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Author/s:

Wang, W;Teel, AR;Nesic, D

Title:

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Date:

2012-06-01

Citation:

Wang, W., Teel, A. R. & Nesic, D. (2012). Analysis for a class of singularly perturbed hybrid systems via averaging. AUTOMATICA, 48 (6), pp.1057-1068. <https://doi.org/10.1016/j.automatica.2012.03.013>.

Persistent Link:

<https://hdl.handle.net/11343/299564>

Analysis for a class of singularly perturbed hybrid systems via averaging [★]

Wei Wang^a, Andrew R. Teel^b, Dragan Nešić^a

^a*Department of Electrical and Electronic Engineering, the University of Melbourne, Parkville, 3010, Victoria, Australia*

^b*Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106-9560, USA*

Abstract

A class of singularly perturbed hybrid dynamical systems is analyzed. The fast states are restricted to a compact set a priori. The continuous-time boundary layer dynamics produce solutions that are assumed to generate a well-defined average vector field for the slow dynamics. This average, the projection of the jump map in the direction of the slow states, and flow and jump sets from the original dynamics define the reduced, or average, hybrid dynamical system. Assumptions about the average system lead to conclusions about the original, higher-dimensional system. For example, forward pre-completeness for the average system leads to a result on closeness of solutions between the original and average system on compact time domains. In addition, global asymptotic stability for the average system implies semiglobal, practical asymptotic stability for the original system. We give examples to illustrate the averaging concept and to relate it to classical singular perturbation results as well as to other singular perturbation results that have appeared recently for hybrid systems. We also use an example to show that our results can be used as an analysis tool to design hybrid feedbacks for continuous-time plants implemented by fast but continuous actuators.

Key words: Singular perturbation; Averaging; Hybrid dynamical systems; Practical stability.

1 Introduction

Systems exhibiting two time scale behavior (or multiple time scale behavior), with fast and slow dynamical variables, arise in all areas of science and engineering; properties of such systems can be analyzed using singular perturbation methods. For continuous-time systems, the celebrated Levinson-Tikonov approach to analyzing singularly perturbed systems relates the dynamical properties of the perturbed system to the properties of two auxiliary systems, fast (boundary layer) and slow (quasi-steady state) systems. The main singular perturbation results can be classified into two main categories: closeness of solutions of the original perturbed system to solutions of its approximation on compact or infinite time intervals [18,32]; and stability results of the origi-

nal system that are based on appropriate stability properties of the slow and the fast systems [34,3]. Analogs of Levinson-Tikonov theorem are established for differential inclusions, such as [9,35] on finite time intervals and [38] on infinite time intervals with the assumptions that the boundary layer system converges to a Lipschitz set-valued map and global asymptotic stability of the reduced system. There are also results on singular perturbations for discrete-time systems [16,20].

The averaging method was developed for continuous-time systems, discrete-time systems and differential inclusions [4,8,26,36], but there exist only a few results for special classes of hybrid systems. For example, results on averaging of switched systems and dither systems to approximate time-varying hybrid systems by non-hybrid systems were considered in [17,23,37]. Recently, asymptotic stability for a class of time-varying hybrid systems via averaging was considered in [33], where states change continuously in a set in the state space and change instantaneously in another set in the state space.

Combining averaging and singular perturbation techniques, the results in [3] consider continuous-time systems when the boundary layer system (obtained in the

[★] Supported by the Australian Research Council under the Discovery Project and Future Fellow program, AFOSR (Grant FA9550-09-1-0203) and NSF (Grants ECCS-0925637 and CNS-0720842). This paper was not presented at any IFAC meeting.

Email addresses: wewang@ee.unimelb.edu.au (Wei Wang), teel@ece.uscb.edu (Andrew R. Teel), dnesic@unimelb.edu.au (Dragan Nešić).

singular perturbation approach by setting the derivative of slow state variables to zero) is time-varying and possesses a time-varying integral manifold on which the derivative of slow state variables can be averaged. The results can be applied to adaptive control systems [25] and extremum seeking control systems [31]. The averaging method is also helpful in considering the singular perturbation problem when the boundary layer system is not time varying. Instead of insisting that trajectories of the boundary layer system converge to an equilibrium manifold, as in the classical singular perturbation theory, a set is used to replace the equilibrium manifold. In particular, trajectories of the boundary layer system are assumed to converge to a family of limit cycles parameterized by slow state variables, which then can be used to average the derivative of slow state variables. This idea can be found in the optimal control results in [11], the work of Artstein in [1,2], the work of Grammel in [14,15] and more recently in a unified framework for studying robustness to slowly-varying parameters, rapidly-varying signals and generalized singular perturbations in [32]. Artstein considered limiting behavior of the slow dynamics and statistical limit behavior of the fast dynamics for differential equations and differential inclusions respectively in [1,2], by replacing the equilibria of the boundary layer system with an invariant probability measure. With the assumption that finite time average of the slow dynamics converges to a continuous limit set, Grammel [14,15] constructed a limit differential inclusion for the slow motion. Singular perturbation theory based on averaging leads to a reduced order system, where fast motions appear implicitly and only their average influence on slow motions is considered. In general, this approach requires the assumption that large time scale behavior of trajectories of the fast dynamics is in some sense independent of its initial values [16], or properties guaranteed by a unique invariant measure [1,2,15] or some stability properties [9,32,35,38].

Hybrid dynamical systems are considered in the present paper, which naturally arise in a range of engineering applications including power electronics, robotics, manufacturing, automated highway systems, air traffic management systems, chemical process, and so on [29,21,10]. Moreover, even for a continuous-time plant, the capabilities of nonlinear feedback control can be enhanced by using a hybrid controller [12,24,22]. For hybrid dynamical systems, singular perturbation results appear in [28,27]. Robustness to measurement noise and unmodeled dynamics of stability in hybrid systems was considered in [28] with the assumption that the boundary layer system converges to a quasi-steady state equilibrium manifold. The paper [27] presents results on stability of hybrid control systems singularly perturbed by fast but continuous actuators, where a more general set-valued mapping used to approximate the limiting behavior of the boundary layer system replaces the equilibrium manifold in the classical singular perturbation theory. It shows that if a hybrid control system has a compact set that is glob-

ally asymptotically stable when the actuator dynamics are omitted, or equivalently, are infinitely fast, then the same compact set is semi-globally practically asymptotically stable in the singular perturbation parameter.

We combine the averaging and the singular perturbation techniques to consider both closeness of solutions of a hybrid system with solutions of its average and stability properties of the actual system based on stability of its average in this paper. We show that each solution of the slow dynamics of the singularly perturbed hybrid system can be made arbitrarily close on compact time domains to some solution of its average system when the average system is forward complete. We also show that a compact set is semi-globally practically asymptotically stable for the actual hybrid system if it is globally asymptotically stable for the average system. Compared to hybrid singular perturbation results in [27], our results give sharper conclusions in some cases and an example is used to illustrate this claim.

The paper is organized as follows. In Section 2, some basic definitions under the hybrid system framework are reviewed. We introduce a class of singularly perturbed hybrid systems in Section 3. The main results are given in Section 4. The proofs of main results are listed in Section 5 and Section 6 contains the conclusions.

2 Preliminaries

The singularly perturbed hybrid systems that we consider are based on two time scales, (τ, j) and (t, j) with $\tau = \varepsilon t$, with the notations $x' = \frac{dx}{d\tau}$, $\dot{x} = \frac{dx}{dt}$. $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$. $\mathbb{B}$ is the closed unit ball in an Euclidean space, the dimension of which should be clear from the context. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous at $x \in \mathbb{R}^n$ if for all sequences $x_i \rightarrow x$ and $y_i \in M(x_i)$ such that $y_i \rightarrow y$ we have $y \in M(x)$, and M is outer semi-continuous (OSC) if it is outer semi-continuous at each $x \in \mathbb{R}^n$. A set-valued mapping $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is locally bounded if for any compact set $\mathcal{A} \subset \mathbb{R}^n$ there exists $r > 0$ such that $M(\mathcal{A}) := \bigcup_{x \in \mathcal{A}} M(x) \subset r\mathbb{B}$; if M is OSC and locally bounded, then $M(\mathcal{A})$ is compact for any compact set $\mathcal{A}$. A function $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ is locally absolutely continuous if its derivative is defined almost everywhere and we have $x(t) - x(t_0) = \int_{t_0}^t \dot{x}(s)ds$ for all $t \geq t_0 \geq 0$. Given a set S , $\overline{\text{con}}S$ denotes the closed convex hull of a set S . Given a compact set $\mathcal{A} \subset \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, define $|x|_{\mathcal{A}} := \min_{y \in \mathcal{A}} |x - y|$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- $\mathcal{L}$ if it is non-increasing and converging to zero as its argument grows unbounded. A continuous function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- $\mathcal{G}$ if it is zero at zero and non-decreasing. It is of class- $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class- $\mathcal{KL}$ if it is class of $\mathcal{K}$ in its first argument and class of $\mathcal{L}$ in its second argument.

Consider a hybrid system with states $\xi \in \mathbb{R}^n$ specified as:

$$H \quad \begin{aligned} \xi' &\in F(\xi), & \xi &\in C, \\ \xi^+ &\in G(\xi), & \xi &\in D, \end{aligned} \quad (1)$$

and a hybrid system with $x \in \mathbb{R}^n$ that depends on a small parameter $\mu > 0$:

$$H_\mu \quad \begin{aligned} x' &\in F_\mu(x), & x &\in C_\mu, \\ x^+ &\in G_\mu(x), & x &\in D_\mu. \end{aligned} \quad (2)$$

Solutions for hybrid system H are defined on hybrid time domains in [6]. A set $S \subset \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is called a compact hybrid time domain if $S = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_J$. The set S is a hybrid time domain if for all $(T, J) \in S$, $S \cap ([0, T] \times \{0, 1, \dots, J\})$ is a compact hybrid time domain.

A hybrid signal is a function defined on a hybrid time domain. hybrid signal $\xi : \text{dom } \xi \mapsto \mathbb{R}^n$ is called a hybrid arc if $\xi(\cdot, j)$ is locally absolutely continuous for each j . A hybrid arc $\xi : \text{dom } \xi \mapsto \mathbb{R}^n$ is a solution to the hybrid system H in (1) if $\xi(0, 0) \in C \cup D$ and:

- (1) for all $j \in \mathbb{N}$ and almost all τ such that $(\tau, j) \in \text{dom } \xi$, $\xi(\tau, j) \in C$ and $\xi'(\tau, j) \in F(\xi(\tau, j))$;
- (2) for all $(\tau, j) \in \text{dom } \xi$ such that $(\tau, j+1) \in \text{dom } \xi$, $\xi(\tau, j) \in D$ and $\xi(\tau, j+1) \in G(\xi(\tau, j))$.

A solution is maximal if it cannot be extended.

The closeness of solutions of the singularly perturbed system to the solutions of its average system on compact time domains is considered as one of the main results. We require the following concepts of forward pre-completeness, that implies solutions are contained in a compact set on compact time domains, and (T, J, ρ) -closeness that defines graphical convergence of hybrid arcs, see details in [13, Section 4].

