

Minerva Access is the Institutional Repository of The University of Melbourne

Author/s:

Nesic, D;Teel, AR

Title:

A Matrosov theorem with an application to model reference adaptive control via approximate discrete-time models

Date:

2003-01-01

Citation:

Nesic, D. & Teel, A. R. (2003). A Matrosov theorem with an application to model reference adaptive control via approximate discrete-time models. Proceedings of the 42nd IEEE Conference on Decision and Control, 2, pp.1556-1561. IEEE. <https://doi.org/10.1109/CDC.2003.1272833>.

Persistent Link:

<https://hdl.handle.net/11343/299632>

A Matrosov theorem with an application to Model Reference Adaptive Control via approximate discrete-time models

Dragan Nešić¹ and Andrew R. Teel²

Abstract

A version of Matrosov's theorem for parameterized discrete-time time-varying systems is presented. The theorem is a discrete-time version of the continuous-time result in [1]. Our result facilitates controller design for sampled-data nonlinear systems via their approximate discrete-time models. An application of the theorem to establishing uniform asymptotic stability of systems controlled by model reference adaptive controllers designed via approximate discrete-time plant models is presented.

1 Introduction

The prevalence of computer controlled systems and the fact that the nonlinearities in the plant model can often not be neglected strongly motivate consideration of the class of nonlinear sampled-data models. One of the main difficulties in dealing with this class of models is that it is not clear what is the model one should use when designing the controller. For example, the exact discrete-time model of this class of systems is typically not available for the controller design and one can only use an approximate model for this purpose.

This has motivated us to present checkable conditions [4, 6] on the controller, approximate model and the continuous-time plant model that guarantee that if the controller stabilizes the approximate model then it would also stabilize the exact discrete-time model in an appropriate sense (semiglobal-practical) for sufficiently small sampling periods. These results provide a framework for controller design for sampled-data nonlinear systems via their approximate discrete-time models. In particular, [4, Theorem 2] gives Lyapunov like conditions on the approximate model to provide such a framework. It is the purpose of this paper to extend the results in [4, 6] in the following way. We prove a version of Matrosov's theorem for establishing uniform asymptotic stability using several Lyapunov like functions that typically have negative semi-definite first difference (in [4] and [6]

we required a negative definite first difference of the Lyapunov function; moreover in [4] we considered only time-invariant systems).

The classical Matrosov theorem establishes UGAS via two Lyapunov like functions (see [3], [9, Theorem 5.5], [10, Theorem 2.5] and [7, Appendix]). Certain extensions of Matrosov theorem can be found in [12] and more recently in [1] where it was shown how it is possible to combine an arbitrary number of Lyapunov like functions to test UGAS of time-varying continuous-time systems. Our main result extends the main result from [1] to parameterized families of time-varying discrete-time systems that naturally arise when an approximate discrete-time model is used to design a controller for a sample-data system. An application to systems for which a model reference adaptive controller is designed via an approximate discrete-time model is presented (a continuous-time analogue of this result can be found in [2]).

The paper is organized as follows. First, we present preliminaries and definitions in Section 2. The main result is presented and proved in Section 3. Finally, in Section 4 we show how our result can be used to analyze systems controlled by model reference adaptive controllers that are designed via their approximate discrete-time plant models.

2 Preliminaries

We denote sets of real and integer numbers respectively as $\mathbb{R}$ and $\mathbb{Z}$. A function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$), if it is continuous, strictly increasing and zero at zero; $\alpha \in \mathcal{K}_{\infty}$ if, in addition, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for all $t > 0$, $\beta(\cdot, t) \in \mathcal{K}$, for all $s > 0$, $\beta(s, \cdot)$ is decreasing to zero. We denote by $|\cdot|$ the Euclidean norm of vectors. Given arbitrary $L \geq 0$ and $T > 0$ we use the following notation:

$$\ell_{L,T} := \left\lfloor \frac{L}{T} \right\rfloor = \max \left\{ z \in \mathbb{Z} : z \leq \frac{L}{T} \right\}.$$

Also, we denote $B_{\Delta} := \{x : |x| \leq \Delta\}$ and $\mathcal{H}(\delta, \Delta) := \{x : \delta \leq |x| \leq \Delta\}$.

Consider the class of parameterized time-varying discrete-time systems:

$$x(k+1) = F_T(k, x(k)). \quad (1)$$

These systems arise naturally when an approximate discrete-time model of a sampled-data plant is used

¹Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, 3010, Victoria, Australia; d.nesic@ee.mu.oz.au. Research supported by the Australian Research Council under the large grants scheme.

²CCEC, Electrical and Computer Engineering Department, University of California, Santa Barbara, CA, 93106-9560, USA; teel@cce.ucsba.edu. Research supported in part by the AFOSR under grants F49620-00-1-0106 and -03-1-0203 and the NSF under grant ECS-9988813.

to design a controller (see Section 4 and [4, 6] for more details). The trajectory of the system (1) at a time k starting at an initial time k_o and emanating from the initial state x_o is denoted $\phi_T(k, k_o, x_o)$. Often k_o and x_o are clear from the context and we use the shorthand notation $\phi_T(k)$. $T > 0$ is a parameter that can be arbitrarily assigned (most of the time T represents the sampling period).

