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A framework for the observer design for networked control systems

Romain Postoyan and Dragan Nešić

Abstract—This paper provides a general framework for the observer design for networked control systems (NCS) affected by disturbances, via an emulation-like approach. The proposed model formulation allows us to consider various static and dynamic time-scheduling protocols, in-network processing implementations and encompasses sampled-data systems as a particular case. Provided that the continuous-time observer is robust to measurement errors (in an appropriate sense) we derive bounds on the maximum allowable transmission interval (MATI) that ensure the convergence of observation errors under network-induced communication constraints. The stability analysis is trajectory-based and utilizes small-gain arguments. It has to be emphasized that this paper also furnishes new tools for the observer design for sampled-data systems. Various observers can be combined and used within our approach to obtain estimators for NCS.

I. INTRODUCTION

Networked control systems (NCS) are systems in which sensors and actuators are spatially distributed and communicate with the control structure via a network. At each transmission instant, only a subset of sensors and/or actuators, collocated into a network *node*, is chosen to transmit their data over the communication channel according to a scheduling rule called *protocol*. The growing interest for NCSs is motivated by the fact that they have lower costs, easier maintenance and installation, greater flexibility as well as lower weight and volume in comparison to the classical control systems. On the other hand, communication constraints induced by the use of a serial channel cannot be ignored and require novel observation and control design methods.

In this paper, we focus on the observer design problem for NCS. Related available studies in the literature have only addressed this problem for particular classes of systems or protocols. In [3], a methodology for the mutual design of protocol and observer gains is developed for linear systems. Considering a weighted dynamic protocol, sufficient conditions are expressed in terms of matrix inequalities that ensure the existence of an observer-protocol pair that asymptotically reconstructs the plant states for a given *maximum allowable transmission interval* (MATI). The observer design originally developed for sampled-data systems in [8] has been extended to NCS in [15]. Assuming a continuous-time observer is known and satisfies some robustness properties with respect

to output disturbances, observation error convergence is ensured under network-induced constraints by replacing the unavailable continuous-time output by an auxiliary variable that flows along the same vector fields between transmission instants and is reset when measurements are received. The existence of a round-robin protocol that preserves the observability of linear discrete-time NCS is ensured in [17] under mild conditions. Using linear time-varying periodic systems analysis tools, a linear observer is then derived for such a protocol. Stochastic protocols and observers that minimize an upper bound on the estimation error covariance are synthesized for linear discrete-time systems in [4].

In this study, we propose a framework for the observer design for nonlinear NCS via an emulation-like approach, for plants whose dynamics are affected by disturbances. Modeling the problem like in [14], our approach allows to study various types of observers, time-scheduling protocols and in-network processing implementations. The stability analysis is trajectory-based and carried out using small-gain arguments that allow to derive easy computable bounds on MATI. We believe that this is the first time that this problem is addressed with such generality and that our model formulation can be the starting point for other observer designs. An important remark is that our framework is also new for the observer design for sampled-data systems. In that way, we provide an alternative to [2] where a framework for the observer design for sampled-data systems is developed based on discrete-time approximate models. Compared to [15], we allow larger classes of protocols and in-network processing implementations and consider perturbed plants.

The paper is organized as follows. After having defined the notations and recalled some stability definitions in Section II, the problem is stated in Section III and a general model formulation is developed. Main results are given in Section IV and applied to linear observers in Section V.

II. PRELIMINARIES

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_{\geq 0} = [0, \infty)$, $\mathbb{R}_{> 0} = (0, \infty)$, $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_{> 0} = \{1, 2, \dots\}$. Let $a \in \mathbb{R}_{> 0} \cup \{\infty\}$, a function $\gamma : [0, a) \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero and strictly increasing. By extension, for $a, b \in \mathbb{R}_{> 0} \cup \{\infty\}$, $\gamma : [0, a) \times [0, b) \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}\mathcal{K}$ if, for any $(s_1, s_2) \in [0, a) \times [0, b)$, $\gamma(s_1, \cdot)$ and $\gamma(\cdot, s_2)$ are of class $\mathcal{K}$. A continuous function $\gamma : [0, a) \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}\mathcal{L}$ if for each $t \in \mathbb{R}_{\geq 0}$, $\gamma(\cdot, t)$ is of class $\mathcal{K}$, and, for each $s \in [0, a)$, $\gamma(s, \cdot)$ is decreasing to zero, moreover γ is said to be of class $\exp - \mathcal{K}\mathcal{L}$ if there exist $\lambda_1, \lambda_2 \in \mathbb{R}_{> 0}$ such that $\gamma(s, t) = \lambda_1 \exp(-\lambda_2 t)s$, for $(s, t) \in [0, a) \times \mathbb{R}_{\geq 0}$. The initial time is denoted $t_0 \in \mathbb{R}_{\geq 0}$ and the initial condition of

