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Title:

L/sub p/ stability of networked control systems

Date:

2003

Citation:

NESIC, D. & TEEL, A. R. (2003). L/sub p/ stability of networked control systems. Proceedings of 42nd IEEE conference on decision and control, 2, pp.1188-1193. IEEE - Institute of Electrical and Electronic Engineers. <https://doi.org/10.1109/CDC.2003.1272769>.

Persistent Link:

<https://hdl.handle.net/11343/299633>

$\mathcal{L}_p$ Stability of Networked Control Systems

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Abstract

Results on input-output $\mathcal{L}_p$ stability of networked control systems (NCS) are presented for a large class of network scheduling protocols. It is shown that static protocols and a recently considered dynamical protocol called Try-Once-Discard (TOD) belong to this class. Our results provide a unifying framework for generating new scheduling protocols that preserve $\mathcal{L}_p$ stability properties of the system if a design parameter is chosen sufficiently small. Our results are always guaranteed to be less conservative than existing results in the literature and we illustrate this via an example of a batch reactor.

1 Introduction

Networked Control Systems (NCS) are currently receiving considerable attention in the literature as illustrated by recent articles [2, 6, 8, 13, 14, 15, 16, 18] and references listed therein. Most existing literature considers only stabilization of linear NCS whereas nonlinear NCS have received little attention (with few exceptions, such as [14]). Moreover, most results treat NCS without disturbances and we are aware only of limited results on stability of NCS with disturbances, such as the $\mathcal{L}_\infty$ to root-mean-square stability of a class of NCS considered in [7] and results on input-output stability of linear jump parameter systems in [4] that can be exploited for certain NCS with static protocols. Also, in some cases it is possible to use tools for linear sampled-data systems [5] for analysis and design of certain classes of linear NCS.

In this paper we consider input-output $\mathcal{L}_p$ stability of *nonlinear NCS with disturbances*. We follow the method proposed in [14, 15], in which one first designs the controller without taking into account the network and then in the second step one determines a design parameter called the maximum allowable transfer interval (MATI) so that the closed loop remains stable when some control and sensor signals are transmitted via the network. This approach was shown to produce stabilizing controllers for linear NCS in [14] and nonlinear NCS in [15].

Our main result states for a large class of protocols that if the controller designed in the first step of the above described procedure achieves $\mathcal{L}_p$ stability in an appro-

priate sense, then for sufficiently small values of MATI the NCS preserves $\mathcal{L}_p$ stability properties. Our result is presented for a very general class of static and dynamic protocols in a unified manner. Indeed, both static and TOD protocols are special cases of the general class of protocols we consider. Moreover, our results can serve as a *framework* for generating new classes of protocols that have good properties. The bound on MATI that we obtain has a very simple form and we prove that it is better than the bounds obtained in [15] when our result is specialized to the case of NCS without disturbances. In particular, we show that $\mathcal{L}_p$ stability of the system is preserved if MATI is smaller than (see Theorem 2)

$$\frac{1}{L} \ln \left(\frac{L + \gamma}{\rho L + \gamma} \right),$$

where ρ characterizes the stability properties of the protocol and it typically depends on the number of nodes ℓ in the network; L characterizes the possible growth of the error between the real values of inputs and outputs and their last transmitted values via the network; γ is the $\mathcal{L}_p$ disturbance gain that captures robustness properties of the system without the network.

We show that, even when using an estimate of the $\mathcal{L}_p$ gain that is the most conservative one possible, the above bound is at least $2(\ell + 1)$ times better than the bound in [15, 18] for linear NCS with TOD protocol and without disturbances, where ℓ is the number of nodes. Similarly, for the so called Round Robin (RR) static protocol our bound is at least $2\sqrt{\ell + 1}$ times better than the bound in [15, 18] for linear NCS. Our bounds compare in a similar manner to the bounds from [14] for nonlinear NCS. Hence, the larger the number of nodes in the network, the less conservative our bound is when compared to the bound from [15]. This is also illustrated via an example of an unstable batch reactor with the TOD protocol considered in [15] where our results provide 1000 times larger bound on MATI than the bound obtained in [15] even though $\ell = 2$.

The paper is organized as follows. In Section 2 we present mathematical preliminaries. The model of the NCS that we use is formulated in Section 3. A large class of protocols is defined in Section 4 and it is shown that the recently considered TOD protocol and static protocols belong to this class. Section 5 contains the main results of the paper results. Comparisons of our bounds with existing bounds in the literature is presented in Section 6 and an example is given in Section 7. Summary is given in the last section.

