Author/s:
Deghat, M;Nesic, D;Teel, AR;Manzie, C

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Observing the Slow States of General Singularly Perturbed Systems

Mohammad Deghat, Dragan Nešić, Andrew R. Teel and Chris Manzie

Abstract— **This paper studies the behaviour of observers for the slow states of a *general singularly perturbed system* – that is a singularly perturbed system which has boundary-layer solutions that do not necessarily converge to a slow manifold. The solutions of the boundary-layer system are allowed to exhibit persistent (e.g. oscillatory) steady-state behaviour which are averaged to obtain the dynamics of the approximate slow system. It is shown that if an observer has certain properties such as asymptotic stability of its error dynamics on average, then it is practically asymptotically stable for the original singularly perturbed system.**

I. Introduction

Estimation of physical variables that cannot be measured is crucial in a range of engineering problems. This includes problems in which the system exhibits a two-time-scale behaviour. Physical systems that exhibit two time-scale behaviour can be modeled as so-called singularly perturbed systems and their stability can be analyzed using singular perturbation techniques which were first developed by Tikhonov in [1].

Singularly perturbed systems have been widely studied after the early work of Tikhonov; see for example [2] for the results on the classical singular perturbation techniques. State estimation and observer design for classical singularly perturbed linear systems are also well studied. Such results can be found in, e.g., [3] and references cited therein. Recently, a more general class of singularly perturbed systems has been studied in which the solutions to the fast subsystem do not converge to a slow manifold, but converge to a bounded set; see [4]–[15].

Observer design for general nonlinear systems is a difficult problem which still attracts research attention in the control literature. This problem is further complicated when the system exhibits multiple time scales. Some of the estimation results in the literature that study the classical singularly perturbed nonlinear systems are [16]–[21]. These results use Lyapunov techniques to study the stability behaviour of observer error dynamics.

This paper studies the observer design of a general singularly perturbed system in which the solutions to the so-called boundary-layer system do not necessarily converge to a slow manifold. Unlike most of the above-mentioned observer design results, we use a trajectory-based approach as opposed to Lyapunov approaches for stability analysis. This will allow us to directly work with the system solutions and obtain the quantitative results that we desire.

The results in this paper are inspired from the results in [13] where an input-to-state stability result is developed for two-time-scale systems. A challenge in using the results from [13] is to formulate the problem in a way that the results are valid for a wide range of nonlinear observers. The seemingly natural approach of "first averaging the model and then designing observer for the averaged model" is not actually the most general approach, and by carefully formulating the problem, we are able to use results from [13] almost off the shelf; see Remark 2 for more details.

The paper is organised as follows. In Section II, the general class of singular perturbation systems is explained and the problem is formulated. The main result of the paper is given in Section III. An illustrative example is presented in Section IV, and finally some concluding remarks are given in Section V.

Notation:

- A class-$\mathcal{KL}$ function from $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0}$ is a continuous function which is zero at zero and strictly increasing in its first argument, and decreasing to zero in its second argument.
- $\mathcal{B}$ denotes a closed unit ball, $\sigma\mathcal{B}$ a closed ball of radius σ, and $\mathcal{X}_s + \sigma\mathcal{B}$ the union of all sets obtained by taking a closed ball of radius σ around each point in the set $\mathcal{X}_s$.

II. Problem Formulation

A. General singularly perturbed systems

We will focus on singularly perturbed systems with the following state-space model

$$\dot{x} = f(x, z, \varepsilon), \quad x(0) = x_0 \tag{1a}$$

$$\varepsilon\dot{z} = g(x, z, \varepsilon), \quad z(0) = z_0 \tag{1b}$$

$$y = h(x, z, \varepsilon), \tag{1c}$$

where ε is the time-scale parameter, $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$ are respectively the slow and fast variables, and $y \in \mathbb{R}^p$ is the measured output.

Define the fast-time variable τ as $\tau := t/\varepsilon$. Then the singularly perturbed system (1) can be written in the τ time scale as

$$\frac{dx}{d\tau} = \varepsilon f(x, z, \varepsilon) \tag{2a}$$

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M. Deghat is with the School of Mechanical and Manufacturing Engineering, University of New South Wales, Sydney, NSW 2052 Australia (e-mail: m.deghat@unsw.edu.au)

D. Nešić and C. Manzie are with the Department of Electrical and Electronic Engineering, The University of Melbourne, Parkville, VIC 3010 Australia (e-mail: {dnesic;manziec}@unimelb.edu.au)

A.R. Teel is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: teel@ece.ucsb.edu)

$$\frac{dz}{d\tau} = g(x, z, \varepsilon) \tag{2b}$$
$$y = h(x, z, \varepsilon). \tag{2c}$$

