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Point-wise Extremum Seeking Control Scheme Under Repeatable Control Environment

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Abstract— **In this paper, a point-wise extremum seeking control scheme is proposed to seek an optimal trajectory of a time-varying static map under a repeatable control environment. At each time instant, the extremum seeking scheme seeks an optimal value in the optimal trajectory. Under some conditions, it is shown that by tuning the controller parameters properly, the output of the system semi-globally practically uniformly converges to an arbitrarily small neighborhood of the optimal trajectory from an arbitrary initial trajectory. It also shows when a weaker assumption is used, the point-wise extremum seeking controller still ensures that the output converges to the neighborhood of the optimal trajectory when the initial trajectory is sufficiently close to the optimal one.**

I. Introduction

The objective of many control applications is to regulate the output to a desired reference value. In some applications, the desired reference value is unknown and is always given by an extremum of a static input-to-reference map. Extremum seeking control (ES) is a mechanism that can drive the output of the system to the unknown optimal value. This method dates back to the early 1950's and 1960's and there are many schemes available in literature [1], [2], [3], [4], [5]. There has been a renewed interest in this research area [6], [7], [8], [9], [10] recently that leads to numerous practical implementations of the scheme, as well as its better theoretical understanding.

In this note, for simplicity, a particular scalar extremum seeking scheme is considered. This simple scheme was first introduced in [10]. As can be seen in Figure 1, the output y and the state x are both scalar. The nonlinear map $h(\cdot)$ has an *unknown* unique global maximum y^*. The excitation signal $a\sin(t)$ is added to nonlinear mapping $h(\cdot)$ to get probing while the multiplication (modulation) of output and the excitation signal ($sin(t)$) extracts the gradient of the unknown mapping $h(\cdot)$. Given any initial condition x_0, by tuning parameters (a,ϵ) properly, as $t\to\infty$, the output of the system $y(t)$ approaches to an arbitrarily small neighborhood of the optimal value y^*.

However, there are many applications in which the nonlinear mapping h is time-varying or the optimal value is shifting with respect to time. For example, in chemical processes, the quality of some raw material may decay with time. Another example is the optimal traffic flow in a city which always varies with time. When the optimal value is time-varying instead of constant, the extremum seeking algorithm shown in Figure 1 cannot work well. In this paper, we focus on the design of the extremum seeking scheme that is able to find the time-varying optimal value (we call it optimal trajectory).

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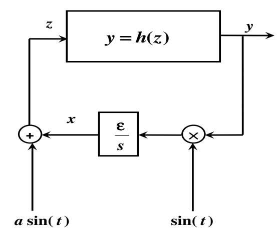

Fig. 1. A simple extremum seeking feedback scheme

In general, it is very difficult to solve an optimization problem that is time dependant. However, when a special problem is considered, for example, when the system is under a repeatable control environment, it is possible to seek the optimal trajectory. A repeatable control environment implies that the control task in a finite time interval $[0,T]$ repeats itself along the iteration domain [13].

Designing a controller under a repeatable control environment is frequently encountered in many industrial processes such as robot arms in assembly lines and chemical batch processes. Intuitively, the information obtained from the last trial can be used to improve the performance of the current trial. Iterative learning control (ILC) is well-known for its ability to improve the performance along the iteration domain (or trials) as it takes advantage of the repeatable control environment. ILC has been successfully applied in many engineering applications [11], [12], [13] with simple updating laws. For example a simple and widely used ILC updating law is

$$x_{i+1}(t)=x_i(t)+q_l(y_i(t)-y_d(t)), \tag{1}$$

where $y_d(t)$ is the desired output, $y_i(t)$ and $x_i(t)$ are the output and input at the i^{th}-trial respectively. q_l is the learning gain. By tuning q_l properly, the output of the system (static/dynamic) will converge to a time-varying desired trajectory $y_d(t)$. However, when the desired (optimal) trajectory is unknown, it is difficult to apply such a simple mechanism.

Although the problem setting for the ES and the ILC is different, they share some similarities. From Figure 1, there is an integrator in the ES diagram. On the other hand, most learning mechanisms also perform an integral action along the iteration domain as can be seen in (1). Moreover, the ES scheme is an adaptive mechanism, while learning algorithms are always treated as a "point-wise adaptation" [14]. When an optimal trajectory is considered, at each time instant along the iteration domain, it is an optimal *value*. By applying the ES along the iteration domain, such an optimal value could be sought.

This paper aims at finding an extremum seeking algorithm that can learn the each point along optimal trajectory. The point-wise extremum seeking scheme is thus proposed. That is, the extreme seeking algorithm performs at every time instant in the interval $[0, T]$. With the help of averaging technique in iteration domain, it is shown that under some conditions, the output of the system converges to an arbitrarily small neighborhood of the optimal trajectory by tuning the controller parameters appropriately no matter where the initial trajectory is. We also show that under a weaker condition, the output of the system can converge to an arbitrarily small neighborhood of the optimal trajectory by tuning the controller parameters provided that the initial trajectory is close to the optimal one.

