_{i=1,\ldots,10000}$ represents the ordered set of simulated forecasts for the mortality rate, with i denoting the i-th smallest value in the sequence. We compute the average CRPS across all forecasts produced by each model, as given by:

$$\text{CRPS} = \frac{1}{J \times H \times (41-H)} \sum_{j=1}^{J} \sum_{h=1}^{H} \sum_{x=50+H}^{90} \text{CRPS}_{x,T+h,j}. \tag{2.12}$$

Table 2.11 presents the out-of-sample RMSE and CRPS for each model fitted to the two-population and four-population data. The S-vine model consistently yields the lowest RMSE and CRPS for both two- and four-population examples. It is also worth noting that all four time series vine copula models outperform existing models.

A formal test for the statistical significance of the differences between the S-vine, VAR, ARIMA-copula and sVAR models is not pursued. These models are non-nested, so likelihood ratio tests do not apply. Predictive accuracy tests, such as the Diebold-Mariano test (Diebold & Mariano, 1995), are also poorly suited to this setting since we consider only a single fitting window and due to the presence of dependence in forecast errors across ages and years. Instead, we assess the robustness of the models through the expanding window analysis in Section 2.7.3.

Model	GBR, AUS		GBR, AUS, USA, DNK	
	RMSE	CRPS	RMSE	CRPS
VAR	5.9070	2.3196	7.8564	4.1206
ARIMA-copula	6.4213	2.6375	8.0733	4.1760
sVAR	6.3060	2.4631	7.9293	3.9963
D-vine	5.8173	2.3390	7.4044	3.8481
M-vine	5.7932	2.3186	7.3541	3.8255
S-vine	**5.4246**	**2.1838**	**7.3074**	**3.7975**
S-COPAR	5.6511	2.2727	7.4999	3.8839

Table 2.11: Out-of-sample RMSE ($\times 10^{-3}$) and CRPS ($\times 10^{-3}$) of various models for the two-population and four-population examples, based on mortality forecasts for ages 69–90 from 2001 to 2019.

2.7.3 Expanding window analysis

We also perform an expanding window analysis on the four-population example to provide a more robust evaluation of forecasting accuracy, measured by RMSE and CRPS. In this analysis, the first window uses data from 1950 to 1996 for fitting and 1997 to 2015 for forecasting. The second window expands to 1950–1997 for fitting and 1998–2016 for forecasting, continuing this process until the final window, which uses data from 1950 to 2000 for fitting and 2001 to 2019 for forecasting.

We evaluate four models: VAR, sVAR, ARIMA-copula, and S-vine (the best-performing time series vine copula). For each window and model, we compute the RMSE and CRPS values as defined in Equations (2.11) and (2.12), respectively. Note that the final year of model fitting shifts with each expanding window. The RMSE and CRPS values for each model for the expanding windows are presented in Tables 2.12 and 2.13, respectively.

Fitting window	Forecasting window	VAR	ARIMA-copula	sVAR	S-vine
1950-1996	1997-2015	7.2633	8.0843	7.3137	**6.4161**
1950-1997	1998-2016	6.8489	8.1961	7.3067	**6.4740**
1950-1998	1999-2017	6.7400	7.4093	7.0432	**6.2705**
1950-1999	1999-2018	7.2187	8.0550	7.7389	**6.9185**
1950-2000	2001-2019	7.8564	8.0733	7.9293	**7.3074**

Table 2.12: RMSE ($\times 10^{-3}$) for the four-population example using an expanding window analysis.

The results presented in these tables demonstrate that the time series vine copula model consistently exhibits better forecasting performance compared to the other models. It consistently performs well across all windows for RMSE and CRPS, indicating the robustness of its forecasting performance. Combined with the goodness-of-fit results discussed earlier, these findings suggest that the time series vine copula model is well suited for both inference and forecasting in mortality modeling.

Fitting window	Forecasting window	VAR	ARIMA-copula	sVAR	S-vine
1950-1996	1997-2015	3.8480	4.1829	3.6935	**3.4540**
1950-1997	1998-2016	3.5930	4.2433	3.6498	**3.3908**
1950-1998	1999-2017	3.5527	3.8286	3.5521	**3.3614**
1950-1999	1999-2018	3.8015	4.1616	3.9227	**3.6815**
1950-2000	2001-2019	4.1206	4.1760	3.9963	**3.7975**

Table 2.13: CRPS ($\times 10^{-3}$) for the four-population example using an expanding window analysis.

2.8 Longevity bond application

A key advantage of time series vine copulas is their ability to capture complex dependence. Accurate modeling of mortality dependence is crucial for developing effective longevity hedging instruments and strategies that often involve multiple populations. In this section, we apply the previously investigated models to price a longevity bond and assess how the choice of model impacts bond pricing.

2.8.1 A longevity bond for hedging mortality divergence between multiple populations

The KORTIS bond, issued by Swiss Re in December 2010, is linked to the divergence in mortality improvement between the United States and the England and Wales populations. It was designed to hedge Swiss Re's exposure to uncertainty in mortality improvements. Specifically, Swiss Re had reinsurance exposure to death benefit policies in the United States and life annuity policies in the United Kingdom (Hunt & Blake, 2015). Although there is a natural hedge between these two lines of business, mortality improvements in the two populations are not perfectly correlated, and the payment structures of these business lines differ, leading to basis risk that requires additional hedging. The principal payment of the bond is reduced when the mortality divergence exceeds a pre-specified threshold, compensating Swiss Re for their mortality losses due to significant divergence.

In this chapter, we consider a longevity bond with a payment structure similar to that of the KORTIS bond, but with the principal payment linked to the divergence between the average mortality improvements of two sets of populations, each comprising multiple populations. This type of bond can be useful for reinsurers exposed to life insurance and annuity portfolios in multiple countries. From an investor's perspective, the improvement averaged across several populations is likely to be less volatile than that of a single country, making the bond less risky.

Assume that this longevity bond is issued at time T with a face value of \$1 and makes annual coupon payments at the LIBOR rate plus a spread. Principal repayment at maturity is associated with the "Longevity Divergence Index Value" (LDIV), defined as the difference between the mortality improvement rates of two sets of populations, J_1 and J_2. The improvement rate observed between ages x_1 to x_2 during the n-year period leading to time $T+n$ in population j is defined as:

$$\text{IPM}(n,j) = \frac{1}{x_2 - x_1 + 1} \sum_{x=x_1}^{x_2} \left(1 - \left(\frac{m_{x,T+n,j}}{m_{x,T,j}}\right)^{1/n}\right). \tag{2.13}$$

The LDIV is then expressed as follows:

$$\text{LDIV} = \frac{1}{|J_1|} \sum_{j \in J_1} \text{IPM}(n,j) - \frac{1}{|J_2|} \sum_{j \in J_2} \text{IPM}(n,j) \tag{2.14}$$

where $|.|$ denotes the cardinality of a set. Here, LDIV is constructed by taking a simple average of the improvement rates for each population set. A weighted average could also be used if the exposure to each population within a set is not equal.

When LDIV is below the attachment point (ap), the full principal of \$1 is repaid at maturity. If LDIV exceeds the detachment point (dp), no principal is repaid. For LDIV values between the attachment and detachment points, the repayment decreases linearly from \$1 to zero. Therefore, the principal reduction factor (PRF) at maturity can be expressed as follows:

$$\text{PRF} = \max\left(\min\left(\frac{\text{LDIV} - ap}{dp - ap}, 1\right), 0\right).$$

As a result, the principal repayment at maturity is $1 - \text{PRF}$.

2.8.2 Simulated distributions of LDIV and PRF for longevity bond pricing

We use simulations to analyze LDIV and PRF, both of which are key in pricing longevity bonds, under different model choices. Although we do not conduct a full pricing exercise, these distributions directly influence the principal repayment, and thus bond prices. For this analysis, we define two population sets $J_1 = \{\text{GBR}, \text{DNK}\}$ and $J_2 = \{\text{USA}, \text{AUS}\}$. Mortality models are fitted using data from 1950 to 2019 for ages 50 to 90. We then simulate 10,000 mortality paths for individuals aged 60 to 90 over the years 2020-2029. As a result, we set $x_1 = 60$, $x_2 = 90$, $n = 10$, and $T = 2019$ in Equation (2.13). The attachment and detachment points are set at $ap = 2.75\%$ and $dp = 3.25\%$, respectively.

Based on simulated LDIV and PRF, we examine two additional risk measures of the longevity bond, including expected loss (EL) and conditional expected loss (CEL) at the specified attachment and detachment points. EL and CEL are calculated as follows:

$$\begin{aligned} \text{EL} &= \mathbb{E}\left[\text{PRF}\right], \\ \text{CEL} &= \mathbb{E}\left[\text{PRF} | \text{PRF} > 0\right]. \end{aligned}$$

Figure 2.12 shows the violin plot of LDIV based on simulated mortality paths under various models. The attachment and detachment points are marked as dashed lines. We observe that the density plots of LDIV from sVAR and VAR show a slightly more concentrated distribution, while those for ARIMA-copula and S-vine have higher variability. The median value from the ARIMA-copula is obviously lower than the median from the other models.

Since principal repayment depends on whether LDIV exceeds specific thresholds, the upper tail of the LDIV distribution plays a critical role in determining the likelihood of principal reductions. More specifically, the densities of the values between the attachment and detachment points determine the distribution of PRF. As shown in Figure 2.12, S-vine exhibits the highest density above the attachment point, while VAR has the lowest. A similar pattern is observed for the density above the detachment point. Therefore, it is reasonable to expect that the S-vine model will lead to the highest PRF, EL, and CEL, while VAR will result in the lowest.

Table 2.14 presents the probabilities that LDIV exceeds various thresholds and the values of EL and CEL for various models. In the upper tail, LDIV has a one-to-one correspondence with PRF. For example, LDIV $> 2.85\%$ corresponds to PRF $> 20\%$. The S-vine model has the highest probability of reaching the attachment point, at 1.77%, while the VAR model shows the lowest, at 0.42%. The S-vine model also has the highest probability of exceeding the detachment point, at 0.31%, while the VAR model shows the lowest, 0.04%. This observation aligns with the density patterns observed in Figure 2.12, where the S-vine model exhibits the highest density above both the attachment and the detachment points, and VAR the lowest. Furthermore, the probabilities of exceeding various thresholds for ARIMA-copula and sVAR are close.

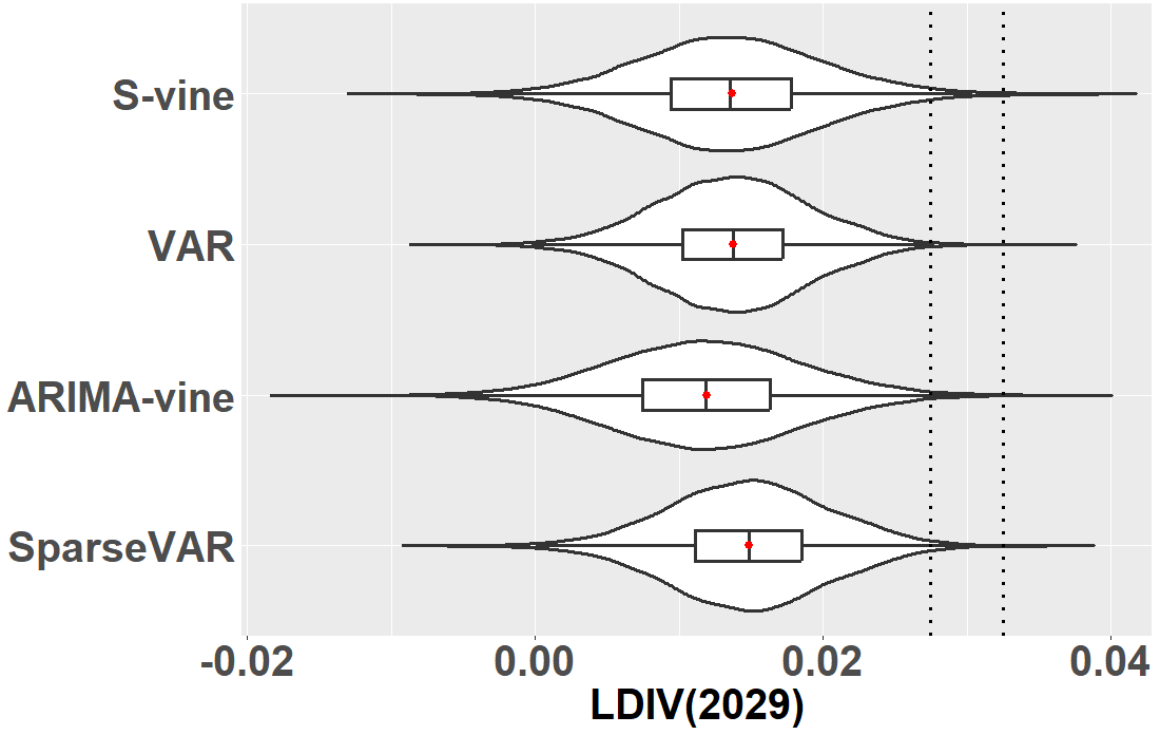


Figure 2.12: Violin plot of LDIV using various models based on simulated mortality paths. The width of each violin reflects the density of the LDIV values, with the embedded boxplot highlighting the median (black line), interquartile range (box), and mean (red dot). The dashed lines represent the attachment and detachment points.

Consistent with observations on the exceeding probabilities, the S-vine model results in the highest EL (0.87%) and CEL (49.17%), while the VAR model yields the lowest EL (0.13%) and CEL (30.14%). The significant difference in EL has important implications for longevity bond pricing. Since existing pricing methods, such as the risk neutral pricing approach (Xu et al., 2020) and the risk cubes method (H. Chen & Cummins, 2010), depend heavily on EL and CEL, the choice of model can lead to substantial variation in bond prices.

The differences in the exceeding probabilities can be explained by the structural properties of the models, including their treatment of tail dependence. Copula-based methods provide greater flexibility in capturing asymmetric and nonlinear dependence structures by selecting bivariate copulas that allow for tail dependence, whereas sVAR and VAR assume no tail dependence and capture linear dependence only. In the case of LDIV, which represents the difference between two averaged mortality improvement rates, tail behavior depends on how the dependence structure is modeled. Copula-based models, that incorporate tail dependence, may exhibit a greater probability of extreme LDIV values than VAR-based models, which assume no tail dependence.

Although the ARIMA-copula model also uses a vine copula and thus incorporates nonlinear and asymmetric dependence, its distribution differs from that of the S-vine model. Specifically, although its overall dispersion is similar to that of the S-vine, its center mass, as indicated by the median in the boxplot, is visibly shifted to the left. As a result, the densities above the attachment and detachment points are lower than those of S-vine, leading to lower exceeding probabilities.

The visibly lower median of ARIMA-copula is an interesting observation. Since ARIMA-copula employs a two-step estimation approach—first modeling the marginal distribution and then separately modeling cross-sectional dependence—the lower median could result from either component. To iden-

Model	Probability						EL	CEL
	LDIV>							
	2.75%	2.85%	2.95%	3.05%	3.15%	3.25%	($ap = 2.75\%$	
	PRF>						$dp = 3.25\%$)	
	0%	20%	40%	60%	80%	100%		
VAR	0.42%	0.20%	0.12%	0.06%	0.05%	0.04%	0.13%	30.41%
ARIMA-vine	1.00%	0.68%	0.44%	0.26%	0.15%	0.10%	0.41%	40.86%
SparseVAR	1.10%	0.61%	0.34%	0.20%	0.14%	0.09%	0.37%	34.01%
S-vine	1.77%	1.28%	0.91%	0.64%	0.48%	0.31%	0.87%	49.17%

Table 2.14: Probabilities of LDIV exceeding various thresholds under different models, and expected loss (EL) and conditional expected loss (CEL) with $ap = 2.75\%$ and $dp = 3.25\%$.

tify the source of this shift in central mass, we compare the ARIMA-copula model with independent ARIMA models in Figure 2.13, where each period effect time series is modeled separately using an ARIMA model without incorporating cross-sectional dependence.

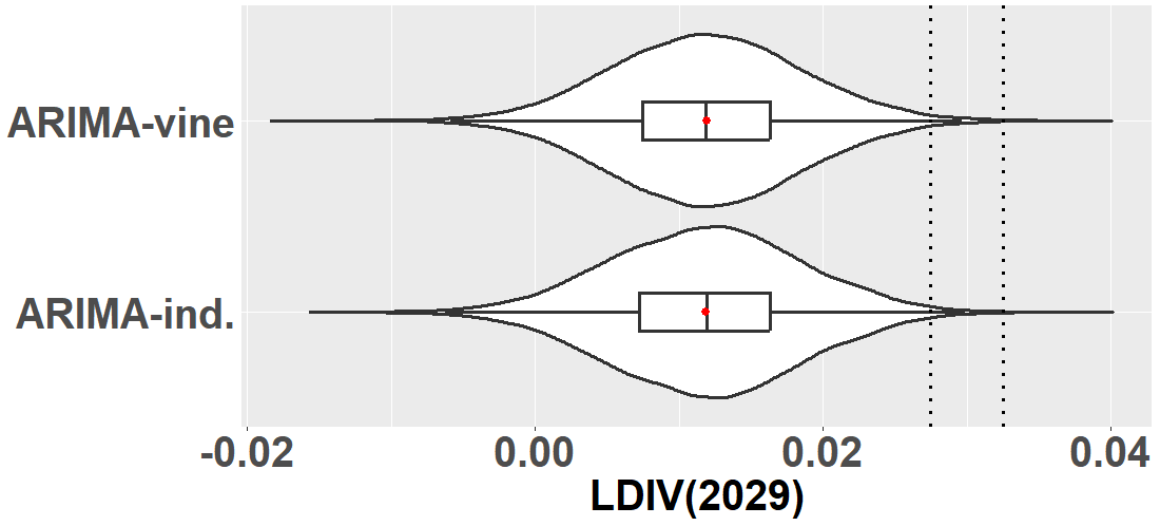


Figure 2.13: Violin plots of LDIV based on simulated mortality paths under ARIMA-copula and independent ARIMA models. The width of each violin reflects the density of the LDIV values, with the embedded boxplot highlighting the median (black line), interquartile range (box), and mean (red dot). The dashed lines represent the attachment and detachment points.

The violin plots in Figure 2.13 suggest that ARIMA-copula and ARIMA-independent exhibit overall similar density patterns, with only small differences in the tail. The probabilities of LDIV$>$ 2.75% and LDIV$>$ 3.25% are 0.77% and 0.07%, respectively, using independent ARIMA models. This indicates that the serial dependence structure obtained in the first step dominates the overall density of LDIV values, whereas the cross-sectional dependence, captured by the vine copula, primarily influences the tail behavior. Among the models we consider, the ARIMA-copula is unique in that it first captures serial dependence before modeling cross-sectional dependence, whereas the other models estimate both

simultaneously. The shift in central mass may result from the two-step modeling approach, which could introduce potential limitations or inefficiencies.

We do not consider full-pricing of the KORTIS bond in our work. We refer the interested readers to Cairns et al. (2006a) for a risk-neutral framework for pricing longevity-based securities. For pricing the KORTIS bond specifically, see H. Chen et al. (2017).

2.9 Concluding remarks

In this chapter, we present an approach to apply time series vine copulas to model mortality dependence. A time series vine copula simultaneously captures both serial and cross-sectional dependence in a single step, allowing for nonlinearity and asymmetry in both types of dependencies. We focus on S-vines and its special cases: D-vine, M-vine, and stationary COPAR models.

We conduct a comprehensive comparison of time series vine copula models with commonly used approaches in the mortality dependence modeling literature. Specifically, we compare these models with VAR, sparse VAR, and ARIMA-copula models, applying them to a two-population example to illustrate their underlying intuition, and a four-population example to assess their performance in capturing more complex dependence structures.

We adopt a modified version of the M6 model (log-M6) as our base mortality structure. For this model, we find that the fitted period components exhibit non-normality due to their large excess kurtosis. Furthermore, we find evidence of asymmetric tail dependence between period components in each dependence type — serial, cross-sectional, and lagged cross-sectional. This is demonstrated by the copulas selected in the ARIMA-vine and time series vine copula models, which exhibit only lower tail dependence. Comparison of the fitted VAR and time series vine copula models with the fitted ARIMA-vine model reveals that the ARIMA-vine model may overlook lagged dependencies between period components, resulting in less accurate inferences about the overall dependence structure.

For both the two-population and four-population applications, the time series vine copula models outperform VAR, sparse VAR, and ARIMA-vine models in both in-sample performance, measured using BIC, and out-of-sample performance, measured using RMSE and CRPS. We perform a robustness check using an expanding window analysis and find that the S-vine model exhibits greater robustness compared to the VAR, sparse VAR, and ARIMA-vine models. These results demonstrate the suitability of the time series vine copula models to model mortality.

To illustrate the practical implications of model selection for mortality dependence, we analyze a KORTIS bond where the principal payment depends on the divergence in average mortality improvements between two sets of populations. The sparse VAR and VAR models yield a more concentrated distribution of divergence in mortality improvement between the two population sets, whereas the ARIMA-vine and S-vine models exhibit greater variability. The median divergence in mortality improvement is lower in the ARIMA-vine model compared to all other models, aligned with the ARIMA-independent model. This indicates that the serial dependence structure from the first step of the ARIMA-vine model has a stronger influence on the distribution than the cross-sectional dependence captured in the second step, highlighting potential limitations of the two-step modeling approach. Due to these factors, the S-vine model implies significantly higher expected and conditional expected losses than other models, leading to substantial variation in longevity bond prices, which rely heavily on these metrics.

Although this study contributes valuable insights into modeling mortality dependence, several avenues for future research could further enhance our understanding. First, time series vine copula models could be generalized to incorporate dynamic dependence structures that evolve over time, addressing previously documented shifts in mortality dependence patterns (Zhou, 2019; Zhou & Ji, 2021). Such generalizations would likely yield improved predictive accuracy. Second, extending time

series vine copula models to accommodate non-stationary time series would offer substantial benefits. This direction has already been explored to some extent by Beare and Seo (2015), who applied a Hodrick–Prescott filter to remove predictable trending behavior from the time series. Finally, integrating the estimation of marginal distributions and the vine copula structure into a unified single-step procedure could reduce the estimation bias, thereby improving overall model performance.

Chapter 3

Compensation-based risk-sharing

3.1 Introduction

Consider a group of n individuals observed at time 0, each of them exposed to a non-negative random loss at time 1. This loss can be related to a well-defined peril (e.g., hospitalization-related or critical illness-related expenses), or it can be expressed as a deterministic claim payment contingent on the occurrence of a well-defined event (e.g., a deterministic payment contingent on death or survival).

In the *classical insurance* approach, individuals engage an insurer and each of them buys an insurance contract at time 0, which entitles them to reimbursement for their respective observed losses at time 1. Every policyholder is compensated ex-post (at time 1) for their realized loss. In return for this coverage, the insurer charges each policyholder an ex-ante insurance premium (at time 0). In such an arrangement, the aggregate risk (randomness) of the insurance portfolio is taken over by the insurer. This risk transfer is possible if the pool consists of a sufficient number of mutually independent and homogeneous risks, with premiums being calculated in a conservative way. In addition, the insurer sets up solvency capital for the case that the collected premiums turn out to be insufficient to cover the guaranteed claims. Premiums and solvency capital are chosen so that the probability of the event that the sum of all accumulated premiums and solvency capital exceeds the aggregate claims of the insurance portfolio is sufficiently large (e.g., 99.5%). Classical insurance as described above is a form of *centralized risk transfer*, meaning that it is a risk-transfer mechanism in which individual losses faced by policyholders are transferred to a central insurer who guarantees that losses will be paid. In exchange, policyholders pay a risk premium and compensate the insurer's shareholders by providing a return on the capital they set aside to maintain the insurer's guarantee.

Instead of a transfer of the aggregate risk of the pool to a central insurer (guarantor), individuals can opt for a so-called *decentralized risk-sharing* approach, where individuals do not transfer the aggregate losses to a guarantor but keep the aggregate risk within the pool, without creating any solvency risk. Examples of such approaches can be found in Abdikerimova and Feng (2022) and the references therein. One way to achieve this goal is that the premiums at time 0 of the classical centralized approach are replaced by time 1 contributions: Each participant in the risk-sharing pool is fully compensated for their loss at time 1, but in return a contribution must be paid to the pool at time 1. These contributions are time 1 measurable random variables, chosen at time 0. The risk-sharing scheme is set up such that the sum of all contributions paid at time 1 by the participants is exactly equal to the sum of all losses covered by the pool. This constraint is called the full allocation condition. In other words, participants contribute at time 1, by sharing total losses once they have been observed. Following Dhaene and Milevsky (2024), we call such a decentralized approach *contribution-based risk-sharing*.

A simple example of such a risk-sharing approach is the uniform risk-sharing rule, where each participant's contribution is set equal to the observed aggregate losses of the pool, divided by the

number of participants. Other examples of contribution-based risk-sharing are the conditional-mean risk sharing scheme, introduced in the actuarial literature in Denuit and Dhaene (2012), and the quantile risk-sharing scheme introduced in Denuit et al. (2022). The properties of contribution-based risk-sharing rules have been investigated in detail in Denuit et al. (2022) and Denuit et al. (2025), among others. An axiomatic characterization of the conditional mean risk-sharing rule is given in Jiao et al. (2022), while an axiomatic characterization of the quantile risk-sharing rule is considered in Dhaene et al. (2025). Axiomatic characterizations of some simple risk-sharing rules, including the uniform rule, are presented in Dhaene et al. (2026).

Decentralized risk-sharing can also be constructed in another way. Indeed, suppose again that all participants are exposed to a random non-negative loss at time 1. Each of them invests an amount at time 0 to set up a so-called 'endowment contingency fund'. At time 1, the total fund value is shared among all participants. The relative part that each participant will receive is a time 1 observable random variable, determined at time 0 as a well-defined function of the claims and possibly also of other information that will be observable at time 1. Solvency is guaranteed by the full allocation condition which states that the sum of all payments to the participants at time 1 is equal to the fund value at time 1. The aim of an 'endowment contingency fund' is to provide participants with a cheaper and more effective protection, compared to commercial insurance. The term 'endowment' indicates that it is an investment portfolio with initial capital deriving from cash inflows, whereas the term 'contingency' means that the payments out of this fund are contingent on the realization of certain random events. For each individual participant, the coverage ratio (i.e., compensation over claim) depends on the initial investments and the observed losses of all participants. A participant may not receive their observed loss, i.e., they might not be fully compensated for their loss, or even receive more than their loss. In this setting, the time 1 payments can be considered as a kind of compensation for the losses that occurred. Therefore, Dhaene and Milevsky (2024) call the time 1 payments from the fund to the participants the compensations, and they baptize this decentralized risk-sharing approach *compensation-based risk-sharing*.

Note that in this setting, there is no solvency risk due to the full allocation condition. As a result, there is no need for solvency monitoring. However, this does not mean that regulation is no longer needed.

The administrator oversees the running of the scheme. This includes collecting survival and death reports, calculating investments, calculating allocations, and distributing payouts. Each of these steps creates scope for manipulation or fraud, which can still harm participants. For this reason, oversight of how each aspect is handled is an important consumer protection issue requiring some regulatory scrutiny.

As the pool becomes larger, the actuarially fair investment of an administrator π_{n+1} tends toward zero. This means the administrator has less at stake and weaker incentives to monitor the pool carefully. Regulation needs to address this, ensuring that administrators remain accountable even when their own financial exposure is small.

More broadly, the administrator represents a new kind of financial intermediary. It is neither an insurer nor a standard fund manager. A classical insurer can be seen as a special 'asymmetric' case of the administrator: one that keeps any surplus when claims are low but must provide additional capital when claims are high. From this perspective, the regulatory framework used for insurance is not made redundant by compensation-based schemes. Instead, it needs to be adapted. The focus should shift away from solvency and capital requirements and toward transparency, conduct, and the integrity of information.

As a simple example of the compensation-based approach, consider the risk-sharing scheme where at time 1, each participant receives a compensation which is proportional to their observed loss. The proportionality factor, which is observable at time 1, is assumed to be equal for all participants and follows from the full allocation condition.

Special cases of compensation-based risk-sharing schemes have been investigated in several actuarial papers, including Denuit and Robert (2025), Dhaene and Milevsky (2024) and Bernard et al. (2025), where only two-point distributed losses (indicator random variables) have been considered. In this chapter, we generalize the approach set out in these papers to include general losses and further explore properties of general compensation-based risk-sharing schemes. Our generalization broadens the applicability of such schemes to a wider range of insurance types with random claim severity, for example, homeowners and automobile insurance. We analyze the role of an administrator in such schemes and compare frameworks with an active and a passive administrator.

All random variables considered in this chapter are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The set of all non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by L_+^0. Throughout this chapter, the term 'positive' is used for 'strictly larger than zero'. The set of non-negative real numbers is denoted by $\mathbb{R}^+$.

3.2 Compensation-based risk-sharing with an active administrator

In this section, we introduce and investigate a general type of fully-funded risk-sharing (further abbreviated as RS) mechanisms, where n participants decide to mutually invest and set up a so-called 'endowment contingency fund'. At the beginning of the investment period, each participant i makes an initial (non-negative) investment π_i in the fund (similar to paying an initial insurance premium). Throughout this chapter, we always implicitly assume that at least one of these investments is positive, which means that for at least one participant i, $i = 1, 2, \ldots, n$, one has that $\pi_i > 0$. The beginning of the investment period is denoted by time 0, and also referred to as 'now'. The end of the investment period is denoted by time 1, also referred to as 'the end of the year'. In practice, these investments will be supplemented with fees to cover expenses, but we assume that the investments π_i are net investments, after fees for expenses have been paid. In this case, we do not have to take these fees further into account in our analysis.

Our objective is to investigate fair methods for the participants to divide the total initial investment among themselves at time 1. To be more precise, at time 0, the time 1 observable non-negative random variables W_i are chosen, where W_i stands for the part of the total fund value available at time 1 that will be attributed to the participant i at this time. Following Dhaene and Milevsky (2024), we call these amounts to be paid at time 1 the 'compensations' to the participants. In general, there is the possibility that the time 1 realizations of all compensations W_i, $i = 1, 2, \ldots, n$, are equal to 0. At time 0, we have to clearly specify what happens to the fund's proceeds in this case.

Apart from the n participants, there is another agent, denoted by $n+1$ and called the administrator. The role of the administrator is to collect the amounts π_i at time 0, to invest them, and to distribute the compensations W_i to the n participants at time 1. Seen from time 0, the compensations to the n participants are random variables defined and agreed upon by the participants at that time. As mentioned above, it might happen that all compensations to the participants are equal to 0. We assume that in case this situation occurs, the administrator receives the full proceeds of the fund. On the other hand, in case the time 1 realization of the compensation of at least one participant is positive, the administrator does not receive anything. The administrator's compensation at time 1 is denoted by W_{n+1}. We assume that the administrator also contributes an initial (non-negative) investment π_{n+1} to the fund, in return for receiving the total proceeds of the fund when $W_i = 0$, $i = 1, 2, \ldots, n$.

When referring to the '$n+1$ agents' of a compensation-based RS scheme, we mean the n participants and the administrator. The sum of the initial investments of all agents, that is, $\sum_{j=1}^{n+1} \pi_j$, is equal to the total value of the fund at the beginning of the investment period. For simplicity, we assume a zero interest rate and no expenses, but our results can easily be generalized to the case of a (deterministic) investment return and expenses by assuming $\sum_{j=1}^{n+1} \pi_j$ is the accumulated value of the investments

net of expenses. Under this assumption, the time 1 value of the investment fund is equal to $\sum_{j=1}^{n+1} \pi_j$. Notice that our implicit assumption that at least one of the participants' investments is positive implies that the time 1 value of the fund is always positive.

The active administrator's time 1 claim W_{n+1} on the fund can be expressed as follows:

$$W_{n+1} = \begin{cases} \sum_{j=1}^{n+1} \pi_j & : W_1 = W_2 = \cdots = W_n = 0 \\ 0 & : \text{ otherwise} \end{cases} \tag{3.1}$$

This means that the administrator receives the full proceeds of the fund if and only if all other participants receive zero compensation. Taking into account that all compensations are non-negative, the event $'W_1 = W_2 = \ldots = W_n = 0'$ is equivalent to the event $' \sum_{j=1}^{n} W_j = 0'$. The random variables $\sum_{j=1}^{n} W_j$ and W_{n+1} exhibit a dependency structure which is a special case of 'countermonotonicity', called 'mutual exclusivity', meaning that both random variables are non-negative, with one of them being positive implying that the other one is equal to 0. The concept of 'mutual exclusivity' is considered in several papers in the actuarial literature, see for example, Cheung and Lo (2014), Dhaene and Denuit (1999), and Lauzier et al. (2024).

While a compensation-based risk-sharing scheme with an active administrator may appear similar to classical insurance, the two are materially different. The differences lie in their motivation, the risks held by each party, their solvency requirements, and their respective functions.

A key distinction is the transfer versus sharing of risk. In a classical insurance setting, the insurer assumes the risks borne by policyholders in exchange for a premium sufficient to cover expected losses, plus a loading for profit and expenses. Risk is transferred away from the policyholder entirely. In a compensation-based scheme, the active administrator assumes no such role. The risk remains shared between participants. The losses incurred by participants are covered by each other, with the administrator's investment serving only to allow for actuarial fairness and not to absorb individual risk.

The motivation of a classical insurer and an active administrator also differs fundamentally. A classical insurer seeks to generate profit for shareholders, who provide solvency capital in return for this residual claim. Because the insurer has made a promise to policyholders to cover their losses, it is exposed to solvency risk. This is why regulators require insurers to hold capital. The active administrator, by contrast, is motivated by the smooth and orderly operation of the pool. Their incentive is the prospect of receiving the entire fund if all participants die. The administrator risks only their own investment and faces no solvency risk, since the scheme is self-financing by construction.

Introducing the random variable P_{n+1}, with $0 \leq P_{n+1} \leq 1$, for the proportion of the aggregate investment that will be attributed to the administrator, the compensation W_{n+1} can be expressed as follows:

$$W_{n+1} = \left(\sum_{j=1}^{n+1} \pi_j \right) \times P_{n+1}, \tag{3.2}$$

with

$$P_{n+1} = \begin{cases} 1 & : W_1 = W_2 = \cdots = W_n = 0 \\ 0 & : \text{otherwise} \end{cases} \tag{3.3}$$

Introducing the random variable P_i, with $0 \leq P_i \leq 1$, for the random proportion of the aggregate investment attributed to participant i, we have that the compensations of the participants can be expressed as

$$W_i = \left(\sum_{j=1}^{n+1} \pi_j \right) \times P_i \qquad i = 1, 2, \ldots, n. \tag{3.4}$$

Taking into account Equation 3.4, the random relative compensation P_{n+1} attributed to the administrator can be expressed as follows:

$$P_{n+1} = \begin{cases} 1 & : P_1 = P_2 = \cdots = P_n = 0 \\ 0 & : \text{otherwise,} \end{cases} \tag{3.5}$$

or equivalently,

$$P_{n+1} = 1\left(\sum_{j=1}^{n} P_j = 0\right), \tag{3.6}$$

where $1(A)$ stands for the indicator function, which is 1 if the event A occurs and 0 otherwise. This expression for P_{n+1} also shows that $\sum_{j=1}^{n} P_j$ and P_{n+1} are mutually exclusive.

Hereafter, we always assume that at time 1, the total amount of the available funds is fully distributed to the participants and the administrator, that is,

$$\sum_{j=1}^{n+1} P_j = 1. \tag{3.7}$$

This condition is called the full allocation condition for the compensation-based RS scheme. We also always implicitly assume that

$$0 < \Pr[P_{n+1} = 0] < 1, \tag{3.8}$$

or equivalently,

$$0 < \Pr[P_{n+1} = 1] < 1. \tag{3.9}$$

The assumption in Equation 3.8 means that the events 'at least one participant receives a non-zero compensation' and 'all participants receive a zero-compensation' have a positive probability. Taking into account that P_{n+1} is Bernoulli distributed, from Equation 3.9 we find that

$$0 < \mathbb{E}[P_{n+1}] = \Pr[P_{n+1} = 1] < 1. \tag{3.10}$$

From Equation 3.2 it follows then that

$$0 < \mathbb{E}[W_{n+1}] < \sum_{j=1}^{n+1} \pi_j. \tag{3.11}$$

Finally, from Equations 3.7 and 3.8 we have that

$$0 < \sum_{k=1}^{n} \mathbb{E}[P_k] = \Pr[P_{n+1} = 0] < 1. \tag{3.12}$$

We will call the administrator as described above an 'active administrator', where 'active' means that the administrator is the 'owner' of the compensation W_{n+1}. Further in the chapter, we will also introduce the 'passive administrator' whose compensation W_{n+1} will be redistributed to the agents.

Let us now introduce the (deterministic) investment vector $\boldsymbol{\pi}$, which is defined by

$$\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{n+1}), \tag{3.13}$$

as well as the time 1 measurable compensation vector $\mathbf{W}$, defined by

$$\mathbf{W} = (W_1, W_2, \ldots, W_{n+1}), \tag{3.14}$$

and the time 1 measurable relative compensation vector $\mathbf{P}$, defined by

$$\boldsymbol{P} = \big(P_1, P_2, \ldots, P_{n+1}\big). \tag{3.15}$$

Taking into account the introduced vector notations, we can rewrite Equations 3.2 and 3.4 in the following way:

$$\boldsymbol{W} = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \boldsymbol{P}. \tag{3.16}$$

In the following definition, we introduce the set of all relative compensation vectors for the $n+1$ agents.

Definition 1 *The set $\mathcal{R}_{n+1}$ is defined by*

$$\mathcal{R}_{n+1} = \left\{ (P_1, P_2, \ldots, P_{n+1}) \in (L_+^0)^n \mid \sum_{j=1}^{n+1} P_j = 1 \text{ and } P_{n+1} = 1 \left(\sum_{j=1}^{n} P_j = 0 \right) \right\}. \tag{3.17}$$

The set $\mathcal{R}_{n+1}$ consists of all $(n+1)$-dimensional relative compensation vectors. Any $\boldsymbol{W} = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \boldsymbol{P}$, where $\boldsymbol{P} \in \mathcal{R}_{n+1}$ is a reallocation (or redistribution) of the time 1 value of the fund between the n participants and the administrator, such that $\sum_{j=1}^{n} W_j$ and W_{n+1} are mutually exclusive. The latter condition means that the events 'at least one W_i is positive' and 'W_{n+1} is positive' are mutually exclusive.

Compensation-based RS is a two-stage process. At time 0, any agent makes an initial investment π_i and their aggregate investment $\left(\sum_{j=1}^{n+1} \pi_j\right)$ is reallocated by transforming $\boldsymbol{\pi}$ into random vector $\boldsymbol{W}$ defined in Equation 3.16, with $\boldsymbol{P} \in \mathcal{R}_{n+1}$. At time 1, the participants and the administrator receive the respective realizations of the compensations that were attributed to them. Hereafter, we denote the compensation W_i by $W_i[\boldsymbol{\pi}]$ and the compensation vector $\boldsymbol{W}$ by $\boldsymbol{W}[\boldsymbol{\pi}]$ in case we want to emphasize that the underlying investment vector is $\boldsymbol{\pi}$. Similarly, we write the relative compensation P_i by $P_i[\boldsymbol{\pi}]$ and the relative compensation vector $\boldsymbol{P}$ by $\boldsymbol{P}[\boldsymbol{\pi}]$ to denote their dependence on $\boldsymbol{\pi}$. When the meaning is clear, we will omit the $[\boldsymbol{\pi}]$ in the notation.

