Output Stabilization of Nonlinear Systems: Linear Systems with Positive Outputs as a Case Study

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Abstract

The problem of stabilization of linear systems for which only the magnitudes of outputs are measured is studied. For systems that are controllable and observable, a stabilizing controller, which is ISS-robust with respect to observation noise, is constructed.

Keywords: Input-to-state stability, nonlinear systems, sampling, measurement disturbances.

1 Introduction

This work is ultimately motivated by the need to further understand the following three issues, which are at the core of current control theory research: stabilization using partial state information; how to make (especially discontinuous) stabilizers robust with respect to noise; and what are reasonable types of discontinuous ("hybrid") feedback. These questions, which are well-understood for linear systems, are extremely difficult to deal with in a general nonlinear situation. We choose to explore them here in the context of a "toy" problem which, while, on the one hand, being general enough to be of interest in itself and to capture important characteristics of the generic problem, is, on the other hand, simple enough to be amenable to explicit solution. This is the problem of stabilization of linear systems $\dot{x} = Ax + Bu$, $y = Cx$ with loss of information at the output due to an output nonlinearity (quantization, etc). The simplest such case, where one bit of information is lost (if y is scalar) results when we take the absolute value of the output, $|Cx|$. Observation noise can act before or after the nonlinearity: $|Cx + d|$ or $|Cx| + d$, where $d = d(\cdot)$ is a function that models noise on measurements. (The second case is much easier to deal with, so we concentrate on the first.)

In contrast to linear systems, for general nonlinear systems it is in general not true that controllability and observability suffice for the existence of a (dynamic) output stabilizer. This issue was studied in detail in [12], where appropriate necessary and sufficient conditions were obtained. These conditions, however, are of an abstract nature, and except for certain classes of systems (such as bilinear systems, cf. [13]), they are hard to check. (For more recent work, see also e.g. [1, 2, 3, 8, 18].) On the other hand, sufficient conditions which are easy, to check (see e.g. [15], Chapter 6) are typically too conservative in the nonlinear case. In this paper, for the class described above, a stabilizing controller is shown to exist.

In addition, when stabilizing nonlinear systems, the effect of disturbances on actuators and/or on measurements cannot be disregarded in the design. For linear systems, small or bounded disturbances produce small or bounded steady state errors, if stability of the undisturbed system holds. Disturbances on *actuators* are somewhat easier to handle than those on measurements. Indeed, while a controller, even a feedback-linearizing one, can easily become unstable in the presence of actuator noise, the paper [14] showed how a simple feedback redesign can render the control system robust to such noise, in the robustness sense formalized by means of the concept of input to state stability (ISS).

On the other hand, disturbances on *measurements* can lead to serious instability, and one of the most interesting open problems in nonlinear control theory concerns the formulation of output controllers which are robust, in an ISS sense, with respect to disturbances. In a remarkable contribution, Randy Freeman produced in [4]

[1]This work was supported by the AFOSR/PRET 442530-25380.

[2]This work was supported in part by US Air Force Grant F49620-97-1-0159.

a counterexample showing that it is in general impossible to stabilize, using a state feedback, a simple system with state measurements $\dot{x} = f(x,u)$, $y = x$ (h is the identity), if ISS behavior with respect to disturbances is required, that is, a feedback law $k(\cdot)$ such that $\dot{x} = f(x, k(x+d))$ is ISS with respect to $d(\cdot)$. On the other hand, Freeman showed in [5] that, for his example, a *dynamic* (time-varying) controller can be used to obtain ISS behavior in this sense. The challenge now is to understand how general this situation is, and to see how explicit the constructions can be made.

As noted in [5], some of the most important features of the counterexample in [4] are captured by the study of ISS design for linear systems "with positive noisy outputs" $\dot{x} = Ax + Bu$, $y = |Cx + d|$, which are the subject of the present work. We will describe an ISS stabilizing procedure based on a sampled hybrid dynamic controller.

