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# Averaging for nonlinear systems on Riemannian manifolds

Farzin Taringoo\*, Dragan Nešić, Ying Tan and Peter M. Dower

**Abstract**—This paper provides a derivation of the averaging methods for nonlinear time-varying dynamical systems defined on Riemannian manifolds. We extend the results on  $\mathbb{R}^n$  to Riemannian manifolds by employing the language of differential geometry.

## I. INTRODUCTION

The state spaces of many dynamical systems constitute Riemannian manifolds (see [1], [3], [5], [6], [21]) and consequently their analyses require differential geometric tools. Examples of such systems can be found in many mechanical systems, see [5], [6].

Perturbation theory is widely used when solutions of dynamical systems cannot be obtained directly, see, for example, [9], [20] and references therein. By providing an approximation of the original dynamical system, the stability analysis of the original system can be simplified. One of the key techniques is the closeness of solutions between the actual system and the approximation. Averaging is a kind of perturbation based method. It has been used to analyze the stability properties of time-varying systems by using their time-invariant approximations that come from the averaged system, see [17], [18], [24]. Averaging results have been studied for different classes of dynamical systems and differential inclusions, see e.g. [4], [7], [23], [25]. This includes dynamical systems on Lie groups where by definition Lie groups are a special class of Riemannian manifolds, see [14]–[16].

In this paper, averaging is extended to a particular class of dynamical systems evolving on Riemannian manifolds. Such systems arise naturally in classical mechanics (see [3], [5], [6]) where the state space of the dynamical system is restricted to such a manifold. A version of averaging methods for dynamical systems on Lie groups is introduced in [14]–[16]. We address the problem of closeness of solutions in a finite time horizon on Riemannian manifolds which is a generalization of the results presented in [9], Chapter 10.

This paper is organized as follows. Section II presents some mathematical preliminaries needed for the analyses of the paper. Section III presents the main averaging results

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for dynamical systems on Riemannian manifolds in the finite time horizon. In Section IV we present an illustrative example to show the closeness of solutions on a torus as a Riemannian manifold.

## II. PRELIMINARIES AND PROBLEM FORMULATION

In this section we provide the differential geometric material which is necessary for the analyses presented in the rest of the paper. Table 1 provides a list of the frequently used symbols in this paper:

TABLE I  
 SYMBOLS AND THEIR DESCRIPTIONS

Symbol	Description
$\mathbb{R}_{>0}$	$(0, \infty)$
$\mathbb{R}_{\geq 0}$	$[0, \infty)$
$M$	Riemannian manifold
$\mathfrak{X}(M)$	space of smooth vector fields on $M$
$\mathfrak{X}(M \times \mathbb{R})$	space of smooth time-varying vector fields on $M$
$\mathfrak{X}(\mathbb{R} \times M)$	space of smooth parameter-varying vector fields on $M$
$T_x M$	tangent space at $x \in M$
$T_x^* M$	cotangent space at $x \in M$
$TM$	tangent bundle of $M$
$T^*M$	cotangent bundle of $M$
$\frac{\partial}{\partial x_i}$	basis tangent vectors at $x \in M$
$dx_i$	basis cotangent vectors at $x \in M$
$f(x, t)$	vector fields on $M$
$\ \cdot\ $	Riemannian norm
$g_x(\cdot, \cdot)$	Riemannian metric on $M$
$d(\cdot, \cdot)$	Riemannian distance on $M$
$\nabla$	Levi-Civita Connection on $M$
$\Phi_f$	flow associated with $f$
$TF$	push-forward of $F$
$T_x F$	push-forward of $F$ at $x$
$C^\infty(M)$	Space of smooth functions on $M$

This paper focuses on time-varying dynamical systems that evolve on finite dimensional Riemannian manifolds. Next, we introduce some standard concepts and results in differential geometry (see for example [1], [8], [10]–[13], [19], etc).

