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Title:
Corrigendum to "On eigenvalues of Laplacian matrix for a class of directed signed graphs" (vol 523, pg 281, 2017)

Date:
2017-10-01

Citation:
Ahmadizadeh, S., Shames, I., Martin, S. & Nesic, D. (2017). Corrigendum to "On eigenvalues of Laplacian matrix for a class of directed signed graphs" (vol 523, pg 281, 2017). Linear Algebra and Its Applications, 530, pp.541-557. https://doi.org/10.1016/j.laa.2017.06.023.

Persistent Link:
https://hdl.handle.net/11343/297963

On Eigenvalues of Laplacian Matrix for a Class of Directed Signed Graphs

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Abstract

This note corrects an error in the results of Subsection 3.1 in authors' paper "On Eigenvalues of Laplacian Matrix for a Class of Directed Signed Graphs", which appeared in Linear Algebra and its Applications 523 (2017), 281 – 306.

Keywords: Directed signed graph, Eigenvalues of Laplacian matrix.

1. Preliminaries and Graph Notation

Throughout this paper, $I_N \in \mathbb{R}^{N\times N}$, $\mathbf{1}_N \in \mathbb{R}^N$, and $\mathbf{0}_N \in \mathbb{R}^N$ denote the $N \times N$ identity matrix, the N-dimensional vectors containing 1, and 0 in every entry, respectively. The standard bases in $\mathbb{R}^N$ are represented by $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$ where $\mathbf{e}_i$ is the i^{th} column of I_N. The 2-norm of a vector $x \in \mathbb{R}^N$ is shown by $\|x\|$. For a complex variable, vector or matrix, $\Re(\cdot)$ and $\Im(\cdot)$ stand for the real and imaginary parts. For a matrix $A \in \mathbb{R}^{N\times N}$, $\text{Spec}(A) = \{\lambda_i(A)\}_{i=1}^N$ denotes the set of eigenvalues of A where $\Re(\lambda_1) \le \Re(\lambda_2) \le \cdots \le \Re(\lambda_N)$. An eigenvalue $\lambda_i(A)$ is called semisimple if its algebraic and geometric multiplicities are equal. The operator $\text{diag}(\cdot)$ constructs a block diagonal matrix from its arguments. The entry in the i^{th} row and j^{th} column of a matrix A is represented by $[A]_{ij}$, while the i^{th} entry of a vector x is denoted by $[x]_i$. For a set $\mathcal{A}$, its cardinality is denoted by $|\mathcal{A}|$.

A weighted directed signed graph $\mathcal{G}$ is represented by the triple $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ where $\mathcal{V} = \{1, \dots, N\}$ is the nodes set, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set, and $\mathcal{W} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a weight function that maps each $(i, j) \in \mathcal{E}$ to a nonzero scalar a_{ij} and returns 0 for all other $(i, j) \notin \mathcal{E}$. The adjacency matrix $A \in \mathbb{R}^{N\times N}$ captures the interconnection between the nodes in the graph where $[A]_{ij} = a_{ij} \neq 0$ iff $(i, j) \in \mathcal{E}$. For the edge (i, j), we follow the definition corresponding to a sensing convention which indicates that node i receives information form node j or equivalently, the node j influences the node i; see [13] for more information. For each node $i \in \mathcal{V}$, $\mathcal{N}_i$ denotes the set of its neighbors, i.e., $\mathcal{N}(i) = \{j \mid a_{ij} \neq 0\}$.

For a given graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ and a set $\overline{\mathcal{V}} \subseteq \mathcal{V}$, the corresponding induced subgraph is denoted by $\mathcal{G}(\overline{\mathcal{V}}, \overline{\mathcal{E}}, \overline{\mathcal{W}})$, where the set $\overline{\mathcal{E}}$ is defined as $\overline{\mathcal{E}} = \{(i, j) \in \mathcal{E} \mid i, j \in \overline{\mathcal{V}}\}$, and $\overline{\mathcal{W}} : \overline{\mathcal{V}} \times \overline{\mathcal{V}} \to \mathbb{R}$ is defined as $\overline{\mathcal{W}}(i, j) = \mathcal{W}(i, j)$. In order to categorize edges in terms of the sign of their values, we define the sets $\mathcal{E}^+ = \{(i, j) \mid a_{ij} > 0\}$, and $\mathcal{E}^- = \mathcal{E} \backslash \mathcal{E}^+ = \{(i, j) \mid a_{ij} < 0\}$. We call the edges in $\mathcal{E}^+$ and $\mathcal{E}^-$ positive edges and negative edges, respectively. Subsequently, for a signed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$, we denote the subgraph with non-negative weights by $\mathcal{G}(\mathcal{V}, \mathcal{E}^+, \mathcal{W}^+)$ where $\mathcal{W}^+ : \mathcal{V} \times \mathcal{V} \to \mathbb{R}_{\ge 0}$ is defined as $\mathcal{W}^+(i, j) = \mathcal{W}(i, j)$ for all $(i, j) \in \mathcal{E}^+$ and $\mathcal{W}^+(i, j) = 0$ for all $(i, j) \notin \mathcal{E}^+$. Similarly, for a signed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$, we denote the subgraph with non-positive weights by $\mathcal{G}(\mathcal{V}, \mathcal{E}^-, \mathcal{W}^-)$. The superposition of two signed directed graphs $\mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1) \bigoplus \mathcal{G}_2(\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2)$ is a new graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and, $\mathcal{W}(i, j) = \mathcal{W}_1(i, j) + \mathcal{W}_2(i, j)$ for every $(i, j) \in \{\mathcal{V} \times \mathcal{V}\}$.

For a directed graph, the in-degree and out-degree of node i are defined as $d_i^{in} = \sum_j a_{ji}$ and $d_i^{out} = \sum_j a_{ij}$ respectively. The Laplacian matrix* $L \in \mathbb{R}^{N\times N}$ is defined by $L = D - A$ where $D = \text{diag}\{d_1^{out}, \dots, d_N^{out}\}$. Since the rows of the

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*The current definition of Laplacian matrix has been inspired from a large variety of applications in control such as consensus [15], security analysis of complex networks [11; 12], and synchronization in networks of oscillators [2; 10]. However, there is another way to define the Laplacian matrix for weighted signed graphs in which the in-degree and out-degree of node i are defined as $d_i^{in} = \sum_j |a_{ji}|$ and $d_i^{out} = \sum_j |a_{ij}|$ respectively [5].

Laplacian matrix add to zero, $\mathbf{1}_N$ is always one of its eigenvector that corresponds to the eigenvalue 0. This eigenvalue is called the trivial eigenvalue while the rest of eigenvalues is called non-trivial eigenvalues.