As for related work, we remark that the controllers presented here are somewhat similar to the stabilizers designed for linear systems with output saturating nonlinearity in [7]. They are also closely related to the time-optimal dead-beat controllers for linear systems with positive inputs presented in [9, 10]. However, due to the different nonlinearities, the designs are notably different and, more importantly, neither of the cited references investigates the performance of the designed controllers in the presence of disturbances. Also, an alternative method for stabilization of linear systems with positive outputs was presented in the journal version of this paper [11].

2 Preliminaries

Consider the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= |Cx| = |y| , \end{aligned} \tag{1}$$

where $y = Cx$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and the matrices A, B, C are of appropriate dimensions. The notation $|Cx|$ is used for the vector whose entries are the absolute values of the entries of Cx. The notation $|\cdot|$ is also used to denote the Euclidean norm, but no confusion should arise, since the meaning of the notation is usually clear from the context. In this extended abstract, we will assume that p, the number of outputs, is one. This simplifies matters considerably, but the full paper [11] describes the general case. Given a matrix $M \in \mathbb{R}^{n\times n}$ and an arbitrary vector $\zeta \in \mathbb{R}^n$ we use the notation $M\zeta > 0$ to denote that all the entries of the vector $M\zeta$ are strictly positive. For each nonnegative $r \in \mathbb{R}$, the open hyperball centered at the origin and of the radius r is denoted as $B_r = \{x \mid |x| < r\}$. $\mathbf{O}(F,C)$ is the observability matrix for the pair (F,C).

3 ISS controller

We now turn to producing a controller with ISS-like behavior. Strictly speaking, the definition of ISS is given in the literature only for systems of the type $\dot{x} = f(x,u)$, with f reasonably regular and can not be used for sampled-data controllers which we use. Instead, we adapt a linear version of one of the equivalent definitions of ISS in the usual case (cf. [17]), and use the corresponding concept in the present context.

Definition 1 *We will say that a controller is an* ultimately-linear L^∞- gain controller *if, for the system*

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= |Cx + d| \end{aligned} \tag{2}$$

the following two properties hold:

1. *There is some constant $c > 0$ such that, for each initial state $x(0)$, and for each real number D, if $|d(t)| \le D$ for all $t \in \mathbb{R}$, then the closed-loop state satisfies that*

$$\limsup_{t\to\infty} |x(t)| \le cD\,.$$

2. *For each $\varepsilon > 0$ there is some $\delta > 0$ such that, if $|x(0)| \le \delta$ and $|d(t)| \le \delta$ for all $t \in \mathbb{R}$, then the closed-loop state satisfies that*

$$|x(t)| \le \epsilon \;\forall\, t \ge 0.$$

In informal discussions, we use the term "ISS" in what follows to mean the above-defined property.

Remark 1 *A better definition will also ask that all signals in the loop be also ultimately bounded in the same manner, which amounts to asking that controls being applied to the system have that property. Also, in the spirit of ISS, the ultimate bound should be allowed to be possibly of the form $\gamma(D)$, where γ is a function of class $\mathcal{K}$, not necessarily linear. Finally we note that the precise definition of "controller" and "closed-loop behavior" is omitted, in order to avoid unnecessary formalism (see also [11]).*

The controller to be designed is a time-varying sampled-data scheme which in the absence of disturbances produces a dead-beat response. It acts by cycling through four basic steps or "modes", each of the same duration and proportional to the dimension n of the system: In the first mode, one applies a zero control and uses the measured outputs $z(\cdot)$ to obtain an estimate of the combined magnitude of the state and disturbances. The second mode drives the state to a region of the state space in which the sign of $y = Cx + d$ can be unambiguously determined. To be precise, it does so provided that the disturbance is small in comparison to the state; otherwise, if the observation noise is large, nothing interesting is accomplished. The third mode makes the assumption that the sign of y is known and reconstructs the state using a linear filter. If the disturbance was large, so that step 2 did not guarantee that the sign was known, this step does not achieve any useful goals. Finally, the controller in the last stage computes a control that drives the estimated state to the origin. If the noise was large in comparison to the initial state, then the final state is not necessarily small, but it is still small with respect to the magnitude of the disturbance. We now formalize this procedure.