### A. Riemannian manifolds

**Definition 1:** (see [13], Chapter 3) A Riemannian manifold  $(M, g)$  is a differentiable manifold  $M$  together with a Riemannian metric  $g$ , where  $g$  is defined for each  $x \in M$  via an inner product  $g_x : T_x M \times T_x M \rightarrow \mathbb{R}$  on the tangent space  $T_x M$  (to  $M$  at  $x$ ) such that the function defined

by  $x \mapsto g_x(X(x), Y(x))$  is smooth for any vector fields  $X, Y \in \mathfrak{X}(M)$ . In addition,

- (i)  $(M, g)$  is  $n$ -dimensional if  $M$  is  $n$ -dimensional;
- (ii)  $(M, g)$  is connected if for any  $x, y \in M$ , there exists a piecewise smooth curve connecting them. ■

*Remark 1:* In the special case where  $M \doteq \mathbb{R}^n$ , the Riemannian metric  $g$  is defined everywhere by  $g_x = \sum_{i=1}^n dx_i \otimes dx_i$ , where  $\otimes$  is the tensor product on  $T_x^*M \times T_x^*M$ , see [13].

As stated before connected Riemannian manifolds possess the property that any pair of points  $x, y \in M$  can be connected via a path  $\gamma \in \mathcal{P}(x, y)$ , where

$$\mathcal{P}(x, y) \doteq \left\{ \gamma : [a, b] \rightarrow M \left| \begin{array}{l} \gamma \text{ piecewise smooth,} \\ \gamma(a) = x, \gamma(b) = y \end{array} \right. \right\} \quad (1)$$

*Theorem 1:* ([11], p. 94) Suppose  $(M, g)$  is an  $n$ -dimensional connected Riemannian manifold. Then, for any  $x, y \in M$ , there exists a piecewise smooth path  $\gamma \in \mathcal{P}_{x,y}$  that connects  $x$  to  $y$ . ■

The existence of connecting paths (via Theorem 1) between pairs of elements of an  $n$ -dimensional connected Riemannian manifold  $(M, g)$  facilitates the definition of a corresponding Riemannian distance. In particular, the Riemannian distance  $d : M \times M \rightarrow \mathbb{R}$  is defined as the infimal path length between any two elements of  $M$ , with

$$d(x, y) \doteq \inf_{\gamma \in \mathcal{P}(x, y)} \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt. \quad (2)$$

Note that in the special case where  $M \doteq \mathbb{R}^n$ , the Riemannian distance (2) simplifies to  $d(x, y) = \|x - y\|_e$ . Using the definition of Riemannian distance  $d$  of (2), it can be shown that  $(M, d)$  defines a metric space.

*Theorem 2:* ([11], p. 94) Any  $n$ -dimensional connected Riemannian manifold  $(M, g)$  defines a metric space  $(M, d)$  via the Riemannian distance  $d$  of (2). Furthermore, the induced topology of  $(M, d)$  is same as the manifold topology of  $(M, g)$ . ■

*Definition 2:* For a given smooth mapping  $F : M \rightarrow N$  from manifold  $M$  to manifold  $N$  the pushforward operator  $TF$  is defined as the following linear map:

$$TF : TM \rightarrow TN, \quad (3)$$

where

$$T_x F : T_x M \rightarrow T_{F(x)} N, \quad (4)$$

and

$$T_x F(X_x) \circ f = X_x(f \circ F), \quad X_x \in T_x M, f \in C^\infty(N). \quad (5)$$

*Definition 3:* ([11]) A linear connection on a manifold  $M$  is a mapping  $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ , denoted by  $(X, Y) \mapsto \nabla_X Y$  for any smooth vector fields  $X, Y \in \mathfrak{X}(M)$ , satisfying the following properties:

- (i)  $\nabla_X Y$  is linear over  $C^\infty(M)$  in  $X$ , i.e.,

$$\nabla_{fX_1+hX_2} Y = f \nabla_{X_1} Y + h \nabla_{X_2} Y, \quad (6)$$

for all  $f, h \in C^\infty(M)$ , where  $X_1, X_2, Y \in \mathfrak{X}(M)$ ;

- (ii)  $\nabla_X Y$  is linear over  $\mathbb{R}$  in  $Y$ , i.e.,

$$\nabla_X (aY_1 + bY_2) = a \nabla_X Y_1 + b \nabla_X Y_2, \quad (7)$$

for all  $a, b \in \mathbb{R}$ ,  $X, Y_1, Y_2 \in \mathfrak{X}(M)$ ;

- (iii)  $\nabla$  satisfies the product rule, i.e.,

$$\nabla_X (fY) = f \nabla_X Y + (Xf)Y, \quad (8)$$

for all  $f \in C^\infty(M)$ ,  $X, Y \in \mathfrak{X}(M)$ . ■

Note that linear connections are also sometimes referred to as *affine* connections. Linear connections can be further specialized in the case where  $(M, g)$  is a Riemannian manifold.