Definition 1 (Forward completeness) *A hybrid solution is said to be forward complete if its domain is unbounded. A hybrid solution is said to be forward pre-complete if its domain is compact or unbounded. System H is said to be forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$ if all maximal solutions ξ with $\xi(0, 0) \in K_0$ are forward pre-complete.* $\square$

Definition 2 (Closeness of hybrid signals) *Two hybrid signals $\xi_1 : \text{dom } \xi_1 \mapsto \mathbb{R}^n$ and $\xi_2 : \text{dom } \xi_2 \mapsto \mathbb{R}^n$ are said to be (T, J, ρ) -close if:*

- (1) for each $(t, j) \in \text{dom } \xi_1$ with $t \leq T$ and $j \leq J$ there exists s such that $(s, j) \in \text{dom } \xi_2$, with $|t - s| \leq \rho$ and $|\xi_1(t, j) - \xi_2(s, j)| \leq \rho$,

- (2) for each $(t, j) \in \text{dom } \xi_2$ with $t \leq T$ and $j \leq J$ there exists s such that $(s, j) \in \text{dom } \xi_1$, with $|t - s| \leq \rho$ and $|\xi_2(t, j) - \xi_1(s, j)| \leq \rho$. $\square$

We also consider the stability properties of the perturbed system under the assumption that the average system has a global asymptotic stability property. Global asymptotic stability for the system H in (1) and semi-global asymptotic stability for the system H_μ in (2) are defined as the follows.

Definition 3 *For the hybrid system H in (1), the compact set $\mathcal{A}$ is said to be globally asymptotically stable with respect to $\beta \in \mathcal{KL}$ if*

$$|\xi(\tau, j)|_{\mathcal{A}} \leq \beta(|\xi(0, 0)|_{\mathcal{A}}, \tau + j), \quad \forall (\tau, j) \in \text{dom } \xi. \quad \square$$

Note that the existence of the class- $\mathcal{KL}$ function β in Definition 3 for the compact set $\mathcal{A}$ is equivalent to stability and pre-attractivity of the set $\mathcal{A}$ as defined in [7]. For more details see [13, Theorem 6.5].

Definition 4 *For the hybrid system H_μ in (2), the compact set $\mathcal{A}$ is said to be semi-globally practically asymptotically stable (SGP-AS) with respect to $\beta \in \mathcal{KL}$ if, for each compact set K_0 and positive real number $\nu > 0$, there exists $\mu^* > 0$ such that, for each $\mu \in (0, \mu^*]$, each solution x with $x(0, 0) \in K_0 + \mu\mathbb{B}$ satisfies*

$$|x(\tau, j)|_{\mathcal{A}} \leq \beta(|x(0, 0)|_{\mathcal{A}}, \tau + j) + \nu, \quad \forall (\tau, j) \in \text{dom } x. \quad \square$$

3 Singularly perturbed hybrid systems

Consider a class of singularly perturbed hybrid systems with the time variables (τ, j) :

$$H_\varepsilon \quad \left. \begin{aligned} x' &= f(x, z, \varepsilon) \\ z' &= \frac{1}{\varepsilon} \psi(x, z, \varepsilon) \end{aligned} \right\}, \quad (x, z) \in C \times \Psi, \quad (3)$$

$$(x, z)^+ \in G(x, z), \quad (x, z) \in D \times \Psi,$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, $C, D \subset \mathbb{R}^n$, $\Psi \subset \mathbb{R}^m$, $f : C \times \Psi \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$, $\psi : C \times \Psi \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^m$, $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$, and $\varepsilon > 0$ is a small parameter reflecting that the flow dynamic of z are much faster than x . Let $f_0(x, z) := f(x, z, 0)$ and $\psi_0(x, z) := \psi(x, z, 0)$. We assume that system H_ε satisfies the following conditions.

Assumption 1 *The sets C and D are closed and the set Ψ is compact. G is outer semi-continuous and locally bounded, for each $(x, z) \in D \times \Psi$, $G(x, z)$ is nonempty. $f_0 : C \times \Psi \rightarrow \mathbb{R}^n$ and $\psi_0 : C \times \Psi \rightarrow \mathbb{R}^m$ are continuous, and for each $\delta > 0$ and compact $K \subset \mathbb{R}^n$ there exists*

$\varepsilon^* := \varepsilon^*(K, \delta) > 0$ such that

$$\left. \begin{aligned} |f(x, z, \varepsilon) - f_0(x, z)| &\leq \delta \\ |\psi(x, z, \varepsilon) - \psi_0(x, z)| &\leq \delta \end{aligned} \right\} \quad \forall ((x, z), \varepsilon) \in ((C \cap K) \times \Psi) \times (0, \varepsilon^*]. \quad (4)$$

The set Ψ is assumed to be compact as we wish to deal with compact attractors for the fast state z and without any assumption on the set-valued map G ; if (3) admits solutions with a purely discrete-time domain then a jump rule like $z^+ = z$ will not allow z to converge to a compact set unless it is constrained to a compact set a priori.

Remark 1 While the averaging results presented here can be extended to the situation where the flow map is a set-valued mapping, resulting in a differential inclusion, for simplicity of exposition we have chosen to focus on the single-valued case here. Indeed, for inclusions, the definition of an average is slightly more involved. Compare with [1, 14] where averaging with inclusions has been addressed, we consider hybrid systems of form (3), for which the flow mapping is a continuous function.

To facilitate the definition of the boundary layer system (see (6) below), the system H_ε is also expressed with the time variables (t, j) with $t := \tau/\varepsilon$:

$$H_\varepsilon \quad \left. \begin{aligned} \dot{x} &= \varepsilon f(x, z, \varepsilon) \\ \dot{z} &= \psi(x, z, \varepsilon) \end{aligned} \right\}, \quad (x, z) \in C \times \Psi, \quad (5)$$

$$(x, z)^+ \in G(x, z), \quad (x, z) \in D \times \Psi.$$

We define the boundary layer system of the system H_ε as

$$H_{bl} \quad \left. \begin{aligned} \dot{x}_{bl} &= 0 \\ \dot{z}_{bl} &= \psi_0(x_{bl}, z_{bl}) \end{aligned} \right\}, \quad (x_{bl}, z_{bl}) \in C \times \Psi, \quad (6)$$

which is obtained by ignoring the jump mapping and setting $\varepsilon = 0$ in (5).

Definition 5 For functions $f_0 : C \times \Psi \rightarrow \mathbb{R}^n$ and $\psi_0 : C \times \Psi \rightarrow \mathbb{R}^m$, the function $f_{av} : C \rightarrow \mathbb{R}^n$ is said to be an average of f_0 with respect to ψ_0 on $C \times \Psi$ if for each compact set $K \subset \mathbb{R}^n$ there exists a class- $\mathcal{L}$ function σ_K such that, for each $L > 0$, $x \in C \cap K$ and each function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $\dot{z}_{bl} = \psi_0(x, z_{bl})$, the following holds:

$$\left| \frac{1}{L} \int_0^L [f_0(x, z_{bl}(s)) - f_{av}(x)] ds \right| \leq \sigma_K(L). \quad (7)$$

To analyze the singularly perturbed system H_ε through its average system H_{av} , we need the assumption that a well-defined average is admitted by H_ε .

Assumption 2 The function $f_0 : C \times \Psi \rightarrow \mathbb{R}^n$ admits a continuous average function $f_{av} : C \rightarrow \mathbb{R}^n$ with respect to $\psi_0 : C \times \Psi \rightarrow \mathbb{R}^m$ on the set $C \times \Psi$. $\square$

Before we give examples to show that Assumption 2 holds, but not limited to, classical singular perturbation problems and periodic disturbances, a lemma on existence of averages for systems H_ε in (3) is first presented. This idea is implicit in the results of [32]. The proof is provided in the Appendix.

Assumption 3 For a given compact set $\Omega \subset C \times \Psi$, there exists a class- $\mathcal{L}$ function σ_Ω such that, for each $L > 0$ and function $z_{bl} : [0, L] \mapsto \Psi$ satisfying $(x, z_{bl}(0)) \in \Omega$ and $\dot{z}_{bl} = \psi_0(x, z_{bl})$, the following holds:

$$\left| \frac{1}{L} \int_0^L [f_0(x, z_{bl}(s)) - f_{av}(x)] ds \right| \leq \sigma_\Omega(L).$$

Lemma 1 Suppose that the singularly perturbed system H_ε in (3) satisfies Assumption 1. Assumption 2 holds if for each compact set $K \subset \mathbb{R}^n$ there exists a compact set $\Omega \subset (C \cap K) \times \Psi$ such that Assumption 3 holds and Ω is globally asymptotically stable for the boundary layer system in (6) with C replaced with $C \cap K$. $\square$

With Lemma 1 in hand, we come to some examples to show where Assumption 2 holds. For instance, it holds in the case when the boundary layer system H_{bl} has a globally asymptotically stable quasi-steady state equilibrium manifold $h : C \rightarrow \Psi$, which is the essential assumption for classical singular perturbation theory. The existence of the average $f_{av} : C \rightarrow \mathbb{R}^n$ of the function $f_0 : C \times \Psi \rightarrow \mathbb{R}^n$ with respect to $\psi_0 : C \times \Psi \rightarrow \mathbb{R}^m$ for the perturbed system H_ε in (5) is considered as an example.

Example 1 For continuous functions f_0 and ψ_0 , assume that there exists a continuous function $h : C \rightarrow \Psi$ such that, for each compact set $K \subset \mathbb{R}^n$, the compact set $\Omega := \{(x, z_{bl}) : x \in C \cap K, z_{bl} = h(x)\}$ is globally asymptotically stable for the boundary layer system H_{bl} formed in (6) with C replaced by $C \cap K$. Then, we show that the function $x \mapsto f_0(x, h(x))$ is the average of f_0 with respect to ψ_0 .

Based on Lemma 1, we just need to show that Assumption 3 holds. From the global asymptotic stability of Ω , in particular, its forward invariance, we have that if $(x, z_{bl}(0)) \in \Omega$ then $z_{bl}(s) = h(x)$ for all $s \geq 0$. It is then immediate that Assumption 3 holds for $f_{av}(x) = f_0(x, h(x))$ using $\sigma_\Omega(L) \equiv 0$.

The following example illustrates the existence of the average of f_0 with respect to ψ_0 if the dynamics of the boundary layer system follow an oscillator that converges to a stable limit cycle.

Example 2 Consider a singularly perturbed system with $x \in \mathbb{R}$ and $z \in \mathbb{R}^2$:

$$\left. \begin{aligned} \dot{x} &= \varepsilon f(x, z) \\ \dot{z} &= \psi(z) + \varepsilon \varphi(x, z) \end{aligned} \right\}, \quad (x, z) \in C \times \Psi, \quad (8)$$

where $C := \{x : x \geq 0\}$, Ψ is a compact set satisfying $\mathbb{S}^1 \subset \Psi \subset \mathbb{R}^2 \setminus \{0\}$ with $\mathbb{S}^1$ being the unit circle, $\varphi : C \times \Psi \rightarrow \Psi$ is locally bounded, and

$$\begin{aligned} f(x, z) &:= (-0.5x + xz_1), \\ \psi(z) &:= \begin{bmatrix} z_1 - z_2 - z_1 \sqrt{z_1^2 + z_2^2} \\ z_1 + z_2 - z_2 \sqrt{z_1^2 + z_2^2} \end{bmatrix}. \end{aligned} \quad (9)$$

It is more straightforward to consider the dynamics $\dot{z} = \psi(z)$ in polar coordinates with $z_1 = \rho \sin \theta$ and $z_2 = \rho \cos \theta$: $\dot{\rho} = \rho(1 - \rho)$ and $\dot{\theta} = 1$, which shows that the solution of $\dot{z} = \psi(z)$ is an oscillator that asymptotically converges to the unit circle $\mathbb{S}^1 \subset \Psi$. For each compact $K \subset \mathbb{R}$, let $\Omega := (C \cap K) \times \mathbb{S}^1$. Noting that Ω is globally asymptotically stable for the system H_{bl} of (8) with C replaced by $(C \cap K)$, the solution z_1 of system H_{bl} with $(x, z(0)) \in \Omega$ satisfies $z_1(t) = \sin(t + \phi)$, where the parameter ϕ is determined by the initial condition of z .