Let $\Pi = I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T$ denote the orthogonal projection matrix onto the subspace of $\mathbb{R}^N$ perpendicular to $\mathbf{1}_N$. The matrix Π is symmetric and since $L\mathbf{1}_N = \mathbf{0}$, $L\Pi = L$ and $\Pi L^T = L^T$ for any graph. We define a matrix $Q \in \mathbb{R}^{(N-1)\times N}$ whose rows are the orthonormal bases for $\text{span}\{\mathbf{1}_N\}^\perp$ where $\perp$ denotes the orthogonal complement of the space. Hence, Q^T is a full column rank matrix. On $\text{span}\{\mathbf{1}_N\}^\perp$, the Laplacian matrix is equivalent to the so-called *reduced Laplacian* $\bar{L} \in \mathbb{R}^{(N-1)\times(N-1)}$ which is defined by [13],

$$\bar{L} := QLQ^T. \tag{1}$$

A *path* of length r from $i_1 \in \mathcal{V}$ to $i_r \in \mathcal{V}$ in graph $\mathcal{G}$ is a sequence $(i_1, i_2, \ldots, i_r)$ of distinct nodes in $\mathcal{V}$ where i_{j+1} is a neighbor of i_j for all $j = 1, \ldots, r-1$. If there exists a path (no path) from the node j to the node i, then the node i is (not) reachable from node j. We use $j \rightarrowtail i$ ($j \not\rightarrowtail i$) to show the existence (absence) of path from j to i. A node i is a globally reachable node if it is reachable from all other nodes of the graph. Similar to [13], we say that two nodes i and j are connected if the graph contains two paths, one starting from node i and the other one from j that both terminates at the same node. A graph $\mathcal{G}$ is connected if every pair of nodes is connected. This notion of a connected graph corresponds to the scrambling matrices [9]. It has been shown that the graph is connected if and only if there exists at least one *globally reachable node*; the node, to which, there exists at least one path from every node in the graph [13]. A graph $\mathcal{G}$ is strongly connected if for every $i \in \mathcal{V}$ and $j \in \mathcal{V}$, $i \rightarrowtail j$. Hence, the graph $\mathcal{G}$ is strongly connected if and only if every node of the graph is a globally reachable node.

2. Adding an extra (un)directed negative edge to directed signed graphs

In this section, we correct the main result of Subsection 3.1 in [3] that aimed at answering the following question:

Question 1. *Consider a signed graph $\mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1)$ with the Laplacian matrix L_1. Construct a new graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W}) = \mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1) \oplus \mathcal{G}_2(\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2)$, where $\mathcal{E}_2 = \{(u,v),(v,u)\}$ and $\mathcal{W}_2(u,v) = -\delta_{uv} \le 0$ and $\mathcal{W}_2(v,u) = -\delta_{vu} \le 0$. Denote L the Laplacian matrix of $\mathcal{G}$. Find conditions on $\mathcal{G}_1$ and a bound on δ_{uv} and δ_{vu} such that*

$$0 = \lambda_1(L) < \Re(\lambda_2(L)) \le \cdots \le \Re(\lambda_N(L)). \tag{2}$$

Unfortunately, Theorem 1 in [3, Subsection 3.1] and Lemma 6 in [3, Appendix *A*] are not correct in general. We used continuity of eigenvalues of a matrix with respect to a single parameter $\delta_{uv} = \delta_{vu} = \delta$ to prove the lemma and, subsequently, the sufficiency of the theorem. In the proof of Lemma 6 in [3, Appendix *A*], we claimed that the continuity of eigenvalues of $\bar{L} = QLQ^T$ with respect to δ states that

$$\exists\, \delta^* \text{ such that } \forall\ 0 < \delta < \delta^*,\ \Re\{\lambda_i(\bar{L})\}_{i=1}^{N-1} > 0, \tag{3}$$

and also with $\delta = \delta^*$, $\Re\{\lambda_i(\bar{L})\} = 0$ for some i (this statement is true). We then claimed that $\det\left(\bar{L}_1^{-1}\bar{L}\right) = 0$ for $\delta = \delta^*$. This statement is true if and only if there exists at least one zero eigenvalue, i.e. $\lambda_i(\bar{L}) = 0$. However, there can be a case where all eigenvalues of $\bar{L}$ with $\Re\{\lambda_i(\bar{L})\} = 0$ are complex. In this case, Theorem 1 in [3, Subsection 3.1] and Lemma 6 in [3, Appendix *A*] are not valid. Here, we point out that the necessity of Theorem 1 in [3, Subsection 3.1] is true since we used the result of Lemma 5 in [3, Appendix *A*] in the proof. We first state the following definition and lemma that are used to prove the main results of this subsection.

Definition 1. *Consider a signed graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ with the Laplacian matrix L. For given two arbitrary nodes u and v, and variables $\delta_{uv}, \delta_{vu}, \omega \ge 0$, we define $r(\omega, \delta_{uv}, \delta_{vu})$ as follows*

$$r(\omega, \delta_{uv}, \delta_{vu}) := (\mathbf{e}_u - \mathbf{e}_v)^T Q^T (\bar{L} - j\omega I)^{-1} Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v), \tag{4}$$

where $\bar{L} = QLQ^T$ is the reduced Laplacian matrix.

Lemma 1. *Assume for a given directed graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W})$ with Laplacian matrix L,*

$$spec\{L\} = \{\underbrace{0,\ldots,0}_{d},\Lambda\},$$

where the set Λ contains the non-zero eigenvalues of L. Then,

1. *$spec\{\bar{L}\} = \{\underbrace{0,\ldots,0}_{d-1},\Lambda\}$. Furthermore, if $d = 1$, then $\bar{L}$ is invertible.*
2. *Suppose further that all weights are non-negative. $\bar{L}$ is invertible and $\Re\{\lambda_i(\bar{L})\} > 0$ for $i = 1,\ldots,N-1$ if and only if $\mathcal{G}$ is connected* [†]*.*

Proof. The proof of the first part can be found in [14, Lemma1]. Even though the lemma deals with directed graphs with non-negative weights, the same arguments are applicable to directed signed graphs, since $L\mathbf{1}_N = \mathbf{0}$. If $\mathcal{G}$ is connected with non-negative weights, then $\text{Spec}(L) = \{0,\lambda_2,\ldots,\lambda_N\}$ with $\Re\{\lambda_2\} > 0$ [1]. Taking into account this point along with the first part of the lemma proves the second part. □

We now present a necessity result for the non-zero eigenvalues of Laplacian matrix to have positive real parts in the presence of a negative weight.

Theorem 1. *Consider a signed graph $\mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)$ with the Laplacian matrix L_1. Assume that L_1 has only one zero eigenvalue and the rest of its eigenvalues have positive real parts. Construct a new graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W}) = \mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)\oplus\mathcal{G}_2(\mathcal{V},\mathcal{E}_2,\mathcal{W}_2)$, where $\mathcal{E}_2 = \{(u,v),(v,u)\}$ and $\mathcal{W}_2(u,v) = -\delta_{uv}$, $\mathcal{W}_2(v,u) = -\delta_{vu}$ with $\delta_{uv} \geq 0$, $\delta_{vu} \geq 0$. Denote L the Laplacian matrix of $\mathcal{G}$. If the eigenvalues of L satisfy (2), then*

$$r(0,\delta_{uv},\delta_{vu}) < 1, \tag{5}$$

Proof. Define $\bar{L} = QLQ^T$ and suppose that the eigenvalues of L satisfy (2). We show the condition (5) holds. In view of the first part of Lemma 3 in Appendix A, if $r(0,\delta_{uv},\delta_{vu}) > 1$, then $\det\left(\bar{L}_1^{-1}\bar{L}\right) = \det\left(\bar{L}_1^{-1}\right)\det\left(\bar{L}\right) < 0$. Hence $\bar{L}$ has at least one non-positive eigenvalue. If $r(0,\delta_{uv},\delta_{vu}) = 1$, then $\det\left(\bar{L}_1^{-1}\bar{L}\right) = 0$. Thus, $\bar{L}$ has at least one zero eigenvalue. This means that the condition (5) is necessary for the eigenvalues of L satisfy (2). □

We now present a sufficiency theorem for the non-zero eigenvalues of Laplacian matrix to have positive real parts in the presence of a negative weight.