*Definition 4:* ([11]) A linear connection  $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$  on a Riemannian manifold  $(M, g)$  is

- 1) *compatible* with Riemannian metric  $g$  if

$$\nabla_X g(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad (9)$$

for all  $X, Y, Z \in \mathfrak{X}(M)$ ;

- 2) *symmetric* if it is torsion-free, i.e.

$$\nabla_X Y - \nabla_Y X = [X, Y], \quad (10)$$

for all  $X, Y \in \mathfrak{X}(M)$ , where

$$[X, Y](f) \doteq X(Y(f)) - Y(X(f)), \quad f \in C^\infty(M). \quad \blacksquare$$

On Riemannian manifolds, a unique linear connection which satisfies the properties above may be characterized by the following theorem.

*Theorem 3:* (Fundamental Lemma of Riemannian Geometry, [11], p.68) Given a Riemannian manifold  $(M, g)$ , there exists a *unique* linear connection  $\nabla$  on  $M$  that is both

- (i) *compatible* with the Riemannian metric  $g$ ; and
- (ii) *symmetric*. ■

The unique, linear, compatible and symmetric connection specified by Theorem 3 is known as the *Levi-Civita* connection. We employ the Levi-Civita connection to analyze the variation of piecewise smooth curves under different vector fields on  $(M, g)$  to relate the closeness of solutions for the state trajectory of a nominal time-varying system and its averaged system.

*Definition 5:* ([11]) An *admissible family* of curves on  $M$  is a continuous map

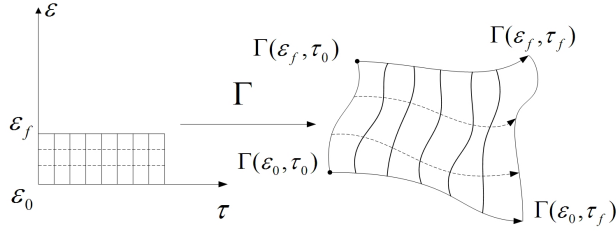


Fig. 1. Admissible Family of Curves

$\Gamma : (\epsilon_0, \epsilon_f) \times [\tau_0, \tau_f] \rightarrow M$ ,  $\epsilon, \tau \in \mathbb{R}$  such that  $\Gamma$  is smooth with respect to  $\epsilon$  and  $\tau$  (see Figure 1).

Let us denote the tangent vectors obtained by differentiating  $\Gamma$  with respect to  $\epsilon$  and  $\tau$  by

$$\partial_\tau \Gamma(\epsilon, \tau) \doteq \frac{\partial}{\partial \tau} \Gamma(\epsilon, \tau), \quad \partial_\epsilon \Gamma(\epsilon, \tau) \doteq \frac{\partial}{\partial \epsilon} \Gamma(\epsilon, \tau). \quad (11)$$

Note that in general  $\partial_\tau \Gamma(\epsilon, \tau)$  and  $\partial_\epsilon \Gamma(\epsilon, \tau)$  do not define vector fields on  $M$  since the image of  $\Gamma$  may not cover  $M$ . However, Lemma 4.1 in [13] enables us to employ the Levi-Civita connection,  $\nabla$ , of  $M$  in order to analyze the variation of  $\partial_\tau \Gamma(\epsilon, \tau)$  and  $\partial_\epsilon \Gamma(\epsilon, \tau)$  with respect to vector fields on  $M$ .

### B. Systems evolving on Riemannian manifolds

This paper focuses on a dynamical system governed by differential equations on  $M$  defined by

$$\begin{aligned} \dot{x}(t) &= f(x(t), t), \\ f(x(t), t) &\in T_{x(t)}M, \quad x(0) = x_0 \in M, t \in [t_0, t_f]. \end{aligned} \quad (12)$$

The time dependent flow associated with a differentiable time dependent vector field  $f$  is a map  $\Phi_f$  satisfying :

$$\begin{aligned} \Phi_f : [t_0, t_f] \times [t_0, t_f] \times M &\rightarrow M, \\ (t_0, s, x) &\mapsto \Phi_f(s, t_0, x) \in M, \end{aligned} \quad (13)$$

and

$$\left. \frac{d\Phi_f(s, t_0, x)}{ds} \right|_{s=t} = f(\Phi_f(t, t_0, x), t) \in T_{\Phi_f(t, t_0, x)}M. \quad (14)$$

One may show for a smooth vector field  $f$ , the integral flow  $\Phi_f(s, t_0, \cdot)$  is a local diffeomorphism, see [13].