The following ϵ - δ stability definitions will be used in the sequel.

Definition 1 Let $\gamma \in \mathcal{K}_\infty$ be given. The system (1) is uniformly semiglobally practically asymptotically stable (USPAS) if for each pair of strictly positive real numbers (r, ϵ) there exists $L = L(r, \epsilon) > 0$ and for each strictly positive (Δ, ν) there exists $T^* = T^*(\Delta, \nu) > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$, $T \in (0, T^*)$ and $r \leq \Delta$ we have

$$|x_o| \leq \Delta \Rightarrow |\phi_T(k)| \leq \gamma(|x_o|) + \nu, \forall k \geq k_o \quad (2)$$

$$|x_o| \leq r \Rightarrow |\phi_T(k)| \leq \epsilon + \nu, \forall k \geq k_o + \ell_{L,T} \quad (3)$$

The system (1) is uniformly semiglobally asymptotically stable (USAS) if for each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that $k_o \geq 0$, $x(k_o) = x_o$, $T \in (0, T^*)$ and $r \leq \Delta$ the inequalities (2) and (3) hold with $\nu = 0$. The system (1) is uniformly globally asymptotically stable (UGAS) if there exists $T^* > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$, $T \in (0, T^*)$ the inequality (2) holds for all $x_o \in \mathbb{R}^n$ with $\nu = 0$ and (3) holds for all $r > 0$ and with $\nu = 0$.

The stability definitions in Definition 1 are suitable for the purposes of our paper. However, in [4, 6] we have used definitions that make use of $\mathcal{KL}$ functions. These are presented next. As Proposition 1 below points out, the properties in Definitions 1 and 2 are equivalent.

Definition 2 Let $\beta \in \mathcal{KL}$ be given. The system (1) is β -uniformly semiglobally practically asymptotically stable (β -USPAS) if for each strictly positive (Δ, ν) there exists $T^* = T^*(\Delta, \nu) > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$, with $|x_o| \leq \Delta$ and $T \in (0, T^*)$ we have

$$|\phi_T(k)| \leq \beta(|x_o|, (k - k_o)T) + \nu, \forall k \geq k_o \quad (4)$$

The system (1) is β -uniformly semiglobally asymptotically stable (β -USAS) if for each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$, with $|x_o| \leq \Delta$ and $T \in (0, T^*)$ the inequality (4) holds with $\nu = 0$. The system (1) is β -uniformly globally asymptotically stable (β -UGAS) if there exists $T^* > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$, with $x_o \in \mathbb{R}^n$ and $T \in (0, T^*)$ the inequality (4) holds with $\nu = 0$.

The proof of the following result follows very similar steps to the proof of [11, Proposition 1] and is therefore omitted.

Proposition 1 The system (1) is:

1. UGAS iff there exists a continuous $\beta \in \mathcal{KL}$ such that (1) is β -UGAS;
2. USAS iff there exists a continuous $\beta \in \mathcal{KL}$ such that (1) is β -USAS;
3. USPAS iff there exists a continuous $\beta \in \mathcal{KL}$ such that (1) is β -USPAS.

We also need the following definition for the statement of our main result.

Definition 3 Let $\gamma, \varphi \in \mathcal{K}_\infty$ be given. The system (1) is said to be semiglobally bounded on compact time intervals (SB) if for each triple of strictly positive real numbers (Δ, L, δ) there exists $T^* = T^*(\Delta, L, \delta) > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$ with $|x_o| \leq \Delta$ and $T \in (0, T^*)$ we have that trajectories of (1) satisfy:

$$|\phi_T(k)| \leq \gamma(|x_o|) + \varphi(\delta T(k - k_o)), \quad (5)$$

for all $k \in [k_o, k_o + \ell_{L,T}]$. If for each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$ with $|x_o| \leq \Delta$ and $T \in (0, T^*)$ the bound (5) holds with $\delta = 0$ and for all $k \geq k_o$, then the system is uniformly semiglobally stable (USS). If there exists $T^* > 0$ such that for all $k_o \geq 0$, $x(k_o) = x_o$ with $x_o \in \mathbb{R}^n$ and $T \in (0, T^*)$ the bound (5) holds, with $\delta = 0$, then the system (1) is uniformly globally stable (UGS).

3 A type of Matrosov Theorem

The following theorem is an adaptation of the Matrosov Theorem given in [1] to parameterized discrete-time time-varying systems that naturally arise when one uses an approximate discrete-time model of the sampled-data plant to design a discrete-time controller. A comparison between the statement of this theorem and the statement of Corollary 2, the latter of which is essentially a translation of the result in [1] to discrete-time, indicates the complications that arise from considering parameterized systems.