2 Preliminaries

$\mathbb{R}$ and $\mathbb{N}$ denote, respectively, the sets of real and natural numbers. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative in-

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tegers. All vector norms, denoted as $|\cdot|$, are Euclidean norms unless otherwise stated. The same notation $|\cdot|$ is used for the induced 2-norm of a matrix. $\lambda_{\max}(A)$ denotes the maximum eigenvalue of a symmetric matrix A . Given a measurable, locally integrable signal $\varphi : [t_0, t] \rightarrow \mathbb{R}^n$ and $p \in [1, \infty)$, we denote its $\mathcal{L}_p$ norm as follows $\|\varphi[t_0, t]\|_{\mathcal{L}_p} := \left(\int_{t_0}^t |\varphi(s)|^p ds \right)^{\frac{1}{p}}$. If $p = \infty$, we denote the $\mathcal{L}_\infty$ norm as follows $\|\varphi[t_0, t]\|_{\mathcal{L}_\infty} := \text{ess sup}_{s \in [t_0, t]} |\varphi(s)|$. If $\varphi(\cdot)$ is defined on $[t_0, \infty)$ and for some $p \in [1, \infty]$ there exists $K \geq 0$ such that $\|\varphi[t_0, t]\|_{\mathcal{L}_p} \leq K, \forall t \geq t_0 \geq 0$, then we write $\varphi \in \mathcal{L}_p$. For the systems we consider in this paper, a monotonically increasing sequence of times $t_{s_i} \in \mathbb{R}_{\geq 0}$ is given where $i \in \mathbb{N}$ and $t_{s_0} = 0$. Moreover, we assume¹ that there exists $\epsilon > 0$ such that $\epsilon \leq t_{s_{i+1}} - t_{s_i}, \forall i \in \mathbb{N}$. The systems are governed by the evolution equations

$$\begin{aligned} \dot{x} &= f(t, x, w) & t \in [t_{s_i}, t_{s_{i+1}}] \\ x(t_{s_i}^+) &= h(i, x(t_{s_i})), \end{aligned} \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ and $w \in \mathbb{R}^{n_w}$ are respectively the state and disturbance input of the system. The meaning of solution for this class of systems is explained in [12]. We use $x(\cdot, t_0, x_0, w)$ to denote such a solution. When (t_0, x_0) and $w(\cdot)$ are clear from the context we use the shorthand notation $x(\cdot)$. If for all x_0, t_0 and $w(\cdot)$, all corresponding solutions are right maximally defined on $[t_0, \infty)$, then we say that the system (1) is *forward complete*. We also consider an output function

$$y = H(t, x, w), \quad (2)$$

where $y \in \mathbb{R}^{n_y}$. We use notation $y(\cdot, t_0, x_0, w) := H(\cdot, x(\cdot, t_0, x_0, w), w(\cdot))$ or shortly $y(\cdot)$ when (t_0, x_0) and $w(\cdot)$ are clear from the context.

We use the following definitions for the system (1)-(2).

Definition 1 Let $p \in [1, \infty]$ and $\gamma \geq 0$ be given. The system (1)-(2) is said to be $\mathcal{L}_p$ stable from w to y (with gain γ) if there exists $K > 0$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{L}_p$ and each corresponding solution $x(\cdot)$, we have that $\|y[t_0, t]\|_{\mathcal{L}_p} \leq K|x_0| + \gamma\|w[t_0, t]\|_{\mathcal{L}_p}$, for all $t \in [t_0, t_0 + T)$, where $[t_0, t_0 + T)$ is the maximal interval of definition of $x(\cdot)$. ■

Definition 2 Let $p, q \in [1, \infty]$ and $\gamma \geq 0$ be given. The state x of the system (1)-(2) is said to be $\mathcal{L}_p$ to $\mathcal{L}_q$ detectable from output y (with gain γ) if there exists $K > 0$ such that for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$, $w \in \mathcal{L}_p$ and each corresponding solution $x(\cdot)$, we have that $\|x[t_0, t]\|_{\mathcal{L}_q} \leq K|x_0| + \gamma\|y[t_0, t]\|_{\mathcal{L}_p} + \gamma\|w[t_0, t]\|_{\mathcal{L}_p}$ for all $t \in [t_0, t_0 + T)$, where $[t_0, t_0 + T)$ is the maximal interval of definition of $x(\cdot)$. ■

Definition 3 Consider system (1) and suppose that $w \equiv 0$. Let $K, c > 0$ be given. We say that the system is uniformly globally exponentially stable (UGES)

if for all $x_0 \in \mathbb{R}^{n_x}$ and all corresponding solutions $x(\cdot)$, we have that $|x(t, t_0, x_0)| \leq K|x_0|\exp(t - t_0)$ for all $t \geq t_0 \geq 0$. ■

Definition 4 The origin of the system (1) with $w \equiv 0$ is said to be uniformly globally fixed time interval stable (UGFTIS) with linear gain if there exist $T > 0$ and $\bar{\rho} > 0$ such that, for all $t_0 \geq 0$, $x_0 \in \mathbb{R}^{n_x}$ and the corresponding solutions $x(\cdot)$, we have that $t - t_0 \in [0, T]$ implies $|x(t, t_0, x_0)| \leq \bar{\rho}|x_0|$. ■

The following result relates $\mathcal{L}_p$ stability for some $p \in [1, \infty)$ and UGES².