Then, for each $L > 0$ and solution $z_1 : [0, L] \mapsto \Psi$ of system H_{bl} satisfying $(x, z(0)) \in \Omega$, it follows that,

$$\begin{aligned} & \left| \frac{1}{L} \int_0^L [(-0.5x + xz_1(s)) + 0.5x] ds \right| \\ & \leq \frac{|x|}{L} \left| \int_0^L \sin(s + \phi) ds \right| \leq \frac{2 \max_{x \in K} |x|}{L} := \sigma_\Omega(L), \end{aligned}$$

where the last inequality holds since the integration of a sinusoid function over a finite time is less than 2. From Lemma 1 and the fact that σ_Ω is of class- $\mathcal{L}$, we have the average of system (8) is:

$$\xi' = -0.5\xi, \quad \xi \in C. \quad (10)$$

Note that global asymptotic stability of the boundary layer system in Example 1 or the condition that solutions of the boundary layer system converge to stable limit cycle in Example 2 is not necessary for the existence of an average. To illustrate this, we revisit Example 2 and redefine $\psi : \Psi \rightarrow \Psi$ such that the boundary layer system contains equilibria that are neither stable nor attractive but the average is equal to what would be obtained by

restricting the initial conditions of the boundary layer system to these equilibria.

Example 3 Consider the singularly perturbed system (8) with replacing ψ in (9) as

$$\psi(z) := \begin{bmatrix} -\frac{z_1(\sqrt{z_1^2 + z_2^2} - 1)^3}{\sqrt{z_1^2 + z_2^2}} + z_1^2 z_2 + z_2(1 - \sqrt{z_1^2 + z_2^2})^2 \\ -\frac{z_2(\sqrt{z_1^2 + z_2^2} - 1)^3}{\sqrt{z_1^2 + z_2^2}} - z_1^3 - z_1(1 - \sqrt{z_1^2 + z_2^2})^2 \end{bmatrix}. \quad (11)$$

Like in Example 2, the set Ψ is a compact set satisfying $\mathbb{S}^1 \subset \Psi \subset \mathbb{R}^2 \setminus \{0\}$.

Noting the dynamics $\dot{z} = \psi(z)$ in polar coordinates:

$$\begin{aligned} \dot{\rho} &= -(\rho - 1)^3, \\ \dot{\theta} &= \rho^2 \sin^2 \theta + (1 - \rho)^2, \end{aligned} \quad (12)$$

we know that θ is unbounded when the solution of $\dot{z} = \psi(z)$ starts off the unit circle $\mathbb{S}^1$, since the first term of righthand of dynamics of θ is positive and second term is not integrable. Considering that $\dot{\theta} > 0$ and $\theta(t)$ is unbounded when t grows, we know that the equilibria of the boundary layer system H_{bl} of system (8) are neither stable nor attractive for any solution of system H_{bl} that starts in $\Psi \setminus \mathbb{S}^1$.

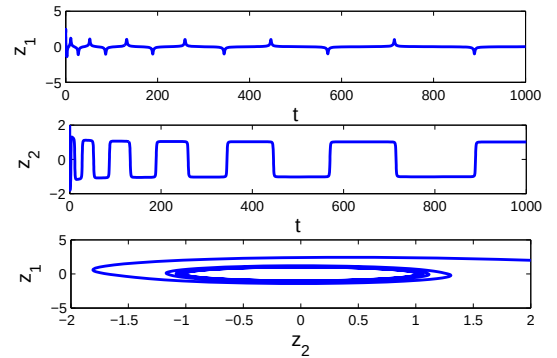


Fig. 1. Trajectories of the solution z of $\dot{z} = \psi_0(z)$.

From Fig. 1, we can see that solutions of $\dot{z} = \psi(z)$ starting off the unit circle $\mathbb{S}^1$, tend toward $\mathbb{S}^1$ while rotating in the counterclockwise direction with motion that becomes arbitrarily slow at points arbitrarily close to $(0, 1)$ or $(0, -1)$. Let $K = [-M, M]$ and defining $\Omega := (C \cap K) \times \mathbb{S}^1$. It is clear that Ω is globally asymptotically stable for the boundary layer system with C replaced by $C \cap K$. We next consider if there exists a $\sigma_\Omega \in \mathcal{L}$ such that Assumption 3 holds to invoke Lemma 1.

Note that system (12) degenerates into $\dot{\theta} = \sin^2 \theta$ when $z(0) \in \mathbb{S}^1$, for which the solution is $\theta(t) = \text{arccotg}(\text{ctg}(\theta(0)) - t)$ with $\theta(0)$ being the initial condition. Noting that cotangent function is periodic with period π , we can consider the solution $\theta(t)$ in a period $[0, \pi]$. For

arbitrary $\delta \in (0, 1/2]$, let

$$\begin{aligned} \theta(0) &:= \arcsin(\delta) \in [0, \pi/6], \\ T(\delta) &:= \{t : \theta(t) = \pi - \theta(0)\}, \end{aligned} \quad (13)$$

and through the solution $\theta(t) = \arccos(\cos(\theta(0)) - t)$, we have $T(\delta) = 2\sqrt{1-\delta^2}$. Noting that $T : (0, 0.5] \rightarrow \mathbb{R}_{>0}$ is continuous, strictly decreasing, bounded away from zero and $\lim_{\delta \rightarrow 0} T(\delta) = \infty$, there exists a $\alpha \in \mathcal{K}_\infty$ such that $\alpha(1/\delta) = T(\delta)$ for $\delta \in (0, 0.5]$.

For each solution $\theta(t)$ of $\dot{\theta} = \sin^2 \theta$, let $T_1 := \min\{t \geq 0 : \theta(t) = \arcsin(\delta)\}$ and $T_1 := 0$ if $\sin(\theta(t)) \leq \delta$ for all $t \geq 0$. Let $T_2 \geq T_1 := \min\{t \geq 0 : \theta(t) = \arcsin(\delta)\}$ and $T_2 := T_1$ if $\sin(\theta(t)) \leq \delta$ for all $t \geq T_1$. Then, we can get that $T_2 - T_1 \leq T(\delta)$ with $T(\delta)$ in (13) and for each $L > 0$:

$$\begin{aligned} & \frac{1}{L} \int_0^L |\sin(\theta(s))| ds \\ & \leq \frac{1}{L} \left(\int_0^{T_1} \delta ds + \int_{T_1}^{T_2} 1 ds + \int_{T_2}^L \delta ds \right) \\ & \leq \frac{T(\delta)}{L} + \delta = \frac{\alpha(1/\delta)}{L} + \delta. \end{aligned} \quad (14)$$

Noting that (14) holds for arbitrary $\delta \in (0, 1/2]$, it holds for

$$\delta = \min \left\{ 0.5, \frac{1}{\alpha^{-1}(\sqrt{L})} \right\}. \quad (15)$$

Recall that the solution $z_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ of system H_{bl} satisfies $z_1(t) = \sin(\theta(t))$ when $(x, z(0)) \in \Omega$. Then, for each $M > 0$, $L > 0$, $x \in [-M, M]$ and solution z_1 of system H_{bl} with $(x, z(0)) \in \Omega$, we have

$$\begin{aligned} & \left| \frac{1}{L} \int_0^L [(-0.5x + xz_1(s)) + 0.5x] ds \right| \\ & \leq \frac{|x|}{L} \int_0^L |\sin(\theta(s))| ds, \\ & \leq M \left(\frac{\max \left\{ 2, \alpha(\sqrt{L}) \right\}}{L} + \min \left(0.5, \frac{1}{\alpha^{-1}(\sqrt{L})} \right) \right), \\ & := \sigma_\Omega(L). \end{aligned}$$

The function σ_Ω defined by the inequalities above is of class- $\mathcal{L}$, which together with Lemma 1 shows that (10) is also the average of system (8) when ψ in (9) is replaced with (11). $\square$

For the singularly perturbed system H_ε modeled in (3) or (5) with a well-defined average, we have its average

system $H_{av} := (F_{av}, G_{av}, C, D)$ formed as (1) with

$$\begin{aligned} F_{av}(x) &:= f_{av}(x), \quad \forall x \in C, \\ G_{av}(x) &:= \{v_1 \in \mathbb{R}^n : (v_1, v_2) \in G(x, z), (z, v_2) \in \Psi \times \mathbb{R}^m\} \end{aligned} \quad (16)$$

where f_{av} comes from the definition of the average, the jump mapping $G_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the projection of G on the x direction. To illustrate how to get the jump mapping G_{av} of the averaged system from G of the actual hybrid system, a simple example is given.

Example 4 Consider the hybrid system H_ε with the data (f, ψ, G, C, D, Ψ) formed as (5) with f, C, Ψ given in (8) and ψ in (9) or (11), and for some $\gamma > 0$ with G, D being defined as

$$G(x, z) := [-\gamma x + z_1^2, g(x, z)], \quad D := \{x : x \leq 0\}, \quad (18)$$

where $g(x, z)$ is an arbitrary function. Noting the definition of G_{av} in (17) and the average of f with respect to ψ on $C \times \Psi$ from Examples 2 and 3, we get the average of the hybrid system H_ε with $\xi \in \mathbb{R}$:

$$\begin{aligned} \xi' &= -0.5\xi, & \xi &\in C, \\ \xi^+ &\in -\gamma\xi + [c_3, c_4], & \xi &\in D, \end{aligned} \quad (19)$$

where the positive real numbers $c_3 := \min_{z \in \Psi} \{z_1^2\}$ and $c_4 := \max_{z \in \Psi} \{z_1^2\}$. $\square$

4 Main results

First in Theorem 1 we present results on closeness of the slow solutions x of the singularly perturbed system H_ε to the solutions of its average system H_{av} on compact time domains, under the assumption that the system H_{av} is forward pre-complete from a given compact set.

Theorem 1 Suppose that the singularly perturbed system H_ε in (3) satisfies Assumptions 1 and 2 and that its average system H_{av} defined in (1), (16) and (17) is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J , there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$ and each solution x to system H_ε with $x(0, 0) \in K_0$, there exists some solution ξ to system H_{av} with $\xi(0, 0) \in K_0$ such that x and ξ are (T, J, ρ) -close. $\square$

Next, the stability properties of system H_ε are considered in Theorem 2 under a global asymptotic stability assumption on the average system H_{av} .

Theorem 2 Suppose that the singularly perturbed system H_ε in (3) satisfies Assumptions 1 and 2 and the compact set $\mathcal{A}$ is globally asymptotically stable for its average system H_{av} defined in (1), (16) and (17) with respect to $\beta \in \mathcal{KL}$. Then, the compact set $\mathcal{A} \times \Psi$ is SGP-AS for system H_ε with respect to β . $\square$

In the following example, we consider a continuous-time plant with a hybrid controller implemented through a fast actuator and with additive disturbances that are fast but have zero average. Note that the hybrid model in Example 5 is general enough to include the motivation example in [27, Section 3] as a special case and the stability analysis based on such models can find applications in areas such as robot motion control systems [30] and rotational-translational actuator systems [19]. In this case, the averaging approach to singular perturbations, as studied in the current paper, allows treating both disturbances within one framework to assess stability properties of the closed-loop system.

Example 5 Consider a continuous-time system with states $\xi \in \mathbb{R}^n$, disturbances $w \in \mathbb{R}^m$ and parameter $\varepsilon > 0$:

$$\begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)u + \ell(\xi)Qw, \\ \varepsilon \dot{w} &= Sw, \end{aligned} \quad (20)$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times l}$, $\ell : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$, $u \in \mathbb{R}^l$ is the control vector, Q is a constant matrix, the matrix S is stable such that $\dot{w} = Sw$ generates sinusoids or exponentially decaying sinusoids.