Following the standard approach in the analysis of singularly perturbed systems, we will define two limiting systems, namely the *boundary-layer system* and the *reduced system*, that arise from the original system (1). The boundary-layer system is defined in the fast, i.e., τ, time domain, for $\varepsilon = 0$, as

$$\frac{dx_b}{d\tau} = 0 \tag{3a}$$
$$\frac{dz_b}{d\tau} = g(x_b, z_b, 0) \tag{3b}$$
$$y_b = h(x_b, z_b, 0), \tag{3c}$$

where the subscript b denotes the state of the boundary-layer system and x_b is constant for all $\tau \geq 0$. We denote $\varphi_b(\tau, x_b, z_0)$ as the solution to the boundary-layer system (3b), when $z_b(0) = z_0$, which is assumed to be unique for each x_b. Unlike the classical singular perturbation problem, we assume the solutions to the boundary-layer system, denoted by φ_b, do not necessarily converge to a unique equilibrium, but may converge to a bounded set which is possibly parameterised by the slow variable x_b. For example, the solutions to the boundary-layer system may converge to a family of limit cycles parameterised by the slow variables.

The second limiting system arising from (1) is the reduced-average system. We will refer to this system as the *reduced system* so that our terminology is consistent with the classical singular perturbation literature. We now define some of the sets which will be used in the paper and then define the reduced system. The notation and presentation in this paper will be similar to [13].

The set $\mathcal{K}_s$ is a compact set of initial conditions for x, $\mathcal{K}_f$ is a compact set of initial conditions for z, and $\mathcal{R}_x$ and $\mathcal{R}_z$ are respectively, the sets over which the solutions of (1a) and (1b), when initialised in $\mathcal{K}_s$ and $\mathcal{K}_f$, will range.

Definition 1 (Admissible average)**.** *The function $f_{av}(\cdot)$ provides an admissible average if for any given $\rho > 0$, there exist $T^* > 0$ and $\varepsilon^* > 0$ such that for all $T > T^*$, $\varepsilon \in (0, \varepsilon^*]$, $x \in \mathcal{R}_x$ and $z_0 \in \mathcal{K}_f$, the following condition holds*

$$\left\| \frac{1}{T} \int_0^T \Big[f\big(x, \varphi_b(s, x, z_0), \varepsilon\big) - f_{av}(x) \Big] ds \right\| \leq \rho. \tag{4}$$

Any admissible average, $f_{av}(\cdot)$, that satisfies the condition in (4) can be used to generate a reduced system which is defined as

$$\frac{dx_{av}}{d\tau} = \varepsilon f_{av}(x_{av}). \tag{5}$$

Remark 1. *In general, the reduced system is a differential inclusion (see e.g. [5], [10]) of the form*

$$\frac{dx_{av}}{d\tau} \in \varepsilon F_{av}(x_{av}),$$

where the set-valued mapping F_{av} is defined for all $x \in \mathcal{R}_x$ as

$$F_{av}(x) := \mathrm{conv}\left(\bigcup_{z_0 \in \mathcal{K}_f} \left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T f\big(x, \phi_b(s, x, z_0), \varepsilon\big) ds \right\} \right)$$

with $\mathrm{conv}(S)$ denoting the closed convex hull of a set S. This is due to the fact that f_{av} in (5) *is in general a function of z_0 which can take values from the set $\mathcal{K}_f$. We however assume in this paper that the set-valued map F_{av} is a singled-valued function, i.e., $F_{av}(x) = \{f_{av}(x)\}$. This is a more restrictive assumption compared to [5], [10] and more general conditions will be the topic for further research. Observer design for differential inclusions is a largely open topic and the above problem formulation strongly motivates observer design for general differential inclusions.*

B. Observer

We aim to show in this paper that any observer which has certain properties, including asymptotic stability on average for the reduced system, has practical asymptotic stability property for the slow states of the original (non-averaged) system in (1). Any function Φ that satisfies the observer conditions, listed below in Definitions 2 and 3, can be used to construct an admissible observer in the form of

$$\frac{d\hat{x}}{d\tau} = \varepsilon \Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big), \tag{6}$$

where $\hat{x} \in \mathbb{R}^n$ denotes the estimate of x.

We now define some other sets and then define admissible observers. Let $\hat{x}(0) \in \mathcal{K}_s$ and let $\mathcal{R}_{\hat{x}}$ be the set over which the solutions of (6), when initialised in $\mathcal{K}_s$, will range.