Theorem 1 Suppose that the system (1) is $\mathcal{L}_p$ stable from w to x for some $p \in [1, \infty)$ and the origin of (1) with $w \equiv 0$ is UGFTIS with linear gain. Then, the system (1) with $w \equiv 0$ is UGES. ■

3 Definition of Networked Control Systems

In this section we introduce a class of models with jumps that we will use to describe NCS. We augment the model of NCS proposed in [15] with an equation that describes the operation of the scheduling protocol and we show that two protocols (RR and TOD protocols) that were studied in [15] can be modelled in this manner. Let the sequence $t_{s_i}, i \in \mathbb{N}$ of monotonically increasing transmission times satisfy $\epsilon \leq t_{s_{j+1}} - t_{s_j} \leq \tau$ for all $j \in \mathbb{N}$ and some fixed $\epsilon, \tau > 0$. At each t_{s_i} , the protocol gives access to the network to one of the nodes $i \in \{1, 2, \dots, \ell\}$. We adopt terminology from [15] and refer to τ as the *maximum allowable transmission interval* (MATI). We consider general nonlinear NCS with disturbances of the following form

$$\begin{aligned} \dot{x}_P &= f_P(t, x_P, \hat{u}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\ y &= g_P(t, x_P) \\ \dot{x}_C &= f_C(t, x_C, \hat{y}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\ u &= g_C(t, x_C) \\ \dot{\hat{y}} &= \hat{f}_P(t, x_P, x_C, \hat{y}, \hat{u}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\ \dot{\hat{u}} &= \hat{f}_C(t, x_P, x_C, \hat{y}, \hat{u}, w) & t \in [t_{s_{i-1}}, t_{s_i}] \\ \hat{y}(t_{s_i}^+) &= y(t_{s_i}) + h_y(i, e(t_{s_i})) \\ \hat{u}(t_{s_i}^+) &= u(t_{s_i}) + h_u(i, e(t_{s_i})) \end{aligned} \quad (3)$$

where x_P and x_C are respectively states of the plant and the controller; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of most recently transmitted plant and controller output values via the network; e is the network induced error defined as $e = (e_y, e_u)$ where $e_y := \hat{y} - y$ and $e_u := \hat{u} - u$; Note that if NCS has ℓ links, then the error vector can be partitioned as follows $e = (e_1, e_2, \dots, e_\ell)$. The functions h_u and h_y are typically such that, if the j th link gets access to the network at some transmission time t_{s_i} we have that the corresponding part of the error vector

¹We use this assumption to rule out a possibility of Zeno solutions.

²Theorem 1 was proved in [10] for systems without jumps. However, the proof extends with minor changes to systems with jumps and it is therefore omitted.

has a jump. For several protocols, such as the RR and TOD protocols (see below), we typically assume that e_j is reset to zero at time $t_{s_j}^+$, that is $e_j(t_{s_j}^+) = 0$. However, we emphasize that this assumption is not needed in general and this will become clear in the next section.

We combine the controller and plant states into a vector $x := (x_P, x_C)$ and using the error vector defined earlier $e = (e_y, e_u)$ we can rewrite (3) as a system with jumps that is more amenable for analysis:

$$\dot{x} = f(t, x, e, w) \quad \forall t \in [t_{s_{i-1}}, t_{s_i}] \quad (4)$$

$$\dot{e} = g(t, x, e, w) \quad \forall t \in [t_{s_{i-1}}, t_{s_i}] \quad (5)$$

$$e(t_{s_i}^+) = h(i, e(t_{s_i})) , \quad (6)$$

where $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_e}$, $w \in \mathbb{R}^{n_w}$ and formulas for functions f, g, h can be found in [12].

We refer to (6) as a protocol. The protocol determines the algorithm by which access to the network is assigned to different nodes in the system. Results of this paper are applicable to a large class of static and dynamic protocols but we will concentrate on two particular examples described below (see [15]).