Let $\mathbb{Z}_1$ be a finite subset of the integers and $\kappa : \mathbb{R}^n \times \mathbb{Z}_1 \rightarrow \mathbb{R}^l$ be continuous. We consider the case that the hybrid state feedback u has an internal state q that takes values in a finite subset $\mathbb{Z}_1$ of the integers. Such input signal u is generated by fast actuators through $\varepsilon \dot{\zeta} = A\zeta + B\kappa(\xi, q)$ and $u = C\zeta$ with a Hurwitz matrix A and matrices C and B of proper dimensions satisfying $-CA^{-1}B = 1$. Then, the overall dynamical system is

$$\left. \begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)C\zeta + \ell(\xi)Qw \\ \dot{q} &= 0 \\ \varepsilon \dot{w} &= Sw \\ \varepsilon \dot{\zeta} &= A\zeta + B\kappa(\xi, q) \end{aligned} \right\} ((\xi, q), (w, \zeta)) \in \bar{C} \times \Psi, \\ \left. \begin{aligned} \xi^+ &= \xi \\ q^+ &\in G(\xi, q) \\ w^+ &= w \\ \zeta^+ &= \zeta \end{aligned} \right\} ((\xi, q), (w, \zeta)) \in \bar{D} \times \Psi, \quad (21)$$

where $\zeta \in \mathbb{R}^l$, $\bar{C}$, $\bar{D}$ are closed subsets of $\mathbb{R}^n \times \mathbb{Z}_1$, Ψ is compact, G is outer semicontinuous, locally bounded and not empty on $\bar{D}$.

Considering the results of Examples 1 and 3 and the fact that the sum of sinusoids has zero mean [18, Exercise 10.12], we get that the average of system (21) is the closed-loop of continuous-time plant (20) with ignored

dynamics of the actuators and disturbances:

$$\left. \begin{aligned} \dot{\xi} &= f(\xi) + g(\xi)\kappa(\xi, q) \\ \dot{q} &= 0 \end{aligned} \right\} (\xi, q) \in \bar{C}, \\ \left. \begin{aligned} \xi^+ &= \xi \\ q^+ &\in G(\xi, q) \end{aligned} \right\} (\xi, q) \in \bar{D}. \quad (22)$$

Note that for the closed-loop original system (21), regularity conditions for $(G, \bar{C}, \bar{D})$ and continuity of the flow mapping required in Assumption 1 are satisfied. Then, we can directly apply Theorem 2 to conclude that if the hybrid feedback $\kappa(\xi, q)$ guarantees that there exists a set $\mathcal{A} := A_1 \times \mathbb{Z}_1$, where $A_1 \subset \mathbb{R}^n$ is compact, such that it is globally asymptotically stable for the average system (22), then the set $\mathcal{A}$ is semi-globally practically asymptotically stable for the actual system (21) under the same κ . In other words, this results demonstrates that ignoring the dynamics of stable fast actuators is justified for hybrid closed loop systems. $\square$

In classical singular perturbation theory, say [34, 3, 18], the boundary layer system H_{bl} is assumed to have a globally asymptotically stable equilibrium manifold. Such an assumption is formulated as follows.

Assumption 4 For the boundary layer system H_{bl} in (6), the function $h : C \rightarrow \Psi$ is continuous and for each compact set $K \subset \mathbb{R}^n$, the compact set

$$\mathcal{M}_K := \{(x, z_{bl}) : x \in C \cap K, z_{bl} = h(x)\}$$

is globally asymptotically stable with respect to $\beta \in \mathcal{KL}$.

As shown in Example 1, Assumption 4 is sufficient to guarantee Assumption 2. On the other hand, as shown in Example 3, Assumption 4 is not necessary to guarantee Assumption 2. From Example 1, we know that the function $x \mapsto f_{av}(x) := f_0(x, h(x))$ is the average of f_0 with respect to ψ_0 for the system H_ε based on Assumption 4. Then, the average system $H_{av} := \{F_{av}, G_{av}, C, D\}$ of the perturbed system H_ε is formed as (1) with same G_{av} in (17) and

$$F_{av}(x) := f_0(x, h(x)), \quad \forall x \in C \quad (23)$$

and the following two corollaries follow directly from our main results. Note that the results in the following corollaries are more general than [34, 3, 18, 32], where both the closeness of solutions between the actual continuous time system with its average system and the stability properties of the actual system are considered, since the assumption of Lipschitz continuity for the functions f_0 and ψ_0 in [34, 3, 18, 32] are not needed in the current paper.

Corollary 1 Suppose that the singularly perturbed system H_ε in (3) satisfies Assumptions 1 and 4 and its av-

erage system H_{av} defined in (1), (23) and (17) is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J there exists $\varepsilon^* > 0$ such that, for each $\varepsilon \in (0, \varepsilon^*]$ and each solution x to system H_ε with $x(0, 0) \in K_0$ there exists some solution ξ to system H_{av} with $\xi(0, 0) \in K_0$ such that x and ξ are (T, J, ρ) -close. $\square$

Corollary 2 Suppose that the singularly perturbed system H_ε in (3) satisfies Assumptions 1 and 4 and the compact set $\mathcal{A}$ is globally asymptotically stable for its average system H_{av} defined in (1), (23) and (17) with respect to $\beta \in \mathcal{KL}$. Then, the compact set $\mathcal{A} \times \Psi$ is SGP-AS for system H_ε with respect to β . $\square$

We next compare our results with [27], which considers a class of hybrid control systems singularly perturbed by fast but continuous actuators, where a reduced system that omits the actuator dynamics is used in analysis of stability properties of the actual system. To extend the classical singular perturbation theory to the hybrid setting, the equilibrium manifold in Assumption 4 is replaced by a set-valued mapping $\mathcal{H} : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ in [27]. The closed-loop of the hybrid control system considered in [27] is formed as

$$\begin{aligned} \text{diag}(I_n, \varepsilon I_m) y' &\in F_1(y), & y &\in C \times \Psi, \\ y^+ &\in G_1(y), & y &\in D \times \Psi, \end{aligned} \quad (24)$$

where $y := (x, z) \in \mathbb{R}^n \times \mathbb{R}^m$, I_n and I_m respectively denote the $n \times n$ and $m \times m$ identity matrices, $F_1 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ and $G_1 : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$. In [27], the reduced system $H_r := \{F_r, G_r, C, D\}$ of the perturbed system (24) is defined as (1) with the set-valued mapping $\mathcal{H}$ and

$$\begin{aligned} F_r(x) &:= \overline{\text{con}}\{v_1 \in \mathbb{R}^n : \\ &\quad (v_1, v_2) \in F_1(x, z), z \in \mathcal{H}(x), v_2 \in \mathbb{R}^m\}, \\ G_r(x) &:= \{v_1 \in \mathbb{R}^n : (v_1, v_2) \in G_1(x, z), (z, v_2) \in \Psi \times \Psi\}. \end{aligned} \quad (25)$$

Note that G_r defined in (25) is a projection of G to the subspace of the slow state x , which is same as the definition of (17) for the average system, except that the fast states z are constrained to the compact set Ψ in (25) and it is not required for our main results. Compared with [27], the current paper gives sharper results in some cases and we revisit Examples 2 and 3 with the result of Example 4 to illustrate this.

Example 2 (continued) Consider the hybrid system $H_\varepsilon := (f, \psi, G, C, D, \Psi)$ formed as (5) with f, C, Ψ given in (8), ψ in (9) or (11), and G, D in (18). From the definition of the reduced system in (25) given in [27], noting the fact that the boundary layer system of system (8) converges to the unit circle $\mathbb{S}^1$ and letting c_3 and c_4 come from Example 4, the reduced system of [27] is

$$\begin{aligned} \xi' &\in -0.5\xi + |\xi|\mathbb{B}, & \xi &\in C, \\ \xi^+ &\in -\gamma\xi + [c_3, c_4] & \xi &\in D. \end{aligned} \quad (26)$$

Note that there are solutions for the reduced system (26) that exponentially grow unbounded.

From the average definition presented in this paper, we get that the average system of system H_ε is formed as (19) in Example 4. Recalling that $C := \{\xi : \xi \geq 0\}$ in (8), the jump mapping make all solutions starting from the set D go back to the flow set C and we know that the flow dynamics globally exponentially converge to the origin from (19). Then, we can analyze the stability properties of system H_ε with the stability of its average system (19) using Theorem 2, but we cannot draw this stability conclusion from the reduced system (26) that was used in [27]. $\square$

5 Proofs

The proofs of Theorems 1 and 2 are given in this section. Applying the coordinate transformation technique, we show in Subsection 5.2 that for any compact set, solutions of the actual hybrid system included in this set are actually the solutions of its average system under small perturbations when the parameter ε is sufficiently small. Then, with the preliminary results on properties of general hybrid systems listed in Subsection 5.1, we show Theorems 1 and 2 in Subsection 5.3 and 5.4 respectively.

5.1 Preliminary results

To apply some useful results for the general hybrid system H formed in (1), some mild conditions in Assumption 5 are assumed to hold for the system H . We also consider a hybrid system H_δ inflated from the system H to recall the robustness properties of H in Lemmas 2-3.

Assumption 5 The sets $C, D \subset \mathbb{R}^n$ are closed; $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded and for each $\xi \in C$, $F(\xi)$ is nonempty and convex; $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semi-continuous and locally bounded, for each $\xi \in D$, $G(\xi)$ is nonempty.

Consider the hybrid system H_δ inflated from system H :

$$\begin{aligned} H_\delta \quad \bar{x}' &\in F_\delta(\bar{x}), & \bar{x} &\in C_\delta, \\ \bar{x}^+ &\in G_\delta(\bar{x}), & \bar{x} &\in D_\delta, \end{aligned} \quad (27)$$

where $\bar{x} \in \mathbb{R}^n$, and for a parameter $\delta > 0$, the data $\{F_\delta, G_\delta, C_\delta, D_\delta\}$ are defined as

$$\begin{aligned} F_\delta(\bar{x}) &:= \overline{\text{con}}F((\bar{x} + \delta\mathbb{B}) \cap C) + \delta\mathbb{B}, \\ G_\delta(\bar{x}) &:= G((\bar{x} + \delta\mathbb{B}) \cap D) + \delta\mathbb{B}, \\ C_\delta &:= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap C \neq \emptyset\}, \\ D_\delta &:= \{\bar{x} : (\bar{x} + \delta\mathbb{B}) \cap D \neq \emptyset\}. \end{aligned}$$

For a given compact set $K_0 \subset \mathbb{R}^n$, let $S(K_0)$ denote the set of maximal solutions ξ to system H in (1) with

$\xi(0,0) \in K_0$. Then, we have Proposition 1, see details in [13, Corollary 4.7], that discusses the compactness of the reachable set for the system H based on its forward pre-completeness.

Proposition 1 *Suppose that system H in (1) satisfies Assumption 5 and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $T, J \geq 0$ the reachable set $R(K_0, T, J) := \{\xi(\tau, j) : \xi \in S(K_0), \tau \leq T, j \leq J\}$ is compact.* $\square$

Lemma 2, also as [13, Corollary 5.5], is about closeness between solutions of the system H and solutions of system H_δ that is inflated from H by a small parameter $\delta > 0$. The robust stability properties of the hybrid system H under small perturbations, also discussed in [13, Theorem 6.6], are given in Lemma 3.