We restrict attention to SISO ($m = p = 1$) systems. The discrete-time version of the system is:

$$\begin{aligned} x(k+1) &= Fx(k) + Gu(k) \\ z(k) &= |Cx(k) + d(k)| \end{aligned} \tag{3}$$

where

$$F = e^{AT} \text{ and } G = \int_0^T e^{As} B\, ds\,, \tag{4}$$

and $|d(k)| \leq D$, $\forall k$. The sampling period T is chosen so that the triple (F, G, C) is minimal. Without loss of generality we also assume below that $T = 1$ in order to simplify the notation.

The modes of operation are first described for one cycle and then we summarize the controller. The notation $Z_{k_1}^{k_2}$ denotes a vector whose entries are stacked measurements $z(i), i = k_1, k_1 + 1, \ldots, k_2$. Similarly, the notation $D_{k_1}^{k_2}$ denotes a vector whose entries are the stacked disturbances $d(i), i = k_1, k_1+1, \ldots, k_2$ at sample times. Note that $\left|D_{k_1}^{k_2}\right| \leq \sqrt{1 + k_2 - k_1} D$ for all k_1, k_2.

3.1 Mode 1:

We apply the controls $u(k+t) = 0, k = 0, 1, 2, \ldots, n-1$, $t \in [0, 1[$ over the time interval $[0, n[$, and take the measurements $z(k), k = 0, 1, \ldots, n-1$. Note that we have:

$$\pm z(k) = CF^k x(0) + d(k)\,,\ k = 0, 1, \ldots, n-1\,.$$

For an arbitrary $L > 0$, we introduce the notation

$$B(Z_0^{n-1}) = (1 + L) \left\|\mathbf{O}(F, C)^{-1}\right\| \left|Z_0^{n-1}\right|\,. \tag{5}$$

The number $B(Z_0^{n-1})$ can be interpreted as an estimate for a bound on the norm of the initial state $x(0)$, and is obtained from the measurements Z_0^{n-1}. Similarly, we introduce:

$$\|F^n\|\, B(Z_0^{n-1}) \tag{6}$$

as an estimate for a bound of the norm of the state $x(n)$. The following Lemma shows that the above estimates are correct if the norm of the initial state is large enough.

Lemma 1 *Consider the bound (5) with an arbitrary $L > 0$. Then, there exists $K > 0$ such that, if $|x(0)| \geq KD$, then*

$$|x(0)| \leq (1 + L) \left\|\mathbf{O}(F, C)^{-1}\right\| \left|Z_0^{n-1}\right| = B(Z_0^{n-1})\,. \tag{7}$$

Proof of Lemma 1: Consider (3) with $u(k) = 0$, $k = 0, 1, \ldots, n-1$, and suppose that we recorded the following measurements:

$$\begin{aligned} z(0) &= |Cx(0) + d(0)| \\ z(1) &= |CFx(0) + d(1)| \\ &\vdots \\ z(n-1) &= \left|CF^{n-1}x(0) + d(n-1)\right|\,. \end{aligned} \tag{8}$$

Rewrite the set of equations (8) as follows:

$$\mathbf{O}(F, C)x(0) + d = \pm Z_0^{n-1}\,, \tag{9}$$

where the matrices d and Z_0^{n-1} have the obvious interpretation. (We use the notation $\pm X$, for a vector $X = (x_1, \ldots, x_n)^T$, to denote a vector each of whose entries is $\pm x_i$.) From equation (9) we obtain:

$$|x(0)| \leq \left\|\mathbf{O}(F, C)^{-1}\right\| \left(\left|Z_0^{n-1}\right| + \sqrt{n} D\right)\,. \tag{10}$$