This paper is organized as follows. In Section 2, the problem formulation and preliminaries are provided. Main results are given in Section 3, followed by a simple simulation result in Section 4. Section 5 draws conclusions. Proofs for propositions and lemmas are presented in the Appendix.

II. PROBLEM FORMULATION AND PRELIMINARIES

In this paper, the set of integers is denoted as $\mathcal{N}$. Let $\mathcal{B}[0, T]$ be the space of any bounded function on the finite time interval $[0, T]$ with the norm $\|x\|_{\mathcal{B}[0,T]} = \sup_{t\in[0,T]} |x(t)|$. Let $\mathcal{C}[0, T]$ be the space of any continuous function on the finite time interval $[0, T]$. Obviously $\mathcal{C}[0, T] \subset \mathcal{B}[0, T]$.

In this paper, for simplicity, we consider the following general nonlinear time-varying single-input-single-output static system.

$$y = h(t, x), \qquad t \in [0, T]. \tag{2}$$

The $h(\cdot, \cdot)$ is an unknown nonlinear mapping and smooth with respect to all its arguments. The system (2) is under a repeatable control environment.

Remark 1: When a dynamic system is considered, it is intuitively clear that if the dynamics dies out quickly, the results for static systems can be extended to dynamic systems with the help of the singular perturbation methodology. How to extend the point-wise extremum seeking scheme to dynamic systems will be our future work.

The system (2) satisfies the following assumptions.

Assumption 1: For each $t \in [0, T]$, there exists a unique $x^*(t)$ maximizing the output $y(t)$. That is, there exists a positive constant α_0 such that the following holds[1]:

$$\partial_x h(t, x^*(t)) = 0 \qquad \Longleftrightarrow \qquad x(t) = x^*(t) \tag{3}$$

$$\partial_{xx} h(t, x^*(t)) \le -\alpha_0. \tag{4}$$

[1]Without loss of generality we assume that the extremum is a maximum.

Moreover $x^*(t) \in \mathcal{C}[0, T]$.

Remark 2: Assumption 1 indicates that for each time instant t in the interval $[0, T]$, there exists a maximum for the nonlinear mapping $h(\cdot, t)$. When we fixed a time instant t_0, this assumption is exactly the same as in [10, Assumption 3]. There are cases where such a point-wise optimal trajectory $x^*(t)$ in Assumption 1 does not exist. In most of applications, the following weaker assumption is always satisfied.

Assumption 1a There exists a unique $x^*(t) \in C[0, T]$ maximizing $h(t, x)$, that is

$$h(t, x^*(t) + \alpha u(t)) \le h(t, x^*(t)), \tag{5}$$

where $\alpha \in R$ is a constant and $u(t) \in \mathcal{B}[0, T]$.

In order to apply the averaging technique, we have to choose a proper excitation signal such that the averaging along the whole Banach space $\mathcal{B}[0, T]$ is well-defined. The well-defined averaging means the information of a Banach space can be persistently excited by a properly chosen excitation signal. How to find such an excitation signal is a quite challenging. We are currently working on this problem. In this paper, Assumption 1 is used to simplify the design of the excitation signal.

Assumption 2: We have that (3) and (4) hold and there exists $\alpha_1 > 0$ such that the following holds

$$\partial_{xx} h(t, x^*(t) + \zeta(t)) \le -\alpha_1 < 0, \tag{6}$$

for all $\zeta(t) \in \mathcal{B}[0, T]$ and $\zeta(t) \neq 0, \forall t \in [0, T]$.

Remark 3: The inequality (6) in Assumption 2 seems strong. This assumption guarantees the non-local convergence of the point-wise extremum seeking (see Theorem 1). However, without Assumption 2, the point-wise extremum seeking controller still achieves local convergence (see Theorem 2).

The main idea of this paper is to extend the results in [10] to seek an optimal trajectory in the finite time interval $[0, T]$ under the repeatable control environment. Under Assumption 1, at each time instant t_0, there exists an optimal value $x^*(t_0)$. By applying an extremum seeking mechanism along the iteration domain, $x^*(t_0)$ can be found. In the sequel, the point-wise extremum seeking mechanism can learn every optimal value in the optimal trajectory, though the continuity of the output may not be guaranteed due to the point-wise adaptation.

