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Title:
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Date:
2000-12-01

Citation:
Nešić, D. & Teel, A. R. (2000). Set stabilization of sampled-data nonlinear differential inclusions via their approximate discrete-time models. Proceedings of the IEEE Conference on Decision and Control, 3, pp.2112-2117. IEEE. https://doi.org/10.1109/cdc.2000.914106.

Set stabilization of sampled-data nonlinear differential inclusions via their approximate discrete-time models

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Abstract

We present conditions under which a family of controllers that semiglobally-practically asymptotically (SPA) stabilizes a set for a family of discrete-time approximations of a nonlinear differential inclusion also SPA stabilizes that set for the inclusion's family of exact discrete-time models, for sufficiently small sampling periods. The result has the following important features: it does not require any regularity assumptions on the designed controllers; it is applicable to dynamic control laws; and it is stated for stability with respect to sets that are not necessarily compact.

1 Introduction

A main obstacle in feedback synthesis for sampled-data nonlinear systems is the absence of a good model for controller design. The approach that we take in this paper is to use an approximate discrete-time model of the plant for the controller design and then implement it using fast sampling (see, for instance, [1, 2, 3, 6, 8]). The results in [8] are particularly important for this approach since they give a set of rather general conditions on the continuous-time plant model, the approximate discrete-time model and the designed controller that guarantee that a controller family (parameterized by sampling period) that semiglobally practically asymptotically (SPA) stabilizes (see Definition 1) the origin of a family of approximate discrete-time models also SPA stabilizes the origin of the family of exact discrete-time models, for sufficiently small sampling periods. In [8] we used tools from numerical analysis literature [9] to obtain the main results. The results in [8] are applicable to: plants whose model is described by a continuous nonlinear differential equation with unique solutions; static state feedbacks; investigation of stability of an equilibrium (the origin).

In this paper, we extend the results of [8] in three directions: 1. we consider plants that can be modeled by a differential inclusion; 2. the results are applicable to dynamic control laws; 3. the considered stability is with respect closed sets that are not necessarily compact.

More precisely, we present a framework for understanding the design of sampled-data control laws for differential inclusions

$$\dot{x} \in F(x,u) \ , \tag{1}$$

based on *approximate* discrete-time models for (1), that SPA stabilize a given closed set $\mathcal{A}$ for sufficiently fast sampling. We make the following assumption for the set-valued map $F(\cdot,u)$:

Assumption 1 *For each $u \in \mathbb{R}^m$, the set-valued map $F(\cdot,u)$ satisfies the following basic conditions: 1) it is upper semi-continuous, i.e., for each $x \in \mathbb{R}^n$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $|\xi - x| \leq \delta$ we have $F(\xi,u) \subseteq F(x,u) + \varepsilon\overline{\mathcal{B}}_n$, where $\overline{\mathcal{B}}_n$ denotes the closed unit ball in $\mathbb{R}^n$, 2) for each $x \in \mathbb{R}^n$ the set $F(x,u)$ is nonempty, compact and convex.*

This assumption guarantees that for each fixed u there exists at least one solution to (1). We will use $\mathcal{S}(x,u)$ to denote the set of solutions to (1) starting at x with u held constant.

With Assumption 1 and assuming that u is held constant between sampling periods, we can define the exact discrete-time model for (1), parameterized by sampling period $T > 0$, as

$$x_{k+1} \in F^e_T(x_k,u_k) \tag{2}$$

where, for each $(x,u) \in \mathbb{R}^n \times \mathbb{R}^m$ and $T > 0$, $F^e_T(x,u) := \{\xi \in \mathbb{R}^n : \xi = \phi(T,x,u), \phi \in \mathcal{S}(x,u)\}$.

[1]Research supported in part by the AFOSR under grant F49620-00-1-0106.

By convention if, for a given triple (T, x, u), there is a solution $\phi \in \mathcal{S}(x, u)$ right maximally defined on $[0, T')$ with $T' \leq T$ then we take $F_T^e(x, u) = \mathbb{R}^n$. In general, it is an impossible task to compute $F_T^e(x, u)$. Instead, control design may be carried out for a family of approximate discrete-time models

$$x_{k+1} \in F_T^a(x_k, u_k) \ . \tag{3}$$

We will suppose that we can find a family of controllers for (3), given by

$$\begin{array}{rcl} z_{k+1} & \in & G_T(z_k, x_k) \\ u_k & \in & H_T(z_k, x_k) \ , \end{array} \tag{4}$$

and a closed set $\mathcal{C}$ such that the set $\mathcal{A} \times \mathcal{C}$ for the system (3)-(4) is semiglobally practically asymptotically (SPA) stable in the parameter T (see Definition 1). The purpose of this paper is to present natural conditions on the relationship between F_T^e and F_T^a and the uniformity of the stability required for the set $\mathcal{A} \times \mathcal{C}$ for the system (3)-(4) to guarantee that the controller (4) renders the set $\mathcal{A} \times \mathcal{C}$ for the system (2),(4) SPA stable for sufficiently fast sampling.

