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Trajectory-based proofs for sampled-data extremum seeking control

Sei Zhen Khong, Dragan Nešić, Ying Tan, and Chris Manzie

***Abstract*— Extremum seeking of nonlinear systems based on a sampled-data control law is revisited. It is established that under some generic assumptions, semi-global practical asymptotically stable convergence to an extremum can be achieved. To this end, trajectory-based arguments are employed, by contrast with Lyapunov-function-type approaches in the existing literature. The proof is simpler and more straightforward; it is based on assumptions that are in general easier to verify. The proposed extremum seeking framework may encompass more general optimisation algorithms, such as those which do not admit a state-update realisation and/or Lyapunov functions. Multi-unit extremum seeking is also investigated within the context of accelerating the speed of convergence.**

***Index Terms*— Extremum seeking, sampled-data control, trajectory properties, robustness, multi-unit systems**

I. Introduction

In the absence of the knowledge of a mathematical model of a dynamical plant and its steady-state input-output map, *extremum seeking* is a real-time optimisation method that drives the system into a vicinity of an extremum of its steady-state behaviour [1], [2], [3], [4]. Teel and Popović made a significant contribution to the area of extremum seeking in [5], [6], where it is shown using Lyapunov-type arguments that under generic assumptions on the asymptotic stability of both the plant and discrete-time nonlinear programming method, extremum seeking can be achieved within a periodic sampled-data framework. This gives rise to powerful capacity to utilise a wide class of optimisation algorithms on the task of steady-state extremum seeking of dynamical systems, where complexity of implementation, speed of convergence, etc. of the algorithms may be taken into account in the control design stage. A Lyapunov stability proof of the scheme based on interconnected systems' theory is examined in [7] using stronger conditions.

In this paper, the sampled-data framework of [5], [6] is reinvestigated and generalised to a multi-unit setting. Semi-global practical asymptotical stability of the proposed extremum seeking schemes is established using the notion of multi-step consistency/robustness. The proof provided is trajectory-based, and serves as a straightforward alternative to the Lyapunov-based proof given in [5], [6]. The latter exploits the closeness of solutions to a differential inclusion form over a single time step. The similarities and differences between the two are clearly identified. In particular, it is demonstrated that by restricting the abstract trajectory-based convergence result of this paper to one where the extremum seeking algorithms take a differential inclusion form as in [5], [6], the underlying technical assumptions employed herein are no stronger than those in [5], [6]. The latter are contingent on the existence of a Lyapunov function for the extremum seeking algorithm. Not only are the trajectory-based assumptions stated here generally easier to verify (or speculated via simulations/experiments), the proof of the main result, having no recourse to Lyapunov-type arguments, is also simpler than that in [5], [6]. To use the results in [5], [6], one would typically need to construct a Lyapunov function or resort to converse Lyapunov theorems [8], [9], whereas the results in this paper apply directly once asymptotic stability of extremum seeking algorithms is established.

Efficiency of extremum seeking can be increased by exploiting parallelism in computations. This paper considers generalisation of the asymptotically stable convergence result to multiple plants of similar but *non-identical* dynamics are available for probing as a means of accelerating the speed of convergence. The aforementioned trajectory-based approach proves to be a valuable asset in the endeavour in terms of the simplicity of proof. Much of the material of this paper has been expanded into a full journal version [10].

The paper is organised as follows. The type of dynamical plants considered for extremum seeking is described in the next section. In Section III, a trajectory-based proof for asymptotically stable convergence is given. A discussion about the relation with the work [5], [6] is provided in Section IV. A gradient descent method based extremum seeking algorithm that satisfies the main assumption employed is provided in Section V. Extremum seeking with multiple units/plants is considered in Section VI and simulation examples in Section VII. Conclusions are given in Section VIII.

II. Dynamical systems

The class of nonlinear, possibly infinite-dimensional, systems that we consider in this paper is introduced in this section. We begin with the following notational definitions.

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ (denoted $\gamma \in \mathcal{K}$) if it is continuous, strictly increasing, and $\gamma(0) = 0$. If γ is also unbounded, then $\gamma \in \mathcal{K}_\infty$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if for each fixed t, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed s, $\beta(s, \cdot)$ is decreasing to zero [8]. The Euclidean norm is denoted $\|\cdot\|_2$.