Round Robin (RR) protocol: We consider RR protocol as an illustrative example for static protocols. Let there be $\ell \geq 1$ nodes in NCS and let the protocol grant access to the network to the node $i \in \{1, \dots, \ell\}$ at $t_{s_{i+j\ell}}$, for all $j \in \mathbb{N}$, that is $e(t_{s_{i+j\ell}}^+) = 0, \forall j \in \mathbb{N}$. In this case we can write

$$h(i, e) = (I - \Delta(i))e , \quad (7)$$

where $\Delta(i) = \text{diag}\{\Delta_1(i), \Delta_2(i), \dots, \Delta_\ell(i)\}$. The square matrices $\Delta_k(i)$ have the dimension n_k , with $\sum_{k=1}^\ell n_k = n_e$ and $\Delta_k(i) = \delta_k(i)I_{n_k}$, where I_{n_k} are identity matrices of dimension n_k and

$$\delta_k(i) = \begin{cases} 1, & \text{if } i = k + j\ell, j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

TOD protocol: TOD protocol was recently introduced in [15] and it operates as follows. Suppose that there are ℓ nodes competing for access to the network. We can partition the error vector as $e = (e_1, e_2^T, \dots, e_\ell)$. The node i with the greatest weighted error at time t_{s_j} will be granted access to network at $t_{s_j}^+$ and hence we have that $e_i(t_{s_j}^+) = 0$. We assume that the weights are already incorporated into the model. If a data packet fails to win access to the network, it is discarded and new data is used at the next transmission time $t_{s_{j+1}}$. If two or more nodes have equal priority, a pre-specified ordering of the nodes is used to resolve the collision. This verbal description can be converted into the model of the form (6) where

$$h(e) = (I - \Psi(e))e ,$$

and $\Psi(e) := \text{diag}\{\psi_1(e)I_{n_1}, \psi_2(e)I_{n_2}, \dots, \psi_\ell(e)I_{n_\ell}\}$. I_{n_j} are identity matrices of dimension n_j with

$$\sum_{j=1}^\ell n_j = n_e \text{ and}$$

$$\psi_j(e) := \begin{cases} 1, & \text{if } j = \min(\arg \max_j |e_j|) \\ 0, & \text{otherwise.} \end{cases}$$

4 Lyapunov UGES protocols

In this section we define a large class of protocols and show that the RR and TOD protocols belong to this class. Hence, our presentation is unifying for seemingly different existing protocols. On the other hand, we will show that all protocols that belong to this large class will preserve important $\mathcal{L}_p$ stability properties of the system for sufficiently small values of MATI. Therefore, our results provide a *framework* for generating genuinely new classes of static and dynamic protocols whose performance is guaranteed for sufficiently small MATI. We find it useful to introduce an auxiliary discrete-time system of the form:

$$e(i+1) = h(i, e(i)) \quad i \in \mathbb{N} , \quad (8)$$

and refer to it as a *discrete-time system induced by the protocol (6)*. Sometimes we abuse the terminology and refer to (8) simply as a protocol. Central to this paper is the following class of protocols:

Definition 5 Let $W : \mathbb{N} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$ be given and suppose that there exist $\rho \in [0, 1)$ and $a_1, a_2 > 0$ such that the following conditions hold for the discrete-time system (8) for all $i \in \mathbb{N}$ and all $e \in \mathbb{R}^{n_e}$:

$$a_1 |e| \leq W(i, e) \leq a_2 |e| \quad (9)$$

$$W(i+1, h(i, e)) \leq \rho W(i, e) . \quad (10)$$

Then, we say that the protocol (6) is uniformly globally exponentially stable (UGES) with Lyapunov function W . ■

The RR and TOD protocols are UGES with appropriate Lyapunov functions as formally stated below.

Proposition 1 The RR protocol is UGES with the Lyapunov function $W(i, e) := \sqrt{\sum_{k=i}^\infty |\phi(k, i, e)|^2}$. In particular, we can take $a_1 = 1$, $a_2 = \sqrt{\ell}$ and $\rho = \sqrt{\frac{\ell-1}{\ell}}$. ■

Proposition 2 The TOD protocol is UGES with the Lyapunov function $W(i, e) := |e|$. In particular, we can take $a_1 = a_2 = 1$ and $\rho = \sqrt{\frac{\ell-1}{\ell}}$. ■

5 Main results

In this section we first show (Proposition 3) that protocols that are UGES with Lyapunov function W induce $\mathcal{L}_p$ stability for the error dynamics subsystem (5) under relatively mild conditions on W and error dynamics (5). Then, we present the main result of this paper (Theorem 2) which shows that under mild conditions UGES protocols induce $\mathcal{L}_p$ stability of NCS for

sufficiently small values of MATI. The proofs follow directly from the Small Gain Theorem (see [12]). Finally, in Corollary 1 we present a result that can be used to conclude UGES of the system with $w(\cdot) \equiv 0$ if the system is $\mathcal{L}_p$ stable.