Theorem 1 The system (1) is USPAS if the following conditions hold:

1. The system (1) is SB.
2. There exist integers $j, m > 0$ and a real number $\tilde{T} > 0$ such that for each $T \in (0, \tilde{T})$ and $i \in \{1, 2, \dots, j\}$ there exist functions $V_T^i : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, $\chi_T : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^m$ and continuous functions $Y_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and for each $\Delta > 0$ there exists $\mu > 0$ such that

(a) for all $T \in (0, \tilde{T})$, $|x| \leq \Delta$ and $k \in \mathbb{Z}$:

$$\max\{|V_T^i(k, x)|, |\chi_T(k, x)|\} \leq \mu. \quad (6)$$

(b) For each integer $k \in \{1, \dots, j\}$ we have that

$$(z, \xi) \in B_\Delta \times B_\mu, Y_i(z, \xi) = 0, \forall i \in \{1, \dots, k-1\}$$

imply $Y_k(z, \xi) \leq 0$. (c) We have that

$$(z, \xi) \in B_\Delta \times B_\mu, Y_i(z, \xi) = 0, \forall i \in \{1, \dots, j\}$$

imply $z = 0$.

3. For each pair of strictly positive real numbers (Δ, ν) there exists $T^* > 0$ such that for all $T \in (0, T^*)$, $i \in \{1, \dots, j\}$, $|x| \leq \Delta$ and $k \in \mathbb{Z}$ we have that:

$$\frac{\Delta V_T}{T} \leq Y_i(x, \chi_T(k, x)) + \nu,$$

where $\Delta V_T := V_T^i(k+1, F_T(k, x)) - V_T^i(k, x)$.

Note that Condition 2 of Theorem 1 concerns only the properties of the functions V_T^i , χ_T and Y_i whereas Conditions 1 and 3 also involve the system (1). The significance of Condition 2 is the consequence given in the next proposition. (The proof is the same as that for continuous-time systems, as in [1], and thus is omitted.) This consequence is typically much more difficult to check than Condition 2 (see Example 1 which follows the proposition, and also the discussion in [1]):

Proposition 2 *Let $\Delta > 0$ be given and let it generate $\mu > 0$ via Condition 2 of Theorem 1. Then, for each $\delta \in (0, \Delta)$ there exist $\alpha > 0$ and numbers $K_i > 0, i = 1, \dots, j-1$ such that the function*

$$Z(z, \xi) := \sum_{i=1}^{j-1} K_i Y_i(z, \xi) + Y_j(z, \xi)$$

satisfies for all $(z, \xi) \in \mathcal{H}(\delta, \Delta) \times B_\mu$:

$$Z(z, \xi) \leq -\frac{\alpha}{2^{j-1}}.$$

Example 1 *The following functions Y_i will arise in the analysis of the model reference adaptive control problem $Y_1(z, \xi) = -\alpha_3(z_1)$; $Y_2(z, \xi) = |z_1| \rho_3(|z|) + [\rho_1(|z_1|) + \rho_2(|z_1|)] \rho_3^2(|z|) - |\xi|^2$; $Y_3(z, \xi) = |\xi| + \exp(1) \rho_3(|z|) \rho_1(|z_1|) - \frac{1}{2} \alpha(|z_2|)$; where $\rho_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ are continuous, nondecreasing, $\rho_i(0) = 0$ for $i = 1, 2$, and α_3, α are continuous, positive definite. We note that, from the properties of α_3 , $Y_1(z, \xi) \leq 0$. Moreover, $Y_1(z, \xi) = 0$ implies $z_1 = 0$. Thus, from the properties of ρ_1 and ρ_2 , $Y_1(z, \xi) = 0$ implies $Y_2(z, \xi) = -|\xi|^2 \leq 0$. In turn, $Y_1(z, \xi) = Y_2(z, \xi) = 0$ imply $z_1 = 0$ and $\xi = 0$ and thus imply $Y_3(z, \xi) = -\frac{1}{2} \alpha(z_2) \leq 0$. Thus Condition 2(b) of the Theorem is satisfied. Finally, from the properties of α , $Y_1(z, \xi) = Y_2(z, \xi) = Y_3(z, \xi) = 0$ imply $z = 0$. So Condition 2(c) of the Theorem is satisfied. On the other hand, it is a nontrivial task to solve for the numbers $K_i > 0$ and $\alpha > 0$ indicated in the Proposition.*

The following results are corollaries of the proof used to establish Theorem 1.

Corollary 1 *The system (1) is USAS if:*

1. The system (1) is USS.
2. Condition 2 of Theorem 1 holds.
3. For each $\Delta > 0$ there exists $T^* = T^*(\Delta) > 0$

such that for all $T \in (0, T^*)$, $i \in \{1, \dots, j\}$, $|x| \leq \Delta$ and $k \in \mathbb{Z}$ the bound in Condition 3 of Theorem 1 holds with $\nu = 0$.

The following corollary is applicable to non-parameterized discrete-time time-varying systems. Indeed, we can write any non-parameterized system $x(k+1) = f(k, x(k))$ as a particular system obtained from a parameterized family of systems (1) with a fixed value of T , such as $T = 1$. If all conditions of the below given theorem hold for the non-parameterized system then we can conclude its UGAS. We are not aware of a reference that presents Matrosov theorem for non-parameterized systems. This result is interesting in situations when the exact discrete-time model of the system is known. Hence, the following result can be also regarded as a discrete-time (non-parameterized) version of the continuous-time result [1, Theorem 1].