Now suppose that

$$|x(0)| > \frac{(1 + L)}{L} \left\|\mathbf{O}(F, C)^{-1}\right\| \sqrt{n} D =: KD\,.$$

Using (10), we obtain that $\sqrt{n} D \leq L \left|Z_0^{n-1}\right|$, and, substituting this estimate of D back into (10), we obtain (7), as desired. ∎

Remark 2 *With $M := (1 + L) \left\|\mathbf{O}(F, C)^{-1}\right\|$, we can summarize the above conclusions as: There are constants M and K such that*

$$|x(0)| \leq \max\left\{M \left|Z_0^{n-1}\right|, K \left|D_0^{n-1}\right|\right\}\,.$$

3.2 Mode 2:

Before we describe Mode 2 in more detail, we need to construct a cone which is used to compute the controls that are applied to the system over the time interval $[n, 2n[$. The cone $\mathcal{C}_0 = \{x : CF^i x > 0 \,|\, i = 0, 1, \ldots, n-1\}$ is not used for the construction of the ISS controller, as opposed to the controller 1. Instead, we introduce a subcone of $\mathcal{C}_0$, denoted as $\mathcal{C}_1$. The cone $\mathcal{C}_1$ is used to "robustify" the controller 1 so that ISS of the closed loop system can be proved. The lemma given below specifies the construction.

Notice that, since (F, C) is observable, $\mathcal{C}_0 \neq \emptyset$. For each number $D \in [0, \infty[$, we introduce the following sets:

$$H_j(D) := \{x : |CF^j x| \leq D\},\ j \in \{0, 1, \dots, n-1\}.$$

For any $r \geq 0$, B_r denotes the ball of radius r.

Lemma 2 *There exists an open cone $\mathcal{C}_1 \subseteq \mathbb{R}^n$ contained in $\mathcal{C}_0$, and there is a positive number $\lambda > 0$ such that*

$$\left(\bigcup_j H_j(D)\right) \bigcap \mathcal{C}_1 \subset B_{\lambda D}$$

for every $D \in [0, \infty[$. In particular, the set $(\cup_j H_j(D)) \cap \mathcal{C}_1$ is bounded, for each D.

Proof of Lemma 2: We define $\mathcal{C}_1$ as the set of vectors $x \in \mathbb{R}^n$ which satisfy all the following inequalities:

$$\begin{aligned} Cx &> 0 \\ 2Cx > CFx &> Cx \\ &\vdots \\ 2CF^{n-2}x > CF^{n-1}x &> CF^{n-2}x. \end{aligned} \tag{11}$$

It is clear that $\mathcal{C}_1$ is a nonempty open cone contained in $\mathcal{C}_0$, and, for all $i, j \in \{0, \dots, n-1\}$ and all $x \in \mathcal{C}_1$,

$$i < j \Rightarrow CF^i x < CF^j x$$

and

$$i > j \Rightarrow CF^i x < 2^{i-j} CF^j x.$$

Now take any j, D and any $x \in H_j(D) \cap \mathcal{C}_1$. Thus all $CF^i x > 0$, and $0 < CF^j x \leq D$. It follows from the above properties that $CF^i x < 2^{n-1} CF^j x \leq 2^{n-1} D$, so also $\|x\| \leq \lambda D$, where

$$\lambda := \left\|\mathbf{O}(F, C)^{-1}\right\| \sqrt{n} 2^{n-1} D. \quad \blacksquare$$

Now we show how to compute the control sequence in Mode 2.