For the system (2), the following point-wise updating law is considered.

$$\begin{aligned} x_{i+1}(t) &= x_i(t) + \epsilon h(t, x_i(t) + a\sin(\omega\pi i))\sin(\omega\pi i) \\ &= x_i(t) + \epsilon f(i, t, x_i(t)), \quad x_0(t) \in \mathcal{C}[0, T], \forall i \in \mathcal{N}, \end{aligned} \tag{7}$$

where ϵ is a small positive constant, $\frac{2}{\omega} = M$ is an integer. $\sin(\omega\pi i)$ is an excitation signal along the iteration domain. It is apparent that the nonlinear smooth function $f(\cdot, \cdot, \cdot)$ satisfies the following condition

$$f(i + M, t, x(t)) = f(i, t, x(t)), \qquad \forall t \in [0, T]. \tag{8}$$

The following definitions and propositions are used in this note.

Definition 1: Consider an iterative process shown as follows

$$x_{i+1}(t) = f(i, t, x_i(t)), \qquad x_0(t) \in \mathcal{C}[0, T] \tag{9}$$

where the nonlinear function $f(\cdot, \cdot, \cdot)$ is smooth with respect to its all arguments. $x_i(t)$ is called a uniformly convergent process, if the following holds. For any $\delta > 0$, there exists a positive integer N (dependent only on δ) such that, for all $i \geq N$, the following inequality holds

$$|x_i(t)| < \delta, \qquad \forall t \in [0, T]. \tag{10}$$

Remark 4: If an iterative process is uniformly convergent, it converges to a continuous function $x(t) \equiv 0$ for all $t \in [0, T]$. Moreover, for all $i \in \mathcal{N}$, we have $x_i(t) \in \mathcal{C}[0, T]$.

Definition 2: The iterative process (9) is uniformly bounded for all $t \in [0, T]$, if $x_i(t) \in \mathcal{B}[0, T]$ for all $i \in \mathcal{N}$. Namely there exists a positive constant Δ such that

$$\|x_i\|_{\mathcal{B}[0,T]} \leq \Delta, \qquad \forall\ i \in \mathcal{N}. \tag{11}$$

From Definition 1, it is obvious that a uniform convergent iterative process is uniformly bounded. The uniform convergent process is highly desirable because at each iteration, $x_i(t)$ is continuous and uniformly bounded. However, there are some applications when the uniform convergence property is hard to achieve. Therefore, a weaker convergence process is defined as follows.

Definition 3: The iterative process (9) is practically uniformly convergent for all $t \in [0, T]$ if there exists a positive pair (Δ, ν), where $\Delta \geq \nu$ such that the following holds

$$\|x_i\|_{\mathcal{B}[0,T]} \leq \Delta, \forall\ i \in \mathcal{N} \quad \text{and} \quad \limsup_{i \to \infty} \|x_i\|_{\mathcal{B}[0,T]} < \nu. \tag{12}$$

The relationship between a uniform bounded process and a practically uniformly convergent process is given in the following proposition.

Proposition 1: Consider the following iterative process

$$x_{i+1}(t) = \alpha x_i(t) + \beta_i(t), \qquad x_0(t) \in \mathcal{C}[0, T] \tag{13}$$

If $\alpha \in (0, 1)$ and $\beta_i \in \mathcal{B}[0, T]$, then this iterative process is practically uniformly convergent.

Proof: See Appendix.

Definition 4: Consider a parameterized iterative process:

$$x_{i+1}(t) = f(i, t, x_i(t), \epsilon_1, \epsilon_2, \ldots, \epsilon_\ell), \qquad x_0(t) \in \mathcal{C}[0, T] \tag{14}$$

where the nonlinear mapping $f(\cdot, \cdot, \cdot, \cdot, \cdot, \cdots, \cdot)$ is smooth with respect to all arguments. ϵ_j is a positive constant for any $j = 1, 2, \ldots, \ell$. The process (14) is said to be semi-globally practically uniformly convergent (SPUC) in $[\epsilon_1, \epsilon_2, \cdots, \epsilon_\ell]$, if for a strictly positive pair (Δ, ν), there exists $\epsilon_j^*(\Delta, \nu) > 0$ and for any $\epsilon_j \in (0, \epsilon_j^*(\Delta, \nu))$, $j = 1, \ldots, \ell$, the process (14) is practically uniformly convergent for all $\|x_0\|_{\mathcal{B}[0,T]} \leq \Delta$.

Remark 5: Definition 4 is different from [10, Definition 1] since the convergence speed is neglected. Definition 1 in [10] showed that by choosing smaller parameters $[\epsilon_1, \epsilon_2, \cdots, \epsilon_\ell]$, we have a larger Δ and a smaller ν, however, a much slower convergence speed. In Definition 4, the convergence speed does not appear implicitly since it is not an easy task to analyze the convergence speed for an iterative process. It is intuitively clear that by choosing smaller parameters $[\epsilon_1, \epsilon_2, \cdots, \epsilon_\ell]$, the convergence speed will be slower. Even though the convergence speed at each time instant can be computed, it is hard to evaluate analytically a *uniform* convergence speed for a convergent process in the finite time interval since there are cases in which a uniform convergence speed does not exist.