Proposition 3 Consider NCS (4)-(6) and suppose:

1. The protocol (6) is UGES with Lyapunov function W that is locally Lipschitz in e , uniformly in i .
2. There exists $L \geq 0$ such that for every $i \in \mathbb{N}$, all t, x, w and almost all e we have that the following holds:

$$\left\langle \frac{\partial W(i, e)}{\partial e}, g(t, x, e, w) \right\rangle \leq LW(i, e) + |\tilde{y}|, \quad (11)$$

where $\tilde{y} : \mathbb{R}^{n_e} \times \mathbb{R}^{n_w} \rightarrow \mathbb{R}$ is continuous in (x, w) .

3. $\tau \in (\epsilon, \tau^*)$, where $\tau^* = \frac{1}{L} \ln\left(\frac{1}{\rho}\right)$, $\rho > 0$ comes from item 1 and $\epsilon \in (0, \tau^*)$ is arbitrary. In particular, if $L = 0$ we have $\tau^* = \lim_{L \rightarrow 0} \frac{1}{L} \ln\left(\frac{1}{\rho}\right) = \infty$.

Then, for each $p \in [1, \infty]$ the system (5) is $\mathcal{L}_p$ stable from $\tilde{y}$ to W with the gain $\gamma = \frac{\exp(L\tau)-1}{L(1-\rho \exp(L\tau))}$. In particular, if $L = 0$, then $\gamma = \lim_{L \rightarrow 0} \frac{\exp(L\tau)-1}{L(1-\rho \exp(L\tau))} = \frac{\tau}{1-\rho}$.

Item 2 of Proposition 3 holds for linear systems with RR and TOD protocols if Lyapunov functions are taken respectively from Propositions 1 and 2. Indeed, consider the linear NCS system:

$$\dot{x} = A_{11}x + A_{12}e; \quad \dot{e} = A_{21}x + A_{22}e \quad (12)$$

$$e(t_{s_i}^+) = h(i, e(t_{s_i})) \quad (13)$$

Then, we can prove (see [12] for details):

RR Protocol: Let (13) be the RR protocol and let the Lyapunov function $W(i, e)$ come from Proposition 1. Denote in this case $\tilde{y} := \sqrt{\ell} A_{21}x$. Then we can show that item 2 of Proposition 3 holds with $L = \sqrt{\ell} |A_{22}|$. Moreover, if A_{22} is block diagonal then we can obtain a tighter bound with $L = |A_{22}|$ (this bound is needed in the case study).

TOD Protocol: Consider the system (12) where (13) is the TOD protocol. Let $W(i, e) = |e|$ and $\tilde{y} = A_{21}x$. Then, we can show that item 2 of Proposition 3 holds with $L = \max \left\{ \frac{1}{2} \lambda_{\max}(A_{22}^T + A_{22}), 0 \right\}$.

The main result of this paper is presented next.

Theorem 2 Consider NCS (4)-(6). Suppose that the following conditions hold:

1. Inequalities (9), (10) and (11) with $\tilde{y} := H(x) + w$ hold.
2. System (4) is $\mathcal{L}_p$ stable from (W, w) to $H(x)$ with gain γ for some $p \in [1, \infty]$.
3. MATI satisfies $\tau \in (\epsilon, \tau^*)$ where $\tau^* := \frac{1}{L} \ln\left(\frac{L+\gamma}{\rho L+\gamma}\right)$, $\epsilon \in (0, \tau^*)$ is arbitrary, $L \geq 0$ comes from (11) and $\rho > 0$ comes from (10). In particular, if $L = 0$ then

$$\tau^* = \lim_{L \rightarrow 0} \frac{1}{L} \ln\left(\frac{L+\gamma}{\rho L+\gamma}\right) = \frac{1-\rho}{\gamma}.$$

Then, the NCS is $\mathcal{L}_p$ stable from w to $(H(x), W(i, e))$. ■

A direct corollary of Theorems 2 and 1 and the Small Gain Theorem is:

Corollary 1 Suppose that for some $p \in [1, \infty]$:

1. All conditions of Theorem 2 are satisfied and consequently the NCS is $\mathcal{L}_p$ stable from w to $(H(x), W(i, e))$;
2. The NCS is $\mathcal{L}_p$ to $\mathcal{L}_p$ detectable from $(H(x), W(i, e))$ to (x, e) ;
3. The origin of the NCS is UGFTIS with linear gain. Then, the NCS with $w \equiv 0$ is UGES.

Remark 1 We emphasize that several other generalizations and different versions of Theorem 2 can be found in [12]. Indeed, a generalization of Theorem 2 can be found in [12] that applies to the important case of data packet dropouts (see [18]). Moreover, a version of Theorem 2 for the property of input-to-state stability (ISS) with linear gains is given in [12]. Further generalizations of this result for ISS with nonlinear gains can be found in [11]. It is also worth remarking that our proof technique appears to be novel and it is based on different types of small gain theorem, such as a version [9] for systems with jumps.