Let $\mathcal{X}$ be a Banach space whose norm is denoted $\|\cdot\|$. Given any subset $\mathcal{Y}$ of $\mathcal{X}$ and a point $x \in \mathcal{X}$, define the distance of x from $\mathcal{Y}$ as $\|x\|_{\mathcal{Y}} := \inf_{a \in \mathcal{Y}} \|x - a\|$. Also let

$$\mathcal{U}_\epsilon(\mathcal{Y}) := \{x \in \mathcal{X} \mid \|x\|_{\mathcal{Y}} < \epsilon\}.$$

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Definition 2.1: Let the state of a time-invariant dynamical system be represented by $x : \mathbb{R}_{\geq 0} \to \mathcal{X}$, where $\mathcal{X}$ is a Banach space. The input to and output from the system are denoted, respectively, by $u : \mathbb{R}_{\geq 0} \to \Omega \subset \mathbb{R}^m$ and $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$. The set Ω denotes the input space of interest, and is taken to be a compact subset of $\mathbb{R}^m$ in this paper. Note that this is not a stringent assumption given the ubiquity of control input saturation constraints in physical systems [8]. Given any $u \in \Omega$ and $x_0 \in \mathcal{X}$, let $x(\cdot, x_0, u)$ be the state of the dynamical system starting at x_0 with input u.

The following assumption is based to a large extent on [5, Assum. 1].

Assumption 2.2: Given a system described in Definition 2.1, the following hold:

(i) There exists a function $\mathcal{A}$ mapping from Ω to subsets of $\mathcal{X}$ such that for each constant $u \in \Omega$, $\mathcal{A}(u)$ is a nonempty closed set and a *global attractor* [11]:
  a) Given any $x_0 \in \mathcal{X}$ and $\epsilon > 0$, there exists a sufficiently large $t > 0$ such that $x(t, x_0, u) \in \mathcal{U}_\epsilon(\mathcal{A}(u))$;
  b) If $x(t_0, x_0, u) \in \mathcal{A}(u)$, then $x(t, x_0, u) \in \mathcal{A}(u)$ for all $t \geq t_0$;
  c) There exists no proper subset of $\mathcal{A}(u)$ having the first two properties above.

  Furthermore,
$$\sup_{u\in\Omega} \sup_{x\in\mathcal{A}(u)} \|x\| < \infty. \tag{1}$$

(ii) Given any constant input $u \in \Omega$, there exists a locally Lipschitz function $h : \mathcal{X} \to \mathbb{R}$ such that the system output
$$y(t) = h(x(t, x_0, u)) \quad \forall t \geq 0.$$
  Moreover, $h(x_a) = h(x_b)$ for every $x_a, x_b \in \mathcal{A}(u)$. Since $\mathcal{A}(u)$ is a global attractor and h is locally Lipschitz, for any $u \in \Omega$ and $x_0 \in \mathcal{X}$,
$$\begin{aligned} Q(u) &:= \lim_{t\to\infty} h(x(t, x_0, u)) \\ &= h\left(\lim_{t\to\infty} x(t, x_0, u)\right) \\ &= h(x_l), \quad \text{where } x_l \in \mathcal{A}(u) \end{aligned}$$
  is a well-defined steady-state input-output map that is Lipschitz on Ω.

(iii) Q takes its global minimum value in a nonempty, compact set $\mathcal{C} \subset \Omega$.

(iv) There exists a class-$\mathcal{KL}$ function β such that
$$\|x(t, x_0, u)\|_{\mathcal{A}(u)} \leq \beta(\|x_0\|_{\mathcal{A}(u)}, t) \quad \forall t \geq 0, u \in \Omega.$$

Remark 2.3: The thesis [6] considers also a closed subset of $\mathcal{X}$ on which the system is not allowed to operate. Such a consideration can be straightforwardly accommodated in this paper by strengthening Assumption 2.2 according to [6, Assum. 2.4(4)].

III. Sampled-data extremum seeking control

The main sampled-data extremum seeking framework based on [5] is described in this section. The convergence proof in this section is established via the use of trajectory-type properties, which is more straightforward than the Lyapunov method employed in [5], [6]. The merits of doing so and the comparison between these results are provided in the succeeding sections.