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a variable x is denoted $x_0 = x(t_0)$. Considering a function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, $n \in \mathbb{Z}_{>0}$, the notation $f(t^+)$ is used to denote $\lim_{s \rightarrow t^+} f(s)$, $t \in \mathbb{R}_{\geq 0}$ (if it exists). The Euclidean norm of a vector or a matrix is denoted by $\|\cdot\|$. Let $f : \mathbb{R} \rightarrow \mathbb{R}^n$, $n \in \mathbb{Z}_{>0}$, be a (Lebesgue) measurable function and define, for $t_1 \leq t_2 \in \mathbb{R}$, $\|f\|_{[t_1, t_2]} = \text{ess. sup}_{\tau \in [t_1, t_2]} |f(\tau)|$ and $\|f\|_\infty = \text{ess. sup}_{\tau \in [t_0, \infty)} |f(\tau)|$. The set $\mathcal{L}_\infty^n$ denotes the set of functions $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ such that $\|f\|_\infty < r$, $r \in \mathbb{R}_{>0}$. For $(x, y) \in \mathbb{R}^{n+m}$, the notation (x, y) stands for $[x^T, y^T]^T$.

Consider system:

$$\dot{x} = f(x, u) \quad (1)$$

$$y = h(x) \quad (2)$$

where $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, $u \in \mathbb{R}^{n_u}$ are, respectively, the state, the output and the input, $n_x, n_y, n_u \in \mathbb{Z}_{>0}$.

Definition 1. System (1)-(2) is input-to-output stable (IOS) from u to y with gain γ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any $x_0 \in \mathbb{R}^{n_x}$, $u \in \mathcal{L}_\infty^{n_u}$:

$$|y(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|u\|_\infty) \quad \forall t \geq t_0 \geq 0. \quad (3)$$

If $y = x$, then system (1) is input-to-state stable (ISS) w.r.t. u . Moreover if γ is a linear function, we say that system (1)-(2) is IOS/ISS with a linear gain γ .

Definition 2. System (1) is said to be bounded-input-bounded-state (BIBS) with input u if there exist $\alpha, \gamma \in \mathcal{K}$, such that, for any $x_0 \in \mathbb{R}^{n_x}$, $u \in \mathcal{L}_\infty^{n_u}$:

$$|x(t)| \leq \alpha(|x_0|) + \gamma(\|u\|_\infty) \quad \forall t \geq t_0 \geq 0. \quad (4)$$

When no input acts on system (1) (i.e. $u = 0$), we say that the system is globally stable (GS).

III. PROBLEM STATEMENT

A. System models

The analysis follows the emulation approach adopted for the control of NCS in [16], [14]. The approach consists in first solving the problem while ignoring communication constraints, that is to design an observer for the continuous-time plant. Thus, for plants modeled by equations:

$$\dot{x} = f_P(x, w) \quad (5)$$

$$y = h_P(x), \quad (6)$$

where $x \in \mathbb{R}^{n_x}$ is the plant state, $y \in \mathbb{R}^{n_y}$ is the plant output, $w \in \mathbb{R}^{n_w}$ is an exogenous disturbance input, $n_x, n_y, n_w \in \mathbb{Z}_{>0}$, an observer is synthesized:

$$\dot{z} = f_O(z, y, w) \quad (7)$$

$$\bar{x} = h_O(z), \quad (8)$$

where $z \in \mathbb{R}^{n_z}$ is the observer state and $\bar{x} \in \mathbb{R}^{n_x}$ is the system state estimate, $n_z \in \mathbb{Z}_{>0}$. Although in practice, system output is likely to be corrupted by noises, (6) is disturbance-free here. This model can be justified by the use of a filter whose dynamics are already embedded into (5).

Remark 1. To model reduced-order observers, we have to change (7)-(8) to:

$$\dot{z} = f_O(z, y, w) \quad (9)$$

$$\bar{x} = h_O(z, y). \quad (10)$$

In this case, the formulation of the network problem does not have a closed form in the coordinates ξ, e, z introduced in the following, but results given Section IV still apply.