We are now ready to define a compensation-based RS scheme with an active administrator.

Definition 2 *A compensation-based risk-sharing scheme for a given group of n participants and an active administrator is a pair $(\boldsymbol{\pi}, \boldsymbol{P})$, where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{n+1})$ is the initial investment vector, while $\boldsymbol{P} = (P_1, P_2, \ldots, P_{n+1})$ is a relative compensation vector, that is $\boldsymbol{P} \in \mathcal{R}_{n+1}$. Moreover, the compensations attributed to the $n+1$ agents are expressed by the compensation vector $\boldsymbol{W}$, which is defined by*

$$W_i = \left(\sum_{j=1}^{n+1} \pi_j\right) \times P_i \qquad i = 1, 2, \ldots, n+1. \tag{3.18}$$

A compensation-based RS *scheme* with an active administrator is specified by its investment vector $\boldsymbol{\pi}$ and its relative compensation vector $\boldsymbol{P}$ (or its compensation vector $\boldsymbol{W}$). The administrator is called 'active' in the sense that they are entitled to a random compensation at time 1. Further in this chapter, we will also consider a 'passive' administrator, who is not entitled to any random compensation at time 1. We remark that investments are known at time 0, whereas the random compensations remain unknown until they become observable at time 1. As $\boldsymbol{P} \in \mathcal{R}_{n+1}$, a compensation-based RS scheme with an active administrator satisfies the following **full allocation condition**:

$$\sum_{j=1}^{n+1} W_j = \sum_{j=1}^{n+1} \pi_j, \tag{3.19}$$

which means that such an RS scheme is a fully funded system and has no insolvency issues: the total amount that will be distributed at time 1 will also be available at that time.

After having introduced compensation-based RS *schemes*, we can now define a compensation-based RS *rule* as an appropriate set of compensation-based RS schemes.

Definition 3 *A compensation-based risk-sharing rule for a given group of n participants and an active administrator is a mapping* $\boldsymbol{P} : (\mathbb{R}^+)^{n+1} \to \mathcal{R}_{n+1}$ *which transforms any investment vector* $\boldsymbol{\pi}$ *in* $(\mathbb{R}^+)^{n+1}$ *into a relative compensation vector* $\boldsymbol{P}[\boldsymbol{\pi}]$*:*

$$\boldsymbol{\pi} \in \left(\mathbb{R}^+\right)^{n+1} \to \boldsymbol{P}[\boldsymbol{\pi}] \in \mathcal{R}_{n+1}. \tag{3.20}$$

In this chapter, when we consider an 'RS scheme' or an 'RS rule', we always mean a 'compensation-based RS scheme' or a 'compensation-based RS rule'.

3.3 Some examples

In this section, we illustrate the concept of compensation-based RS with an active administrator with some examples.

Example 3.3.1 *Suppose that each participant i of a group of n individuals is exposed to a (random) non-negative loss* X_i *at time* 1*. The n participants decide to share the risk related to these losses among themselves. Therefore, they appoint an administrator, called agent* $n+1$*, and set up an RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$*, with*

$$P_i = \frac{X_i}{\sum_{j=1}^{n+1} X_j}, \qquad i = 1, 2, \ldots, n+1, \tag{3.21}$$

with X_{n+1} *defined by*

$$X_{n+1} = 1\left(\sum_{j=1}^{n} X_j = 0\right). \tag{3.22}$$

It is easy to see that X_{n+1} *and* $\sum_{j=1}^{n} X_j$ *are mutually exclusive. This mutual exclusivity property implies that the denominator in Equation 3.21 is never equal to* 0*, so that the* P_i*'s are always well-defined. Obviously, the mutual exclusivity of* $\left(X_{n+1}, \sum_{j=1}^{n} X_j\right)$ *is equivalent to the mutual exclusivity of* $\left(P_{n+1}, \sum_{j=1}^{n} P_j\right)$*, which is equivalent to the mutual exclusivity of* $\left(W_{n+1}, \sum_{j=1}^{n} W_j\right)$*.*
The participants in the RS scheme with relative compensation vector defined by Equation 3.21 share the proceeds of the fund proportionally, where each participant's proportion is equal to the proportion that they contribute to aggregate claims $\sum_{j=1}^{n+1} X_j$*. On the other hand, the administrator receives the full proceeds of the fund in case all participants experience a zero loss, whereas the administrator receives nothing in any other case. Typically, insurance losses have a strictly positive probability mass at zero, implying that our assumption that* $\Pr[P_{n+1} = 1] > 0$ *is reasonable.*
Note that the compensations W_i *defined via the relative compensations in Equation 3.21 can be expressed as follows:*

$$W_i = \frac{\sum_{j=1}^{n+1} \pi_j}{\sum_{j=1}^{n+1} X_j} \times X_i, \qquad i = 1, 2, \ldots, n+1. \tag{3.23}$$

This means that the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ *is such that each participant i is compensated the same time* 1 *observable proportion of their loss* X_i*, while the administrator receives the full proceeds of the fund in case each participant has a zero claim. This seems to be a reasonable way to distribute the total fund over the* $n+1$ *agents, provided each agent's initial investment* π_i *is 'reasonable' or 'fair'. In Section 4, we introduce and investigate 'actuarially fair' initial investments.* $\triangleleft$

Example 3.3.2 *Consider a group of n participants. Each of them may experience a particular event of a given type in the observation period $[0,1]$. Possible events include the participant's death, survival, being hospitalized, being diagnosed with a critical illness, etc. For simplicity, hereafter we will assume that all predefined events are 'survival to time 1', but any other choice for the predefined events is possible. For each participant i, we introduce the indicator variable I_i, which is defined by*

$$I_i = \begin{cases} 1 & : i \text{ survives until time } 1 \\ 0 & : i \text{ dies before time } 1 \end{cases} \tag{3.24}$$

The participants appoint an active administrator and attach the following indicator variable I_{n+1} to the administrator:

$$I_{n+1} = \prod_{j=1}^{n} (1 - I_j) \,. \tag{3.25}$$

We further introduce the notation $p_i = \Pr[I_i = 1]$ and $q_i = \Pr[I_i = 0]$, $i = 1, \ldots, n+1$. Obviously, $\sum_{j=1}^{n} I_j$ and I_{n+1} are mutually exclusive. The participants decide to set up an RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$, with the relative compensation vector $\boldsymbol{P}$ given by

$$P_i = \frac{f_i \times I_i}{\sum_{j=1}^{n+1} f_j \times I_j}, \qquad i = 1, 2, \ldots, n+1, \tag{3.26}$$

where f_i, for $i = 1, 2, \ldots, n+1$, are strictly positive real numbers.
The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ defined via the relative compensations in Equation 3.26 is a special case of the RS scheme considered in Example 3.3.1. Such RS schemes with an active administrator have been investigated in detail in Dhaene and Milevsky (2024), who investigate fair methods for the surviving participants to share the total investment among themselves if one or more survive. In case all participants pass away, the administrator receives the full proceeds of the fund. An RS scheme of this type is often called a tontine fund. Dhaene and Milevsky (2024) call f_i the number of tontine shares invested in the tontine fund, and describe the RS scheme defined by Equation 3.26 as a scheme where the proceeds of the fund are equally shared among all surviving tontine shares, where the total number of surviving tontine shares is given by the denominator in Equation 3.26. They consider the situation where initial investments (wealth) and survival probabilities (health) vary among participants, which is called the heterogeneous case. As a special case, they also examine the situation where all participants invest the same amount and the random variables $I_1, I_2, \ldots, I_n$ are i.i.d., which they refer to as the homogeneous case. Denuit and Robert (2025) consider a similar scheme (with a passive administrator, see further), where the f_i are related to (what they call) protection units from the investment fund, and where the total value of the tontine fund is divided equally among all claiming units. An essential difference between the two approaches is that Dhaene and Milevsky (2024) introduce the active administrator who contributes to the investments and will own the proceeds of the fund in case no person survives, whereas Denuit and Robert (2025) assume a passive administrator who does not contribute to the investments, and in case nobody experiences the event under consideration, participants receive their initial investment back. We will come back to this essential difference between the two approaches in a further section of this chapter.
Special cases of the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ defined via the relative compensations in Equation 3.26 have been considered in several papers. Dhaene and Milevsky (2024) consider the following choices for the claiming units f_i:

$$f_i^{DM} = \frac{\pi_i}{p_i}, \qquad i = 1, 2, \ldots, n+1. \tag{3.27}$$

Tavin (2023) propose the following set of f_i:

$$f_i^{T} = \pi_i, \qquad i = 1, 2, \ldots, n+1. \tag{3.28}$$

Denuit and Robert (2023) also consider the case of uniform f_i:

$$f_i^{DR} = 1, \qquad i = 1, 2, \ldots, n+1. \tag{3.29}$$

Finally, in Dhaene and Milevsky (2024), the RS scheme with

$$f_i = \frac{1}{p_i}, \qquad i = 1, 2, \ldots, n+1$$

is also considered. Interpretations and motivations for any of these choices for the number of claiming units f_i can be found in the above-mentioned papers.
We remark that the compensation vector $\boldsymbol{W}$ with relative compensation vector determined by Equation 3.26 can be rewritten as

$$W_i = \left(\frac{\sum_{k=1}^{n+1} \pi_k}{\sum_{j=1}^{n+1} f_j \times I_j} \right) \times f_i \times I_i, \qquad i = 1, 2, \ldots, n+1. \tag{3.30}$$

This means that under the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ defined by Equation 3.26, each surviving participant i is compensated the same time 1 observable proportion of their surviving shares or protection units f_i. The proportion is the random payment per claiming protection unit, which is determined such that the full allocation condition is fulfilled. ◁

Example 3.3.3 *Consider the loss vector $(X_1, X_2, \ldots, X_n)$, describing the non-negative losses of the n participants in the observation period $[0, 1]$. The i-th order statistic $X_{(i)}$ of $(X_1, X_2, \ldots, X_n)$ is the i-th smallest value in $(X_1, X_2, \ldots, X_n)$. Hence,*

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}.$$

Furthermore, we define $X_{(n+1)}$ as follows:

$$X_{(n+1)} = 1 \left(\sum_{j=1}^{n} X_j = 0 \right). \tag{3.31}$$

Notice that we only consider the order statistics for the losses of the n participants. The random variable $X_{(n+1)}$ defined above is not an order statistic. The notation $X_{(n+1)}$ is introduced only to make notations uniform and simple.
Suppose that the participants set up a fund in which each participant i invests an amount π_i. Moreover, they appoint an administrator, who contributes the amount π_{n+1} to the fund. The $n+1$ agents determine the compensations according to the risk-sharing scheme $(\boldsymbol{\pi}, \boldsymbol{P})$, with relative compensation vector $\boldsymbol{P}$ determined by

$$P_i = \frac{X_{(i)}}{\sum_{j=1}^{n+1} X_{(j)}}, \qquad i = 1, 2, \ldots, n+1. \tag{3.32}$$

The interpretation of this compensation-based RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is as follows. For participants who are ordered in decreasing risk-bearing capacity (e.g., decreasing wealth or decreasing age), a lower risk-bearing capacity leads to a higher compensation. In case all participants have a zero-claim, the proceeds of the fund are fully transferred to the administrator. ◁

Example 3.3.4 *Consider the vector $(X_1, X_2, \ldots, X_n)$, describing the non-negative losses of the n participants in the observation period $[0, 1]$. The participants set up a fund in which each participant i invests an amount π_i. They also appoint an administrator, who contributes the amount π_{n+1} to the fund. The participants and the administrator decide to share the proceeds of the fund according to the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$, with the relative compensation vector $\boldsymbol{P}$ being $(X_1, X_2, \ldots, X_n)$ - measurable. This*

means that the randomness of $\boldsymbol{P}$ *is only due to the randomness of the vector* $(X_1, X_2, \ldots, X_n)$*. Hence, there exist functions* $g_i : (\mathbb{R}^+)^n \to \mathbb{R}^+$ *such that*

$$P_i = g_i(X_1, X_2, \ldots, X_n), \qquad i = 1, \ldots, n+1. \tag{3.33}$$

Notice that the relative compensation vector will in general also depend on the initial investment vector $\boldsymbol{\pi}$ *of the RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ *under consideration. But as* $\boldsymbol{\pi}$ *is fixed, we do not explicitly indicate the dependence of* $\boldsymbol{\pi}$ *in the notation of the relative compensation vector. The full allocation condition in Equation 3.7 can now be expressed as follows:*

$$\sum_{j=1}^{n+1} g_j(X_1, X_2, \ldots, X_n) = 1.$$

Moreover, the relative compensation vector is assumed to satisfy

$$g_{n+1}(X_1, X_2, \ldots, X_n) = 1\left(\sum_{j=1}^{n} g_j(X_1, X_2, \ldots, X_n) = 0\right), \tag{3.34}$$

which means that $\sum_{i=1}^{n} g_i(X_1, X_2, \ldots, X_n)$ *and* $g_{n+1}(X_1, X_2, \ldots, X_n)$ *are mutually exclusive.*
The RS scheme considered in Example 3.3.1 is a special case of Equation 3.33, with the relative compensation vector determined from

$$P_i = g_i(X_1, X_2, \ldots, X_n) = \frac{X_i}{\sum_{j=1}^{n+1} X_j}, \qquad i = 1, \ldots, n+1,$$

with X_{n+1} *given by Equation 3.22.*
The RS scheme considered in Example 3.3.2 is also a special case, with the relative compensations defined by

$$P_i = g_i(X_1, X_2, \ldots, X_n) = \frac{f_i \times I_i}{\sum_{j=1}^{n+1} f_j \times I_j}, \qquad i = 1, \ldots, n+1,$$

with the Bernoulli random variables I_i *as defined in Equations 3.24 and 3.25.*
Another special case of Equation 3.33 arises by making the following choice for the relative compensations:

$$P_i = g_i(X_1, X_2, \ldots, X_n) = \frac{X_{(i)}}{\sum_{j=1}^{n+1} X_{(j)}}, \qquad i = 1, \ldots, n+1,$$

where $X_{(n+1)}$ *is defined in Equation 3.31. This special case was considered in Example 3.3.3.* ◁

3.4 Actuarial fairness of compensation-based risk-sharing with an active administrator

3.4.1 Actuarially fair risk-sharing schemes

In this subsection, we consider actuarially fair risk-sharing schemes. While actuarial fairness is not a strict requirement for a functioning tontine or compensation-based risk-sharing scheme, the idea gives a useful and practically important benchmark for several reasons.

First, it provides a natural starting point for setting investments. When individual actuarial fairness holds, each participant's investment is tied to their expected payout, which depends on their loss probability. This gives scheme designers a clear and internally consistent way to determine π_i, rather

than relying on arbitrary judgment. Without such a benchmark, the investment vector $\boldsymbol{\pi}$ is underdetermined, and different choices will tend to favor or disadvantage identifiable subgroups. That presents an ethical concern for certain subgroups. Such a setting can also create a legal risk, as illustrated by the Aboriginal Australian pension case (Dhaene & Milevsky, 2024).

Second, actuarial fairness matters for trust and legitimacy. If participants can see that the group expects to receive back what it contributes, they have a clear and verifiable assurance that no value is being quietly extracted by the administrator or redistributed in a non-transparent way. This becomes especially important in smaller, more heterogeneous pools, where differences in wealth and health are more pronounced. In these cases, perceived unfairness is more likely to make participants unwilling to join the pool.

Third, actuarial fairness offers a useful reference point for intentional departures. We show that there are several plausible allocation rules that trade off individual fairness against broader social objectives such as favoring younger participants or redistributing toward poorer members. These choices are only convincing if it is clear what is being traded off. In this sense, actuarial fairness provides a common reference point through which participants, administrators, and regulators can understand and discuss the value in a pool design.

An RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is said to be actuarially fair for the participants if the time 1 value of each participant's initial investment π_i in the fund is equal to the expected value of the compensation W_i that is received at time 1. This means that no participant experiences a gain or loss on average by joining the pool. Actuarial fairness of particular compensation-based RS schemes has been investigated in Bernard et al. (2025), Denuit and Robert (2025), and Dhaene and Milevsky (2024), among others.

Definition 4 *The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with an active administrator is actuarially fair for each participant if the following conditions hold:*

$$\pi_i = \mathbb{E}\left[W_i\right], \qquad i = 1, 2, \ldots, n.$$

Taking into account Equation 3.18, we can rewrite the actuarial fairness conditions for the n participants as

$$\pi_i = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \mathbb{E}\left[P_i\right], \qquad i = 1, 2, \ldots, n. \tag{3.35}$$

Notice that in a real-world context, the particular RS scheme that is chosen by a group of participants may depend on the social cohesion between them, ranging from solidarity to pure individualism. Especially in small pools of connected participants (e.g., family members), actuarial fairness may not be the first concern and may be replaced by a form of organized transfer, e.g., from the older participants to the younger, or from the richer to the poorer ones. In this section, however, we will further investigate actuarial fairness of RS schemes.

In our general compensation-based RS set-up with an active administrator, all proceeds of the fund are transferred to the administrator in case no participant receives a positive compensation. In return, the administrator pays an initial investment π_{n+1} for this benefit. Let us now define actuarial fairness for the administrator.

Definition 5 *The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with an active administrator is actuarially fair for the active administrator if*

$$\pi_{n+1} = \mathbb{E}\left[W_{n+1}\right].$$

Taking into account Equations 3.10 and 3.18, we can rewrite this actuarial fairness condition for the administrator as follows:

$$\pi_{n+1} = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \Pr\left[P_{n+1} = 1\right]. \tag{3.36}$$

As $\Pr[P_{n+1}=0]$ is strictly positive by assumption, see Equation 3.8, the actuarial fairness condition for the administrator can also be expressed as follows:

$$\pi_{n+1} = \left(\sum_{j=1}^{n} \pi_j\right) \times \frac{\Pr[P_{n+1}=1]}{\Pr[P_{n+1}=0]}. \tag{3.37}$$

We can summarize the relations in Equations 3.36 and 3.37 in the following lemma.

Lemma 1 *If the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is actuarially fair for the active administrator, then we have that*

$$\sum_{j=1}^{n+1} \pi_j = \frac{1}{\Pr[P_{n+1}=0]} \times \sum_{j=1}^{n} \pi_j = \frac{1}{\Pr[P_{n+1}=1]} \times \pi_{n+1} \tag{3.38}$$

In case the relative compensations P_i of the participants are i.i.d., we have that

$$\Pr[P_{n+1}=1] = \Pr[P_1 = P_2 = \cdots = P_n = 0] = (\Pr[P_1=0])^n.$$

This means that if the number of participants n is sufficiently large, we find that $\Pr[P_{n+1}=1] \approx 0$ and $\Pr[P_{n+1}=0] \approx 1$, which implies that the actuarially fair initial investment π_{n+1} of the administrator is close to zero.

Let us now consider the relation between actuarial fairness for the participants and actuarial fairness for the active administrator.

Proposition 1 *If the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is actuarially fair for the n participants, then it is also actuarially fair for the active administrator.*

Proof: From the actuarial fairness conditions in Equation 3.35 for the n participants, we find that

$$\sum_{k=1}^{n} \pi_k = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \sum_{k=1}^{n} \mathbb{E}[P_k].$$

Taking into account Equation 3.12 leads to the actuarial fairness condition in Equation 3.37 for the administrator. ■

From Proposition 1, we can conclude that if the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is not actuarially fair for the active administrator, then it can also not be actuarially fair for all participants. In particular, this situation will occur in case the administrator makes an investment of zero. This observation is further explored in the following proposition.

Proposition 2 *If the active administrator of the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ makes a zero initial investment, i.e., $\pi_{n+1}=0$, then there must be at least one participant i whose investment π_i exceeds the expected compensation, i.e., $\pi_i > \mathbb{E}[W_i]$.*

Proof: In case $\pi_{n+1}=0$, we find from the full allocation condition in Equation 3.19 that

$$\sum_{j=1}^{n} \pi_j = \sum_{j=1}^{n+1} \mathbb{E}[W_j].$$

Furthermore, from Equation 3.11, we have that $\mathbb{E}[W_{n+1}] > 0$. This means that for at least one participant i, their investment exceeds the expected compensation, that is, for at least one participant i we must have that $\pi_i > \mathbb{E}[W_i]$. ■

In the following proposition, we consider several necessary and sufficient conditions for actuarial fairness for the $n+1$ agents involved in the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$.

Proposition 3 *The RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ *with active administrator is actuarially fair for the* $n+1$ *agents if and only if any of the following conditions is satisfied:*
Condition 1*: The RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ *satisfies*

$$\pi_i = \left(\sum_{j=1}^{n+1} \pi_j \right) \times \mathbb{E}\left[P_i\right], \qquad i = 1, 2, \ldots, n+1. \tag{3.39}$$

Condition 2*: The RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ *satisfies*

$$\pi_i = \left(\sum_{j=1}^{n} \pi_j \right) \times \frac{\mathbb{E}\left[P_i\right]}{\Pr\left[P_{n+1} = 0\right]}, \qquad i = 1, 2, \ldots, n+1. \tag{3.40}$$

Condition 3*: The RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ *satisfies*

$$\pi_i = \pi_{n+1} \times \frac{\mathbb{E}\left[P_i\right]}{\Pr\left[P_{n+1} = 1\right]}, \qquad i = 1, 2, \ldots, n+1. \tag{3.41}$$

Proof: In order to prove the implications

$$\text{Condition 1} \Rightarrow \text{Condition 2} \Rightarrow \text{Condition 3} \Rightarrow \text{Condition 1},$$

it suffices to prove that each of the three conditions implies actuarial fairness for the active administrator, see Lemma 1.
Obviously, Condition 1 implies actuarial fairness for the active administrator. From Condition 2, for agent $n+1$, we find that

$$\pi_{n+1} = \left(\sum_{j=1}^{n} \pi_j \right) \times \frac{\Pr[P_{n+1} = 1]}{\Pr[P_{n+1} = 0]},$$

which is equivalent to actuarial fairness for the active administrator.
Summing over all agents, Condition 3 leads to:

$$\sum_{j=1}^{n+1} \pi_j = \pi_{n+1} \times \frac{1}{\Pr[P_{n+1} = 1]},$$

which is again equivalent to actuarial fairness for the active administrator.
This ends the proof. ∎

3.4.2 Risk-sharing rules and actuarial fairness

So far, we have considered the actuarial fairness of a given *RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ with active administrator. Let us now consider an *RS rule* $\boldsymbol{P}$ with an active administrator, which transforms any investment vector $\boldsymbol{\pi}$ into a relative compensation vector $\boldsymbol{P}\left[\boldsymbol{\pi}\right]$:

$$\boldsymbol{\pi} \to \boldsymbol{P}\left[\boldsymbol{\pi}\right]. \tag{3.42}$$

This means that any investment vector $\boldsymbol{\pi}$ leads to the compensation vector $\boldsymbol{W}\left[\boldsymbol{\pi}\right]$, with

$$\boldsymbol{W}\left[\boldsymbol{\pi}\right] = \left(\sum_{j=1}^{n+1} \pi_j \right) \times \boldsymbol{P}\left[\boldsymbol{\pi}\right], \tag{3.43}$$

(see Definition 3). Let us now consider a special type of RS rules $\boldsymbol{P}$, which satisfy the following indifference property:

$$\boldsymbol{P}\left[c \times \boldsymbol{\pi}\right] = \boldsymbol{P}\left[\boldsymbol{\pi}\right], \qquad \text{for any } c > 0 \text{ and any investment vector } \boldsymbol{\pi}. \tag{3.44}$$

This means that the RS rule $\boldsymbol{P}$ is such that if all participants increase their initial investment by a constant proportion, e.g., by 20%, then their relative compensation vector remains unchanged, which seems to be a reasonable property. Obviously any RS rule of which the relative compensation vector $\boldsymbol{P}$ is independent of the initial investment vector satisfies the indifference property in Equation 3.44. In the following proposition, we consider compensation vectors of an RS rule $\boldsymbol{P}$ that satisfies the indifference property in Equation 3.44.

Proposition 4 *Consider the RS rule $\boldsymbol{P}$ with active administrator that satisfies the indifference property in Equation 3.44. Then for any positive c and any investment vector $\boldsymbol{\pi}$, one has that*

$$\mathbf{W}\left[c \times \boldsymbol{\pi}\right] = c \times \mathbf{W}\left[\boldsymbol{\pi}\right]$$

Proof: For any investment vector $\boldsymbol{\pi}$, any $c > 0$ and any participant i, we find that

$$\begin{aligned} W_i\left[c \times \boldsymbol{\pi}\right] &= c \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times P_i\left[c \times \boldsymbol{\pi}\right] \\ &= c \times \left(\sum_{j=1}^{n+1} \pi_j\right) \times P_i\left[\boldsymbol{\pi}\right] \\ &= c \times W_i\left[\boldsymbol{\pi}\right]. \end{aligned}$$

This ends the proof. ∎

Suppose now that a group of participants decides to use the RS rule $\boldsymbol{P}$ with an active administrator, which satisfies the indifference property in Equation 3.44. The next question they have to answer is then what particular initial investment vector $\boldsymbol{\pi}$ to choose. It may be reasonable to select an investment vector $\boldsymbol{\pi}$ such that the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}\left[\boldsymbol{\pi}\right])$ is actuarially fair. In the following proposition, we show that this problem has in general no unique solution.

Proposition 5 *Consider the RS rule $\boldsymbol{P}$ with an active administrator satisfying the indifference property in Equation 3.44. Suppose that the RS scheme $(\boldsymbol{\pi}^*, \boldsymbol{P}\left[\boldsymbol{\pi}^*\right])$ is actuarially fair for all participants, then for any $c > 0$, also the RS scheme $(c \times \boldsymbol{\pi}^*, \boldsymbol{P}\left[c \times \boldsymbol{\pi}^*\right])$ is actuarially fair for all participants.*

Proof: From Proposition 4, we find for any participant i that

$$\mathbb{E}\left[W_i\left[c \times \boldsymbol{\pi}^*\right]\right] = c \times \mathbb{E}\left[W_i\left[\boldsymbol{\pi}^*\right]\right].$$

The actuarial fairness of $(\boldsymbol{\pi}^*, \boldsymbol{P}\left[\boldsymbol{\pi}^*\right])$ can be expressed as

$$\mathbb{E}\left[W_i\left[\boldsymbol{\pi}^*\right]\right] = \pi_i^*, \qquad i = 1, 2, \ldots, n+1.$$

Combining both expressions leads to

$$\mathbb{E}\left[W_i\left[c \times \boldsymbol{\pi}^*\right]\right] = c \times \pi_i^*, \qquad i = 1, 2, \ldots, n+1,$$

which are the actuarial fairness conditions for $(c \times \boldsymbol{\pi}^*, \boldsymbol{P}\left[c \times \boldsymbol{\pi}^*\right])$. ∎

We can conclude that for any RS rule $\boldsymbol{P}$ satisfying the indifference property in Equation 3.44, one has that if the RS scheme $(\boldsymbol{\pi^*}, \boldsymbol{P}\left[\boldsymbol{\pi}^*\right])$ is actuarially fair for all participants, then also the RS scheme

$(c \times \boldsymbol{\pi}^*, \boldsymbol{P}[c \times \boldsymbol{\pi}^*])$ is actuarially fair for all participants. This means that for such RS rules, actuarially fair investment vectors $\boldsymbol{\pi}$ are only defined up to a positive constant factor. For RS rules satisfying the indifference property in Equation 3.44, Proposition 3 may be a guide for choosing the appropriate actuarially fair set of initial investments: One can either first determine the level of the aggregate investments of the $n+1$ agents, or the level of the aggregate investment of the n participants, or the level of the individual investment of the administrator, and then determine the individual actuarially fair investments π_i by the corresponding Conditions 1, 2 or 3.

Example 3.4.1 *In order to illustrate the previous propositions, consider a group of n participants who are exposed to the losses $X_1, X_2, \ldots, X_n$, respectively. Suppose they agree to use the RS rule $\boldsymbol{P}$, where for any RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$, the relative compensation vector $\boldsymbol{P}[\boldsymbol{\pi}]$ is given by Equation 3.21, as considered in Example 3.3.1:*

$$P_i[\boldsymbol{\pi}] = \frac{X_i}{\sum_{k=1}^{n+1} X_k}, \qquad i = 1, 2, \ldots, n+1,$$

with X_{n+1} defined by Equation 3.22. As the random losses $X_1, X_2, \ldots, X_n$ are assumed to be independent of $\boldsymbol{\pi}$, we have that the relative compensation vectors of the RS rule $\boldsymbol{P}$ satisfy the indifference property in Equation 3.44.
Let us assume that the participants decide to choose an investment vector $\boldsymbol{\pi}$ such that the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$ is actuarially fair for any of them. This means that the investment vector follows from the set of Equations 3.39:

$$\pi_i = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \mathbb{E}\left[\frac{X_i}{\sum_{k=1}^{n+1} X_k}\right], \qquad i = 1, 2, \ldots, n+1. \tag{3.45}$$

In this case an actuarially fair investment vector is only defined up to a positive constant factor, that is, if $\boldsymbol{\pi}^*$ is a solution of Equation 3.45, then also $c \times \boldsymbol{\pi}^*$ is a solution, for any $c > 0$. Applying this RS rule in practice, we could first determine a reference solution $\boldsymbol{\pi}$ of Equation 3.45, and in a second step, determine the investment vector $c \times \boldsymbol{\pi}$ with c such that $\Pr\left[c \times \sum_{j=1}^{n+1} \pi_j > \sum_{j=1}^{n+1} X_k\right]$ is sufficiently large. As an extreme case, suppose that c is chosen such that $\Pr\left[c \times \sum_{j=1}^{n+1} \pi_j > \sum_{j=1}^{n+1} X_k\right] = 1$, then we have that

$$W_i = \left(c \times \sum_{j=1}^{n+1} \pi_j\right) \times \frac{X_i}{\sum_{k=1}^{n+1} X_k} \geq X_i, \qquad i = 1, 2, \ldots, n+1,$$

which means that each compensation W_i is always larger than its corresponding claim X_i in this case. ◁

Example 3.4.2 *Consider the setting of Example 3.3.2 with the number of protection units f_i chosen to be equal to the initial investment π_i. This means that we consider the RS rule $\boldsymbol{P}$, where for any RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$, each participant and the administrator receive a level of compensation proportional to their initial contribution in case the predefined event occurs:*

$$W_i[\boldsymbol{\pi}] = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \frac{\pi_i \times I_i}{\sum_{j=1}^{n+1} \pi_j \times I_j}, \qquad i = 1, 2, \ldots, n+1. \tag{3.46}$$

In these equations, the I_i are Bernoulli-distributed random variables with $\Pr[I_i = 1] = p_i$ and $\Pr[I_i = 0] = q_i$.

We call this rule that was proposed in Tavin (2023) the Tavin RS rule, see Equation 3.28. Obviously, the RS rule $\boldsymbol{P}$ satisfies the indifference property in Equation 3.44, implying that actuarially fair

investment vectors are only determined up to a positive constant factor. From Equation 3.46 it follows that the actuarial fairness conditions for all participants in this RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$ *can be written as follows:*

$$1 = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \mathbb{E}\left[\frac{I_i}{\sum_{j=1}^{n+1} \pi_j \times I_j}\right], \qquad i = 1, 2, \ldots, n+1.$$

Obviously, for actuarially fair initial investments the expectations $\mathbb{E}\left[\frac{I_i}{\sum_{j=1}^{n+1} \pi_j \times I_j}\right]$ *have to be equal for all agents. It is important to note that an RS scheme which is not actuarially fair is not necessarily a 'wrong' choice. Suppose that two participants of different age each pay the same initial investment. Then, upon survival, they will receive the same compensation. However, the one with the higher survival probability (the younger one, let us say) is more likely to survive and hence, is favored. As clearly discussed in Tavin (2023), this RS scheme accommodates a reallocation of wealth that is favorable to those who are likely to survive longer. Such a reallocation can be a valuable objective, for example, in a given small community of family members.*

Example 3.4.3 *Consider the Tavin RS rule* $\boldsymbol{P}$ *based on the RS schemes* $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$ *of Example 3.4.2, with two participants and an active administrator.*

From Equation 3.46 it follows that the actuarial fairness conditions for all participants can be written as follows:

$$1 = (\pi_1 + \pi_2 + \pi_3) \times \mathbb{E}\left[\frac{I_i}{\pi_1 I_1 + \pi_2 I_2 + \pi_3 I_3}\right], \quad i = 1, 2, 3.$$

Let us assume that I_1 *and* I_2 *are mutually independent. In this setting, we find that*

$$\mathbb{E}\left[\frac{I_1}{\pi_1 I_1 + \pi_2 I_2 + \pi_3 I_3}\right] = \left(\frac{p_1 p_2}{\pi_1 + \pi_2} + \frac{p_1 q_2}{\pi_1}\right),$$

while

$$\mathbb{E}\left[\frac{I_2}{\pi_1 I_1 + \pi_2 I_2 + \pi_3 I_3}\right] = \left(\frac{p_1 p_2}{\pi_1 + \pi_2} + \frac{p_2 q_1}{\pi_2}\right)$$

and

$$\mathbb{E}\left[\frac{I_3}{\pi_1 I_1 + \pi_2 I_2 + \pi_3 I_3}\right] = \frac{q_1 q_2}{\pi_3}.$$

This results in the following system of equations which characterizes the set of all initial investments (π_1, π_2, π_3) *which lead to an actuarially fair Tavin RS scheme in Equation 3.46:*

$$\begin{cases} 1 = (\pi_1 + \pi_2 + \pi_3) \times \left(\frac{p_1 p_2}{\pi_1 + \pi_2} + \frac{p_1 q_2}{\pi_1}\right) & (3.47) \\ 1 = (\pi_1 + \pi_2 + \pi_3) \times \left(\frac{p_1 p_2}{\pi_1 + \pi_2} + \frac{p_2 q_1}{\pi_2}\right) & (3.48) \\ 1 = (\pi_1 + \pi_2 + \pi_3) \times \left(\frac{q_1 q_2}{\pi_3}\right). & (3.49) \end{cases}$$

From Equations 3.47 and 3.48, we find that

$$\pi_2 = \frac{p_2 q_1}{p_1 q_2} \pi_1, \tag{3.50}$$

while Equation 3.49 leads to

$$\pi_1 + \pi_2 = \pi_3 \times \frac{1 - q_1 q_2}{q_1 q_2}. \tag{3.51}$$

Substituting Equation 3.50 in Equation 3.51 gives rise to

$$\pi_1 = \pi_3 \times \frac{p_1}{q_1} \times \frac{1 - q_1 q_2}{p_1 q_2 + p_2 q_1}, \tag{3.52}$$

and also

$$\pi_2 = \pi_3 \times \frac{p_2}{q_2} \times \frac{1 - q_1 q_2}{p_1 q_2 + p_2 q_1}. \tag{3.53}$$

It is easy to verify that any (π_1, π_2, π_3) with $\pi_3 > 0$ and π_1 and π_2 given by Equations 3.52 and 3.53, respectively, satisfies the system of Equations 3.47, 3.48 and 3.49. We conclude that the set of all initial investments (π_1, π_2, π_3) that lead to an actuarially fair Tavin RS scheme in Equation 3.46 with two participants is characterized by Equations 3.52 and 3.53, where π_3 can be any positive real number.

Example 3.4.4 *We end this subsection by considering the RS rule $\boldsymbol{P}$ based on the RS schemes $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$ of Example 3.3.2, with constant number of protection units for each participant:*

$$f_i = 1, \qquad i = 1, 2, \ldots, n+1.$$

This means that the RS rule $\boldsymbol{P}$ is such that for any RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$, the relative compensation vector $\boldsymbol{P}[\boldsymbol{\pi}]$ is given by

$$P_i[\boldsymbol{\pi}] = \frac{I_i}{\sum_{j=1}^{n+1} I_j}, \qquad i = 1, 2, \ldots, n+1. \tag{3.54}$$

Obviously, $\boldsymbol{P}$ satisfies the indifference property in Equation 3.44. The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P}[\boldsymbol{\pi}])$ is actuarially fair for all participants if and only if the conditions in Equation 3.35 are satisfied. Inspired by the approach proposed in Denuit and Robert (2025), one can verify that the expected proportions follow from

$$\mathbb{E}[P_i[\boldsymbol{\pi}]] = \sum_{k=1}^{n} \frac{1}{k} \Pr\left[I_i = 1, \ \sum_{j=1}^{n} I_j = k \right], \qquad i = 1, 2, \ldots, n. \tag{3.55}$$

In case the indicator variables I_i of the n participants are mutually independent, the expressions in Equation 3.55 can be transformed into

$$\mathbb{E}[P_i[\boldsymbol{\pi}]] = \Pr[I_i = 1] \sum_{k=1}^{n} \frac{1}{k} \Pr\left[\sum_{j=1}^{n} I_j - I_i = k - 1 \right] \qquad i = 1, 2, \ldots, n. \tag{3.56}$$

In this case, the expectations of the relative compensations $P_i[\boldsymbol{\pi}]$ follow from probabilities of events related to sums of independent Bernoulli random variables. The actuarially fair initial investments of this RS scheme follow then from Proposition 3 and from Equation 3.56. Notice that for this RS rule, the actuarially fair initial investments are only defined up to a positive constant factor.
The calculations above can in a straightforward way be generalized to the case that all f_i are positive (not necessarily equal) integers, rather than all equally 1. We refer to Denuit and Robert (2025) for more details on this case. ◁

3.5 Compensation-based risk-sharing with a passive administrator

3.5.1 Introducing the passive administrator

Consider an RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with n participants and an active administrator as defined above. The time 0 investments of the $n+1$ agents are summarized in the investment vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{n+1})$, while the relative compensation vector is given by $\boldsymbol{P} = (P_1, P_2, \ldots, P_{n+1})$. The compensations are summarized in the compensation vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_{n+1})$. The latter vector is given by

$$W_i = \left(\sum_{j=1}^{n+1} \pi_j\right) \times P_i \qquad i = 1, 2, \ldots, n+1. \tag{3.57}$$

This setup of a compensation-based RS scheme with an active administrator is a generalization of the setup in Dhaene and Milevsky (2024), who consider the RS schemes related to tontine funds, as described in Example 3.3.2.

Denuit and Robert (2025) also consider the setting of Dhaene and Milevsky (2024), but with a passive administrator who does not join the group of participants in investing. Hence, these authors assume that $\pi_{n+1} = 0$, implying that (without any adaptation of the compensations), such an RS scheme can never be actuarially fair, see Proposition 2. In order to include the possibility that the RS scheme is actuarially fair for all participants, Denuit and Robert (2025) assume that in case no participant receives a positive compensation, that is, in case $\sum_{j=1}^{n} P_j = 0$, or equivalently, $P_{n+1} = 1$, each participant i is returned their original investment. Most papers on tontine funds in the last few years have added this element to restore expected payoffs and, hence, actuarial fairness. Hereafter, we will generalize the approach of Denuit and Robert (2025) in the same way as we generalized the approach of Dhaene and Milevsky (2024) in the first part of this chapter. That is, we will consider risk-sharing with a passive administrator for a general class of compensations, instead of restricting to the tontine fund case of Example 3.3.2. Let us start by defining a compensation-based RS scheme with a passive administrator.