Let $\{u_k\}_{k=0}^{\infty}$ be a sequence of vectors in Ω and define the zero-order hold (ZOH) operation
$$u(t) := u_k \quad \text{for all } t \in [kT, (k+1)T) \tag{2}$$
and $k = 0, 1, 2, \ldots$, where $T > 0$ denotes the sampling period or waiting time. Furthermore, let the state and output of a dynamical system in Definition 2.1 with respect to the input u be respectively x and y and define the ideal periodic sampling operation $x_k := x(kT)$;
$$y_k := y(kT) \quad \text{for all } k = 1, 2, \ldots. \tag{3}$$

The following lemma on dynamical systems is needed to establish the main result of this section. The proof is based on ideas from [12, Prop. 1], where finite-dimensional state-space systems with asymptotically stable equilibrium points are considered. Note that infinite-dimensional systems with general attractors are accommodated here.

Lemma 3.1: Given any dynamical system described in Definition 2.1 that satisfies Assumption 2.2, $\Delta > 0$, and $\nu > 0$, there exists a $T > 0$ such that for any $\{u_k\}_{k=0}^{\infty} \subset \Omega$ and $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta$,
$$|y_k - Q(u_{k-1})| \leq \nu \quad \text{for all } k = 1, 2, \ldots,$$
where y_k is as in (3) with y being the output of the system for the input u given by (2).

Proof: Assumption 2.1(iv) implies that for each triplet (ϵ_1, ϵ_2, Δ) of strictly positive real numbers, there exists a waiting time $T > 0$ such that if $\|x_0\|_{\mathcal{A}(u)} \leq \Delta$,
$$\|x(t, x_0, u)\|_{\mathcal{A}(u)} \leq \epsilon_1 \|x_0\|_{\mathcal{A}(u)} + \epsilon_2 \quad \forall t \geq T, u \in \Omega.$$
The rest of the proof then follows from [13, Lem. 13]. ■

Remark 3.2: The proof of Lemma 3.1 exploits the assumption that the set of attractors is uniformly bounded with respect to all constant inputs in Ω; see (1). Alternatively, an assumption as in [5, Assum. 4(1)] can be made to ensure the conclusion of the lemma holds.

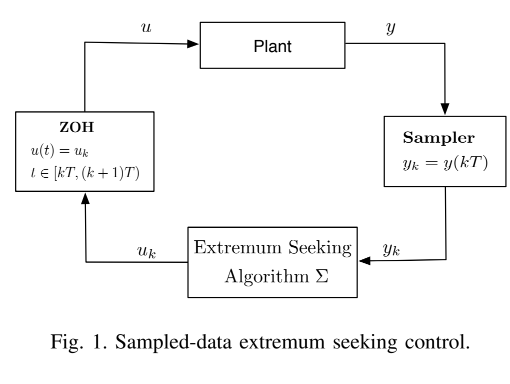

Fig. 1. Sampled-data extremum seeking control.

Figure 1 shows an extremum seeking scheme based on a sampled-data control law with period T. We show here that when the extremum seeking algorithms Σ satisfy a convergence and robustness property, specifically, asymptotic stability and multi-step consistency, it is possible to establish semi-global practical asymptotic stability for the feedback system in Figure 1.

Consider the problem of finding y^*, where

$$y^* := \min_{u \in \Omega} Q(u). \tag{4}$$

Assumption 3.3: The discrete-time extremum seeking controller Σ in Figure 1, when applied to (4), satisfies the following conditions:

(i) Σ is time-invariant. Denote by $\{\hat{u}_k\}_{k=0}^{\infty} \subset \Omega$ the output sequence Σ generates based on input to Σ, $\{\hat{y}_k\}_{k=1}^{\infty}$, where $\hat{y}_k := Q(\hat{u}_{k-1})$. Σ is causal in the sense that the output at any time $N \in \mathbb{N}$, i.e. $\hat{u}_N$, is determined based only on $\hat{u}_k$ and $\hat{y}_{k+1}$ for $k = 0, 1, \ldots, N-1$, that is the past probe values to Q and the corresponding measurements.