Lemma 2 *Suppose that the system H in (1) satisfies Assumption 5, and it is forward pre-complete from a compact set $K_0 \subset \mathbb{R}^n$. Then, for each $\rho > 0$ and any strictly positive real numbers T, J there exists $\delta^* > 0$ such that for all $\delta \in (0, \delta^*]$ and any solution $\bar{x}$ of the inflated system H_δ formed as (27) with $\bar{x}(0,0) \in K_0 + \delta\mathbb{B}$ there exists a solution ξ to the system H with $\xi(0,0) \in K_0$ such that $\bar{x}$ and ξ are (T, J, ρ) -close.* $\square$

Lemma 3 *Suppose that system H in (1) satisfies Assumption 5, and the compact set $\mathcal{A}$ is globally asymptotically stable with respect to $\beta \in \mathcal{KL}$ for system H . Then, the compact set $\mathcal{A}$ is SGP-AS for the system H_δ .* $\square$

5.2 State augmentation and coordinate transformation

To employ a coordinate transformation, a continuous function reflects accumulating errors between the actual system with its average is usually constructed to facilitate averaging techniques, see early works in [5] for adaptive control systems, general averaging theory for continuous-time systems [18] and for hybrid dynamical systems [33].

We next augment H_ε with a state η that facilitates the averaging technique and is to be used in coordinate transformation. Intersect the sets C, D with K and augment the perturbed system H_ε in (5) with $\mu \geq 0$ and the state $\eta \in \mathbb{R}^n$ into the form:

$$H_K \quad \left. \begin{aligned} x' &= f(x, z, \varepsilon) \\ z' &= \frac{1}{\varepsilon} \psi(x, z, \varepsilon) \\ \eta' &= \frac{1}{\varepsilon} [f(x, z, \varepsilon) - f_{av}(x) - \mu\eta] \end{aligned} \right\}, \quad \forall (x, z, \eta) \in (C \cap K) \times \Psi \times \mathbb{R}^n, \quad (28)$$

$$\left. \begin{aligned} (x, z)^+ &\in G(x, z) \\ \eta^+ &= 0 \end{aligned} \right\}, \quad \forall (x, z, \eta) \in (D \cap K) \times \Psi \times \mathbb{R}^n.$$

For each solution (x, z, η) of H_K , letting $x = \bar{x} + \varepsilon\eta$ and noting $\eta^+ = 0$, it follows that

$$\begin{aligned} \bar{x}' &= f_{av}(x) + \mu\eta, \\ \bar{x}^+ &= x^+. \end{aligned}$$

Considering the definition of G_{av} in (17), for any given $\delta > 0$, we can write

$$\begin{aligned} \bar{x}' &= f_{av}(\bar{x} + \varepsilon\eta) + \mu\eta, & (\bar{x} + \mu\eta) &\in C, \\ \bar{x}^+ &\in G_{av}(\bar{x} + \varepsilon\eta), & (\bar{x} + \mu\eta) &\in D. \end{aligned} \quad (29)$$

Note that all solutions x of the system H_K are constrained to the compact set $K \cup G_{av}(K \cap D)$, with the compactness coming from the semi-continuity of the jump mapping G . From (29), we know that if μ, ε and η can be chosen sufficiently small such that both $\mu|\eta|$ and $\varepsilon|\eta|$ are bounded by any given $\delta > 0$, then we get a system that is inflated from the average system of H_K by δ :

$$\begin{aligned} \bar{x}' &\in f_{av}(\bar{x} + \delta\mathbb{B}) + \delta\mathbb{B}, & \bar{x} &\in C_\delta, \\ \bar{x}^+ &\in G_{av}(\bar{x} + \delta\mathbb{B}), & \bar{x} &\in D_\delta. \end{aligned} \quad (30)$$

Using Lemma 2 and the forward pre-completeness of the average system H_{av} of the perturbed system H_ε , we can consider the closeness of solutions of H_ε and H_{av} in Theorem 1 through the coordinate transformation. Similarly, with Lemma 3, we can consider the stability properties of H_ε with the global asymptotic stability of its average system H_{av} in Theorem 2. Note that the compact set K that is used to define the augmented system H_K can be constructed from the forward pre-completeness of H_{av} in the proof of Theorem 1 and global asymptotic stability of H_{av} in the proof of Theorem 2. The requirements on $\mu|\eta|$ and $\varepsilon|\eta|$ are established in the following results. The proof of Lemma 4 is given in the Appendix, while Corollary 3 follows from Lemma 4 and the discussion above.

Lemma 4 *Suppose that Assumption 1 holds for the singularly perturbed system H_ε in (5). Then, for any $\nu > 0$ and compact set $K \subset \mathbb{R}^n$ there exists $(\mu, \varepsilon^*) > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$, each solution (x, z, η) of the system H_K in (28) with $\eta(0,0) = 0$ satisfies*

$$\mu|\eta(t, j)| \leq \nu, \quad \forall (t, j) \in \text{dom}(x, z). \quad \square$$

Corollary 3 *Suppose that Assumption 1 holds for the singularly perturbed system H_ε in (5). Then, for any $\delta > 0$ and compact set $K \subset \mathbb{R}^n$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0, \varepsilon^*]$ and each solution (x, z, η) of the system H_K in (28) with $\eta(0,0) = 0$, $\bar{x} := x - \varepsilon\eta$ is the solution of the system H_δ in (27) inflated from the average system H_{av} of H_ε .* $\square$

5.3 Proof of Theorem 1

Let the compact set $K_0 \subset \mathbb{R}^n$, $T, J > 0$ and $\rho \in (0, 1)$ be given. From Lemma 2, the set K_0 with T and $\frac{\rho}{2}$ generate a $\delta^* \in (0, \rho/2)$ such that, for all $\delta \in (0, \delta^*]$ and for any solution $\bar{x}$ with $\bar{x}(0, 0) \in K_0 + \delta\mathbb{B}$ of system H_δ in (27) inflated from the average system H_{av} defined in (1), (16) and (17), there exists a solution ξ with $\xi(0, 0) \in K_0$ of system H_{av} such that ξ and $\bar{x}$ are $(T, J, \frac{\rho}{2})$ -close for any $\varepsilon > 0$. Consider a $\delta \in (0, \delta^*]$.

Let $S_{av}(K_0)$ denote the set of maximal solutions to the average system H_{av} in (1), (16) and (17) with $\xi(0, 0) \in K_0$. Define

$$\begin{aligned} R_{av}(K_0, T, J) &:= \{\xi(\tau, j) : \xi \in S_{av}(K_0), \tau \leq T, j \leq J\}, \\ K &:= (R_{av}(K_0, T, J) + \mathbb{B}) \cup G_{av}((R_{av}(K_0, T, J) + \mathbb{B}) \cap D), \end{aligned} \quad (31)$$

where K is compact from Proposition 1 and the outer semi-continuity and local boundedness of the jump map G_{av} .

Let $\varepsilon_1^* > 0$ and $\mu > 0$ be generated by Lemma 4 with the compact set K and δ . Let $\varepsilon^* := \min\{\varepsilon_1^*, \mu\}$ and consider a $\varepsilon \in (0, \varepsilon^*]$. Then, Corollary 3 shows that, for each solution (x, z, η) of system H_K with $(x(0, 0), z(0, 0)) \in K_0 \times \Psi$ and $\eta(0, 0) = 0$, $\bar{x}(\tau, j) = x(\tau, j) - \varepsilon\eta(\tau, j)$ is a solution of the system H_δ in (30) inflated from H_{av} . From the fact that $\delta < \rho/2$, x is $\frac{\rho}{2}$ close to $\bar{x}$ and then it is (T, J, ρ) -close to ξ , the latter being a solution of the average system starting in K_0 .

Now, consider a solution $\tilde{x}$ of the system H_ε in (3) with $(\tilde{x}(0, 0), \tilde{z}(0, 0)) \in K_0 \times \Psi$. According to the discussion above, if $\tilde{x}(\tau, j) \in K$ for all $(\tau, j) \in \text{dom}(\tilde{x}, \tilde{z})$ with $\tau \leq T$ and $j \leq J$, then there exists a solution ξ of the average system H_{av} such that $\tilde{x}$ is also (T, J, ρ) -close to ξ . Otherwise, suppose that there exists $(\tau, j) \in \text{dom}(\tilde{x}, \tilde{z})$ such that $\tilde{x}(s, i) \in K$ for all $(s, i) \in \text{dom}(\tilde{x}, \tilde{z})$ satisfying $s \leq \tau$, $i \leq j$ and either

Case 1 $(\tau, j+1) \in \text{dom}(\tilde{x}, \tilde{z})$ and $\tilde{x}(\tau, j+1) \notin K$ or else,

Case 2 there exist a monotonically decreasing sequence $r_i > 0$ satisfying $\lim_{i \rightarrow \infty} r_i = \tau$, $(r_i, j) \in \text{dom}(\tilde{x}, \tilde{z})$ and $\tilde{x}(r_i, j) \notin K$ for each i .

Note that the solution $\tilde{x}(s, i)$ must agree with a solution of system H_K up to time (τ, j) . Because of the relationship, stated in Corollary 3, between H_K and the average system, $\tilde{x}$ must satisfy $\tilde{x}(\tau, j) \in R_{av}(K_0, T, J) + \rho\mathbb{B}$. Then, for Case 1, using the definition of K above, $\tilde{x}(\tau, j+1) \in G_{av}((R_{av}(K_0, T, J) + \rho\mathbb{B}) \cap D) \subset K$, which is a contradiction since $\rho < 1$, while, for Case 2, there exists $\bar{\rho} \in (\rho, 1)$ such that, for large i , $\tilde{x}(r_i, j) \in R_{av}(K_0, T, J) + \bar{\rho}\mathbb{B} \subset K$, which is a contradiction. These observations establish the result.

5.4 Proof of Theorem 2

Let $\nu \in (0, 1)$ and the compact set $K_0 \subset \mathbb{R}^n$ be given. Let $\beta \in \mathcal{KL}$ and the compact set $\mathcal{A} \subset \mathbb{R}^n$ come from the definition of global asymptotic stability for the average system H_{av} . Using Lemma 3, let $\frac{\nu}{3}$ and the compact set K_0 generate a $\delta > 0$ such that each solution $\bar{x}$ of system H_δ inflated from H_{av} with $\bar{x}(0, 0) \in K_0 + \delta\mathbb{B}$ satisfies

$$|\bar{x}(\tau, j)|_{\mathcal{A}} \leq \beta(|\bar{x}(0, 0)|_{\mathcal{A}}, \tau + j) + \frac{\nu}{3}, \quad \forall (\tau, j) \in \text{dom } \bar{x}. \quad (32)$$

Define

$$\begin{aligned} K_1 &:= \left\{ x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \beta \left(\max_{\bar{x} \in K_0} |\bar{x}|_{\mathcal{A}}, 0 \right) + 1 \right\}, \\ K &:= K_1 \cup G_{av}(K_1 \cap D). \end{aligned} \quad (33)$$

The set K is compact because of continuity of the β , compactness of the set $\mathcal{A}$ and outer semi-continuity of the set mapping $G_{av} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$.