Definition 6 *A compensation-based risk-sharing scheme for a given group of n participants and a passive administrator is a pair $(\boldsymbol{\pi}, \boldsymbol{P})$, where $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{n+1})$ is the initial investment vector with $\pi_{n+1} = 0$. Further, $\boldsymbol{P} = (P_1, P_2, \ldots, P_n, P_{n+1})$ is the relative compensation vector, that is $\boldsymbol{P} \in \mathcal{R}_{n+1}$ defined in Equation 3.17, and where the compensations attributed to the n participants follow from the compensation vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_{n+1})$, which is defined by*

$$W_i = \left(\sum_{j=1}^{n} \pi_j\right) \times P_i + \pi_i \times P_{n+1} \qquad : i = 1, 2, \ldots, n, \tag{3.58}$$

whereas $W_{n+1} = 0$.

From this definition, we see that an RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator is one where each participant i invests an amount π_i and receives a relative compensation P_i of the available fund at time 1, whereas the administrator makes an investment $\pi_{n+1} = 0$ and receives a zero-compensation $W_{n+1} = 0$ at time 1. Moreover, in case the relative compensations P_i of all participants are 0, or equivalently, $P_{n+1} = 1$, every participant is returned their investment π_i. The administrator is 'passive' in the sense that they neither invest in the fund nor receive any compensation based on the participants' initial investments.

In order to be able to clearly distinguish between the 'active' and the 'passive' administrator case, see Equations 3.57 and 3.58, hereafter we will sometimes write W_i^{active} or W_i^{passive} instead of W_i for the compensation of agent i.

Taking into account that $\sum_{j=1}^{n+1} P_j = 1$, it is a straightforward exercise to prove the following full allocation property for a compensation-based RS scheme with a passive administrator:

$$\sum_{i=1}^{n} W_i = \sum_{j=1}^{n} \pi_j. \tag{3.59}$$

This means that the total amount of compensations paid at time 1 is exactly equal to the fund value at that time. Hereafter, when considering an 'RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator', we mean a 'compensation-based RS scheme with a passive administrator'.

Example 3.5.1 *Denuit and Robert (2025) and Dhaene and Milevsky (2024) discuss the above-mentioned approach with a passive administrator for the setting of Example 3.3.2, that is, in the framework of a tontine fund with $\pi_{n+1} = 0$. In particular, they consider the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator, where the compensation vector $\boldsymbol{W}$ is defined by Equation 3.58, with relative compensation vector $\boldsymbol{P}$ given by*

$$P_i = \frac{f_i \times I_i}{\sum_{j=1}^{n+1} f_j \times I_j}, \qquad i = 1, 2, \ldots, n+1, \tag{3.60}$$

where the protection units f_i and the indicator variables I_i are defined in Example 3.3.2. Dhaene and Milevsky (2024) write the following about this approach: 'While the above-mentioned approach (i.e., without an (active) administrator, and returning the investments if no participant receives a positive compensation) resolves the mathematical problem, we believe that this isn't why people buy tontines. Indeed, it violates the spirit of the (historical) tontine in which all rights and ownership benefits are lost at death. Furthermore, some members may not have any beneficiaries, leading to yet another unintended redistribution of wealth. In extreme cases, when there is only one person surviving, this may create a moral hazard. In other words, and for many reasons, while adding a death benefit refund or payout 'solves' the math, it 'ruins' the elegance of the tontine ideal.' As far as we are aware, Dhaene and Milevsky (2024) are the first to introduce an active tontine administrator as both a technical and real-world solution to some of the above-mentioned issues, instead of artificially adding legacy or bequest payouts to participants.

3.5.2 Actuarially fair risk-sharing schemes with a passive administrator

Similar to actuarial fairness for an RS scheme with an active administrator, the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator is said to be actuarially fair for the participants if the time 1 value of each participant's initial investment is equal to the expected value of the compensation received at time 1.

Definition 7 *The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator is actuarially fair for all participants if the following conditions hold:*

$$\pi_i = \mathbb{E}[W_i], \qquad i = 1, 2, \ldots, n. \tag{3.61}$$

Taking into account that $\pi_{n+1} = W_{n+1} = 0$, we find that the RS scheme is always actuarially fair for the passive administrator.

In the next proposition, another characterization is given for the actuarial fairness of an RS scheme with a passive administrator.

Proposition 6 *The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator is actuarially fair for its n participants if and only if any of the following conditions is satisfied:*

***Condition 1**: The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ satisfies*

$$\pi_i = \left(\sum_{j=1}^{n} \pi_j\right) \times \mathbb{E}\left[P_i\right] + \pi_i \times \mathbb{E}[P_{n+1}], \qquad i = 1, 2, \ldots, n. \tag{3.62}$$

***Condition 2**: The RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ satisfies*

$$\pi_i = \left(\sum_{j=1}^{n} \pi_j\right) \times \frac{\mathbb{E}\left[P_i\right]}{\Pr\left[P_{n+1} = 0\right]}, \qquad i = 1, 2, \ldots, n. \tag{3.63}$$

Proof: Actuarial fairness for all participants is, by definition, equivalent to Condition 1. Taking into account Equation 3.10, the actuarial fairness conditions in Equation 3.62 can be transformed into the Equations 3.63. ■

In the following corollary, we show that under appropriate conditions, actuarially fair investments π_i of the participants in an RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with an active administrator are also actuarially fair in the corresponding RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator.

Corollary 1 *Consider the RS scheme $((\pi_1, \ldots, \pi_n, \pi_{n+1}), \boldsymbol{P})$ with an active administrator (as defined in Equation 3.57) and the RS scheme $((\pi_1, \ldots, \pi_n, 0), \boldsymbol{P})$ with a passive administrator (as defined in Equation 3.58). Then we have that the following statements are equivalent:*

(a) $\pi_i = \mathbb{E}[W_i^{active}], \quad i = 1, 2, \ldots, n.$

(b) $\pi_i = \mathbb{E}[W_i^{passive}], \quad i = 1, 2, \ldots, n \quad$ *and* $\quad \pi_{n+1} = \mathbb{E}[W_{n+1}^{active}].$

Proof: Taking into account Proposition 1 and Condition 2 in Proposition 3, we can rewrite the equations (a) as follows:

$$\pi_i = \left(\sum_{j=1}^{n} \pi_j\right) \times \frac{\mathbb{E}[P_i]}{\Pr[P_{n+1} = 0]}, \qquad i = 1, 2, \ldots, n$$

and

$$\pi_{n+1} = \left(\sum_{j=1}^{n+1} \pi_j\right) \times \mathbb{E}[P_{n+1}].$$

From Condition 2 in Proposition 6, we can conclude the equations in (a) are equivalent to the equations in (b).

■

Notice that the only if statement of the corollary above can also be proven by rewriting Equations 3.57 and 3.58 as

$$W_i^{\text{active}} = \left(\sum_{j=1}^{n} \pi_j\right) \times P_i + \pi_{n+1} \times P_i, \qquad i = 1, 2, \ldots, n+1$$

and

$$W_i^{\text{passive}} = \left(\sum_{j=1}^{n} \pi_j\right) \times P_i + \pi_i \times P_{n+1}, \qquad i = 1, 2, \ldots, n.$$

When the participants' contributions are actuarially fair in case of an active administrator, then from Condition 3 of Proposition 3, we have that

$$\pi_{n+1} \times \mathbb{E}[P_i] = \pi_i \times \mathbb{E}[P_{n+1}], \qquad i = 1, \ldots, n+1.$$

This implies that

$$\mathbb{E}[W_i^{\text{active}}] = \mathbb{E}[W_i^{\text{passive}}], \qquad i = 1, \ldots, n,$$

which proves the stated implication.

Suppose that the participants in an RS scheme decide to first fix their initial investments and only afterwards make the choice between an active or a passive administrator. Corollary 1 states that from an actuarial fairness point of view, both schemes are equivalent, provided the active administrator makes an actuarially fair initial investment.

3.5.3 Risk-sharing rules with a passive administrator

Similar to Section 3.4.2, where we considered RS rules with an active administrator, we introduce RS rules with a passive administrator. An RS rule $\boldsymbol{P}$ with a passive administrator transforms any investment vector $\boldsymbol{\pi}$ into a relative compensation vector $\boldsymbol{P}[\boldsymbol{\pi}]$. This means that any investment vector leads to the compensation vector $\boldsymbol{W}[\boldsymbol{\pi}]$, with

$$W_i[\boldsymbol{\pi}] = \left(\sum_{j=1}^{n} \pi_j \right) \times P_i\left[\boldsymbol{\pi}\right] + \pi_i \times P_{n+1}\left[\boldsymbol{\pi}\right], \qquad i = 1, 2, \ldots, n. \tag{3.64}$$

As before, we consider the indifference property in Equation 3.44 for RS rules with a passive administrator. It is then a straightforward exercise to prove that Propositions 4 and 5 can easily be adapted to hold for RS rules with a passive administrator. This means that for RS rules with a passive administrator, which satisfy the indifference property in Equation 3.44, actuarially fair investment vectors are only defined up to a positive constant factor.

3.6 Transforming RS schemes

In this section, we compare RS schemes with active and passive administrators. In order to establish the link between the two approaches, we define the RS scheme with an active administrator as $(\boldsymbol{\pi}^a, \boldsymbol{P})$ and define the RS scheme with a passive administrator as $(\boldsymbol{\pi}^p, \boldsymbol{P})$. The relative compensation vector $\boldsymbol{P} = (P_1, P_2, \ldots, P_{n+1})$ for the RS schemes with the active and the passive administrator is identical. The investment and compensation vectors of the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator are given by

$$\boldsymbol{\pi}^a = \left(\pi_1^a, \pi_2^a, \ldots, \pi_{n+1}^a\right)$$

and

$$\boldsymbol{W}^a = \left(W_1^a, W_2^a, \ldots, W_{n+1}^a\right),$$

with the participants' compensations following from

$$W_i^a = \left(\sum_{j=1}^{n+1} \pi_j^a \right) \times P_i \qquad i = 1, 2, \ldots, n+1. \tag{3.65}$$

On the other hand, the investment and compensation vectors of the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator are given by

$$\boldsymbol{\pi}^p = (\pi_1^p, \pi_2^p, \ldots, \pi_n^p, 0)$$

and

$$\boldsymbol{W}^p = (W_1^p, W_2^p, \ldots, W_n^p, 0)$$

with the participants' compensations determined by

$$W_i^p = \left(\sum_{j=1}^{n} \pi_j^p\right) \times P_i + \pi_i^p \times P_{n+1} \qquad i = 1, 2, \ldots, n. \tag{3.66}$$

3.6.1 Active-to-passive transformation

Let us start from the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator, with the compensations W_i^a of the $(n+1)$ participants given by Equation 3.65. The first step in the transformation of an RS scheme with an active administrator to an RS scheme with a passive administrator is that the administrator of the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ does not pay π_{n+1}^a themselves, but asks each participant i to invest the amount $\pi_{n+1,i}^a$ in addition to their original investment π_i^a, with

$$\pi_{n+1}^a = \sum_{i=1}^{n} \pi_{n+1,i}^a. \tag{3.67}$$

This means that after this transformation, the total time 0 investment effort of each participant, notation π_i^p, is given by

$$\pi_i^p = \begin{cases} \pi_i^a + \pi_{n+1,i}^a & : i = 1, 2, \ldots, n \\ 0 & : i = n+1 \end{cases} \tag{3.68}$$

From Equations 3.67 and 3.68, it is easy to verify that

$$\sum_{j=1}^{n} \pi_j^p = \sum_{j=1}^{n+1} \pi_j^a, \tag{3.69}$$

which means that the transfer of the time 0 investment of the administrator to the participants does not change the aggregate investment effort.

Similar to spreading the investment π_{n+1}^a over all participants, let us suppose that the time 1 payout W_{n+1}^a to the administrator is also shared among the n participants. To be more precise, in addition to the original compensation W_i^a, each participant i receives an extra compensation $W_{n+1,i}^a$ which is given by

$$W_{n+1,i}^a = \pi_i^p \times P_{n+1}, \qquad i = 1, 2, \ldots, n. \tag{3.70}$$

Taking into account Equation 3.65 for $i = n+1$, as well as Equation 3.69, we find that

$$W_{n+1}^a = \sum_{i=1}^{n} W_{n+1,i}^a. \tag{3.71}$$

The total time 1 compensation of each participant i in the RS scheme with a passive administrator, notation W_i^p, is then given by

$$W_i^p = \begin{cases} W_i^a + \pi_i^p \times P_{n+1} & : i = 1, 2, \ldots, n \\ 0 & : i = n+1, \end{cases} \tag{3.72}$$

From Equation 3.69 and the fact that $W_i^a = \left(\sum_{j=1}^{n+1} \pi_j^a\right) \times P_i$ for $i = 1, 2, \ldots, n+1$, it follows that the compensations W_i^p of the participants can be expressed as follows:

$$W_i^p = \left(\sum_{j=1}^{n} \pi_j^p\right) \times P_i + \pi_i^p \times P_{n+1}, \qquad i = 1, 2, \ldots, n, \tag{3.73}$$

which is identical to the participants' compensations with a passive administrator in Equation 3.66.

From Equations 3.69, 3.73 and taking into account the full allocation condition $\sum_{j=1}^{n+1} P_j = 1$, it is easy to verify that

$$\sum_{i=1}^{n} W_i^p = \sum_{i=1}^{n+1} W_i^a. \tag{3.74}$$

This means that the transformation from the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with active administrator into the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with passive administrator does not change the aggregate compensations.

Summarizing, by applying Equations 3.68 and 3.72, we have transformed the original RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator into the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator, with $\pi_{n+1}^p = 0$ and $W_{n+1}^p = 0$. This transformation redistributes the active administrator's investment π_{n+1}^a and W_{n+1}^a over the n participants. Let us now define an actuarially fair transformation.

Definition 8 *Consider the actuarially fair RS scheme with an active administrator $(\boldsymbol{\pi}^a, \boldsymbol{P})$. The transformation from $(\boldsymbol{\pi}^a, \boldsymbol{P})$ into the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator is said to be actuarially fair in case the following conditions hold:*

$$\pi_{n+1,i}^a = \mathbb{E}\left[W_{n+1,i}^a\right], \qquad i = 1, 2, \ldots, n. \tag{3.75}$$

The definition above means that a transformation is actuarially fair in case the extra investment $\pi_{n+1,i}^a$ made by each participant i is equal to the expected value of the extra compensation $W_{n+1,i}^a$ received. Taking into account Equation 3.70 and the fact that $\mathbb{E}\left[P_{n+1}\right] = \Pr\left[P_{n+1} = 1\right]$, the fairness conditions in Equation 3.75 can be expressed as follows:

$$\pi_{n+1,i}^a = \pi_i^p \times \Pr\left[P_{n+1} = 1\right], \qquad i = 1, 2, \ldots, n. \tag{3.76}$$

From Equations 3.67 and 3.71, we find that the conditions in Equation 3.75 imply that

$$\pi_{n+1}^a = \mathbb{E}\left[W_{n+1}^a\right],$$

which corresponds to the actuarial fairness of the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ for the active administrator.

We can now investigate the consequences of assuming actuarial fairness of the transformation of an actuarially fair RS scheme with an active administrator into one with a passive administrator.

Proposition 7 *Consider the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator, which is actuarially fair for all participants. Let $(\boldsymbol{\pi}^p, \boldsymbol{P})$ be the actuarially fair transformation of $(\boldsymbol{\pi}^a, \boldsymbol{P})$. In this case, we have that*

$$\pi_{n+1,i}^a = \pi_i^a \times \frac{\Pr\left[P_{n+1} = 1\right]}{\Pr\left[P_{n+1} = 0\right]}, \qquad i = 1, 2, \ldots, n. \tag{3.77}$$

and

$$\pi_i^p = \frac{\pi_i^a}{\Pr\left[P_{n+1} = 0\right]}, \qquad i = 1, 2, \ldots, n. \tag{3.78}$$

Proof: These expressions follow immediately from Equations 3.68, 3.69 and 3.76. ∎

In the following proposition we investigate the relation between the actuarial fairness of $(\boldsymbol{\pi}^a, \boldsymbol{P})$ and the actuarial fairness of $(\boldsymbol{\pi}^p, \boldsymbol{P})$.

Proposition 8 *Consider the RS scheme with an active administrator $(\boldsymbol{\pi}^a, \boldsymbol{P})$, which is actuarially fair for its participants, and let the RS scheme with a passive administrator $(\boldsymbol{\pi}^p, \boldsymbol{P})$ be its actuarially fair transformation. Then we have that $(\boldsymbol{\pi}^p, \boldsymbol{P})$ is also actuarially fair for all its participants.*

Proof: By construction, $\pi^p_{n+1} = W^p_{n+1} = 0$, implying that $(\boldsymbol{\pi}^p, \boldsymbol{P})$ is actuarially fair for the administrator. It remains to prove that $(\boldsymbol{\pi}^p, \boldsymbol{P})$ is also actuarially fair for all its participants.
We have that $(\boldsymbol{\pi}^a, \boldsymbol{P})$ is actuarially fair for its participants if and only if the following conditions hold:

$$\pi^a_i = \left(\sum_{j=1}^{n+1} \pi^a_j\right) \times \mathbb{E}\left[P_i\right], \qquad i = 1, 2, \ldots, n.$$

Taking into account Equations 3.69 and 3.78, this set of equations can be transformed into

$$\pi^p_i = \left(\sum_{j=1}^{n} \pi^p_j\right) \times \frac{\mathbb{E}\left[P_i\right]}{\Pr\left[P_{n+1} = 0\right]}, \qquad i = 1, 2, \ldots, n.$$

These relations, together with the earlier observed relation $\pi^p_{n+1} = 0$, are equivalent to actuarial fairness for all the participants of the RS scheme with a passive administrator $(\boldsymbol{\pi}^p, \boldsymbol{P})$. ■

The proposition above states that if we transform an RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator into the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator as explained in Equations 3.68 and 3.72 and if the transformation is actuarially fair in the sense of Equation 3.75, then actuarial fairness of the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator implies actuarial fairness of the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator.

3.6.2 Passive-to-active transformation

In this section, we transform the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator into an RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator. Let us start from the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator, with the compensations W^p_i of the participants given by Equation 3.73. The first step in the transformation is that the participants of the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ agree that, in the event $P_1 = P_2 = \ldots = P_n = 0$, the total investments are given to an administrator, instead of being returned to the participants. That is, we have that

$$W^a_{n+1} = \begin{cases} \left(\sum_{j=1}^{n} \pi^p_j\right) & : P_1 = P_2 = \ldots = P_n = 0 \\ 0 & : \text{otherwise.} \end{cases} \tag{3.79}$$

From Equation 3.79 and from the definition of P_{n+1}, we have that

$$W^a_{n+1} = \left(\sum_{j=1}^{n} \pi^p_j\right) \times P_{n+1}. \tag{3.80}$$

This means that after this transformation, using Equation 3.66, the participants' compensations are determined by

$$W^a_i = \begin{cases} W^p_i - \pi^p_i \times P_{n+1} & i = 1, 2, \ldots, n \\ \left(\sum_{j=1}^{n} \pi^p_j\right) \times P_{n+1} & i = n+1. \end{cases} \tag{3.81}$$

From Equation 3.81 it is easy to verify that

$$\sum_{j=1}^{n+1} W^a_j = \sum_{j=1}^{n} W^p_j, \tag{3.82}$$

which means that the transformation from the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator into the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator does not change the aggregate compensations.

For $i = 1, 2, \ldots, n$, we define the reduction in the compensation to participant i appearing in Equation 3.81 as

$$W^a_{n+1,i} = \pi^p_i \times P_{n+1}. \tag{3.83}$$

Using Equation 3.81 we directly find that

$$W^a_{n+1} = \sum_{i=1}^{n} W^a_{n+1,i}. \tag{3.84}$$

Since the administrator now receives a compensation, they must invest a positive amount $\pi^a_{n+1} > 0$ in the fund. Similarly, because the participants' compensations are now reduced, their investment in the fund must also reduce. We assume that this reduction of investment into the fund by the participants is invested by the administrator instead. To be precise, each participant i invests an amount $\pi^a_{n+1,i}$ less than their original investment π^p_i. As a result, the administrator invests the amount

$$\pi^a_{n+1} = \sum_{i=1}^{n} \pi^a_{n+1,i}.$$

This means that after transformation, the total time 0 investment of each participant, denoted π^a_i, is given by:

$$\pi^a_i = \begin{cases} \pi^p_i - \pi^a_{n+1,i} & : i = 1, 2, \ldots, n \\ \sum_{j=1}^{n} \pi^a_{n+1,j} = \pi^a_{n+1} & : i = n+1 \end{cases} \tag{3.85}$$

From (3.85) it is easy to verify that

$$\sum_{j=1}^{n} \pi^p_j = \sum_{j=1}^{n+1} \pi^a_j. \tag{3.86}$$

Summarizing, by applying Equations 3.81 and 3.85, we have transformed the original RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator into the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator, with $\pi^a_{n+1} = \sum_{j=1}^{n} \pi^a_{n+1,j}$ and $W^a_{n+1} = \left(\sum_{j=1}^{n} \pi^p_j\right) \times P_{n+1}$. This transformation reallocates a part of both the participants' investments and their compensations to the administrator.

Definition 9 *Consider the actuarially fair RS scheme with a passive administrator $(\boldsymbol{\pi}^p, \boldsymbol{P})$. The transformation from $(\boldsymbol{\pi}^p, \boldsymbol{P})$ into the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator is said to be actuarially fair in case the following conditions hold:*

$$\pi^a_{n+1,i} = \mathbb{E}\left[W^a_{n+1,i}\right], \qquad i = 1, 2, \ldots, n. \tag{3.87}$$

The definition above means that a transformation is actuarially fair in case the investment of the participant π^p_i is reduced by an amount $\pi^a_{n+1,i}$ that equals the expected value of the compensation $W^a_{n+1,i}$ that the participant will no longer receive, and is received by the administrator instead. Taking into account Equation 3.83 and the fact that $\mathbb{E}[P_{n+1}] = \Pr[P_{n+1} = 1]$, the fairness conditions (3.87) can be expressed as follows:

$$\pi^a_{n+1,i} = \pi^p_i \times \Pr[P_{n+1} = 1], \qquad i = 1, 2, \ldots, n. \tag{3.88}$$

Note that Equations 3.88 and 3.76 are identical.

Proposition 9 *Consider the RS scheme* $(\boldsymbol{\pi}^p, \boldsymbol{P})$ *with a passive administrator, which is actuarially fair for all participants. Let* $(\boldsymbol{\pi}^a, \boldsymbol{P})$ *be the actuarially fair transformation of* $(\boldsymbol{\pi}^p, \boldsymbol{P})$*. In this case, we have that*

$$\pi^a_{n+1,i} = \pi^a_i \times \frac{\Pr[P_{n+1} = 1]}{\Pr[P_{n+1} = 0]}, \qquad i = 1, 2, \ldots, n. \tag{3.89}$$

and

$$\pi^a_i = \pi^p_i \times \Pr[P_{n+1} = 0], \qquad i = 1, 2, \ldots, n. \tag{3.90}$$

Proof: These expressions follow immediately from (3.85) and (3.88). ∎

Once again, note that Equations 3.89 and 3.77 are identical, as are Equations 3.90 and 3.78.

Proposition 10 *Consider the RS scheme with a passive administrator* $(\boldsymbol{\pi}^p, \boldsymbol{P})$*, which is actuarially fair for its participants, and let the RS scheme with an active administrator* $(\boldsymbol{\pi}^a, \boldsymbol{P})$ *be its actuarially fair transformation. Then we have that* $(\boldsymbol{\pi}^a, \boldsymbol{P})$ *is also actuarially fair for all its participants.*

Proof: Since $(\boldsymbol{\pi}^p, \boldsymbol{P})$ is fair, we have that $\mathbb{E}[\pi^p_i] = \mathbb{E}[W^p_i]$. It remains to prove that $(\boldsymbol{\pi}^a, \boldsymbol{P})$ is also actuarially fair for all participants. For $i = 1, \ldots, n$, taking into account Equations 3.85, 3.83 and 3.87 we get

$$\mathbb{E}[\pi^a_i] = \mathbb{E}[W^p_i - \pi^p_i \times P_{n+1}] \tag{3.91}$$

and for $i = n + 1$, taking into account Equations 3.85, 3.83 and 3.87 we get that

$$\mathbb{E}\left[\pi^a_{n+1}\right] = \mathbb{E}\left[\sum_{j=1}^{n} \pi^p_j \times P_{n+1}\right] \tag{3.92}$$

The conditions in Equations 3.91 and 3.92 together are equivalent to the actuarial fairness for all the participants of the RS scheme with an active administrator $(\boldsymbol{\pi}^a, \boldsymbol{P})$. ∎

The above proposition states that if we transform the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator to the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator as explained in Equations 3.81 and 3.85 and if the transformation is actuarially fair in the sense of Equation 3.88, then the actuarial fairness of the RS scheme $(\boldsymbol{\pi}^p, \boldsymbol{P})$ with a passive administrator implies the actuarial fairness of the RS scheme $(\boldsymbol{\pi}^a, \boldsymbol{P})$ with an active administrator.

3.7 The two-participant tontine fund

In this section, we investigate an RS scheme related to a tontine fund with only two participants. We consider both cases of an active and a passive administrator. We investigate actuarial fairness conditions for the heterogeneous case where survival probabilities of the two participants may be different.

3.7.1 The two-participant tontine fund with an active administrator

Consider two participants who set up a tontine fund. The initial investment made by participant i, $i = 1, 2$, is denoted by π_i, while the survival probability is given by $p_i = 1 - q_i$. In addition to the two participants, a third agent, called the active administrator, is also involved. The administrator's contribution to the investment pool is denoted by π_3. As before, for simplicity, we assume a zero return over the observation period.

Suppose that the participants agree on an RS rule with an active administrator, with relative compensation vector as described in Example 3.3.2, that is

$$P_i = \frac{f_i \times I_i}{\sum_{j=1}^{3} f_j \times I_j}, \qquad i = 1, 2, 3, \tag{3.93}$$

and compensations determined from

$$W_i = (\pi_1 + \pi_2 + \pi_3) \times P_i, \qquad i = 1, 2, 3. \tag{3.94}$$

The protection units f_i and the indicator variables I_i are as defined in Example 3.3.2. In particular, we have that I_i is equal to 1 in case participant $i = 1, 2$ survives, while it is equal to 0 otherwise. Furthermore, I_3 is equal to $(1 - I_1) \times (1 - I_2)$.

The relative compensation vector $\boldsymbol{P} = (P_1, P_2, P_3)$ of the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ defined by Equation 3.93 is assumed to be independent of the initial investments in the sense that the f_i's are given real numbers, independent of π_i. This vector can be expressed as follows:

$$\boldsymbol{P} = \begin{cases} (1, 0, 0) & : \text{if } I_1 = 1 \text{ and } I_2 = 0 \\ (0, 1, 0) & : \text{if } I_1 = 0 \text{ and } I_2 = 1 \\ (0, 0, 1) & : \text{if } I_1 = 0 \text{ and } I_2 = 0 \\ (\beta, 1 - \beta, 0) & : \text{if } I_1 = 1 \text{ and } I_2 = 1 \end{cases} \tag{3.95}$$

with β given by

$$\beta = \frac{f_1}{f_1 + f_2}. \tag{3.96}$$

If participant 1 survives and participant 2 dies, the total amount of $(\pi_1 + \pi_2 + \pi_3)$ is awarded to person 1 at time 1. Similarly, if participant 1 dies while participant 2 survives, the total amount is awarded to person 2 at time 1. If no participant survives, the total proceeds $(\pi_1 + \pi_2 + \pi_3)$ belong to the administrator. If both participants survive, $(\pi_1 + \pi_2 + \pi_3)$ is shared by the two participants: The first participant takes a relative share β of the available funds, while the second participant receives a relative share $(1 - \beta)$ of these funds.

When setting up the tontine RS scheme, the two participants and the administrator must agree on the value of the relative share β of the total fund that participant 1 will receive if both participants survive, as well as on the investments π_1, π_2 and π_3. Given the relative share β, the choice of the initial investments by the participants may reflect considerations about their survival probabilities. Consider, for example, uniform risk-sharing, that is $\beta = \frac{1}{2}$ or, equivalently, $f_1 = f_2$, then equal investments may be considered as 'unfair' as this choice does not take into consideration that health (survival probabilities) may be different for both participants. However, notice that once the relative share β and the investments are chosen, knowledge of the survival probabilities is no longer required to be able to further manage the tontine fund.

We assume that the remaining lifetimes of the two participants are mutually independent. The probabilities of the different relevant events related to the tontine fund payouts can then be expressed as

$$\Pr[I_1 = i \text{ and } I_2 = j] = \begin{cases} p_1 \times q_2 & : \text{if } i = 1 \text{ and } j = 0 \\ q_1 \times p_2 & : \text{if } i = 0 \text{ and } j = 1 \\ q_1 \times q_2 & : \text{if } i = 0 \text{ and } j = 0 \\ p_1 \times p_2 & : \text{if } i = 1 \text{ and } j = 1. \end{cases} \tag{3.97}$$

In this subsection, we want to find out what a reasonable choice for the investments π_1, π_2 and π_3 is, once the relative share β is chosen. Note that in case 'reasonability' is translated into 'actuarial

fairness', the participants have to agree on a common choice for the survival probabilities p_1 and p_2 to determine the compensations (also called the tontine fund payouts) W_i.

Taking into account Equations 3.95 and 3.97, we find that the expected relative compensations $\mathbb{E}[P_i]$ of the participants are given by

$$\mathbb{E}[P_1] = p_1 \times (q_2 + \beta \times p_2) \tag{3.98}$$

and

$$\mathbb{E}[P_2] = p_2 \times (q_1 + (1-\beta) \times p_1), \tag{3.99}$$

while the administrator's expected relative compensation $\mathbb{E}[P_3]$ is given by

$$\mathbb{E}[P_3] = q_1 \times q_2.$$

Let us now determine the actuarially fair investments π_1, π_2 and π_3 for the three participants. From Equation 3.39, we find that the actuarial fairness conditions, which state that each agent's investment is equal to their expected compensation, are given by

$$\begin{cases} \pi_1 &= (\pi_1 + \pi_2 + \pi_3) \times p_1 \times (q_2 + \beta \times p_2) \\ \pi_2 &= (\pi_1 + \pi_2 + \pi_3) \times p_2 \times (q_1 + (1-\beta) \times p_1) \\ \pi_3 &= (\pi_1 + \pi_2 + \pi_3) \times q_1 \times q_2. \end{cases} \tag{3.100}$$

Any solution (π_1, π_2, π_3) of Equation 3.100 is a set of actuarially fair initial investments. Determining such a set of investments requires a common choice for the survival probabilities of the two participants. It is important to note that even if a 'wrong' choice (for instance a too conservative choice) is made, there will not be an issue of insolvency, due to the full allocation condition in Equation 3.19, which guarantees that the sum of all tontine fund payouts (compensations) is exactly equal to the available funds.

Taking into account Equation 3.40 and the fact that $\Pr[P_3 = 0] = 1 - q_1 \times q_2$, we can rewrite the actuarial fairness conditions in Equation 3.100 as follows:

$$\begin{cases} \pi_1 &= (\pi_1 + \pi_2) \times p_1 \times \frac{q_2 + \beta \times p_2}{1 - q_1 \times q_2} \\ \pi_2 &= (\pi_1 + \pi_2) \times p_2 \times \frac{q_1 + (1-\beta) \times p_1}{1 - q_1 \times q_2} \\ \pi_3 &= (\pi_1 + \pi_2) \times \frac{q_1 \times q_2}{1 - q_1 \times q_2} \end{cases} \tag{3.101}$$

From Equation 3.41, we find that the actuarial fairness conditions can also be expressed as follows:

$$\begin{cases} \pi_1 = \pi_3 \times p_1 \times \frac{q_2 + \beta \times p_2}{q_1 \times q_2} \\ \pi_2 = \pi_3 \times p_2 \times \frac{q_1 + (1-\beta) \times p_1}{q_1 \times q_2} \end{cases} \tag{3.102}$$

Equations 3.100, 3.101 and 3.102 provide three different ways to determine actuarially fair initial investments, once there is an agreement on the relative share β and the survival probabilities p_i. One can first choose the aggregate investments $(\pi_1 + \pi_2 + \pi_3)$ of all participants, or the total investments $(\pi_1 + \pi_2)$ of all participants, or the investment π_3 of the administrator, and then use either Equation 3.100, 3.101 or 3.102 to derive the actuarially fair investments for the three participants.

Example 3.7.1 ***(A game of chance with a coin and a die)*** *Consider the combined game of chance with two participants, described in the online appendix of Dhaene and Milevsky (2024). To enter the game, participant 1 pays an amount π_1, while participant 2 pays π_2. Participant 1 tosses a two-sided coin, while participant 2 rolls a six-sided die. In this game, participant 1 is successful if heads*

is the outcome, while participant 2 is successful if a 1 is rolled. In addition to the two participants, an active administrator is involved. The active administrator contributes to the prize pool by paying an amount π_3.
The payouts for this game of chance are defined as follows: If the coin lands on heads and the die does not land on 1, the total amount of $(\pi_1 + \pi_2 + \pi_3)$ *is awarded to participant 1. Similarly, if the coin does not land on heads but the die lands on 1, the total amount of* $(\pi_1 + \pi_2 + \pi_3)$ *is awarded to participant 2. If both participants are successful (i.e., heads and 1 appear after the respective throws), the total proceeds of* $(\pi_1 + \pi_2 + \pi_3)$ *are shared by person 1 and person 2. In this case, participant 1 receives a relative share* β, *while participant 2 receives a relative share* $(1 - \beta)$ *of the available fund. Finally, if both participants are not successful (i.e., neither heads nor 1 appear after their respective throws), the total proceeds of* $(\pi_1 + \pi_2 + \pi_3)$ *go to the administrator. Let us assume that the outcomes of the coin and the die are independent. The probability that the administrator will receive the entire prize pool is then given by* $\frac{5}{12}$, *so it seems reasonable to require the administrator to contribute to the prize pool in exchange for this chance of winning.*

This game of chance can be described within the framework of the two-participant tontine fund considered above in this subsection:

$$W_i = (\pi_1 + \pi_2 + \pi_3) \times P_i \qquad i = 1, 2, 3,$$

with the relative compensations P_i *defined by Equation 3.95.*
From Equation 3.100, we find that the game of chance is actuarially fair for the three participants if and only if the amounts π_1, π_2 *and* π_3 *satisfy the following set of equations:*

$$\begin{cases} \pi_1 &= (\pi_1 + \pi_2 + \pi_3) \times \frac{5+\beta}{12} \\ \pi_2 &= (\pi_1 + \pi_2 + \pi_3) \times \frac{2-\beta}{12} \\ \pi_3 &= (\pi_1 + \pi_2 + \pi_3) \times \frac{5}{12} \end{cases} \tag{3.103}$$

In this case, the actuarially fair initial investments π_1, π_2 *and* π_3 *are expressed as proportions of the total payment* $(\pi_1 + \pi_2 + \pi_3)$ *of the three participants. One can choose the magnitude of* $(\pi_1 + \pi_2 + \pi_3)$ *first, and then determine the corresponding actuarially fair initial payments of the participants and the administrator by Equation 3.103.*
From Equation 3.101, we find that the actuarial fairness conditions can also be expressed in the following way:

$$\begin{cases} \pi_1 &= (\pi_1 + \pi_2) \times \frac{5+\beta}{7} \\ \pi_2 &= (\pi_1 + \pi_2) \times \frac{2-\beta}{7} \\ \pi_3 &= (\pi_1 + \pi_2) \times \frac{5}{7} \end{cases} \tag{3.104}$$

In this case, the actuarially fair investments π_1, π_2 *and* π_3 *are expressed as proportions of the total payment* $(\pi_1 + \pi_2)$ *of the two participants. One can choose the magnitude of* $(\pi_1 + \pi_2)$ *first, and then determine the corresponding initial payments of the participants and the administrator by Equation 3.104. Alternatively, one can first choose* π_1 *and* π_2, *and then determine* π_3 *and set* β *equal to* $\frac{2\pi_1 - 5\pi_2}{\pi_1 + \pi_2}$, *such that the game is actuarially fair for all participants. However, notice that in this case, not every choice of* π_1 *and* π_2 *will lead to a relative share* β *between* 0 *and* 1.
Finally, from Equation 3.102 one finds that the actuarially fair initial investments also follow from

$$\begin{cases} \pi_1 = \pi_3 \times \frac{5+\beta}{5} \\ \pi_2 = \pi_3 \times \frac{2-\beta}{5} \end{cases} \tag{3.105}$$

Here, π_1 *and* π_2 *are expressed as proportions of the investment* π_3 *of the administrator. One can choose the magnitude of* π_3 *first, and then determine the investments of the two participants by Equation 3.105.*
◁

3.7.2 The two-participant tontine fund with a passive administrator

Let us now replace the tontine RS scheme with an active administrator defined in the previous subsection by the tontine RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator, as explained before. The investment vector and the compensation vector are now given by $\boldsymbol{\pi} = (\pi_1, \pi_2, 0)$ and $\boldsymbol{W} = (W_1, W_2, 0)$. From Equation 3.58, it follows that the compensations W_i of the participants are given by

$$W_i = (\pi_1 + \pi_2) \times P_i + \pi_i \times P_3, \qquad i = 1, 2, \tag{3.106}$$

with the relative compensation vector $\boldsymbol{P} = (P_1, P_2, P_3)$ given by Equation 3.95.

From Equation 3.63, we find that the actuarial fairness conditions for the two participants are given by

$$\begin{cases} \pi_1 &= (\pi_1 + \pi_2) \times \frac{p_1 \times (q_2 + \beta \times p_2)}{1 - q_1 \times q_2} \\ \pi_2 &= (\pi_1 + \pi_2) \times \frac{p_2 \times (q_1 + (1-\beta) \times p_1)}{1 - q_1 \times q_2} \end{cases} \tag{3.107}$$

Expressions in Equation 3.107 provide a way to determine actuarially fair initial investments for the two-participant tontine fund with a passive administrator. Indeed, once there is an agreement on the relative share β and the survival probabilities p_i, one can first choose the aggregate initial payments $(\pi_1 + \pi_2)$ of the two participants, and then use Equation 3.107 to derive actuarially fair investments π_i. Another possibility consists of first choosing π_1, and then determining π_2 from Equation 3.107:

$$\pi_2 = \pi_1 \times \frac{p_2 \times (q_1 + (1 - \beta) \times p_1)}{p_1 \times (q_2 + \beta \times p_2)}. \tag{3.108}$$

Example 3.7.2 ***(A game of chance with a coin and a die)*** *Let us revisit the 'coin and die' game of chance with active administrator considered in Example 3.7.1. We replace this game of chance with one with a passive administrator and denote it by $(\boldsymbol{\pi}, \boldsymbol{P})$. From Definition 6, we have that the payouts (compensations) of the transformed RS scheme are given by Equation 3.58 with the relative compensation vector $\mathbf{P}$ defined by Equation 3.95.*
From Equation 3.107, we find that this game of chance is actuarially fair for all its participants if and only if the initial investments π_1 and π_2 follow from the following set of equations:

$$\begin{cases} \pi_1 &= (\pi_1 + \pi_2) \times \frac{5+\beta}{7} \\ \pi_2 &= (\pi_1 + \pi_2) \times \frac{2-\beta}{7}. \end{cases} \tag{3.109}$$

This set of equations expresses the actuarially fair payments π_1 and π_2 as a proportion of the total investment $(\pi_1 + \pi_2)$ of the two participants. One can choose the magnitude of $(\pi_1 + \pi_2)$ first and then determine the investment efforts of the participants by Equation 3.109. Alternatively, one can also first choose π_1 and π_2, and then set β equal to $\frac{2\pi_1 - 5\pi_2}{\pi_1 + \pi_2}$, so that the game is actuarially fair. However, notice that in this case, not every choice of π_1 and π_2 will lead to a relative share β between 0 and 1. ◁

3.8 The homogeneous tontine fund

3.8.1 The homogeneous tontine fund with an active administrator

In this subsection, we investigate a special case of the tontine RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with an active administrator, described in Example 3.3.2. We consider n participants and an active administrator. We assume that all protection units f_i are equal to 1. The $n + 1$ agents each invest an initial amount π_i. The components of the relative compensation vector are given by

$$P_i = \frac{I_i}{\sum_{j=1}^{n+1} I_j}, \qquad i = 1, 2, \ldots, n+1. \tag{3.110}$$

The compensations W_i of the $n+1$ agents follow then from Equation 3.18. We consider a homogeneous tontine RS scheme, which means that we assume that the indicator variables I_i of the n participants are i.i.d., with $\Pr[I_i = 1] = p = 1 - q$.