**Definition 6:** A vector field  $f$  is *complete* if the flow  $\Phi_f(t, t_0, x_0)$  defined by (13) exists for all  $t \in [t_0, \infty)$ . (That is, if  $t_f = \infty$ .)

## III. AVERAGING ON RIEMANNIAN MANIFOLDS

The analysis of time-varying systems evolving on Riemannian manifolds can be substantially simplified by averaging which is the case for dynamical systems on Euclidean spaces [17], [18], [23], [25].

**Definition 7:** A time varying vector field  $f \in \mathfrak{X}(M \times \mathbb{R})$  is *T periodic* if

$$f(x, t + T) = f(x, t). \quad (15)$$

For a complete  $T$  periodic vector field  $f \in \mathfrak{X}(M \times [t_0; \infty))$ , the averaged system  $\hat{f} \in \mathfrak{X}(M)$  is defined as  $\dot{x}(t) = \hat{f}(x(t))$ , where

$$\hat{f}(x) \doteq \frac{1}{T} \int_0^T f(x, s) ds. \quad (16)$$

We derive the propagation equations for a single point under two different vector fields in order to bound the variation of the distance between different state trajectories.

**Assumption 1:** We assume that the vector fields  $f_1, f_2 \in \mathfrak{X}(M \times \mathbb{R})$  are complete (as per Definition 6).

**Theorem 4:** (Perturbation Theorem, [22]) Consider the following time varying dynamical systems on  $M$ :

$$\begin{aligned} \dot{x}(t) &= f_1(x(t), t), \\ \dot{y}(t) &= f_2(y(t), t), \\ x(t_0) = y(t_0) &= x_0, \quad f_1, f_2 \in \mathfrak{X}(M \times \mathbb{R}). \end{aligned} \quad (17)$$

Then,

$$\begin{aligned} d(\Phi_{f_1}(t, t_0, x_0), \Phi_{f_2}(t, t_0, x_0)) &\leq \\ K(t_1 - t_0) \exp[C(t - t_0)], \quad t &\in [t_0, t_1], \end{aligned} \quad (18)$$

for some  $K, C \in \mathbb{R}_{\geq 0}$ .

Now consider the perturbed vector fields  $f_1^\epsilon$  and  $f_2^\epsilon$  defined by

$$f_1^\epsilon(x, t) = \epsilon f_1(x, t), \quad f_2^\epsilon(x, t) = \epsilon f_2(x, t), \quad \epsilon \in \mathbb{R}_{\geq 0}. \quad (19)$$

The following lemma is an extension of the results of Theorem 4 to perturbed dynamical systems on  $M$ .

**Lemma 1:** Consider the dynamical systems

$$\begin{aligned} \dot{x}(t) &= \epsilon f_1(x(t), t), \\ \dot{y}(t) &= \epsilon f_2(y(t), t), \\ x(t_0) = y(t_0) &= x_0, \quad f_1, f_2 \in \mathfrak{X}(M \times \mathbb{R}), \end{aligned} \quad (20)$$

on  $M$ . Suppose there exists  $\epsilon_1 \in \mathbb{R}_{> 0}$  such that the integral flows  $\Phi_{\epsilon f_i}(\cdot, t_0, x_0)$ ,  $i = 1, 2$ , exist on  $[t_0, t_1]$  for  $\epsilon \in (0, \epsilon_1]$ . Then for a time interval of order  $O(1)$ ,

$$d(\Phi_{\epsilon f_1}(t, t_0, x_0), \Phi_{\epsilon f_2}(t, t_0, x_0)) = O(\epsilon), \quad t \in [t_0, t_1]. \quad (21)$$