Corollary 2 *The system (1) is UGAS if:*

1. The system (1) is UGS.
2. Condition 2 of Theorem 1 holds.
3. There exists $T^* > 0$ such that for all $T \in (0, T^*)$, $i \in \{1, \dots, j\}$, $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$ the bound in Condition 3 of Theorem 1 holds with $\nu = 0$.

Proof of Theorem 1: We will establish the characterization of USPAS given in Definition 1. Namely, we will construct $\gamma \in \mathcal{K}_\infty$ (in fact it comes directly from Condition 1 of Theorem 1), and for each pair of strictly positive real numbers (r, ϵ) we will construct $L > 0$ and for each pair of strictly positive numbers (Δ, ν) we will construct $T^* > 0$ so that (2)-(3) hold. While the construction of $L > 0$ and $T^* > 0$ is somewhat technical (see the calculations below) the main idea is this: We use Condition 1 of Theorem 1 to derive a bound like (2) on finite time intervals, and we use Conditions 2 and 3 of Theorem 1 (and their consequence in Proposition 2) to establish that trajectories converge to a small neighborhood of the origin at the end of the finite horizon. After this, an induction argument can be used to derive the bounds (2)-(3) for all appropriate time steps.

Let γ come from Condition 1 of Theorem 1. For arbitrary $a > b > 0$ we define:

(a) $\tilde{\delta}(b) := 1/2\gamma^{-1}(b/2)$; $\tilde{r}(a) := \gamma(a) + 1$. Note that since $\gamma(s) \geq s$, we have that $\tilde{\delta}(b) \leq b/4$, $\forall b \geq 0$;

(b) Let $\tilde{r}(a)$ generate $\tilde{\mu}(a)$ via Condition 2 of Theorem 1, with $\Delta = \tilde{r}(a)$. Let $\tilde{r}(a), \tilde{\mu}(a)$ and $\tilde{\delta}(b)$ generate $\tilde{K}_i(a, b) > 0$ and $\tilde{\alpha}(a, b)$ via Proposition 2;

(c) Let

$$\tilde{\eta}(a, b) := \tilde{\mu}(a, b) \left(1 + \sum_{i=0}^{j-1} \tilde{K}_i(a, b) \right);$$

(d) Let $\tilde{L}(a, b)$ be such that

$$\tilde{L}(a, b) > \frac{2^{j+1}\tilde{\eta}(a, b)}{\tilde{\alpha}(a, b)} + 1.$$

Note: For any positive numbers satisfying $0 < b_1 \leq b_2 < a_2 \leq a_1$ we have that $\mathcal{H}(b_2, a_2) \subseteq \mathcal{H}(b_1, a_1)$ and consequently we can choose the above given numbers so that:

$$\begin{aligned} \tilde{\alpha}(a_1, b_1) &\leq \tilde{\alpha}(a_2, b_2) \\ \tilde{\eta}(a_1, b_1) &\geq \tilde{\eta}(a_2, b_2) \\ \tilde{L}(a_1, b_1) &\geq \tilde{L}(a_2, b_2). \end{aligned} \quad (7)$$

Using the above given functions we generate all the numbers needed in the proof.

(i) Let (r, ϵ) be strictly positive given numbers and without loss of generality assume $r > \epsilon$;

(ii) Let $\delta_1 := \tilde{\delta}(\epsilon)$; $r_1 := \tilde{r}(r)$, $\mu_1 := \tilde{\mu}(r)$, $\alpha_1 := \tilde{\alpha}(r, \epsilon)$, $K_i^1 := \tilde{K}_i(r, \epsilon)$, $\eta_1 := \tilde{\eta}(r, \epsilon)$ and $L_1 := \tilde{L}(r, \epsilon)$;

(iii) Let (Δ, ν) be given and let $\nu_1 := \nu/2$;

(iv) Let $\delta_2 := \tilde{\delta}(\nu_1)$; $\tilde{\Delta} := \tilde{r}(\Delta)$, $\mu_2 := \tilde{\mu}(\Delta)$, $\alpha_2 := \tilde{\alpha}(\Delta, \nu_1)$, $K_i^2 := \tilde{K}_i(\Delta, \nu_1)$, $\eta_2 := \tilde{\eta}(\Delta, \nu_1)$ and $L_2 := \tilde{L}(\Delta, \nu_1)$;

(v) Define $\tilde{\nu} := \min\{\alpha_2/2^j, \nu_1/2, 1\}$;

(vi) Let $(\tilde{\Delta}, \tilde{\nu})$ generate $T_1^* > 0$ via Condition 3 of the theorem;

(vii) Let $L := L_2 + 1$ and $\tilde{\nu}_1$ be such that $\tilde{\nu}_1 \leq \varphi^{-1}(\tilde{\nu})/L$, where φ comes from Condition 1 of the theorem;

(viii) Let $(\tilde{\Delta}, L, \tilde{\nu}_1)$ generate $T_2^* > 0$ via Condition 1 of the theorem;

(ix) Let $T^* := \min\{T_1^*, T_2^*, \tilde{T}, 1\}$, where $\tilde{T}$ comes from Condition 2 of the theorem.