Due to symmetry reasons, we must have that $\mathbb{E}[P_i]$ is equal for all n participants. Furthermore, we have that

$$\mathbb{E}[P_{n+1}] = \Pr[I_{n+1} = 1] = q^n. \tag{3.111}$$

Taking into account that $\sum_{i=1}^{n+1} \mathbb{E}[P_i] = 1$ leads to

$$\mathbb{E}[P_i] = \mathbb{E}\left[\frac{I_i}{\sum_{j=1}^{n+1} I_j}\right] = \frac{1-q^n}{n}, \qquad i = 1, 2, \ldots, n. \tag{3.112}$$

According to Equation 3.40, the homogeneous tontine RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is actuarially fair for all agents if and only if

$$\pi_i = \frac{1}{n}\sum_{j=1}^{n} \pi_j, \qquad i = 1, 2, \ldots, n \tag{3.113}$$

and

$$\pi_{n+1} = \frac{q^n}{1-q^n}\sum_{j=1}^{n} \pi_j. \tag{3.114}$$

This implies that all initial investments π_i of the participants are equal, which we denote hereafter by π. This result was to be expected because of the inherent symmetry of the problem. Actuarially fair initial investments can then be determined by first choosing the administrator's initial investment π_{n+1} and then determining the participants' initial investments by Equations 3.113 and 3.114.

3.8.2 The homogeneous tontine fund with a passive administrator

Let us now replace the homogeneous tontine RS scheme with an active administrator, considered in Section 7.1, by the corresponding homogeneous tontine RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator. The compensations W_i are then given by Equation 3.58, with the relative compensations P_i as defined in Equation 3.110.
From Equations 3.63, 3.111 and 3.112, we find that $(\boldsymbol{\pi}, \boldsymbol{P})$ is actuarially fair for its n participants if and only if all initial investments π_i of the participants are equal:

$$\pi_i = \frac{1}{n}\sum_{j=1}^{n} \pi_j, \qquad i = 1, 2, \ldots, n.$$

Again, this result was to be expected because of the inherent symmetry of the problem.

3.8.3 Comparing the homogeneous tontine fund with an active administrator and a centralized insurance approach

Consider a group of n persons (participants). The one-year survival of person i is described by the Bernoulli random variable I_i, which equals 1 if they survive until time 1, while it equals 0 otherwise. The group is assumed to be homogeneous in the sense that all Bernoulli random variables are i.i.d. with a common survival probability p. Suppose that each person is willing to invest an initial amount π which will entitle them to a survival benefit at time 1.

The participants could opt for *centralized insurance*, by each buying a pure endowment with a premium π. Suppose that the premiums are pure premiums. In case person i chooses an insurance with pure premium π, then, assuming a zero-discount rate, their payout W_i^c at time 1 is given by

$$W_i^c = \frac{\pi}{p} \times I_i, \tag{3.115}$$

where we use the superscript 'c' to indicate that this is the payout in a 'centralized' approach. The expected payoff in case of buying insurance is given by

$$\mathbb{E}\left[W_i^c\right] = \pi.$$

On average, the premium is equal to the expected insurance payment. This means that the insurance approach is actuarially fair for each person.

Instead of buying a pure endowment insurance, the participants could decide to take part in a homogeneous *RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ with an active administrator. Any participant makes an initial investment π in the tontine fund. The active administrator's investment is, as usual, denoted by π_{n+1}. The initial investment vector is given by $\boldsymbol{\pi} = (\pi, \pi, \ldots, \pi, \pi_{n+1})$. Suppose that the relative compensation vector $\boldsymbol{P}$ of the homogeneous RS scheme is given by Equation 3.110. The compensation $W_i(n)$ that participant i will receive at time 1 in a fund with n participants in a decentralized approach with an active administrator is then given by

$$W_i(n) = (n \times \pi + \pi_{n+1}) \times \frac{I_i}{\sum_{j=1}^{n+1} I_j}, \qquad i = 1, 2, \ldots, n+1, \tag{3.116}$$

where we added 'n' in the notation $W_i(n)$ to indicate the number of participants in the RS scheme.

Let us now assume that the RS scheme $(\boldsymbol{\pi}, \boldsymbol{P})$ is actuarially fair for all participants and for the administrator. Then from Equations 3.113 and 3.114 we find that the compensations $W_i(n)$ can be written as follows:

$$W_i(n) = \frac{n \times \pi}{1 - q^n} \times \frac{I_i}{\sum_{j=1}^{n+1} I_j}, \qquad i = 1, 2, \ldots, n+1. \tag{3.117}$$

From Equation 3.117, one can prove that for any participant we have

$$W_i(n) = \frac{\pi}{1 - q^n} \times \frac{I_i}{\frac{1}{n}\sum_{j=1}^{n} I_j + \frac{1}{n}\prod_{j=1}^{n}(1 - I_j)} \xrightarrow{a.s.} \frac{\pi}{p} \times I_i.$$

Intuitively, this means that when the number of participants in the homogeneous tontine fund with an active administrator becomes infinitely large, then with probability 1, the contribution of each participant i becomes equal to the corresponding insurance payment W_i^c.

Similarly, for the contribution of the administrator we have

$$W_{n+1}(n) = \frac{n \times \pi}{1 - q^n} \times \prod_{j=1}^{n}(1 - I_j) \xrightarrow{a.s.} 0.$$

This means that when the number of participants in the homogeneous tontine fund with an active administrator becomes infinitely large, then with probability 1, the contribution of the administrator becomes equal to 0. We can conclude that the homogeneous tontine fund approach with an active administrator converges to the centralized insurance approach when the number of participants goes to infinity.

3.8.4 Comparing the homogeneous tontine fund with a passive administrator and a centralized insurance approach

Consider again the group of n persons as described in Section 7.3. Suppose that the participants decide to take part in a homogeneous *RS scheme* $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator. Assuming again actuarially fair (and hence equal) contributions, any participant makes an initial investment, denoted by π, in the tontine fund. The initial investment vector is then given by $\boldsymbol{\pi} = (\pi, \pi, \ldots, \pi, 0)$. The compensation that participant i will receive at time 1 in a fund with n participants in a decentralized approach with a passive administrator then follows from

$$W_i(n) = n \times \pi \times \frac{I_i}{\sum_{j=1}^{n+1} I_j} + \pi \times I_{n+1}, \quad i = 1, 2, \ldots, n. \tag{3.118}$$

From Equation 3.118, for any participant i one has that

$$W_i(n) = \pi \times \frac{I_i}{\frac{1}{n}\sum_{j=1}^{n} I_j + \frac{1}{n}\prod_{j=1}^{n}(1 - I_j)} + \pi \times \prod_{j=1}^{n} (1 - I_j) \xrightarrow{a.s.} \frac{\pi}{p} \times I_i,$$

which means that when the number of participants in the homogeneous tontine fund with a passive administrator becomes infinitely large, then with probability 1, the contribution of participant i becomes equal to the insurance payment W_i^c. We can conclude that the tontine fund with a passive administrator also converges to the centralized insurance approach.

3.9 Concluding remarks

In this chapter, we considered compensation-based RS schemes $(\boldsymbol{\pi}, \boldsymbol{P})$ with an active administrator. At time 0, each participant i invests an initial amount π_i in the fund, while the administrator invests the amount π_{n+1} in the same fund. These initial investments are summarized in the investment vector $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{n+1})$. At time 1, each participant and also the administrator receives compensation from the fund, summarized in the compensation vector $\boldsymbol{W} = (W_1, W_2, \ldots, W_{n+1})$, which is given by

$$W_i = \left(\sum_{j=1}^{n+1} \pi_j \right) \times P_i \qquad i = 1, 2, \ldots, n+1,$$

where $\boldsymbol{P} = (P_1, P_2, \ldots, P_{n+1})$ is the relative compensation vector of the RS scheme under consideration. This setup is a generalization of the setup in Denuit and Robert (2025) and Dhaene and Milevsky (2024), who consider RS schemes with relative compensation vector given by Equation 3.26 as described in Example 3.3.2 on tontine funds.

Apart from the case of an active administrator, we also considered RS schemes $(\boldsymbol{\pi}, \boldsymbol{P})$ with a passive administrator. In this case, the investment vector is given by $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n, 0)$, while the compensation vector is given by $\boldsymbol{W} = (W_1, W_2, \ldots, W_n, 0)$, with

$$W_i = \left(\sum_{j=1}^{n} \pi_j \right) \times P_i + \pi_i \times P_{n+1} \qquad i = 1, 2, \ldots, n \tag{3.119}$$

where $\boldsymbol{P} = (P_1, P_2, \ldots, P_{n+1})$ is again the relative compensation vector of the RS scheme under consideration. This setup is a generalization of the setup in Denuit and Robert (2025), who consider RS schemes with relative compensation vector given by Equation 3.26 as described in Example 3.3.2 on tontine funds.

For both types of compensation-based RS schemes, we derived actuarial fairness conditions, that is, conditions under which the initial investment of each agent is equal to the expected compensation received at time 1.

We considered two particular cases of tontine funds in some detail. First, the two-participant tontine fund was investigated. In this case, two participants decide to set up a tontine fund and share the terminal fund value among the survivors. We also considered the homogeneous tontine fund, where all participants have i.i.d. survival indicator variables and each hold equal numbers of protection units. Finally, we also considered the relation between the homogeneous tontine fund approach with passive and active administrators and the centralized insurance approach.

Chapter 4

Information-constrained fair dynamic valuation of hybrid insurance liabilities

4.1 Introduction

Modern solvency frameworks, such as Solvency II and the Swiss Solvency Test, mandate that insurance liabilities be valued in a manner appropriate for their specific characteristics. Liabilities arising from claims that can be hedged using tradable assets (hedgeable claims) must be valued at the current market price of the replicating portfolio, ensuring market-consistency. Liabilities arising from claims that are independent of tradable assets (orthogonal claims) must be valued at a best estimate plus a risk margin, ensuring model-consistency. Most insurance liabilities, however, are neither fully hedgeable nor independent of the tradable assets. The valuation of such hybrid claims must integrate both approaches: market-consistency for the hedgeable component and model-consistency for the orthogonal component. Moreover, since insurers must report their liability valuations to stakeholders at regular intervals, the valuations of hybrid claims must remain economically consistent across multiple reporting dates, ensuring time-consistency (Salahnejhad Ghalehjooghi & Pelsser, 2021). The question addressed in this chapter is how to construct valuations that satisfy all three requirements simultaneously in a discrete multi-period setting, where financial and actuarial risks are dependent and the insurer observes more information than the market.

The framework of Barigou et al. (2019) comes closest to answering this question. The authors introduce fair dynamic valuations: valuations that are simultaneously market consistent, model consistent and time consistent. This framework, however, rests on two assumptions that often fail in practice. First, Barigou et al. (2019) do not distinguish between diversifiable policyholder-specific risk and non-diversifiable systematic risk. Systematic risks, such as catastrophe risk, simultaneously affect the mortality of multiple policyholders and the prices of tradable assets, thereby inducing dependence between policyholder-specific risks and the financial market. The two types of risks are inherently different and call for different actuarial principles for their valuation: policyholder-specific risk can be reduced by pooling policies, whereas systematic risk affects all policyholders simultaneously and persists regardless of portfolio size. A canonical example is a unit-linked life insurance contract, whose payoff depends simultaneously on the performance of a traded investment fund, a systematic longevity index affecting the survival of the entire portfolio, and the survival of the individual policyholder. Valuing such a contract correctly requires treating these three risk types differently. Second, the framework of Barigou et al. (2019) is constructed within a single information filtration, which implicitly permits

trading strategies in risky assets that depend on policyholder-specific information. Since this information is not publicly available, such strategies constitute insider trading and, as shown by Artzner et al. (2024), can give rise to an insurance–finance arbitrage. The three-step valuations of Linders (2023) resolve the first issue by separating the valuation of diversifiable policyholder-specific risk from that of non-diversifiable systematic risk, but do so only in a single period setting.

This chapter closes these gaps. We develop a discrete-time, multi-period financial–actuarial framework built on three nested information filtrations, each corresponding to a distinct type of risk: the asset-price filtration $\mathcal{F}^{\boldsymbol{Y}}$, generated by the prices of traded assets and used to characterize the no-arbitrage prices; the financial filtration $\mathcal{F}$, which additionally incorporates non-traded systematic risks and constitutes the largest filtration the insurer may use to construct trading strategies in risky assets; and the insurer-relevant filtration $\mathcal{G}$, which further incorporates policyholder-specific risks and is reserved for the actuarial valuation of claims independent of the financial filtration. Systematic risks such as catastrophe or longevity indices are treated as publicly available since they are observable by all market participants. Our contributions are fourfold. First, we define dynamic valuations and dynamic hedgers in this filtration-sensitive setting, where compliance with the no-insider trading requirement of Artzner et al. (2024) is enforced by restricting the holdings in the risky assets to depend only on publicly available information in $\mathcal{F}$, while the holdings in the risk-free asset may depend on all information in $\mathcal{G}$. Second, we introduce conditional valuations in this filtration-sensitive setting, which extend the frameworks in Dhaene et al. (2017) and Pelsser and Stadje (2014). Third, we propose an extension of the three-step valuations of Linders (2023) to a multi-period framework via a backward recursion and prove our main result: in an incomplete market, the class of three-step hedge-based valuations coincides with the class of strong-fair dynamic valuations. Fourth, we propose a discrete multi-period adaptation of Pelsser and Stadje (2014) and establish that this class also coincides with the class of strong-fair dynamic valuations. Together these results unify different valuation methodologies and establish their equivalence in a multi-period incomplete-market setting. We present a simple numerical example that compares the proposed valuations with the hedge based valuations of Barigou et al. (2019) and the two-step regression valuations of Barigou et al. (2022), and illustrate the impact of accounting for the dependence between asset-price and policyholder-specific risks.

This chapter builds on the growing literature of fair valuation in insurance. Market-consistent valuation was formalized by Cont (2006) and Malamud et al. (2008) was among the first to apply market-consistent valuation in an insurance context. Model-consistency extends the mark-to-model principle embedded in Solvency II (European Parliament and Council of the European Union, 2009) and was adapted to a financial–actuarial setting by Barigou et al. (2023) and Dhaene et al. (2017). Dhaene et al. (2017) introduced the class of fair valuations in a single-period setting and showed that this class coincides with hedge-based valuations and a modified version of the two-step market evaluations of Pelsser and Stadje (2014) (see also the course notes of Dhaene (2022)). The multi-period extension of fair valuations proposed by Barigou and Dhaene (2019) is not time consistent; Barigou et al. (2019) introduce time consistency through backward recursion by iterating single-period fair valuation backward from maturity (see also Jobert and Rogers (2008)). Related developments include continuous-time fair valuation frameworks (Delong et al., 2019a, 2019b), alternative loss functions (Z. Chen et al., 2020), efficient continuous-time hedging approaches (Motte & Hainaut, 2025), and two-step regression valuations based on sequential expected-value and Value-at-Risk matching (Barigou et al., 2022). The three-step valuation of Linders (2023) generalizes the additive construction of Deelstra et al. (2020) to allow for dependence between financial and actuarial risks. The present chapter embeds the single-period insights of Linders (2023) into the dynamic setting of Barigou et al. (2019), while departing from the latter through the nested-filtration structure and the no-insider-trading restriction on trading strategies. We focus on a discrete-time multi-period framework, which not only reflects the valuation practices of insurers in the industry (since regulatory reporting occurs at discrete intervals) but also avoids the distributional assumptions required by the continuous-time frameworks.

This chapter is organized as follows. Section 2 introduces the combined financial–actuarial framework. In Section 3, we define fair valuations and fair hedgers at a given time point. We distinguish

between the information sets used to construct the hedging strategies and characterize the relationship between valuations and hedgers. Section 4 extends these concepts to a dynamic multi-period setting, incorporating time-consistency, and characterizing the relationship between dynamic valuations and dynamic hedgers. In Section 5, we introduce conditional valuations in an information-sensitive multi-period framework. Section 6 proposes three-step hedge-based valuations and shows that the class of fair valuations coincides with the class of three-step hedge-based valuations. Section 7 presents a discrete multi-period adaptation of the two-step market evaluations of Pelsser and Stadje (2014) and establishes that this class of valuations is also equivalent to the class of fair valuations. Finally, Section 8 compares the valuation methods introduced in Sections 6 and 7 with the hedge-based valuations of Barigou et al. (2019) and the two-step regression valuations of Barigou et al. (2022). The comparison is illustrated using a simple numerical example.

4.2 The combined financial-actuarial world

In this section, we introduce the setting, notation, assumptions, types of risks, and the general ideas used in subsequent sections.

4.2.1 Setting

We consider a combined financial–actuarial setting in which both tradable and non-tradable claims coexist. The analysis is conducted in discrete time, starting at time-0 and extending over a finite horizon $T \in \{1, 2, \ldots\}$, with trading dates indexed by $\tau = \{0, 1, \ldots, T\}$. This setting is modeled on a probability space $(\Omega, \mathcal{H}, \mathbb{P})$. All random variables and stochastic processes in this chapter are defined on $(\Omega, \mathcal{H}, \mathbb{P})$. The σ-algebra $\mathcal{H}$ is never used as an information set in any valuation or hedging argument. Its sole role is to provide the common measurable space on which the underlying random variables are defined, so that their natural filtrations can be used to construct nested filtrations. The probability measure $\mathbb{P}$ assigns physical probabilities to events in the combined world. All equalities are understood in the $\mathbb{P}$-almost sure sense. Moreover, throughout this chapter, we assume that all random variables have finite second moments under $\mathbb{P}$.

Within this combined framework, we consider a financial market consisting of $n^a + 1$ tradable, non-dividend-paying[1] assets. These assets may include stocks, bonds, mutual funds, options, and similar instruments. We denote by $Y^{(m)}(t)$ the time-t market price of asset $m \in \{0, 1, \ldots, n^a\}$. Trading is assumed to take place in deep, liquid, and frictionless markets, allowing assets to be bought and sold in arbitrary quantities without transaction costs. Asset 0 is taken to be the risk-free bank account. An investment of one unit in this account grows deterministically according to

$$Y^{(0)}(t) = e^{rt},$$

where $r \geq 0$ denotes the constant risk-free interest rate.

Let $\boldsymbol{Y}(t)$, $t \in \tau$, denote the vector of time-t prices of the tradable assets, given by

$$\boldsymbol{Y}(t) = \big(Y^{(0)}(t), Y^{(1)}(t), \ldots, Y^{(n^a)}(t)\big).$$

The evolution of traded asset prices over time is described by the $(n^a + 1)$-dimensional stochastic process $\boldsymbol{Y} = \{\boldsymbol{Y}(t)\}_{t\in\tau}$.

We define a filtration $\mathbb{F}^{\boldsymbol{Y}} = \{\mathcal{F}_t^{\boldsymbol{Y}}\}_{t\in\tau}$, where $\mathcal{F}_t^{\boldsymbol{Y}} = \sigma(\boldsymbol{Y}(0), \boldsymbol{Y}(1), \ldots, \boldsymbol{Y}(t))$. The σ-algebra $\mathcal{F}_t^{\boldsymbol{Y}}$, $t \in \tau$, represents the asset price information available up to and including time-t. By definition, we

[1] Without loss of generality, we assume that there are no dividends. One may equivalently replace the traded asset by its gain process, defined as the sum of its price process and the accumulated dividend process.

write $\mathbb{F} \subseteq \mathbb{G}$ to mean $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all t. Although it may be the case that $\mathbb{F}^{\boldsymbol{Y}} = \mathbb{G}$, in this chapter, we consider a general setting where $\mathbb{G}$ is richer than $\mathbb{F}^{\boldsymbol{Y}}$, containing additional information on non-traded systematic risks and policyholder-specific risks.

The combined financial-actuarial world $(\Omega, \mathcal{H}, \mathbb{P})$ is home to n^s non-traded systematic risks. A non-traded systematic risk can, for example, arise from exposure to a non-traded longevity index that impacts the survival of a life insurer's policyholders. We denote the random variables representing the state of the non-traded systematic risk $m \in \{1, \dots, n^s\}$ at time-$t \in \tau$ by $Z^{(m)}(t)$.

We define a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\tau}$, where $\mathcal{F}_t = \sigma(\boldsymbol{Y}(0), \dots, \boldsymbol{Y}(t), \boldsymbol{Z}(0), \dots, \boldsymbol{Z}(t))$. The σ-algebra $\mathcal{F}_t$, $t \in \tau$, represents the **publicly available** information up to and including time-t.

The combined financial-actuarial world $(\Omega, \mathcal{H}, \mathbb{P})$ is home to n^p policyholder-specific risks. We assume that the insurer holds the liabilities for the insurance contracts issued to n^p policyholders. A policyholder-specific risk may, for example, arise from the mortality risk of an individual insured under a life insurance contract. We denote the T-claim that represents the state of the policyholder-specific risk $m \in \{1, \dots, n^p\}$ at time-$t \in \tau$ by $X^{(m)}(t)$. The distinction between policyholder-specific risks $X^{(m)}(t)$ and systematic risks $Z^{(m)}(t)$ is a modeling assumption: a random variable is treated as systematic when it is publicly observable and common to many policyholders, so it cannot be eliminated by pooling, whereas it is treated as a policyholder-specific risk when it is observed only by the insurer and arises from an insurance contract, so its impact on the insurer's risk diminishes as the portfolio size increases. In most cases, the policyholder-specific risks $X^{(m)}(t)$ cannot be replicated using tradable assets $\boldsymbol{Y}(t)$. However, these risks are not assumed to be independent of the tradable assets, due to the presence of systematic risks $\boldsymbol{Z}(t)$. In this chapter, we assume that dependence between policyholder-specific risk and the asset-prices arises solely due to systematic risks.

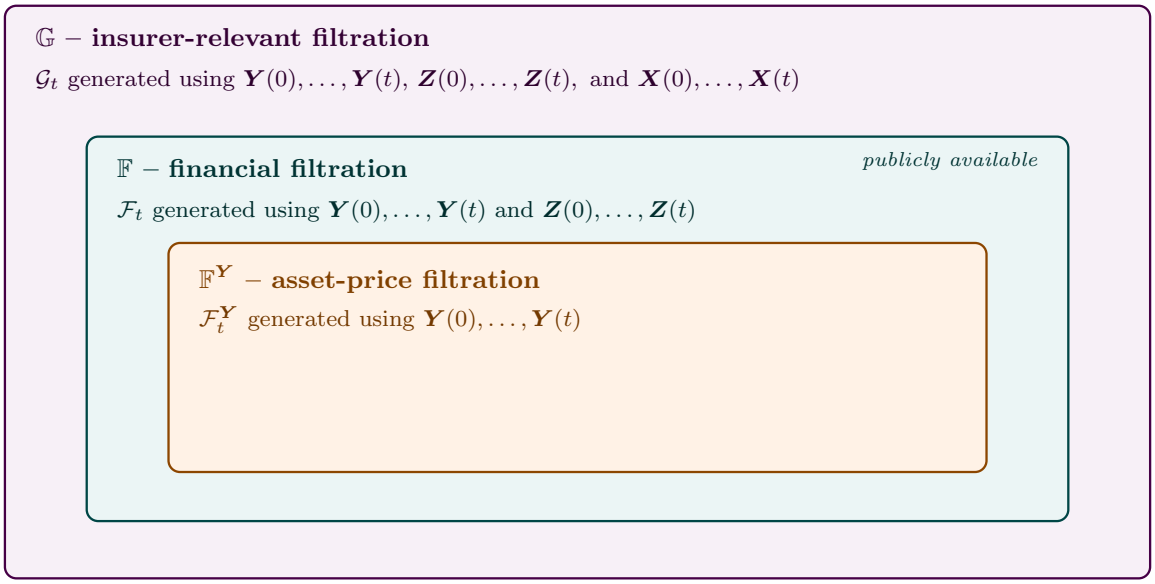
Let $\boldsymbol{Z}(t)$ and $\boldsymbol{X}(t)$, $t \in \tau$, be the vector of time-t states of non-traded systematic risks and states of policyholder-specific risks, respectively. That is, $\boldsymbol{Z}(t) = \left(Z^{(1)}(t), Z^{(2)}(t), \dots, Z^{(n^s)}(t)\right)$ and $\boldsymbol{X}(t) = \left(X^{(1)}(t), X^{(2)}(t), \dots, X^{(n^p)}(t)\right)$. The state processes of the non-traded systematic risks and the policyholder-specific risks are then described by the n^s-dimensional stochastic process $\boldsymbol{Z} = \{\boldsymbol{Z}(t)\}_{t\in\tau}$ and the n^p-dimensional stochastic process $\boldsymbol{X} = \{\boldsymbol{X}(t)\}_{t\in\tau}$, respectively.

Similarly, we define a filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in\tau}$, where $\mathcal{G}_t = \sigma(\boldsymbol{Y}(0), \dots, \boldsymbol{Y}(t), \boldsymbol{Z}(0), \dots, \boldsymbol{Z}(t), \boldsymbol{X}(0), \dots, \boldsymbol{X}(t))$. The σ-algebra $\mathcal{G}_t$, $t \in \tau$, represents the insurer-relevant information available up to and including time-t, which is the enlargement of $\mathcal{F}_t$ by the realizations of the policyholder-specific risks $\boldsymbol{X}(0), \dots, \boldsymbol{X}(t)$, which are observed only by the insurer and are not publicly available. Since the time-0 values $\boldsymbol{X}(0), \boldsymbol{Y}(0), \boldsymbol{Z}(0)$ are deterministic, the initial σ-algebras are trivial: $\mathcal{F}_0^{\boldsymbol{Y}} = \mathcal{F}_0 = \mathcal{G}_0 = \{\emptyset, \Omega\}$. By construction, we have

$$\mathbb{F}^{\boldsymbol{Y}} \subseteq \mathbb{F} \subseteq \mathbb{G}.$$

The three filtrations $\mathbb{F}^{\boldsymbol{Y}}, \mathbb{F}$ and $\mathbb{G}$ play distinct roles in our valuation framework. The asset-price filtration $\mathbb{F}^{\boldsymbol{Y}}$ is used to characterize the no-arbitrage prices of the traded assets: under an equivalent martingale measure $\mathbb{Q}$, discounted asset prices are $\mathbb{F}^{\boldsymbol{Y}}$-martingales. The financial filtration $\mathbb{F}$ is the largest filtration the insurer may use to construct trading strategies in risky assets, since using policyholder-specific information to construct these strategies would be considered insider-trading (Artzner et al., 2024). The insurer-relevant filtration $\mathbb{G}$ is reserved for actuarial valuation of orthogonal claims, where policyholder-specific information is used to apply diversification arguments. Figure 4.1 illustrates this setting. This separation departs from Barigou et al. (2019), whose multi-period framework is constructed within a single information filtration $\mathbb{G}$. Hedgers, valuations, and martingale conditions are all stated with respect to $\mathbb{G}$, which implicitly permits trading strategies measurable with respect to policyholder-specific risks. The single-period framework of Linders (2023) introduced an analogous distinction between asset-price, financial, and insurer-relevant filtrations. The present chapter extends this distinction to a discrete multi-period setting.

The σ-algebra $\mathcal{H}$ contains all information—even information that is not relevant to an insurer. For example, the policyholder-specific risks arising from the individuals insured by a different insurer.



Role of each filtration

- $\mathbb{F}^{\boldsymbol{Y}}$: used for no-arbitrage condition (discounted prices are $\mathbb{Q}$-martingales)
- $\mathbb{F}$: largest filtration admissible for trading strategies in risky assets
- $\mathbb{G}$: filtration for the actuarial valuation of orthogonal claims

Figure 4.1: Nested information filtrations $\mathbb{F}^{\boldsymbol{Y}} \subseteq \mathbb{F} \subseteq \mathbb{G}$ in the combined financial–actuarial setting, with the role played by each filtration in our valuation framework.

Since we focus only on the valuation for a single insurer, for the remainder of this chapter we focus only on filtrations $\mathbb{F}^{\boldsymbol{Y}}, \mathbb{F}$ and $\mathbb{G}$. Since $\mathbb{F}^{\boldsymbol{Y}}$, $\mathbb{F}$, and $\mathbb{G}$ are the natural filtrations of the stochastic processes $\{\boldsymbol{Y}(t)\}_{t\in\tau}$, $\{(\boldsymbol{Y}(t),\boldsymbol{Z}(t))\}_{t\in\tau}$, and $\{(\boldsymbol{Y}(t),\boldsymbol{Z}(t),\boldsymbol{X}(t))\}_{t\in\tau}$, respectively, we have that

$$\boldsymbol{Y}(t) \text{ is } \mathcal{F}_t^{\boldsymbol{Y}}\text{-measurable}, \qquad \text{for any } t = 0,1,\ldots,T,$$

$$\boldsymbol{Z}(t) \text{ is } \mathcal{F}_t\text{-measurable}, \qquad \text{for any } t = 0,1,\ldots,T$$

and

$$\boldsymbol{X}(t) \text{ is } \mathcal{G}_t\text{-measurable}, \qquad \text{for any } t = 0,1,\ldots,T.$$

The set of T-claims defined on $(\Omega,\mathcal{G},\mathbb{G},\mathbb{P})$, that is, a set of $\mathcal{G}_T$-measurable r.v.'s, is denoted by $\mathcal{C}_T$. These are the claims of a specific insurer. Similarly, the set of t-claims defined on $(\Omega,\mathcal{G},\mathbb{G},\mathbb{P})$, that is a set of $\mathcal{G}_t$-measurable r.v.'s, is denoted by $\mathcal{C}_t$.

The combined financial-actuarial world hosts a number of insurance liabilities that are due at time-T. An insurance liability due at time-T will generally be denoted by $S(T)$ or simply S if no confusion is possible. A simple example of an insurance liability that is observed at time-0 and is related to the death of an insured (x) is the r.v. $1_{(x)} \in \mathcal{C}_T$ defined by

$$1_{(x)} = \begin{cases} 1 & : (x) \text{ dies before or at time-}T \\ 0 & : (x) \text{ dies after time-}T. \end{cases} \tag{4.1}$$

4.2.2 Hedging strategies

We consider a general information structure represented by a filtration $\mathbb{J} = \{\mathcal{J}_t\}_{t\in\mathcal{T}}$, where $\mathcal{J}_t$ denotes the σ-algebra representing the information available up to time-t. Let $\big(\mathbb{J}, \mathcal{J}\big) \in \big\{(\mathbb{F}^{\boldsymbol{Y}}, \mathcal{F}^{\boldsymbol{Y}}), (\mathbb{F}, \mathcal{F}), (\mathbb{G}, \mathcal{G})\big\}$.

A *time-t $\mathcal{J}$-trading strategy*, $t \in \{0, 1, \ldots, T-1\}$, is an (n^a+1)-dimensional $\mathcal{J}_t$-adapted stochastic process $\boldsymbol{\theta}_t = \{\theta_t(u)\}_{u=\{t+1,\ldots,T\}}$, where $\boldsymbol{\theta}_t(u)$ denotes the portfolio held over the period $(u-1, u]$. We write

$$\theta_t(u) = \Big(\theta_t^{(0)}(u), \theta_t^{(1)}(u), \ldots, \theta_t^{(n^a)}(u)\Big),$$

where $\theta_t^{(0)}(u)$ represents the number of units invested (holdings) in the risk free asset and $\theta_t^{(m)}(u)$, $m = 1, \ldots, n^a$, represent the number of units invested (holdings) in the risky assets during the period $(u-1, u]$. We denote the vector of the holdings in the risky assets during the period $(u-1, u]$ by

$$\boldsymbol{\theta}_t^{(1:n^a)}(u) := \big(\theta_t^{(1)}(u), \ldots, \theta_t^{(n^a)}(u)\big).$$

Similarly, we denote the vector of the time-t prices of the risky assets by

$$\boldsymbol{Y}^{(1:n^a)}(u) := \big(Y^{(1)}(u), \ldots, Y^{(n^a)}(u)\big).$$

In our framework, information restrictions are imposed component-wise. Specifically, the holdings in the risky assets are chosen using the information filtration $\mathbb{J}$, while the holdings in the risk-free asset may depend on the larger filtration $\mathbb{G}$. That is, for each $u = t+1, \ldots, T$, we require that

$$\theta_t^{(1:n^a)}(u) \text{ is } \mathcal{J}_{u-1}\text{-measurable} \quad \text{and} \quad \theta_t^{(0)}(u) \text{ is } \mathcal{G}_{u-1}\text{-measurable.}$$

Throughout this chapter, the term *time-t $\mathbb{J}$-trading strategy* is understood component-wise: holdings in the risky assets are $\mathbb{J}$-predictable, while the holdings in the risk-free asset may depend on the larger filtration $\mathbb{G}$. This asymmetric measurability requirement reflects the fundamentally distinct roles of the two asset classes within the combined financial-actuarial framework. As established by Artzner et al. (2024), policyholder-specific information is not publicly available and its use in determining risky-asset holdings would constitute insider trading, potentially giving rise to an insurance-finance arbitrage. Insurers must therefore be restricted to $\mathbb{F}$-trading strategies or $\mathbb{F}^{\boldsymbol{Y}}$-trading strategies when determining their holdings in the risky assets. No analogous constraint applies to the risk-free asset, whose deterministic return grants no informational advantage to any market participant. This definition departs from that adopted in the existing multi-period fair-valuation literature: Barigou et al. (2019) and Barigou and Dhaene (2019) impose $\mathbb{G}$-predictability uniformly across all asset holdings, implicitly permitting risky-asset trading strategies to depend on policyholder-specific information.

The *initial investment* at time-t of a $\mathcal{J}$-trading strategy $\boldsymbol{\theta}_t$ can be expressed as

$$\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = \sum_{m=0}^{n^a} \theta_t^{(m)}(t+1) \times Y^{(m)}(t).$$

The value of the $\mathcal{J}$-trading strategy $\boldsymbol{\theta}_t$ at time-u, just before rebalancing, is given by

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \sum_{m=0}^{n^a} \theta_t^{(m)}(u) \times Y^{(m)}(u), \qquad \text{for any } u = t+1, t+2, \ldots, T,$$

whereas its value at time-u, just after rebalancing, is given by

$$\boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u) = \sum_{m=0}^{n^a} \theta_t^{(m)}(u+1) \times Y^{(m)}(u), \qquad \text{for any } u = t+1, t+2, \ldots, T-1.$$

By the component-wise measurability requirements imposed earlier, the value of the risky-asset holdings in the $\mathcal{J}$-trading strategy $\boldsymbol{\theta}_t$ at time-u, just before rebalancing, $\boldsymbol{\theta}_t^{(1:n^a)}(u) \cdot \boldsymbol{Y}^{(1:n^a)}(u)$, is $\mathcal{J}_{u-1}$-measurable. This contrasts with Barigou et al. (2019) where $\boldsymbol{\theta}_t^{(1:n^a)}(u) \cdot \boldsymbol{Y}^{(1:n^a)}(u)$ is $\mathcal{G}_{u-1}$-measurable.

A time-t $\mathcal{J}$-trading strategy is said to be *self-financing* if no capital is injected or withdrawn at any rebalancing time:

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{Y}(u) = \boldsymbol{\theta}_t(u+1) \cdot \boldsymbol{Y}(u) \qquad \text{for any } u = t+1, t+2, \ldots, T-1. \tag{4.2}$$

The linear space of all self-financing time-t $\mathcal{J}$-trading strategies is denoted by $\boldsymbol{\Theta}_t^{\mathbb{J}}$. A self-financing time-$t$ $\mathcal{J}$-trading strategy is understood component-wise: the holdings in the risky assets are $\mathbb{J}$-predictable, while the holdings in the risk-free asset are $\mathbb{G}$-predictable. From Definition 4.2, the time-T value of any self-financing time-t $\mathcal{J}$-trading strategy $\boldsymbol{\theta}_t \in \boldsymbol{\Theta}_t^{\mathbb{J}}$ may be expressed as

$$\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) = \boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) + \sum_{u=t+1}^{T} \boldsymbol{\theta}_t(u) \cdot \boldsymbol{\Delta Y}(u), \tag{4.3}$$

where $\boldsymbol{\Delta Y}(u)$ is the price increment vector in period $(u-1, u]$, that is, $\boldsymbol{\Delta Y}(u) = \boldsymbol{Y}(u) - \boldsymbol{Y}(u-1)$.

Example 4.2.1 *Two examples of static self-financing $\mathcal{J}$-trading strategies are:*

- *The null investment:*
$$\mathbf{0}_t = \left(\mathbf{0}_t(t+1), \mathbf{0}_t(t+2), \ldots, \mathbf{0}_t(T)\right),$$
where $\mathbf{0}_t(u) = (0, 0, \ldots, 0)^\intercal$ *for all* $u = t+1, t+2, \ldots, T$.
- *Buying* 1 *unit of the asset* m *at time-*t *and holding it until time-*T*:*
$$\mathbf{1}_t^{(m)} = \left(\mathbf{1}_t^{(m)}(t+1), \mathbf{1}_t^{(m)}(t+2), \ldots, \mathbf{1}_t^{(m)}(T)\right),$$
where $\mathbf{1}_t^{(m)}(u) = (0, \ldots, 0, 1, 0, \ldots, 0)^\intercal$ *for all* $u = t+1, t+2, \ldots, T$.