Definition 2 (Average observer)**.** *The function Φ_{av} provides an admissible average for $(\hat{x}, x, z, \varepsilon) \mapsto \Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big)$ if for any given $\rho > 0$, there exist $T^* > 0$ and $\varepsilon^* > 0$ such that for all $T > T^*$, $\varepsilon \in (0, \varepsilon^*]$, $x \in \mathcal{R}_x$, $\hat{x} \in \mathcal{R}_{\hat{x}}$ and $z_0 \in \mathcal{K}_f$, the following condition holds*

$$\left\| \frac{1}{T} \int_0^T \Big[\Phi\Big(\hat{x}, h\big(x, \phi_b(s, x, z_0), \varepsilon\big)\Big) - \Phi_{av}(\hat{x}, x) \Big] ds \right\| \leq \rho. \tag{7}$$

Definition 3 (Admissible observer)**.** *The function $(\hat{x}, x, z, \varepsilon) \mapsto \Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big)$ provides an admissible observer for the slow states x on average if the set*

$$\Omega := \Big\{ (x_{av}, \hat{x}_{av}) : x_{av} = \hat{x}_{av} \Big\} \tag{8}$$

is asymptotically stable for the system

$$\frac{dx_{av}}{d\tau} = \varepsilon f_{av}(x_{av}) \tag{9a}$$
$$\frac{d\hat{x}_{av}}{d\tau} = \varepsilon \Phi_{av}(\hat{x}_{av}, x_{av}). \tag{9b}$$

This condition holds if there exists a class-$\mathcal{KL}$ function β_s such that for $x_{av}(0), \hat{x}_{av}(0) \in \mathcal{K}_s$, the solutions of (9) *exist and satisfy*

$$\|x_{av}(\tau) - \hat{x}_{av}(\tau)\| \leq \beta_s\big(\|x_{av}(0) - \hat{x}_{av}(0)\|, \varepsilon\tau\big). \tag{10}$$

Remark 2. *There are different ways of constructing an observer and defining its average. The approach in this paper is to design an admissible observer so that when we use the actual output in the observer map in* (7) *and then average, the averaged observer satisfies the desired stability condition given in Definition 3. An alternative approach is to first average f and h, given in* (1)*, and then design observer for averaged f_{av}, h_{av} and then find conditions under which the stability of the averaged estimation error gives us practical stability of the estimation error of the original system when the observer designed for the averaged system is applied to the original system, that is, when the observer uses h instead of h_{av}. This approach is used in [17]–[20] for observer design for classical singularly perturbed systems. The results of these two cases will coincide in situations when Φ is affine in h. The alternative approach would require stronger conditions to ensure that the solutions still converge, but in some sense, it is an easier approach to design an observer.*

C. Assumptions

In this subsection, we state the assumptions needed for establishing the main result of the the paper. But before that, we define the set $\mathcal{X}_s$ which will be used in the following Assumptions and also in the statement of the main result of the paper.

$$\mathcal{X}_s = \big\{(x_{av}, \hat{x}_{av}) : \|x_{av} - \hat{x}_{av}\| \leq \beta_s(\sup_{x_{av}, \hat{x}_{av} \in \mathcal{K}_s} \|x_{av} - \hat{x}_{av}\|, 0)\big\}. \tag{11}$$

Assumption 1 (Regularity and boundedness)**.** *There exist $L > 0$, $M > 0$ and $\sigma > 0$ such that the following hold*

1) $\mathcal{X}_s + \sigma\mathcal{B} \subseteq \mathcal{R}_x \times \mathcal{R}_{\hat{x}}$
2) $$\sup_{x, \hat{x} \in \mathcal{K}_s} \|x - \hat{x}\| := c_s < \infty \tag{12}$$
3) *For each $\rho > 0$, there exist $\varepsilon^* > 0$ such that for all $z \in \mathcal{R}_z$, $(x, \hat{x}), (x_{av}, \hat{x}_{av}) \in \mathcal{X}_s + \sigma\mathcal{B}$ and $\varepsilon \in (0, \varepsilon^*]$ the following hold:*

$$\left\| \begin{matrix} f(x, z, \varepsilon) \\ \Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big) \end{matrix} \right\| \leq M \tag{13}$$

$$\left\| \begin{matrix} f_{av}(x_{av}) \\ \Phi_{av}(\hat{x}_{av}, x_{av}) \end{matrix} \right\| \leq M \tag{14}$$

$$\|f(x, z_1, \varepsilon) - f(x, z_2, \varepsilon)\| \leq L\|z_1 - z_2\| + \rho. \tag{15}$$

Assumption 2. *For any given $\hat{\rho} > 0$ and all $z_b(0) = z(0) \in \mathcal{K}_f$ and $x_b \in \mathcal{X}_s + \sigma\mathcal{B}$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and $\tau \in [0, \infty)$,*

$$\|z(\tau) - \varphi_b(\tau, x_b, z_b(0))\| \leq \hat{\rho}, \tag{16}$$

where $z(\tau)$ is the solution to the fast part of the original system (1) *and $\varphi_b(\tau, x_b, z_b(0))$ is the boundary-layer solution.*

Assumption 2 guarantees closeness of solutions of the boundary-layer system and the fast part of the original singularly perturbed system.