Theorem 2. *Consider a signed graph $\mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)$ with the Laplacian matrix L_1. Assume that L_1 has only one zero eigenvalue and the rest of its eigenvalues have positive real parts. Construct a new graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W}) = \mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)\oplus\mathcal{G}_2(\mathcal{V},\mathcal{E}_2,\mathcal{W}_2)$, where $\mathcal{E}_2 = \{(u,v),(v,u)\}$ and $\mathcal{W}_2(u,v) = -\delta q_{uv} \leq 0$, $\mathcal{W}_2(v,u) = -\delta q_{vu} \leq 0$ with $\delta > 0$ and given $q_{uv}, q_{vu} \geq 0$. Denote L the Laplacian matrix of $\mathcal{G}$. Let δ^* be obtained by,*

$$\begin{aligned} \min_{\delta_1\in\mathbb{R}_{>0},\omega\in\mathbb{R}_{\geq 0}} \quad & \delta_1 \\ \textit{subject to} \quad & r(\omega,\delta_1 q_{uv},\delta_1 q_{vu}) = 1. \end{aligned} \tag{6}$$

Then, the eigenvalues of L satisfy (2) for all $\delta \in [0,\delta^*)$.

Proof. Denote L_2 the Laplacian matrix of $\mathcal{G}_2$. Since $\mathcal{E}_2 = \{(u,v),(v,u)\}$, L_2 can be expressed as $L_2 = -(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^T$ with $\delta_{uv} = \delta q_{uv}$ and $\delta_{vu} = \delta q_{vu}$. Under conditions of the theorem, L can be written as $L = L_1 + L_2$, leading to the following expression for $\bar{L}$,

$$\bar{L} = \underbrace{QL_1Q^T}_{\bar{L}_1} - Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^T Q^T. \tag{7}$$

Since L_1 has only one zero eigenvalue and the rest of its eigenvalues have positive real parts, all eigenvalues of $\bar{L}_1 - j\omega I$ have positive real parts according to Property 1 in Lemma 1.

[†] For a graph with non-negative weights, the definition of connectivity, falls somewhere between the definitions of strong and weak connectivity [1; 13].

We now prove the theorem for the case with $q_{uv}, q_{vu} > 0$. The proof of theorem for the cases with $q_{uv} > 0,\ q_{vu} = 0$ or $q_{uv} = 0,\ q_{vu} > 0$ follows the same lines. Since $q_{uv}, q_{vu} > 0$, we have $\delta_{uv} = \delta q_{uv}$ and $\delta_{vu} = \delta q_{vu}$. Note that for $\delta = 0$, $\bar{L} = \bar{L}_1$ and for sufficiently large δ, i.e. $\delta > \sum_{i=1}^{N} [L_1]_{ii}$, the sum of the diagonal entries of L becomes negative. This leads L to have at least one eigenvalue with a negative real part, and consequently, in view of Property 1 in Lemma 1, $\bar{L}$ has that eigenvalue with negative real part. Hence, the continuity of eigenvalues of $\bar{L}$ with respect to δ states that

$$\exists\ \delta^* > 0 \text{ such that } \forall\ \delta \in [0, \delta^*),\ \Re\{\lambda_i(\bar{L})\}_{i=1}^{N-1} > 0, \tag{8}$$

and also with $\delta = \delta^*$, $\Re\{\lambda_i(\bar{L})\} = 0$ for some i. This means there exists at least one $\omega \in \mathbb{R}_{\geq 0}$ such that $\lambda_i(\bar{L}) = j\omega$ for $\delta = \delta^*$, or equivalently $\det(\bar{L} - j\omega I) = 0$. Using the expression of $\bar{L}$ in (7) and taking into account that $\bar{L}_1 - j\omega I$ is invertible, we have

$$\begin{aligned}\det(\bar{L} - j\omega I) &= \det\left(\bar{L}_1 - j\omega I - Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^T Q^T\right)\\ &= \det(\bar{L}_1 - j\omega I)\left(1 - (\mathbf{e}_u - \mathbf{e}_v)^T Q^T (\bar{L}_1 - j\omega I)^{-1} Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)\right)\\ &= \det(\bar{L}_1 - j\omega I)\left(1 - \delta^*(\mathbf{e}_u - \mathbf{e}_v)^T Q^T (\bar{L}_1 - j\omega I)^{-1} Q(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v)\right)\\ &= 0,\end{aligned} \tag{9}$$

where the last two equalities are obtained by applying Lemma 2 and taking into account $\delta_{uv} = \delta q_{uv}$ and $\delta_{vu} = \delta q_{vu}$. Since $\bar{L}_1 - j\omega I$ is invertible, we observe that $\det(\bar{L} - j\omega I) = 0$ if and only if $\delta^*(\mathbf{e}_u - \mathbf{e}_v)^T Q^T (\bar{L}_1 - j\omega I)^{-1} Q(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v) = 1$. By solving (6), we find the minimum value of δ^* such that the eigenvalues of L satisfy (2) for all $\delta \in (0, \delta^*)$. The optimization problem (6) always has a solution since the continuity argument above guarantees that the eigenvalues of $\bar{L}$ cross the imaginary axis for some $\delta \in (0, \delta_1]$. This completes the proof of the case $q_{uv}, q_{vu} > 0$. □

Remark 1. *Theorem 2 includes three different cases that correspond to perturbing the edge (u, v) ($q_{uv} > 0, q_{vu} = 0$), the edge (v, u) ($q_{uv} = 0, q_{vu} > 0$), or both edges (u, v) and (v, u) ($q_{uv} > 0, q_{vu} > 0$). The variables q_{uv}, q_{vu} allow to incorporate all these cases in only one optimization problem as stated in (6).*

The sufficiency condition (6) in Theorem 2 correspond to the minimum value of δ^* such that the eigenvalues of L satisfy (2) for all $\delta \in (0, \delta^*)$. One of the following statements holds for δ^* obtained from solving (6):

1. δ^* is obtained with $\omega_1, \ldots, \omega_k$ where $\omega_i \neq 0$ for $i = 1, \ldots, k$ and k is the number of solutions for (6) which is finite since the constraint is a non trivial polynomial equation in ω by definition of $r(\omega, \delta_{uv}, \delta_{vu})$.;
2. or, there exists at least one zero ω_i.

In the second case, the matrix $\bar{L}$ has at least one zero eigenvalue with $\delta = \delta^*$ (in the view of (8)) which means that $\det\left(\bar{L}_1^{-1}\bar{L}\right) = 0$ for $\delta = \delta^*$. In this case, the condition (5) becomes necessary and sufficient for the non-zero eigenvalues of Laplacian matrix have positive real parts in the presence of a negative weight according to the following theorem.