4.2.3 Assumptions for the financial markets

We assume that the financial market is *free of arbitrage*, meaning that there exists no self-financing trading strategy $\boldsymbol{\theta}_0 \in \boldsymbol{\Theta}_0^{\mathbb{F}}$ such that:

$$\boldsymbol{\theta}_0(1) \cdot \boldsymbol{Y}(0) = 0, \qquad \mathbb{P}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) \geq 0\right] = 1 \quad \text{and} \quad \mathbb{P}\left[\boldsymbol{\theta}_0(T) \cdot \boldsymbol{Y}(T) > 0\right] > 0. \tag{4.4}$$

In our discrete-time framework, the no-arbitrage condition is equivalent to the existence of an equivalent martingale measure (EMM) $\mathbb{Q}$. Under $\mathbb{Q}$, the price process of any traded asset $Y^{(m)}(t)$ at trading date t admits the representation

$$Y^{(m)}(t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}(T) \mid \mathcal{F}_t\right], \qquad \text{for any } t = 1, 2, \ldots, T, \tag{4.5}$$

where the discounted price process $\{e^{-rt} Y^{(m)}(t)\}$ is an $\mathbb{F}$-martingale under $\mathbb{Q}$. This implies that no arbitrage can be constructed at time-t from buying or selling tradable assets using publicly available financial information up to and including time-t. Financial information includes the asset prices and the non-traded systematic risk states. This assumption aligns with that of Artzner et al. (2024), who incorporate all publicly available information in $\mathbb{F}$. Note that in this chapter we do not assume the market to be complete. In other words, the EMM $\mathbb{Q}$ is not unique. Multiple EMMs may exist. For

the risky-assets $m \in \{1, \ldots, n^a\}$, we further assume that the immersion property, also known as the hypothesis $(\boldsymbol{H})$, holds (Brémaud & Yor, 1978; Kusuoka, 1999):

$$Y^{(m)}(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}(T) \mid \mathcal{F}_t\right] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[Y^{(m)}(T) \mid \mathcal{G}_t\right]. \tag{4.6}$$

In other words, every $\mathbb{F}$-martingale under $\mathbb{Q}$ is also a $\mathbb{G}$-martingale under $\mathbb{Q}$. This is a standard assumption in the mathematical finance literature, see Coculescu et al. (2012). The immersion property ensures that any risk-neutral measure for the asset-price process with respect to $\mathbb{F}$ remains risk-neutral with respect to $\mathbb{G}$, so that enlarging the information flow from $\mathbb{F}$ to $\mathbb{G}$ does not create new arbitrage opportunities. Without the immersion property, additional information present in $\mathbb{G}$ that is not present in $\mathbb{F}$—specifically information about policyholder-specific risks—could give rise to arbitrage opportunities. The immersion property is stated with respect to the risk-neutral measure $\mathbb{Q}$ rather than the physical measure $\mathbb{P}$. No analogous condition on $\mathbb{P}$-martingales is assumed.

In this setting, even if the insurer used a $\mathbb{G}$-predictable strategy for the risky assets, instead of an $\mathbb{F}$-predictable strategy, no arbitrage would be created due to the immersion property. Restricting to $\mathbb{F}$-predictable strategies for the risky assets is motivated by the no-insider-trading requirement. In general, an $\mathbb{F}$-predictable strategy for the risky assets will differ from a $\mathbb{G}$-predictable strategy and will lead to a larger hedging error. In our framework, the additional hedging error arising from using an $\mathbb{F}$-predictable strategy for the risky assets is subsequently valued using a model-consistent valuation.

Following earlier work in this area of Barigou et al. (2019) and Dhaene et al. (2017), we also assume that the price increments $\boldsymbol{\Delta Y}(t)$ satisfy the following non-redundancy condition:

$$\boldsymbol{\theta}_t(u) \cdot \boldsymbol{\Delta Y}(u) = 0 \quad \Rightarrow \quad \boldsymbol{\theta}_t(u) = \mathbf{0}_t(u). \tag{4.7}$$

A consequence of this assumption is that for any $\boldsymbol{\theta}_t \in \boldsymbol{\Theta}_t^{\mathbb{F}}$ one has $\boldsymbol{\theta}_t(T) \cdot \boldsymbol{\Delta Y}(T) = 0 \Leftrightarrow \boldsymbol{\theta}_t \equiv \mathbf{0}_t$, and for any $\boldsymbol{\mu}_t, \boldsymbol{\nu}_t \in \boldsymbol{\Theta}_t^{\mathbb{F}}$ one has $\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) = \boldsymbol{\nu}_t(T) \cdot \boldsymbol{Y}(T) \Leftrightarrow \boldsymbol{\mu}_t \equiv \boldsymbol{\nu}_t$. See Dhaene et al. (2017) for the proofs.

4.2.4 Types of claims

Definition 10 ***(Hedgeable claims).*** *Observed at time-t, a t-hedgeable T-claim, denoted S^h, is an $\mathcal{F}_T$-measurable element of $\mathcal{C}_T$, which can be replicated by a time-t self-financing strategy $\boldsymbol{\theta}_t \in \boldsymbol{\Theta}_t^{\mathbb{F}}$:*

$$S^h = \boldsymbol{\theta}_{t,S^h}(T) \cdot \boldsymbol{Y}(T).$$

We denote the linear space of all t-hedgeable T-claims as $\mathcal{C}_T^{t,h}$. The term $\boldsymbol{\theta}_{t,S^h}(T)$ in the above definition is known as the 'hedge' at time-t for the T-claim S^h. By definition, we have that the set of t-hedgeable claims $\mathcal{C}_T^{t,h} \subseteq \mathcal{C}_T$.

From an insurer's perspective, at time-t a t-hedgeable T-claim can be managed by investing the amounts $\boldsymbol{\theta}_{t,S^h}(t+1)$ and then rebalancing the investments in the tradable assets at each future time $u = t+1, ..., T-1$ according to $\boldsymbol{\theta}_{t,S^h}(u+1)$, leading to a value of exactly $S^h = \boldsymbol{\theta}_{t,S^h}(T) \cdot \boldsymbol{Y}(T)$ at time-T. As a result, hedgeable claims should always be valued using the current market price of the portfolio. Since we assumed the existence of an EMM $\mathbb{Q}$, this method is equal to discounted risk-neutral expectation pricing.

The time-t price of a t-hedgeable T-claim $S^h = \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) \in \mathcal{C}_T^{t,h}$ for $u = t, t+1, \ldots, T$ is given by:

$$\boldsymbol{\theta}_t(t+1) \cdot \boldsymbol{Y}(t) = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{F}_t\right], \tag{4.8}$$

where $\mathbb{Q}$ is an EMM.

Next we introduce the set of orthogonal claims. The idea of orthogonal claims was formalized in the literature by Dhaene et al. (2017).

Definition 11 ***(Orthogonal claims).*** *Observed at time-t, a t-orthogonal T-claim, denoted $S^{\perp}$, is an $\mathcal{G}_T$-measurable element of $\mathcal{C}_T$, which is $\mathbb{P}$-independent of $\sigma\big(\boldsymbol{Y}(t+1),\ldots,\boldsymbol{Y}(T),\boldsymbol{Z}(t+1),\ldots,\boldsymbol{Z}(T)\big)$.*

A t-orthogonal T-claim $S^{\perp}$ is defined as a claim that is independent of all future traded asset-price movements $\big(\boldsymbol{Y}(t+1),\ldots,\boldsymbol{Y}(T)\big)$ and all future non-traded systematic risk movements $\big(\boldsymbol{Z}(t+1),\ldots,\boldsymbol{Z}(T)\big)$. This independence eliminates exposure to both asset-price and non-traded systematic risks, ensuring that the claim depends solely on policyholder-specific risk, for which no market price exists and an actuarial valuation is therefore appropriate. We denote the linear space of such claims by $\mathcal{C}_T^{t,\perp}$. Whereas Barigou et al. (2019) impose orthogonality only with respect to future traded asset-price movements $\big(\boldsymbol{Y}(t+1),\ldots,\boldsymbol{Y}(T)\big)$, we follow Linders (2023) in also excluding claims from this definition that depend on the future non-traded systematic risk movements $\big(\boldsymbol{Z}(t+1),\ldots,\boldsymbol{Z}(T)\big)$.

Definition 12 ***(Hybrid claims).*** *Observed at time-t, a t-hybrid T-claim S is a $\mathcal{G}_T$-measurable element of $\mathcal{C}_T$ that is neither t-hedgeable nor t-orthogonal:*

$$S \in \mathcal{C}_T \setminus \left(\mathcal{C}_T^{t,h} \cup \mathcal{C}_T^{t,\perp}\right).$$

An example of a t-hybrid T-claim is $S = S^h \times S^{\perp}$, where $S^h \in \mathcal{C}_T^{t,h}$ and $S^{\perp} \in \mathcal{C}_T^{t,\perp}$. A simple example of hedgeable and hybrid claims is considered next in a two-period setting.

Example 4.2.2 ***(Examples of hedgeable and hybrid claims).*** *Consider a two-period financial-actuarial world for a specific insurer $(\Omega, \mathcal{G}_2, \mathbb{G}, \mathbb{P})$. In this setting, the insurer may be exposed to a combination of asset-price risk, non-traded catastrophe risk, and policyholder-specific risk, depending on the contract sold.*

Suppose there exists a financial market containing one risk-free and one risky asset. The asset prices evolve as follows:

$$\begin{pmatrix} 1 & e^{r} & e^{2r} \\ Y^{(1)}(0) & Y^{(1)}(1) & Y^{(1)}(2) \end{pmatrix},$$

a non-traded earthquake index that evolves as follows $(Z^{(1)}(0), Z^{(1)}(1), Z^{(1)}(2))$ with

$$Z^{(1)}(t) = \begin{cases} 0, & \text{no earthquake before } t \\ 1, & \text{earthquake before } t, \end{cases}$$

and policyholder survival process that evolves as follows $(X^{(1)}(0), X^{(1)}(1), X^{(1)}(2))$ with

$$X^{(1)}(t) = \begin{cases} 0, & \text{alive at } t, \\ 1, & \text{dies before } t. \end{cases}$$

The following examples illustrate the differences between 2-claims (i.e. claims with payoff at time-2) and their classification as hedgeable, orthogonal, or hybrid at different time points:

- *0-hedgeable: $S^{(i)} = 50 + Y^{(1)}(2)$. This represents a unit-linked endowment policy that guarantees a fixed payment of 50 plus the value of an equity index at maturity. Since the entire payoff is driven by the traded risky asset, it can be perfectly replicated at time-0.*
- *1-hedgeable: $S^{(ii)} = Y^{(1)}(1)\,Y^{(1)}(2)$. This represents a path-dependent equity-linked benefit whose payoff is scaled by the asset price at an intermediate date. It is not hedgeable at time-0 since $Y^{(1)}(1)$ is unknown, but it becomes fully replicable at time-1 when $Y^{(1)}(1)$ is known.*
- *0-hybrid: $S^{(v)} = Y^{(1)}(2) \times \mathbf{1}_{\{X^{(0)}(2)=1\}} + 100 \times \mathbf{1}_{\{X^{(0)}(2)=0\}}$. This represents a unit-linked life insurance contract paying the asset value on survival and a guaranteed death benefit of 100 otherwise. This example combines exposure to asset-prices and policyholder survival.*

Definition 13 ***(Diversifiable claims).*** *Observed at time-t, S^d is a t-diversifiable T-claim if there exist i.i.d. $\mathcal{G}_T$-measurable random variables $V_1, V_2, \ldots, V_n$ such that*

$$S^d = \frac{1}{n}\sum_{i=1}^{n} V_i.$$

We denote the set of t-diversifiable T-claims by $\mathcal{C}_T^{t,d}$.

A t-diversifiable T-claim is a combination of n independent claims V_i. For t-diversifiable T-claims that are not hedgeable, S^d converges to $\mathbb{E}^{\mathbb{P}}[V_1 \mid \mathcal{G}_t]$ as $n \to \infty$, by the law of large numbers. Similarly, the variance of S^d shrinks as n increases. For sufficiently large n, the central limit theorem shows that S^d is normally distributed.

Following Linders (2023), we impose the following modeling assumption that, at time-t, policyholder-specific risks $X^{(i)}(t)$, $i = 1, \ldots, n^p$ are $\mathcal{F}_t$-conditionally independent:

$$\mathbb{P}\left[\boldsymbol{X}(t) \leq \boldsymbol{x} \mid \mathcal{F}_t\right] = \prod_{i=1}^{n^p} \mathbb{P}\left[X^{(i)}(t) \leq x_i \mid \mathcal{F}_t\right] \tag{4.9}$$

The conditional independence of the policyholder-specific risks is a modeling assumption ensuring that, once common risks arising from asset-price and non-traded systematic risks are fixed, the remaining uncertainty is idiosyncratic and therefore is a diversifiable claim. This assumption is adopted for our framework to ensure tractability, under which the conditional actuarial valuation (introduced later) is mathematically well-defined.

Example 4.2.3 ***(Example of a diversifiable claim).*** *Consider the same two-period financial-actuarial world as in Example 4.2.2. Now suppose the insurer holds a portfolio of n^p policyholders, each with an independent survival process $X^{(i)}(t)$, $i = 1, \ldots, n^p$, defined as in that example. Consider the per-policy death benefit*

$$S^{(vi)} = \frac{1}{n^p}\sum_{i=1}^{n^p} \mathbf{1}_{\{X^{(i)}(2)=1\}},$$

which pays 1 for each policyholder who dies before time-2, averaged over the portfolio. Since the policyholder-specific risks $X^{(i)}(2)$ are $\mathcal{F}_2$-conditionally independent, the indicators $\mathbf{1}_{\{X^{(i)}(2)=1\}}$ are i.i.d. given $\mathcal{F}_2$, so $S^{(vi)}$ is a 0-diversifiable 2-claim. Note that $S^{(vi)}$ is not 0-orthogonal: the survival probabilities depend on the earthquake index $Z^{(1)}(2)$, which is a future systematic risk movement, so $S^{(vi)}$ is not independent of $\sigma\big(Z^{(1)}(1), Z^{(1)}(2)\big)$.

Insurers typically hold portfolios of multiple policies whose payoffs are hybrid claims, combining exposure to asset-price movements, non-traded systematic risk, and policyholder-specific risk. Since the policyholder-specific risks $X^{(i)}(t)$, $i = 1, \ldots, n^p$, are assumed to be $\mathcal{F}_t$-conditionally independent, these risks are diversifiable at the portfolio level. The insurer can reduce the overall risk arising from these components by increasing the number of policies held. Here, we assume that the insurer can always increase the number of policies in its portfolio. In other words, we assume that there exist more copies of X_i than the insurer currently holds in its portfolio.

4.3 Fair valuations and fair hedgers

In this section, we define t-valuations and t-hedgers. For $t = 0, 1, \ldots, T$, t-valuations assign a $\mathcal{G}_t$-measurable random variable to each T-claim (by definition $\mathcal{G}_T$-measurable). The t-valuation of a

T-claim represents the 'value' at time-t of a claim based on the information available up to time-t. For $t = 0, 1, \ldots, T-1$, t-hedgers assign a self-financing $\mathcal{J}$-trading strategy to each T-claim. The t-hedger of a T-claim represents the 'best possible' trading strategy (for a given criterion) for the T-claim based on the information in $\mathcal{J}_t$. The definitions in this section have been inspired by Barigou et al. (2019). We highlight any differences in definitions.

4.3.1 t-valuations

In this subsection, we introduce t-valuations. Valuations for hybrid insurance claims were first formalized by Dhaene et al. (2017) in a single-period setting. This framework was subsequently extended to a multi-period setting by Barigou et al. (2019), who introduced t-valuations defined at arbitrary intermediate time points $t \in \tau$.

Definition 14 ***(t-valuations).*** *A t-valuation is a mapping $\rho_t : \mathcal{C}_T \to \mathcal{C}_t$, attaching a t-claim to a T-claim $S \in \mathcal{C}_T$:*

$$S \to \rho_t[S]$$

such that

- *ρ_t is normalized:*

$$\rho_t[0] = 0$$

- *ρ_t is translation invariant:*

$$\rho_t(S + a) = \rho_t(S) + e^{-r(T-t)}\, a, \quad \text{for all } S \in \mathcal{C}_T \text{ and } a \in \mathcal{C}_t.$$

The value $\rho_t[S]$ is itself a t-claim and hence, seen from time-0, is a random variable which is 'realized' at time-t. $\rho_t[S]$ is $\mathcal{G}_t$-measurable. In other words, $\rho_t[S]$ is known and deterministic at time-t if one has all the insurer-relevant information up to that point.

Definition 15 ***(Market-consistency for valuations).*** *A t-valuation ρ_t is market consistent if the t-hedgeable part of every T-claim is marked-to-market:*

$$\rho_t\left[S + S^h\right] = \rho_t[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S^h \mid \mathcal{F}_t\right], \tag{4.10}$$

where $S \in \mathcal{C}_T$ and $S^h = \boldsymbol{\theta}_t(T) \cdot \boldsymbol{Y}(T) \in \mathcal{C}_T^{t,h}$, with $\boldsymbol{\theta}_t \in \boldsymbol{\Theta}_t^{\mathbb{F}}$.

Market-consistency requires that any t-hedgeable component of a T-claim be valued according to the current market price of the replicating hedge. Accordingly, a hybrid T-claim $S \in \mathcal{C}_T$ is valued at time-t as the sum of the discounted risk-neutral expectation of its t-hedgeable component, and the valuation ρ_t applied to the residual component.

Definition 16 ***(Actuarial t-valuation).*** *Given information up to and including time-t, a mapping $\pi_t : \mathcal{C}_T \to \mathcal{C}_t$ is an* actuarial t-valuation *if for any $S \in \mathcal{C}_T$:*

(i) $\pi_t[S]$ is $\mathcal{G}_t$-measurable;

(ii) $\pi_t[S]$ depends on S only through its conditional distribution given $\mathcal{G}_t$: if $\mathbb{P}[S \leq x \mid \mathcal{G}_t] = \mathbb{P}[S' \leq x \mid \mathcal{G}_t]$ for all $x \in \mathbb{R}$, then $\pi_t[S] = \pi_t[S']$;

(iii) for any $S^{\perp} \in \mathcal{C}_T^{t,\perp}$, $\pi_t[S^{\perp}]$ is $\mathbb{P}$-independent of

(a) $\{\boldsymbol{Y}(u)\}_{u\in\{t+1,\dots,T\}}$ *and*
(b) $\{\boldsymbol{Z}(u)\}_{u\in\{t+1,\dots,T\}}$

Condition (iii) is the key actuarial requirement in this definition, and is both economically natural and consistent with actuarial practice. Since an orthogonal claim $S^{\perp} \in \mathcal{C}_T^{t,\perp}$ is by definition independent of future market movements, it would be economically undesirable for its actuarial value $\pi_t[S^{\perp}]$ to depend on those same movements: condition (iii) ensures the actuarial valuation does not reintroduce a market dependence that the claim itself does not possess.

Definition 17 ***(Model-consistency for valuations).*** *A t-valuation ρ_t is* model consistent *if any t-orthogonal T-claim is marked-to-model:*

$$\rho_t\big[S^{\perp}\big] = \pi_t\big[S^{\perp}\big], \quad \textit{for all } S^{\perp} \in \mathcal{C}_T^{t,\perp}, \tag{4.11}$$

for some actuarial t-valuation π_t.

Model-consistency means that the value of any t-orthogonal T-claim is determined by an actuarial valuation. Note that the set of actuarial t-valuations is a subset of model-consistent t-valuations.

Example 4.3.1 ***(Examples of model-consistent valuations).*** *The following are actuarial t-valuations, and hence model consistent when applied to t-orthogonal claims:*

- $\rho_t[S] = (1+\gamma)\, e^{-r(T-t)}\mathbb{E}^{\mathbb{P}}\left[S \mid \mathcal{G}_t\right]$,
- $\rho_t[S] = e^{-r(T-t)}\left(\mathbb{E}^{\mathbb{P}}\left[S \mid \mathcal{G}_t\right] + \alpha\sqrt{Var^{\mathbb{P}}[S \mid \mathcal{G}_t]}\right)$.

Each satisfies conditions (i) and (ii) of Definition 16: the output is $\mathcal{G}_t$-measurable, and depends on S only through its $\mathcal{G}_t$-conditional distribution. Condition (iii) is also satisfied: for any $S^{\perp} \in \mathcal{C}_T^{t,\perp}$, independence of $S^{\perp}$ from $\{\boldsymbol{Y}(u)\}_{u>t}$ and $\{\boldsymbol{Z}(u)\}_{u>t}$ means that conditioning on $\mathcal{G}_t$ does not introduce any dependence of $\pi_t[S^{\perp}]$ on these future values, so the output remains $\mathbb{P}$-independent of $\{\boldsymbol{Y}(u)\}_{u>t}$ and $\{\boldsymbol{Z}(u)\}_{u>t}$.

The above examples allow the insurer to use all available information at time-t, including past asset prices and systematic risk realizations, when forming its actuarial estimate.

Next, we define a fair t-valuation. The concept of fair-valuations and model-consistent valuations were originally introduced by Dhaene et al. (2017) in a single-period setting and were subsequently generalized to a multi-period framework by Barigou et al. (2019).

Definition 18 ***(Fair valuation).*** *A t-valuation ρ_t is a fair valuation if it is both market consistent and model consistent.*

4.3.2 t-hedgers

In this subsection, we introduce t-hedgers. Hedgers for insurance claims were first formalized by Dhaene et al. (2017) in a single-period setting. This framework was subsequently extended to a multi-period setting by Barigou et al. (2019), who introduced t-hedgers defined at arbitrary intermediate time points $t \in \tau$.

Definition 19 ***(t-hedgers)****. Let $\mathbb{J} \in \{\mathbb{F}^{\boldsymbol{Y}}, \mathbb{F}, \mathbb{G}\}$. A t-$\mathbb{J}$-hedger is a function $\boldsymbol{\theta}_t : \mathcal{C}_T \to \boldsymbol{\Theta}_t^{\mathbb{J}}$ which maps any T-claim S into a self-financing $\mathcal{J}$-trading strategy:*

$$S \to \boldsymbol{\theta}_{t,S}$$

such that

- *$\boldsymbol{\theta}_t$ is normalized:*

$$\boldsymbol{\theta}_{t,0} = \mathbf{0}_t$$

- *$\boldsymbol{\theta}_t$ is translation invariant:*

$$\boldsymbol{\theta}_{t,S+a} = \boldsymbol{\theta}_{t,S} + e^{-rT} a \times \mathbf{1}_t^{(0)}, \text{ for any } S \in \mathcal{C}_T \text{ and any } a \in \mathcal{C}_t \text{ payable at } T.$$

A hedger produces a self-financing trading strategy $\left(\boldsymbol{\theta}_{t,S}^{\mathbb{J}}(t+1), \ldots, \boldsymbol{\theta}_{t,S}^{\mathbb{J}}(T)\right)$, where each vector $\boldsymbol{\theta}_{t,S}^{\mathbb{J}}(u) = \left(\theta_{t,S}^{(0)\,\mathbb{J}}(u), \ldots, \theta_{t,S}^{(n^a)\,\mathbb{J}}(u)\right)^{\intercal}$ for $u = t+1, \ldots, T$ specifies the holdings in each asset over time.

The definition of a t-hedger in Barigou et al. (2019) is formulated with respect to a single filtration $\mathbb{G}$. In contrast, our framework introduces t–$\mathbb{J}$-hedgers for different information filtrations $\mathbb{J} \in \{\mathbb{F}^Y, \mathbb{F}, \mathbb{G}\}$, allowing the hedging strategy to be adapted to the specific information set available to the trader. This leads to a more general formulation in which the same claim may result in different hedgers depending on the information used to determine the hedge.

Definition 20 ***(Market-consistency for hedgers).*** *A t-$\mathbb{J}$-hedger $\boldsymbol{\theta}_t$ is market consistent if:*

$$\boldsymbol{\theta}_{t,S+S^h} = \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h}, \quad \text{for any } S \in \mathcal{C}_T \text{ and any } S^h \in \mathcal{C}_T^{t,h}, \tag{4.12}$$

where $\boldsymbol{\theta}_{t,S^h}$ is a t-$\mathbb{F}$-hedge of S^h.

Market-consistency for hedgers requires that the hedgeable component of any T-claim S be hedged using its replicating t-$\mathbb{F}$-hedge. Accordingly, for any t-hybrid T-claim, the associated hedger is given by the sum of the replicating hedge for the hedgeable component and the hedger calculated for the residual component. Whether a hedger satisfies market-consistency depends on the underlying definition of hedgeability.

Definition 21 ***(Model-consistency for hedgers).*** *A t-$\mathbb{J}$-hedger $\boldsymbol{\theta}_t$ is model consistent if there exists an actuarial t-valuation π_t such that:*

$$\boldsymbol{\theta}_{t,S^\perp} = \frac{\pi_t[S^\perp]}{Y^{(0)}(t)} \times \mathbf{1}^{(0)}, \quad \text{for any } S^\perp \in \mathcal{C}_T^\perp, \tag{4.13}$$

where π_t is called the underlying actuarial t-valuation of the t-$\mathbb{J}$-hedger $\boldsymbol{\theta}_t$.

Model-consistency means that any t-orthogonal T-claim is hedged via an actuarial t-valuation invested solely in the risk-free asset. The definition of model-consistency in our framework also differs from that of Barigou et al. (2019) because the underlying set of orthogonal claims is not the same. In our framework, a narrower class of claims is orthogonal, so model-consistency requirement for a hedger must be satisfied for a smaller class of claims than in Barigou et al. (2019).

Remark 1 *We note that for an orthogonal claim with a long time horizon, an insurer may in principle wish to invest in risky assets in order to benefit from higher expected returns, thereby reducing the amount that must be set aside at the initial time. In this chapter, however, insurers are not permitted to base their* ***investment strategy for risky assets*** *on non-publicly available information. Consequently, we assume that the insurer invests the funds associated with orthogonal claims exclusively in the risk-free asset, guaranteeing a deterministic return to exactly meet the obligation at time T, rather than adopting a risky asset trading strategy that cannot fully use the available information.*

Definition 22 *A t-$\mathbb{J}$-hedger $\boldsymbol{\theta}_t$ is a fair hedger if it is both market consistent and model consistent.*

4.3.2.1 Mean-variance hedger

In this subsection, we introduce the mean-variance hedger. We further show that this hedger is a fair hedger.

Hereafter, for $\mathbb{J} \in \left\{\mathbb{F}^{\boldsymbol{Y}}, \mathbb{F}, \mathbb{G}\right\}$, the notation $\boldsymbol{\Theta}_t^{\mathbb{J}'}$ denotes a linear subspace of $\boldsymbol{\Theta}_t^{\mathbb{J}}$ such that the set $\left\{\boldsymbol{\mu}_t(T) \cdot \boldsymbol{Y}(T) \mid \boldsymbol{\mu} \in \boldsymbol{\Theta}_t^{\mathbb{J}'}\right\}$ is closed. The prime in $\mathbb{J}'$ indicates this closed-subspace restriction and does not denote a separate filtration. Following Barigou and Dhaene (2019), we restrict minimization to closed linear spaces to ensure that the mean-variance hedger has a solution.

We define the mean-variance hedger $\boldsymbol{\theta}_{t,S}^{MV}$ as follows

$$\boldsymbol{\theta}_{t,S}^{MV} = \begin{cases} \arg\min_{\boldsymbol{\mu} \in \boldsymbol{\Theta}_t^{\mathbb{F}'}} \mathbb{E}^{\mathbb{P}}\left[\left(S - \boldsymbol{\mu}(T) \cdot \boldsymbol{Y}(T)\right)^2 \mid \mathcal{F}_t\right] & \text{for any } S \notin \mathcal{C}_T^{t,\perp} \\ \frac{\pi_t[S^{\perp}]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} & \text{for any } S \in \mathcal{C}_T^{t,\perp}, \end{cases} \tag{4.14}$$

where π_t is an actuarial valuation. Here $\boldsymbol{\Theta}_t^{\mathbb{F}}$ denotes the class of self-financing time-t trading strategies for which the holdings in the risky assets are $\mathbb{F}$-predictable, while the holding in the risk-free asset is $\mathbb{G}$-predictable.

We find $\boldsymbol{\theta}_{t,S}$ in the linear space $\boldsymbol{\Theta}_t^{\mathbb{F}'}$ instead of $\boldsymbol{\Theta}_t^{\mathbb{G}'}$, since the latter would also allow holdings in the risky assets to depend on policyholder-specific risks $\left(\boldsymbol{X}(0), \ldots, \boldsymbol{X}(t)\right)$. This information would be regarded as 'insider information' and therefore cannot be used to inform the hedging decision for the risky assets of an insurer.

Adaptations of the mean-variance hedger defined above have been studied by many authors in the fair-valuation literature (Barigou & Dhaene, 2019; Dhaene et al., 2017). From Barigou and Dhaene (2019), Černỳ and Kallsen (2009), and Föllmer and Schweizer (1988), we have that

$$\boldsymbol{\theta}_{t,S}^{(1:n^a),MV} = \begin{cases} \Sigma_t^{-1} \times \mathrm{Cov}_t\left[\boldsymbol{Y}^{(1:n^a)}(T), S \mid \mathcal{F}_t\right] & \text{for any } S \notin \mathcal{C}_T^{t,\perp}, \\ 0 & \text{for any } S \in \mathcal{C}_T^{t,\perp} \end{cases}$$

and

$$\boldsymbol{\theta}_{t,S}^{(0),MV} = \begin{cases} \frac{\left(\mathbb{E}^{\mathbb{P}}[S|\mathcal{F}_t] - \sum_{m=1}^{n^a} \mathbb{E}^{\mathbb{P}}[Y^{(m)}(T)|\mathcal{F}_t] \times \theta_{t,S}^{(m),MV}\right)}{\mathbb{E}^{\mathbb{P}}[Y^{(0)}(T)|\mathcal{F}_t]} & \text{for any } S \notin \mathcal{C}_T^{t,\perp}, \\ \frac{\pi_t[S^{\perp}]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} & \text{for any } S \in \mathcal{C}_T^{t,\perp}, \end{cases}$$

where Σ_t is the conditional covariance matrix of the risky asset prices at time-T, $\boldsymbol{Y}(T)$, given the information $\mathcal{F}_t$. We have assumed that Σ_t is invertible. Precisely, for $i, j \in \{1, 2, \ldots, n^a\}$,

$$\begin{aligned} (\Sigma_t)_{ij} &= \mathrm{Cov}\left[Y^{(i)}(T), Y^{(j)}(T) \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[Y^{(i)}(T) \times Y^{(j)}(T) \mid \mathcal{F}_t\right] - \mathbb{E}\left[Y^{(i)}(T) \mid \mathcal{F}_t\right] \times \mathbb{E}\left[Y^{(j)}(T) \mid \mathcal{F}_t\right] \end{aligned}$$

and $\mathrm{Cov}_t\left[\boldsymbol{Y}^{(1:n^a)}(T), S\right]$ is the conditional covariance vector between the traded asset-price at time-T and the T-claim S given the information $\mathcal{F}_t$. Precisely, for $i, j \in \{1, 2, \ldots, n^a\}$,

$$\begin{aligned} \left(\mathrm{Cov}[\mathbf{Y}(T), S \mid \mathcal{F}_t]\right)_i &= \mathrm{Cov}\left[Y^{(i)}(T), S \mid \mathcal{F}_t\right] \\ &= \mathbb{E}\left[Y^{(i)}(T) \times S \mid \mathcal{F}_t\right] - \mathbb{E}\left[Y^{(i)}(T) \mid \mathcal{F}_t\right] \times \mathbb{E}\left[S \mid \mathcal{F}_t\right]. \end{aligned}$$

In the following lemma, we show that the mean-variance hedger defined in Equation 4.14 is a fair hedger.

Lemma 2 *The mean-variance hedger $\boldsymbol{\theta}_{t,S}^{MV}$ is fair.*

Proof: For market-consistency, since $S + S^h \notin \mathcal{C}_T^{t,\perp}$, we have

$$\begin{aligned}\boldsymbol{\theta}_{t,S+S^h}^{(1:n^a),MV} &= \Sigma_t^{-1} \times \mathrm{Cov}_t\left[\boldsymbol{Y}^{(1:n^a)}(T), S + S^h \mid \mathcal{F}_t\right] \\ &= \boldsymbol{\theta}_{t,S}^{(1:n^a),MV} + \Sigma_t^{-1} \times \mathrm{Cov}\left[\boldsymbol{Y}^{(1:n^a)}(T), S^h \mid \mathcal{F}_t\right] \\ &= \boldsymbol{\theta}_{t,S}^{(1:n^a),MV} + \boldsymbol{\theta}_{t,S_h}^{(1:n^a),MV},\end{aligned}$$

where in the final step, we use the fact that $S^h \notin \mathcal{C}_T^{t,\perp}$. For $\boldsymbol{\theta}_{t,S+S^h}^{(0),MV}$, we have

$$\begin{aligned}\boldsymbol{\theta}_{t,S+S^h}^{(0),MV} &= \frac{\left(\mathbb{E}^{\mathbb{P}}[S + S^h \mid \mathcal{F}_t] - \sum_{m=1}^{n^a} \mathbb{E}^{\mathbb{P}}[Y^{(m)}(T) \mid \mathcal{F}_t] \times \theta_{t,S+S^h}^{(m),MV}\right)}{\mathbb{E}^{\mathbb{P}}[Y^{(0)}(T) \mid \mathcal{F}_t]} \\ &= \frac{\left(\mathbb{E}^{\mathbb{P}}[S \mid \mathcal{F}_t] - \sum_{m=1}^{n^a} \mathbb{E}^{\mathbb{P}}[Y^{(m)}(T) \mid \mathcal{F}_t] \times \theta_{t,S}^{(m),MV}\right)}{\mathbb{E}^{\mathbb{P}}[Y^{(0)}(T) \mid \mathcal{F}_t]} \\ &\qquad + \frac{\left(\mathbb{E}^{\mathbb{P}}[S^h \mid \mathcal{F}_t] - \sum_{m=1}^{n^a} \mathbb{E}^{\mathbb{P}}[Y^{(m)}(T) \mid \mathcal{F}_t] \times \theta_{t,S^h}^{(m),MV}\right)}{\mathbb{E}^{\mathbb{P}}[Y^{(0)}(T) \mid \mathcal{F}_t]} \\ &= \boldsymbol{\theta}_{t,S}^{(0),MV} + \boldsymbol{\theta}_{t,S^h}^{(0),MV}\end{aligned}$$

where in the second step, we use the fact that $\boldsymbol{\theta}_{t,S+S^h}^{(1:n^a),MV} = \boldsymbol{\theta}_{t,S}^{(1:n^a),MV} + \boldsymbol{\theta}_{t,S^h}^{(1:n^a),MV}$. So we get that

$$\boldsymbol{\theta}_{t,S+S^h}^{MV} = \boldsymbol{\theta}_{t,S}^{MV} + \boldsymbol{\theta}_{t,S^h}^{MV}.$$

For model-consistency, from the definition of $\boldsymbol{\theta}_{t,S}^{MV}$, we directly have that

$$\boldsymbol{\theta}_{t,S^\perp}^{MV} = \frac{\pi_t[S^\perp]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}.$$

■

This lemma shows that the mean-variance hedger is a fair hedger in our framework. Furthermore, the definition in Equation 4.14 is $\mathbb{F}$-predictable for the risky assets and $\mathbb{G}$-predictable for the risk-free asset.

4.3.3 Characterization of t-valuations

In this subsection, we present results characterizing t-valuations in terms of t-$\mathbb{J}$-hedgers. These results generalize those of Barigou et al. (2019) and Dhaene et al. (2017), and form the basis for the theorems developed in later sections. The proofs largely follow the methodology of Barigou et al. (2019), and we explicitly highlight the points at which the arguments differ from the existing proofs.

Lemma 3 *Consider a t-$\mathbb{F}$-hedger $\boldsymbol{\theta}_t$ and a t-valuation ρ_t. For any $S \in \mathcal{C}_T$, define $\boldsymbol{\mu}_t$ by*

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}, \tag{4.15}$$

then $\boldsymbol{\mu}_t$ is a t-$\mathbb{G}$-hedger.

Proof:

a) First, we check for normalization. For the 0-claim we have

$$\begin{aligned}\boldsymbol{\mu}_{t,0} &= \boldsymbol{\theta}_{t,0} + \frac{\rho_t\left[0 - \boldsymbol{\theta}_{t,0}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} \\ &= \mathbf{0}_t + \frac{\rho_t\left[0 - \mathbf{0}_t\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} \\ &= \mathbf{0}_t,\end{aligned}$$

where we use the fact that both $\boldsymbol{\theta}_t$ and ρ_t are normalized.

b) Next, we check for translation invariance. For any $S \in \mathcal{C}_T$ and any $a \in \mathcal{C}_t$ payable at T we have

$$\begin{aligned}\boldsymbol{\mu}_{t,S+a} &= \boldsymbol{\theta}_{t,S+a} + \frac{\rho_t\left[S + a - \boldsymbol{\theta}_{t,S+a}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} \\ &= \boldsymbol{\theta}_{t,S} + e^{-rT}a \times \mathbf{1}_t^{(0)} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} \\ &= \boldsymbol{\mu}_{t,S} + e^{-rT}a \times \mathbf{1}_t^{(0)}\end{aligned}$$

where we use the fact that both $\boldsymbol{\theta}_t$ and ρ_t are translation invariant.

c) For self-financing, since $\boldsymbol{\theta}_{t,S}$ is self-financing and $\mathbf{1}_t^{(0)}(u) = \mathbf{1}_t^{(0)}(u+1)$ by definition, $\boldsymbol{\mu}_{t,S}(u)\cdot\boldsymbol{Y}(u) = \boldsymbol{\mu}_{t,S}(u+1)\cdot\boldsymbol{Y}(u)$ for any $u = t+1, t+2, \ldots, T-1$.

d) We note that $\boldsymbol{\theta}_{t,S}$ and $\mathbf{1}_t^{(0)}$ are $\mathbb{F}$-predictable, and since $\mathbb{F} \subseteq \mathbb{G}$, this makes them $\mathbb{G}$-predictable. The t-valuation ρ_t is $\mathcal{G}_t$-measurable and $Y^{(0)}(t)$ is $\mathcal{F}_t$-measurable and hence also $\mathcal{G}_t$-measurable. Hence, $\boldsymbol{\mu}_{t,S}$ is $\mathbb{G}$-predictable.

We can conclude that $\boldsymbol{\mu}_t$ is a t-$\mathbb{G}$-hedger. ■

The previous lemma shows that a t-$\mathbb{G}$-hedger can be constructed from a t-$\mathbb{F}$-hedger and a t-valuation ρ_t. This result generalizes Lemma 1 in Barigou et al. (2019), where the authors construct a t-$\mathbb{G}$-hedger from a t-$\mathbb{G}$-hedger and a t-valuation ρ_t.