III. Main result

Theorem 1. *If f_{av} and Φ_{av} provide admissible averages according to Definitions 1 and 2, Φ provides an admissible observer for the slow states x according to Definition 3, and if Assumptions 1 and 2 hold then for each $\delta > 0$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ and all initial conditions $x(0), \hat{x}(0) \in \mathcal{K}_s$ and $z(0) \in \mathcal{K}_f$, the solutions of*

$$\frac{dx}{d\tau} = \varepsilon f(x, z, \varepsilon) \tag{17a}$$

$$\frac{dz}{d\tau} = g(x, z, \varepsilon) \tag{17b}$$

$$\frac{d\hat{x}}{d\tau} = \varepsilon\Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big) \tag{17c}$$

exist and satisfy

$$\|x(\tau) - \hat{x}(\tau)\| \leq \beta_s\big(\|x(0) - \hat{x}(0)\|, \varepsilon\tau\big) + \delta, \tag{18}$$

where β_s is defined in Definition 3.

Proof.
The proof follows similar idea and technique as the proof of Theorem 1 and Proposition 2 in [13].
A. Summary: The proof is decomposed into three claims which establish the following:

1) By choosing ε small, $\big(x(\tau), \hat{x}_{av}(\tau)\big)$ can be made to stay arbitrarily close to $\big(x_{av}(\tau), \hat{x}_{av}(\tau)\big)$ and also close to the set $\mathcal{X}_s$, on finite, but arbitrarily large, time intervals with length of order $1/\varepsilon$.
2) Because of the closeness of solutions over a finite time given above, $\|x - \hat{x}\|$ stays close to $\|x_{av} - \hat{x}_{av}\|$ over the same finite time interval.
3) The above two claims are combined to establish practical stability of the set Ω given in (8).

B. Claims:
Claim 1: For each $\delta > 0$ and $T > 0$ there exists $\varepsilon^* > 0$ such that

$$x(0), \hat{x}(0), x_{av}(0), \hat{x}_{av}(0) \in \mathcal{K}_s; \quad z(0) \in \mathcal{K}_f \tag{19}$$

$$\left\| \begin{matrix} x(0) - x_{av}(0) \\ \hat{x}(0) - \hat{x}_{av}(0) \end{matrix} \right\| \leq \varepsilon^*; \quad \varepsilon \in (0, \varepsilon^*] \tag{20}$$

imply for $\tau \in [0, T/\varepsilon]$

$$\big(x(\tau), \hat{x}(\tau)\big) \in \mathcal{X}_s + \tilde{\sigma}\mathcal{B}, \tag{21}$$

$$\left\| \begin{matrix} x(\tau) - x_{av}(\tau) \\ \hat{x}(\tau) - \hat{x}_{av}(\tau) \end{matrix} \right\| \leq \delta. \tag{22}$$

Proof.

Step 1. Definition of ε^*: Let $\delta > 0$ and $T > 0$ be given. Assume without loss of generality that $\tilde{\sigma}$ satisfies $\delta < \tilde{\sigma}$. Define

$$\varepsilon_1^* := \frac{\delta}{2\exp(LT)}, \tag{23}$$

$$\rho := \frac{\delta L}{8[\exp(LT) - 1]}, \quad \text{and} \quad \hat{\rho} := \frac{\rho}{L}. \tag{24}$$

For this ρ, assume there exist admissible averages, f_{av} and Φ_{av}, according to Definitions 1 and 2, and let T^* and ε_2^* be respectively the maximum and minimum values generated by Definitions 1 and 2. Let Assumption 2 generate ε_3^* and let the last item of Assumption 1 generate ε_4^*. Next, define

$$\varepsilon_5^* := \frac{\delta}{4MT^*}. \tag{25}$$

Then define ε^* as

$$\varepsilon^* = \max\{\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*, \varepsilon_4^*, \varepsilon_5^*\}. \tag{26}$$

Step 2. Behaviour of trajectories: From (10), the definition of the set $\mathcal{X}_s$ in (11) and c_s in (12), it follows that

$$\begin{aligned}\|x_{av}(\tau) - \hat{x}_{av}(\tau)\| &\leq \beta_s\big(\|x_{av}(0) - \hat{x}_{av}(0)\|, \tau\varepsilon\big) \\ &\leq \beta_s\big(c_s, \tau\varepsilon\big) \\ &\leq \beta_s\big(c_s, 0\big).\end{aligned} \tag{27}$$

Therefore,

$$\big(x_{av}(\tau), \hat{x}_{av}(\tau)\big) \in \mathcal{X}_s, \qquad \forall \tau \in [0, \infty). \tag{28}$$