Communication constraints are then taken into account. The variable y is no longer available but $\hat{y}$, that is the generated variable from the most recent system output sent through the network. The problem can be modeled in the following form:

$$\left. \begin{aligned} \dot{x} &= f_P(x, w) & \forall t \in [t_{i-1}, t_i] \\ y &= h_P(x) \\ \dot{z} &= f_O(z, \hat{y}, w) & \forall t \in [t_{i-1}, t_i] \\ \bar{x} &= h_O(z) \\ \dot{\hat{y}} &= \hat{f}_P(\hat{y}, z, w) & \forall t \in [t_{i-1}, t_i] \\ \hat{y}(t_i^+) &= \hat{y}(t_i) + h(i, e(t_i), z(t_i)) \end{aligned} \right\} \quad (11)$$

where the sequence t_i , $i \in \mathbb{Z}_{>0}$, of monotonically increasing transmission times satisfies $v \leq t_i - t_{i-1} \leq \tau$ for some fixed $\tau \in [v, \infty)^1$, $t_0 \in \mathbb{R}_{\geq 0}$ being the initial time. We refer to τ as the MATI. The network implementation can be described as follows. Grouping sensors into l sensor nodes, where $l \in \{1, \dots, n_y\}$, system output is partitioned into l corresponding subvectors (that are reordered if necessary). At each transmission instant, exactly one sensor node is chosen to transmit its packet according to the protocol. Between transmission instants, $\hat{y}$ is generated according to the in-network processing implementation. The network may also contain an arbitrary number of passive nodes that can only receive packets. They may be used for hosting distant observers and/or actuators for plants with inputs. Without loss of generality, it is assumed that there is only one passive node in the network where an observer is run.

Model (11) can be written in a more compact way that is appropriate to our study:

$$\dot{\xi} = f_\xi(\xi, e, z, w) \quad \forall t \in [t_{i-1}, t_i] \quad (12)$$

$$\dot{z} = f_z(\xi, e, z, w) \quad \forall t \in [t_{i-1}, t_i] \quad (13)$$

$$\dot{e} = g(\xi, e, z, w) \quad \forall t \in [t_{i-1}, t_i] \quad (14)$$

$$\xi(t_i^+) = \xi(t_i) \quad (15)$$

$$z(t_i^+) = z(t_i) \quad (16)$$

$$e(t_i^+) = h(i, e(t_j), z(t_i)), \quad (17)$$

where $\xi = x - \bar{x} \in \mathbb{R}^{n_\xi}$ ($n_\xi = n_x$) is the observation error, $e = \hat{y} - y \in \mathbb{R}^{n_e}$ ($n_e = n_y$) is the network-induced error and

$$\begin{aligned} f_\xi(\xi, e, z, w) &= f_P(\xi + h_O(z), w) \\ &\quad - \frac{\partial h_O}{\partial z}(z) f_O(z, e + h_P(\xi + h_O(z)), w) \\ f_z(\xi, e, z, w) &= f_O(z, e + h_P(\xi + h_O(z)), w) \\ g(\xi, e, z, w) &= \hat{f}_P(e + h_P(\xi + h_O(z)), z, w) \\ &\quad - \frac{\partial h_P}{\partial x}(\xi + h_O(z)) f_P(\xi + h_O(z), w). \end{aligned}$$

¹Note that v is arbitrary and it is used to prevent Zeno solutions in (11).

Equation (12) represents the dynamics of the observation error and (13) those of the observer state z in the new coordinates. Variables ξ and z could have been embedded into one single vector to be consistent with [14], however we will need to distinguish them in the sequel since we are not interested in the same properties. Indeed, contrary to ξ , no convergence property is desired for z but only some well definition or bounded behaviour for all time. The function h defines the protocol and is typically such that, if the node $j \in \{1, \dots, l\}$ has been granted access to the network, the corresponding error subvector is reset to zero. Throughout the study, attention is focused on uniformly globally exponentially stable (UGES) protocols as initially defined in [14].

Definition 3. Protocol (17) is said to be UGES with Lyapunov function W if there exist $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$, $\rho \in [0, 1)$ and $(a_1, a_2) \in \mathbb{R}_{\geq 0}^2$ such that, for all $i \in \mathbb{Z}_{\geq 0}$, $(e, z) \in \mathbb{R}^{n_e + n_z}$:

$$a_1|e| \leq W(i, e) \leq a_2|e| \quad (18)$$

$$W(i+1, h(i, e, z)) \leq \rho W(i, e). \quad (19)$$

The main problem of this study is to ensure the convergence of the observation error when plant measures are transmitted through a serial communication channel ruled by a UGES protocol. Like in [14], our framework allows to consider various types of protocols, in-network processing implementations and covers sampled-data systems as shown in the following.

B. In-network processing implementations

We give two examples but many others can be considered.