Lemma 4 *Consider a t-$\mathbb{F}$-hedger $\boldsymbol{\theta}_t$ and a t-valuation ρ_t. Define the t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t$ by:*

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}, \qquad \text{for any } S \in \mathcal{C}_T.$$

1. *If $\boldsymbol{\theta}_t$ is a model-consistent t-$\mathbb{F}$-hedger and ρ_t is a model-consistent t-valuation with underlying actuarial t-valuation π_t, then $\boldsymbol{\mu}_t$ is a model-consistent t-$\mathbb{G}$-hedger with underlying actuarial t-valuation π_t.*

2. *If $\boldsymbol{\theta}_t$ is a market-consistent t-$\mathbb{F}$-hedger, then $\boldsymbol{\mu}_t$ is a market-consistent t-$\mathbb{G}$-hedger. Moreover, $\boldsymbol{\mu}_{t,S^h} = \boldsymbol{\theta}_{t,S^h}$ for any t-hedgeable T-claim S^h implying that $\boldsymbol{\mu}_t$ is in that case a t-$\mathbb{F}$-hedger.*

3. *If $\boldsymbol{\theta}_t$ is a fair t-$\mathbb{F}$-hedger and ρ_t is a fair t-valuation with underlying t-actuarial valuation π_t, then $\boldsymbol{\mu}_t$ is a fair t-$\mathbb{G}$-hedger with underlying actuarial t-valuation π_t.*

Proof:

1. Suppose $\boldsymbol{\theta}_{t,S}$ is a model-consistent t-$\mathbb{F}$-hedger and has the underlying actuarial t-valuation ϕ_t. Further suppose that ρ_t is a model-consistent t-valuation with an underlying actuarial t-valuation π_t. For $S^{\perp} \in \mathcal{C}_T^{t,\perp}$, we have

$$\begin{aligned}\boldsymbol{\mu}_{t,S^{\perp}} &= \boldsymbol{\theta}_{t,S^{\perp}} + \frac{\rho_t\left[S^{\perp} - \boldsymbol{\theta}_{t,S^{\perp}}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} \\ &= \frac{\phi_t[S^{\perp}]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} + \frac{\left(\rho_t\left[S^{\perp}\right] - \phi_t[S^{\perp}]\right)}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} \\ &= \frac{\pi_t[S^{\perp}]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)},\end{aligned}$$

   where in the second step we use the fact that $\boldsymbol{\theta}$ is model consistent and that ρ_t is translation invariant; in the final step we use the fact that ρ_t is model consistent with underlying actuarial valuation π_t.

2. Suppose $\boldsymbol{\theta}_{t,S}$ is a market-consistent t-$\mathbb{F}$-hedger and ρ_t is a market-consistent t-valuation. For $S \in \mathcal{C}_T$ and $S^h \in \mathcal{C}_T^{t,h}$ we have:

$$\boldsymbol{\mu}_{t,S+S^h} = \boldsymbol{\theta}_{t,S+S^h} + \frac{\rho_t\left[S + S^h - \boldsymbol{\theta}_{t,S+S^h}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$

   since $S^h = \boldsymbol{\theta}_{t,S^h}(T)\cdot\boldsymbol{Y}(T)$ and since $\boldsymbol{\theta}_{t,S}$ is market-consistent,

$$= \boldsymbol{\theta}_{t,S} + \boldsymbol{\theta}_{t,S^h} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$

   since $\rho_t\left[S^h - \boldsymbol{\theta}_{t,S^h}(T)\cdot\boldsymbol{Y}(T)\right] \times \mathbf{1}_t^{(0)} = 0$,

$$= \boldsymbol{\mu}_{t,S} + \boldsymbol{\mu}_{t,S^h},$$

   It is clear that $\boldsymbol{\mu}_{t,S^h} = \boldsymbol{\theta}_{t,S^h}$.

3. Using the results in 1. and 2., we have that if $\boldsymbol{\theta}_{t,S}$ is a fair t-$\mathbb{F}$-hedger and ρ_t is a fair t-valuation, then $\boldsymbol{\mu}_{t,S}$ is both model consistent and market consistent, which means that $\boldsymbol{\mu}_{t,S}$ is fair.

■

As a result, we have that a t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_{t,S}$ defined in Equation 4.15 is fair if the t-$\mathbb{F}$-hedger $\boldsymbol{\theta}_{t,S}$ used to construct it is fair.

Lemma 5

1. *Consider a model-consistent t-valuation $\rho_t^{MOC} : \mathcal{C}_T \to \mathcal{C}_t$. The t-valuation ρ_t^{MOC} is model consistent if and only if there exists a model-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MOC}$ such that:*

$$\rho_t^{MOC}[S] = \boldsymbol{\mu}_{t,S}^{MOC}(t+1)\cdot\boldsymbol{Y}(t), \quad \textit{for any } S \in \mathcal{C}_T.$$

2. *Consider a market-consistent t-valuation $\rho_t^{MAC} : \mathcal{C}_T \to \mathcal{C}_t$. The t-valuation ρ_t^{MAC} is market-consistent if and only if there exists a market-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MAC}$ such that:*

$$\rho_t^{MAC}[S] = \boldsymbol{\mu}_{t,S}^{MAC}(t+1)\cdot\boldsymbol{Y}(t), \quad \textit{for any } S \in \mathcal{C}_T.$$

3. *Consider a fair t-valuation* $\rho_t^{fair} : \mathcal{C}_T \to \mathcal{C}_t$. *The t-valuation* ρ_t^{fair} *is fair if and only if there exists a fair t-*$\mathbb{G}$*-hedger* $\boldsymbol{\mu}_t^{fair}$ *such that:*

$$\rho_t^{fair}[S] = \boldsymbol{\mu}_{t,S}^{fair}(t+1) \cdot \boldsymbol{Y}(t), \quad \textit{for any } S \in \mathcal{C}_T.$$

Proof:

1. a) If ρ_t^{MOC} is a model-consistent t-valuation then there exists a model-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MOC}$ such that
$$\rho_t^{MOC}[S] = \boldsymbol{\mu}_{t,S}^{MOC}(t+1) \cdot \boldsymbol{Y}(t).$$
For any model-consistent t-valuation, we have: $\rho_t^{MOC}\left[S^{\perp}\right] = \pi_t[S^{\perp}]$. We may write
$$\begin{aligned}\rho_t^{MOC}[S] &= \frac{\rho_t^{MOC}[S]}{Y^{(0)}(t)} \times Y^{(0)}(t) \\ &= \frac{\rho_t^{MOC}[S]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}(t+1) \times \boldsymbol{Y}(t) \\ &= \boldsymbol{\mu}_{t,S}^{MOC}(t+1) \cdot \boldsymbol{Y}(t),\end{aligned}$$
where $\boldsymbol{\mu}_{t,S}^{MOC} = \frac{\rho_t^{MOC}[S]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$. $\boldsymbol{\mu}_t^{MOC}$ defined this way is indeed a model-consistent t-$\mathbb{G}$-hedger because by the model-consistency of ρ_t^{MOC}, for any $S^{\perp} \in \mathcal{C}_T^{t,\perp}$ we have
$$\boldsymbol{\mu}_{t,S^{\perp}}^{MOC} = \frac{\rho_t^{MOC}[S^{\perp}]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)} = \frac{\pi_t[S^{\perp}]}{Y^{(0)}(t)} \times \mathbf{1}^{(0)}.$$

   b) If $\boldsymbol{\mu}_t^{MOC}$ is a model-consistent t-$\mathbb{G}$-hedger then the t-valuation ρ_t^{MOC} given by
$$\rho_t^{MOC}[S] = \boldsymbol{\mu}_{t,S}^{MOC}(t+1) \cdot \boldsymbol{Y}(t),$$
is also model consistent. This is because, since $\boldsymbol{\mu}_t^{MOC}$ is model consistent, for any $S^{\perp} \in \mathcal{C}_T^{t,\perp}$ we have
$$\begin{aligned}\rho_t^{MOC}[S^{\perp}] &= \boldsymbol{\mu}_{t,S^{\perp}}^{MOC}(t+1) \cdot \boldsymbol{Y}(t) \\ &= \pi_t[S^{\perp}].\end{aligned}$$

2. a) If ρ_t^{MAC} is a market-consistent t-valuation then there exists a market-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MAC}$ such that
$$\rho_t^{MAC}[S] = \boldsymbol{\mu}_{t,S}^{MAC}(t+1) \cdot \boldsymbol{Y}(t).$$
For any T-claim S, let S^h denote the hedgeable part of S. In other words, $S^h = \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)$, where $\boldsymbol{\theta}_t$ is a market-consistent t-$\mathbb{F}$-hedger. Since ρ_t^{MAC} is market consistent, for any $S \in \mathcal{C}_T$ and any $S^h \in \mathcal{C}_T^{t,h}$, we have
$$\begin{aligned}\rho_t^{MAC}[S] &= \rho_t^{MAC}\left[S - S^h + S^h\right] \\ &= \rho_t^{MAC}\left[S - S^h\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S^h \mid \mathcal{F}_t\right] \\ &= \frac{\rho_t^{MAC}\left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}(t+1) \cdot \boldsymbol{Y}(t) + \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) \\ &= \boldsymbol{\mu}_{t,S}^{MAC}(t+1) \cdot \boldsymbol{Y}(t),\end{aligned}$$
where in the second step, we use the fact that ρ^{MAC} is market consistent and we define $\boldsymbol{\mu}_{t,S}^{MAC} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t^{MAC}[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$. By Lemmas 3 and 4, we have that $\boldsymbol{\mu}_t^{MAC}$ is a market-consistent t-$\mathbb{G}$-hedger.

b) If $\boldsymbol{\mu}_t^{MAC}$ is a market-consistent t-$\mathbb{G}$-hedger then the t-valuation ρ_t^{MAC} given by

$$\rho_t^{MAC}[S] = \boldsymbol{\mu}_{t,S}^{MAC}(t+1) \cdot \boldsymbol{Y}(t),$$

is also market consistent. This is because, since $\boldsymbol{\mu}_t^{MAC}$ is market consistent, we have: $\boldsymbol{\mu}_{t,S+S^h}^{MAC} = \boldsymbol{\mu}_{t,S}^{MAC} + \boldsymbol{\mu}_{t,S^h}^{MAC} = \boldsymbol{\mu}_{t,S}^{MAC} + \boldsymbol{\theta}_{t,S^h}^{MAC}$, where we invoke the second part of Lemma 4. So for any $S \in \mathcal{C}_T$ and any $S^h \in \mathcal{C}_T^{t,h}$, we get

$$\begin{aligned}\rho_t^{MAC}[S+S^h] &= \boldsymbol{\mu}_{t,S+S^h}^{MAC}(t+1) \cdot \boldsymbol{Y}(t) \\ &= \left(\boldsymbol{\mu}_{t,S}^{MAC}(t+1) + \boldsymbol{\theta}_{t,S^h}^{MAC}(t+1)\right) \cdot \boldsymbol{Y}(t) \\ &= \rho_t^{MAC}[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} S^h \mid \mathcal{F}_t\right].\end{aligned}$$

3. a) If ρ_t^{fair} is a fair t-valuation then there exists a fair t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{fair}$ such that

$$\rho_t^{fair}[S] = \boldsymbol{\mu}_{t,S}^{fair}(t+1) \cdot \boldsymbol{Y}(t).$$

Take ρ_t^{fair} to be a fair t-valuation. Consider a fair t-$\mathbb{F}$-hedger $\boldsymbol{\theta}_t$ with underlying actuarial valuation φ_t. From 2.(a) and Lemma 4, we can write

$$\rho_t^{fair}[S] = \boldsymbol{\mu}_{t,S}^{MAC}(t+1) \cdot \boldsymbol{Y}(t),$$

with $\boldsymbol{\mu}_{t,S}^{MAC} = \boldsymbol{\theta}_{t,S} + \frac{\pi_t[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$. From the third part of Lemma 4 and since we assumed $\boldsymbol{\theta}_t$ is a fair t-$\mathbb{F}$-hedger, we have that $\boldsymbol{\mu}_t^{MAC}$ is also fair. So, if ρ_t^{fair} is a fair t-valuation, then $\boldsymbol{\mu}_{t,S}^{MAC}$ is not only market consistent but also model consistent, and therefore $\boldsymbol{\mu}_{t,S}^{MAC}$ is fair.

b) If $\boldsymbol{\mu}_t^{fair}$ is a fair t-$\mathbb{G}$-hedger then the t-valuation ρ_t^{fair} defined as

$$\rho_t^{fair}[S] = \boldsymbol{\mu}_{t,S}^{fair}(t+1) \cdot \boldsymbol{Y}(t),$$

is also fair. If $\boldsymbol{\mu}_t^{fair}$ is a fair t-$\mathbb{G}$-hedger, from 1. and 2., we can conclude that the t-valuation ρ_t is model consistent and market consistent, which means that it is fair.

■

As a result, we have that a fair t-valuation of a T-claim can be expressed as a fair t-$\mathbb{G}$-hedger for S as defined in Equation 4.15.

Lemma 6 *For a t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t \in \boldsymbol{\Theta}_t^{\mathbb{G}'}$ and a t-valuation ρ_t, we obtain the following results.*

1. *Consider a model-consistent t-valuation ρ_t^{MOC} of any claim $S \in \mathcal{C}_T$. ρ_t^{MOC} can be written in terms of a model-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MOC}$ such that:*

$$\rho_t^{MOC}[S] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}^{MOC}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{G}_t\right]$$

2. *Consider a market-consistent t-valuation ρ_t^{MAC} of any claim $S \in \mathcal{C}_T$. ρ_t^{MAC} can be written in terms of a market-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MAC}$ such that:*

$$\rho_t^{MAC}[S] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}^{MAC}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{G}_t\right]$$

3. *Consider a fair t-valuation ρ_t^{fair} of any claim $S \in \mathcal{C}_T$. ρ_t^{fair} can be written in terms of a fair t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{fair}$ such that:*

$$\rho_t^{fair}[S] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}^{fair}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{G}_t\right]$$

Proof:

1. Since ρ_t^{MOC} is a model-consistent valuation by the first part of Lemma 5, there exists a model-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t^{MOC}$ such that

$$\rho_t^{MOC}[S] = \boldsymbol{\mu}_{t,S}^{MOC}(t+1)\cdot\boldsymbol{Y}(t), \quad \text{for any } S \in \mathcal{C}_T.$$

   By the immersion property for $\mathbb{F}$-martingales and the $\mathbb{G}$-predictability of $\boldsymbol{\mu}_t^{MOC}$ the product $\boldsymbol{\mu}_t^{MOC}\cdot\boldsymbol{Y}$ is a $\mathbb{G}$-martingale. Since $\boldsymbol{\mu}_t^{MOC}$ is also a self-financing trading strategy, we have

$$\boldsymbol{\mu}_{t,S}^{MOC}(t+1)\cdot\boldsymbol{Y}(t) = \boldsymbol{\mu}_{t,S}^{MOC}(t)\cdot\boldsymbol{Y}(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}^{MOC}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{G}_t\right].$$

   Combining these, we get

$$\rho_t^{MOC}[S] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}^{MOC}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{G}_t\right].$$

2. Replacing ρ_t^{MOC} and $\boldsymbol{\mu}_{t,S}^{MOC}$ in the proof of 1. with a market-consistent valuation ρ_t^{MAC} and a market-consistent t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_{t,S}^{MAC}$ respectively, we obtain the same result. Note we apply the second part of Lemma 5 instead.

3. Replacing ρ_t^{MOC} and $\boldsymbol{\mu}_{t,S}^{MOC}$ in the proof of 1. with a fair valuation ρ_t^{fair} and a fair t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_{t,S}^{fair}$ respectively, we obtain the same result. Note we apply the third part of Lemma 5 instead.

■

As a result, the fair t-valuation of a T-claim can be expressed as a conditional expectation with respect to $\mathcal{G}_t$, under an EMM $\mathbb{Q}$, of a fair t-$\mathbb{G}$-hedger for S. This is a powerful result, as it shows that any hybrid T-claim can be transformed into the terminal value of a fair $\mathbb{G}$-trading strategy. This terminal value is replicable within the insurer-relevant filtration $\mathbb{G}$, which can then be valued using discounted risk-neutral expectation conditioned on $\mathcal{G}_t$ under an EMM. Importantly, since the fair t-valuation ρ_t is model consistent, actuarial judgment is implicitly incorporated in the transformation from a hybrid claim to a hedgeable claim.

4.4 Dynamic valuations and hedgers

In this section, we extend the ideas of t-valuations and t-hedgers to a multi-period setting involving filtrations. Previously, we introduced t-valuations and t-hedgers calculated only at a certain point $t \in \{0, 1, \ldots, T-1\}$ for a T-claim. Since insurers value their liabilities over multiple time periods, it is important that these valuations satisfy an appropriate notion of time-consistency. In this section, we link the valuations and hedgers through strong time-consistency. The results in this section are extensions of the results presented by Barigou et al. (2019) to our generalized setting with t-$\mathbb{J}$-hedgers.

4.4.1 Dynamic valuations

Definition 23 ***(Dynamic valuations).*** *A dynamic valuation is a sequence $\{\rho_t\}_{t=0}^{T-1}$ where for each $t = 0, 1, \ldots, T-1$, ρ_t is a t-valuation.*

Since t-valuations are $\mathcal{G}_t$-measurable, we have that the dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ is $\mathbb{G}$-adapted.

To define a 'fair' dynamic valuation, we begin by defining model-consistency, market-consistency, and strong time-consistency. Strong time-consistency allows us to link t-valuations together to create a dynamic valuation.

Definition 24 ***(Market-, model-, and strong time-consistency for dynamic valuations).*** *A dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ is:*

- *model consistent if any t-valuation ρ_t is model consistent for $t = 0, 1, \ldots, T-1$.*
- *market consistent if any t-valuation ρ_t is market consistent for $t = 0, 1, \ldots, T-1$.*
- *strong time-consistent if all t-valuations are connected in the following way:*

$$\rho_t[S] = \rho_t\left[\rho_{t+1}[S]\right]$$

for any $S \in \mathcal{C}_T$ and $t = 0, 1, \ldots, T-2$.

Definition 25 ***(Strong-fair dynamic valuations).*** *A dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ is a strong-fair dynamic valuation if it is model consistent, market consistent, and strong time-consistent. This definition is analogous to that in Barigou et al. (2019).*

Note that the definition of strong time-consistency is the same as that used by Barigou et al. (2019). For more details on this type of time-consistency, see the references therein. Some examples of strong-fair dynamic valuations are presented in Section 5 of Barigou et al. (2019), and Section 3 of Barigou et al. (2022). An example of a dynamic valuation that is model consistent and market consistent but not strong time consistent is presented in Section 4 of Barigou and Dhaene (2019).

4.4.2 Dynamic hedgers

Similarly, we define dynamic hedgers and the corresponding notions of model-consistency, market-consistency, and strong time-consistency.

Definition 26 ***(Dynamic hedgers).*** *Let $\mathbb{J} \in \left\{\mathbb{F}^{\boldsymbol{Y}}, \mathbb{F}, \mathbb{G}\right\}$. A dynamic $\mathbb{J}$-hedger is a sequence $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ where for each $t = 0, 1, \ldots, T-1$, $\boldsymbol{\theta}_t$ is a t-$\mathbb{J}$-hedger.*

Definition 27 ***(Market-, model-, and strong time-consistency for dynamic hedgers).*** *A dynamic $\mathbb{J}$-hedger $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is:*

- *model consistent if any t-$\mathbb{J}$-hedger $\boldsymbol{\theta}_t$ is model consistent.*
- *market consistent if any t-$\mathbb{J}$-hedger $\boldsymbol{\theta}_t$ is market consistent.*
- *strong time consistent if all t-$\mathbb{J}$-hedgers $\boldsymbol{\theta}_t$ are connected in the following way:*

$$\boldsymbol{\theta}_{t,S}(t+1) = \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1),$$

for any $S \in \mathcal{C}_T$ and $t = 0, 1, \ldots, T-2$, where $\rho_{t+1}[S]$ is the time-$(t+1)$ value of the hedge $\boldsymbol{\theta}_{t+1}$

$$\rho_{t+1}[S] = \boldsymbol{\theta}_{t+1,S}(t+2) \cdot \boldsymbol{Y}(t+1).$$

This definition is more general than the one considered in Barigou et al. (2019) as we make a distinction based on the predictability of the hedger $\boldsymbol{\theta}_t$.

Definition 28 ***(Strong-fair dynamic hedgers).*** *A dynamic $\mathbb{J}$-hedger $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is a 'strong-fair dynamic hedger' if it is model consistent, market consistent, and strong time-consistent.*

For examples of dynamic hedgers that are model consistent and market consistent, see Barigou and Dhaene (2019) and Barigou et al. (2019).

4.4.3 Characterizations of dynamic valuations

In this subsection, we present a lemma characterizing dynamic valuations in terms of dynamic $\mathbb{J}$-hedgers, generalizing Theorem 2 of Barigou et al. (2019) to our richer filtration structure. The main differences relative to Barigou et al. (2019) are: (i) our framework involves three nested filtrations whereas Barigou et al. (2019) use only a single filtration; (ii) the definition of hedgers is different in our setup, where the risk-free asset is $\mathbb{G}$-predictable and the risky assets are $\mathbb{F}$-predictable to enforce the no-insider-trading requirements; (iii) the classes of hedgeable and orthogonal claims differ in our setting, which changes the domain over which market- and model-consistency must be verified. This lemma forms the basis for the results in later sections.

Theorem 1 *A dynamic valuation* $\{\rho_t\}_{t=0}^{T-1}$ *is strong-fair if and only if there exists a strong-fair dynamic* $\mathbb{G}$*-hedger* $\{\boldsymbol{\mu}_t\}_{t=0}^{T-1}$ *such that*

$$\rho_t[S] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \text{ , for any } S\in\mathcal{C}_T$$

Proof:

1. Suppose we have a dynamic strong-fair valuation $(\rho_t)_{t=0}^{T-1}$ with π_t as the underlying actuarial t-valuation according to Lemma 4. From Lemma 5, we have that for any $t=0,1,\ldots,T-1$ there exists a fair t-$\mathbb{G}$-hedger $\boldsymbol{\eta}_t$ such that:

$$\rho_t[S] = \boldsymbol{\eta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t), \quad \text{for any } S\in\mathcal{C}_T.$$

The dynamic $\mathbb{G}$-hedger $(\boldsymbol{\eta}_t)_{t=1}^{T-1}$ is both model consistent and market consistent, but it is not necessarily strong time consistent. We construct another dynamic $\mathbb{G}$-hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ which is model consistent, market consistent, and strong time consistent. First, we set $\boldsymbol{\mu}_{T-1}=\boldsymbol{\eta}_{T-1}$. By construction, $\boldsymbol{\mu}_{T-1}$ is a fair $(T-1)$-$\mathbb{G}$-hedger and it holds that

$$\rho_{T-1}[S] = \boldsymbol{\eta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T) = \boldsymbol{\mu}_{T-1,S}(T)\cdot\boldsymbol{Y}(T), \quad \text{for any } S\in\mathcal{C}_T.$$

We then define the t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_{t,S}=\boldsymbol{\eta}_{t,\rho_{t+1}[S]}$ for $S\in\mathcal{C}_T$ for $t=0,1,\ldots,T-2$, where

$$\rho_{t+1}[S] = \boldsymbol{\eta}_{t+1,S}(t+2)\cdot\boldsymbol{Y}(t+1), \quad \text{for any } S\in\mathcal{C}_T.$$

Now, we prove that $\boldsymbol{\mu}_t$ for $t=0,1,\ldots,T-2$ is fair:

(a) For model-consistency with π_t as underlying actuarial valuation, we need to prove $\boldsymbol{\mu}_{t,S^\perp} = \frac{\pi_t[S^\perp]}{Y^{(0)}(t)}\times\mathbf{1}_t^{(0)}$ for any t-orthogonal T-claim $S^\perp$. We have

$$\begin{aligned}\boldsymbol{\mu}_{t,S^\perp} &= \boldsymbol{\eta}_{t,\rho_{t+1}[S^\perp]}\\ &= \boldsymbol{\eta}_{t,\pi_{t+1}[S^\perp]}\\ &= \frac{\pi_t[\pi_{t+1}[S^\perp]]}{Y^{(0)}(t)}\times\mathbf{1}_t^{(0)}\\ &= \frac{\pi_t[S^\perp]}{Y^{(0)}(t)}\times\mathbf{1}_t^{(0)},\end{aligned}$$

where in the first step, we have used the fact that $\boldsymbol{\eta}_t$ is itself model consistent and we have that $\pi_t[S^\perp]$ is itself a t-orthogonal $(t+1)$-claim because an actuarial valuation is $\mathbb{P}$-independent of time-t and future asset-prices and the non-traded systematic risk processes; for the final step, since ρ_t is strong time consistent, we have that $\pi_t\left[\pi_{t+1}[S^\perp]\right]=\pi_t\left[S^\perp\right]$

(b) For market-consistency, we need $\boldsymbol{\mu}_{t,S+S^h} = \boldsymbol{\mu}_{t,S} + \boldsymbol{\mu}_{t,S^h}$ for any T-claim S and t-hedgeable T-claim S^h. We have:

$$\begin{aligned}\boldsymbol{\mu}_{t,S+S^h} &= \boldsymbol{\eta}_{t,\rho_{t+1}[S+S^h]} \\ &= \boldsymbol{\eta}_{t,\rho_{t+1}[S]+\rho_{t+1}[S^h]} \\ &= \boldsymbol{\eta}_{t,\rho_{t+1}[S]} + \boldsymbol{\eta}_{t,\rho_{t+1}[S^h]} \\ &= \boldsymbol{\mu}_{t,S} + \boldsymbol{\mu}_{t,S^h},\end{aligned}$$

where in the second step, we use the fact that ρ_t is market consistent; in the penultimate step we have that $\boldsymbol{\eta}_t$ is market consistent and $\rho_{t+1}[S^h]$ is itself a t-hedgeable $(t+1)$-claim.

(c) For strong time-consistency, we obtain for any T-claim $S \in \mathcal{C}_T$

$$\begin{aligned}\boldsymbol{\mu}_{t,S}(t+1) &= \boldsymbol{\eta}_{t,\rho_{t+1}[S]}(t+1) \\ &= \boldsymbol{\eta}_{t,\rho_{t+1}[\rho_{t+1}[S]]}(t+1) \\ &= \boldsymbol{\mu}_{t,\rho_{t+1}[S]}(t+1),\end{aligned}$$

where in the second step we use the fact that $\rho_{t+1}[S] \in \mathcal{C}_{t+1}$ and also that ρ_{t+1} is translation invariant.

Finally, we must show the stated expression for ρ_t holds. To this end, we derive

$$\begin{aligned}\rho_t[S] &= \rho_t[\rho_{t+1}[S]] \\ &= \boldsymbol{\eta}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t) \\ &= \boldsymbol{\mu}_{t,S}(t+1) \cdot \boldsymbol{Y}(t),\end{aligned}$$

where in the first step we use the fact that ρ_t is strong time consistent.

Hence, we can conclude that $\{\boldsymbol{\mu}_t\}_{t=0}^{T-1}$ is a fair dynamic $\mathbb{G}$-hedger for $\{\rho_t\}_{t=0}^{T-1}$.

2. Suppose that there exists a fair dynamic $\mathbb{G}$-hedger $(\boldsymbol{\mu}_t)_{t=0}^{T-1}$ such that

$$\rho_t[S] = \boldsymbol{\mu}_{t,S}(t+1) \cdot \boldsymbol{Y}(t), \quad \text{for any } S \in \mathcal{C}_T.$$

From Lemma 5, we know that the t-valuation ρ_t is fair for all $t \in \{0, \ldots, T\}$. Moreover, we find

$$\begin{aligned}\rho_t[\rho_{t+1}[S]] &= \boldsymbol{\mu}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t) \\ &= \boldsymbol{\mu}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) \\ &= \rho_t[S],\end{aligned}$$

where in the second step we use the fact that $(\boldsymbol{\mu}_t)_{t=1}^{T-1}$ is strong time consistent. This proves that ρ_t is also strong time consistent.

This concludes the proof. ∎

The forward implication of the previous lemma shows that any strong-fair dynamic valuation admits a dynamic hedging representation, wherein the hedger itself is a strong-fair hedger. This means that the dynamic valuation process can be equivalently generated by a self-financing trading strategy. The backward implication of the previous lemma establishes that the dynamic valuation induced by a strong-fair dynamic hedger is necessarily strong-fair itself.

Using Theorem 1 we can obtain the following corollaries.

Corollary 2 *A dynamic valuation* $\{\rho_t\}_{t=0}^{T-1}$ *is fair if and only if there exists a fair dynamic* $\mathbb{G}$*-hedger* $\{\boldsymbol{\mu}_t\}_{t=0}^{T-1}$ *such that for any* $S \in \mathcal{C}_T$*:*

$$\rho_t[S] = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{G}_t\right] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t).$$

Proof:
The proof is clear from Lemma 6. From the definition of a strong-fair dynamic valuation and strong-fair dynamic hedgers, all t-valuations $\{\rho_t\}_{t=0}^{T-1}$ and all t-$\mathbb{G}$-hedgers $\boldsymbol{\mu}_t$ must be fair individually. ■

The previous corollary strengthens Theorem 1 by combining it with Lemma 6. In particular, this corollary shows that when valuation and hedging are based on the full insurer-relevant filtration, the time-t strong-fair value of a claim can be represented as the discounted risk-neutral expectation applied to terminal value of a dynamic strong-fair $\mathbb{G}$-hedger.

The following corollary extends the result of Lemma 4 by showing that the t-$\mathbb{G}$-hedger sequence obtained by repeating Equation 4.15 for $t = 1, \ldots, T-1$ is also strong time consistent, thereby making it a strong-fair hedger.

Corollary 3 *Consider a fair dynamic* $\mathbb{F}$*-hedger* $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ *and a fair dynamic valuation* $\{\rho_t\}_{t=0}^{T-1}$ *for a claim* $S \in \mathcal{C}_T$*. Then the dynamic* $\mathbb{G}$*-hedger* $\{\boldsymbol{\eta}_t\}_{t=0}^{T-1}$ *defined by*

$$\boldsymbol{\eta}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}, \qquad t = 0, 1, \ldots, T-1$$

is not only model consistent and market consistent but also strong time consistent and hence fair, and

$$\rho_t[S] = \boldsymbol{\eta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t).$$

Proof:
Consider a fair dynamic $\mathbb{F}$-hedger $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$. From Lemma 4, we know that there exists a model-consistent and market-consistent sequence of t-$\mathbb{G}$-hedgers $\{\boldsymbol{\eta}_t\}_{t=0}^{T-1}$ of the form

$$\boldsymbol{\eta}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)},$$

where ρ_t is a fair t-valuation with underlying t-actuarial valuation π_t. Lemma 4 also states that the sequence of t-$\mathbb{G}$-hedgers $\{\boldsymbol{\eta}_t\}_{t=0}^{T-1}$ has the underlying actuarial valuation π_t.

Since ρ_t is market consistent and because $\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)$ is a hedgeable claim, we have

$$\boldsymbol{\eta}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S\right] - \rho_t\left[\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$

$$\boldsymbol{\eta}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$

We now show that the sequence $\{\boldsymbol{\eta}_t\}_{t=0}^{T-1}$ is also strong time consistent and hence strong-fair. We have that

$$\boldsymbol{\eta}_{t,\rho_{t+1}[S]} = \boldsymbol{\theta}_{t,\rho_{t+1}[S]} + \frac{\rho_t\left[\rho_{t+1}[S]\right] - \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t)}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$

since $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ and $\{\rho_t\}_{t=0}^{T-1}$ are both strong time consistent,

$$= \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S\right] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$

since ρ_t is market consistent and since $\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)$ is a hedgeable claim,

$$= \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}$$
$$= \boldsymbol{\eta}_{t,S}.$$

■

The fair dynamic hedger $\{\boldsymbol{\eta}_t\}_{t=0}^{T-1}$ in the previous corollary is a strong-fair dynamic $\mathbb{G}$-hedger. From Theorem 1, we know that there exists a strong-fair dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ such that

$$\rho_t[S] = \boldsymbol{\eta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t).$$

The previous corollary shows that the dynamic $\mathbb{G}$-hedger constructed using Equation 4.15 from a strong-fair dynamic $\mathbb{F}$-hedger and a strong fair dynamic valuation is itself strong-fair.

4.5 Conditional valuations in a multi-period setting

In this section, we introduce $\mathcal{F}_u$-conditional and $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional valuations, extending the single-period framework of Dhaene et al. (2017) and the continuous-time framework of Pelsser and Stadje (2014) to a discrete multi-period setting. Conditioning is taken with respect to the financial filtration and the asset-price filtration, respectively, at the terminal time.

Denote the space of all $\mathcal{F}_u$-measurable random variables by $\mathcal{C}_u^{\mathcal{F}}$ and the space of all $\mathcal{F}_u^{\boldsymbol{Y}}$-measurable random variables by $\mathcal{C}_u^{\mathcal{F}^{\boldsymbol{Y}}}$ for any $u \in \{1, 2, \ldots, T\}$. We have that $\mathcal{C}_u^{\mathcal{F}^{\boldsymbol{Y}}} \subseteq \mathcal{C}_u^{\mathcal{F}} \subseteq \mathcal{C}_u$. These conditional valuations are then used to define three-step hedge-based valuations and two-step market evaluations in further sections.

Example 4.5.1 ***(Claims in $\mathcal{C}_T^{\mathcal{F}}$ and $\mathcal{C}_T^{\mathcal{F}^{\boldsymbol{Y}}}$).***

- *Consider the claim*
$$S^{(i)} = \max\left(Y^{(1)}(T), K\right),$$
where $K \in \mathbb{R}_+$ represents a guaranteed minimum maturity benefit paid at time-T, independently of the survival of the policyholder. This payoff provides downside protection on the fund investment $Y^{(1)}$ and depends exclusively on traded asset prices. Hence, $S^{(i)} \in \mathcal{C}_T^{\mathcal{F}^{\boldsymbol{Y}}}$.

- *Consider the claim*
$$S^{(ii)} = \mathbf{1}_{\{Z^{(1)}(T)>K\}} \cdot L,$$
where $K, L \in \mathbb{R}_+$. This payoff represents a payment triggered by an extreme mortality event, as measured by the systematic mortality index $Z^{(1)}(T)$. Such a claim is an example of a mortality catastrophe bond. Since the payoff depends on non-traded systematic risk, we have $S^{(ii)} \in \mathcal{C}_T^{\mathcal{F}}$.

We first define both $\mathcal{F}_u$-conditional and $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional t-valuations and then explain the intuition underlying each.

Definition 29 ***($\mathcal{F}_u$-conditional t-valuations).*** *For any $t < u \in \{1, 2, \dots, T\}$ an $\mathcal{F}_u$-conditional t-valuation is a mapping $\pi_t[\cdot \mid \mathcal{F}_u] : \mathcal{C}_u \to \mathcal{C}_u^{\mathcal{F}}$, attaching the claim $\pi_t[S \mid \mathcal{F}_u] \in \mathcal{C}_u^{\mathcal{F}}$ to a hybrid $\mathcal{G}_u$-measurable u-claim $S \in \mathcal{C}_u$ such that:*

- *$\pi_t[\cdot \mid \mathcal{F}_u]$ is normalized:*
$$\pi_t[0 \mid \mathcal{F}_u] = 0$$
- *$\pi_t[\cdot \mid \mathcal{F}_u]$ is conditionally translation invariant:*
$$\pi_t[S + S^h \mid \mathcal{F}_u] = \pi_t[S \mid \mathcal{F}_u] + S^h,$$
for any $S \in \mathcal{C}_u$ and any $S^h \in \mathcal{C}_u^{t,h}$.

Definition 30 ***($\mathcal{F}_u^{\boldsymbol{Y}}$-conditional t-valuations).*** *For any $t < u \in \{1, 2, \dots, T\}$ an $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional t-valuation is a mapping $\pi_t\left[\cdot \mid \mathcal{F}_u^{\boldsymbol{Y}}\right] : \mathcal{C}_u \to \mathcal{C}_u^{\mathcal{F}^{\boldsymbol{Y}}}$, attaching the claim $\pi_t[S \mid \mathcal{F}_u^{\boldsymbol{Y}}] \in \mathcal{C}_u^{\mathcal{F}^{\boldsymbol{Y}}}$ to a hybrid $\mathcal{G}_u$-measurable u-claim $S \in \mathcal{C}_u$ such that:*

- *$\pi_t[\cdot \mid \mathcal{F}_u^{\boldsymbol{Y}}]$ is normalized:*
$$\pi_t[0 \mid \mathcal{F}_u^{\boldsymbol{Y}}] = 0$$
- *$\pi_t[\cdot \mid \mathcal{F}_u^{\boldsymbol{Y}}]$ is conditionally translation invariant for $t < u$:*
$$\mathbb{E}^{\mathbb{Q}}\left[\pi_t[S + S^h \mid \mathcal{F}_u^{\boldsymbol{Y}}] \mid \mathcal{G}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\pi_t[S \mid \mathcal{F}_u^{\boldsymbol{Y}}]\right] + \mathbb{E}^{\mathbb{Q}}\left[S^h \mid \mathcal{G}_t\right],$$
for any $S \in \mathcal{C}_u$ and any $S^h \in \mathcal{C}_u^{t,h}$.

Normalization ensures that a zero payoff has a zero conditional t-valuation. Conditional translation invariance for $\mathcal{F}_u$-conditional t-valuations reflects the principle that the hedgeable component of a u-claim $S \in \mathcal{C}_u$ is left unchanged by the conditional valuation operator. Indeed, hedgeable claims are terminal payoffs of self-financing strategies based on $\mathcal{F}$-information and are therefore $\mathcal{F}_u$-measurable. As a result, they already belong to $\mathcal{C}_u^{\mathcal{F}}$, and so the $\mathcal{F}_u$-conditional t-valuation acts as the identity for hedgeable claims.

For $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional t-valuations, conditional translation invariance takes a modified form. Since the operator $\pi_t[\cdot \mid \mathcal{F}_u^{\boldsymbol{Y}}]$ maps t-hedgeable u-claims $S^h \in \mathcal{C}_u^{t,h}$ to $\mathcal{C}_u^{\mathcal{F}^{\boldsymbol{Y}}}$, the standard conditional translation invariance requirement is reformulated. Adding a t-hedgeable u-claim S^h to a hybrid u-claim S shifts the $\mathbb{Q}$-expectation of $\pi_t[\cdot \mid \mathcal{F}_u^{\boldsymbol{Y}}]$ by precisely $\mathbb{E}^{\mathbb{Q}}[S^h \mid \mathcal{G}_t]$, leaving the contribution of S through the conditional valuation otherwise unchanged.

An $\mathcal{F}_u$-conditional ($\mathcal{F}_u^{\boldsymbol{Y}}$-conditional) t-valuation is a mapping from the set of claims defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ to the set of claims defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ (respectively, $(\Omega, \mathcal{F}^{\boldsymbol{Y}}, \mathbb{F}^{\boldsymbol{Y}}, \mathbb{P})$). In particular, it maps a $\mathcal{G}_u$-measurable random variable to an $\mathcal{F}_u$-measurable (respectively, $\mathcal{F}_u^{\boldsymbol{Y}}$-measurable) random variable, thereby representing the original claim using only the randomness contained in the smaller filtration $\mathcal{F}_u \subseteq \mathcal{G}_u$ (respectively, $\mathcal{F}_u^{\boldsymbol{Y}} \subseteq \mathcal{F}_u \subseteq \mathcal{G}_u$). Importantly, at time-$t$, a conditional t-valuation is itself a random variable that becomes observable only at time-u. However, its realization depends only on the information contained in the conditional filtration and no information outside of it.

Example 4.5.2 ***($\mathcal{F}_u$-conditional t-valuation).*** *For $t < u \in \{1, 2, \dots, T\}$, an example of a $\mathcal{F}_u$-conditional t-valuation is (Pelsser & Stadje, 2014):*

$$\pi_t[S \mid \mathcal{F}_u] = \mathbb{E}^{\mathbb{P}}[S \mid \mathcal{F}_u] + \beta\sqrt{Var^{\mathbb{P}}[S \mid \mathcal{F}_u]} \qquad \text{for any } S \in \mathcal{C}_u,$$

where β is any non-negative real number.