(ii) Denote by $\mathcal{S}(\hat{u}_0)$ the set of all admissible output sequences of Σ with respect to the initial point $\hat{u}_0$. There exists a class-$\mathcal{KL}$ function β such that for any initial point $\hat{u}_0 \in \Omega$, all outputs $\hat{u} \in \mathcal{S}(\hat{u}_0)$ satisfy

$$\|\hat{u}_k(\hat{u}_0)\|_{\mathcal{C}} \leq \beta(\|\hat{u}_0\|_{\mathcal{C}}, k) + \delta \quad \forall k \geq 0. \tag{5}$$

(iii) As depicted in Figure 2, let $y_k := Q(u_{k-1}) + w_k$, where $w_k \in \mathbb{R}$ and denote by $\{u_k\}_{k=0}^{\infty}$ the sequence Σ generates based on $\{y_k\}_{k=1}^{\infty}$. The pair (u, y) is multi-step consistent/close [14] with $(\hat{u}, \hat{y})$, in the sense that for any positive (Δ, η) and $N \in \mathbb{N}$, there exists a $\nu > 0$ such that if $\|u_0\|_{\mathcal{C}} \leq \Delta$ and $|w_k| \leq \nu$ for $k = 1, \ldots, N$, then there exists a $\hat{u} \in \mathcal{S}(u_0)$ satisfying

$$\|u_k - \hat{u}_k\|_2 \leq \eta \quad \text{for } k = 0, 1, \ldots, N.$$

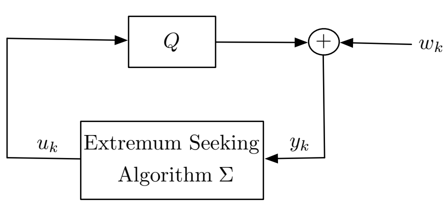

Fig. 2. Extremum seeking algorithm with noisy output measurement.

Remark 3.4: The set of outputs $\mathcal{S}(\hat{u}_0)$ arises, for example, from modelling the optimisation algorithm with a difference inclusion involving a set-value 'state-update' map F by $\hat{u}^+ \in F(\hat{u}, G(\hat{u}))$; see [9] and Section IV. In the simplest case, $\mathcal{S}(\hat{u}_0)$ is a singleton, i.e. there is only one possible output sequence given a fixed initial condition. For example, one modelled by a difference equation $\hat{u}^+ = F(\hat{u}, G(\hat{u}))$.

Theorem 3.5: The closed-loop system depicted in Figure 1, consisting of a dynamical plant satisfying Definition 2.1 and Assumption 2.2, T-periodic sampler (3), zero-order hold (2), and an extremum seeking algorithm satisfying Assumption 3.3, is semi-globally practically asymptotically stable in the following sense: Given any (Δ, μ) such that $\Delta, \mu > \delta$, where $\delta \geq 0$ is described in Assumption 3.3(ii), there exist a sampling/waiting period $T > 0$ and a $\bar{\beta} \in \mathcal{KL}$ such that for any $\|x_0\|_{\mathcal{A}(u_0)} \leq \Delta$ and $\|u_0\|_{\mathcal{C}} \leq \Delta$,

$$\|u_k\|_{\mathcal{C}} \leq \bar{\beta}(\|u_0\|_{\mathcal{C}}, k) + \mu \tag{6}$$

for all $k = 0, 1, \ldots$, where $\mathcal{C}$ is the set of global minimisers for $Q : \Omega \subset \mathbb{R}^m \to \mathbb{R}$, the steady-state map of the plant, as in Assumption 2.2.

Proof: See [10]. ■

Remark 3.6: Since Q is Lipschitz-continuous, Theorem 6.5 implies that there exists a $\hat{\mu} > 0$ corresponding to μ such that $Q(u_k) \to y^* + \hat{\mu}\bar{\mathcal{B}}$ as $k \to \infty$, where $\bar{\mathcal{B}}$ denotes the closed unit ball (interval) in $\mathbb{R}$. In other words, the output of the plant converges to a $\hat{\mu}$-neighbourhood of the global minimum y^*.

IV. Relation to the Work by Teel and Popović

This section discusses the link between this paper and the predecessor work by Teel and Popović [5], [6]. The stability and consistency assumptions used are clarified and the differences and similarities identified. The conclusion is that when restricted to extremum seeking algorithms taking a difference inclusion form, the technical assumptions utilised in [5], [6] imply Assumption 3.3 of this paper.