We can represent $\mathcal{C}_1$ by means of some set of M inequalities, as follows:

$$r_i x > 0, \quad i = 1, 2, \dots, M \tag{12}$$

where the r_i, $i = 1, 2, \dots, M$, are row vectors. We now choose an arbitrary vector $v \in \mathcal{C}_1$ with unit norm $|v| = 1$; by definition, $r_i v > 0$ for $i = 1, \dots, M$. Since the pair (F, G) is controllable, there exist $a_i \in \mathbb{R}^m$, $i = 0, 1, \dots, n-1$, such that

$$v = \sum_{i=0}^{n-1} F^{n-1-i} G a_i. \tag{13}$$

Assume now that the following control sequence is applied to the system:

$$u(k+t) = \alpha a_{k-n} \tag{14}$$

for $k = n, n+1, \dots, 2n-1$, $t \in [0, 1[$, where $\alpha > 0$ is a positive number to be specified below. The state of the system, under the control sequence (14) and starting from the state $x(nT)$, is given by:

$$\begin{aligned} x(2n) &= F^n x(n) + \sum_{i=0}^{n-1} F^{n-1-i} G(\alpha a_i) \\ &= F^n x(n) + \alpha v. \end{aligned} \tag{15}$$

The particular choice of α which we use in the design is given by:

$$\alpha(Z_0^{n-1}) = \max_{i=1,\dots,M} \frac{\left\|F^{2n}\right\| + \delta + 1}{r_i v} \|r_i\| B(Z_0^{n-1}), \tag{16}$$

where $\delta > 0$ is a fixed but arbitrary constant, and the control sequence applied in Mode 2 is given by (14) with (13) and (16).

The following Lemma shows that any "large" initial state is transferred to the cone $\mathcal{C}_1$ at the end of Mode 2:

Lemma 3 *Consider the Modes 1 and 2 of the ISS controller over the time interval $[0, 2n[$. Suppose that $|x(0)| > KD$, where K is satisfies the conditions of Lemma 1. Then we have that:*

1. *$x(2n) \in \mathcal{C}_1$,*
2. *$|x(2n)| \geq |x(0)|$.*

Proof of Lemma 3: If $\alpha > 0$ is chosen so that all the inequalities:

$$\begin{aligned} r_1 F^n x(n) + \alpha r_1 v &> 0 \\ r_2 F^n x(n) + \alpha r_2 v &> 0 \\ &\vdots \\ r_M F^n x(n) + \alpha r_M v &> 0 \end{aligned} \tag{17}$$

are satisfied, then $x(2n) \in \mathcal{C}_1$. It can be verified that, if

$$\alpha > \alpha^* := \max_{i=1,\dots,M} \frac{\|r_i\| \left\|F^{2n}\right\| |x(0)|}{r_i v},$$

then all the inequalities are indeed satisfied. Since for our choice given in (16), and for those initial states so that $|x(0)| > KD$ we have that $B(Z_0^{n-1}) \geq |x(0)|$, it follows that $\alpha(Z_0^{n-1}) \geq \alpha^*$ and $x(2n) \in \mathcal{C}_1$.

For the second statement, we proceed as follows. Since $|v| = 1$ and $r_i v > 0, \forall i$, we have that $r_i v = |r_i v| \leq |r_i|\,|v| = |r_i|$. Together with $|x(0)| \leq B\left(Z_0^{n-1}\right)$, which follows because $|x(0)| \geq KD$, we have:

$$\begin{aligned}
\alpha\left(Z_0^{n-1}\right) &\geq \left(\left\|F^{2n}\right\|+1\right)\, B\left(Z_0^{n-1}\right) \qquad (18)\\
&\geq \left(\left\|F^{2n}\right\|+1\right)\, |x(0)| \;\geq\; \left|F^{2n}x(0)\right| + |x(0)|
\end{aligned}$$

and so

$$\begin{aligned}
|x(2n)| &= \left|\alpha\left(Z_0^{n-1}\right) v - F^{2n}x(0)\right| \qquad (19)\\
&\geq \left|\alpha\left(Z_0^{n-1}\right) v\right| - \left|F^{2n}x(0)\right| \;\geq\; |x(0)|
\end{aligned}$$

where the last inequality follows from (18) and $\left|\alpha\left(Z_0^{n-1}\right) v\right| = \alpha\left(Z_0^{n-1}\right)$. ∎