Definition 5: The process (14) is said to be locally practically uniformly convergent (LPUC) in $[\epsilon_1, \epsilon_2, \cdots, \epsilon_\ell]$, if the following holds. For a strictly positive real number ν, there exists $\epsilon_j^*(\nu) > 0$ and $\Delta^*(\nu) > 0$ and for any $\epsilon_j \in (0, \epsilon_j^*(\nu))$, $j = 1, \ldots, \ell$ and any $\Delta \in (0, \Delta^*(\nu))$, (12) holds true for all $\|x_0\|_{\mathcal{B}[0,T]} \leq \Delta$.

Remark 6: Let us compare a semi-globally practically uniformly convergent process (see Definition 4) and a locally practically uniformly convergent process (see Definition 5). A SPUC process implies that by tuning parameters $[\epsilon_1, \epsilon_2, \cdots, \epsilon_\ell]$, the process will converge to an arbitrary ν-neighborhood of $x(t) \equiv 0$ uniformly when starting from an arbitrary initial trajectory. A LPUC process implies that the uniform convergence can be guaranteed only when the initial trajectory is close to $x(t) \equiv 0$. Since a SPUC property is stronger than a LPUC property, it is natural that a stronger assumption (Assumption 2) is needed.

Next, an auxiliary average for the point-wise updating law (7) is introduced as follows.

$$\bar{x}_{i+1}(t) = \bar{x}_i(t) + \epsilon f_{av}(t, \bar{x}_i(t)), \bar{x}_0(t) = x_0(t) \in \mathcal{C}[0, T], \tag{15}$$

where

$$f_{av}(t, x(t)) \triangleq \frac{1}{M} \sum_{i=0}^{M-1} f(i, t, x(t)).$$

Proposition 2: Consider the system (7) and its average system (15). If the averaged system (15) is uniformly bounded, then for any positive pair (Δ, ν), there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, we have

$$|x_i(t) - \bar{x}_i(t)| \leq \nu, \qquad \forall t \in [0, T] \ \text{ and } \ \forall i \in \mathcal{N}, \tag{16}$$

for all $\|x_0 - \bar{x}_0\|_{\mathcal{B}[0,T]} \leq \Delta$.

Proof: See Appendix.

Remark 7: Proposition 2 shows a kind of closeness of solution between the trajectory of the actual system (7) and the trajectory of average (15) in the finite time interval $[0, T]$. It is an extension of results in [15] along the iteration domain. Moreover, Proposition 2 implies that the properties of the actual system (7) can be obtained from the properties of the average system (15).

III. Main Results

In this section, our main results will be stated. For convenience, we denote

$$\delta x_i(t) \quad = \quad x_i(t) - x^*(t). \tag{17}$$

The first main result is presented as follows

Theorem 1: Suppose that Assumptions 1 and 2 hold. Then, the iterative process $\delta x_i(t)$, where $\delta x_i(t)$ is from (17) and $x_i(t)$ is from (7), is semi-globally practically uniformly convergent in $[a, \epsilon]$.

Remark 8: The result of Theorem 1 extends that of [10, Corollary 1]. Corollary 1 in [10] showed that the output of a static time-invariant nonlinear system with the ES mechanism semi-globally practically asymptotically converges to a ν-neighborhood of the optimal value y^* from an arbitrarily initial value. By a point-wise extremum seeking algorithm, Theorem 1 states that output *trajectory* of a static *time-varying* nonlinear system with the ES mechanism along the iteration domain converges uniformly to a ν-neighborhood of the optimal *trajectory* $y^*(t)$ from an arbitrarily initial *trajectory*.

Proof: The proof consists three steps.

Step 1. Lemma 1 shows that the average system of (17) takes the following form

$$\begin{aligned}\delta\bar{x}_{i+1}(t) &= (1+q(a,\epsilon,x^*(t),\delta\bar{x}_i(t),\bar{\mu}_i))\,\delta\bar{x}_i(t)\\ &\quad+\frac{a^3}{3!}g(t,x^*(t)),\end{aligned}\tag{18}$$

where $\bar{\mu}_i \in (0,1)$ and

$$q=\frac{a}{2}\partial_{xx}h(t,x^*+\mu_i\delta\bar{x}_i)+\frac{a^3\epsilon}{3!}g(t,x^*+\bar{\mu}_i\delta\bar{x}_i).\tag{19}$$

Step 2. Lemma 2 shows that the above average system is uniformly bounded.