2 Preliminaries

A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ if $\beta(\cdot, \tau)$ is of class-$\mathcal{K}$ for each $\tau \geq 0$ and $\beta(s, \cdot)$ is decreasing to zero for each $s > 0$. Given an arbitrary set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, the following notation is used $|x|_{\mathcal{A}} := \inf_{s \in \mathcal{A}} |x - s|$, where $|x|$ denotes the Euclidean norm of the vector x. We will use the following definition in what follows: for a closed set $\mathcal{A}$ and nonnegative real numbers $0 \leq \delta \leq \Delta$ we define $\mathcal{H}_{\mathcal{A}}(\delta, \Delta) := \{x \in \mathbb{R}^n : \delta \leq |x|_{\mathcal{A}} \leq \Delta\}$. The main stability property we will work with is defined as follows:

Definition 1 *Let $\widetilde{\mathcal{A}}$ be a closed subset of $\mathbb{R}^p$ and let $\mathcal{F}_{T,\epsilon}(\cdot)$ be a family of set-valued maps from $\mathbb{R}^p$ to subsets of $\mathbb{R}^p$. For the parameterized family of systems*

$$\xi_{k+1} \in \mathcal{F}_{T,\epsilon}(\xi_k) \tag{5}$$

the set $\widetilde{\mathcal{A}}$ is said to be semiglobally practically asymptotically (SPA) stable in T uniformly in small ϵ if there exists $\beta \in \mathcal{KL}$, and, for each $\Delta > 0$ and $\delta > 0$, there exists $T^* > 0$ and $\epsilon^* > 0$ such that, for all $T \in (0, T^*)$, $\epsilon \in [0, \epsilon^*]$ and all $\xi \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0, \Delta)$, all solutions $\psi_{T,\epsilon}(\cdot, \xi)$ of (5) satisfy

$$|\psi_{T,\epsilon}(k, \xi)|_{\widetilde{\mathcal{A}}} \leq \beta(|\xi|_{\widetilde{\mathcal{A}}}, kT) + \delta \ . \tag{6}$$

If (5) does not depend on a parameter ϵ, and the above property holds, we simply say that the set $\widetilde{\mathcal{A}}$ is SPA stable in T.

The property we will assume about the family of approximate discrete-time models compared to the family of exact discrete-time models is the following:

Definition 2 *The family F_T^a is said to be* one-step upper semi-consistent *with* F_T^e uniformly with respect to $\mathcal{A}$ *if, for each pair of positive real numbers (Δ, M) there exist $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that, for all $(x, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta) \times M\overline{\mathcal{B}}_m$ and all $T \in (0, T^*)$, we have*

$$F_T^e(x, u) \subseteq F_T^a(x, u) + T\rho(T)\overline{\mathcal{B}}_n \tag{7}$$

A sufficient condition for one-step upper semi-consistency is given in the next lemma, whose proof is omitted for space reasons.

Lemma 1 *If*
1. for each $\widetilde{\Delta} \geq 0$ there exists $M > 0$ such that $\sup_{\left\{(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\widetilde{\Delta}) \times \widetilde{\Delta}\overline{\mathcal{B}}_m \ , \ w \in F(x,u)\right\}} |w| \leq M$, 2. there exists a set-valued map $\widetilde{F}(\cdot, \cdot)$ such that
(a) for each $u \in \mathbb{R}^m$, the set-valued map $\widetilde{F}(\cdot, u)$ satisfies the basic conditions of Assumption 1,
(b) F_T^a is one-step upper semi-consistent with $\widetilde{F}_T^{Euler}(x, u) := x + T\widetilde{F}(x, u)$ uniformly with respect to $\mathcal{A}$,
(c) for each $\widetilde{\Delta} \geq 0$ there exists $\widetilde{\rho} \in \mathcal{K}_\infty$ such that $(x, \xi, u) \in \mathcal{H}_{\mathcal{A}}(0, \widetilde{\Delta}) \times \mathcal{H}_{\mathcal{A}}(0, \widetilde{\Delta}) \times \widetilde{\Delta}\overline{\mathcal{B}}_m$ implies $F(\xi, u) \subseteq \widetilde{F}(x, u) + \rho(|\xi - x|)\overline{\mathcal{B}}_n$, then F_T^a is one-step upper semi-consistent with F_T^e uniformly with respect to $\mathcal{A}$.