3.3 Mode 3:

We apply

$$u(k+t) = 0\,,\; k = 2n, 2n+1, \ldots, 3n-1\,,\; t \in [0,1[\,, \qquad (20)$$

measure $z(k), k = 2n, \ldots, 3n-1$, and then reconstruct the state at time step $3n$ using

$$\hat{x}(3n) = F^n \mathbf{O}(F,C)^{-1} Z_{2n}^{3n-1}\,.$$

Corollary 1 *Suppose that $x(2n) \in \mathcal{C}_1$ is such that $|x(2n)| > \lambda D$, where λ is as in Lemma 2, and let us compute the state estimate of $x(2n)$ as follows:*

$$\hat{x}(2n) = \mathbf{O}(F,C)^{-1} Z_{2n}^{3n-1} \qquad (21)$$

and introduce the notation $E := \hat{x}(2n) - x(2n)$. Then we have that the bound on the norm of the state estimation error E is $|E| \leq \left\|\mathbf{O}(F,C)^{-1}\right\| \sqrt{n} D$ (which is independent of $x(0)$).

Proof of Corollary 1: From the construction of $\mathcal{C}_1$ in Lemma 2 we have that if $|x(2n)| > \lambda D$, then $\operatorname{sign}(z(k)) = \operatorname{sign}(y(k)) = +1$ for $k = 2n, \ldots, 3n-1$. So, for all such k, $z(k) = y(k) = CF^{k-2n}x(2n) + d(k)$, and thus

$$\begin{aligned}
\hat{x}(2n) &= \mathbf{O}(F,C)^{-1} Z_{2n}^{3n-1}\\
&= \mathbf{O}(F,C)^{-1}\left[\mathbf{O}(F,C)x(2n) + D_{2n}^{3n-1}\right]\\
&= x(2n) + \mathbf{O}(F,C)^{-1} D_{2n}^{3n-1}\,, \qquad (22)
\end{aligned}$$

and the estimate is proved. ∎

3.4 Mode 4:

In this mode we fix an integer $N \geq n$ and steer the state of the system estimated at time $3n$, which we take to be $\hat{x}(3n)$, to the origin using the minimum energy control over the time interval $[3n, 3n+N-1[$. For simplicity, we take $N = n$, so that controls are computed using:

$$[u(3n)\;\ldots\; u(4n-1)]^T = -\mathbf{C}(F,G)^{-1} F^{2n}\hat{x}(2n)\,, \qquad (23)$$

where $\mathbf{O}(F,G)$ is the non-singular controllability matrix for the pair (F,G).

The ISS controller then consists of Modes 1-4 which are cyclically applied to the plant. (Thus, the controller that we obtain is periodic.)

The following lemmas are instrumental in the proof of the ISS property for the controller:

Lemma 4 *Suppose that the state $x(2n)$ is estimated with an error E, that is, $\hat{x}(2n) = x(2n) + E$. Then, if we apply the sequence of controls (23), we have the following bound:*

$$|x(4n)| \leq \left\|F^{2n}\right\| |E|\,.$$

Proof of Lemma 4: From $\hat{x}(3n) = F^n\hat{x}(2n) = F^n x(2n) + F^n E = x(3n) + F^n E$, and applying the control u from (23), we obtain $0 = x(4n) + F^{2n}E$. ∎

Corollary 2 *There exist positive numbers K_1, K_2, such that, if $|x(0)| > K_2 D$, then the closed-loop state satisfies $|x(4n, x(0))| \leq K_1 D$.*