Step 3. Conclusion holds true by applying Proposition 2. ■

Remark 9: The proof of Theorem 1 is quite similar to the proof of [16, Theorem 1]. By showing that the average system (18) is uniformly bounded, the closeness of the solutions between the average system (18) and the actual system (17) from Proposition 2 ensures that the SPUC properties of the actual system (17).

As indicated in Remark 6, the second main result shows that when only Assumption 1 holds, a weaker convergence property is obtained.

Theorem 2: Suppose that Assumptions 1 holds. Then, the iterative process $\delta x_i(t)$, where $\delta x_i(t)$ is from (17) and $x_i(t)$ is from (7), is locally practically uniformly convergent in $[a,\epsilon]$.

Sketch of Proof: Using the same computation technique, it follows that

$$\begin{aligned}&f_{av}(t,\bar{x}_i(t))\\ =&\ \frac{a}{2}\partial_x h(t,\bar{x}_i(t))+\frac{a^3}{3!}g(t,\bar{x}_i(t))\\ =&\ \frac{a}{2}\partial_x h(t,x^*(t))+\frac{a}{2}\partial_{xx}h(t,x^*(t))\delta\bar{x}_i(t)\\ &+\frac{a}{2\cdot 3!}\partial_{xxx}h(t,x^*(t)+\bar{\mu}_i\delta\bar{x}_i)\left(\delta\bar{x}_i(t)\right)^2\\ &+\frac{a^3}{3!}\left[g(t,x^*(t))+g(t,x^*(t)+\bar{\mu}_i\delta\bar{x}_i(t))\right]\delta\bar{x}_i(t),\end{aligned}\tag{20}$$

with the following average system

$$\begin{aligned}\delta\bar{x}_{i+1}(t) &= (1+q_1(a,\epsilon,x^*(t),\delta\bar{x}_i(t),\bar{\mu}_i))\,\delta\bar{x}_i(t)\\ &\quad+\frac{a\epsilon}{12}\partial_{xxx}h(t,x^*(t)+\bar{\mu}_i\delta\bar{x}_i)\left|\delta\bar{x}_i(t)\right|^2\\ &\quad+\frac{a^3}{3!}g(t,x^*(t)),\end{aligned}\tag{21}$$

where

$$q_1=\frac{a}{2}\partial_{xx}h(t,x^*(t))+\frac{a^3\epsilon}{3!}g(t,x^*(t)+\bar{\mu}_i\delta\bar{x}_i(t))$$

If ignoring the difference between q and q_1[2], compared with the iterative process (18), there is an additional term

$$\frac{a\epsilon}{12}\partial_{xxx}h(t,x^*(t)+\bar{\mu}_i\delta\bar{x}_i)\left|\delta\bar{x}_i(t)\right|^2$$

appearing in the iterative process (21). The boundedness and the convergence of the iterative process (21) can be guaranteed provided that the initial trajectory is quite close to the optimal one $x^*(t)$, that is, $\|\delta x_0\|_{\mathcal{B}[0,T]}$ is sufficiently small. Or in other words, we show the iterative process (7) is locally practically uniformly convergent. ■

Remark 10: Theorems 1 and 2 are quite preliminary results. The main purpose of this paper is to extend the concept of the extremum seeking to the design of learning control. There are many interesting applications in the learning control when optimization is considered. How to provide a more general framework for optimization problems under repeatable control environment will be our future work.

IV. A Simple Simulation Result

Due to space limitation, we just consider a very simple nonlinear time-varying function

$$y=-(x-t)^2,\quad \forall t\in[0,T],\tag{22}$$

where $x^*(t)=t$. Obviously, both Assumption 1 and Assumption 2 are satisfied in this simple example.

First, we choose $T=1$, $x_0(t)=0$, for all $t\in[0,T]$, $a=0.1$ and $\epsilon=0.01$, it is shown in Figure 2 that the iterative process $\|\delta x_i\|_{\mathcal{B}[0,1]}$ is semi-globally practically uniformly convergent.

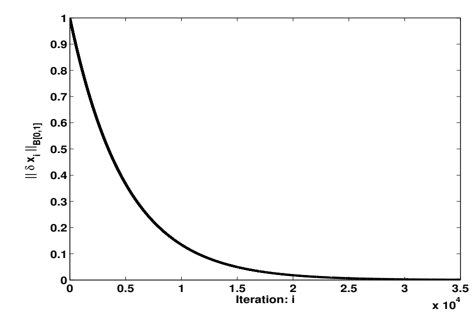

Fig. 2. The performance of the point-wise extremum seeking scheme.

When a larger interval is chosen, $T=10$, keeping $x_0(t)=0,\forall t\in[0,T]$ and $a=0.1$, a smaller $\epsilon=0.0001$ has to be selected in order to achieve the convergence. It is shown in Figure 3 that the iterative process is still semi-globally practically uniformly convergent. Due to a smaller ϵ, the learning process converges much slower.