6 Comparison with existing results

We now show that our results (when specialized to analysis of exponential stability for systems without disturbances) yield much less conservative bounds than those obtained in [14] for nonlinear NCS and in [15, 18] for linear NCS. The following class of systems was considered in [14]

$$\dot{x} = f(t, x, e); \quad \dot{e} = g(t, x, e), \quad (14)$$

where also the shorthand notation is used $\dot{z} = h(t, z)$, with $z = (x, e)$. Lipschitz constants for f , g and h are denoted respectively as k_f , k_g and k_h . The class of linear systems (12) was considered in [15, 18], where also the shorthand notation is used $\dot{z} = Az$, with $z = (x^T e^T)^T$. It is supposed in [14] that there exists a continuously differentiable Lyapunov function V such that the system (14) satisfies for all $x \in \mathbb{R}^{n_x}$ the following: $c_1|x|^2 \leq V(t, x) \leq c_2|x|^2$; $\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) \leq -c_3|x|^2$ and $\left| \frac{\partial V}{\partial x} \right| \leq c_4|x|$, where c_1, c_2, c_3, c_4 are positive constants. A similar condition was used in [15, 18] for the linear system (12). Indeed, it was assumed that for some positive definite and symmetric matrix Q there exists a positive definite and symmetric matrix P that solves the Lyapunov matrix equation³ $A_{11}^T P + P A_{11} = -Q$ and we can let $V(x) = x^T P x$ and $c_1 = \lambda_{\min}(P)$; $c_2 = \lambda_{\max}(P)$; $c_3 = \lambda_{\min}(Q)$; $c_4 = 2\lambda_{\max}(P)$. Moreover, note that, for linear systems, we can (conservatively) let $k_h = k_f = k_g = |A|$. A bound on MATI that

³Results in [15] are only presented for the special case $Q = I$. The result with general Q is presented in [18].

guarantees the stability of the linear system (12) with the RR and TOD protocols was obtained in [15, 18]. We denote bounds computed in [15, 18] respectively as τ_{old}^{RR} and τ_{old}^{TOD} for the RR and TOD protocols. On the other hand, our bounds are denoted respectively as τ_{new}^{RR} and τ_{new}^{TOD} . Results in [15, 18, 14] do not distinguish between different protocols⁴ and, in particular, it is obtained that $\tau_{old}^{RR} = \tau_{old}^{TOD}$. The bounds in [14, 15, 18] can be written in the following way:

$$\tau_{old}^{RR} = \tau_{old}^{TOD} = \frac{c_3}{M\ell(\ell+1)k_h k_f c_4} . \quad (15)$$

where the value of the constant M is different for the linear and nonlinear systems. Using results in [14] we have for nonlinear systems that $M = M_{NL} := 16 \left(\frac{c_2}{c_1} \right)^{3/2} \left(\sqrt{\frac{c_2}{c_1}} + 1 \right)$. On the other hand, in [15, 18] the following is obtained for linear systems $M = M_L := 8 \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + 1 \right)$. A summary of comparisons of bounds on MATI obtained using our results and results in [15, 18, 14] are presented below. All calculations are omitted for space reasons and can be found in [12]. Comparison of our results with results for linear systems in [15, 18] yields for RR Protocol:

$$\frac{\tau_{new}^{RR}}{\tau_{old}^{RR}} \geq 2 \frac{\ell+1}{\sqrt{\ell}} \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + 1 \right)$$

and for TOD protocol:

$$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \geq 2(\ell+1) \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \left(\sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} + 1 \right)$$

Similarly, comparison of our results with results for nonlinear systems in [14] yields for RR Protocol:

$$\frac{\tau_{new}^{RR}}{\tau_{old}^{RR}} \geq 8 \frac{\ell+1}{\sqrt{\ell}} \left(\frac{c_2}{c_1} \right)^{\frac{3}{2}} \left(\sqrt{\frac{c_2}{c_1}} + 1 \right)$$

and for TOD Protocol

$$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \geq 8(\ell+1) \left(\frac{c_2}{c_1} \right)^{\frac{3}{2}} \left(\sqrt{\frac{c_2}{c_1}} + 1 \right) .$$

Note that $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \geq 1$ and in the extreme case we have $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} = 1$. Assume that this “best” case holds. We obtain that $\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \geq 4(\ell+1)$ and, hence, for the simplest nontrivial case $\ell = 2$ we have that our bound is at least 12 times less conservative than the bound in [15]. Moreover, the larger the number of nodes ℓ , the less conservative our bound when compared to the one in [15]. In particular, as $\ell \rightarrow \infty$ we have $\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \rightarrow \infty$. For

⁴Note that we do not use different notation for MATI bounds for linear and nonlinear systems, although they are different in general. This is because it will be always clear from the context which bound we are talking about.