Before we proceed with the proof, note that if $|x_o| \leq \delta_1 + \delta_2 \leq \Delta$, $T \in (0, T^*)$ and $k \in [k_o, k_o + \ell_{L,T}]$ the following holds:

$$\begin{aligned} |\phi_T(k)| &\leq \gamma(|x_o|) + \tilde{\nu} \leq \gamma(\delta_1 + \delta_2) + \tilde{\nu} \\ &\leq \gamma\left(\frac{1}{2}\gamma^{-1}(\epsilon/2) + \frac{1}{2}\gamma^{-1}(\nu_1/2)\right) + \frac{\nu_1}{2} \\ &\leq \frac{\epsilon}{2} + \nu_1, \end{aligned} \quad (8)$$

where the last inequality follows from the weak triangle inequality for the class $\mathcal{K}$ function γ , i.e. $\gamma(a+b) \leq \gamma(2a) + \gamma(2b)$, $\forall a, b \geq 0$.

As in Definition 1, $\Delta \geq r$, $T \in (0, T^*)$, $k_o \geq 0$, $x(k_o) = x_o$ with $|x_o| \leq r$ and the corresponding trajectory is denoted as $\phi_T(k)$. Now we consider two cases $\epsilon \geq \nu_1$ and $\epsilon < \nu_1$.

Case 1: $\epsilon \geq \nu_1$. We consider two sub cases $|x_o| \leq \delta_1 + \delta_2$ and $x_o \in \mathcal{H}(\delta_1 + \delta_2, r)$.

Case 1a: $|x_o| \leq \delta_1 + \delta_2$

From Condition 1 and the definitions (a), (ii), (iv) and (v) and the inequality (8) we can write for all $k \in [k_o, k_o + \ell_{L,T}]$:

$$|\phi_T(k)| \leq \epsilon/2 + \nu_1 \leq \epsilon + \nu. \quad (9)$$

Case 1b: $x_o \in \mathcal{H}(\delta_1 + \delta_2, r)$

From $r \leq \Delta$, Condition 1 of the theorem, (vii) and (viii), we have for all $k \in [k_o, k_o + \ell_{L,T}]$:

$$\begin{aligned} |\phi_T(k)| &\leq \gamma(|x_o|) + \varphi(\tilde{\nu}_1 T(k - k_o)) \\ &\leq \gamma(|x_o|) + \varphi(\tilde{\nu}_1 L) \\ &\leq \gamma(|x_o|) + \tilde{\nu} \\ &\leq \gamma(|x_o|) + \min\left\{\frac{\nu_1}{2}, 1\right\}, \end{aligned} \quad (10)$$

which together with (iv), (v) and (vi) implies that $\phi_T(k) \in B_{r_1}, \forall k \in [k_o, k_o + \ell_{L,T}]$.

We claim that there exists $k_1^* \in [k_o + 1, k_o + \ell_{L_1,T}]$ such that

$$|\phi_T(k_1^*)| \leq \delta_1 + \delta_2. \quad (11)$$

To prove this define the function $W_T(k, x) := \sum_{i=1}^{j-1} K_i^1 V_T^i(k, x) + V_T^j(k, x)$. From Condition 2 and (ii) we have that for all $(k, x) \in \mathbb{Z} \times B_{r_1}$ that the following holds:

$$|W_T(k, x)| \leq \mu_1 \left(1 + \sum_{i=1}^{j-1} K_i^1\right) = \eta_1. \quad (12)$$

Moreover, since $0 < \nu_1 \leq \epsilon < r \leq \Delta$, we have that $\mathcal{H}(\delta_1, r_1) \subseteq \mathcal{H}(\delta_2, \tilde{\Delta})$ and we can always pick $\alpha_1, \alpha_2, L_1, L_2$ such that $\alpha_1 \geq \alpha_2$ and $L_2 \geq L_1$ (see (7) and the preceding text). Hence, for all $(k, x) \in \mathbb{Z} \times \mathcal{H}(\delta_1 + \delta_2, r_1) \subset \mathbb{Z} \times \mathcal{H}(\delta_1, r_1)$ we have $Z(x, \chi_T(k, x)) \leq -\frac{\alpha_1}{2^{j-1}}$ and consequently:

$$\begin{aligned} \frac{\Delta W_T}{T} &\leq Z(x, \chi_T(k, x)) + \tilde{\nu} \\ &\leq -\frac{\alpha_1}{2^{j-1}} + \frac{\alpha_2}{2^j} \\ &\leq -\frac{\alpha_1}{2^j}, \end{aligned} \quad (13)$$

where $\Delta W_T := W_T(k+1, \phi_T(k, x)) - W_T(k, x)$. Suppose for the purpose of showing contradiction to (11) that for all $k \in [k_o, k_o + \ell_{L_1,T}]$ we have $|\phi_T(k)| > \delta_1 + \delta_2 > \delta_1$. Then using (13) along the trajectories of the system we can write:

$$\frac{W_T(k+1, \phi_T(k+1)) - W_T(k, \phi_T(k))}{T} \leq -\frac{\alpha_1}{2^j},$$

for $k \in [k_o, k_o + \ell_{L_1,T}]$. Adding both sides of the above inequality for $k \in [k_o, k_o + \ell_{L_1,T} - 1]$ and rearranging it we obtain:

$$T \frac{\alpha_1}{2^j} \ell_{L_1,T} \leq \widetilde{\Delta W}_T \leq 2\eta_1,$$

where $\widetilde{\Delta W}_T := W_T(k_o + \ell_{L_1,T}, \phi_T(k_o + \ell_{L_1,T})) - W_T(k_o, x_o)$. From our choice $T^* \leq 1$ and definition of $\ell_{L_1,T}$ we have that $L_1 - 1 \leq T \ell_{L_1,T}$ and hence we obtain $\frac{\alpha_1}{2^j}(L_1 - 1) \leq 2\eta_1$, which contradicts our choice of L_1 in (ii) with (d).