1) *Zero-Order-Hold (ZOH)*: The ZOH implementation simply consists in keeping constant the vector $\hat{y}$ between two successive transmission instants as commonly done in the sampled-data literature, namely:

$$\hat{f}_P = 0. \quad (20)$$

When a node sends its packet at a transmission instant, the corresponding subvector of $\hat{y} = (\hat{y}_1, \dots, \hat{y}_l)$ ($\hat{y}_j \in \mathbb{R}^{n_j}$ with $\sum_{j \in \{1, \dots, l\}} n_j = n_e$) takes this new value until this node transmits again, i.e. in the case where node $j \in \{1, \dots, l\}$ has sent its packet at t_i^+ :

$$\hat{y}_k(t_i^+) = \begin{cases} \hat{y}_k(t_i) & \text{if } k \neq j \\ y_j(t_i) & \text{if } k = j \end{cases} \quad \forall k \in \{1, \dots, l\}. \quad (21)$$

2) *Predictive-type*: The predictive-type implementation consists in predicting the unavailable plant output by a variable, $\hat{y}$ here, that flows along the same vector fields. Thus, a kind of ‘copy’ of system output is used by the observer between transmission instants. This gives here:

$$\hat{f}_P(z, w) = \frac{\partial h_P}{\partial x}(h_O(z))f_P(h_O(z), w). \quad (22)$$

The initial condition $\hat{y}(t_0)$ is arbitrary and, at each transmission instant t_i , $i \in \mathbb{Z}_{\geq 0}$, differential equation (22)

is reinitialized with $\hat{y}(t_i^+)$. This technique was originally developed for the observer design for sampled-data systems in [8] and then applied for NCS in [15]. Note that similar ideas are used for the control of quantized systems in [11], [12].

C. Protocols

Examples of UGES protocols are the round-robin (RR) (also referred as token ring in the literature) and try-once-discard (TOD) protocols. We recall their principle and give a Lyapunov function for each of them.

1) *RR protocol*: RR protocol is the simplest network configuration where each node transmits its data with period l . This type of time-scheduling has been widely investigated, see for instance [5], [6], [10], and its stability was analysed in [14]. Decomposing vector e as $(e_1, \dots, e_l)$, function h is defined as, for $i \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{R}^{n_e}$:

$$h(i, e) = (\mathbb{I} - \Delta(i))e, \quad (23)$$

where $\Delta(i) = \text{diag}(\Delta_1(i), \dots, \Delta_l(i))$, $\Delta_j(i) = \delta_j(i)\mathbb{I}_{n_j}$, $j \in \{1, \dots, l\}$ and

$$\delta_j(i) = \begin{cases} 1 & \text{if } i = j + kl, \quad k \in \mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

The RR protocol is UGES with the Lyapunov function $W(i, e) = \sqrt{\sum_{j=i}^{\infty} |\phi(j, i, e)|^2}$ (where $\phi(j, i, e)$ denotes the solution of the discrete-time system (17) at time j starting at time i and initial condition e) with $a_1 = 1$, $a_2 = \sqrt{l}$, $\rho = \sqrt{\frac{l-1}{l}}$, according to Proposition 4 in [14].

2) *TOD protocol*: TOD time-scheduling protocol was introduced in [16] and its stability was analysed in [14]. This protocol grants access to the node $j \in \{1, \dots, l\}$, where the error $|e_j|$ is the biggest, namely, for $e \in \mathbb{R}^{n_e}$:

$$h(e) = (\mathbb{I} - \Psi(e))e, \quad (25)$$

and $\Psi(e) = \text{diag}(\Psi_1(e)\mathbb{I}_{n_1}, \dots, \Psi_l(e)\mathbb{I}_{n_l})$ with:

$$\Psi_j(e) = \begin{cases} 1 & \text{if } j = \min(\arg \max_{i \in \{1, \dots, l\}} |e_i|) \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

At transmission times, all sensor nodes try to transmit their packets consisting of the packet priority and the data part, while the network hardware allows only the node where the error $|e_j|$ is the biggest to transmit its packet to the passive node and all sensor nodes (if several nodes have the same maximum error, we transmit the node with a minimum index). The network hardware configuration makes TOD protocols applicable in Control Area Network (CAN) as explained in [16], but not directly in many wired and any wireless networks. Moreover, contrary to the RR protocol, TOD requires the use of smart sensors that have sufficient computational capacities to run a copy of the observer located in the passive node (see Fig. 1 in [3]) when using the predictive-type in-network processing implementation. At

each node $j \in \{1, \dots, l+1\}$ (where the index $l+1$ denotes the passive node), the observer below is run:

$$\dot{z}^j = f_O(z^j, \hat{y}, w) \quad (27)$$

$$\dot{\bar{x}}^j = h_O(z^j). \quad (28)$$

They are synchronized ($z^j(t) = z^k(t)$ for all t and $j, k \in \{1, \dots, l+1\}$) by assuming that they start with the same initial condition and thank to the assignment procedure, which is modeled by a piecewise constant function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \dots, l\}$ so that when node k has been assigned at time t_i , $\sigma(t) = k$ for $t \in [t_i, t_{i+1})$ and $y_{\sigma(t_i)}(t_i) = y_k(t_i)$. In that way, at time $t \in [t_i, t_{i+1})$, the following signals are available at:

- sensor node $j \in \{1, \dots, l\}$: $\sigma(t_i)$, $y_{\sigma(t_i)}(t_i)$, $y_j(t)$, $z^j(t)$;
- passive node: $\sigma(t_i)$, $y_{\sigma(t_i)}(t_i)$, $z^{l+1}(t)$.