Theorem 3. *Consider a signed graph $\mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1)$ with the Laplacian matrix L_1. Assume that L_1 has only one zero eigenvalue and the rest of its eigenvalues have positive real parts. Construct a new graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W}) = \mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1) \oplus \mathcal{G}_2(\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2)$, where $\mathcal{E}_2 = \{(u, v), (v, u)\}$ and $\mathcal{W}_2(u, v) = -\delta q_{uv} \leq 0$, $\mathcal{W}_2(v, u) = -\delta q_{vu} \leq 0$ with $\delta > 0$ and given $q_{uv}, q_{vu} \geq 0$. Denote L the Laplacian matrix of $\mathcal{G}$. Let δ^* be obtained from (6) with $\omega_1, \ldots, \omega_k$ being the roots of the equality constraint. Assume further that there exists at least one zero ω_i. The eigenvalues of L satisfy (2) if and only if*

$$\underbrace{\delta\left(\mathbf{e}_u - \mathbf{e}_v\right)^T Q^T \bar{L}_1^{-1} Q\left(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v\right)}_{r(0, \delta q_{uv}, \delta q_{vu})} < 1. \tag{10}$$

Furthermore, $\delta^*\left(\mathbf{e}_u - \mathbf{e}_v\right)^T Q^T \bar{L}_1^{-1} Q\left(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v\right) = 1$ and $\left(\mathbf{e}_u - \mathbf{e}_v\right)^T Q^T \bar{L}_1^{-1} Q\left(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v\right) > 0$.

Proof. The necessity part results from Theorem 1, i.e. if the eigenvalues of L satisfy (2), then (10) holds. We now show that the condition (10) is sufficient for the case with $q_{uv}, q_{vu} > 0$. The cases with $q_{uv} > 0,\ q_{vu} = 0$ or $q_{uv} = 0,\ q_{vu} > 0$ can be proven by using the same arguments.
In the proof of Theorem 2, it is observed from the continuity of eigenvalues of $\bar{L}$ with respect to δ that

$$\exists\ \delta^* > 0 \text{ such that } \forall\ \delta \in [0, \delta^*),\ \Re\{\lambda_i(\bar{L})\}_{i=1}^{N-1} > 0, \tag{11}$$

and also with $\delta = \delta^*$, $\Re\{\lambda_i(\bar{L})\} = 0$ for some i. Since δ^* is obtained from (6) with at least one zero ω_i, then there exists at least one zero eigenvalue $\lambda_i(\bar{L}) = 0$ for $\delta = \delta^*$, meaning that $\det\left(\bar{L}_1^{-1}\bar{L}\right) = 0$. From the definition of $r(0, \delta q_{uv}, \delta q_{vu})$ in (10), we have

$$r_\delta := r(0, \delta q_{uv}, \delta q_{vu}) = \alpha\delta, \tag{12}$$

where $\alpha = (\mathbf{e}_u - \mathbf{e}_v)^T Q^T \bar{L}_1^{-1} Q\left(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v\right)$.
Let assume $\alpha > 0$. Using this assumption, (11) can be rewritten as

$$\exists\, r^* > 0 \text{ such that } \forall\, r_\delta \in [0, r^*),\ \Re\{\lambda_i(\bar{L})\}_{i=1}^{N-1} > 0. \tag{13}$$

We now need to show $r^* = 1$. If $r^* < 1$, according to (13), the real part of at least one eigenvalue of $\bar{L}$ is negative for $r_\delta > r^*$ and the real part becomes zero at $r_\delta = r^*$; however, from Lemma 3 in Appendix A, it is observed that $\bar{L}$ can have zero eigenvalue(s) if and only if $r_\delta = 1$. This contradicts the assumption $r^* < 1$. Hence, $r^* = 1$
To complete the proof, we should show the assumption $\alpha > 0$. To do so, we show α cannot be zero or strictly negative by considering two cases.

Case 1: To obtain a contradiction, assume $\alpha = 0$. Then $r_\delta = 0$, and according to (A.2), $\det\left(\bar{L}_1^{-1}\bar{L}\right)$ cannot be zero which means $\bar{L}$ has no zero eigenvalue. This means that $\delta^* = +\infty$ or equivalently all eigenvalues of $\bar{L}$ have positive real parts. However, this is not true, since for sufficiently large δ, i.e. $\delta > \sum_{i=1}^{N} [L_1]_{ii}$, the sum of the diagonal entries of L becomes negative. This leads L to have at least one eigenvalue with a negative real part. Hence, in view of Property 1 in Lemma 1, $\bar{L}$ has the same eigenvalue with negative real part. This is in contradiction with $\delta^* = +\infty$. Therefore, $\alpha \neq 0$.

Case 2: To obtain a contradiction, assume $\alpha < 0$. Thus (11) and (12) imply

$$\exists\, r^* > 0 \text{ such that } \forall\, r_\delta \in (-r^*, 0],\ \Re\{\lambda_i(\bar{L})\}_{i=1}^{N-1} > 0. \tag{14}$$

Furthermore, letting $r_\delta = -r^*$ yields $\Re\{\lambda_i(\bar{L})\} = 0$ for some i as r^* is taken as the maximum value satisfying (14). This means that $\det\left(\bar{L}_1^{-1}\bar{L}\right) = 0$ for $r_\delta = -r^* < 0$. However, according to (A.2), $\det\left(\bar{L}_1^{-1}\bar{L}\right) = 0$ if and only if $r_\delta = 1$. This contradicts $r_\delta < 0$. Hence, α cannot be negative. This completes the proof. □

The key step to apply Theorems 2 and 3 is to numerically solve the optimization problem (6), which has at least one solution as explained in the proof of Theorem 2. The feasible set of this optimization problem can be interpreted in terms of the Nyquist plots of the following system Σ_{uv},

$$\Sigma_{uv} := \begin{cases} \frac{d}{dt}x_{uv} &= \bar{L}_1 x_{uv} + Q(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v)u_{uv} \\ y_{uv} &= (\mathbf{e}_u - \mathbf{e}_v)^T Q^T x_{uv} \end{cases} \tag{15}$$

where $x_{uv} \in \mathbb{R}^{N-1}$, $u_{uv} \in \mathbb{R}$ and $y_{uv} \in \mathbb{R}$ denote the state, the input and the output of the system Σ_{uv}. Consider the optimization problem (6) and the system Σ_{uv} in a negative feedback structure with a proportional controller δ_1. The optimization problem (6) yields the minimum value of the gain for which the Nyqiust diagram of $\delta_1 G_{uv}(s)$ crosses the critical point -1 where $G_{uv}(s)$ is the transfer function from u_{uv} to y_{uv}. Hence, to attain δ^*, one can plot the Nyquist diagram of Σ_{uv} and find frequencies ω_i, $i = 1, \ldots, k$, at which it crosses the real axis. Then, $\delta^* = \frac{1}{|G_{uv}(j\omega^*)|}$ where $G_{uv}(j\omega^*) \leq G_{uv}(j\omega_i)$ for $i = 1, \ldots, k$. If $\omega^* = 0$, then we can use the results of Theorem 3; otherwise we should use the results of Theorem 2. In the following examples, we illustrate how to apply the results of Theorems 2 and 3.