Example 4.5.2 illustrates a simple $\mathcal{F}_u$-conditional t-valuation that maps a $\mathcal{G}_u$- measurable random variable to an $\mathcal{F}_u$-measurable random variable. This particular $\mathcal{F}_u$-conditional t-valuation is known as the 'conditional standard-deviation principle' (Pelsser & Stadje, 2014). The conditional expectation represents the best estimate of the claim given the information in $\mathcal{F}_u$, while the conditional variance introduces a risk loading that accounts for residual uncertainty. The resulting valuation is an $\mathcal{F}_u$-measurable random variable and should be interpreted as an information-conditioned actuarial assessment rather than a market price. This example demonstrates how classical actuarial valuation principles naturally extend to $\mathcal{F}_u$-conditional t-valuations in a multi-period setting.

Example 4.5.3 ***($\mathcal{F}_u^{\boldsymbol{Y}}$-conditional $(u-1)$-valuation).*** *An example of a $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional valuation is:*

$$\pi_{u-1}[S \mid \mathcal{F}_u^{\boldsymbol{Y}}] = \boldsymbol{\theta}_{u-1,S}(u) \cdot \boldsymbol{Y}(u) \qquad \text{for any } S \in \mathcal{C}_u,$$

where $\boldsymbol{\theta}$ is a fair hedger.

Example 4.5.3 shows a simple $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional $(u-1)$-valuation that maps a $\mathcal{G}_u$-measurable random variable to an $\mathcal{F}_u^{\boldsymbol{Y}}$-measurable random variable. The vector $\boldsymbol{\theta}_{u-1,S}(u)$ represents the holdings of a time-$(u-1)$ hedging strategy for the u-claim S. At time-$(u-1)$, the vector $\boldsymbol{\theta}_{u-1,S}(u)$ is deterministic. Consequently, $\boldsymbol{\theta}_{u-1,S}(u) \cdot \boldsymbol{Y}(u)$ is a $\mathcal{F}_u^{\boldsymbol{Y}}$-measurable random variable.

We introduce a notion of model-consistency for $\mathcal{F}_u$-conditional and $\mathcal{F}_u^{\boldsymbol{Y}}$-conditional t-valuations, extending the previously introduced concept of t-valuations.

Definition 31 ***(Model-consistent conditional t-valuation).*** *For $t < u \in \{1, 2, \ldots, T\}$, a conditional t-valuation is model-consistent if the conditional t-valuation of a t-orthogonal claim is defined with an actuarial valuation:*

$$\pi_t\left[S^{\perp} \mid \mathcal{F}_u\right] = e^{r(u-t)}\, \varphi_t[S^{\perp}], \qquad \pi_t\left[S^{\perp} \mid \mathcal{F}_u^{\boldsymbol{Y}}\right] = e^{r(u-t)}\, \varphi_t[S^{\perp}],$$

for any $S^{\perp} \in \mathcal{C}_u^{t,\perp}$ and for a given actuarial t-valuation φ_t.

Model-consistency requires that claims which are orthogonal to future asset-prices and non-traded systematic risk states, be valued using an actuarial valuation principle. Since such claims cannot be influenced by hedging or conditioning on future market information, their valuation should depend solely on actuarial principles.

4.6 Three-step hedge-based valuations

In this section, we extend the three-step hedge-based valuations proposed by Linders (2023) to a multi-period framework. We show that the class of three-step hedge-based valuations coincides with the class of strong-fair dynamic valuations. We further ensure that the insurer must not set up a trading strategy for risky assets using non-publicly available information.

Three-step hedge-based valuations use backward recursion to value a portfolio claim $S = \sum_{i=1}^{n^p} S_i$, where each $S_i \in \mathcal{C}_T$ denotes the time-T claim made by policyholder i. For unit-linked insurance contracts each claim takes the form $S_i = f\big(\boldsymbol{Y}(T), X^{(i)}(T)\big)$ for some measurable function f. Since $X^{(i)}(T)$ and $\boldsymbol{Y}(T)$ are dependent through the systematic risk process $\boldsymbol{Z}$, each S_i is $\mathcal{G}_T$-measurable but not $\mathcal{F}_T$-measurable. The value of S is calculated at time-$(T-1)$ and then recursively backward to time-t.

At time-$(T-1)$, the first step finds the best hedge $\boldsymbol{\theta}_{T-1,S}$ of S using a strong-fair dynamic $\mathbb{F}$-hedger. The amount $\boldsymbol{\theta}_{T-1,S}$ is invested in the financial market. We specifically use an $\mathbb{F}$-hedger to ensure that

trading in the financial markets does not allow for insider trading. The second step at time-$(T-1)$ applies a model-consistent $\mathcal{F}_T$-conditional $(T-1)$-valuation to the residual $\left(S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)\right)$ from the first step. Using the assumption in Equation 4.9, the policyholder-specific risks $X^{(i)}(T)$, $i = 1, \ldots, n^p$, are $\mathcal{F}_T$-conditionally independent. As a result, conditioning on $\mathcal{F}_T$ removes all randomness in the residual arising from financial and systematic risks. The residual $\left(S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)\right)$ is then diversifiable in the sense of Definition 13. The output of the $\mathcal{F}_T$-conditional $(T-1)$-valuation is an $\mathcal{F}_T$-measurable random variable that is free of diversifiable risk. This output is valued in third step at time-$(T-1)$ using a model-consistent $(T-1)$-valuation to account for the uncertainty about the financial and systematic scenario $(\boldsymbol{Y}(T), \boldsymbol{Z}(T))$ that will unfold.

The three steps outlined in the previous paragraph are iteratively repeated at times $T-2, T-3, \ldots, t$. However, at each time $u \in \{T-2, T-3, \ldots, t\}$, we apply the three-step hedge-based valuation to the $(u+1)$-claim $\rho_{u+1}[S] \in \mathcal{C}_{u+1}$ calculated previously instead of S. This is to ensure strong time-consistency of the three-step hedge-based valuations.

Definition 32 ***(Three-step hedge-based valuations).*** *A dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ is said to be a three-step hedge-based valuation if for any claim $S \in \mathcal{C}_T$, the value at time-$(T-1)$ is defined as:*

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[\rho_{T-1}^{CMCV}\left[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{F}_T\right]\right]$$

and the value at time-t is determined backward iteratively by:

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right],$$

where $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is a strong-fair dynamic $\mathbb{F}$-hedger, $\{\rho_t^{MCV}\}_{t=0}^{T-1}$ is a model-consistent dynamic t-valuation, and $\rho_t^{CMCV}[\cdot \mid \mathcal{F}_{t+1}]$ is a model-consistent $\mathcal{F}_{t+1}$-conditional t-valuation for $t = 0, 1, \ldots, T-1$.

4.6.1 Characterization

In this subsection, we present two theorems which together establish that a valuation is a strong-fair dynamic valuation if and only if it is a three-step hedge-based valuation.

The following theorem shows that the three-step hedge-based valuation is a strong-fair dynamic valuation.

Theorem 2 *A three-step hedge-based valuation is a strong-fair dynamic valuation.*

Proof:
Consider a strong-fair dynamic $\mathbb{F}$-hedger $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$, a model-consistent dynamic t-valuation $\{\rho_t^{MCV}\}_{t=0}^{T-1}$ and a model-consistent $\mathcal{F}_{t+1}$-conditional t-valuation $\rho_t^{CMCV}[\cdot \mid \mathcal{F}_{t+1}]$ for all $t = 0, 1, \ldots, T-1$.

1. Market-consistency:

(a) For time-$(T-1)$ and $S^h \in \mathcal{C}_T^{T-1,h}$, we have that

$$\rho_{T-1}[S + S^h] = \boldsymbol{\theta}_{T-1,S+S^h}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[\rho_{T-1}^{CMCV}\left[S + S^h - \boldsymbol{\theta}_{T-1,S+S^h}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{F}_T\right]\right]$$

since $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is a market-consistent hedger,

$$\begin{aligned}
&= \left(\boldsymbol{\theta}_{T-1,S}(T) + \boldsymbol{\theta}_{T-1,S^h}(T)\right) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\Big[\rho_{T-1}^{CMCV}\Big[S + S^h \\
&\qquad - \left(\boldsymbol{\theta}_{T-1,S}(T) + \boldsymbol{\theta}_{T-1,S^h}(T)\right) \cdot \boldsymbol{Y}(T) \mid \mathcal{F}_T\Big]\Big] \\
&= \rho_{T-1}[S] + \boldsymbol{\theta}_{T-1,S^h}(T) \cdot \boldsymbol{Y}(T-1)
\end{aligned}$$

since $S^h \in \mathcal{C}_T^{T-1,h}$ is a $(T-1)$-hedgeable T-claim,

$$= \rho_{T-1}[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\boldsymbol{\theta}_{T-1,S^h}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{F}_{T-1}\right].$$

(b) For time $t = 0, 1, \ldots, T-2$ and noting that, in particular, any t-claim is also a T-claim, we have that

$$\begin{aligned}\rho_t[S+S^h] &= \boldsymbol{\theta}_{t,S+S^h}(t+1)\cdot\boldsymbol{Y}(t)\\ &\quad + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S+S^h] - \boldsymbol{\theta}_{t,S+S^h}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right]\end{aligned}$$

since we are using backward recursion and because we have already proved this result for $\rho_{T-1}[S+S^h]$, we have $\rho_{t+1}[S+S^h] = \rho_{t+1}[S] + \rho_{t+1}[S^h]$,

$$\begin{aligned}&= \left(\boldsymbol{\theta}_{t,S}(t+1) + \boldsymbol{\theta}_{t,S^h}(t+1)\right)\cdot\boldsymbol{Y}(t)\\ &\quad + \rho^{MCV}\left[\rho^{CMCV}\left[\rho_{t+1}[S+S^h]\right.\right.\\ &\quad\quad \left.\left. - \left(\boldsymbol{\theta}_{t,S}(t+1) + \boldsymbol{\theta}_{t,S^h}(t+1)\right)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right]\\ &= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \boldsymbol{\theta}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t)\\ &\quad + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S] + \rho_{t+1}[S^h]\right.\right.\\ &\quad\quad \left.\left. - \left(\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) + \boldsymbol{\theta}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t+1)\right) \mid \mathcal{F}_{t+1}\right]\right]\end{aligned}$$

since $S^h = \boldsymbol{\theta}_{t,S^h}(T)\cdot\boldsymbol{Y}(T)$ and because of the self-financing condition, we have $\boldsymbol{\theta}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t+1) = \boldsymbol{\theta}_{t,S^h}(t+2)\cdot\boldsymbol{Y}(t+1) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-(t+1))}S^h \mid \mathcal{F}_{t+1}\right]$,

$$\begin{aligned}&= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \boldsymbol{\theta}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S]\right.\right.\\ &\quad\quad \left.\left. - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right]\\ &= \rho_t[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\cdot S^h \mid \mathcal{F}_t\right].\end{aligned}$$

2. Model-consistency:

(a) For time-$(T-1)$ and $S^{\perp} \in \mathcal{C}_T^{T-1,\perp}$, we have that

$$\rho_{T-1}[S^{\perp}] = \boldsymbol{\theta}_{T-1,S^{\perp}}(T)\cdot\boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[\rho_{T-1}^{CMCV}\left[S^{\perp} - \boldsymbol{\theta}_{T-1,S^{\perp}}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{F}_T\right]\right]$$

since $\{\boldsymbol{\theta}\}_{t=0}^{T-1}$ is model consistent for all $t = 0, 1, \ldots, T-1$,

$$= \pi_{T-1}[S^{\perp}] + \rho_{T-1}^{MCV}\left[\rho_{T-1}^{CMCV}\left[S^{\perp} - \pi_{T-1}[S^{\perp}]e^r \mid \mathcal{F}_T\right]\right]$$

since $\pi_{T-1}[S^{\perp}]e^r \in \mathcal{C}_T^{T-1,h}$ and that ρ_{T-1}^{CMCV} is conditionally translation invariant,

$$\begin{aligned}
&= \pi_{T-1}[S^{\perp}] + \rho_{T-1}^{MCV}\left[\rho_{T-1}^{CMCV}\left[S^{\perp} \mid \mathcal{F}_T\right] - \pi_{T-1}[S^{\perp}]e^r\right] \\
&= \pi_{T-1}[S^{\perp}] + \rho_{T-1}^{MCV}\left[e^r\varphi_{T-1}[S^{\perp}] - \pi_{T-1}[S^{\perp}]e^r\right] \\
&= \varphi_{T-1}[S^{\perp}].
\end{aligned}$$

(b) For time $t = 0, 1, \ldots, T-2$, we have that

$$\begin{aligned}
\rho_t[S^{\perp}] &= \boldsymbol{\theta}_{t,S^{\perp}}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S^{\perp}] - \boldsymbol{\theta}_{t,S^{\perp}}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right] \\
&= \boldsymbol{\theta}_{t,S^{\perp}}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S^{\perp}] \mid \mathcal{F}_{t+1}\right] - \boldsymbol{\theta}_{t,S^{\perp}}(t+1)\cdot\boldsymbol{Y}(t+1)\right]
\end{aligned}$$

since we are using backward recursion and because we have already proved this result for $\rho_{T-1}[S^{\perp}]$, we have $\rho_{t+1}[S^{\perp}] = \varphi_{t+1}[S^{\perp}]$,

$$= \pi_t[S^{\perp}] + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\varphi_{t+1}[S^{\perp}] \mid \mathcal{F}_{t+1}\right] - \pi_t[S^{\perp}]e^r\right]$$

since $\varphi[S^{\perp}]$ is an orthogonal claim itself because an actuarial $(t+1)$-valuation is $\mathbb{P}$-independent of $\{\boldsymbol{Y}(u), \boldsymbol{Z}(u)\}_{u\in\{t+1,\ldots,T\}}$,

$$= \rho_t^{MCV}\left[e^r\varphi_t\left[\varphi_{t+1}[S^{\perp}]\right]\right]$$

since $\rho_t^{CMCV}[\cdot \mid \mathcal{F}_{t+1}]$ is model consistent for all $t = 0, 1, \ldots, T-1$, we have that $\varphi_t[S^{\perp}] = \varphi_t[\varphi_{t+1}[S^{\perp}]]$,

$$= \varphi_t[S^{\perp}].$$

3. Strong time-consistency. For $t = 0, 1, \ldots, T-2$, we have that

$$\rho_t[\rho_{t+1}[S]] = \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[\rho_{t+1}[S]] - \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right]$$

since $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is strong time consistent,

$$\begin{aligned}
&= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[\rho_{t+1}[S]] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right] \\
&= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right] \\
&= \rho_t[S].
\end{aligned}$$

This completes the proof. ∎

The previous theorem shows that the three-step hedge-based valuation constructed from any strong-fair dynamic $\mathbb{F}$-hedger $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$, any model-consistent dynamic t-valuation $\{\rho_t^{MCV}\}_{t=0}^{T-1}$, and any t-model-consistent $\mathcal{F}_{t+1}$-conditional valuation $\rho^{CMCV}[\cdot \mid \mathcal{F}_{t+1}]$ (for $t = 0, 1, \ldots, T-1$) in Definition 32 is itself market-consistent, model-consistent, and strongly time-consistent.

Theorem 3 *Consider a strong-fair dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$. Then, there exists a strong-fair dynamic $\mathbb{F}$-hedger $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$, a model-consistent t-valuation ρ_t^{MCV} and a model-consistent $\mathcal{F}_{t+1}$-conditional t-valuation $\rho_t^{CMCV}[\cdot \mid \mathcal{F}_{t+1}]$ for $t = 0, 1, \ldots, T-1$ such that the value of a claim $S \in \mathcal{C}_T$ at time-$(T-1)$ is:*

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[\rho_{T-1}^{CMCV}\left[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{F}_T\right]\right]$$

and the value at time-t is determined backward iteratively by:

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}\left[\rho_t^{CMCV}\left[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}\right]\right].$$

Proof:
Consider a strong-fair dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ and a strong-fair dynamic $\mathbb{F}$-hedger $\{\boldsymbol{\theta}\}_{t=0}^{T-1}$. Since ρ_t is market consistent, we can write that

$$\begin{aligned}\rho_{T-1}[S] &= \rho_{T-1}[S - S^h + S^h] \quad \text{with } S^h = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) \\ &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)]\end{aligned}$$

and for $t = 0, 1, \ldots, T-2$, by also using strong time-consistency of ρ_t ($\rho_t[S] = \rho_t[\rho_{t+1}[S]]$) and $\boldsymbol{\theta}_t(t+1)$ ($\boldsymbol{\theta}_{t,S} = \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1)$), we have that

$$\rho_t[S] = \rho_t[\rho_{t+1}[S]] = \rho_t[\rho_{t+1}[S] - S^h + S^h],$$

where $S^h = \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t+1) = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)$. So, we have that

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)].$$

As a result we have that

$$\begin{aligned}\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)] \\ &\quad \text{and for time } t = 0, 1, \ldots, T-2, \\ \rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)].\end{aligned}$$

Applying Theorem 1 to the right-hand sides of the expressions for $\rho_{T-1}[S]$ and $\rho_t[S]$ above, we get that

$$\begin{aligned}\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{G}_{T-1}] \\ &\quad \text{and for time } t = 0, 1, \ldots, T-2, \\ \rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1) \cdot \boldsymbol{Y}(t+1) \mid \mathcal{G}_t],\end{aligned}$$

where the second equation is because $\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1) \in \mathcal{C}_{t+1}$.

Since $\{\boldsymbol{\mu}_t\}_{t=0}^{T-1}$ is a dynamic strong-fair $\mathbb{G}$-hedger and $\{\boldsymbol{\theta}\}_{t=0}^{T-1}$ is a dynamic strong-fair $\mathbb{F}$-hedger, for an orthogonal claim $S \in \mathcal{C}_{t+1}^{t,\perp}$, we have that

$$\rho_{T-1}[S^\perp] = \boldsymbol{\theta}_{T-1,S^\perp}(T) \cdot \boldsymbol{Y}(T-1) + e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{T-1,S^\perp-\boldsymbol{\theta}_{T-1,S^\perp}(T)\cdot\boldsymbol{Y}(T)}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{G}_{T-1}]$$

since $\boldsymbol{\theta}_t$ is model consistent,

$$= \boldsymbol{\theta}_{T-1,S^\perp}(T) \cdot \boldsymbol{Y}(T-1) + e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{T-1,S^\perp-e^r\varphi_{T-1}[S^\perp]}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{G}_{T-1}]$$

since $(S^\perp - e^r\varphi_{T-1}[S^\perp])$ is itself an orthogonal claim and because $\boldsymbol{\mu}_t$ is model consistent,

$$\begin{aligned}&= \boldsymbol{\theta}_{T-1,S^\perp}(T) \cdot \boldsymbol{Y}(T-1) + e^{-r}\mathbb{E}^{\mathbb{Q}}\left[e^r\pi_{T-1}\left[S^\perp - e^r\varphi_{T-1}[S^\perp]\right] \mid \mathcal{G}_{T-1}\right] \\ &= \boldsymbol{\theta}_{T-1,S^\perp}(T) \cdot \boldsymbol{Y}(T-1) + \pi_{T-1}\left[S^\perp - \boldsymbol{\theta}_{T-1,S^\perp}(T) \cdot \boldsymbol{Y}(T)\right],\end{aligned}$$

Similarly, we have that for time $t = 0, 1, \ldots, T-2$

$$\rho_t[S^\perp] = \boldsymbol{\theta}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t) + e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{t,\rho_{t+1}[S^\perp]-\boldsymbol{\theta}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1)\cdot\boldsymbol{Y}(t+1)\mid\mathcal{G}_t]$$

since $\boldsymbol{\theta}_t$ is model consistent,

$$= \boldsymbol{\theta}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t) + e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{t,\rho_{t+1}[S^\perp]-e^r\varphi_t[S^\perp]}(t+1)\cdot\boldsymbol{Y}(t+1)\mid\mathcal{G}_t]$$

since $(\rho_{t+1}[S^\perp] - e^r\varphi_t[S^\perp])$ is itself an orthogonal claim and because $\boldsymbol{\mu}_t$ is model consistent,

$$\begin{aligned}&= \boldsymbol{\theta}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t) + e^{-r}\mathbb{E}^{\mathbb{Q}}\left[e^r\pi_t\left[\rho_{t+1}[S^\perp] - e^r\varphi_t[S^\perp]\right]\mid\mathcal{G}_t\right]\\ &= \boldsymbol{\theta}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t) + \pi_t\left[\rho_{t+1}[S^\perp] - \boldsymbol{\theta}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t+1)\right]\end{aligned}$$

Combining the above results, we get that

$$\rho_{T-1}[S] = \begin{cases}\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) & \\ \quad +\pi_{T-1}\left[S - \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)\right], & \text{if } S\in\mathcal{C}_T^{T-1,\perp}\\ \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) & \\ \quad +e^{-r}\,\mathbb{E}^{\mathbb{Q}}\Big[\boldsymbol{\mu}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)}(T)\cdot\boldsymbol{Y}(T)\mid\mathcal{G}_{T-1}\Big], & \text{if } S\notin\mathcal{C}_T^{T-1,\perp}\end{cases}$$

and for $t = 0, 1, \ldots, T-2$, we have that

$$\rho_t[S] = \begin{cases}\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) & \\ \quad +\pi_t\left[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)\right], & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}\\ \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) & \\ \quad +e^{-r}\mathbb{E}^{\mathbb{Q}}[\boldsymbol{\mu}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1)\cdot\boldsymbol{Y}(t+1)\mid\mathcal{G}_t], & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}\end{cases}$$

Now define a model-consistent t-valuation as follows:

$$\rho_t^{MCV}[S] = \begin{cases}\pi_t[S], & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}\\ e^{-r}\mathbb{E}^{\mathbb{Q}}\left[S\mid\mathcal{G}_t\right] & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}.\end{cases}$$

This is a valid model-consistent t-valuation because it is

- Normalized:
$$\rho_t^{MCV}[0] = 0.$$
- Translation invariant:
$$\begin{aligned}\rho_t^{MCV}[S+a] &= \begin{cases}\pi_t[S+a], & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}\\ e^{-r}\mathbb{E}^{\mathbb{Q}}\left[S+a\mid\mathcal{G}_t\right], & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}\end{cases}\\ &= \begin{cases}\pi_t[S] + ae^{-r}, & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}\\ e^{-r}\mathbb{E}^{\mathbb{Q}}\left[S\mid\mathcal{G}_t\right] + ae^{-r}, & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}\end{cases}\\ &= \rho_t^{MCV}[S] + ae^{-r}.\end{aligned}$$
- Model-consistent:
$$\rho_t^{MCV}\left[S^\perp\right] = \pi_t[S^\perp].$$

Returning to our earlier result, we then have that

$$\begin{aligned}\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}[\boldsymbol{\mu}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)}(T)\cdot\boldsymbol{Y}(T)]\\ &\quad \text{and for time } t = 0,1,\ldots,T-2,\\ \rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}[\boldsymbol{\mu}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1)\cdot\boldsymbol{Y}(t+1)]\end{aligned}$$

Now define a model-consistent $\mathcal{F}_{t+1}$-conditional t-valuation as follows

$$\rho_t^{CMCV}[S \mid \mathcal{F}_{t+1}] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1), \qquad \text{for } t = 0,1,\ldots,T-1.$$

This is a valid $\mathcal{F}_{t+1}$-conditional t-valuation. Firstly, since $\boldsymbol{\mu}_{t,S}(t+1)$ is known at time-t, we have that $\rho_t^{CMCV}[S \mid \mathcal{F}_{t+1}] = \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) \in \mathcal{C}_{t+1}^{\mathcal{F}}$. Further, it is also

- Normalized:
$$\rho_t^{CMCV}[0 \mid \mathcal{F}_{t+1}] = \boldsymbol{\mu}_{t,0}(t+1)\cdot\boldsymbol{Y}(t+1) = 0,$$
since $\boldsymbol{\mu}_t$ is a hedger and hence normalized.

- Conditionally translation invariant:
$$\begin{aligned}\rho_t^{CMCV}\left[S + S^h \mid \mathcal{F}_{t+1}\right] &= \boldsymbol{\mu}_{t,S+S^h}(t+1)\cdot\boldsymbol{Y}(t+1)\\ &= \boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) + \boldsymbol{\mu}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t+1)\\ &= \rho_t^{CMCV}\left[S \mid \mathcal{F}_{t+1}\right] + S^h.\end{aligned}$$

- Model consistent:
$$\begin{aligned}\rho_t^{CMCV}\left[S^\perp \mid \mathcal{F}_{t+1}\right] &= \boldsymbol{\mu}_{t,S^\perp}(t+1)\cdot\boldsymbol{Y}(t+1)\\ &= \frac{\pi_t[S^\perp]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}(t+1)\cdot\boldsymbol{Y}(t+1)\\ &= e^r\pi_t[S^\perp].\end{aligned}$$

Using this, we finally get that

$$\begin{aligned}\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}[\rho_{T-1}^{CMCV}[S - \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T) \mid \mathcal{F}_T]]\\ &\quad \text{and for } t = 0,1,\ldots,T-2,\\ \rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) + \rho_t^{MCV}[\rho_t^{CMCV}[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1) \mid \mathcal{F}_{t+1}].\end{aligned}$$

This completes the proof. ■

Theorem 2 shows that a three-step hedge-based valuation is a strong-fair valuation. Theorem 3 shows that a strong-fair valuation is a three-step hedge-based valuation. Combining these two theorems, we have that the class of strong-fair valuations is equivalent to the class of three-step hedge-based valuations in a multi-period setting.

This result provides a link between the class of strong-fair valuations proposed by Barigou et al. (2019) in a multi-period setting and the multi-period extension of the three-step hedge-based valuations proposed by Linders (2023). Furthermore, we have specifically disallowed an insurer from trading in the financial markets on non-publicly available information by using an $\mathbb{F}$-hedger at each re-balancing time-point. Our work further shows that despite this added constraint, the three-step hedge-based valuation is a strong-fair valuation.

Corollary 4 *Consider a fair dynamic valuation* $\{\rho_t\}_{t=0}^{T-1}$ *and a fair dynamic* $\mathbb{F}$*-hedger* $\{\boldsymbol{\theta}\}_{t=0}^{T-1}$*. Then, the value of a claim* $S \in \mathcal{C}_T$ *at time-*$(T-1)$ *is:*

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)]$$

and the value at time $t = 0, 1, \ldots, T-2$ *is determined backward iteratively by:*

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t\left[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)\right].$$

Proof:
Starting from the proof of Theorem 3, we have that for a model-consistent t-valuation ρ_t^{MCV} for all $t = 0, 1, \ldots, T-1$ and a t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_t$

$$\begin{aligned}
\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}[\boldsymbol{\mu}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)}(T) \cdot \boldsymbol{Y}(T)] \\
&\quad \text{and for } t = 0, 1, \ldots, T-2, \\
\rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}[\boldsymbol{\mu}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1) \cdot \boldsymbol{Y}(t+1)].
\end{aligned}$$

From Lemma 3 and Lemma 4, we may write that

$$\boldsymbol{\mu}_{t,S} = \boldsymbol{\theta}_{t,S} + \frac{\rho_t\left[S - \boldsymbol{\theta}_{t,S}(T) \cdot \boldsymbol{Y}(T)\right]}{Y^{(0)}(t)} \times \mathbf{1}_t^{(0)}.$$

Combining these equations, we get that

$$\begin{aligned}
\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[\boldsymbol{\theta}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)}(T) \cdot \boldsymbol{Y}(T) + \right. \\
&\qquad \left. e^r \rho_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) - \boldsymbol{\theta}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)}(T) \cdot \boldsymbol{Y}(T)]\right] \\
&\quad \text{and for } t = 0, 1, \ldots, T-2, \\
\rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}\left[\boldsymbol{\theta}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1) \cdot \boldsymbol{Y}(t+1) + \right. \\
&\qquad \left. e^r \rho_t[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1) - \boldsymbol{\theta}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)}(t+1) \cdot \boldsymbol{Y}(t+1)]\right].
\end{aligned}$$

Since $\boldsymbol{\theta}_t$ is a fair hedger for all $t = 0, 1, \ldots, T-1$ and since $\boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)$ and $\boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)$ are both hedgeable, we have that

$$\begin{aligned}
\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[\boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) + \right. \\
&\qquad \left. e^r \rho_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) + \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)]\right] \\
&\quad \text{and for } t = 0, 1, \ldots, T-2, \\
\rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}\left[\boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t+1) - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1) \right. \\
&\qquad \left. + e^r \rho_t[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1) - \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t+1) + \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)]\right].
\end{aligned}$$

Due to the strong time-consistency of $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$, we get that

$$\begin{aligned}
\rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[e^r \rho_{T-1}[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)]\right] \\
&\quad \text{and for } t = 0, 1, \ldots, T-2, \\
\rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}\left[e^r \rho_t[\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)]\right].
\end{aligned}$$

Applying the following definition of ρ_t^{MCV} defined in Theorem 3,

$$\rho_t^{MCV}[S] = \begin{cases} \pi_t[S], & \text{if } S \in \mathcal{C}_{t+1}^{t,\perp}, \\ e^{-r}\mathbb{E}^{\mathbb{Q}}\left[S \mid \mathcal{G}_t\right], & \text{if } S \notin \mathcal{C}_{t+1}^{t,\perp}, \end{cases}$$

to the earlier equation we get that

$$\rho_{T-1}[S] = \begin{cases} \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) \\ \quad +\pi_{T-1}\left[e^r\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)]\right], & \text{if } S\in\mathcal{C}_T^{T-1,\perp}, \\ \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) \\ \quad +\mathbb{E}^{\mathbb{Q}}\left[\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)]\mid\mathcal{G}_{T-1}\right], & \text{if } S\notin\mathcal{C}_T^{T-1,\perp} \end{cases}$$

and for $t=0,1,\ldots,T-2$, we have that

$$\rho_t[S] = \begin{cases} \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \\ \quad +\pi_t\left[e^r\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)]\right], & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}, \\ \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \\ \quad +\mathbb{E}^{\mathbb{Q}}\left[\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)]\mid\mathcal{G}_t\right], & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}. \end{cases}$$

Since $\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)]$ is $\mathcal{G}_{T-1}$-measurable and $\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)]$ is $\mathcal{G}_t$-measurable, we have that

$$\rho_{T-1}[S] = \begin{cases} \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) \\ \quad +\pi_{T-1}\left[e^r\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)]\right], & \text{if } S\in\mathcal{C}_T^{T-1,\perp}, \\ \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1) \\ \quad +\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)], & \text{if } S\notin\mathcal{C}_T^{T-1,\perp} \end{cases}$$

and for $t=0,1,\ldots,T-2$, we have that

$$\rho_t[S] = \begin{cases} \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \\ \quad +\pi_t\left[e^r\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)]\right], & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}, \\ \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t) \\ \quad +\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)], & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}. \end{cases}$$

Similarly, since an actuarial valuation π_t is translation invariant, we have that

$$\rho_{T-1}[S] = \begin{cases} \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1)+\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)], & \text{if } S\in\mathcal{C}_T^{T-1,\perp}, \\ \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1)+\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)], & \text{if } S\notin\mathcal{C}_T^{T-1,\perp}, \end{cases}$$

and

$$\rho_t[S] = \begin{cases} \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)+\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)], & \text{if } S\in\mathcal{C}_{t+1}^{t,\perp}, \\ \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)+\rho_t[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)], & \text{if } S\notin\mathcal{C}_{t+1}^{t,\perp}. \end{cases}$$

Using this result, we get that

$$\begin{aligned} \rho_{T-1}[S] &= \boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1)+\rho_{T-1}[S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)] \\ &\quad \text{and for } t=0,1,\ldots,T-2, \text{ we have that} \\ \rho_t[S] &= \boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t)+\rho_t\left[\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)\right]. \end{aligned}$$

This completes the proof. ∎

The previous corollary shows that the three-step hedge-based valuation for a T-claim S may also be calculated in two steps. At time-$(T-1)$, first calculate the $\mathbb{F}$-hedge $\boldsymbol{\theta}_{t,S}$. Second, apply the three-step hedge-based valuation to the residual claim $(S-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T))$. The value of the claim S at time-$(T-1)$ is then the market value of the hedge $\boldsymbol{\theta}_{t,S}$ at time-$(T-1)$ plus the value of the three-step hedge-based valuation applied to the residual. At time-t for $t\in\{0,1,\ldots,T-2\}$, first calculate the $\mathbb{F}$-hedge $\boldsymbol{\theta}_{t,S}$. Second, apply the three-step hedge-based valuation to a separate claim $(\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T))$. The value of the claim S at time-t is then the market value of the hedge $\boldsymbol{\theta}_{t,S}$ at time-t plus the value of the three-step hedge-based valuation applied to $(\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T))$. Importantly, the $\mathbb{F}$-hedge is always calculated for the claim S for all $t\in\{0,1,\ldots,T-1\}$. However, the three-step hedge-based valuation is applied to a different residual $(\rho_{t+1}[S]-\boldsymbol{\theta}_{t,S}(T)\cdot\boldsymbol{Y}(T))$ at times $t\in\{0,1,\ldots,T-1\}$.

Example 4.6.1 *In a single-period setting, the three-step hedge-based valuation from Definition 32 reduces to*

$$\rho_0[S]=\boldsymbol{\theta}_{0,S}(1)\cdot\boldsymbol{Y}(0)+\rho_0^{MCV}\left[\rho_0^{CMCV}\left[S-\boldsymbol{\theta}_{0,S}(1)\cdot\boldsymbol{Y}(1)\mid\mathcal{F}_1\right]\right].$$

This expression coincides with the single-period three-step hedge-based valuation introduced in Linders (2023). Importantly, the $\mathbb{F}$-hedger $\boldsymbol{\theta}_{0,S}(1)$ is also a $\mathbb{G}$-hedger, since at time-0 we assume that $\mathcal{G}_0=\mathcal{F}_0=\{\varnothing,\Omega\}$.

4.7 Two-step market-evaluations

In this section, we propose a discrete multi-period modification of the two-step market evaluations of Pelsser and Stadje (2014). This extends the modification proposed in Dhaene et al. (2017) to a multi-period setting. We show that this class of two-step market evaluations coincides with the class of strong-fair dynamic valuations.

Two-step market evaluation also uses a backward recursion method to value a T-claim S. At time-$(T-1)$, the two-step market evaluation is defined as the conditional expectation, given $\mathcal{G}_{T-1}$, of the discounted $\mathcal{F}_T^{\boldsymbol{Y}}$-conditional $(T-1)$-valuation of S. At time $t\in\{0,1,\ldots,T-2\}$, the two-step market evaluation is defined as the conditional expectation, given $\mathcal{G}_t$, of the discounted $\mathcal{F}_{t+1}^{\boldsymbol{Y}}$-conditional t-valuation of $\rho_{t+1}[S]$. Intuitively, the valuation involves an 'information-reduction step', in which the future value is replaced by an $\mathcal{F}_{u+1}^{\boldsymbol{Y}}$-measurable random variable. The randomness in this $\mathcal{F}_{u+1}^{\boldsymbol{Y}}$-measurable random variable is then valued using the discounted risk-neutral expectation given $\mathcal{G}_u$.

Definition 33 ***(Two-step market evaluations).*** *A dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$ is said to be a two-step market evaluation if for any claim $S\in\mathcal{C}_T$, the value at time-$(T-1)$ is defined as:*

$$\rho_{T-1}[S]=\mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_{T-1}^{CMCV}[S\mid\mathcal{F}_T^{\boldsymbol{Y}}]\mid\mathcal{G}_{T-1}\right]$$

and the value at time-t is determined backward iteratively by:

$$\rho_t[S]=\mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_t^{CMCV}[\rho_{t+1}[S]\mid\mathcal{F}_{t+1}^{\boldsymbol{Y}}]\mid\mathcal{G}_t\right],$$

where $\rho_t^{CMCV}[\cdot\mid\mathcal{F}_{t+1}^{\boldsymbol{Y}}]$ is a model-consistent $\mathcal{F}_{t+1}^{\boldsymbol{Y}}$-conditional t-valuation for $t=0,1,\ldots,T-1$.

4.7.1 Characterization

In this subsection, we present two theorems which together establish that a valuation is a strong-fair dynamic valuation if and only if it is a two-step market-evaluation.

The following theorem shows that the two-step market evaluation is a strong-fair dynamic valuation.

Theorem 4 *A two-step market evaluation is a strong-fair dynamic valuation.*

Proof:
Consider a model-consistent $\mathcal{F}^{\boldsymbol{Y}}_{t+1}$-conditional t-valuation $\rho^{CMCV}_t[\cdot \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}]$ for $t = 0, 1, \ldots, T-1$.