Proof of Corollary 2: Choose $K_2 = \max(K, \lambda)$, where K is taken as in Lemma 1 and λ is taken from Lemma 2. From Lemma 3 it follows that if $|x(0)| > K_2 D$, then $|x(2n, x(0))| \geq |x(0)|$, since $K_2 \geq K$ and from Corollary 1 it follows that since $|x(2n, x(0))| > \lambda D$ we have that $|E| \leq \left\|\mathbf{O}(F,C)^{-1}\right\| \sqrt{n} D$. Finally, Lemma 4 guarantees that with $K_1 = \left\|F^{2n}\right\| \left\|\mathbf{O}(F,C)^{-1}\right\| \sqrt{n}$ the statement of corollary holds. ∎

Let us denote the motion of the continuous time system, under the closed loop action due to to the controller described in this section, as $x(t, x(0))$.

Lemma 5 *There exist positive numbers K_3, K_4 such that, for each initial state $x(0) \in \mathbb{R}^n$, we have that*

$$|x(t, x(0))| \leq K_3\,|x(0)| + K_4 D, \quad t \in [0, 4n[\,.$$

Proof of Lemma 5: Notice that for all Modes 1-4, the system is a linear system with different piecewise constant inputs. The following bound holds in continuous time for an arbitrary time interval $[t_1, t_2]$:

$$\begin{aligned}
|x(t, x(t_1))| &\leq \max_{t\in[t_1,t_2]} \left\|e^{At}\right\| |x(t_1)| \qquad (24)\\
&+ \left[\int_{t_1}^{t_2} \left\|e^{A(t_2-s)}\right\| \|B\|\; ds\right] \max_{t\in[t_1,t_2]} |u(t)|\,.
\end{aligned}$$

Also, notice that the control input for Modes 1 and 3 is identically equal to zero, whereas for Modes 2 and 4 it is computed based on the measurement vectors Z_0^{n-1} and Z_{2n}^{3n-1} respectively. Also, it is easy to see that we have, for suitable constants P_i:

$$\begin{aligned} \left|Z_0^{n-1}\right| &\leq P_1\,|x(0)| + P_2 D\,, \\ \left|Z_{2n}^{3n-1}\right| &\leq P_3\,|x(2nT)| + P_4 D\,, \end{aligned} \qquad (25)$$

from which one may derive bounds on the control $u(t)$ over the entire time interval $[0, 4n[$. Finally, using (25), we conclude as desired. ∎

Corollary 3 *There exists $K_5 > 0$ such that given an arbitrary $x(0) \in \mathbb{R}^n$, we have that $|x(4n, x(0))| \leq K_5 D$.*

Proof of Corollary 3: From Corollary 2 we have that if $|x(0)| > K_2 D$ then $|x(4n, x(0))| \leq K_1 D$. If on the other hand we have that $|x(0)| \leq K_2 D$, from Lemma 5 it follows that $|x(4n, x(0))| \leq (K_3 K_2 + K_4) D$. The claim follows with $K_5 = \max\{K_1, K_2 K_3 + K_4\}$. ∎

Corollary 4 *There exists $K_6 > 0$ such that given an arbitrary $x(0) \in \mathbb{R}^n$, we have that*

$$|x(t, x(0))| \leq K_6 D\,,\ t \geq 4n\,.$$

Proof of Corollary 4: From Corollary 3 it follows that $|x(4n)| \leq K_5 D$. Also, note that the bound given in Lemma 5 can be rewritten as follows:

$$|x(t, x(4nj))| \leq K_3\,|x(4nj)| + K_4 D$$

$t \in [4nj, 4nj+4n[$, $\forall\, j = 0, 1, \ldots$, and the proof follows by induction since we also have from Corollary 3 that

$$|x(4nj, x(4nj - 4n))| \leq K_5 D\,,\ \forall\, j = 1, \ldots\,.$$

So we may use $K_6 = K_3 K_5 + K_4$. ∎

Theorem 1 *The controller presented by Modes 1-4 is an ultimately-linear L^∞- gain controller for the system (2).*

Proof of Theorem 1: Just combine the bounds given in Corollary 4 and Lemma 5. ∎

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