Proof of Lemma 1: Let (Δ, M) be given. Define $\Delta_1 := \max\{\Delta, M\}$. Let item 1 of the lemma generate $T_1^* > 0$ and $\rho_1 \in \mathcal{K}_\infty$. Define $\widetilde{\Delta} = \Delta_1 + 1$ and let items 2 and 3 of the lemma generate $M_1 > 0$ and $\widetilde{\rho} \in \mathcal{K}_\infty$. Define $T^* := \min\left\{T_1^*, M_1^{-1}\right\}$ and $\rho(s) = \rho_1(s) + \widetilde{\rho}(M_1 s)$. It follows from item 2 of the lemma that, for all $(x, u) \in \mathcal{H}_{\mathcal{A}}(0, \Delta) \times \Delta\overline{\mathcal{B}}_m$, $\{\phi \in \mathcal{S}(x, u) \ , \ t \in [0, T^*]\}$ implies $\left\{\phi(t, x, u) \in \mathcal{H}_{\mathcal{A}}(0, \widetilde{\Delta}) \ , \ |\phi(t, x, u) - x| \leq M_1 t\right\}$. For each $v \in \mathbb{R}^n$ and $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, define $g_{(x,u)}(v)$ to be the unique (since $\widetilde{F}(x, u)$ is closed and convex) closest point in $\widetilde{F}(x,u)$ to v. Since $\widetilde{F}(x,u)$ is closed and convex, the function $g_{(x,u)}(\cdot)$ is continuous. Let $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be a measurable function such that $w(t) \in F(\phi(t,x,u),u) \quad \forall t \geq 0$ and, for almost all $t \geq 0$, $w(t) = \overbrace{\phi(t,x,u)}^{\cdot} \in F(\phi(t,x,u),u)$. Then the function $g_{(x,u)}(w(\cdot))$ enjoys the following properties:

1. $g_{(x,u)}(w(\cdot))$ is measurable (since $w(\cdot)$ is measurable and $g_{(x,u)}(\cdot)$ is continuous),
2. for all $t \in [0,T^*]$, $|w(t) - g_{(x,u)}(w(t))| \leq \widetilde{\rho}(|\phi(t,x,u)-x|) \leq \widetilde{\rho}(M_1 t)$,
3. from convexity of $\widetilde{F}(x,u)$ and the fact that $g_{(x,u)}(w(t)) \in \widetilde{F}(x,u)$ for all t, $\int_0^T g_{(x,u)}(w(t))dt \in T\widetilde{F}(x,u)$.

It follows that, for all $T \in (0,T^*)$

$$\begin{aligned}
\phi(T,x,u) &= x + \int_0^T w(t)dt \\
&= x + \int_0^T g_{(x,u)}(w(t))dt \\
&\quad + \textstyle\int_0^T \left[w(t) - g_{(x,u)}(w(t))\right] dt \qquad (8)\\
&\in x + T\widetilde{F}(x,u) + T\widetilde{\rho}(M_1 T)\overline{\mathcal{B}}_n \\
&\subseteq F_T^a(x,u) + T\left(\rho_1(T) + \widetilde{\rho}(M_1 T)\right)\overline{\mathcal{B}}_n \\
&= F_T^a(x,u) + T\rho(T)\overline{\mathcal{B}}_n \ .
\end{aligned}$$

It follows that, for all $(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta) \times M\overline{\mathcal{B}}_m$ and all $T \in (0,T^*)$,

$$F_T^e(x,u) \subseteq F_T^a(x,u) + T\rho(T)\overline{\mathcal{B}}_n \ , \qquad (9)$$

i.e., F_T^a is one-step upper semi-consistent with F_T^e uniformly with respect to $\mathcal{A}$. ■