Example 1. *Consider the graphs $\mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1)$, $\mathcal{G}_2(\mathcal{V}, \mathcal{E}_2, \mathcal{W}_2)$ and $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ depicted in Figure 1 with the Laplacian matrices L_1, L_2, L respectively. In this figure, the solid arrows represent the positive weights which are set equal to 2, and the dashed arrows represent the negative weights with $a_{36} = a_{63} = a_{38} = -1$. The graph $\mathcal{G}$ is constructed from the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ by adding negative weights between two pairs of nodes, i.e. $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$. We now add the negative directed edge $a_{38} = -\delta$. The Nyquist diagram plotted in Figure 2 crosses the real axis at $\omega_1 = 0$ and $\omega_2 = \infty$. Furthermore, the magnitude of the Nyquist diagram at ω_1 is smaller than of that at ω_2. Hence, in view of Theorem 3, the condition (10) is necessary and sufficient condition, leading to $\delta^* = 1.94285$. This means that the eigenvalues of L satisfy (2) if and only if $\delta < 1.94285$.* △

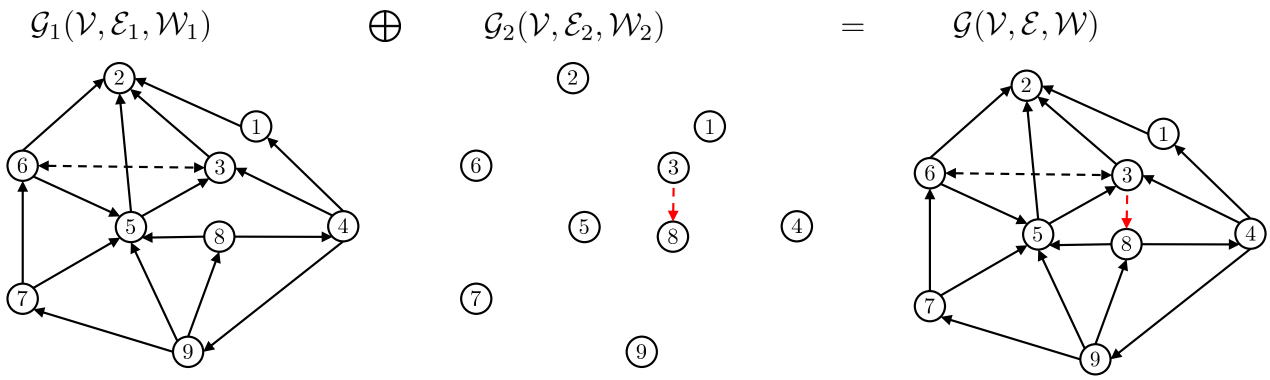


Figure 1: A directed graph studied in Example 1. The solid arrows represent the positive weight edges while the dashed arrows show the edges with negative weights. All positive and negative weights are set equal to 2 and -1, respectively.

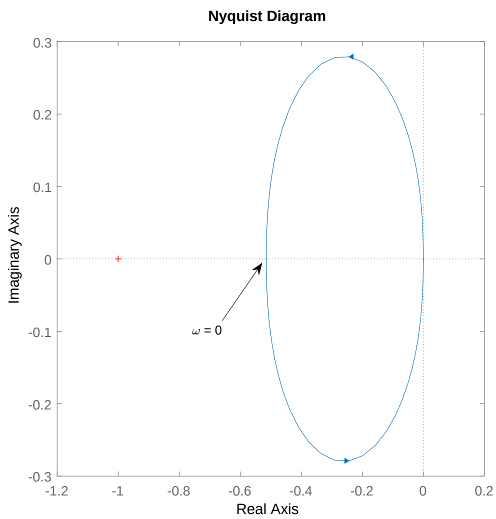


Figure 2: The Nyquist diagram in Example 1.

Example 2. *Consider the graphs $\mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)$, $\mathcal{G}_2(\mathcal{V},\mathcal{E}_2,\mathcal{W}_2)$ and $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W})$ depicted in Figure 3 with the Laplacian matrices L_1, L_2, L respectively. In this figure, the solid black arrows, the dashed red arrow and the solid read arrows represent the positive weights, the negative weight and the perturbed edges, respectively. We assign the following weights to the edges of the graph $\mathcal{G}_1$: $a_{12} = 1$, $a_{14} = 1$, $a_{15} = 1$, $a_{21} = 1$, $a_{23} = 1$, $a_{24} = 1$, $a_{31} = -1$, $a_{32} = 1$, $a_{35} = -0.8$, $a_{42} = -0.3$, $a_{43} = 1.5$, $a_{45} = -2$, $a_{51} = 2$, $a_{52} = 1$, $a_{53} = 2$, $a_{54} = 1$.*
In the first scenario, the graph $\mathcal{G}$ is constructed from the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ by perturbing positive weights between two pairs of nodes 1 and 2. According to the left Nyquist diagram plotted in Figure 4, the Nyquist diagram crosses the real axis at $\omega_1 = 0$, $\omega_{2,3} = \pm 0.6$ and $\omega_{4,5} = \pm\infty$. Furthermore, the magnitude of the Nyquist diagram at $\omega_{2,3}$ is smaller than others. Hence, in view of Theorem 3, the condition (10) is sufficient, leading to $\delta^* = 0.52$. This means that if $\delta < 0.52$ the eigenvalues of L meet (2). On the other hand, the necessary condition (5) holds for $\delta < 1.8$.
In the second scenario, the graph $\mathcal{G}$ is constructed from the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ by perturbing positive weights between two pairs of nodes 2 and 5. According to the right Nyquist diagram plotted in Figure 4, the Nyquist diagram crosses the real axis at $\omega_1 = 0$, $\omega_{2,3} = \pm 0.63$, $\omega_{4,5} = \pm 0.8$ and $\omega_{6,7} = \pm\infty$. Furthermore, the magnitude of the Nyquist diagram at ω_1 is smaller than others. Hence, in view of Theorem 3, the condition (10) is necessary and sufficient condition,

leading to $\delta^* = 2.3239$*. This means that the eigenvalues of* L *meet (2) if and only if* $\delta < 2.3239$*.* $\triangle$

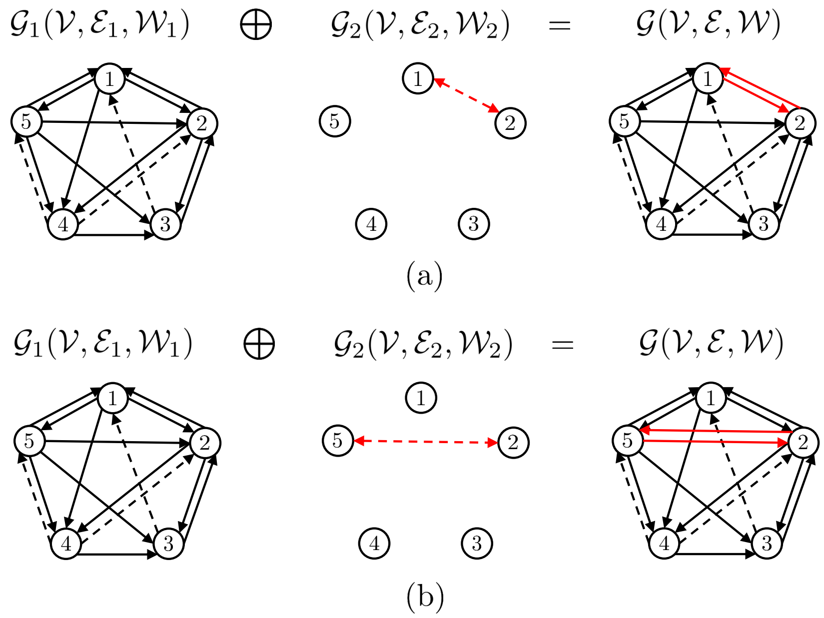


Figure 3: A directed graph studied in Example 2. The solid arrows represent the positive weight edges while the dashed arrows show the edges with negative weights. All positive and negative weights are set equal to 2 and -1, respectively.