With the fact that $\beta(m, s)$ converges to zero as $s \geq 0$ approaches infinity for all $m \geq 0$, let $\varepsilon_1^* > 0$ be such that, for all $x \in K$ and $\bar{x} \in K + \varepsilon_1^*\mathbb{B}$ satisfying $|x - \bar{x}| \leq \varepsilon_1^*$, the following holds:

$$\begin{aligned} |x|_{\mathcal{A}} &\leq |\bar{x}|_{\mathcal{A}} + \frac{\nu}{3}, \\ \beta(|\bar{x}|_{\mathcal{A}}, s) &\leq \beta(|x|_{\mathcal{A}}, s) + \frac{\nu}{3}, \quad \forall s \in \mathbb{R}_{\geq 0}. \end{aligned} \quad (34)$$

System H_K defined in (28) is introduced. Let Lemma 4 with the δ and the set K generate $\varepsilon_2^* > 0$ and $\mu > 0$. Let $\varepsilon^* := \min\{\varepsilon_1^*, \varepsilon_2^*, \mu\}$ and consider a $\varepsilon \in (0, \varepsilon^*]$. Then, Corollary 3 shows that for each $\varepsilon \in (0, \varepsilon_2^*]$ and solution (x, z, η) of system H_K with $(x(0, 0), z(0, 0)) \in K_0 \times \Psi$ and $\eta(0, 0) = 0$, $\bar{x} := x - \varepsilon\eta$ is the solution to the inflated system H_δ , and then (32) holds.

Using (34), for all solutions (x, z) to system H_K with $(x(0, 0), z(0, 0)) \in K_0 \times \Psi$ and $(\tau, j) \in \text{dom}(x, z)$, we have

$$\begin{aligned} |x(\tau, j)|_{\mathcal{A}} &\leq |\bar{x}(\tau, j)|_{\mathcal{A}} + \frac{\nu}{3}, \\ &\leq \beta(|\bar{x}(0, 0)|_{\mathcal{A}}, \tau + j) + \frac{2\nu}{3}, \\ &\leq \beta(|x(0, 0)|_{\mathcal{A}}, \tau + j) + \nu. \end{aligned} \quad (35)$$

In particular, since $\nu < 1$, each solution to system H_K starting in K_0 remains in the compact set

$$K_\nu := \left\{ x \in \mathbb{R}^n : |x|_{\mathcal{A}} \leq \beta \left(\max_{\bar{x} \in K_0} |\bar{x}|_{\mathcal{A}}, 0 \right) + \nu \right\}.$$

With $\nu < 1$, K_ν is contained in K defined in (33). Considering the solutions $(\tilde{x}, \tilde{z})$ of the perturbed system H_ε

in (3) with $(\tilde{x}(0,0), \tilde{z}(0,0)) \in (C \cap K) \times \Psi$, we show that for the solution $\tilde{x}$ such that $\tilde{x}(s,i) \in K$ up to $s \leq \tau$, $i \leq j$ and Cases 1-2 in the proof of Theorem 1 are assumed occur. As $\tilde{x}$ must agree with a solution of H_K up to time (τ, j) and then satisfies (35). Noting the definition of K in (33), neither of Cases 1-2 can occur, which establishes the conclusion.

6 Conclusions

We considered a class of hybrid dynamical systems with the singular perturbations theory and the averaging method. We showed that if there exists a well defined average for the actual perturbed hybrid system, the slow solutions of the actual system on compact time domains are arbitrarily close to the solution of the average system that approximates the slow dynamics of the actual system for arbitrarily small values of the singular perturbation parameter. We also showed that the global asymptotic stability of a compact set for the average system implies that the set is semi-globally practically asymptotically stable for the actual perturbed system. Through several examples, we showed that the condition to guarantee the existence of the average is not stringent and our results are more general than the classical singular perturbation theory, where the asymptotic stability of the boundary layer system or local Lipschitz continuity of the vector fields is assumed. The continuity assumption on the slow vector field can be relaxed to an assumption that small perturbations to the solutions of the boundary layer system lead to small changes in the integral that defines the average vector field. These generalizations are useful for recovering the averaging results of [33]. Also, one can consider set-valued averages and set-valued boundary layer dynamics. These generalizations are useful for recovering the singular perturbation results in [27]. These generalizations are possible directions for further research.

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Appendix

To present the proofs of Lemmas 1 and 4, we first define functions $\mathcal{Y}_{bl} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n \times \mathbb{R}^m$, which are constructed by piece-wise concatenating the solutions (x_{bl}, z_{bl}) of the system H_{bl} . Consider an arbitrary compact set $K \subset \mathbb{R}^n$. Let $S_{bl}(K)$ denote the set of maximal solutions $(x_{bl}, z_{bl}) : \text{dom}(x_{bl}, z_{bl}) \mapsto C \times \Psi$ of the boundary layer system in (6) for $(x_{bl}, z_{bl}(0)) \in (C \cap K) \times \Psi$.

Definition 6 For any $L > 0$, $T \geq 0$ and compact set K , let $n := \lceil \frac{T}{L} \rceil$ and $\mathcal{F}(L, K, T)$ be a set of functions $\mathcal{Y}_{bl} : [0, T] \rightarrow C \times \Psi$ with $\mathcal{Y}_{bl} := (\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ such that for each integer $k \in \{0, \dots, n\}$ there exists $y^k := (x_{bl}^k, z_{bl}^k) \in S_{bl}(K)$ with $L \in \text{dom}(x_{bl}^k, z_{bl}^k)$:

$$\mathcal{Y}_{bl}(s + kL) = y^k(s), \quad \forall s \in [0, L] \text{ s.t. } (s + kL) \leq T.$$

A Proof of Lemma 1

We need two technical lemmas to prove the conclusion of Lemma 1. For a compact set $\Omega \subset C \times \Psi$, let $\tilde{S}_{bl}(\Omega)$ denote the set of maximal solutions $(x_{bl}, z_{bl}) : \text{dom}(x_{bl}, z_{bl}) \rightarrow C \times \Psi$ of the boundary layer system H_{bl} for $(x_{bl}, z_{bl}(0)) \in \Omega$. Let $\mathcal{F}(L, \Omega, T)$ be defined same as $\mathcal{F}(L, K, T)$ in Def. 6 with replacing $S_{bl}(K)$ by $\tilde{S}_{bl}(\Omega)$.

Lemma 5 Suppose that Assumption 3 holds for a compact set $\Omega \subset C \times \Psi$, $f_0 : C \times \Psi \rightarrow \mathbb{R}^n$, $\psi_0 : C \times \Psi \rightarrow \Psi$ and $f_{av} : C \rightarrow \mathbb{R}^n$ are continuous. Then, for the set Ω there exists $\alpha_\Omega \in \mathcal{K}_\infty$ such that, for each $\nu \in (0, 1]$ there exists $L > 0$ such that, for each $T \geq 0$ and each functions $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, \Omega, T)$, the following holds for all $t \in [0, T]$:

$$\frac{1}{t} \left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{av}(\mathcal{X}_{bl}(s))] ds \right| \leq \nu + \frac{\alpha_\Omega(1/\nu)}{t}.$$

□

Proof of Lemma 5: Let $\sigma_\Omega \in \mathcal{L}$ come from Assumption 3. Let arbitrary $\nu \in (0, 1]$ be given and let $L = \sigma_\Omega^{-1}(\nu)$. Let $\alpha_\Omega \in \mathcal{K}_\infty$ be such that $\sigma_\Omega^{-1}(1/s) \leq \frac{1}{\sigma_\Omega(0)} \alpha_\Omega(s)$ for $s \in [1, \infty)$. The existence of α_Ω is implied by the fact that $\sigma_\Omega^{-1}(\cdot)$ is non-decreasing and $\sigma_\Omega^{-1}(1/s)$ is bounded for $s \in [1, \infty)$. Then, we have $L\sigma_\Omega(0) \leq \alpha_\Omega(1/\nu)$, which with Claim 1 implies that $\left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{av}(\mathcal{X}_{bl}(s))] ds \right| \leq t\sigma_\Omega(L) + L\sigma_\Omega(0)$, which completes the proof. □

Lemma 6 Suppose that $f_0 : C \times \Psi \rightarrow \mathbb{R}^n$, $\psi_0 : C \times \Psi \rightarrow \Psi$ and $f_{av} : C \rightarrow \mathbb{R}^n$ are continuous. Then, for each triple of strictly positive real numbers (ν, L, ρ) there exists $\delta > 0$ such that, for each compact set $\Omega \subset C \times \Psi$ and solution (x, z) of the boundary layer system (6) with $(x(0), z(0)) \in \Omega + \delta\mathbb{B}$ there exists a function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, \Omega, T)$ where $T \geq \tilde{T} - \rho$ and $[0, \tilde{T}] = \text{dom}(x, z)$ such that

$$\begin{aligned} |f_0(x(t), z(t)) - f_0(\mathcal{X}_{bl}(t), \mathcal{Z}_{bl}(t)) - f_{av}(x(t)) \\ + f_{av}(\mathcal{X}_{bl}(t))| \leq \nu \quad t \in [0, \min\{T, \tilde{T}\}]. \end{aligned} \quad \square$$

Proof of Lemma 6: Let a compact set $\Omega \subset C \times \Psi$ be given. Let $\nu, L, \rho > 0$ be given. Considering continuity property of the functions f_0 and f_{av} , it follows that for any $\nu > 0$ there exists $\delta_1 \in (0, 1)$ such that

$$\begin{aligned} \max\{|x_1 - x_2|, |z_1 - z_2|\} \leq \delta_1 \Rightarrow \\ \begin{cases} |f_0(x_1, z_1) - f_0(x_2, z_2)| \leq \frac{\nu}{2}, \\ |f_{av}(x_1) - f_{av}(x_2)| \leq \frac{\nu}{2}. \end{cases} \end{aligned} \quad (\text{A.1})$$

Let $\rho_1 := \min\{\delta_1, \rho\}$. Let $L+1$ and ρ_1 generate a $\delta_2 > 0$ by using continuous dependence of ψ_0 on initial conditions to guarantee that solutions of the boundary layer system H_{bl} satisfy $|z(t, z_1) - z(t, z_2)| \leq \rho_1$ for all $t \in [0, L+1-\rho_1]$ and $|z_1 - z_2| \leq \delta_2$. Let $\delta := \min\{\delta_1, \delta_2, \rho\}$. For each solution $(x, z) \in \tilde{S}_{bl}(\Omega)$ and $\tilde{T} > 0$ such that $[0, \tilde{T}] = \text{dom}(x, z)$, consider any $T \geq \tilde{T} - \rho$.

For the given L , the determined $\tilde{T}$ and T , let $n := \left\lceil \frac{\min\{T, \tilde{T}\}}{L} \right\rceil$. Noting the definition of δ_2 , for each $k \in \{1, \dots, n-1\}$ and solution (x, z) of system H_{bl} with $(x(0), z(0)) \in \Omega + \delta\mathbb{B}$, it follows that there exists a solution $(x_{bl}^k, z_{bl}^k) \in \tilde{S}_{bl}(\Omega)$ of system H_{bl} and $t \in \text{dom}(x_{bl}^k, z_{bl}^k)$ with $t \in [0, L+1-\rho]$, $|x(t+kL) - x_{bl}^k(t)| \leq \delta$ and $|z(t+kL) - z_{bl}^k(t)| \leq \delta$. When $k = n$, we get that $|x(t+nL) - x_{bl}^n(t)| \leq \delta$ and $|z(t+nL) - z_{bl}^n(t)| \leq \delta$ for all $t+nL \leq \min\{T, \tilde{T}\}$. Considering the definition of function $(\mathcal{X}_{bl}, \mathcal{Y}_{bl}) \in \mathcal{F}(L, \Omega, T)$, we get that

$$\left. \begin{aligned} |x(t) - \mathcal{X}_{bl}(t)| &\leq \delta \\ |z(t) - \mathcal{Z}_{bl}(t)| &\leq \delta \end{aligned} \right\}, \quad \forall t \in [0, \min\{T, \tilde{T}\}],$$

which with (A.1) completes the proof. $\square$

Proof of Lemma 1: Let a compact set $K \subset \mathbb{R}^n$ be given. Let the compact set $\Omega \subset (C \cap K) \times \Psi$ satisfy Assumption 3 and be GAS for the boundary layer system H_{bl} with C replaced by $C \cap K$. Let $\beta \in \mathcal{KL}$ and $\sigma_\Omega \in \mathcal{L}$ come from Definition 3 and Assumption 3, respectively. Let the set K with Assumption 1 generate $M_K > 0$ such that $|f_0(x, z)| + |f_{av}(x)| \leq M_K$ for all $(x, z) \in (C \cap K) \times \Psi$. For each $\nu \in (0, 1]$, let $\alpha_\Omega \in \mathcal{K}_\infty$ be generated by the set Ω from Lemma 5.