*Proof:* We define  $\Gamma(\tau, t, \epsilon)$  as an admissible family of curves defined by

$$X(\tau, t, x, \epsilon) = \epsilon f_2(x, t) + \epsilon \tau (f_1(x, t) - f_2(x, t)) \in T_x M, \\ \tau \in [0, 1], t \in [t_0, t_1], x \in M, \quad (22)$$

such that

$$\Gamma(\tau, t, \epsilon) \doteq \Gamma_\epsilon(\tau, t) = \Phi_X(t, t_0, \gamma(\tau)) \in M, \\ \gamma(\tau) = x_0, \tau \in [0, 1], \quad (23)$$

where  $\Gamma_\epsilon(\tau, t)$  is an admissible family of curves. By construction,  $\Gamma$  is continuous with respect to  $(\tau, t)$ . Employing the results of [1], it can be shown that  $\Gamma$  is continuous with respect to  $\epsilon$ . Hence,  $\hat{D}_\Gamma$  is compact, where

$$\hat{D}_\Gamma \doteq \bigcup_{\tau \in [0, 1], t \in [t_0, t_1], \epsilon \in [0, \epsilon_1]} \Gamma(\tau, t, \epsilon). \quad (24)$$

Following the results of [22], we define  $K_\Gamma$  and  $C_i$ ,  $i = 1, 2$ , as follows:

$$\hat{K}_\Gamma \doteq \sup_{(x, t) \in \hat{D}_\Gamma \times [t_0, t_1]} \|f_1(x, t) - f_2(x, t)\|, \\ \hat{C}_i \doteq \sup_{(x, t) \in \hat{D}_\Gamma \times [t_0, t_1]} \|\nabla f_i(x, t)\|, \quad i = 1, 2, \quad (25)$$

where  $\|\nabla f_i(x, t)\|$  is the operator norm of  $\nabla f_i(x, t) : \mathfrak{X}(M \times \mathbb{R}) \rightarrow \mathfrak{X}(M \times \mathbb{R})$ . Therefore, by Theorem 4 and the results of [22],

$$d(\Phi_{f_1}(t, t_0, x_0), \Phi_{f_2}(t, t_0, x_0)) \\ \leq \epsilon \hat{K}_\Gamma (t_1 - t_0) \exp[\epsilon_1 (\hat{C}_1 + 2\hat{C}_2)(t - t_0)] = O(\epsilon), \quad (26)$$

which completes the proof.  $\blacksquare$

*Theorem 5 (Averaging Theorem):* For a smooth  $n$  dimensional compact Riemannian manifold  $(M, g)$ , let  $f \in \mathfrak{X}(M \times \mathbb{R})$  be a  $T$ -periodic smooth vector field. Then for any given  $t_1 \in [t_0, \infty)$ , such that  $t_1 - t_0 = O(\frac{1}{\epsilon})$ ,  $\epsilon \in (0, \epsilon_1)$ ,  $0 < \epsilon_1$ . Then we have

$$d(\Phi_{\epsilon f}(t, t_0, x_0), \Phi_{\hat{f}}(t, t_0, x_0)) = O(\epsilon). \quad (27)$$

In order to prove the theorem above we need to use the notion of pullbacks of vector fields along diffeomorphisms on  $M$ . Let  $X, Y \in \mathfrak{X}(M \times \mathbb{R})$  be smooth time varying vector fields on  $M$ , where it is shown that  $\Phi_Y(t, t_0, \cdot) : M \rightarrow M$  is a local diffeomorphism (see [1]). Define

$$\Phi_Y^{(t, t_0)*} : T_{\Phi_Y(t, t_0, x_0)} M \rightarrow T_{x_0} M, \\ \Phi_Y^{(t, t_0)*} X(x_0, s) = T\Phi_Y^{(t, t_0)*-1} X(\Phi_Y(t, t_0, x_0), s), \\ t, s \in \mathbb{R}, \quad (28)$$

where  $T\Phi_Y^{(t, t_0)*-1}$  is the push-forward of  $\Phi_Y^{-1}(t, t_0, \cdot) : M \rightarrow M$  defined in the standard framework of differential geometry (see [13], Chapter 3). We have the following lemma for

the variation of smoothly varying vector fields with respect to the parameter variable.

*Lemma 2:* ([2], p. 40, [6], p. 451) Consider a smooth parameter-varying vector field  $Y \in \mathfrak{X}(\mathbb{R} \times M)$  with the associated flow  $\Phi_Y(t, t_0, \cdot) : M \rightarrow M$ , then

$$\frac{\partial}{\partial \lambda} \Phi_Y(t, t_0, x_0) \\ = T_{x_0} \Phi_Y^{(t, t_0)} \times \int_{t_0}^t \Phi_Y^{(s, t_0)*} \frac{\partial}{\partial \lambda} Y(\lambda, \Phi_Y(s, t_0, x_0)) ds \\ = \int_{t_0}^t (\Phi^{-1})_{\epsilon Y}^{(t, s)*} \frac{\partial}{\partial \lambda} Y(\lambda, \Phi_Y(s, t_0, x_0)) ds \\ \in T_{\Phi_Y(t, t_0, x_0)} M. \quad (29)$$

Here we give the proof of Theorem 5 by extending the results of [9], Theorem 10.4 by following the methodology presented in [6] for dynamical systems in  $\mathbb{R}^n$ .