[2]The difference can be ignored by choosing a and ϵ properly.

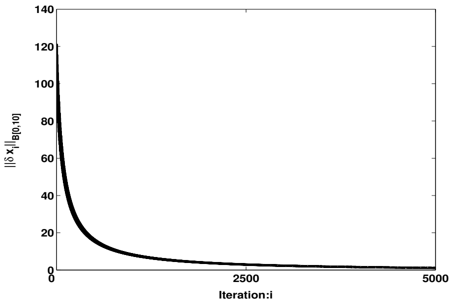

Fig. 3. The performance of the point-wise extremum seeking scheme.

V. CONCLUSION

In this paper, a point-wise extremum seeking controller is proposed to find every optimal value of an optimal trajectory under the repeatable control environment. It shows that given any initial trajectory, by tuning the parameters of the point-wise extremum seeking controller, the output of the system can converge to the optimal trajectory. A simple example illustrates our results.

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APPENDIX

Sketch of Proof of Proposition 1. Since $x_0(t) \in \mathcal{C}[0,T]$, there exists a positive constant Δ such that $\|x_0\|_{\mathcal{B}[0,T]} \leq \Delta$. Assume that $\|\beta_i\|_{\mathcal{B}[0,T]} \leq \beta$, where β is a positive constant. At $(i+1)^{th}$ iteration, it follows that

$$
\begin{aligned}
|x_{i+1}(t)| &\leq \alpha|x_i(t)| + \beta \leq \alpha^{i+1}\|x_0(t)\|_{\mathcal{B}[0,T]} + \sum_{k=0}^{i} \alpha^k \beta \\
&\leq \alpha^{i+1}\Delta + \beta \sum_{k=0}^{i} \alpha^k.
\end{aligned}
$$

Taking the norm on both sides yields $\|x_{i+1}\|_{\mathcal{B}[0,T]} \leq \alpha^{i+1}\Delta + \beta \sum_{k=0}^{i} \alpha^k$. Note that $\limsup_{i\to\infty} \|x_i\|_{\mathcal{B}[0,T]}$ exists for the bounded process $\|x_i\|_{\mathcal{B}[0,T]}$. In the sequel, we have

$$
\limsup_{i\to\infty} \|x_i\|_{\mathcal{B}[0,T]} \leq \frac{\beta}{1-\alpha} = \nu,
$$

which completes the proof. ■

Proof of Proposition 2. We prove it by Induction.

1) At the first iteration, we have

$$
\begin{aligned}
& |x_1(t) - \bar{x}_1(t)| \\
= \;& \epsilon\left[|x_0(t) + f(0,t,x_0(t)) - \bar{x}_0(t) - f_{av}(t,x_0(t))|\right] \\
\leq \;& \epsilon\left\{\left|\frac{1}{M}\sum_{j=0}^{M-1}\left[f(j,t,x_0(t)) - f(0,t,x_0(t))\right]\right|\right\} \\
& +\epsilon\|x_0 - \bar{x}_0\|_{\mathcal{B}[0,T]} \\
\leq \;& \epsilon\left\{\frac{1}{M}\sum_{j=0}^{M-1}|\partial_j f(M_j,t,x_0(t))j| + \Delta\right\},
\end{aligned}
$$

where $M_j = \zeta j$, where $\zeta \in [0,1]$. Note that $x_0(t) \in \mathcal{C}[0,T]$ and M_j is bounded. The nonlinear function $f(\cdot,\cdot,\cdot)$ is smooth with respect to all its arguments. It ensures that $|\partial_j f(M_j,t,x_0(t))j|$ is bounded by a positive constant α_{f_0}. For any given positive pair (Δ,ν), there exists $\epsilon_1^* > 0$ such that for any $\epsilon \in (0,\epsilon^*)$ (16) holds true for any $\|x_0 - \bar{x}_0\|_{\mathcal{B}[0,T]} \leq \Delta$.