Let $L := \sigma_\Omega^{-1}(\nu/2)$ and $(\nu/2, L, 1)$ generate a $\delta > 0$ from Lemma 6. Noting that $L(\cdot)$ is non-increasing function, it follows from the proof of Lemma 6 that $\delta(\nu)$ increases when ν grows. From the continuity condition of ψ_0 in Assumption 1 and the set Ω being GAS for system H_{bl} with respect to β , it follows that there exists a continuous, strictly decreasing function $T_K : (0, 1] \rightarrow \mathbb{R}_{\geq 0}$ that satisfies

$$\beta \left(\max_{(x, z) \in (C \cap K) \times \Psi} |(x, z)|_\Omega, T_K(\delta) \right) \leq \delta, \quad (\text{A.2})$$

which together with the fact that $\lim_{\delta \rightarrow 0} T_K(\delta) = \infty$ and $\delta(\cdot)$ is a continuous increasing function, shows that there exists $\hat{\alpha}_K \in \mathcal{K}_\infty$ satisfying $T_K(\delta(\nu)) \leq \hat{\alpha}_K(1/\nu)$.

Noting the definition of $M_K > 0$, for each solution $(x, z) \in (C \cap K) \times \Psi$ of system H_{bl} , we have:

$$\begin{aligned} & \frac{1}{L} \left| \int_0^{T_K(\delta(\nu))} [f_0(x, z(s)) - f_{av}(x)] ds \right| \\ & \leq \frac{M_K T_K(\delta(\nu))}{L} \leq \frac{M_K \hat{\alpha}_K(1/\nu)}{L}. \end{aligned} \quad (\text{A.3})$$

Let $\tilde{\mathcal{X}}_{bl}(\cdot) := \mathcal{X}_{bl}(\cdot + T_K(\delta(\nu)))$ and $\tilde{\mathcal{Y}}_{bl}(\cdot) := \mathcal{Y}_{bl}(\cdot + T_K(\delta(\nu)))$. Noting (A.2), all solutions (x, z) of system H_{bl} satisfy $(x(t), z(t)) \in \Omega + \delta\mathbb{B}$ when $t \geq T_K(\delta)$. Then, using the results of Lemmas 5 and 6 gives

$$\begin{aligned} & \frac{1}{L} \left| \int_{T_K(\delta(\nu))}^L [f_0(x, z(s)) - f_{av}(x)] ds \right| \\ & \leq \frac{1}{L} \left| \int_{T_K(\delta(\nu))}^{L-1} [f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}(s)) - f_{av}(\mathcal{X}_{bl})] ds \right| \\ & \quad + \frac{1}{L} \int_{T_K(\delta(\nu))}^{L-1} |f_0(x, z(s)) - f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}(s)) + f_{av}(\mathcal{X}_{bl}) \\ & \quad - f_{av}(x)| ds + \frac{1}{L} \left| \int_{L-1}^L [f_0(x, z(s)) - f_{av}(x)] ds \right|, \\ & \leq \frac{1}{L} \left| \int_0^{L-1-T_K(\delta(\nu))} [f_0(\tilde{\mathcal{X}}_{bl}, \tilde{\mathcal{Z}}_{bl}(s)) - f_{av}(\tilde{\mathcal{X}}_{bl})] ds \right| \\ & \quad + \frac{\nu}{2} + \frac{M_K}{L} \leq \nu + \frac{\alpha_\Omega(2/\nu)}{L} + \frac{M_K}{L}. \end{aligned} \quad (\text{A.4})$$

Combing (A.3) and (A.4), we have

$$\begin{aligned} & \frac{1}{L} \left| \int_{T_K(\delta(\nu))}^L [f_0(x, z(s)) - f_{av}(x)] ds \right| \\ & \leq \nu + \frac{\alpha_K(1/\nu)}{L} + \frac{M_K}{L}, \end{aligned} \quad (\text{A.5})$$

where $\alpha_K(s) = \alpha_\Omega(2s) + M_K \hat{\alpha}_K(s)$ is of class- $\mathcal{K}_\infty$ with α_Ω and $\hat{\alpha}_K$ of class- $\mathcal{K}_\infty$. Noting that (A.5) holds for arbitrary $\nu \in (0, 1]$, let $\nu = \min \left\{ 1, \frac{1}{\hat{\alpha}_K(\sqrt{L})} \right\}$ and substitute it in (A.5), we get for all solutions $(x, z) \in (C \cap K) \times \Psi$ of system H_{bl} :

$$\begin{aligned} & \frac{1}{L} \left| \int_0^L [f_0(x, z(s)) - f_{av}(x)] ds \right| \\ & \leq \min \left\{ 1, \frac{1}{\hat{\alpha}_K(\sqrt{L})} \right\} + \frac{\max \{ \alpha_K(1), \sqrt{L} \}}{L} + \frac{M_K}{L}, \\ & := \sigma_K(L), \end{aligned}$$

where σ_K is of class- $\mathcal{L}$. Noting that the definition of average holds with this σ_K , we know that f_{av} is the average of f_0 with respect to ψ_0 on $C \times \Psi$ and which gives the conclusion. $\square$

B Proof of Lemma 4

To prove Lemma 4, we need the following technical results in Claims 1-4.

Claim 1 Suppose that f_{av} is a continuous function that is an average of f_0 with respect to ψ_0 on $C \times \Psi$. Then, for each compact set $K \subset \mathbb{R}^n$, $L > 0$, $T \geq 0$ and all functions $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$, the following holds for all $t \in [0, T]$:

$$\left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{av}(\mathcal{X}_{bl}(s))] ds \right| \leq t\sigma_K(L) + L\sigma_K(0). \quad \square$$

Proof of Claim 1. Let a compact set $K \subset \mathbb{R}^n$, $T \geq 0$ and $L > 0$ be given and the $\mathcal{L}$ -class function σ_K be generated by the set K from the definition of average of f_0 with respect to ψ_0 . Using the definition of average in (7), we have:

$$\left| \int_0^L [f_0(x_{bl}, z_{bl}(s)) - f_{av}(x_{bl})] ds \right| \leq L\sigma_K(L).$$

For any $t \in [0, T]$ and given $L > 0$, let $n := \lfloor \frac{t}{L} \rfloor$ and then we have $t := nL + \tilde{t}$ with $0 \leq \tilde{t} < L$. With the definition of the function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl})$ in Def. 6, it follows that:

$$\begin{aligned} & \left| \int_0^t [f_0(\mathcal{X}_{bl}(s), \mathcal{Z}_{bl}(s)) - f_{av}(\mathcal{X}_{bl}(s))] ds \right| \\ & \leq \left| \sum_{k=0}^{n-1} \int_{kL}^{(k+1)L} [f_0(x_{bl}^k, z_{bl}^k(s)) - f_{av}(x_{bl}^k)] ds \right| \\ & + \left| \int_{nL}^{nL+\tilde{t}} [f_0(x_{bl}^n, z_{bl}^n(s)) - f_{av}(x_{bl}^n)] ds \right|, \\ & \leq nL\sigma_K(L) + \tilde{t}\sigma_K(\tilde{t}) \leq t\sigma_K(L) + L\sigma_K(0), \quad \forall t \in [0, T]. \end{aligned}$$

Claim 2 Suppose that f_{av} is a continuous function that is an average of f_0 with respect to ψ_0 on $C \times \Psi$. Then, for each compact set $K \subset \mathbb{R}^n$ and $\nu > 0$ there exist $L, \mu > 0$ such that, for each $T \geq 0$ and function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ and η satisfying

$$\dot{\eta} = -\mu\eta + f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{av}(\mathcal{X}_{bl}), \quad \eta(0) = 0,$$

the following holds

$$\mu|\eta(t)| \leq \nu, \quad \forall t \in [0, T]. \quad \square$$

Proof of Claim 2. Let a compact set $K \subset \mathbb{R}^n$ and $\nu > 0$ be given. Let $L > 0$ be large enough such that $\sigma_K(L) \leq \frac{\nu}{2(\exp(-1)+1)}$, where σ_K is a $\mathcal{L}$ -class function generated by the set K from the definition of average of f_0 with respect to ψ_0 . Let $\mu = \frac{\nu}{4L\sigma_K(0)}$.

Let the function $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ be such that $\dot{\phi} = f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{av}(\mathcal{X}_{bl})$ with $\phi(0) = 0$. Then, we have that $\dot{\eta} - \dot{\phi} = -\mu(\eta - \phi) - \mu\phi$. Integrating this differential equation, we have

$$\eta(t) = \phi(t) - \int_0^t \exp(\mu(s-t))(\mu\phi(s)) ds. \quad (\text{B.1})$$

Adding and subtracting $\mu\phi(t) \int_0^t \exp(\mu(s-t)) ds$ to the right hand side of (B.1), we obtain $\eta(t) = \exp(-\mu t)\phi(t) + \int_0^t \exp(-\mu(t-s))[\mu(\phi(t) - \phi(s))] ds$. For any $T \geq 0$, we have that $|\phi(t)| \leq t\sigma_K(L) + L\sigma_K(0)$ for all $t \in [0, T]$ from Claim 1. For any $s \in [0, T]$, let $\tilde{\mathcal{X}}_{bl}(\cdot) := \mathcal{X}_{bl}(s + \cdot)$ and $\tilde{\mathcal{Z}}_{bl}(\cdot) := \mathcal{Z}_{bl}(s + \cdot)$. Then, it follows that

$$\begin{aligned} |\phi(t) - \phi(s)| &= \left| \int_s^t [f_0(\mathcal{X}_{bl}(r), \mathcal{Z}_{bl}(r)) - f_{av}(\mathcal{X}_{bl}(r))] dr \right|, \\ &= \left| \int_0^{t-s} [f_0(\mathcal{X}_{bl}(s+y), \mathcal{Z}_{bl}(s+y)) - f_{av}(\mathcal{X}_{bl}(s+y))] dy \right|, \\ &= \left| \int_0^{t-s} [f_0(\tilde{\mathcal{X}}_{bl}(y), \tilde{\mathcal{Z}}_{bl}(y)) - f_{av}(\tilde{\mathcal{X}}_{bl}(y))] dy \right|, \\ &\leq (t-s)\sigma_K(L) + L\sigma_K(0). \end{aligned} \quad (\text{B.2})$$