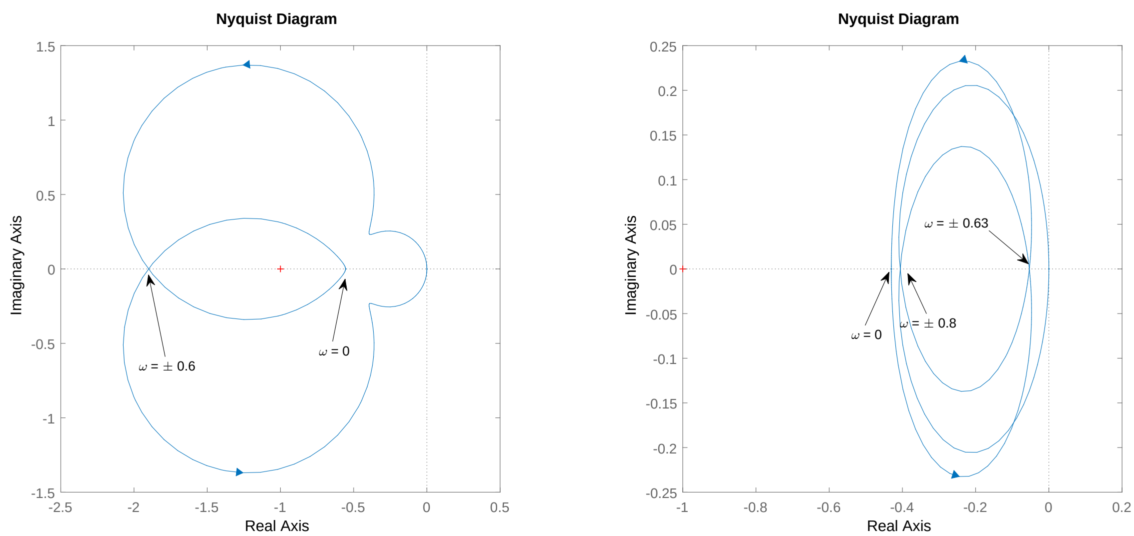


Figure 4: The Nyquist diagrams in Example 2. The left Nyquist diagram corresponds to the first scenario with the graph in Figure 3(a), while the right Nyquist diagram corresponds to the second scenario with the graph in Figure 3(b).

Remark 2. *The results of this subsection are different from those in [7] where sufficient conditions for the upper bound* δ *has been derived via Nyquist stability criteria. First, we provide the necessary result which considers the case in which both edges between any arbitrary pairs of nodes are perturbed with negative weights. Secondly, our* *sufficiency result is more general than the main result of [7] since we also allow perturbing two edges between two nodes with the same negative weight. Our results cover a more general set of graphs as a graph with multiple negative edges might satisfy the assumption of the theorem, while [7, Theorem 1] only is applied to graphs with no negative edges. Even though the results of this paper are interpreted via Nyquist criteria, the definitions of systems are different from [7]. Finally, we highlight the case where the condition becomes necessary and sufficient.*

In all theorems above, it is assumed that the original graph $\mathcal{G}_1$ satisfy (2). If $\mathcal{G}_1$ is connected and all of its weight are non-negative, the second part of Lemma 1 provides the necessary and sufficient conditions for $\mathcal{G}_1$ to satisfy (2). In this case, the following corollaries are obtained.

Corollary 1. *Consider a signed graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W})$ with the Laplacian matrix L where it is decomposed into two subgraphs $\mathcal{G}(\mathcal{V},\mathcal{E}^+,\mathcal{W}^+)$ and $\mathcal{G}(\mathcal{V},\mathcal{E}^-,\mathcal{W}^-)$ with the corresponding Laplacian matrices L^+ and L^-, respectively. Assume $\mathcal{E}^- = \{(u,v),(v,u)\}$ with $\mathcal{W}^-(u,v) = -\delta q_{uv} \leq 0$ and $\mathcal{W}^-(v,u) = -\delta q_{vu} \leq 0$ with $\delta > 0$ and given $q_{uv}, q_{vu} \geq 0$. Assume also $\mathcal{G}(\mathcal{V},\mathcal{E}^+,\mathcal{W}^+)$ is connected, and define $\bar{L}^+ = QL^+Q^T$. Let δ^* be obtained by,*

$$\begin{aligned} \min_{\omega\in\mathbb{R}_{\geq 0}} \quad & \delta_1 \\ \text{subject to} \quad & \delta_1(\mathbf{e}_u-\mathbf{e}_v)^T Q^T(\bar{L}^+ - j\omega I)^{-1}Q(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v) = 1, \quad \delta_1 > 0. \end{aligned} \tag{16}$$

Then, the eigenvalues of L satisfy (2) for all $\delta \in [0,\delta^*)$.

Corollary 2. *Consider a signed graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W})$ with the Laplacian matrix L where it is decomposed into two subgraphs $\mathcal{G}(\mathcal{V},\mathcal{E}^+,\mathcal{W}^+)$ and $\mathcal{G}(\mathcal{V},\mathcal{E}^-,\mathcal{W}^-)$ with the corresponding Laplacian matrices L^+ and L^-, respectively. Assume $\mathcal{E}^- = \{(u,v),(v,u)\}$ with $\mathcal{W}^-(u,v) = -\delta q_{uv} \leq 0$ and $\mathcal{W}^-(v,u) = -\delta q_{vu} \leq 0$ with $\delta > 0$ and given $q_{uv}, q_{vu} \geq 0$. Assume also $\mathcal{G}(\mathcal{V},\mathcal{E}^+,\mathcal{W}^+)$ is connected, and define $\bar{L}^+ = QL^+Q^T$. Let δ^* be obtained from (6) with $\omega_1,\ldots,\omega_k$ for $i = 1,\ldots,k$. Assume further that there exists at least one zero ω_i. The eigenvalues of L satisfy (2) if and only if*

$$\delta \underbrace{(\mathbf{e}_u-\mathbf{e}_v)^T Q^T (\bar{L}^+)^{-1} Q(q_{uv}\mathbf{e}_u - q_{vu}\mathbf{e}_v)}_{r(0,q_{uv},q_{vu})} < 1. \tag{17}$$

Furthermore, $\delta^*(\mathbf{e}_u-\mathbf{e}_v)^T Q^T (\bar{L}^+)^{-1} Q(\mathbf{e}_u-\mathbf{e}_v) = 1$ and $(\mathbf{e}_u-\mathbf{e}_v)^T Q^T (\bar{L}^+)^{-1} Q(\mathbf{e}_u-\mathbf{e}_v) > 0$.

We end this subsection by commenting on the relationship between the aforementioned results and the concept of effective resistance introduced in the literature e.g. see [15] and references there in. First, consider an undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E}^+,\mathcal{W}^+)$ with non-negative weights and the Laplacian matrix $\bar{L}^+$. Assume that the graph is connected. We construct an electrical network from the graph $\mathcal{G}(\mathcal{V},\mathcal{E}^+,\mathcal{W}^+)$ by replacing each weighted edge (i,j) with a resistor $R_{ij} = a_{ij}^{-1}$ as shown in Figure 5a. Then, a constant current source I is connected between nodes u and v, and the voltage at its terminal is calculated. The effective resistance between nodes u and v is computed by $r_{uv} = \frac{V}{I}$. Using electrical circuit theory, it was shown that r_{uv} can be obtained by [6]

$$r_{uv} = (\mathbf{e}_u-\mathbf{e}_v)^T Q^T (\bar{L}^+)^{-1} Q(\mathbf{e}_u-\mathbf{e}_v), \tag{18}$$

which has the same expression as (17) with $q_{uv} = q_{vu} = 1$.
Now consider the case where a negative undirected edge is added between the nodes u and v to construct the undirected graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W})$ with the Laplacian matrix L. The corresponding equivalent electrical circuit contains two parallel resistors r_{uv} and $R_- = -\delta^{-1}$[‡], which correspond to the aforementioned effective resistance and the added negative edge, respectively (Figure 5b). In this case, the equivalent resistance between nodes u and v is calculated by

$$R_{th} = \frac{-\delta^{-1} r_{uv}}{r_{uv} - \delta^{-1}}.$$

[‡]Negative resistance has a practical meaning in electrical circuit theory as there exist electrical components with this property (see https://en.wikipedia.org/wiki/Negative_resistance).