A. Difference inclusions

The following difference inclusion form describing the extremum seeking/optimisation algorithm Σ is assumed in [5]:

$$u^+ \in F(u, G(u)), \tag{7}$$

where F is an upper semi-continuous (cf. Definition 4.1) set-valued map (the update u^+ can be any element of the set) and G is a function that carries information regarding the estimate of the gradient of Q around u. In particular, F maps from $\mathbb{R}^m \times \mathbb{R}^p$ to subsets of $\mathbb{R}^m$,

$$G(u_k) := \begin{bmatrix} Q(u_k + d_1(u_k)) \\ \vdots \\ Q(u_k + d_p(u_k)) \end{bmatrix},$$

and $d_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, p$ are dither/perturbation functions. For each $u \in \Omega$, the set $F(u, G(u))$ is nonempty and compact. See [15] for a class of Lyapunov-based nonsmooth optimisation algorithms of the form described above which employ the notion of Clarke generalised gradient. It is apparent that (7) is time-invariant and causal, i.e. it satisfies Assumption 3.3(i).

B. Asymptotic stability

By the Lyapunov stability result [9, Thm. 2.7], the existence of a continuous Lyapunov function V for (7) presumed in [5, Assum. 2] or [6, Assum. 2.7] implies that Assumption 3.3(ii) is true; see also [16]. In particular, $V : \Omega \subset \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ satisfies

$$\max_{w \in F(u, G(u))} V(w) - V(u) \leq -\|u\|_{\mathcal{C}} + \delta \quad \forall u \in \Omega.$$

Albeit not explicitly stated in [5, Assum. 2] or [6, Assum. 2.7], the Lyapunov function therein should actually be bounded below and above two $\mathcal{K}_\infty$ functions α_1 and α_2 as in [9], [16], i.e.,

$$\alpha_1(\|u\|_{\mathcal{C}}) \leq V(u) \leq \alpha_2(\|u\|_{\mathcal{C}}).$$

This is necessary for the main extremum seeking results of Teel and Popović [5, Rem. 5] and [6, Cor. 2.15] to hold, by the stability results in [9], [16]. Indeed, [6, Rem. 2.9] notes that the existence of the required Lyapunov function is ascertained by a converse Lyapunov theorem in [16].

C. Consistency

Definition 4.1: $F(u, G(u))$ is said to be an upper semi-continuous function of $u \in \Omega$ if for every $u_a \in \Omega$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $u_b \in \Omega$,

$$\|u_a - u_b\|_2 \leq \delta \implies F(u_b, G(u_b)) \subseteq F(u_a, G(u_a)) + \epsilon\mathcal{B},$$

where $\mathcal{B}$ denotes the open unit ball in $\mathbb{R}^m$.

The following one-step consistency (i.e. closeness of solution over the next time step) assumption is taken from [5, Assum. 4(4)] or [6, Assum. 2.12(5)].

Assumption 4.2: Given any $u_H \in F(u, H)$, let u_G be its closest point in the set $F(u, G(u))$, i.e.

$$u_G := \underset{u \in F(u,G(u))}{\arg\min} \|u - u_H\|_2.$$

Then for all $\Delta > 0$, there exists $L_F > 0$ such that if $\|u\|_{\mathcal{C}} \leq \Delta$ and $\|H\|_2 \leq \Delta$, then

$$\|u_H - u_G\|_2 = \|u_H\|_{F(u,G(u))} \leq L_F \|H - G(u)\|_2.$$

The next result demonstrates that given upper semi-continuity of the set-valued function F, single-step consistency in [5] implies multi-step consistency used in this paper.

Lemma 4.3: Given a difference inclusion $\hat{u}^+ \in F(\hat{u}, G(\hat{u}))$ and its perturbed form $u^+ \in F(u, H(u))$, where $\hat{u}_0 = u_0$,

$$H(u_k) := G(u_k) + W(u_k) = G(u_k) + [w_1(u_k) \cdots w_p(u_k)]^T$$

with

$$|w_i(u)| \leq \nu \text{ for all } i = 1, \ldots, p,\ u \in \Omega, \qquad (8)$$

and $F(\cdot, G(\cdot))$ is upper semi-continuous, one-step consistency in Assumption 4.2 implies multi-step consistency in Assumption 3.3(iii).

Proof: See [10]. ■

V. An example of extremum seeking algorithm

A gradient-based example of extremum seeking controller that satisfies Assumption 3.3 is given here. It is often the case that gradient-based extremum seeking algorithms can be serially decomposed into a derivative[1] estimator and a nonlinear programming method as shown in Figure 3. This paradigm is analogous to its *continuous-time* counterpart in [17], [18], where the singular perturbation technique and time-scale separation are used to establish convergence of the extremum seeking scheme therein.