Remark 1 A candidate choice for $\widetilde{F}$ is $\widetilde{F}(x,u) = F(x,u)$. In this case, item 2c becomes a *uniform* upper condition on $F(\cdot,u)$. If $F(\cdot,u)$ has this uniform continuity property and item 1 of the lemma holds then the Euler approximation $F_T^a(x,u) = x + TF(x,u)$ is one-step upper semi-consistent with F_T^e. Unfortunately, it is not sufficient to take $\widetilde{F}(x,u) = F(x,u)$ and assume that $F(x,u)$ is only upper semi-continuous. Indeed, consider $F(x,u)$ which has value 1 when $x < 0$, value 10 when $x > 0$ and the set value $[1,10]$ when $x = 0$, which satisfies Assumption 1 and let $F_T^a(x,u) = x + TF(x,u)$. Let $\mathcal{A} = \{1\}$, let $\Delta = 2$ and suppose there exist $\rho \in \mathcal{K}_\infty$ and $T^* > 0$ such that (7) holds for all $(x,u) \in \mathcal{H}_{\mathcal{A}}(0,\Delta) \times \Delta\overline{\mathcal{B}}_m$. Let $T > 0$ be such that $T < \min\left\{T^*, 1, \rho^{-1}\left(\frac{1}{2}\right)\right\}$. Let $x = -\frac{T}{2}$, so that $x \in \mathcal{H}_{\mathcal{A}}(0,\Delta)$, and $u = 0$. Then we have $F_T^a(x,u) + \frac{1}{2}T\overline{\mathcal{B}}_1 = [0,T]$ and $F_T^e(x,u) = 5T$. It follows from the fact that $\rho(T) \leq \frac{1}{2}$ that $F_T^e(x,u) \not\subseteq F_T^a(x,u) + T\rho(T)\overline{\mathcal{B}}_1$. This contradicts (7).

In the absence of this uniform continuity condition, one option is to search for an inclusion $\widetilde{F}$ satisfying basic conditions:
1. $F(x,u) \subseteq \widetilde{F}(x,u)$,
2. $\widetilde{F}(\cdot,u)$ is uniformly continuous, or uniformly locally Lipschitz,
3. the Euler approximation of $\dot{x} \in \widetilde{F}(x,u)$ can be stabilized in the appropriate sense.

In this case, $x + T\widetilde{F}(x,u)$ will be one-step upper semi-consistent with F_T^e uniformly with respect to $\mathcal{A}$ and we will be in a position to apply the main theorem of the next section. ■

3 Stability results

In this section we state and prove two stability results. First, we state a stability theorem that follows directly from the definition of one-step upper semi-consistency. Then, we state and prove the main result of the paper which assumes the existence of a Lyapunov function for the closed loop discrete-time approximate model to prove SPA stability of the closed loop discrete-time exact model.

Theorem 1 *Suppose:*
1. the family F_T^a is one-step upper semi-consistent with F_T^e with respect to $\mathcal{A}$,
2. For each $\Delta > 0$ there exist M and $T^* > 0$ such that

$$\sup_{\left\{\begin{array}{c}(x,z) \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0,\Delta) \\ w \in H_T(z,x)\ ,\ T \in (0,T^*)\end{array}\right\}} |w| \leq M \ , \qquad (10)$$

3. the parameterized family of systems:

$$\begin{aligned}
x_{k+1} &\in F_T^a(x_k,u_k) + T\epsilon\mathcal{B}_n \\
z_{k+1} &\in G_T(z_k,x_k) \\
u_k &\in H_T(z_k,x_k) \ , \qquad (11)
\end{aligned}$$

is SPA stable in T with respect to $\mathcal{A}\times\mathcal{C}$, uniformly in ϵ.
Under these conditions the system (2), (4) is SPA stable with respect to $\mathcal{A}\times\mathcal{C}$.

Proof of Theorem 1: Let $\psi_{T,\epsilon}^a(\cdot,\xi)$ represent a solution of (11) starting at ξ and let $\psi_T^e(\cdot,\xi)$ represent a solution of (2), (4) starting at ξ.