2) Assume that at k^{th} iteration, given any positive pair (Δ,ν) there exists $\epsilon_2^* > 0$ such that (16) holds true for any $\|x_0 - \bar{x}_0\|_{\mathcal{B}[0,T]} \leq \Delta$. At $(k+1)^{th}$ iteration, we have

$$
\begin{aligned}
& |x_{k+1}(t) - \bar{x}_{k+1}(t)| \\
= \;& \epsilon\left|x_k(t) + f(k,t,x_k(t)) - \bar{x}_k(t) - f_{av}(t,\bar{x}_k(t))\right| \\
\leq \;& \epsilon\left|\frac{1}{M}\sum_{j=0}^{M-1} f(k+j,t,\bar{x}_k(t)) - f(k,t,\bar{x}_k(t))\right| \\
& +\epsilon\left|f(k,t,x_k(t)) - f(k,t,\bar{x}_k(t))\right| + \epsilon\nu \\
\leq \;& \epsilon\left\{\frac{1}{M}\sum_{j=0}^{M-1}|\partial_j f(M_j,t,\bar{x}_k(t))j|\right\} + \epsilon\nu \\
& +\epsilon\left|\partial_x f(k,t,\bar{x}_k(t) + \zeta(x_k(t) - \bar{x}_k(t))\right|\nu,
\end{aligned}
$$

here $M_j = \zeta j$, where $\zeta \in [0,1]$, is bounded. Note that the average system (15) is uniformly bounded, i.e., $\|\bar{x}_k\|_{\mathcal{B}[0,T]}$ is bounded, for all $k \in \mathcal{N}$ and $t \in [0,T]$. $|\partial_j f(M_j, t, \bar{x}_k(t))j|$ can be bounded by α_{f_1} while $|\partial_x f(k, t, \bar{x}_k(t) + \zeta(x_k(t) - \bar{x}_k(t))|$ can be bounded by α_{f_2}. It leads to the following inequality

$$|x_{k+1}(t) - \bar{x}_{k+1}(t)| \quad \le \quad \epsilon\left(\alpha_{f_1} + \nu + \nu\alpha_{f_2}\right).$$

Therefore, given any positive constant ν, there exists $\epsilon^* > 0$, which is smaller than ϵ_1^*, such that for all $\epsilon \in (0, \epsilon^*)$, (16) holds true for any $\|x_0 - \bar{x}_0\|_{\mathcal{B}[0,T]} \le \Delta$. It completes the proof. ■

Lemma 1: The average of (7) exists and takes the form as in (18).

Proof: Substituting (17) into the updating law (7) yields

$$\begin{aligned}
& \delta x_{i+1} \\
= \; & \delta x_i + \epsilon h(t, x_i(t) + a\sin(\omega\pi i))\sin(\omega\pi i) \\
= \; & \delta x_i(t) + \epsilon h(t, x_i(t))\sin(\omega\pi i) \\
& + \epsilon\left[\partial_x h(t, x_i(t)) a \sin^2(\omega\pi i)\right] \\
& + \frac{\epsilon}{2}\left[\partial_{xx} h(t, x_i(t)) a^2 \sin^3(\omega\pi i)\right] \\
& + \frac{\epsilon}{3!}\left[\partial_{xxx} h(t, x_i(t) + a\mu_i \sin(\omega\pi i)) a^3 \sin^4(\omega\pi i)\right] \\
\triangleq \; & \delta x_i(t) + \epsilon f(i, t, x_i(t)), \qquad (23)
\end{aligned}$$

where $\mu_i \in (0,1)$ and

$$\begin{aligned}
& f(i, t, x(t)) \\
\triangleq \; & \epsilon\left[h(t, x(t))\sin(\omega\pi i) + \partial_x h(t, x(t)) a \sin^2(\omega\pi i)\right] \\
& + \frac{\epsilon}{2}\left[\partial_{xx} h(t, x(t)) a^2 \sin^3(\omega\pi i)\right] \\
& + \frac{\epsilon}{3!}\left[\partial_{xxx} h(t, x(t) + a\mu_i \sin(\omega\pi i)) a^3 \sin^4(\omega\pi i)\right].
\end{aligned}$$

Obviously, we have

$$f(i, t, x(t)) = f(i + M, t, x(t)), \qquad \forall t \in [0,T]. \qquad (24)$$

For every $t \in [0,T]$ and $\delta\bar{x}_0(t) = x_0(t) - x^*(t)$, we introduce an auxiliary average system for (23) along the iteration domain as follows

$$\begin{aligned}
\delta\bar{x}_{i+1}(t) &= \delta\bar{x}_i(t) + \epsilon f_{av}(t, \bar{x}_i(t)) \\
f_{av}(t, \bar{x}_i(t)) &= \frac{1}{M}\sum_{j=0}^{M-1} f(j, t, \bar{x}_i(t)).
\end{aligned}$$