[1]The first or higher derivatives of an objective function.

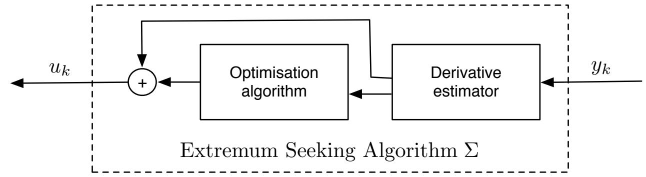

Fig. 3. An extremum seeking controller paradigm.

One of the most well-known methods [19], [20] in operations research is the *gradient descent method*:

$$u_{k+1} = u_k - \lambda_k \nabla Q(u_k),$$

where λ_i denotes the step size which can be computed by, say, the Armijo method [19, Alg. 1.3.3], and $\nabla Q(\cdot)$ denotes the Jacobian of Q. That this satisfies Assumption 3.3 is shown in [10]; see also [3].

VI. A multi-unit paradigm

The problem of driving a finite number of almost identical systems/units to an optimal steady-state input-output behaviour is prevalent in engineering [21], [22]. In the presence of more than a single unit, measurement collection for the purpose of extremum seeking can be made more efficient in an appropriate fashion via parallel computation. Figure 4 shows a generalisation of the sampled-data extremum seeking control scheme of Figure 1 to multiple units. Within this context, each of these units may be driven with a different input concurrently and the corresponding outputs collected/sampled after a chosen waiting time. This results in several function evaluations of the (perturbed) steady-state behaviour within the same sampling/waiting period.

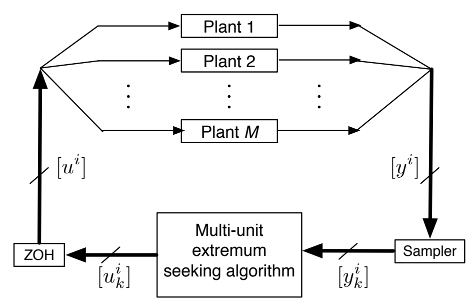

Fig. 4. Extremum seeking based on modified DIRECT

We consider in this paper systems or units that may exhibit different dynamics *and* steady-state input-output maps. The performance measure of extremum seeking is dependent on an auxiliary system, which is taken to be the 'average' of all the available ones. Semi-global asymptotically stable convergence that is practical with respect to the infinity-norm error bound on the discrepancy between units is established within the unified framework of Section III.

Definition 6.1: Given M number of dynamical plants P_1 to P_M each satisfying Definition 2.2, let the average system

$$P_a := \frac{1}{M} \sum_{i=1}^{M} P_i.$$

Denote respectively the corresponding steady-state input-output maps by $Q_1, \ldots, Q_M$ and Q_a, all mapping from $\Omega \subset \mathbb{R}^m$ into $\mathbb{R}$. Let $\mathcal{C} \subset \Omega$ be the set of global minimisers of Q_a.

Throughout, we use superscripts to label the inputs/outputs corresponding to a particular system. For instance, the input, state, and output of P_i are u^i, x^i, and y^i, respectively, for $i = 1, \ldots, M$. The T-periodic sampling and hold operations of Figure 4 are described similarly to (3) and (2) as

$$u^i(t) := u^i_k \quad \text{for all } t \in [kT, (k+1)T) \tag{9}$$

and $x^i_k := x^i(kT)$,

$$y^i_k := y^i(kT) \quad \text{for all } k = 1, 2, \ldots. \tag{10}$$

Assumption 6.2: There exists a $\gamma > 0$ such that

$$\|Q_i - Q_a\|_\infty := \sup_{u \in \Omega} |Q_i(u) - Q_a(u)| \leq \gamma$$

for all $i = 1, \ldots, M$.

Remark 6.3: Note that by Definition 6.1 the plants do not need to exhibit identical dynamics. However, it is required in Assumption 6.2 that their steady-state behaviours differ by no more than some bound γ in the infinity-norm sense.