Since each observer receives $y_{\sigma(t_i)}(t_i)$, they all have the same input signal $\hat{y}(t)$ and are synchronised for all time (unless a computational glitch occurs) and the superscripts j can be omitted in (27)-(28).

In Proposition 5 in [14], it was proved that TOD protocols are UGES with Lyapunov function $W(e) = |e|$ and $a_1 = a_2 = 1$, $\rho = \sqrt{\frac{l-1}{l}}$.

D. Observer design for sampled-data systems

Model (12)-(17) can be used to analyse observer emulation for sampled-data systems by simply setting the number of nodes l to 1. In that way, our study offers alternative tools to [2]. It can be noticed that observers developed in [8] can be written in the form (12)-(17) by seeing that variable w in [8] corresponds to $\hat{y}$ with the predictive-type implementation. As a consequence, we recover the approach of [8] but considering multi-output systems affected by perturbations. It has to be noted that our trajectory-based conditions will be different from [8].

In our formulation, the sampled-data case would be modelled with function

$$h = 0, \quad (29)$$

since the network-induced error is reset to zero at each transmission time. Therefore, Lyapunov function $W(e) = |e|$ shows that this protocol is UGES with $a_1 = a_2 = 1$ and $\rho = 0$.

IV. MAIN RESULTS

In this section, we give sufficient conditions on system (12)-(17) and explicit bounds on MATI that ensure the convergence of the observation error under network-induced constraints.

It is needed that the continuous-time observer (7)-(8) is robust to measurement errors in the following sense.

Assumption 1. System (12)-(13) is IOS from e, w to ξ with linear gains γ_1^e, γ_1^w .

This type of condition was already needed for the observer design for sampled-data systems and NCS respectively in [8], [15] and is similar to IOS (or ISS) assumptions for the

control of NCS in [14] (condition 2 in Theorem 7) and NCS with quantization in [13] (condition (i) in Theorem 1).

Attention is focused on the following class of UGES protocols.

Assumption 2. Protocol (17) is UGES with Lyapunov function $W : \mathbb{Z}_{\geq 0} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}_{\geq 0}$ that is locally Lipschitz in e and uniformly in i .

The following assumption is reminiscent of condition (27) in [14] for the control of NCS and (12) in [13] for the control of quantized NCS. By combining it with Assumption 2, it is shown that system (14), (17) is ISS w.r.t. ξ, z, w with linear gains.

Assumption 3. There exist $L, \gamma_2^\xi, \gamma_2^z, \gamma_2^w \in \mathbb{R}_{\geq 0}$, such that, for all $(i, t, \xi, z, w) \in \mathbb{Z}_{\geq 0} \times [t_0, \infty) \times \mathbb{R}^{n_\xi + n_z + n_w}$ and almost all $e \in \mathbb{R}^{n_e}$:

$$\left\langle \frac{\partial W(i, e)}{\partial e}, g(\xi, e, z, w) \right\rangle \leq LW(i, e) + \gamma_2^\xi |\xi| + \gamma_2^z |z| + \gamma_2^w |w|.$$

The following result is a direct consequence of Proposition 6 in [14].

Proposition 1. If Assumptions 2 and 3 hold and $\tau \in (v, \tau_0)$ where $\tau_0 = \frac{1}{L} \ln(\frac{1}{\rho})$ ($\tau_0 = \lim_{L \rightarrow 0} \frac{1}{L} \ln(\frac{1}{\rho}) = \infty$ if $L = 0$), then system (14), (17) is ISS w.r.t. ξ, z, w with linear gains $\zeta(\tau)\gamma_2^\xi, \zeta(\tau)\gamma_2^z, \zeta(\tau)\gamma_2^w$ where $\zeta(\tau) = \frac{\exp(L\tau)-1}{a_1 L(1-\rho \exp(L\tau))}$. $\square$

Remark 2. It can be noticed that $\zeta \in \mathcal{K}$ since it is a strictly increasing continuous function on $[0, \frac{1}{L} \ln(\frac{1}{\rho}))$ and $\zeta(0) = 0$.

The following assumption will be used to guarantee the boundedness of the states of system (12)-(17).