Now the proof of (3) for $\epsilon \geq \nu_1$ is completed using induction. We let for all initial states with $|x_o| \leq \delta_1 + \delta_2$ that $k_1^* = 0$. There are two possibilities,

either $|\phi_T(k)| \leq \delta_1 + \delta_2 \leq \epsilon/4 + \nu_1/4$ for all $k \geq k_1^*$ or there exists $\bar{k}_2 > k_1^*$ such that $|\phi_T(\bar{k}_2)| > \delta_1 + \delta_2$. For definiteness assume that $\bar{k}_2$ is minimum such integer. Hence, we have that $|\phi_T(k)| \leq \delta_1 + \delta_2, \forall k \in [k_1^*, \bar{k}_2 - 1]$ and $|\phi_T(\bar{k}_2)| > \delta_1 + \delta_2$. Then there exists $k_2^* \in [\bar{k}_2 + 1, \bar{k}_2 + \ell_{L_1, T}]$ such that the following hold:

$$|\phi_T(k)| \leq \gamma(\delta_1 + \delta_2) + \nu_1/2 \leq \epsilon/2 + \nu_1, \quad \forall k \in [\bar{k}_2, \bar{k}_2 + \ell_{L_1, T}] \quad (14)$$

$$|\phi_T(k_2^*)| \leq \delta_1 + \delta_2. \quad (15)$$

Indeed, (14) is proved using the same calculations as in Case 1a starting at the initial state $\phi_T(\bar{k}_2 - 1)$ and the fact that the bound in (9) holds on intervals of length $\ell_{L, T}$ (note also that since $L = L_2 + 1 \geq L_1 + 1$ and $T^* \leq 1$, we have that $\ell_{L, T} \geq \ell_{L_1, T} + 1$). Inequality (15) is proved using the same calculations as in Case 1b starting from the initial state $\phi_T(\bar{k}_2)$. One can generate a sequence of such intervals and by induction we prove that:

$$x_o \in B_r, \quad k \geq k_o + \ell_{L_1, T} \Rightarrow |\phi_T(k)| \leq \epsilon/2 + \nu_1 \leq \epsilon + \nu, \quad (16)$$

which shows that (3) holds since $k_1^* \leq \ell_{L_1, T}$ and L_1 depends only on (r, ϵ) .

Since (16) holds for all $\epsilon \geq \nu_1$ and $r \leq \Delta$, it holds in particular for $r = \Delta$ and $\epsilon = \nu_1$. Hence, for this choice we have from previous calculations that: $|x_o| \leq \Delta, k \in [k_o, k_o + \ell_{L_2, T}]$ imply $|\phi_T(k)| \leq \gamma(|x_o|) + \nu_1 < \gamma(|x_o|) + \nu$; and $|x_o| \leq \Delta, k \geq k_o + \ell_{L_2, T}$ imply $|\phi_T(k)| \leq 2\nu_1 = \nu$ which shows that (2) holds.

Case 2: $\epsilon < \nu_1$. Again since the above calculations hold for all $\epsilon \geq \nu_1$, they hold, in particular, for $\epsilon = \nu_1$. Let $L_3 := \bar{L}(r, \nu_1)$. Then, since $\epsilon < \nu_1$ we can write using (8) that for all $k \geq k_o + \ell_{L_3, T}$

$$|\phi_T(k)| \leq \epsilon/2 + \nu_1 \leq \nu_1/2 + \nu_1 < 2\nu_1 = \nu.$$

Note that the time $\ell_{L, T}$ needed for $\phi_T(k)$ to converge to the ball $B_{\nu+\epsilon}$ is smaller than the time $\ell_{L_3, T}$ needed to converge to the ball $B_{2\nu_1} = B_\nu$. This completes the proof of (3).