Let $\widetilde{\mathcal{A}} := \mathcal{A} \times \mathcal{C}$. Let (Δ, δ) be given. Since the family (11) is SPA stable in T with respect to $\mathcal{A} \times \mathcal{C}$, uniformly in ϵ, there exists $\beta \in \mathcal{KL}$ and, for the given (Δ, δ), there exist $T_1^* > 0$ and $\epsilon^* > 0$ such that, for all $\xi = (x, z) \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \Delta)$, $T \in (0, T_1^*)$ and $\epsilon \in [0, \epsilon^*]$, the solutions of (11) satisfy (6). Let $\widetilde{\Delta} := \beta(\Delta, 0) + \delta$ and let item 2 with $\widetilde{\Delta}$ generate M and $T_2^* > 0$. Then let item 1 with $(\widetilde{\Delta}, M)$ generate $\rho \in \mathcal{K}_\infty$ from one-step upper semi-consistency (see Definition 2). Choose $T_3^* := \rho^{-1}(\epsilon^*)$ and $T^* := \min\{T_1^*, T_2^*, T_3^*\}$. Note that $\xi \in \mathcal{H}_{\widetilde{\mathcal{A}}}(0, \widetilde{\Delta})$ implies $x \in \mathcal{H}_{\mathcal{A}}(0, \widetilde{\Delta})$. It follows that for all $T \in (0, T^*)$ and $\xi \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \widetilde{\Delta})$ that $F_T^e(x, H_T(z, x)) \subseteq F_T^a(x, H_T(z, x)) + T\epsilon^* \mathcal{B}_n$. It follows that, for all $T \in (0, T^*)$ and all $\xi \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \Delta)$, we have $|\psi_T^e(k, \xi)|_{\widetilde{\mathcal{A}}} \leq \widetilde{\Delta}, \forall k$. This, in turn, implies that the solutions of (2), (4) satisfy (6) for all $(x, z) \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \Delta)$ and $T \in (0, T^*)$. Hence, the system (2), (4) is SPA stable. ■

The main assumption of Theorem 1 is that SPA stability for the family closed loop discrete-time approximate models is robust to the types of uncertainties that describe the assumed mismatch between the exact and approximate model. In the next theorem we give Lyapunov type conditions that guarantee this type of robustness. To shorten notation, we introduce the following notation $\tilde{x} = (x^T \; z^T)^T$ and

$$\mathcal{F}_T^\ell(\tilde{x}) := \begin{pmatrix} F_T^\ell(x, H_T(z, x)) \\ G_T(z, x) \end{pmatrix} ,$$

where ℓ can be either "a" (for approximate model) or "e" (for exact model).

Theorem 2 *Suppose that the following conditions hold:*
1. the family F_T^a is one-step upper semi-consistent with F_T^e uniformly with respect to $\mathcal{A}$,
2. for each $\Delta > 0$ there exist $M > 0$ and $T^* > 0$ such that

$$\sup_{\left\{\begin{array}{c} \tilde{x} \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \Delta) \\ w \in H_T(z, x) \;,\; T \in (0, T^*) \end{array}\right\}} |w| \leq M \;, \tag{12}$$

3. there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 \in \mathcal{K}$ and, for each triple of strictly positive numbers $(\delta_1, \delta_2, \Delta)$ with $\delta_2 \leq \Delta$, there exist $T^* > 0$ and $L > 0$ and, for each $T \in (0, T^*)$, there exists a function $V_T : \mathbb{R}^{n+n_c} \to \mathbb{R}_{\geq 0}$ such that, for all $\tilde{x} \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \Delta)$, we have

$$\alpha_1(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) \leq V_T(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) \tag{13}$$

$$\sup_{w \in \mathcal{F}_T^a(\tilde{x})} V_T(w) - V_T(\tilde{x}) \leq -T\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) + T\delta_1 \;, \tag{14}$$

and, for all $(x_1, z), (x_2, z) \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(\delta_2, \Delta)$, we have

$$|V_T(x_1, z) - V_T(x_2, z)| \leq L\,|x_1 - x_2| \;. \tag{15}$$

Then the system (2), (4) is SPA stable.

Remark 2 *We emphasize that no regularity assumptions on H_T and G_T are needed in Theorem 2. In particular, H_T and G_T may be discontinuous in x and z. Moreover, we do not need a continuity assumption on z dependence of V_T. On the other hand, the condition that V_T is locally Lipschitz, uniformly in z, is crucial in the proof.*

Proof of Theorem 2: Given an arbitrary solution $\psi_T^e(k, \tilde{x})$ of (2),(4) at time k starting from an initial condition $\tilde{x}$, we use the notation $V_T(k) := V_T(\psi_T^e(k, \tilde{x}))$.