Applying Assumption 1, noting that $\sum_{j=0}^{M-1}\sin(\omega\pi j) = 0$ and $\sum_{j=0}^{M-1}\sin^3(\omega\pi j) = 0$, a simple calculation yields

$$\begin{aligned}
& f_{av}(t, \bar{x}_i(t)) \\
= \; & \frac{a}{2}\partial_x h(t, \bar{x}_i(t)) + \frac{a^3}{3!} g(t, \bar{x}_i(t)) \\
= \; & \frac{a}{2}\partial_x h(t, x^*(t)) + \frac{a}{2}\partial_{xx} h(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_i(t))\delta\bar{x}_i(t) \\
& + \frac{a^3}{3!} g(t, x^*(t)) + \frac{a^3}{3!} g(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_i(t))\delta\bar{x}_i(t) \\
= \; & \frac{a}{2}\left[q(a, \epsilon, x^*(t), \delta\bar{x}_i(t), \bar{\mu}_i)\right]\delta\bar{x}_i(t) + \frac{a^3}{3!} g(t, x^*(t)), \; (25)
\end{aligned}$$

where q is given in (19), $\bar{\mu}_i \in [0,1]$ for all $i = 1, 2, \ldots$ and $g(t,x) \triangleq \frac{1}{M}\sum_{j=0}^{M-1}\left\{\partial_{xxx} h(t, x + a\mu_j \sin(\omega\pi j))\sin^4(\omega\pi j)\right\}$.

Consequently, the average system takes the form as in (18). ■

Lemma 2: The averaging process (18) is uniformly bounded.

Proof: Taking the absolute value of (18) at each time instant t yields the following inequality

$$\begin{aligned}
& |\delta\bar{x}_{i+1}(t)| \\
\le \; & \left(1 - \frac{a\epsilon}{2}\alpha_1 + \left|\frac{a^3\epsilon}{6} g(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_i(t))\right|\right)|\delta\bar{x}_i(t)| \\
& + \frac{a^3}{3!}|g(t, x^*(t))|, \qquad (26)
\end{aligned}$$

where α_1 comes from Assumption 2 (6). Note that $g(\cdot,\cdot)$ is a continuous function with respect to all it arguments and $x^*(t) \in \mathcal{C}[0,T]$, there exists a positive constant B_{g_0} such that $|g(t, x^*(t))| \le B_{g_0}$.

We prove the result by Induction.

- When $i = 1$. Since both $x_0(t)$ and $x^*(t)$ belong to $\mathcal{C}[0,T]$, $\delta x_0(t) \in \mathcal{C}[0,T]$. Assume $\|\delta x_0\|_{\mathcal{B}[0,T]} \le \Delta$. The smoothness of $g(t,x)$ and $\partial_{xxx} h(t,x)$ ensures that $|g(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_0(t)|$ is bounded by a positive constant B_g and $|\partial_{xxx} h(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_0)|$ is bounded by a positive constant B_3 for all $\|\delta\bar{x}_0\|_{\mathcal{B}[0,T]} \le \Delta$. It follows that

  $$|\delta\bar{x}_1(t)| \quad \le \quad \left(1 - \frac{a\epsilon}{2}\alpha_1 + \frac{a^3\epsilon}{6}B_g\right)\Delta + \frac{a^3}{12}B_{g_0}$$

  It is apparent that there exist constants ϵ_0^* and a_0^* such that for all $\epsilon \in (0, \epsilon_0^*)$ and all $a \in (0, a_0^*)$ such that there exists $\gamma \in (0,1)$, the following inequalities hold

  $$\begin{aligned}
  \frac{a\epsilon}{2}\alpha_1 - \frac{a^3\epsilon}{6}B_g &> \gamma \qquad (27) \\
  \frac{a^3}{12}B_{g_0} &< \gamma\Delta, \qquad (28)
  \end{aligned}$$

  leading to $|\delta\bar{x}_1(t)| < \Delta$ for every $t \in [0,T]$, i.e., $\|\delta\bar{x}_1\|_{\mathcal{B}[0,T]} < \Delta$.
- Assume that at k^{th} iteration, we have $\|\delta\bar{x}_k\|_{\mathcal{B}[0,T]} \le \Delta$. $|g(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_k(t)|$ and $|\partial_{xxx} h(t, x^*(t) + \bar{\mu}_i\delta\bar{x}_k)|$ can be bounded by B_g and B_3 respectively for all $\|\delta\bar{x}_k\|_{\mathcal{B}[0,T]} \le \Delta$. At $(k+1)^{th}$ iteration, a simple computation yields

  $$|\delta\bar{x}_{i+1}(t)| \le \left(1 - \frac{a\epsilon}{2}\alpha_1 + \frac{a^3\epsilon}{6}B_g\right)\Delta + \frac{a^3}{3!}|g(t, x^*(t))|,$$

  for all $t \in [0,T]$. Therefore, for all $a \in (0, a_0^*)$ and $\epsilon \in (0, \epsilon_0^*)$, (27) and (28) hold. Consequently we have

  $$|\delta\bar{x}_{i+1}(t)| < \Delta, \forall t \in [0,T] \Rightarrow \|\delta\bar{x}_{i+1}\|_{\mathcal{B}[0,T]} < \Delta,$$

which completes the proof. ■