The interpretation of the inequality (17)[§] from the electrical circuit perspective is as follows: the equivalent resistance R_{th} is positive as long as (17) holds. Otherwise, R_{th} is either short circuit (if $\delta^{-1} = r_{uv}$) or negative (if $\delta^{-1} < r_{uv}$). By taking into account this point and Corollary 1, it is concluded that the Laplacian matrix L satisfy the condition (2) if and only if the equivalent resistance R_{th} of the corresponding equivalent electrical circuit is positive.

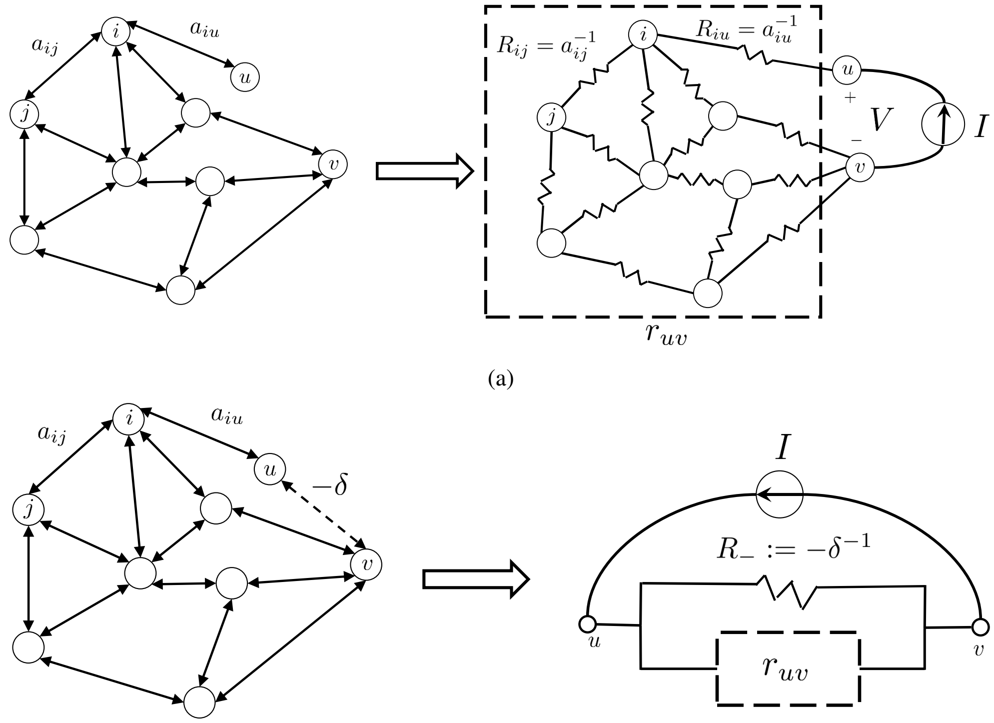
For directed graphs satisfying in the conditions of Corollary 2, one may be interested in interpreting the condition (17) using the concept of effective resistance similar to undirected graphs if directed graph satisfy assumptions of Corollary 2. However, this is not directly possible. Indeed, the notion of effective resistance has been recently introduced for both directed and undirected graphs as [13],

$$r_{uv} = 2\left(\mathbf{e}_u - \mathbf{e}_v\right)^T Q^T \Sigma\, Q\left(\mathbf{e}_u - \mathbf{e}_v\right), \tag{19}$$

where Σ is a symmetric matrix obtained from the Lyapunov equation

$$\bar{L}\Sigma + \Sigma\left(\bar{L}\right)^T = I_{N-1}. \tag{20}$$

For a connected undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}^+, \mathcal{W}^+)$, $\frac{1}{2}\left(\bar{L}^+\right)^{-1} = \frac{1}{2}\left(\bar{L}^+\right)^{-T}$ is a unique solution of (20) and r_{uv} has the same expression as (18). Unlike undirected graphs, the expression of r_{uv} in (17) (even with $q_{uv} = q_{vu} = 1$) cannot be obtained from (19) and (20), since $\left(\bar{L}^+\right)^{-1} \neq \left(\bar{L}^+\right)^{-T}$. This reveals that, unlike undirected graph, the upper bound of δ cannot be interpreted by effective resistance defined in (19). As a result, it is not clear how to generalize the notion of effective resistance for directed graphs with non-negative weights by using electrical circuit theory.



(a)

(b)

Figure 5: (a) An equivalent electrical circuit for a connected undirected graph $\mathcal{G}(\mathcal{V}, \mathcal{E}^+, \mathcal{W}^+)$ with non-negative weights to measure the effective resistance between two nodes u and v. (b) An equivalent electrical circuit for $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{W})$ which is obtained by adding a negative edge to $\mathcal{G}(\mathcal{V}, \mathcal{E}^+, \mathcal{W}^+)$.

[§]Note that inequality (17) is necessary and sufficient for this case as the underlying graph is undirected.

3. Application

In this section, we elucidate the application of our results to the consensus problem in social networks with antagonistic interactions. The dynamic of the whole network with N node is written as

$$\dot{\mathbf{x}} = -L\mathbf{x}, \tag{21}$$

where $\mathbf{x} \in \mathbb{R}^N$, $\mathbf{x} = [x_1 x_2 \dots x_N]^T$ is the stack vector consisting of the state of each node in the network. L is the Laplacian matrix of the network graph. The network achieves consensus, i.e. $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for all $i, j = 1, \dots, N$ and $\mathbf{x}(0) \in \mathbb{R}^N$, if and only if the condition (2) holds [8, Lemma 2].

Consider a network with a graph network $\mathcal{G}_1(\mathcal{V}, \mathcal{E}_1, \mathcal{W}_1)$ in Example 1. In Example 1, we have shown that the Laplacian matrix L_1 with these values for weights satisfies (2), and consequently, the network achieves consensus. We have perturbed the network with negative directed edge $a_{38} = -\delta$. We have pointed out that the eigenvalues of L meet (2) if and only if $\delta < 1.94285$.

With the choice of $\delta = 1.5$, the eigenvalues of L satisfy (2) and network achieves consensus as shown in Figure 6a. However, with $\delta = 1.95$, the eigenvalues of L no longer satisfy (2) and network does not achieve consensus as shown in Figure 6b.