Noting (B.2), the definitions of L and μ , and using the fact that $y \exp(-y) \leq \exp(-1)$ for all $y \geq 0$ and $\int_0^\infty \exp(-y) y dy = 1$, it follows that

$$\begin{aligned} \mu|\eta(t)| &\leq \mu \exp(-\mu t)(t\sigma_K(L) + L\sigma_K(0)) \\ &+ \mu^2 \int_0^t \exp(-\mu(t-s))[(t-s)\sigma_K(L) + L\sigma_K(0)] ds, \\ &= \sigma_K(L)\mu t \exp(-\mu t) + \mu L\sigma_K(0) \exp(-\mu t) \\ &+ \mu^2 \int_0^t \exp(-\mu r)[r\sigma_K(L) + L\sigma_K(0)] dr, \\ &\leq \sigma_K(L) \exp(-1) + \mu L\sigma_K(0) + \sigma_K(L) \int_0^\infty \exp(-y) y dy \\ &+ \mu L\sigma_K(0) \int_0^\infty \exp(-y) dy, \\ &\leq \sigma_K(L)(\exp(-1) + 1) + 2\mu L\sigma_K(0) \leq \nu, \quad \forall t \in [0, T]. \end{aligned}$$

$\square$

Claim 3 For any function η satisfying $\dot{\eta} = -\mu\eta + u_2$ with $\eta(0) = 0$, the following holds:

$$\mu|\eta(t)| \leq \|u_2\|, \quad \forall t \geq 0, \quad (\text{B.3})$$

where $u_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ and $\|u_2\| = \text{ess sup}_{t \geq 0} |u_2(t)|$. $\square$

Proof of Claim 3. Consider the differential equation $\dot{\eta} = -\mu\eta + u_2$ with the Lyapunov function $V(\eta) = \eta^2$. Note that $\dot{V} = 2\eta(-\mu\eta + u_2) < 0$ for all $\mu|\eta| > |u_2|$, which implies that $|\eta(t)| \leq \frac{\|u_2\|}{\mu}$ for all $t \geq 0$ and completes the proof. $\square$

Consider a system H_{bl}^δ inflated from system H_{bl} in (6) by $\delta > 0$ and its flow set intersected with a compact set $K \subset \mathbb{R}^n$:

$$\left. \begin{aligned} \dot{x} &\in \delta\mathbb{B} \\ \dot{z} &\in \overline{\text{con}} \psi_0((x + \delta\mathbb{B}) \cap C) + \delta\mathbb{B} \end{aligned} \right\}, (x, z) \in (C_\delta \cap K) \times \Psi_\delta, \quad (\text{B.4})$$

where $C_\delta := \{x : (x + \delta\mathbb{B}) \cap C \neq \emptyset\}$ and $\Psi_\delta := \{z : (z + \delta\mathbb{B}) \cap \Psi \neq \emptyset\}$. For this H_{bl}^δ , we give the following claim.

Claim 4 For each triple of strictly positive real number (ν, L, ρ) and compact set $K \subset \mathbb{R}^n$ there exists $\delta^* > 0$ such that, for each $\delta \in (0, \delta^*]$, each $\tilde{T} > 0$, and each solution (x, z) of the system H_{bl}^δ in (B.4) with $[0, \tilde{T}] = \text{dom}(x, z)$ there exists a function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ where $T \geq \tilde{T} - \rho$ such that

$$\begin{aligned} |f(x(t), z(t), \delta) - f_0(\mathcal{X}_{bl}(t), \mathcal{Z}_{bl}(t)) - f_{av}(x(t)) \\ + f_{av}(\mathcal{X}_{bl}(t))| \leq \nu \quad t \in [0, \min\{T, \tilde{T}\}]. \end{aligned} \quad (\text{B.5}) \quad \square$$

Proof of Claim 4. Let $\nu, L, \rho > 0$ and a compact set $K \subset \mathbb{R}^n$ be given. Considering the continuity property of the functions f , ψ and f_{av} in Assumption 1, it follows that for any $\nu > 0$ there exists $\nu_1 \in (0, 1)$ such that

$$\begin{aligned} \max\{|x_1 - x_2|, |z_1 - z_2|\} \leq \nu_1 \Rightarrow \\ \begin{cases} |f_0(x_1, z_1) - f_0(x_2, z_2)| \leq \frac{\nu}{3}, \\ |f_{av}(x_1) - f_{av}(x_2)| \leq \frac{\nu}{3}. \end{cases} \end{aligned}$$

Let Assumption 1, $\nu/3$ and the compact set K generate a δ_1^* such that the bounds (4) hold and there exists $M > 0$ such that $|\psi_0(x, z)| \leq M$ for all $(x, z) \in (C \cap K) \times \Psi$.

Let $\nu_2 := \min\left\{\frac{\nu_1}{M+2}, \rho\right\}$. Let δ_2^* be generated by Lemma 2 with the set K and a triple of determined numbers $(L+1, 0, \nu_2)$. Let $\delta^* := \min\{\delta_1^*, \delta_2^*, 1\}$ and consider $\delta \in (0, \delta^*]$. For each solution (x, z) of the system H_{bl}^δ in (B.4) and $\tilde{T} > 0$ such that $[0, \tilde{T}] = \text{dom}(x, z)$, consider any $T \geq \tilde{T} - \rho$. From the definition of δ_1^* , for each solution (x, z) of the system H_{bl}^δ if we can find

$(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ such that

$$\left. \begin{aligned} |x(t) - \mathcal{X}_{bl}(t)| &\leq \nu_1 \\ |z(t) - \mathcal{Z}_{bl}(t)| &\leq \nu_1 \end{aligned} \right\}, \quad \forall t \in [0, \min\{T, \tilde{T}\}], \quad (\text{B.6})$$

then this will complete the proof of the claim.

For the given L , the determined $\tilde{T}$ and T , let $n := \left\lceil \frac{\min\{T, \tilde{T}\}}{L} \right\rceil$. For each $k \in \{1, \dots, n-1\}$, each solution (x, z) of the system H_{bl}^δ and $l \in [0, L+1]$, Lemma 2 with $(L+1, 0, \nu_2)$ guarantees that there exists a solution $(x_{bl}^k, z_{bl}^k) \in S_{bl}(K)$ of the boundary layer system H_{bl} and $t \in \text{dom}(x_{bl}^k, z_{bl}^k)$ with $|t - l| \leq \nu_2$ such that, $|x(l+kL) - x_{bl}^k(t)| \leq \nu_2$ and $|z(l+kL) - z_{bl}^k(t)| \leq \nu_2$.

With the fact that $|\dot{x}| \leq 1$ and $|\dot{z}| \leq M+1$ for the system H_{bl}^δ , we have the solution (x, z) of H_{bl}^δ starting from the point $(x_{bl}^k(t), z_{bl}^k(t))$ such that $|x(0) - x_{bl}^k(t)| \leq \nu_2$ and $|z(0) - z_{bl}^k(t)| \leq \nu_2$ satisfies

$$\left. \begin{aligned} |x(\tilde{t}) - x_{bl}(t)| &\leq 2\nu_2 \leq \nu_1 \\ |z(\tilde{t}) - z_{bl}(t)| &\leq (M+2)\nu_2 \leq \nu_1 \end{aligned} \right\}, \quad \forall |\tilde{t} - t| \leq \nu_2,$$

which with Lemma 2 implies that for each $k \in \{1, \dots, n-1\}$ and solution (x, z) of H_{bl}^δ there exists some solution (x_{bl}^k, z_{bl}^k) of H_{bl} such that $|x(t+kL) - x_{bl}^k(t)| \leq \nu_1$ and $|z(t+kL) - z_{bl}^k(t)| \leq \nu_1$ for all $t \in [0, L+1-\nu_2]$. Noting the fact that $\nu_2 \leq \rho$ from its definition, it follows that the conclusion $|x(t+kL) - x_{bl}^k(t)| \leq \nu_1$ and $|z(t+kL) - z_{bl}^k(t)| \leq \nu_1$ also holds for all $t \in [0, L+1-\rho]$. Similarly, when $k = n$, we get $|x(t+nL) - x_{bl}^n(t)| \leq \nu_1$ and $|z(t+nL) - z_{bl}^n(t)| \leq \nu_1$ for all $t+nL \leq \tilde{T}+1-\rho$ and then holds for $t+nL \leq \min\{T, \tilde{T}\}$. Noting the definition of $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(L, K, T)$ in Def. 6, then (B.6) holds and which establishes the conclusion. $\square$

Proof of Lemma 4. Let a compact set $K \subset \mathbb{R}^n$ and $\nu > 0$ be given. Let $\mu, L > 0$ be generated from Claim 2 with $\frac{\nu}{3}$ and the set K . Let $M > 0$ be such that $\max\{|f_0(x, z)|, |f_{av}(x)|\} \leq M/2$ for all $(x, z) \in (C \cap K) \times \Psi$ from the continuity of f_0 and f_{av} . Let Claim 4, the compact set K with $(\frac{\nu}{3}, L, \frac{\nu}{3(M+1)})$ generate a $\delta_1 > 0$. Let $\delta := \min\{\delta_1, 1\}$ and Assumption 1 with K and δ generate $\varepsilon_1^* > 0$ such that the bounds (4) hold for all $\varepsilon \in (0, \varepsilon_1^*]$. Let $\varepsilon^* := \min\left\{\varepsilon_1^*, \frac{\delta}{M+1}\right\}$ and consider a $\varepsilon \in (0, \varepsilon^*]$.

For each solution (x, z) of the system H_K and $(t, j) \in \text{dom}(x, z)$, let $I_j := \{t : (t, j) \in \text{dom}(x, z)\}$, $t_j^0 := \min\{t, t \in I_j\}$ and $\tilde{T}_j := \max\{t, t \in I_j\}$. From the construction of the augmented system H_K in (28), for each

$j \in \{(t, j) \in \text{dom}(x, z)\}$, the solution $\eta : [t_j^0, T_j] \mapsto \mathbb{R}^n$ of system H_K agrees with $\dot{\eta} = -\mu\eta + u_1 + u_2$ and $\eta(t_j^0) = 0$, where $u_1 := f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{av}(\mathcal{X}_{bl})$ and $u_2 := f(x, z, \varepsilon) - f_0(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) - f_{av}(x) + f_{av}(\mathcal{X}_{bl})$.

For any $\tilde{T}_j \geq \rho$, let $T_j := \tilde{T}_j - \rho$ and the function $(\mathcal{X}_{bl}, \mathcal{Z}_{bl}) \in \mathcal{F}(K, L, T_j)$ come from Claim 4. Consider the response of η separately under inputs u_1 and u_2 on the time interval $[t_j^0, T_j]$. For the effect of u_1 , Claim 2 shows that $\mu|\eta| \leq \nu/3$. Noting the definition of ε and the fact that the solution (x, z) of the system H_K only flows for all $t \in [t_j^0, T_j]$, the solution (x, z) agrees with the system H_{bl}^δ defined in (B.4). Then, Claim 4 shows that $|u_2(t)| \leq \nu/3$ for $t \in [0, T_j]$ and Claim 3 guarantees that $\mu|\eta(t)| \leq \nu/3$ for all $t \in [0, T_j]$ under u_2 . Driven by both u_1 and u_2 , we know that $\mu|\eta(t)| \leq 2\nu/3$ for all $t \in [0, T_j]$ from the superposition principle.

Using the fact that $|u_1(t) + u_2(t)| \leq M + 1$ for all $t \geq 0$ from the definition of ε_1^* and considering $|\eta(t)|$ on $t \in [T_j, \tilde{T}_j]$ from an initial point $|\eta(T_j)| \leq \frac{2\nu}{3\mu}$, we can get $\mu|\eta(t)| \leq \frac{2\nu}{3} + (M + 1)\rho \leq \nu$ for $t \in [T_j, \tilde{T}_j]$. Similarly, we can consider $|\eta(t)|$ when $\tilde{T}_j \leq \rho$ from $\eta(t_j^0) = 0$ and get $\mu|\eta(t)| \leq (M + 1)\rho \leq \frac{\nu}{3}$ for all $t \in [t_j^0, \tilde{T}_j]$, which completes the proof. $\square$