Assumption 4. There exist $\alpha_3 \in \mathcal{K}$ and $\gamma_3^\xi, \gamma_3^e, \gamma_3^w \in \mathbb{R}_{\geq 0}$ such that, for all $z_0 \in \mathbb{R}^{n_z}$, $(\xi, e, w) \in \mathcal{L}_\infty^{n_\xi + n_e + n_w}$, the following holds along solutions to (13):

$$|z(t)| \leq \alpha_3(|(\xi_0, z_0)|, 0) + \gamma_3^\xi \|\xi\|_\infty + \gamma_3^e \|e\|_\infty + \gamma_3^w \|w\|_\infty \quad \forall t \geq t_0 \geq 0. \quad (30)$$

Assumption 4 is weaker than the BIBS property for system (13) with inputs ξ, e, w since α_3 depends on ξ_0 . This condition is very related to the stability of system (5). Indeed, we usually prove it by assuming system (5) is BIBS with input w and under conditions on system (13) as shown on an example in Section V.

Remark 3. Throughout this study, the gains of the exogenous perturbation, γ_i^w with $i \in \{1, \dots, 3\}$, are assumed to be linear only for the clarity of presentation: our results immediately apply to nonlinear $\gamma_i^w \in \mathcal{K}$.

We are now ready to state the main theorem. For space reasons, the proof is omitted. The main idea is to consider system (12)-(17) as the interconnection of three subsystems in ξ, z and e and to apply small-gain arguments to conclude using Assumptions 1, 4 and Proposition 1. Due to the fact that we are dealing with three subsystems (and not two) and that one of them is not expected to converge (z -subsystem (13), (16)), the analysis is non-standard compared to [7].

Theorem 1. Under Assumptions 1-4, if $\tau \in (v, \tau_1)$ where $\tau_1 = \frac{1}{L} \ln \left(\frac{La_1 + [\gamma_1^e(\gamma_2^e + \gamma_2^e \gamma_3^e) + \gamma_2^e \gamma_3^e]}{L\rho a_1 + [\gamma_1^e(\gamma_2^e + \gamma_2^e \gamma_3^e) + \gamma_2^e \gamma_3^e]} \right)$ ($\tau_1 = \frac{a_1(1-\rho)}{\gamma_1^e(\gamma_2^e + \gamma_2^e \gamma_3^e) + \gamma_2^e \gamma_3^e}$ if $L = 0$), then system (12)-(17) is BIBS with w as input and there exist $\beta \in \mathcal{KL}$, $\sigma \in \mathcal{K}$, $\bar{\sigma}, \varepsilon \in \mathcal{KK}$ such that, for any $\Delta \in \mathbb{R}_{\geq 0}$, $(\xi_0, e_0, z_0) \in \mathbb{R}^{n_\xi + n_e + n_z}$ with $|(\xi_0, e_0, z_0)| < \Delta$, $w \in \mathcal{L}_\infty^{n_w}$, this holds:

$$|(\xi(t), e(t))| \leq \beta(|(\xi_0, e_0, z_0)|, t - t_0) + \sigma(\|w\|_\infty) + \bar{\sigma}(\tau, \|w\|_\infty) + \varepsilon(\tau, \Delta) \quad \forall t \geq t_0 \geq 0. \quad (31)$$

□

It can be seen that when $w = 0$ the observation error does not asymptotically converge to the origin but to the ball centered in 0 of radius $\varepsilon(\tau, \Delta)$. It is shown in the sequel that when $\gamma_2^z = 0$ in Assumption 3, asymptotic convergence to the origin is guaranteed (and Assumption 4 may be relaxed). The case where $\gamma_2^z \neq 0$ typically arises when ZOH devices are implemented, as shown in Section V, because the observer receives then measurements delayed by $T = t - t_i$ that affect its convergence in the general case.

In some situations, a stronger robustness property than Assumption 1 is ensured by the observation error.

Assumption 5. There exist $\beta_1 \in \mathcal{KL}$, $\gamma_1^e, \gamma_1^w \in \mathbb{R}_{\geq 0}$ such that, for any $\xi_0 \in \mathbb{R}^{n_\xi}$, $(e, w) \in \mathcal{L}_\infty^{n_e + n_w}$, solutions to (12) satisfy, for all $t \geq t_0 \geq 0$:

$$|\xi(t)| \leq \beta_1(|\xi_0|, t - t_0) + \gamma_1^e \|e\|_\infty + \gamma_1^w \|w\|_\infty. \quad (32)$$

Remark 4. Assumption 5 implies that system (12) is ISS w.r.t. z, e, w with linear gains.

When Assumption 5 holds and Assumption 3 is satisfied with $\gamma_2^z = 0$, stability of ξ - and e -dynamics can be investigated separately from the whole system (12)-(17). In that way, no boundedness condition is necessary anymore to analyse stability of system (12)-(17): Assumption 4 can be relaxed as follows.