Define

$$\bar{\tau} := \sup\left\{\tau \in [0, T/\varepsilon] : \big(x(t), \hat{x}(t)\big) \in \mathcal{X}_s + \tilde{\sigma}\mathcal{B} \quad t \in [0, \tau]\right\}. \tag{29}$$

Since $x(0), \hat{x}(0) \in \mathcal{K}_s$ (see (19)), it follows from the definition of $\mathcal{X}_s$ that $\bar{\tau}$ is well-defined and $\bar{\tau} > 0$. In fact, $\big(x(\tau), \hat{x}(\tau)\big) \in \mathcal{X}_s + \tilde{\sigma}\mathcal{B}$ for all $\tau \in [0, \bar{\tau})$ and if $\bar{\tau} < T/\varepsilon$ then at $\tau = \bar{\tau}$ we have

$$\left\| \begin{array}{c} x(\bar{\tau}) - x_{av}(\bar{\tau}) \\ \hat{x}(\bar{\tau}) - \hat{x}_{av}(\bar{\tau}) \end{array} \right\| \leq \tilde{\sigma}. \tag{30}$$

Suppose $\bar{\tau} < T^*$, where T^* is the maximum T^* generated by Definitions 1 and 2. Then we obtain from (13) and (14) that

$$\varepsilon \left\| \begin{array}{c} f(x, z, \varepsilon) - f_{av}(x) \\ \Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big) - \Phi_{av}(\hat{x}, x) \end{array} \right\| \leq 2\varepsilon M. \tag{31}$$

From (20) and the above inequality, it follows for $\tau \in [0, \bar{\tau}]$ that

$$\begin{aligned}\left\| \begin{array}{c} x(\tau) - x_{av}(\tau) \\ \hat{x}(\tau) - \hat{x}_{av}(\tau) \end{array} \right\| &\leq \varepsilon^* + \int_0^\tau 2\varepsilon M dt \\ &= \varepsilon^* + 2\varepsilon M \tau \\ &\leq \varepsilon^* + 2\varepsilon M T^* \\ \overset{(26)}{\Longrightarrow} \quad &\leq \varepsilon^* + 2\varepsilon_5^* M T^* \\ \overset{(23),(25)}{\Longrightarrow} \quad &\leq \frac{\delta}{2} + \frac{\delta}{2} < \tilde{\sigma}.\end{aligned} \tag{32}$$

Up to here, we showed that for $\tau \in [0, \bar{\tau}]$, the inequality in (32) holds. We need to show the inequality holds for all $\tau \in [0, T/\varepsilon]$. From (28), (32) and also the definition of $\bar{\tau}$ in (29), it follows that $\bar{\tau} \geq \min\{T^*, T/\varepsilon\}$. If $T/\varepsilon \leq T^*$ then the proof is complete. Otherwise, suppose $T^* \leq \bar{\tau} < T/\varepsilon$. Then we have for all $\tau \in [0, \bar{\tau}]$

$$\begin{aligned}&\left\| \begin{array}{c} x(\tau) - x_{av}(\tau) \\ \hat{x}(\tau) - \hat{x}_{av}(\tau) \end{array} \right\| \\ &\leq \varepsilon^* + \varepsilon \left\| \int_{T^*}^\tau \Big(f(x, z, \varepsilon) - f_{av}(x)\Big) ds \right\| \\ &\quad + \varepsilon \left\| \int_{T^*}^\tau \Big(\Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big) - \Phi_{av}(\hat{x}, x)\Big) ds \right\| \\ &\leq \varepsilon^* + \varepsilon \left\| \int_{T^*}^\tau \Big(f(x, z, \varepsilon) - f\big(x, \varphi_b(s, x, z_0), \varepsilon\big)\Big) ds \right\| \\ &\quad + \varepsilon \left\| \int_{T^*}^\tau \Big(f\big(x, \varphi_b(s, x, z_0), \varepsilon\big) - f_{av}(x)\Big) ds \right\| \\ &\quad + \varepsilon \int_{T^*}^\tau \Big(\Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big) - \Phi_{av}(\hat{x}, x)\Big) ds \\ &\overset{(4),(7),(15)}{\Longrightarrow} \quad \leq \varepsilon^* + \varepsilon L \int_{T^*}^\tau \|z(s) - \varphi_b(s, x, z_0)\| \, ds + 3\varepsilon\rho\tau \\ &\overset{\text{Assumption 2}}{\Longrightarrow} \quad \leq \varepsilon^* + \varepsilon L \hat{\rho} \tau + 3\varepsilon\rho\tau.\end{aligned} \tag{33}$$