First, we claim that for each pair of strictly positive numbers (d, D) there exists $T^* > 0$ such that for all $T \in (0, T^*)$ we have that $|\tilde{x}|_{\mathcal{A}\times\mathcal{C}} \leq D$, $w \in \mathcal{F}_T^e(\tilde{x})$ and $\max\{V_T(w),\ V_T(\tilde{x})\} \geq d$ imply

$$V_T(w) - V_T(\tilde{x}) \leq -\frac{T}{4}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}). \tag{16}$$

We will prove this claim later and suppose for now that it is true. Let $\Delta > 0$ be arbitrary, let $D := \alpha_1^{-1} \circ \alpha_2(\Delta)$ and let d be an arbitrary number such that $d \in (0, \alpha_1(D))$.

Using the above claim with the pair (d, D) it follows that for all $T \in (0, T^*)$ and all $\tilde{x} \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0, \Delta)$ all solutions of (2),(4) satisfy:

$$V_T(k) \leq \max\{V_T(0), d\} \tag{17}$$

and

$$\begin{aligned} V_T(k+1) - V_T(k) \quad &\leq \quad -\frac{T}{4}\alpha_3 \circ \alpha_2^{-1}(V_T(k)) \\ &=: \quad -T\alpha(V_T(k)), \end{aligned} \tag{18}$$

whenever $\max\{V_T(k+1), V_T(k)\} \geq d$

Relation (17) follows by induction. Indeed, (17) clearly holds for $k = 0$. Suppose now that (17) holds for some solution $\psi_T^e(k, \tilde{x})$ at time $k \geq 0$.

Then (17), (13) and the definitions of d and D imply

$$|\psi_T^e(k,\tilde{x})| \le \max\left\{\alpha_1^{-1}(V_T(0)), \alpha_1^{-1}(d)\right\} \le D. \tag{19}$$

Hence, either $V_T(k+1) \ge d$ which, from (16) and (19), implies $V_T(k+1) \le V_T(k)$ or else $V_T(k+1) \le d$ so that, in either case, (17) holds for $V_T(k+1)$ as well. With (16), (19) and (13), the relation (18) is immediate.

Consider now an arbitrary solution $\psi_T^e(k,\tilde{x})$ with $T \in (0,T^*)$ and $\tilde{x} \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0,\Delta)$. Introduce $y_T(t) := V_T(k) + (\frac{t}{T} - k)(V_T(k+1) - V_T(k)), \forall t \in [kT,(k+1)T), k \ge 0$. Then $y_T(t)$ is a continuous, non-negative, piecewise linear function of "time" t and hence it is absolutely continuous with derivative given, for almost all t, by

$$\dot{y}_T(t) = \frac{V_T(k+1) - V_T(k)}{T} \ , \tag{20}$$

for $t \in [kT,(k+1)T)$. Using the definition of $y_T(t)$ and (18), we have that $\{y_T(t) \ge d \ , \ t \in [kT,(k+1)T)\}$ implies the following

$$\begin{aligned} \max\{V_T(k+1), V_T(k)\} \ge d \qquad &\Longrightarrow \\ V_T(k+1) - V_T(k) \le -T\alpha(V_T(k)) \quad &\Longrightarrow \\ y_T(t) \le V_T(k) \qquad & \end{aligned} \tag{21}$$

and so, combining with (20), we can write for almost all t that $y_T(t) \ge d$ impies:

$$\begin{aligned} \dot{y}_T(t) &\le -\alpha(V_T(k)), \quad k : t \in [kT,(k+1)T) \\ &\le -\alpha(y_T(t)) \ . \end{aligned} \tag{22}$$

It follows from the arguments in [5, Section VI] that there exists $\beta_1 \in \mathcal{KL}$ that is determined by $\alpha = \alpha_3 \circ \alpha_2^{-1}$, i.e., independent of T and d, such that

$$y_T(t) \le \max\{\beta_1(y_T(0),t), d\} \ , \ \forall t \ge 0 \ .$$

This implies, using $V_T(k) = y_T(kT)$ for all integers $k \ge 0$, that

$$V_T(k) \le \max\{\beta_1(V_T(0),kT), d\} \ , \ k \ge 0.$$

By using (13) we obtain that $\forall \tilde{x} \in \mathcal{H}_{\mathcal{A}\times\mathcal{C}}(0,\Delta), T \in (0,T^*)$, any solution of (2),(4) satisfies:

$$\begin{aligned} |\psi_T^e(k,\tilde{x})|_{\mathcal{A}\times\mathcal{C}} \le \ &\alpha_1^{-1}(\ \beta_1(\alpha_2(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}),kT)\ ) \\ &+\alpha_1^{-1}(d) \ , \ k \ge 0 \ , \end{aligned} \tag{23}$$

where it is obvious that $\beta(s,\tau) := \alpha_1^{-1}(\ \beta_1(\alpha_2(s),\tau)\ )$ is a class-$\mathcal{KL}$ function that is independent of T and d, and if d is chosen so that $d \le \alpha_1(\delta)$, the claim of the theorem is proved.