Assumption 6.4: The extremum seeking controller Σ satisfies the following conditions:

(i) Σ is time-invariant and causal. Denote by $\{\hat{u}^i_k\}_{k=0}^\infty \subset \Omega$ the output sequence Σ generates based on input to Σ, $\{\hat{y}^i_k\}_{k=1}^\infty$, where $\hat{y}^i_k := Q_i(\hat{u}^i_{k-1})$, for $i = 1, \ldots, M$.

(ii) There exists a class-$\mathcal{KL}$ function β and $\delta > 0$ dependent on γ in Assumption 6.2 such that for any initial point $\hat{u}^i_0 \in \Omega$, $i = 1, \ldots, M$, all outputs $\hat{u}^i \in \mathcal{S}(\hat{u}^i_0)$ satisfy

$$\|\hat{u}^i_k(\hat{u}^i_0)\|_{\mathcal{C}} \leq \beta(\|\hat{u}^i_0\|_{\mathcal{C}}, k) + \delta \quad \forall k \geq 0,$$

where $\mathcal{C} \subset \Omega$ is the set of global minimisers for Q_a (cf. Definition 6.1).

(iii) Let $y^i_k := Q_i(u^i_{k-1}) + w^i_k$, where $w^i_k \in \mathbb{R}$ and denote by $\{u^i_k\}_{k=0}^\infty$ the sequence Σ generates based on $\{y^i_k\}_{k=1}^\infty$. The pair (u^i, y^i) is multi-step consistent/close [14] with $(\hat{u}^i, \hat{y}^i)$, $i = 1, \ldots, M$, in the sense that for any positive (Δ, η) and $N \in \mathbb{N}$, there exists a $\nu > 0$ such that if $\|u^i_0\|_{\mathcal{C}} \leq \Delta$ and $|w^i_k| \leq \nu$ for $k = 1, \ldots, N$, then there exists a $\hat{u}^i \in \mathcal{S}(u^i_0)$ satisfying

$$\|u^i_k - \hat{u}^i_k\|_2 \leq \eta \quad \text{for } k = 0, 1, \ldots, N.$$

Theorem 6.5: The closed-loop system depicted in Figure 4, consisting of M dynamical plants satisfying Definition 6.1 and Assumption 6.2, T-periodic sampler (10), zero-order hold (9), and an extremum seeking algorithm satisfying Assumption 6.4, is asymptotically stable in the following sense: Given any (Δ, μ) such that $\Delta, \mu > \delta$, where $\delta \geq 0$ is described in Assumption 6.4(ii), there exist a sampling/waiting period $T > 0$ and a $\bar{\beta} \in \mathcal{KL}$ such that for any $\|x^i_0\|_{\mathcal{A}(u_0)} \leq \Delta$ and $\|u^i_0\|_{\mathcal{C}} \leq \Delta$,

$$\|u^i_k\|_{\mathcal{C}} \leq \bar{\beta}(\|u^i_0\|_{\mathcal{C}}, k) + \mu \tag{11}$$

for all $k = 0, 1, \ldots$ and $i = 1, \ldots, M$.

Proof: The theorem can be established using the same arguments in Theorem 6.5 by appropriately replacing references to the properties in Assumption 3.3 with those in Assumption 6.4. In particular, application of Lemma 3.1 to the i^{th} plant leads to a sampling period $T_i > 0$ such that the required multi-step consistency holds with respect to a desired bound. The overall sampling period for the whole feedback setup can then be taken to be $T := \max_{i=1,\ldots,M} T_i$. ■

Remark 6.6: As demonstrated by the theorem above, another benefit of working with trajectories instead of Lyapunov functions is that generalisation of convergence results to multi-unit systems is rather straightforward. It does not involve constructing aggregate Lyapunov functions.

For nonlinear systems whose steady-state input-output maps are multivariate, the multi-unit framework proposed above equipped with gradient-based extremum seeking controllers delivers great benefits in terms of increasing the efficiency of asymptotic convergence. To be specific, if the number of available units M is no less than the number of dither signals p required to perform a good derivative estimate on a multidimensional space, the required measurements that $(u_k + d_1(u_k), \ldots, u_k + d_p(u_k))$ entail can be all collected within a sampling/waiting period, instead of p-multiples of it in the case of sequential extremum seeking. This is carried out by feeding $u^i_k := u_k + d_i(u_k)$ to the i^{th} plant and sampling the corresponding output y^i after T seconds, for $i = 1, \ldots, p$.