*Proof:* Let us define the following smooth parameter varying vector field  $Y(\lambda, x) \doteq \int_0^\lambda (f(x, s) - \hat{f}(x)) ds$ ,  $0 \leq \lambda$ , then by Lemma 2 we have

$$\frac{\partial}{\partial \lambda} \Phi_{\epsilon Y}(t, t_0, x_0) \\ = \epsilon \int_{t_0}^t (\Phi^{-1})_{\epsilon Y}^{(s, t)*} \frac{\partial}{\partial \lambda} Y(\lambda, s, \Phi_Y(s, t_0, x_0)) ds \\ = \epsilon \int_{t_0}^t (\Phi^{-1})_{\epsilon Y}^{(s, t)*} (f(x(s), \lambda) - \hat{f}(x(s))) ds. \quad (30)$$

For a given initial condition  $y_0 \in M$ , define a perturbed curve  $y : \mathbb{R} \times \mathbb{R} \rightarrow M$  by

$$y(\lambda, \tau) = \Phi_{\epsilon Y}(\tau, 0, y_0), \\ y_0 \in M, \tau \in [0, 1], \lambda \in \mathbb{R}_{\geq 0}. \quad (31)$$

For each  $x \in M$ ,  $Y(\cdot, x)$  is smooth with respect to  $\lambda$ , then  $\Phi_{\epsilon Y}(\lambda, y(\lambda, \tau))(1, 0, y_0)$  has the same degree of regularity with respect to  $\lambda$  (see [1], [6], p. 450). We have

$$D_{\Phi, \epsilon} \doteq \bigcup_{\tau \in [0, 1]} \Phi_{\epsilon Y}(\tau, 0, y_0) \subset M, \quad \lambda \in \mathbb{R}_{\geq 0}. \quad (32)$$

Now we show that

$$d(\Phi_{\epsilon Y}(1, 0, y_0), y_0) = O(\epsilon), \quad t \in [t_0, \infty). \quad (33)$$

By the definition of the length function,  $d(\Phi_{\epsilon Y}(t, t_0, y_0), y_0) \leq l(\Phi_{\epsilon Y}(t, t_0, y_0))$ , therefore

$$d(\Phi_{\epsilon Y}(1, 0, y_0), y_0) \\ \leq l(\Phi_{\epsilon Y}(1, 0, y_0)) \leq \epsilon \int_0^1 \|Y(\lambda, y)\| d\tau. \quad (34)$$

Periodicity of  $Y$  with respect to  $\lambda$ , boundedness of  $y(\lambda, \tau)$ ,  $\lambda \in [0, T]$ , in the sense of precompactness of  $D_{\Phi, \epsilon}$  (i.e.  $D_{\Phi, \epsilon}$  is contained in a compact set  $M$ ) in (32) and smoothness of  $Y$  with respect to  $y$  together yield  $d(\Phi_{\epsilon Y}(1, 0, y_0), y_0) = O(\epsilon)$ .

In order to obtain the statement of the theorem it is sufficient to prove

$$d(\Phi_{\epsilon f}(t, t_0, x_0), \Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0)) = O(\epsilon), \quad (35)$$

since

$$\begin{aligned} & d(\Phi_{\epsilon f}(t, t_0, x_0), \Phi_{\epsilon \hat{f}}(t, t_0, x_0)) \\ & \leq d(\Phi_{\epsilon f}(t, t_0, x_0), \Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0)) \\ & \quad + d(\Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0), \Phi_{\epsilon \hat{f}}(t, t_0, x_0)), \end{aligned} \quad (36)$$

where  $\Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0) \doteq \Phi_{\epsilon Y}(1, 0, \Phi_{\epsilon \hat{f}}(t, t_0, x_0))$ . Here we compute the tangent vector field of  $y(t) = \Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0) \in M$ . Obviously the derivative of  $\Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0)$  with respect to time can be computed by the chain rule as follows:

$$\begin{aligned} \dot{y}(t) &= T_{\Phi_{\epsilon \hat{f}}(t, t_0, x_0)} \Phi_{\epsilon Y}^{(1,0)} \left( \epsilon \hat{f}(\Phi_{\epsilon \hat{f}}(t, t_0, x_0)) \right) \\ & \quad + \frac{\partial}{\partial t} \Phi_{\epsilon Y}^{(1,0)} (\Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0)) \\ &= (\Phi^{-1})_{\epsilon Y}^{(1,0)*} \left( \epsilon \hat{f}(\Phi_{\epsilon \hat{f}}(t, t_0, x_0)) \right) \\ & \quad + \epsilon \int_0^1 (\Phi^{-1})_{\epsilon Y}^{(s,1)*} (f(y(s), t) - \hat{f}(y(s))) ds, \end{aligned} \quad (37)$$

where the second equality is established by the definition of pullbacks in (28) and the equation of parameter variation of flows given by Lemma 2. In a compact form, (37) can be written as

$$\begin{aligned} & \dot{y}(t) \\ &= \epsilon \left[ (\Phi^{-1})_{\epsilon Y}^{(1,0)*} \hat{f} \right. \\ & \quad \left. + \int_0^1 (\Phi^{-1})_{\epsilon Y}^{(s,1)*} (f - \hat{f}) ds \right] \circ y(t) \\ &=: \epsilon G(\epsilon, t, y(t)), \end{aligned} \quad (38)$$

where  $G(\epsilon, t, y) \in T_{y(t)}M$ . Since the vector fields  $f, \hat{f}$  are both smooth, the construction above implies that  $G$  is smooth with respect to  $\epsilon$ . One can see that by setting  $\epsilon = 0$ , the nominal vector field  $f$  is retrieved from  $G$ , i.e.  $G(0, t, x) = f(x, t)$ . This is due to the fact that at  $\epsilon = 0$ ,  $(\Phi^{-1})_{\epsilon Y}^{(1,0)*} = I$  and the state trajectory  $y(\cdot)$  will be independent of  $s$  in the integral term of  $G$  (for an identically zero vector field, the state trajectory stays on the initial point). By applying the Taylor expansion with a remainder we have

$$G(\epsilon, t, x) = f(x, t) + \epsilon h(x, \zeta, t), \quad (39)$$

where  $g(x, \zeta, t) = \frac{\partial}{\partial \epsilon} G(\epsilon, t, x)|_{\epsilon=\zeta}$  and  $\zeta \in [0, \epsilon]$  and both  $G$  and  $g$  are  $T$  periodic. Now let us explore the state variation along the following dynamical equations:

$$\begin{aligned} \dot{x}(t) &= \epsilon f(x(t), t), & x(t_0) &= x_0 \\ \dot{y}(t) &= \epsilon f(y(t), t) + \epsilon^2 h(y, \zeta, t), & y(t_0) &= x_0. \end{aligned} \quad (40)$$

We note that  $G$  is a smooth vector field on a compact Riemannian manifold  $(M, g)$ . Therefore employing the results of the Escape Lemma (see [13], Lemma 17.10) gives the completeness of  $G$  on  $M$ . Following the results of Theorem 4 and [22], we have

$$\begin{aligned} & d(\Phi_{\epsilon f}(t, t_0, x_0), \Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0)) \\ & \leq \epsilon^2 K_{\Gamma, h}(t - t_1) \exp[\epsilon(C + \epsilon \hat{C})(t - t_0)], \end{aligned} \quad (41)$$

where there exist  $0 < K_{\Gamma, h}, C, \hat{C} < \infty$  such that

$$\begin{aligned} K_{\Gamma, h} &= \sup_{(x, t) \in M \times [t_0, t_0 + T]} \|g(x, \zeta, t)\|, \\ C &= \sup_{(x, t) \in M \times [t_0, t_0 + T]} \|\nabla f(x, t)\|, \\ \hat{C} &= \sup_{(x, t) \in M \times [t_0, t_0 + T]} \|\nabla h(x, \zeta, t)\|. \end{aligned} \quad (42)$$

The parameters  $K_{\Gamma, h}, C, \hat{C}$  are all invariant with respect to  $x$  since

$$\hat{D}_{\Gamma}^{\infty} \doteq \bigcup_{\tau \in [0, 1], t \in [t_0, \infty], \epsilon \in [0, \epsilon_1]} \Gamma(\tau, t, \epsilon) \subset M, \quad (43)$$

where  $\Gamma$  is defined by (23). Also  $f$  and  $g$  are both  $T$  periodic therefore the maximization in (42) is taken on  $t \in [t_0, t_0 + T]$ .