4 Application to MRAC via approximate models

We consider the problem of sampled-data, adaptive tracking control (sometimes called model reference adaptive control or MRAC for nonlinear systems in the form

$$\dot{\xi} = f(\xi) + g(\xi) [u + h(\xi)^\top \theta]. \quad (17)$$

The parameter vector θ is unknown. The functions f, g and h are supposed to be locally Lipschitz. We will assume we can find a family of certainty equivalence discrete-time, tracking feedback laws for the Euler approximation of the system (17):

$$\begin{aligned} \xi(k+1) &= \xi(k) + Tf(\xi(k)) \\ &\quad + Tg(\xi(k)) [u(k) + h(\xi(k))^\top \theta]. \end{aligned} \quad (18)$$

In other words, we will assume:

Assumption 1 For a given family of uniformly bounded reference state trajectories $\xi_{r, T}$ satisfying

$$\xi_{r, T}(k+1) = \xi_{r, T}(k) + Ts(\xi_{r, T}(k), k, T) \quad (19)$$

for some function s , there exists a family of feed-backs $\tilde{u}(\xi, \xi_{r, T}, T)$ such that, with the definition, $F_T(k, e) := e + T[f(e + \xi_{r, T}(k)) + g(e + \xi_{r, T}(k))\tilde{u}(e + \xi_{r, T}(k), \xi_{r, T}(k), T) - s(\xi_{r, T}(k), k, T)]$, there exist $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_4 \in \mathcal{K}_\infty$ and a continuous positive definite $\tilde{\alpha}_3 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$ such that for every pair of strictly positive real numbers $(\tilde{\Delta}, \tilde{\nu}, \tilde{\mu})$ there exists $T^* > 0$ and for each $T \in (0, T^*)$ there exists $W_T : \mathbb{Z} \times \mathbb{R}^n$ such that the following hold:

$$\begin{aligned} \tilde{\alpha}_1(|e|) &\leq W_T(k, e) \leq \tilde{\alpha}_2(|e|) \\ \frac{W_T(k+1, F_T(k, e)) - W_T(k, e)}{T} &\leq -\tilde{\alpha}_3(e) + \tilde{\nu} \\ |\nabla_e W_T(k, e)| &\leq \tilde{\alpha}_4(|e|) + \tilde{\mu}, \end{aligned}$$

for all $T \in (0, T^*)$, $k \geq 0$, $|e| \leq \tilde{\Delta}$.

Under this assumption, we will implement the following discrete-time controller

$$u(k) = \tilde{u}(\xi(k), \xi_{r, T}(k)) - h(\xi(k))^\top \hat{\theta}(k) \quad (20)$$

$$\hat{\theta}(k+1) = \hat{\theta}(k) - Th(\xi(k))g^\top(\xi(k))\nabla_e W_T(k) \quad (21)$$

where $\nabla_e W_T(k) := \nabla_e W_T(k, F_T(k, \xi(k) - \xi_{r, T}(k)))$. For the purposes of obtaining USPAS, we will assume the following persistency of excitation condition:

Assumption 2 There exists $T^* > 0$, $L > 0$ and $\mu > 0$ such that, for all $k \in \mathbb{Z}$ and $T \in (0, T^*)$, we have

$$T \sum_{i=k}^{k+\ell_{L, T}} |g(\xi_{r, T}(i))h(\xi_{r, T}(i))^\top x_2| \geq \mu|x_2|. \quad (22)$$

Under Assumptions 1 and 2, and with the definitions: $x_1 := \xi - \xi_{r, T}$; $x_2 := \theta - \hat{\theta}$; $x = [x_1 \ x_2]^\top$; $A := A(T, k, x) := \frac{F_T(k, x_1) - x_1}{T}$; $B := B(T, k, x) := g(x_1 + \xi_{r, T}(k))h(x_1 + \xi_{r, T}(k))^\top x_2$; $B_o := B_o(T, k, x_2) := B(T, k, x)|_{x_1=0}$; $C := C(T, k, x) := -h^\top(x_1 + \xi_{r, T}(k))g(x_1 + \xi_{r, T}(k))^\top \nabla_e W_T(k+1, F_T(k, x_1))$; $B_o^+ := B_o^+(T, k, x_2) := B_o(T, k+1, x_2 + TC(T, k, x))$; the Euler approximation of the closed-loop system consisting of (18), (19), (20) and (21) in (x_1, x_2) coordinates has the form

$$x_1(k+1) = x_1(k) + TA(T, k, x(k)) + TB(T, k, x(k)) \quad (23)$$

$$x_2(k+1) = x_2(k) + TC(T, k, x(k)) \quad (24)$$

and the following three properties hold:

Property 1 There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite $\alpha_3 : \mathbb{R}^{n_1} \rightarrow \mathbb{R}_{\geq 0}$ such that for every pair of strictly positive real numbers (Δ, ν) there exists $T^* > 0$ and for each $T \in (0, T^*)$

there exists $V_T : \mathbb{Z} \times \mathbb{R}^n$ such that the following hold:

$$\begin{aligned} \alpha_1(|x|) &\leq V_T(k, x) \leq \alpha_2(|x|) \\ \frac{V_T(k+1, F_T(k, x)) - V_T(k, x)}{T} &\leq -\alpha_3(x_1) + \nu, \end{aligned}$$

for all $T \in (0, T^*)$, $k \geq 0$, $|x| \leq \Delta$.