VII. Numerical Examples

Consider the following one-dimensional nonlinear system:

$$\dot{x} = -x^3 + u^2, \quad x(0) = 2; \qquad y = x^3. \tag{12}$$

It is apparent that the steady-state input-output map is $Q(u) = u^2$, of which its global minimum is 0. We employ the extremum seeking controller based on the gradient descent method with a fixed step size of 0.2, as described in Section V. The smooth low noise differentiator with a filter length 5 from [23] is used as the derivative estimator:

$$Q'(u) \approx \frac{2(Q_1 - Q_{-1}) + Q_2 - Q_{-2}}{8h}, \tag{13}$$

where $Q_i := Q(u + ih)$ and $h := 0.1$ denotes the step size. We terminate the algorithm when an input $|u| \leq 0.05$ is found. With a sampling period $T := 0.5$s and an initial guess of -3, it takes 66s to locate an input $u = -0.0475$.

In light of the fact that we selected a first-order derivative estimator (13) which requires four function evaluations, suppose there exist three additional similar dynamical units at our disposal, one of which is the same as (12) and the other two are given by:

$$\dot{x}_a = -x_a^3 + (0.95u_a)^2, \quad x_a(0) = 2; \qquad y_a = x_a^3.$$
$$\dot{x}_b = -x_b^3 + (1.05u_b)^2, \quad x_b(0) = 2; \qquad y_b = x_b^3.$$

Using the multi-unit extremum seeking scheme in Section VI within the same parameter setup as the above single-unit scenario, it takes only 5s to locate an input $u = -0.04$. A significant contributor to the acceleration of speed of convergence is that of being able to collect four measurements within a single 0.5s period. If there are more units at hand then a derivative estimator with greater filter length [23] can be deployed to increase robustness to perturbations and hence the accuracy. The reader is referred to [10] for more simulation examples, where dynamical plants with periodic attractors are examined.

VIII. Conclusions

The work by Teel and Popović [5], [6] establishes semi-global practical asymptotic stability of the sampled-data extremum scheme in Figure 1 via the Lyapunov's second method [8], [9], [16] for a particular class of algorithms modelled by the state-update difference inclusion of the form (7). The main regularity condition which ensures robustness to dynamical perturbation of the steady-state map is one-step consistency of the algorithm's state, as described in Assumption 4.2. By contrast, in this paper we adopt a trajectory-based approach to developing the same end result, but in conjunction with consistency of output of the extremum seeking algorithm over multiple time steps (cf. Assumption 3.3(iii)), *without* stipulating a differential inclusion model as in (7). Interestingly, Lemma 4.3 shows that multi-step consistency follows from one-step consistency provided the right-hand side of (7) satisfies the upper semicontinuity property given in Definition 4.1. In view of this result, it can be concluded that the asymptotically stable convergence of extremum seeking in Theorem 6.5 is developed using no assumptions stronger than those made in [5], [6]. To be specific, Assumption 3.3 is a consequence of realising the extremum seeking algorithm with the differential inclusion (7) whose right-hand side is upper semicontinuous, the Lyapunov stability results in [9], [16], and the one-step consistency Assumption 4.2.

It is the authors' belief that the trajectory-based stability proofs presented in Section III are more direct and straightforward than the Lyapunov methods in [5], [6]. They also lead to a natural generalisation to multi-unit setting for the purpose of convergence speedup. Take the well-known descent methods of gradient and Newton for example, their convergence proofs are most readily established via a $\mathcal{KL}$-type trajectory-based argument; see [19, Chapter 1]. As a second example, recall that taking the notion of physical energy into account, there exist many systems for which Lyapunov functions whose derivative are not strictly negative can be found. The principle of Krasovskii-Lasalle [8] is often exploited to conclude asymptotic stability of such systems. To apply the extremum seeking convergence result in [5] to the above cases, one would need to appeal to the converse Lyapunov theorems in [16] to arrive at the required Lyapunov functions for the extremum seeking algorithm. Furthermore, since the main result in [5] is expressed in terms of Lyapunov functions, the user would need to apply the Lyapunov theorems in [9], [16] to conclude the asymptotic stability derived in Theorem 6.5. The results in this paper eliminate the need for such a detour, while accommodating situations where the required Lyapunov theorems are not readily available. This may be the case when, for instance, the extremum seeking algorithms are not realisable by a difference inclusion.

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