Obviously for  $t - t_0 = O(\frac{1}{\epsilon})$  we have

$$d(\Phi_{\epsilon f}(t, t_0, x_0), \Phi_{\epsilon Y}^{(1,0)} \circ \Phi_{\epsilon \hat{f}}(t, t_0, x_0)) = O(\epsilon), \quad (44)$$

which completes the proof.  $\blacksquare$

#### IV. EXAMPLE – A SYSTEM EVOLVING ON A TORUS $\mathbf{T}^2$

In order to show the effectiveness of our results, a system evolving on a Torus  $\mathbf{T}^2$  is considered.

Consider a parametrization of  $\mathbf{T}^2$  which is given by

$$\begin{aligned} x(x_1, x_2) &= (R + r \cos(x_1)) \cos(x_2), \\ y(x_1, x_2) &= (R + r \cos(x_1)) \sin(x_2), \\ z(x_1, x_2) &= r \sin(x_2), \quad x_1, x_2 \in [-\pi, \pi]. \end{aligned} \quad (45)$$

The induced Riemannian metric is given by  $g_{T^2}(x_1, x_2) \doteq (R + r \cos(x_1))^2 dx_2 \otimes dx_2 + r^2 dx_1 \otimes dx_1$ ,  $R = 1, r = 0.5$ .

The dynamical equations are as follows:

$$f : \begin{cases} \dot{x}_1(t) = \epsilon(x_2(t) - \sin(x_1(t)) \cos(t)) \\ \dot{x}_2(t) = \epsilon \left( -\frac{1}{2} x_2(t) - \frac{1}{4} \sin(x_1(t)) + x_2(t) \cos(x_1(t)) \cos(t) - \sin(x_1(t)) \cos(x_1(t)) \cos^2(t) \right) \end{cases} \quad (46)$$

By applying averaging (see 16) to (46), the averaged system is given by

$$\hat{f} : \begin{cases} \dot{x}_1(t) = \epsilon x_2(t) \\ \dot{x}_2(t) = \epsilon \left( -\frac{1}{2} x_2(t) - \frac{1}{4} \sin(x_1(t)) - \frac{1}{4} \sin(2x_1(t)) \right) \end{cases} \quad (47)$$

Figures 2 and 3 show the closeness of solutions for the

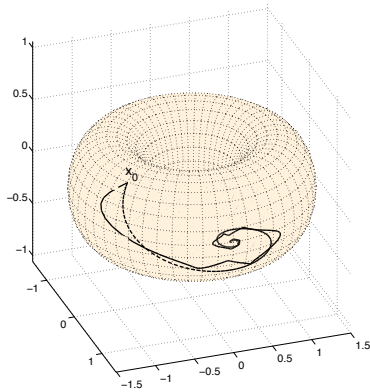


Fig. 2. State trajectories on the torus (Nominal system: solid line, Averaged system: dashed line),  $\epsilon = 0.3$ .

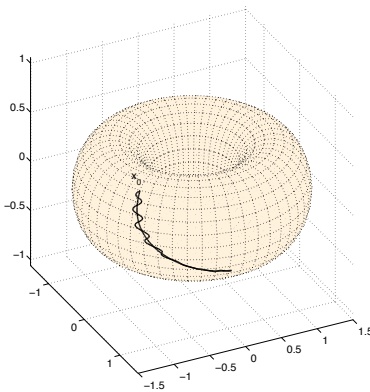


Fig. 3. State trajectories on the torus (Nominal system: solid line, Averaged system: dashed line),  $\epsilon = 0.05$ .

nominal system (46) and the averaged system (47) for  $\epsilon = 0.3$ , and  $0.05$  respectively for  $t \in [0, 50]$  as expected by the results of Theorem 5.

## V. CONCLUSION

In this paper, the averaging techniques that are used for time-varying dynamical systems are extended to dynamical systems evolving on differentiable Riemannian manifolds. This extension is carried out by employing the differential geometric tools to analyze the behavior of dynamical systems on Riemannian manifolds. Future directions of this research

will address the closeness of solutions for non-periodic dynamical systems and dynamical systems with external inputs on Riemannian manifolds and Lie groups.

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