Property 2 *There exist continuous nondecreasing functions $\rho_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, $i = 1, 2, 3$ such that $\rho_i(0) = 0$, $i = 1, 2$ and for any strictly positive numbers $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3$ there exists $\hat{T} > 0$ such that for all $T \in (0, \hat{T})$, $k \geq 0$ and $x \in \mathbb{R}^n$ we have:*

$$\begin{aligned} \max \left\{ |B_o|, |B_o^+|, \frac{|B_o^+ - B_o|}{T} \right\} &\leq \rho_3(|x|) + \tilde{\nu}_1 \\ \frac{|B_o(T, k, x_2 + TC) - B_o|}{T} &\leq \rho_3(|x|)\rho_1(|x_1|) + \tilde{\nu}_2 \\ |B - B_o| &\leq \rho_3(|x|)\rho_1(|x_1|) + \tilde{\nu}_2 \quad (25) \\ \max\{|A|, |C|\} &\leq \rho_3(|x|)\rho_2(|x_1|) + \tilde{\nu}_3 \end{aligned}$$

Property 3 *There exists a continuous, positive definite function $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, and $\hat{T} > 0$ such that for all $x_2 \in \mathbb{R}^{n_2}$, $T \in (0, \hat{T})$ and $k \geq 0$ we have $T \sum_{i=k}^{\infty} e^{(k-i)T} |B_o(T, i, x_2)| \geq \alpha(|x_2|)$.*

The reason why we introduce Properties 1-3 is because the main result of this section (Theorem 2 given below) applies to more general problems than the MRAC problem we introduced in the beginning of this section. Moreover, it is obvious that Assumption 2 implies Property 3 and we can further prove the following:

Proposition 3 *If Assumption 1 holds with W_T such that $\nabla_e W_T$ is locally Lipschitz, uniformly in small T . Then Property 1 holds for the system (18), (19), (20) and (21) (in (x_1, x_2) coordinates) with $V_T(k, x) = W_T(k, x_1) + \frac{1}{2}x_2^T x_2$.*

Proposition 4 *If Assumption 1 holds with W_T such that $\nabla_e W_T$ is locally Lipschitz, uniformly in small T . Then Property 2 holds for the system (18), (19), (20) and (21) (in (x_1, x_2) coordinates).*

Remark 1 *The condition stated in Property 3 is like the so called uniform δ persistency of excitation ($u\delta$ -PE) that was introduced in [8] (see also [1, 2] for more recent developments using this notion).*

We can now establish the following result which covers the solution to the MRAC problem (the proof is omitted for space reasons):

Theorem 2 *Suppose that Properties 1, 2 and 3 hold. Then the system (23), (24) satisfies all conditions of Theorem 1 and hence it is USPAS.*

Finally, using [4, Theorem 2] and conclusion of Theorem 2, we can conclude that the sampled-data system consisting of continuous-time dynamics (17) and discrete-time controller (20) and (21) that are interconnected via a sampler and zero-order-hold is USPAS.

5 Conclusions

We presented a Matrosov theorem for parameterized discrete-time time-varying models that facilitates controller design for sampled-data nonlinear systems via their approximate discrete-time models. Our main theorem is an analogue of the continuous-time result in [1] that generalized the classical Matrosov theorem for continuous-time systems. We presented an application of our results to a class of models that arise when an approximate discrete-time model of the plant is used to design a model reference adaptive controller.

References

- [1] A.Loria, E. Panteley, D. Popović and A. R. Teel, *An extension of Matrosov's theorem with application to stabilization of nonholonomic control systems* 41st IEEE Conf. Decis. Contr., Las Vegas, NV, pp. 1528–1533.
- [2] A.Loria, E. Panteley, D. Popović and A. R. Teel, *δ -Persistency of excitation: a necessary and sufficient condition for uniform attractivity* in Proc. 41st IEEE Conf. Decis. Contr., Las Vegas, NV, pp. 3506–3511, 2002.
- [3] V. M. Matrosov. On the stability of motion. *J. Appl. Math. Mech.*, 26:1337–1353, 1962.
- [4] D. Nešić, A. Teel, and P. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Syst. & Contr. Letters*, 38:259–270, 1999.
- [5] D. Nešić, A. Teel, and E. Sontag. Formulas relating $\mathcal{KL}$ stability estimates of discrete-time and sampled-data nonlinear systems. *Syst. & Contr. Letters*, 38:49–60, 1999.
- [6] D. Nešić and A.R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. submitted *IEEE Trans. Automat. Contr.*, 2002.
- [7] B. Paden and R. Panja Globally asymptotically stable PD+ controller for robot manipulators *Int. J. Contr.*, 47: 1697–1712, 1988.
- [8] E. Panteley, A. Loria. and A.R. Teel. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. Automat. Contr.*, 46(12):1874–1886, 2001.
- [9] N. Rouche and J. Mawhin. *Ordinary differential equations II: Stability and periodical solutions*. Pitman publishing Ltd., London, 1980.
- [10] N. Rouche, P. Habets and M. Laloy *Stability theory by Lyapunov's direct method*. Springer Verlag: New York, 1977.
- [11] A. R. Teel and L. Praly, *A smooth Lyapunov function from a class $\mathcal{KL}$ estimate involving two positive semidefinite functions*, ESAIM: Control, Optimization and Calculus of Variations, vol. 5, pp. 313–367, 2000.
- [12] A.R. Teel, E. Panteley and A. Loria, new-block *Integral characterizations of uniform asymptotic and exponential stability with applications* Math. Contr. Sig. Syst., 12: 177–201, 2002.