Assumption 6. System (13) is forward complete with input $(\xi, e, w) \in \mathcal{L}_\infty^{n_\xi + n_e + n_w}$ i.e. there exist $\nu_1, \nu_2, \nu_3 \in \mathcal{K}$ and $c \in \mathbb{R}_{\geq 0}$ such that, for any $z_0 \in \mathbb{R}^{n_z}$, $(\xi, e, w) \in \mathcal{L}_\infty^{n_\xi + n_e + n_w}$, along solutions to (13), for all $t \geq t_0 \geq 0$:

$$|z(t)| \leq \nu_1(t) + \nu_2(|z_0|) + \nu_3(\|(\xi, e, w)\|_\infty) + c. \quad (33)$$

Remark 5. It is shown in Corollary 2.3 in [1] that forward completeness of systems with inputs of the form of (13) is equivalent to (33).

The following theorem can then be derived. Its proof follows similar lines to the proof of Theorem 1.

Theorem 2. Under Assumptions 2,5,6 and if Assumption 3 holds with $\gamma_2^z = 0$, if $\tau \in (v, \tau_2)$ where $\tau_2 = \frac{1}{L} \ln \left(\frac{La_1 + \gamma_2^e \gamma_1^e}{L\rho a_1 + \gamma_2^e \gamma_1^e} \right)$ ($\tau_2 = \frac{a_1(1-\rho)}{\gamma_1^e \gamma_2^e}$ if $L = 0$), then system (12)-(17) is forward complete with input $w \in \mathcal{L}_\infty^{n_w}$ and there exist $\beta \in \mathcal{KL}$, $\sigma \in \mathcal{K}$, $\bar{\sigma} \in \mathcal{KK}$ such that, for all $(\xi_0, z_0, e_0) \in \mathbb{R}^{n_\xi + n_z + n_e}$, $w \in \mathcal{L}_\infty^{n_w}$:

$$|(\xi(t), e(t))| \leq \beta(|(\xi_0, e_0)|, t - t_0) + \sigma(\|w\|_\infty) + \bar{\sigma}(\tau, \|w\|_\infty) \quad \forall t \geq t_0 \geq 0. \quad (34)$$

□

V. APPLICATIONS

In this section, we illustrate the generality of our approach to the observer emulation for NCS. We show that linear observers satisfy the assumptions of Section IV for a range of network configurations (RR/TOD protocols and ZOH/predictive-type implementations) and derive for the first time explicit MATI bounds². The case where system outputs are simply sampled is also considered. Consider linear systems:

$$\dot{x} = Ax \quad (35)$$

$$y = Cx, \quad (36)$$

where $x \in \mathbb{R}^{n_x}$, $y \in \mathbb{R}^{n_y}$, A and C are real matrices of appropriate dimensions such that (A, C) is detectable. The following continuous-time Luenberger observer is synthesized:

$$\dot{\bar{x}} = A\bar{x} + \Lambda(y - \bar{y}) \quad (37)$$

$$\bar{y} = C\bar{x}, \quad (38)$$

where $\bar{x} \in \mathbb{R}^{n_x}$, $\bar{y} \in \mathbb{R}^{n_y}$ and Λ is a real matrix such that $(A - \Lambda C)$ is Hurwitz. Consider the sequence of monotonically increasing transmission instants t_i , $i \in \mathbb{Z}_{>0}$, that satisfies $v \leq t_i - t_{i-1} \leq \tau$ for all $i \in \mathbb{Z}_{>0}$ and some fixed $v, \tau \in \mathbb{R}_{>0}$. Like model (12)-(17), system (35)-(38) is written as

$$\dot{\xi} = (A - \Lambda C)\xi - \Lambda e \quad \forall t \in [t_{i-1}, t_i] \quad (39)$$

$$\dot{z} = Az + \Lambda(e + C\xi) \quad \forall t \in [t_{i-1}, t_i] \quad (40)$$

$$\dot{e} = \hat{f}_P(z) - CA(\xi + z) \quad \forall t \in [t_{i-1}, t_i] \quad (41)$$

$$\xi(t_i^+) = \xi(t_i) \quad (42)$$

$$z(t_i^+) = z(t_i) \quad (43)$$

$$e(t_i^+) = h(i, e(t_i)), \quad (44)$$

where $z = \bar{x}$, $\hat{f}_P$ is defined by (20) or (22) and h by (23), (25) or (29). We will show that conditions of Theorems 1 and 2 are ensured by system (39)-(44) respectively with the ZOH and the predictive-type implementations.