From (23), $\varepsilon^* \leq \delta/2$. Also since $\tau < T/\varepsilon$, it follows that $\varepsilon\tau < T$ and using (23) and (24), we have $T(L\hat{\rho} + 3\rho) \leq \delta/2$. Then (33) can be written as

$$\left\| \begin{array}{c} x(\tau) - x_{av}(\tau) \\ \hat{x}(\tau) - \hat{x}_{av}(\tau) \end{array} \right\| \leq \delta < \tilde{\sigma}, \quad \tau \in [T^*, \bar{\tau}] \subseteq [T^*, T/\varepsilon]. \tag{34}$$

It follows that $\bar{\tau} = T/\varepsilon$ and the result is established. □

Claim 2: There exists $T^* > 0$ such that for each $T > T^*$ and $\delta > 0$, there exists $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$ and $\tau \in [0, T/\varepsilon]$

$$\big(x(0), \hat{x}(0)\big), \big(x_{av}(0), \hat{x}_{av}(0)\big) \in \mathcal{K}_s; \quad z(0) \in \mathcal{K}_f \tag{35}$$

$$\left\| \begin{array}{c} x(0) - x_{av}(0) \\ \hat{x}(0) - \hat{x}_{av}(0) \end{array} \right\| \leq \varepsilon^*; \quad \varepsilon \in (0, \varepsilon^*] \tag{36}$$

imply

$$\big|\|x(\tau) - \hat{x}(\tau)\| - \|x_{av}(\tau) - \hat{x}_{av}(\tau)\|\big| \leq \delta. \tag{37}$$

Proof. Let Claim 1 generate ε^* for the pair (δ, T). Then from Claim 1,

$$\left\| \begin{array}{c} x(\tau) - x_{av}(\tau) \\ \hat{x}(\tau) - \hat{x}_{av}(\tau) \end{array} \right\| \leq \delta, \qquad \forall \tau \in (0, T/\varepsilon]. \tag{38}$$

Using the properties of the norm function, (38) implies (37). □

Claim 3: For each $\delta > 0$, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, (37) which holds for $\tau \in [0, T/\varepsilon]$ implies for $\tau \in [0, \infty)$ that

$$\|x(\tau) - \hat{x}(\tau)\| \leq \beta_s\big(\|x(0) - \hat{x}(0)\|, \varepsilon\tau\big) + \delta. \tag{39}$$

Proof. The proof is almost identical to the proof of Theorem 1 in [13]. Let $\delta > 0$ be given and let $\bar{\delta} > 0$ be such that

$$\sup_{r \in [0, c_s], \tau \in [0, \infty)} \Big(\beta_s(r + \bar{\delta}, \tau) - \beta_s(r, \tau)\Big) + \bar{\delta} \leq \frac{\delta}{2}. \tag{40}$$

The existence of $\bar{\delta}$ follows from the properties of class-$\mathcal{KL}$ functions. Let T^* come from Definition 1 and let $T > T^*$ be large enough so that

$$\beta_s(c_s, \varepsilon\tau) \leq \frac{\delta}{2}, \qquad \forall \tau \in [T/\varepsilon, \infty]. \tag{41}$$

Let ε^* come from Claim 2 for the pair $(\bar{\delta}, 2T)$. Then from (37), we have for all $\varepsilon \in (0, \varepsilon^*]$ and $\tau \in [0, 2T/\varepsilon]$ that

$$\begin{aligned} \|x(\tau) - \hat{x}(\tau)\| &\leq \|x_{av}(\tau) - \hat{x}_{av}(\tau)\| + \bar{\delta} \\ \overset{(7)}{\Longrightarrow} \quad &\leq \beta_s\big(\|x_{av}(0) - \hat{x}_{av}(0)\|, \varepsilon\tau\big) + \bar{\delta} \\ \overset{(19),(21)}{\Longrightarrow} \quad &\leq \beta_s\big(\|x(0) - \hat{x}(0)\| + \bar{\delta}, \varepsilon\tau\big) + \bar{\delta} \\ \overset{(40)}{\Longrightarrow} \quad &\leq \beta_s\big(\|x(0) - \hat{x}(0)\|, \varepsilon\tau\big) + \frac{\delta}{2}. \end{aligned} \tag{42}$$

From (41) and (42), it follows that for $\tau \in [T/\varepsilon, 2T/\varepsilon]$

$$\|x(\tau) - \hat{x}(\tau)\| \leq \delta. \tag{43}$$

This argument can be applied repeatedly to obtain

$$\|x(\tau) - \hat{x}(\tau)\| \leq \delta, \qquad \tau \in [T/\varepsilon, \infty). \tag{44}$$

Finally, we conclude from (42) and (44) that for $\tau \in [0, \infty)$,

$$\|x(\tau) - \hat{x}(\tau)\| \leq \beta_s\big(\|x(0), \hat{x}(0)\|, \varepsilon\tau\big) + \delta. \tag{45}$$