instance, if $\ell = 10$ we have that our bound is at least 44 times larger than the bound in [15]. Suppose now that ℓ is fixed. In this case we have that the larger the ratio $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}$, the less conservative our bound is compared to the bound in [15]. In particular, as $\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \rightarrow \infty$ we have that $\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \rightarrow \infty$ holds. Similar observations hold for nonlinear systems and/or the RR protocol. Moreover, these general formulas are very conservative and in particular examples we may obtain even better results. Indeed, in the example in the next section we obtain that $\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} = 1000!$

7 Case study: a batch reactor

To further illustrate that our results improve considerably results of [15] even in the case when there are no disturbances, we consider the unstable batch reactor example considered in [15]. The linearized model of an unstable batch reactor is a two-input-two-output NCS of the form (12) (for space reasons the model is omitted and can be found in [15]).

TOD protocol: We first compute⁵ the $\mathcal{L}_2$ gain for the x subsystem from the input e to the “output” $\tilde{y} := A_{21}x$, which is $\gamma \approx 15.9222$. Note that since $W(e) = |e|$ the gain from W to $\tilde{y}$ is bounded by γ . Using Proposition 2 we see that the TOD protocol is UGES with Lyapunov function $W(e) = |e|$, where for two nodes ($\ell = 2$ since we have two outputs transmitted via the network) we have that $\rho = \sqrt{\frac{1}{2}} \approx 0.7071$. Furthermore, using result from Section 5 we have that the error subsystem satisfies the bound (11) with $L = \lambda_{\max}(0.5(A_{22}^T + A_{22})) = 15.73$. Finally, using our Theorem 2 and Corollary 1 we obtain that the NCS is UGES for all MATI $\tau \in (0, \tau_{new}^{TOD})$ where

$$\tau_{new}^{TOD} = \frac{1}{L} \ln \left(\frac{L + \gamma}{\rho L + \gamma} \right) \approx 0.01 .$$

The theoretical bound on MATI obtained using results in [15] for which NCS remains stable is $\tau \in (0, \tau_{old}^{TOD})$, where $\tau_{old}^{TOD} \approx 10^{-5} \text{ sec}$ (see pg. 443). The simulations with randomly generated transmission times t_{s_j} reported in [15] showed that the system remains stable for the values of MATI $\tau \in (0, \tau_{rand}^{TOD})$, where $\tau_{rand}^{TOD} \approx 0.06 \text{ sec}$. Note that

$$\frac{\tau_{new}^{TOD}}{\tau_{old}^{TOD}} \approx 1000 \text{ and } \frac{\tau_{rand}^{TOD}}{\tau_{new}^{TOD}} \approx 6 .$$

Hence, our result yields a theoretical bound on MATI that is about 1000 times less conservative than the bound obtained in [15]! This is only about 6 times more conservative than the real bound on MATI observed in simulations. We have also carried out simulations for the case when transmission times t_{s_j} are equidistant and we obtained a slightly larger value for the MATI

⁵Computation of $\mathcal{L}_p$ gains was done using the μ -synthesis toolbox in MATLAB.

for which stability is preserved $\tau_{const}^{TOD} \approx 0.089 \text{ sec}$ and, hence, our result is slightly more conservative for this situation.

RR protocol: Suppose now that RR protocol is used for the same system. In this case, it turns out that the Lyapunov function from Proposition 1 takes the form $W(i, e) = \sqrt{\sum_{k=i}^{\infty} \phi^2(k, i, e)} = \sqrt{a_1^2(i)e_1^2 + a_2^2(i)e_2^2}$, where $a_j^2(i) = 1$ or 2 for $j = 1, 2$ and all $i \in \mathbb{N}$. This implies that $\max\{a_1(i), a_2(i)\} \leq \sqrt{2}$. Moreover, the error system has the diagonal form:

$$\dot{e}_1 = 15.73e_1 + f_1; \dot{e}_2 = 11.358e_2 + f_2,$$

where $(f_1 \ f_2)^T := A_{21}x$. In this case, we let $\tilde{y} := \sqrt{2}A_{21}x$ (note the difference with the $\tilde{y}$ defined in the case of TOD protocol). As a result, we can show that for almost all e we have that (11) holds, where $L \approx 15.73$ is the same as in the previous example. The new definition of $\tilde{y}$ gives the gain from W to the newly defined $\tilde{y}$ equal to $\gamma_1 = \gamma\sqrt{2}$, where $\gamma \approx 15.2222$ was computed in the previous example as the gain from W to $A_{21}x$. Finally, using our Theorem 2 and Corollary 1 we obtain that the NCS is UGES for all MATI $\tau \in (0, \tau_{new}^{RR})$ where

$$\tau_{new}^{RR} = \frac{1}{L} \ln \left(\frac{L + \gamma_1}{\rho L + \gamma_1} \right) \approx 0.0082 \text{ sec}.$$

Assume now that the transmission times are equidistant. It was computed⁶ for this situation that $\tau_{const}^{RR} = 0.0657 \text{ sec}$. Hence, we have that

$$\frac{\tau_{const}^{RR}}{\tau_{new}^{RR}} \approx 8,$$

which means that our theoretical bound is only about 8 times more conservative than the real bound obtained from analytic calculations.