According to p. 174 in [9], system (39) is ISS w.r.t. e with a linear gain γ_1^e , therefore Assumption 5 (equivalently Assumption 1) is guaranteed. In view of the Lyapunov functions given in Sections III-C.1, III-C.2 and III-D, we can see that Assumption 2 is satisfied. Moreover, we can prove that Assumption 3 holds for the different network configurations with the coefficients given in Table I, using (20) and (22). It has to be noticed that in some special cases, tighter coefficients can be obtained when taking advantage of the possible particular structure of CA (see the discussion in Example 3 in [14]). Before applying Theorem 1, we need to prove that Assumption 4 is satisfied. Using the fact that $\xi = x - z$, system (40) can be written as:

$$\dot{z} = (A - \Lambda C)z + \Lambda(e + Cx). \quad (45)$$

²Similar results can be deduced for high-gain observers as we will show it in the journal version of this paper.

	ZOH	predictive-type
RR protocol	$L = 0, \gamma_2^\xi = \sqrt{l} CA , \gamma_2^z = \sqrt{l} CA $	$L = 0, \gamma_2^\xi = \sqrt{l} CA , \gamma_2^z = 0$
TOD protocol	$L = 0, \gamma_2^\xi = CA , \gamma_2^z = CA $	$L = 0, \gamma_2^\xi = CA , \gamma_2^z = 0$
Sampled-data	$L = 0, \gamma_2^\xi = CA , \gamma_2^z = CA $	$L = 0, \gamma_2^\xi = CA , \gamma_2^z = 0$

TABLE I
COEFFICIENTS OF ASSUMPTION 3 FOR SYSTEM (41)

Since $(A - \Lambda C)$ is Hurwitz, it can be shown that there exists $\beta_3 \in \exp -\mathcal{KL}$ such that for any $z_0 \in \mathbb{R}^{n_z}, (e, x) \in \mathcal{L}_\infty^{n_e+n_x}, t \geq t_0 \geq 0$:

$$|z(t)| \leq \beta_3(|z_0|, t - t_0) + \gamma_1^e \|e\|_{[t_0, t]} + \gamma_1^e |C| \|x\|_{[t_0, t]}. \quad (46)$$

Now let assume that system (35) is GS (that is A has no eigenvalue with strictly positive real part), in view of Definition 2 there exists $\alpha \in \mathcal{K}$ such that $|x(t)| \leq \alpha(|x_0|)$ for any $x_0 \in \mathbb{R}^{n_x}$, consequently from (46) we have:

$$|z(t)| \leq \beta_3(|z_0|, t - t_0) + \gamma_1^e \|e\|_{[t_0, t]} + \gamma_1^e |C| \alpha(|x_0|) \leq \beta_3(|z_0|, t - t_0) + \gamma_1^e \|e\|_{[t_0, t]} + \gamma_1^e |C| \alpha(|z_0| + |x_0|).$$

and Assumption 4 holds with $\alpha_3(s) = \beta_3(s, 0) + \gamma_1^e |C| \alpha(2s), s \in \mathbb{R}_{\geq 0}, \gamma_3^\xi = 0, \gamma_3^z = \gamma_1^e$. We are now ready to apply results of Section IV. The first proposition is a direct consequence of Theorem 1.

Proposition 2. Assume system (35) is GS, consider system (39)-(44) with the ZOH implementation (20), if $\tau \in (v, \tau^*)$ where τ^* is defined in Table II depending on the protocol, then system (39)-(44) is GS and (31) holds. $\square$

Since system (40) is linear, Assumption 6 always applies. The following proposition follows from Theorem 2.

Proposition 3. Consider system (39)-(44) with the predictive-type implementation (22), if $\tau \in (v, \tau^*)$ where τ^* is defined in Table II depending on the protocol, then system (39)-(44) is forward complete and (34) holds. $\square$

	ZOH	predictive-type
RR protocol	$\tau^* = \frac{\sqrt{l} - \sqrt{l-1}}{2l\gamma_1^e CA }$	$\tau^* = \frac{\sqrt{l} - \sqrt{l-1}}{l\gamma_1^e CA }$
TOD protocol	$\tau^* = \frac{\sqrt{l} - \sqrt{l-1}}{2\sqrt{l}\gamma_1^e CA }$	$\tau^* = \frac{\sqrt{l} - \sqrt{l-1}}{\sqrt{l}\gamma_1^e CA }$
Sampled-data	$\tau^* = \frac{1}{2\gamma_1^e CA }$	$\tau^* = \frac{1}{\gamma_1^e CA }$

TABLE II
MATI BOUNDS FOR SYSTEM (39)-(44)

VI. CONCLUSION

In this paper, we have developed a framework for the observer design for NCS affected by disturbances, via an emulation-based approach. A general model is proposed that allows us to study various observers, time-scheduling protocols and in-network processing implementations. Sufficient

conditions on the system are given and explicit MATI bounds are deduced that ensure the convergence of the observation error under network-induced communication constraints. In a forthcoming paper, the linear gains assumption on IOS and ISS conditions will be relaxed and uniformly globally asymptotically stable (UGAS) protocols addressed.

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