□

IV. EXAMPLE

We consider an example which has a stable reduced system and a boundary-layer system whose solution converges to a limit cycle. Consider the following singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - 2x_2 + 0.5x_1^2 + z_1 \\ \varepsilon\dot{z}_1 &= -z_1 + z_2 + \frac{z_1}{\sqrt{z_1^2 + z_2^2}} \\ \varepsilon\dot{z}_2 &= -z_1 - z_2 + \frac{z_2}{\sqrt{z_1^2 + z_2^2}} + \varepsilon x_2 \\ y &= x_1 + z_1. \end{aligned} \tag{46}$$

Note that (46) is not defined for $(z_1, z_2) = 0$. The boundary-layer system is exponentially stable and its solutions converge exponentially fast to the set

$$\left\{z_1, z_2 \in \mathbb{R} : z_1^2 + z_2^2 = 1\right\}. \tag{47}$$

Indeed, the solutions to the boundary-layer system, starting from the initial condition $z_0 = (z_1(0), z_2(0))$, are

$$\begin{aligned} z_1(\tau) &= \big((r_0 - 1)e^{-\tau} + 1\big)\cos(-\tau + \theta_0) \\ z_2(\tau) &= \big((r_0 - 1)e^{-\tau} + 1\big)\sin(-\tau + \theta_0) \end{aligned} \tag{48}$$

where $\theta_0 = \operatorname{atan}(z_2(0)/z_1(0))$ and $r_0 = \|z_0\| = \sqrt{z_1^2(0) + z_2^2(0)}$.

Consider the following observer for the x-component of (46)

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \Phi\big(\hat{x}, h(x, z, \varepsilon), \varepsilon\big) \\ &= \begin{bmatrix} \hat{x}_2 + k(y - \hat{x}_1) \\ -\hat{x}_1 - 2\hat{x}_2 - 0.5\hat{x}_1^2 + \hat{x}_1(y - z_1) + z_1 + k(y - \hat{x}_1) \end{bmatrix} \end{aligned} \tag{49}$$

where $k > 0$ is the estimator parameter to be determined. Define the estimation error as

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 - \hat{x}_1 \\ x_2 - \hat{x}_2 \end{bmatrix}. \tag{50}$$

Then the estimation error dynamics can be written as

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 - k\tilde{x}_1 - kz_1 \\ -(1+k)\tilde{x}_1 - 2\tilde{x}_2 + 0.5\tilde{x}_1^2 - kz_1 \end{bmatrix}. \tag{51}$$

Define $\mathcal{K}_s = \{x : \|x\| \leq 0.5\}$ and $\mathcal{K}_f = \{z : 0.5 \leq \|z\| \leq 1.5\}$. Then from the exponential stability of the boundary-layer solutions and also from (46), it can be shown that Assumption 2 holds.

Using (48), we can find the following admissible average and observer that satisfy the conditions in Definition 1 and 2:

$$f_{av}(x_{av}) = \begin{bmatrix} x_{av_2} \\ -x_{av_1} - 2x_{av_2} + 0.5x_{av_1}^2 \end{bmatrix} \tag{52}$$

$$\begin{aligned} &\Phi_{av}\big(\hat{x}_{av}, x_{av}\big) = \\ &\begin{bmatrix} \hat{x}_{av_2} + k(x_{av_1} - \hat{x}_{av_1}) \\ -\hat{x}_{av_1} - 2\hat{x}_{av_2} - 0.5\hat{x}_{av_1}^2 + \hat{x}_{av_1}x_{av_1} + k(x_{av_1} - \hat{x}_{av_1}) \end{bmatrix} \end{aligned} \tag{53}$$

where $x_{av} = [x_{av_1}, x_{av_2}]^T$.

It follows from (46) and (52) that

$$f\big(x, \varphi_b(\tau, x, z_0), \varepsilon\big) - f_{av}(x) = \begin{bmatrix} 0 \\ z_1(\tau) \end{bmatrix} \tag{54}$$

and from (48), there exists $T^* > 0$ for any given $\rho > 0$ such that for $T > T^*$ the condition in (4) is satisfied. Similarly, the condition in (7) is satisfied for the observer (49) and its admissible average (53).

We now show the stability condition in Definition 3 holds. Define the estimation error for the average system as

$$\tilde{x}_{av} = \begin{bmatrix} \tilde{x}_{av_1} \\ \tilde{x}_{av_2} \end{bmatrix} = \begin{bmatrix} x_{av_1} - \hat{x}_{av_1} \\ x_{av_2} - \hat{x}_{av_2} \end{bmatrix}. \tag{55}$$

The estimation error dynamics for the average system can be written as

$$\begin{bmatrix} \dot{\tilde{x}}_{av_1} \\ \dot{\tilde{x}}_{av_2} \end{bmatrix} = \begin{bmatrix} \tilde{x}_{av_2} - k\tilde{x}_{av_1} \\ -\tilde{x}_{av_1} - 2\tilde{x}_{av_2} + 0.5\tilde{x}_{av_1}^2 - k\tilde{x}_{av_1} \end{bmatrix}. \tag{56}$$