8 Conclusions

We presented results on $\mathcal{L}_p$ stability for Networked Control Systems that apply to a general class of scheduling protocols. Our results generalize those in [14, 15] since they apply to general nonlinear systems with disturbances, as well as a more general class of protocols than those considered in the references. In particular, we show that the static and TOD protocols considered in [14, 15] belong to the class of protocols that we considered. Moreover, our result can serve as a framework for generating new protocols that have good properties. We illustrated via an example considered in [15] that our results produce several orders of magnitude less conservative results than those obtained in [15]. Finally, we remark that several generalizations and different versions of our results can be found in [12, 11].

⁶In this case, the NCS dynamics induces a discrete-time system at transmission times whose stability can be checked explicitly using the standard discrete-time calculations.

References

- [1] O.Beldiman, G.C.Walsh and L.Bushnell, "Predictors for networked control systems", *Proc. Americ. Contr. Conf.*, Chicago, 2000, pp. 2347-2351.
- [2] R.W.Brockett and D.Liberzon, *Quantized stabilization of linear systems*, IEEE Trans. Automat. Contr., 45 (2000), pp. 1279-1289.
- [3] T. Chen and B.Francis, "Input-output stability of sampled-data systems", *IEEE Trans. Automat. Contr.*, vol. 36, No. 1 (1991), pp. 50-58.
- [4] G.E. Dullerud and S. Lall, "A synchronous hybrid systems with jumps analysis: analysis and synthesis problems", *Syst. Contr. Lett.*, vol. 37 (1999), pp. 61-89.
- [5] G.E. Dullerud and S. Lall, "A new approach for analysis and synthesis of time varying systems", *IEEE Trans. Automat. Contr.*, vol. 44 (1999), pp. 1486-1497.
- [6] N.Eilia and S.K.Mitter, *Stability of linear systems with limited information*, IEEE Trans. Automat. Contr., 46 (2001), pp. 1384-1400.
- [7] A.Hassibi, S.P.Boyd and J.P.How, "Control of asynchronous dynamical systems with rate constraints on events", *Proc. IEEE Conf. Decis. Contr.*, Phoenix, AZ, USA, (1999), pp. 1345-1351.
- [8] H.Ishii and B.A.Francis, *Stabilizing a linear system by switching control with dwell time*, IEEE Trans. Automat. Contr., 47 (2002), pp. 1962-1973.
- [9] Z. P. Jiang, A. R. Teel and L. Praly, "Small gain theorem for ISS systems and applications", *Math. Contr. Sign. Syst.*, 7 (1994), pp. 95-120.
- [10] A.R.Teel, E.Panteley and A.Loria, "Integral characterizations of uniform asymptotic and exponential stability with applications", *Math. Contr. Sig. Sys.*, vol. 15 (2002), pp. 177-201.
- [11] A.R. Teel and D. Nešić, "Input to state stability of Networked Control Systems", in preparation, 2003.
- [12] D. Nešić and A.R. Teel, "Input-output stability properties of Networked Control Systems", submitted IEEE Trans. Automat. Contr., 2003.
- [13] G. C. Walsh and H. Ye, "Scheduling of Networked Control Systems", *IEEE Contr. Magazine*, vol. 21, No. 1 (2001), pp. 57-65.
- [14] G. C. Walsh, O. Beldiman and L. G. Bushnell, "Asymptotic behaviour of nonlinear networked control systems", *IEEE Trans. Automat. Contr.*, vol. 46, No. 7 (2001), 1093-1097.
- [15] G. C. Walsh, H. Ye and L. G. Bushnell, "Stability analysis of networked control systems", *IEEE Trans. Contr. Syst. Techn.*, vol. 10, No. 3 (2002), pp. 438-446.
- [16] H. Ye, G. C. Walsh and L. G. Bushnell, "Real-time mixed-traffic wireless networks", *IEEE Trans. Indust. Electr.*, vol. 48, No. 5 (2001), pp. 883-890.
- [17] L. Zaccarian, A.R. Teel and D. Nešić, "On finite gain L_p stability of sampled-data nonlinear systems", *Syst. Contr. Lett.*, vol. 49, No. 3 (2003), pp. 201-212.
- [18] W. Zhang, M.S. Branicky and S.M. Phillips, "Stability of networked control systems", *IEEE Contr. Syst. Magazine*, Vol. 21, No. 1 (2001), pp. 84-99.