1. Market-consistency:

(a) For time-$(T-1)$ and $S^h \in \mathcal{C}^{T-1,h}_T$, we have

$$\rho_{T-1}[S + S^h] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_{T-1}[S + S^h \mid \mathcal{F}^{\boldsymbol{Y}}_T] \mid \mathcal{G}_{T-1}\right]$$

since $\rho^{CMCV}_{T-1}[\cdot \mid \mathcal{F}^{\boldsymbol{Y}}_T]$ is conditionally translation invariant,

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_{T-1}[S \mid \mathcal{F}^{\boldsymbol{Y}}_T]\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r}S^h \mid \mathcal{G}_{T-1}\right]$$

due to the immersion property,

$$\begin{aligned} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_{T-1}[S \mid \mathcal{F}^{\boldsymbol{Y}}_T] \mid \mathcal{G}_{T-1}\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r}S^h \mid \mathcal{F}_{T-1}\right] \\ &= \rho_{T-1}[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r}S^h \mid \mathcal{F}_{T-1}\right]. \end{aligned}$$

(b) For time $t = 0, 1, \ldots, T-2$ and noting that, in particular, any t-claim is also a T-claim, we have that

$$\rho_t[S + S^h] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_t[\rho_{t+1}[S + S^h] \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}] \mid \mathcal{G}_t\right]$$

since $S^h = \boldsymbol{\theta}_{t,S^h}(T) \cdot \boldsymbol{Y}(T)$ and because of the self-financing condition, we have $\boldsymbol{\theta}_{t,S^h}(t+1) \cdot \boldsymbol{Y}(t+1) = \boldsymbol{\theta}_{t,S^h}(t+2) \cdot \boldsymbol{Y}(t+1) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-(t+1))}S^h \mid \mathcal{F}_{t+1}\right]$,

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_t[\rho_{t+1}[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-(t+1))}S^h \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}] \mid \mathcal{G}_t\right]$$

since $\rho^{CMCV}_t[\cdot \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}]$ is conditionally translation invariant, $\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-(t+1))}S^h \mid \mathcal{F}_{t+1}\right] \in \mathcal{C}^{t,h}_{t+1}$,

$$\begin{aligned} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_t[\rho_{t+1}[S] \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S^h \mid \mathcal{F}_{t+1}\right] \mid \mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_t[\rho_{t+1}[S] \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}] \mid \mathcal{G}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S^h \mid \mathcal{F}_{t+1}\right] \mid \mathcal{G}_t\right] \end{aligned}$$

due to the immersion property,

$$\begin{aligned} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_t[\rho_{t+1}[S] \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}] \mid \mathcal{G}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S^h \mid \mathcal{F}_{t+1}\right] \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_t[\rho_{t+1}[S] \mid \mathcal{F}^{\boldsymbol{Y}}_{t+1}] \mid \mathcal{G}_t\right] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S^h \mid \mathcal{F}_t\right] \\ &= \rho_t[S] + \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}S^h \mid \mathcal{F}_t\right]. \end{aligned}$$

2. Model-consistency:

(a) For time-$(T-1)$ and $S^{\perp} \in \mathcal{C}^{T-1,\perp}_T$, we have that

$$\rho_{T-1}[S^{\perp}] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho^{CMCV}_{T-1}[S^{\perp} \mid \mathcal{F}^{\boldsymbol{Y}}_T] \mid \mathcal{G}_{T-1}\right]$$

since $\rho^{CMCV}_{T-1}[\cdot \mid \mathcal{F}^{\boldsymbol{Y}}_T]$ is model consistent for all $t = 0, 1, \ldots, T-1$,

$$\begin{aligned} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r} \times e^{r}\varphi_{T-1}[S^{\perp}] \mid \mathcal{G}_{T-1}\right] \\ &= \varphi_{T-1}[S^{\perp}]. \end{aligned}$$

(b) For time-t, we have that

$$\rho_t[S^\perp] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_t^{CMCV}[\rho_{t+1}[S^\perp] \mid \mathcal{F}_{t+1}^{\boldsymbol{Y}}] \mid \mathcal{G}_t\right]$$

since we are using backward recursion and because we have already proved this result for $\rho_{T-1}[S^\perp]$, we have $\rho_{t+1}[S^\perp] = \varphi_{t+1}[S^\perp]$,

$$= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_t^{CMCV}[\varphi_{t+1}[S^\perp] \mid \mathcal{F}_{t+1}^{\boldsymbol{Y}}] \mid \mathcal{G}_t\right]$$

since $\varphi[S^\perp]$ is an orthogonal claim itself because an actuarial $(t+1)$-valuation is $\mathbb{P}$-independent of $\{\boldsymbol{Y}(u), \boldsymbol{Z}(u)\}_{u\in\{t+1,\ldots,T\}}$,

$$\begin{aligned} &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r} \times e^{r}\varphi_t[\varphi_{t+1}[S^\perp]] \mid \mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\varphi_t[\varphi_{t+1}[S^\perp]] \mid \mathcal{G}_t\right] \end{aligned}$$

since $\rho_t^{CMCV}[\cdot \mid \mathcal{F}_{t+1}]$ is model consistent for all $t = 0, 1, \ldots, T-1$, we have that $\varphi_t[S^\perp] = \varphi_t[\varphi_{t+1}[S^\perp]]$,

$$= \varphi_t[S^\perp].$$

3. Strong time-consistency. For $t = 0, 1, \ldots, T-2$, we have that

$$\begin{aligned} \rho_t[\rho_{t+1}[S]] &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_t^{CMCV}[\rho_{t+1}[\rho_{t+1}[S]] \mid \mathcal{F}_{t+1}^{\boldsymbol{Y}}] \mid \mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_t^{CMCV}[\rho_{t+1}[S] \mid \mathcal{F}_{t+1}^{\boldsymbol{Y}}] \mid \mathcal{G}_t\right] \\ &= \rho_t[S] \end{aligned}$$

■

The previous theorem shows that the two-step market-evaluation constructed from any t-model-consistent $\mathcal{F}_{t+1}^{\boldsymbol{Y}}$-conditional valuation $\rho^{CMCV}[\cdot \mid \mathcal{F}_{t+1}^{\boldsymbol{Y}}]$ (for $t = 0, 1, \ldots, T-1$) in Definition 33 is itself market-consistent, model-consistent, and strongly time-consistent.

Theorem 5 *Consider a fair dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$. Then, there exists a model-consistent $\mathcal{F}_{t+1}^{\boldsymbol{Y}}$-conditional t-valuation $\rho_t^{CMCV}[\cdot|\mathcal{F}_{t+1}^{\boldsymbol{Y}}]$ for all $t = 0, 1, \ldots, T-1$ such that the value of a claim $S \in \mathcal{C}_T$ at time-$(T-1)$ is:*

$$\rho_{T-1}[S] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_{T-1}^{CMCV}[S \mid \mathcal{F}_T^{\boldsymbol{Y}}] \mid \mathcal{G}_{T-1}\right]$$

and the value at time-t is determined backward iteratively by:

$$\rho_t[S] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_t^{CMCV}[\rho_{t+1}[S] \mid \mathcal{F}_{t+1}^{\boldsymbol{Y}}] \mid \mathcal{G}_t\right]$$

Proof:
Consider a fair dynamic valuation $\{\rho_t\}_{t=0}^{T-1}$. From Lemma 6, we can write that

$$\rho_{T-1}[S] = e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{T-1,S}(T) \cdot \boldsymbol{Y}(T) \mid \mathcal{G}_{T-1}\right],$$

for a fair $(T-1)$-$\mathbb{G}$-hedger $\boldsymbol{\mu}_{T-1,S}$. For $t = 0, 1, \ldots, T-2$, by using the strong time-consistency of ρ_t ($\rho_t[S] = \rho_t[\rho_{t+1}[S]]$) and Lemma 6, we have that

$$\rho_t[S] = \rho_t[\rho_{t+1}[S]] = e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t) \mid \mathcal{G}_t\right],$$

for a fair t-$\mathbb{G}$-hedger $\boldsymbol{\mu}_{t,S}$.

So we have that

$$\begin{aligned}\rho_{T-1}[S] &= e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{T-1,S}(T)\cdot\boldsymbol{Y}(T)\mid\mathcal{G}_{T-1}\right]\\ &\text{and for } t=0,1,\ldots,T-2,\\ \rho_t[S] &= e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,\rho_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t)\mid\mathcal{G}_t\right]\end{aligned}$$

Now define a model-consistent $\mathcal{F}^{\boldsymbol{Y}}_{t+1}$-conditional t-valuation as follows:

$$\rho_t^{CMCV}[S\mid\mathcal{F}^{\boldsymbol{Y}}_{t+1}]=\boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1),\quad\text{for } t=0,1,\ldots,T-1.$$

This is a valid $\mathcal{F}^{\boldsymbol{Y}}_{t+1}$-conditional t-valuation. Firstly, we have that since $\boldsymbol{\mu}_{t,S}(t+1)$ is known at time-t, we have that $\rho_t^{CMCV}[S\mid\mathcal{F}^{\boldsymbol{Y}}_{t+1}]=\boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)\in\mathcal{C}^{\mathcal{F}^{\boldsymbol{Y}}}_{t+1}$. Further, it is also

- Normalized:
$$\rho_t^{CMCV}[0\mid\mathcal{F}^{\boldsymbol{Y}}_{t+1}]=\boldsymbol{\mu}_{t,0}(t+1)\cdot\boldsymbol{Y}(t+1)=0,$$
since $\boldsymbol{\mu}$ is a hedger and hence normalized.

- Conditionally translation invariant with $S^h\in\mathcal{C}^{t,h}_{t+1}$:
$$\mathbb{E}^{\mathbb{Q}}\left[\rho_t^{CMCV}\left[S+S^h\mid\mathcal{F}^{\boldsymbol{Y}}_{t+1}\right]\mid\mathcal{G}_t\right]=\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S+S^h}(t+1)\cdot\boldsymbol{Y}(t+1)\mid\mathcal{G}_t\right],$$
since $\boldsymbol{\mu}_{t,S}$ is market consistent,
$$\begin{aligned}&=\mathbb{E}^{\mathbb{Q}}\left[\boldsymbol{\mu}_{t,S}(t+1)\cdot\boldsymbol{Y}(t+1)+\boldsymbol{\mu}_{t,S^h}(t+1)\cdot\boldsymbol{Y}(t+1)\mid\mathcal{G}_t\right]\\ &=\mathbb{E}^{\mathbb{Q}}\left[\rho_t^{CMCV}\left[S\mid\mathcal{F}^{\boldsymbol{Y}}_{t+1}\right]\mid\mathcal{G}_t\right]+\mathbb{E}^{\mathbb{Q}}\left[S^h\mid\mathcal{G}_t\right].\end{aligned}$$

- Model consistent:
$$\begin{aligned}\rho_t^{CMCV}\left[S^{\perp}\mid\mathcal{F}_{t+1}\right]&=\boldsymbol{\mu}_{t,S^{\perp}}(t+1)\cdot\boldsymbol{Y}(t+1)\\ &=e^r\pi_t[S^{\perp}],\end{aligned}$$
where we use the fact that $\boldsymbol{\mu}_{t,S}$ is a fair t-$\mathbb{G}$-hedger.

Returning to our earlier result, we then finally have that

$$\begin{aligned}\rho_{T-1}[S] &= e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\rho_{T-1}^{CMCV}[S\mid\mathcal{F}^{\boldsymbol{Y}}_T]\mid\mathcal{G}_{T-1}\right]\\ &\text{and for } t=0,1,\ldots,T-2,\\ \rho_t[S] &= e^{-r}\mathbb{E}^{\mathbb{Q}}\left[\rho_t^{CMCV}[\rho_{t+1}[S]\mid\mathcal{F}^{\boldsymbol{Y}}_{t+1}]\mid\mathcal{G}_t\right]\end{aligned}$$

This proves our result. ∎

Theorem 4 shows that a two-step market evaluation is a strong-fair valuation. Theorem 5 shows that a strong-fair valuation is a two-step market evaluation. Combining these two theorems, we have that the class of strong-fair valuations is equivalent to the class of two-step market evaluations in a multi-period setting.

This result provides a link between the class of strong-fair valuations proposed by Barigou et al. (2019) in a multi-period setting and the multi-period extension of the modification of the two-step market evaluations of Pelsser and Stadje (2014). From the last section, we further have that the class of two-step market evaluations is equivalent to the class of three-step hedge-based valuations in a multi-period setting.

4.8 Comparison with valuation methods in literature

In this section, we compare the strong-fair dynamic valuations presented in this chapter with other strong-fair valuation methods in the literature. We compare the different methods using a simple example.

4.8.1 Hedge-based valuation of Barigou, Chen, and Dhaene (2019)

Barigou et al. (2019) define the 'hedge-based valuation' $\{\rho_t\}_{t=0}^{T-1}$ for any claim $S \in \mathcal{C}_T$ as follows:

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + \rho_{T-1}^{MCV}\left[S - \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T)\right]$$

and for $t = 0, 1, \ldots, T-1$,

$$\rho_t[S] = \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t) + \rho_t^{MCV}\left[\,\rho_{t+1}[S] - \boldsymbol{\theta}_{t,S}(t+1) \cdot \boldsymbol{Y}(t+1)\right],$$

where $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is a fair dynamic $\mathbb{G}$-hedger and $\{\rho_t^{MCV}\}_{t=0}^{T-1}$ is a model-consistent dynamic t-valuation.

In the framework of Barigou et al. (2019), the hedging strategy is allowed to depend on the full insurer-relevant filtration $\mathbb{G}$, whereas in the three-step hedge-based valuation, trading strategies are restricted to information admissible in financial markets (based on $\mathbb{F}$). This distinction ensures that hedging strategies in the three-step approach comply with no–insider-trading requirements by construction. A further distinction lies in the treatment of risk sources. In Barigou et al. (2019), non-traded systematic and idiosyncratic risks are not separated but instead valued jointly in the residual valuation step. In contrast, the three-step hedge-based valuation explicitly separates financial risk, non-traded systematic risk, and purely actuarial risk, allowing each component to be treated individually using an appropriate valuation principle.

4.8.2 Two-step regression valuation of Barigou, Bignozzi, and Tsanakas (2022)

Barigou et al. (2022) define the 'Two-step regression valuation' $\{\rho_t\}_{t=0}^{T-1}$ for any claim $S \in \mathcal{C}_T$ as follows:

$$\rho_{T-1}[S] = \boldsymbol{\theta}_{T-1,S}(T) \cdot \boldsymbol{Y}(T-1) + i\boldsymbol{\eta}_{T-1,S-\boldsymbol{\theta}_{T-1,S}(T)\cdot\boldsymbol{Y}(T-1)}(T) \cdot \boldsymbol{Y}(T-1)$$

and for $t = 0, 1, \ldots, T-1$,

$$\rho_t[S] = \boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1) \cdot \boldsymbol{Y}(t) + i\boldsymbol{\eta}_{t,\rho_{t+1}[S]-\boldsymbol{\theta}_{t,\rho_{t+1}[S]}(t+1)\cdot\boldsymbol{Y}(t)}(t+1) \cdot \boldsymbol{Y}(t),$$

where $i \in (0,1)$ is the cost-of-capital rate, $\{\boldsymbol{\theta}_t\}_{t=0}^{T-1}$ is a dynamic mean-variance $\mathbb{G}$-hedger defined by

$$\boldsymbol{\theta}_t(S) = \underset{\boldsymbol{\beta}\in\boldsymbol{\Theta}_t^{\mathbb{G}'}}{\arg\min}\, \mathbb{E}_t\left[\left(S - \boldsymbol{\beta}(T) \cdot \boldsymbol{Y}(T)\right)^2\right],$$

and $\{\boldsymbol{\eta}_t\}_{t=0}^{T-1}$ is a dynamic Koenker-–Bassett loss $\mathbb{G}$-hedger defined by

$$\boldsymbol{\eta}_t(S) = \underset{\boldsymbol{\beta}\in\boldsymbol{\Theta}_t^{\mathbb{G}'}}{\arg\min}\, \mathbb{E}_t\left[l_\alpha\left(S - \boldsymbol{\beta}(T) \cdot \boldsymbol{Y}(T)\right)\right],$$

where $l_a(x) = \frac{\alpha}{1-\alpha} \times \max(x,0) + \max(-x,0)$, with $\alpha \in (0,1)$.

Similar to the framework of Barigou et al. (2019), the two-step regression valuation of Barigou et al. (2022) allows the hedging strategy to depend on the full insurer-relevant filtration $\mathbb{G}$. As a result, the construction of the hedge is not restricted to information admissible for trading in financial markets. Moreover, non-traded systematic and idiosyncratic risks are not treated separately, but are instead incorporated jointly through a second hedging step based on the Koenker-–Bassett loss function. By employing this asymmetric loss function, the combined hedging strategies implicitly recover the Value-at-Risk of the T-claim S.

4.8.3 An example

In this subsection, we present a two-period example with the goal to compare the different valuations.

Consider the space $(\Omega, \mathcal{G}_2, \mathbb{G}, \mathbb{P})$. The financial-actuarial world on this space consists of three sources of uncertainty: financial risk, non-traded systematic risk, and policyholder-specific risk. The financial market includes a risk-free bank account growing at a constant rate r and a risky asset with initial price 100 that evolves on a two-period binomial tree. At time $t = 1$, the asset-price is 110 with probability 0.6 and 90 with probability 0.4. Conditional on the realization at time-1, the price at time-2 again moves up with probability 0.6 and down with probability 0.4, so that the transition probabilities are time-homogeneous. In addition, the model includes a non-traded systematic risk factor Z, interpreted as a cold-winter indicator, whose distribution at time-2 depends on its realization at time-1, thereby introducing persistence in the occurrence of cold winters. Finally, policyholder-specific risk is represented by a mortality process X, where death probabilities depend on the non-traded systematic risk factor and past survival, and death is absorbing once it occurs. This setup is described below.

Financial assets. The risk-free bank account grows at the constant rate r, and the risky asset has initial price $Y^{(1)}(0) = 100$, evolving on a binomial tree with time-homogeneous transition probabilities:

$$\begin{aligned}
\mathrm{P}\Big(Y^{(1)}(1) = 110\Big) = 0.6, \quad \mathrm{P}\Big(Y^{(1)}(1) = 90\Big) = 0.4,\\
\mathrm{P}\Big(Y^{(1)}(2) = 120 \mid Y^{(1)}(1) = 110\Big) = 0.6, \quad \mathrm{P}\Big(Y^{(1)}(2) = 100 \mid Y^{(1)}(1) = 110\Big) = 0.4,\\
\mathrm{P}\Big(Y^{(1)}(2) = 100 \mid Y^{(1)}(1) = 90\Big) = 0.6, \quad \mathrm{P}\Big(Y^{(1)}(2) = 80 \mid Y^{(1)}(1) = 90\Big) = 0.4.
\end{aligned}$$

Non-traded systematic risk (cold-winter indicator). The process $Z^{(1)}$ starts at $Z^{(1)}(0) = 0$ and evolves with the following transition probabilities:

$$\begin{aligned}
\mathrm{P}\Big(Z^{(1)}(1) = 1\Big) = 0.5, \quad \mathrm{P}\Big(Z^{(1)}(1) = 0\Big) = 0.5,\\
\mathrm{P}\Big(Z^{(1)}(2) = 1 \mid Z^{(1)}(1) = 1\Big) = 1,\\
\mathrm{P}\Big(Z^{(1)}(2) = 1 \mid Z^{(1)}(1) = 0\Big) = 0.5, \quad \mathrm{P}\Big(Z^{(1)}(2) = 0 \mid Z^{(1)}(1) = 0\Big) = 0.5.
\end{aligned}$$

Policyholder-specific risk (mortality). The process $X^{(1)}$ starts at $X^{(1)}(0) = 0$ (alive), where $X^{(1)}(t) = 1$ indicates that the policyholder has died by time-t. Death is absorbing, so $X^{(1)}(2) = 1$ whenever $X^{(1)}(1) = 1$. The remaining transition probabilities, which depend on the realization of the cold-winter indicator, are:

$$\begin{aligned}
\mathrm{P}\Big(X^{(1)}(1) = 1 \mid Z^{(1)}(1) = 1\Big) = 0.4,\\
\mathrm{P}\Big(X^{(1)}(1) = 1 \mid Z^{(1)}(1) = 0\Big) = 0.2,\\
\mathrm{P}\Big(X^{(1)}(2) = 1 \mid X^{(1)}(1) = 0,\, Z^{(1)}(2) = 1\Big) = 0.4,\\
\mathrm{P}\Big(X^{(1)}(2) = 1 \mid X^{(1)}(1) = 0,\, Z^{(1)}(2) = 0\Big) = 0.2
\end{aligned}$$

We assume that there exists a claim $S = \max\big(Y^{(1)}(2), 100\big)$, payable at time-2 provided that the policyholder dies between time-0 and time-2. We compute the time-0 value $\rho_0[S]$ using four methods. Throughout, we assume a continuously compounded risk-free rate of $r = 0.05$. Model-consistent valuations are based on the cost-of-capital principle (Barigou et al., 2019; Dhaene et al., 2017; Linders, 2023), with perunage $i = 0.2$ (called the cost-of-capital rate) and value-at-risk level $\alpha = 0.15$.

Hedge-based valuation of Barigou et al. (2019): At time-1, we first compute the hedge $\boldsymbol{\theta}_{1,S}(2)$ via mean–variance hedging. Table 4.1 reports the resulting hedge ratios state by state. The hedge is obtained by choosing $\boldsymbol{\theta}_{1,S}(2)$ such that the payoff $\boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)$ is as close to the claim S as possible.

We then value the remaining hedging error $S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)$ with a cost-of-capital ρ_1^{MCV}

$$\rho_1[S] = \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(1) + \rho_1^{MCV}[S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)] .$$

Table 4.2 reports the resulting values $\rho_1[S]$ in each time-1 state.

At time-0, we compute a time-0 hedge $\boldsymbol{\theta}_{0,S}(1)$ by choosing $\boldsymbol{\theta}_{0,\rho_1[S]}(1)$ such that the payoff $\boldsymbol{\theta}_{0,\rho_1[S]}(1) \cdot \boldsymbol{Y}(1)$ is as close to the time-1 values $\rho_1[S]$ as possible. Table 4.3 reports the corresponding hedge. The time-0 valuation then applies the cost-of-capital ρ_0^{MCV} to the residual $\rho_1[S] - \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(1)$:

$$\rho_0[S] = \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(0) + \rho_0^{MCV}[\rho_1[S] - \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(1)] .$$

The resulting time-0 value is $\rho_0[S] = 73.882$.

State (Y_1, Z_1, X_1)	$\theta_1^{(0)}(2)$	$\theta_1^{(1)}(2)$
(110,1,1)	0.000	1.000
(110,1,0)	0.000	0.400
(110,0,1)	0.000	1.000
(110,0,0)	0.003	0.300
(90,1,1)	95.123	0.000
(90,1,0)	49.464	0.000
(90,0,1)	95.123	0.000
(90,0,0)	41.854	0.000

Table 4.1: Time-1 hedge $\boldsymbol{\theta}_{1,S}(2)$ under the hedge-based valuation of Barigou et al. (2019). This hedge determines the traded portfolio used to compute the hedging error $S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)$.

We then value the remaining hedging error $S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)$ with a cost-of-capital ρ_1^{MCV}

$$\rho_1[S] = \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(1) + \rho_1^{MCV}[S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)] .$$

Table 4.2 reports the resulting values $\rho_1[S]$ in each time-1 state.

State (Y_1, Z_1, X_1)	$\rho_1[S]$
(110,1,1)	110.000
(110,1,0)	57.698
(110,0,1)	110.000
(110,0,0)	48.981
(90,1,1)	95.123
(90,1,0)	60.879
(90,0,1)	95.123
(90,0,0)	55.171

Table 4.2: Time-1 values $\rho_1[S]$ for the hedge-based valuation of Barigou et al. (2019) obtained by adding the time-1 value of the hedge $\boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(1)$ to the cost-of-capital valuation of the hedging error under ρ_1^{MCV}. These values are the inputs to the time-0 hedge.

At time-0, we compute a time-0 hedge $\boldsymbol{\theta}_{0,S}(1)$ by choosing $\boldsymbol{\theta}_{0,\rho_1[S]}(1)$ such that the payoff $\boldsymbol{\theta}_{0,\rho_1[S]}(1) \cdot \boldsymbol{Y}(1)$ is as close to the time-1 values $\rho_1[S]$ as possible. Table 4.3 reports the corresponding hedge. The

time-0 valuation then applies the cost-of-capital ρ_0^{MCV} to the residual $\rho_1[S] - \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(1)$:

$$\rho_0[S] = \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(0) + \rho_0^{MCV}[\rho_1[S] - \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(1)]\,.$$

The resulting time-0 value is $\rho_0[S] = 73.882$.

$\theta_0^{(0)}(1)$	$\theta_0^{(1)}(1)$
61.090	0.052

Table 4.3: Time-0 hedge $\boldsymbol{\theta}_{0,S}(1)$ for the hedge-based valuation of Barigou et al. (2019). This hedge defines the residual $\rho_1[S] - \boldsymbol{\theta}_{0,S}(1) \cdot \boldsymbol{Y}(1)$ valued by ρ_0^{MCV}.

Two-step regression valuation of Barigou et al. (2022): This method replaces the explicit actuarial valuation of the hedging error by a second hedging step. At time-1, the mean-variance hedge $\boldsymbol{\theta}_{1,S}(2)$ is computed as before, and a second hedge $\boldsymbol{\eta}_{1,S}(2)$ is computed using the Koenker–Bassett loss function. The time-1 value is then obtained directly from:

$$\rho_1[S] = \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(1) + i\,\boldsymbol{\eta}_{1,S}(2) \cdot \boldsymbol{Y}(1), \tag{4.16}$$

$$\rho_0[S] = \boldsymbol{\theta}_{0,\rho_1[S]}(1) \cdot \boldsymbol{Y}(0) + i\,\boldsymbol{\eta}_{0,\rho_1[S]}(1) \cdot \boldsymbol{Y}(0). \tag{4.17}$$

Table 4.4 reports the time-1 hedges $(\boldsymbol{\theta}_1(2), \boldsymbol{\eta}_1(2))$, and Table 4.5 reports the resulting $\rho_1[S]$ values. The resulting time-0 value is $\rho_0[S] = 71.211$.

State (Y_1, Z_1, X_1)	$\theta_1^{(0)}(2)$	$\theta_1^{(1)}(2)$	$\eta_1^{(0)}(2)$	$\eta_1^{(1)}(2)$
(110,1,1)	0.000	1.000	0.000	0.000
(110,1,0)	0.000	0.400	0.056	0.600
(110,0,1)	0.000	1.000	0.000	0.000
(110,0,0)	0.003	0.300	-0.035	0.700
(90,1,1)	95.123	0.000	0.000	0.000
(90,1,0)	38.049	0.000	8.326	0.513
(90,0,1)	95.123	0.000	0.000	0.000
(90,0,0)	28.537	0.000	-0.016	0.700

Table 4.4: Time-1 hedges for the two-step regression valuation of Barigou et al. (2022). The first hedge $\boldsymbol{\theta}_1(2)$ captures the mean-variance (quadratic) hedging component, while $\boldsymbol{\eta}_1(2)$ provides an additional VaR adjustment. Together they determine $\rho_1[S]$ via Equation 4.16.

State (Y_1, Z_1, X_1)	$\rho_1[S]$
(110,1,1)	110.000
(110,1,0)	57.201
(110,0,1)	110.000
(110,0,0)	48.400
(90,1,1)	95.123
(90,1,0)	48.949
(90,0,1)	95.123
(90,0,0)	41.140

Table 4.5: Time-1 values $\rho_1[S]$ under the two-step regression valuation, computed from the hedges in Table 4.4 using Equation 4.16. These values determine the time-0 hedger step in Equation 4.17.

Three-step hedge-based valuation in a multi-period setting: This method inserts a cost-of-capital $\mathcal{F}_{t+1}$-conditional t-valuation step before applying the cost-of-capital ρ_t^{MCV}. All trading strategies in this example are computed using the mean-variance hedger, to allow for comparison. At time-1, we compute $\boldsymbol{\theta}_{1,S}(2)$ and then evaluate the residual $S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2)$ conditionally. The time-1 and time-0 values are calculated as:

$$\rho_1[S] = \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(1) + \rho_1^{MCV}\left(\rho_1^{CMCV}[S - \boldsymbol{\theta}_{1,S}(2) \cdot \boldsymbol{Y}(2) \mid \mathcal{F}_2]\right), \tag{4.18}$$

$$\rho_0[S] = \boldsymbol{\theta}_{0,\rho_1[S]}(1) \cdot \boldsymbol{Y}(0) + \rho_0^{MCV}\left(\rho_0^{CMCV}\left[\rho_1[S] - \boldsymbol{\theta}_{0,\rho_1[S]}(1) \cdot \boldsymbol{Y}(1) \mid \mathcal{F}_1\right]\right). \tag{4.19}$$

Table 4.6 reports the time-1 hedge $\boldsymbol{\theta}_{1,S}(2)$. Table 4.7 reports both the output $\rho_1^{MCV}(\cdot)$ and the resulting $\rho_1[S]$. Table 4.8 provides the time-0 hedge $\boldsymbol{\theta}_{0,S}(1)$ used in Equation 4.19. The resulting time-0 value is $\rho_0[S] = 75.471$.

State (Y_1, Z_1, X_1)	$\theta_1^{(0)}(2)$	$\theta_1^{(1)}(2)$
(110,1,1)	108.060	0.000
(110,1,0)	56.922	0.000
(110,0,1)	108.060	0.000
(110,0,0)	0.000	0.440
(90,1,1)	95.123	0.000
(90,1,0)	49.464	0.000
(90,0,1)	95.123	0.000
(90,0,0)	41.854	0.000

Table 4.6: Time-1 hedge $\boldsymbol{\theta}_{1,S}(2)$ under the three-step hedge-based valuation.

State (Y_1, Z_1, X_1)	$\rho_1^{MCV}(\cdot)$	$\rho_1[S]$
(110,1,1)	3.653	111.712
(110,1,0)	3.653	60.574
(110,0,1)	3.882	111.941
(110,0,0)	3.882	52.282
(90,1,1)	3.653	98.776
(90,1,0)	3.653	53.117
(90,0,1)	3.348	98.471
(90,0,0)	3.348	45.202

Table 4.7: Time-1 calculation of the second term in Equation 4.18 and resulting $\rho_1[S]$ under the three-step valuation. The column $\rho_1^{MCV}(\cdot)$ is cost-of-capital model-consistent valuation applied to the $\mathcal{F}_2$-conditional 1-valuation in Equation 4.18.

$\theta_0^{(0)}(1)$	$\theta_0^{(1)}(1)$
21.977	0.450

Table 4.8: Time-0 hedge $\boldsymbol{\theta}_{0,S}(1)$ used in the time-0 valuation in Equation 4.19.

Two-step market evaluation of Pelsser and Stadje (2014) in a multi-period setting: This approach evaluates the claim under a risk-neutral measure $\mathbb{Q}$ for the traded asset, while the

cost-of-capital actuarial adjustment is applied through $e^{-r}\rho^{CMCV}(\cdot \mid \mathcal{F}^Y)$ inside the $\mathbb{Q}$-expectation:

$$\rho_1[S] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_1^{CMCV}\left(S \mid \mathcal{F}_2^Y\right)\right], \tag{4.20}$$

$$\rho_0[S] = \mathbb{E}^{\mathbb{Q}}\left[e^{-r}\rho_0^{CMCV}\left(\rho_1[S] \mid \mathcal{F}_1^Y\right)\right]. \tag{4.21}$$

Table 4.9 reports the resulting time-1 values $\rho_1[S]$. The resulting time-0 value is $\rho_0[S] = 71.398$.

State (Y_1, Z_1, X_1)	$\rho_1[S]$
(110,1,1)	110.000
(110,1,0)	57.200
(110,0,1)	110.000
(110,0,0)	48.400
(90,1,1)	95.123
(90,1,0)	49.464
(90,0,1)	95.123
(90,0,0)	41.854

Table 4.9: Time-1 values $\rho_1[S]$ under the two-step market evaluation, computed as $\mathbb{Q}$-expectations of the discounted actuarial step in Equation 4.20.

The results of the two-step regression valuation of Barigou et al. (2022), and the two-step market evaluation of Pelsser and Stadje (2014) are very close, yielding time-0 values of 71.211 and 71.398, respectively. The hedge-based valuation of Barigou et al. (2019) produces a slightly higher value of 73.882. Finally, the three-step hedge-based valuation proposed in the previous sections yields the highest value, 75.471, reflecting its ability to explicitly account for the dependence between financial risk and policyholder-specific risk. Moreover, the three-step hedge-based valuation restricts risky-asset investment to strategies based solely on publicly available information, thereby ensuring compliance with no-insider-trading requirements.

4.9 Concluding remarks

In this chapter, we extend the class of three-step hedge-based valuations proposed by Linders (2023) to a discrete-time, multi-period setting. This valuation principle explicitly distinguishes between systematic risks and diversifiable risks, recognizing that they must be valued differently. In contrast to the multi-period hedge-based valuation framework of Barigou et al. (2019), non-traded systematic risk is treated as a separate layer from diversifiable risk, which cannot be eliminated by pooling across multiple policyholders and instead must be valued through an additional valuation step.

Three-step hedge-based valuations proceed by first pricing the hedgeable component of a hybrid claim using the current market price of the hedge. The remaining residual claim is then valued conditionally on the realization of the traded assets and systematic risk factors, allowing the diversifiable component of the residual risk to be evaluated with an actuarial approach within each systematic scenario. Finally, an actuarial valuation is applied to account for uncertainty across the systematic scenarios. To ensure time-consistency in this valuation methodology, we employ a stepwise backward iteration procedure to construct a fair dynamic extension of the three-step hedge-based valuation. We show that, in a discrete-time, multi-period setting, the resulting class of dynamic three-step hedge-based valuations is equivalent to the class of dynamic fair valuations of Barigou et al. (2019).

A central contribution of this chapter is that these results are developed in an information-sensitive setting with three nested filtrations: an asset-price filtration, a financial filtration, and an insurer-relevant filtration. We impose a no-insider-trading requirement by restricting trading strategies in

the risky assets to depend only on publicly available information contained in the financial filtration. Since the insurer-relevant filtration is not observable by market participants, allowing it to influence risky-asset investment decisions would constitute insider trading (Artzner et al., 2024). Within this setting, we re-derive the characterization results linking static valuations to static hedgers, as well as the characterization results linking dynamic valuations to dynamic hedgers.

Within the same framework, we further show that an adapted version of the class of two-step market evaluations proposed by Pelsser and Stadje (2014) is also equivalent to the class of dynamic fair valuations (Barigou et al., 2019) in a discrete-time, multi-period setting. Using a simple example, we then compare the three-step hedge-based valuation and the two-step market evaluation with the other dynamic fair valuation methodologies proposed in the literature: the hedge-based valuation of Barigou et al. (2019) and the two-step regression valuation of Barigou et al. (2022). Overall, this chapter provides a unifying perspective on dynamic fair valuation methodologies for insurers operating under regulatory and information constraints.

From a practical perspective, the proposed framework offers insurers a realistic methodology to value long-dated hybrid insurance liabilities over multiple periods in a practical environment. The resulting valuations align with no-insider-trading requirements and with regulatory valuation principles of market-consistency and model-consistency, as embedded in frameworks such as Solvency II.

Chapter 5

Outlook

The actuarial profession is undergoing rapid adaptation as insurance systems are reshaped by demographic, financial, technological, and regulatory changes. Demographic developments, particularly the increasing interdependence of longevity trends across populations, have made risk quantification more complex. Financial innovations have created new ways through which insurance risk can be traded or shared. Regulatory changes have resulted in the adoption of more sophisticated frameworks, while technological advancements have made their implementation practically feasible.

Accordingly, the need to address challenges in the area of risk transfer has become a major priority for insurance entities globally. Insurance risk is now spread across a wider network of insurers, reinsurers, capital-market participants, and collective risk-sharing arrangements, each operating with different incentives, constraints, and information sets. As a result, the insurance industry requires an improved understanding of how complex risk dynamics are modeled, how insurance risk is shared or redistributed, and how valuation principles determine the pricing of transferred risks over time.

This thesis is situated within this changing environment and responds to these developments by contributing to an improved actuarial perspective on risk modeling, risk sharing, and risk valuation. The work presented here aligns with the profession's ongoing evolution towards principles that address demographic complexity, accommodate financial innovation, leverage technological advancements, and respond to evolving regulatory requirements.

The remainder of this section provides a brief overview of the chapter-specific contributions and outlines directions for future research.

5.1 Main contribution

This thesis makes three primary contributions to the actuarial and insurance literature. Its three main chapters examine the common theme of risk transfer from complementary perspectives: modeling dependence structures that underlie the transfer of longevity risk, designing equitable compensation-based risk-sharing arrangements, and developing valuation frameworks that ensure consistent pricing of transferred risk under regulatory constraints. The main contributions of each chapter are summarized below.

Chapter 2 makes a methodological contribution to the modeling of period effects by introducing time series vine copulas as a unified tool to jointly capture serial, cross-sectional, and lagged cross-sectional dependence between period effects. This chapter demonstrates that important features of period effects, such as non-normality, asymmetric tail dependence, and lagged dependence across populations, are not fully captured by the commonly used VAR, sparse VAR, and ARIMA-copula

models. Further, it establishes that time series vine copulas provide greater robustness and superior in-sample and out-of-sample performance. Most importantly, this chapter shows that these modeling choices are not just statistical refinements but materially affect the pricing of mortality-linked securities, thereby highlighting the financial relevance of accurately modeling complex dependence between period effects.

Chapter 3 contributes to the theory of compensation-based risk-sharing by developing a unified framework that accommodates both active and passive administrators within a single setting. Extending existing tontine and compensation-based risk-sharing models, this chapter derives explicit conditions under which participants are 'fairly' compensated for their initial investment in the pool. By analyzing both two-participant and homogeneous tontine settings, Chapter 3 clarifies the links between decentralized risk-sharing arrangements and centralized insurance mechanisms. The results also show how administrative design affects the distribution of the fund's value. We identify conditions under which the participants are indifferent between active and passive administrators, in expected value terms. The chapter provides a framework in which contribution-based risk-sharing principles and tontine-style setups can be combined to mitigate counterparty risk and offer an alternative mechanism for sharing risk.

Chapter 4 develops a unifying valuation framework for long-dated hybrid insurance liabilities by extending three-step hedge-based valuations to a discrete-time, multi-period setting within nested information filtrations. The chapter also departs from the existing literature by explicitly imposing a 'no-insider-trading' constraint. Its central contribution is to show that the class of dynamic three-step hedge-based valuations and the class of adapted two-step market evaluations are both equivalent to the class of dynamic fair valuations, thereby reconciling several seemingly distinct valuation methodologies in the literature. By explicitly distinguishing between the types of information available to an insurer, the framework provides a realistic and regulatory-aligned approach to dynamic insurance valuation. The resulting framework is a methodology that is simultaneously market-consistent, model-consistent, and time-consistent.

5.2 Future research

Despite the contributions made in this thesis, several open questions remain. In what follows, a selection of important directions for future research is outlined.

Chapter 2: Modeling mortality dependence with time series vine copulas

- **Evolving dependence dynamics:** Time series vine copulas used in Chapter 2 assume a single, static dependence structure. However, several authors have noted that dependence dynamics evolve over time (see e.g., Zhou (2019) and Zhou and Ji (2021)). Future research could therefore investigate the impact of dynamically evolving time series vine copulas on the modeling and projection of period effects.
- **Single-step estimation:** The time series vine copula approach used in Chapter 2 models dependence between period effects in a single step, improving upon ARIMA-copula models that capture dependence through a two-step procedure. However, in both approaches, period effects are first estimated using a mortality model, such as the Lee-Carter model, before the dependence structure is modeled. A natural direction for future research is to unify these steps within a single framework that jointly estimates the mortality model and the dependence structure of the period effects.

Chapter 3: Compensation-based risk-sharing

- **Multi-period setting:** Contribution-based risk-sharing schemes have been studied in a multi-period setting (Abdikerimova et al., 2024). Extending compensation-based risk-sharing schemes to a similar multi-period framework is a natural direction for future research and would enhance the practical applicability of such schemes.
- **Comparison of compensation-based risk-sharing with traditional insurance:** The shareholders of an insurer can be modeled as a 'special administrator' who contributes capital when the pool's losses exceed investments and receives compensation when the pool's losses fall below investments. This perspective enables a systematic comparison of traditional insurance arrangements and compensation-based risk-sharing schemes. An important direction for future research here is to assess, from a participant's perspective, the relative desirability of compensation-based schemes with an administrator compared with traditional insurance.

Chapter 4: Fair dynamic valuation of hybrid insurance liabilities

- **Weaker forms of time-consistency:** The notion of time-consistency used in Chapter 4 to construct dynamic valuations may be too strong and can lead to high premiums for long-dated insurance contracts (see e.g., Roorda and Schumacher (2007)). Investigating weaker notions of time-consistency is therefore an important direction for future research to ensure that calculated premiums remain competitive and to improve the commercial viability of insurance contracts priced using time-consistent methodologies.
- **Dependence between systematic risks and policyholder-specific risks:** Chapter 4 assumes that, conditional on the systematic risk level, policyholder-specific risks are mutually independent. While this assumption is appropriate for many insurance applications, it does not hold in general. For instance, consider the systematic risk of a pandemic occurring. Mortality outcomes across policyholders will exhibit high dependence if a pandemic occurs. An important direction for future research is to extend the fair valuation framework to accommodate such systematic risks.
- **Application of fair valuation to non-life insurance:** Chapter 4 focuses on the application of fair valuation in an insurance setting, with modeling choices and examples motivated by a life-insurance context. Managing systematic risk is also a key challenge for non-life insurers (Swiss Re Institute, 2025). Since three-step hedge-based valuations explicitly account for systematic risks, this framework is also promising for non-life insurance applications. Extending fair valuation methodologies to the non-life setting therefore represents a promising avenue for future research.

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