Define the Lyapunov function $V(\tilde{x}_{av}) := \frac{1}{2}(\tilde{x}_{av_1}^2 + \tilde{x}_{av_2}^2)$. Then

$$\dot{V} = -k\tilde{x}_{av_1}^2 - 2\tilde{x}_{av_2}^2 + 0.5\tilde{x}_{av_1}^2\tilde{x}_{av_2} - k\tilde{x}_{av_1}\tilde{x}_{av_2}. \tag{57}$$

Let $\|\tilde{x}_{av_2}\|$, $\|\tilde{x}_{av_2}\| < 1$ and define the matrix Q as

$$Q = \begin{bmatrix} k & -\frac{0.5+k}{2} \\ -\frac{0.5+k}{2} & 2 \end{bmatrix}. \tag{58}$$

The matrix Q is positive definite for all $0.036 < k < 6.96$. Therefore for any k in the above range,

$$\dot{V} \leq -\lambda_{\min}(Q)\|\tilde{x}_{av}\|^2 \tag{59}$$

and there exists a class-$\mathcal{KL}$ function β_s such that (10) holds.

From the definition of $\mathcal{K}_s$ and $\mathcal{K}_f$, it is straightforward to demonstrate that for any given $\delta > 0$, there exist $\sigma > 0$,

$L > 0$ and $M > 0$ such that Assumption 1 holds. Therefore from Theorem 1, there exists ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, (45) holds.

We choose $k = 1$ and plot $\|\hat{x}_{av}(t) - x_{av}(t)\|$ in Fig. 1 and $\|\hat{x}(t) - x(t)\|$ in Fig. 2 for two different values of ε to show how the error changes as ε changes. It can be seen in Fig. 1 that for the two values of ε, both plots overlap. This is due to the fact that the estimation error dynamics in (56) is independent of the fast states z_1 and z_2 and also the parameter ε. But from Fig. 2, we see that a smaller ε results in a smaller steady state estimation error.

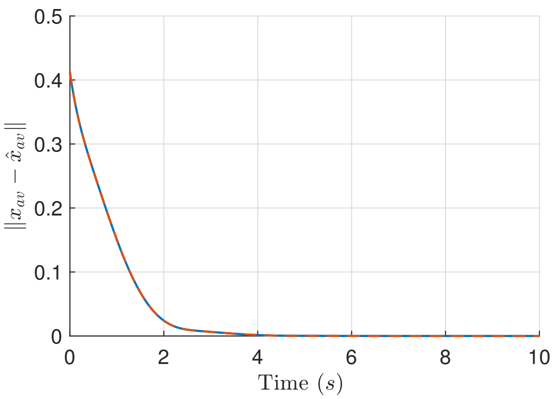

Fig. 1. The estimation error of the average system $\dot{x}_{av} = f_{av}(x_{av})$ with $f_{av}(x_{av})$ defined in (52) when estimated using the observer $\dot{\hat{x}}_{av} = \Phi_{av}(\hat{x}_{av}, x_{av})$ with $\Phi_{av}(\hat{x}_{av}, x_{av})$ defined in (53) for two different values of $\varepsilon = 0.1$ (solid blue line) and $\varepsilon = 0.02$ (dashed brown line)

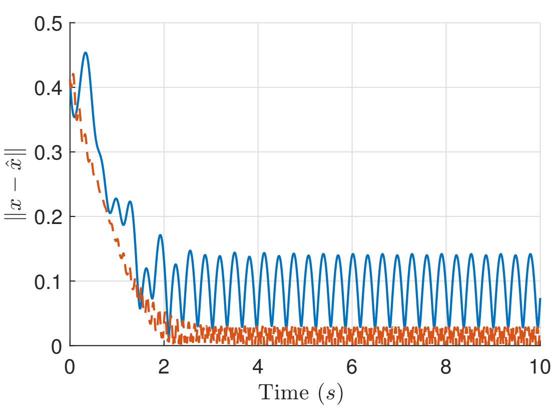

Fig. 2. The estimation error of the slow part of the full-order system (46) when estimated using the observer (49) for two different values of $\varepsilon = 0.1$ (solid blue line) and $\varepsilon = 0.02$ (dashed brown line).

V. Conclusion and Future work

The focus of the paper is on the estimation of the slow states of a general singularly perturbed system. We study conditions under which an observer that is asymptotically stable in average for the reduced system, can be used to estimate the states of the original non-averaged singularly perturbed system. While we do not estimate fast states in this paper, this is an interesting topic for further research. Another future direction is to find stronger conditions that result in semiglobal, global and non-practical stability of the estimation error of the non-averaged system.

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