The only thing that is left to prove is claim (16). Given the pair of strictly positive real numbers (d, D), we make the definitions:

$$\begin{aligned} \delta &:= \frac{1}{2}\alpha_2^{-1}\left(\frac{d}{2}\right) \\ \Delta &:= \alpha_1^{-1}(\ 2\alpha_2(D)\ ) + \frac{1}{2}\alpha_2^{-1}\left(\frac{d}{2}\right) \ . \end{aligned} \tag{24}$$

Let δ_1 and δ_2 be defined as:

$$\begin{aligned} \delta_1 &:= \min\left\{\frac{1}{2}\alpha_1\left(\ \frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right)\ \right),\right. \\ &\qquad \left.\frac{1}{2}\alpha_3 \circ \alpha_2^{-1}\left(\frac{1}{2}\alpha_1(\delta)\right)\right\} \end{aligned} \tag{25}$$

$$\delta_2 := \alpha_2^{-1}\left(\frac{1}{2}\alpha_1(\delta)\right), \tag{26}$$

and let the triple $(\delta_1, \delta_2, \Delta)$ generate from condition 3 of Theorem 2 the numbers $T_1^* > 0$ and $L > 0$. Let Δ generate $M > 0$ and $T_2^* > 0$ from condition 2 of Theorem 2. Let the pair (Δ, M) generate $T_3^* > 0$ and ρ from condition 1 of Theorem 2. Let $T_4^* > 0$ be such that

$$L\rho(T_4^*) \le \frac{1}{4}\alpha_3 \circ \alpha_2^{-1}\left(\frac{1}{2}\alpha_1(\delta)\right) \ . \tag{27}$$

Let $T_5^* > 0$ be such that:

$$T_5^*\rho(T_5^*) \le \frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right) \ . \tag{28}$$

Take $T^* = \min\{T_1^*, T_2^*, T_3^*, T_4^*, T_5^*, 1\}$.

In all subsequent calculations we suppose that $|\tilde{x}|_{\mathcal{A}\times\mathcal{C}} \le D$. Let $w \in \mathcal{F}_T^e(\tilde{x})$ be arbitrary and let $v \in \mathcal{F}_T^a(\tilde{x})$ be such that $|w - v| \le T\rho(T)$ (it is always possible to choose such v because of condition 1 of Theorem). From conditions (13), (14) and (25) and the fact that $T^* \le 1$, we have for any $v \in \mathcal{F}_T^a(\tilde{x})$ and all $T \in (0,T^*)$:

$$\begin{aligned} |v|_{\mathcal{A}\times\mathcal{C}} &\le \alpha_1^{-1}(V_T(v)) \\ &\le \alpha_1^{-1}(V_T(\tilde{x}) + T\delta_1) \\ &\le \alpha_1^{-1}(2\alpha_2(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}})) + \alpha_1^{-1}(2T\delta_1) \\ &\le \alpha_1^{-1}(2\alpha_2(D)) + \frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right) \\ &< \Delta \ . \end{aligned} \tag{29}$$

Then using (29) and from our choice of δ_1 and T^* (in particular the choice of T_5^*) and using that the function $|\cdot|_{\mathcal{A}\times\mathcal{C}}$ is globally Lipschitz with Lipschitz constant equal to one, we can write for all $T \in (0, T^*)$:

$$\begin{aligned} |w|_{\mathcal{A}\times\mathcal{C}} &\leq |v|_{\mathcal{A}\times\mathcal{C}} + |w-v| \\ &\leq \alpha_1^{-1}(2\alpha_2(D)) + \frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right) + T\rho(T) \\ &\leq \alpha_1^{-1}(2\alpha_2(D)) + \frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right) \\ &\quad + \frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right) \\ &= \Delta \; . \end{aligned} \tag{30}$$

Suppose now for an arbitrary $w \in \mathcal{F}_T^e(\tilde{x})$ that $V_T(w) \geq \frac{d}{2}$. This implies that for all $T \in (0, T^*)$:

$$|w|_{\mathcal{A}\times\mathcal{C}} \;\geq\; \alpha_2^{-1}\left(\frac{d}{2}\right) \;\geq\; \delta \; . \tag{31}$$