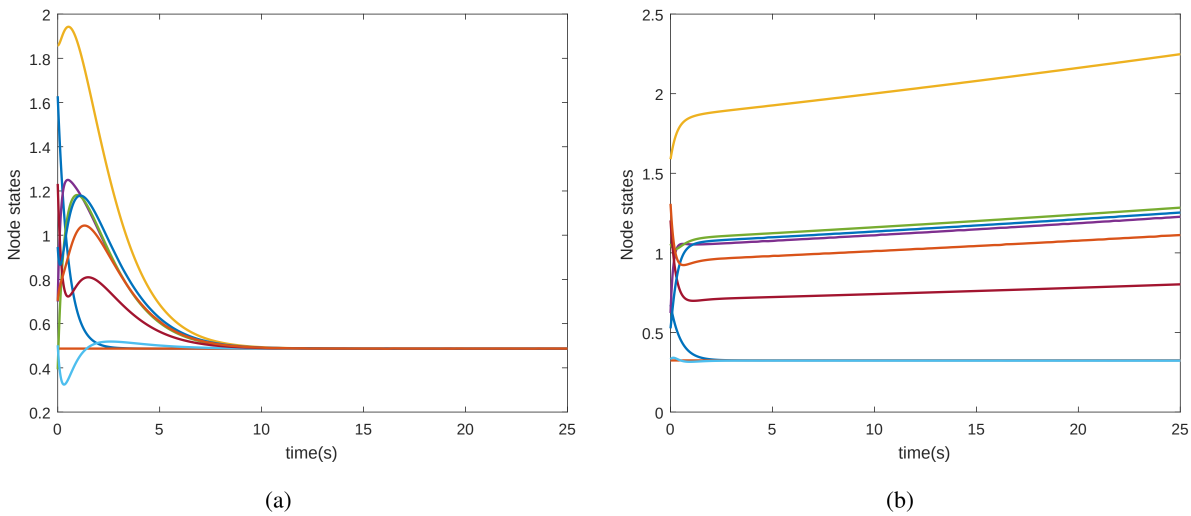


Figure 6: Evolution of node states over time for a network with the graph in Example 1. The perturbation between two nodes 3 and 8 set equal to (a) $\delta_{38} = 1.5$ (b) $\delta_{38} = 1.95$.

Acknowledgement

The error in the original paper was pointed out by Dr. Sei Zhen Khnog from University of Minnesota, as well as Dr. Daniel Zelazo and Dr. Dwaipayan Mukherjee from Israel Institute of Technology. We appreciate their insightful skepticism which enabled us to correct our results.

Appendix A.

The following lemmas are used in the proofs of Theorems 1 and 3.

Lemma 2 (Fact 2.16.3 in [4]). *Let $A \in \mathbb{C}^{n\times n}$, assume that A is nonsingular, and let $c, d \in \mathbb{R}^{n\times 1}$. Then*

$$\det(A + cd^T) = \det(A)(1 + d^T A^{-1} c). \tag{A.1}$$

Lemma 3. *Consider a signed graph $\mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)$ with the Laplacian matrix L_1. Assume that L_1 has only one zero eigenvalue and the rest of its eigenvalues have positive real parts. Construct a new graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{W}) = \mathcal{G}_1(\mathcal{V},\mathcal{E}_1,\mathcal{W}_1)\oplus\mathcal{G}_2(\mathcal{V},\mathcal{E}_2,\mathcal{W}_2)$, where $\mathcal{E}_2 = \{(u,v),(v,u)\}$ and $\mathcal{W}_2(u,v) = -\delta_{uv}$, $\mathcal{W}_2(v,u) = -\delta_{vu}$ with $\delta_{uv} \geq 0$, $\delta_{vu} \geq 0$. Denote L the Laplacian matrix of $\mathcal{G}$. Then,*

$$Spec\{\bar{L}_1^{-1}\bar{L}\} = \{\underbrace{1,\ldots,1}_{N-2}, 1 - r(\omega,\delta_{uv},\delta_{vu})\}, \tag{A.2}$$

where $r(\omega,\delta_{uv},\delta_{vu}) = (\mathbf{e}_u - \mathbf{e}_v)^T Q^T \bar{L}_1^{-1} Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)$, and $\bar{L} = QLQ^T$ and $\bar{L}_1 = QL_1Q^T$ the reduced Laplacian matrices for $\mathcal{G}$ and $\mathcal{G}_1$, respectively.

Proof. Denote L_2 the Laplacian matrix of $\mathcal{G}_2$. Since $\mathcal{E}_2 = \{(u,v),(v,u)\}$, L_2 can be expressed as $L_2 = -(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^T$. Furthermore,

$$\text{Spec}(L_2) = \{\underbrace{0,\ldots,0}_{N-1}, -(\delta_{uv} + \delta_{vu})\}, \tag{A.3}$$

with the set of eigenvectors $\{\mathbf{1}_N, \mathbf{e}_1, \ldots, \mathbf{e}_N\}\backslash\{\mathbf{e}_u, \mathbf{e}_v\}$ that corresponds to the zero eigenvalues, and $\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v$ that corresponds to the eigenvalue $-(\delta_{uv} + \delta_{vu})$. Using the first property in Lemma 1 with (A.3), we have

$$\text{Spec}\,(\bar{L}_2) = \{\underbrace{0,\ldots,0}_{N-2}, -(\delta_{uv} + \delta_{vu})\}, \tag{A.4}$$

where $\bar{L}_2 = QL_2Q^T$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same set of nodes which means L can be written as $L = L_1 + L_2$ leading to the following expression for $\bar{L}$,

$$\bar{L} = \underbrace{QL_1Q^T}_{\bar{L}_1} - Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)(\mathbf{e}_u - \mathbf{e}_v)^T Q^T. \tag{A.5}$$

Since L_1 has only one zero eigenvalue, $\bar{L}_1$ is invertible according to Lemma 1. By multiplying both sides of (A.5) by $\bar{L}_1^{-1}$ and then $(\mathbf{e}_u - \mathbf{e}_v)^T Q^T$, we obtain

$$\begin{aligned}(\mathbf{e}_u - \mathbf{e}_v)^T Q^T \bar{L}_1^{-1}\bar{L} &= (\mathbf{e}_u - \mathbf{e}_v)^T Q^T - \underbrace{(\mathbf{e}_u - \mathbf{e}_v)^T Q^T \bar{L}_1^{-1} Q(\delta_{uv}\mathbf{e}_u - \delta_{vu}\mathbf{e}_v)}_{r_\delta := r(\omega,\delta_{uv},\delta_{vu})}(\mathbf{e}_u - \mathbf{e}_v)^T Q^T \\ &= (\mathbf{e}_u - \mathbf{e}_v)^T Q^T (1 - r_\delta),\end{aligned} \tag{A.6}$$

or equivalently,

$$(\mathbf{e}_u - \mathbf{e}_v)^T Q^T \left(\bar{L}_1^{-1}\bar{L} - (1 - r_\delta)I_{N-1}\right) = \mathbf{0}_{N-1}^T. \tag{A.7}$$

Since Q^T is a full column rank matrix, (A.7) implies that the vector $QE_2 \in \mathbb{R}^{N-1}$ is a left eigenvector of matrix $\bar{L}_1^{-1}\bar{L} \in \mathbb{R}^{(N-1)\times(N-1)}$ that corresponds to the eigenvalue $1 - r_\delta$. Showing (A.2) is equivalent to showing that $X = I_{N-1} - \bar{L}_1^{-1}\bar{L}$ has $N-2$ zero eigenvalues. From (A.5), we obtain

$$\bar{L}_1 X = \bar{L}_1 - \bar{L} = -\bar{L}_2, \tag{A.8}$$

which means $\text{spec}\{\bar{L}_1 X\} = \text{spec}\{-\bar{L}_2\}$. Using (A.4) and noting that $\bar{L}_1$ is non-singular, we conclude X has $N-2$ zero eigenvalues. This completes the proof. □

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