Consider arbitrary $w \in \mathcal{F}_T^e(\tilde{x})$ such that $V_T(w) \geq \frac{d}{2}$. Then, from our choice of T^*, we have for all $T \in (0, T^*)$:

$$\begin{aligned} |v|_{\mathcal{A}\times\mathcal{C}} &\geq -|w-v| + |w|_{\mathcal{A}\times\mathcal{C}} \\ &\geq -T\rho(T) + \alpha_2^{-1}\left(\frac{d}{2}\right) \\ &\geq -\frac{1}{4}\alpha_2^{-1}\left(\frac{d}{2}\right) + \alpha_2^{-1}\left(\frac{d}{2}\right) \\ &\geq \delta \; . \end{aligned} \tag{32}$$

and from our choice of $T^* \leq 1$, δ and δ_1 we write:

$$\begin{aligned} \frac{1}{2}\alpha_1(\delta) &\leq \frac{1}{2}\alpha_1(\delta) + \frac{1}{2}\alpha_1(\delta) - \frac{1}{2}\alpha_1\left(\frac{\delta}{2}\right) \\ &\leq \alpha_1(\delta) - T\delta_1 \\ &\leq \alpha_1(|v|_{\mathcal{A}\times\mathcal{C}}) - T\delta_1 \\ &\leq V_T(\tilde{x}) \\ &\leq \alpha_2(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}), \end{aligned} \tag{33}$$

which implies:

$$|\tilde{x}|_{\mathcal{A}\times\mathcal{C}} \geq \alpha_2^{-1}\left(\frac{1}{2}\alpha_1(\delta)\right) = \delta_2 \; . \tag{34}$$

From (15), (31)-(33), $\delta_2 \leq \delta$ and condition 1 of Theorem it follows that given any $w \in \mathcal{F}_T^e(\tilde{x})$ with $V_T(w) \geq d/2$ there exists $v \in \mathcal{F}_T^a(\tilde{x})$ such that

$$|V_T(w) - V_T(v)| \leq L\,|w-v| \leq TL\rho(T) \; . \tag{35}$$

Then, from the choice of T^* (in particular the choice of T_4^*), the choice of δ_1 in (25) and δ_2 in (26) and the the conditions (29)-(33) we conclude that $|\tilde{x}|_{\mathcal{A}\times\mathcal{C}} \leq D$, $w \in \mathcal{F}_T^e(\tilde{x})$ and $V_T(w) \geq d/2$ imply for all $T \in (0, T^*)$:

$$\begin{aligned} V_T(w) - V_T(\tilde{x}) &\leq V_T(v) - V_T(\tilde{x}) \\ &\quad + |V_T(w) - V_T(v)| \\ &\leq -T\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) + T\delta_1 \\ &\quad + LT\rho(T) \\ &\leq -\frac{T}{4}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) \; . \end{aligned} \tag{36}$$

Suppose now that $w \in \mathcal{F}_T^e(\tilde{x})$, $V_T(w) \leq d/2$ and $V_T(\tilde{x}) \geq d$. We also know from (14) that $V_T(\tilde{x}) \geq T\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) - T\delta_1$ and from our choice of δ_1 in (25) it follows for all $T \in (0, T^*)$ that (37) holds, which completes the proof.

$$\begin{aligned} V_T(w) - V_T(\tilde{x}) &\leq \tfrac{1}{2}(d - V_T(\tilde{x}) - V_T(\tilde{x})) \\ &\leq -\tfrac{T}{2}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) + T\tfrac{\delta_1}{2} \\ &\leq -\tfrac{T}{4}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) - \tfrac{T}{4}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) \\ &\quad + \tfrac{T}{4}\alpha_3 \circ \alpha_2^{-1}\left(\tfrac{1}{2}\alpha_1(\delta)\right) \\ &\leq -\tfrac{T}{4}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) - \tfrac{T}{4}\alpha_3 \circ \alpha_2^{-1}\left(\tfrac{1}{2}\alpha_1(\delta)\right) \\ &\quad + \tfrac{T}{4}\alpha_3 \circ \alpha_2^{-1}\left(\tfrac{1}{2}\alpha_1(\delta)\right) \\ &= -\tfrac{T}{4}\alpha_3(|\tilde{x}|_{\mathcal{A}\times\mathcal{C}}) \; . \end{aligned} \tag{37}$$

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