Author/s:
Li, X;Mak-Hau, V;Zhou, S

Title:
The L(2,1)-labelling problem for cubic Cayley graphs on dihedral groups

Date:
2013-05-01

Citation:
Li, X., Mak-Hau, V. & Zhou, S. (2013). The L(2,1)-labelling problem for cubic Cayley graphs on dihedral groups. Journal of Combinatorial Optimization, 25 (4), pp.716-736. https://doi.org/10.1007/s10878-012-9525-4.

Persistent Link:
https://hdl.handle.net/11343/282617

The $L(2,1)$-labelling problem for cubic Cayley graphs on dihedral groups *

Xiangwen Li[a] , Vicky Mak-Hau[b] , Sanming Zhou[c]

[a] *Department of Mathematics, Central China Normal University, Wuhan 430079, China*

[b] *School of Information Technology, Deakin University, VIC 3125, Australia*

[c] *Department of Mathematics and Statistics, The University of Melbourne, VIC 3010, Australia*

June 20, 2012

Abstract

A k-$L(2,1)$-labelling of a graph G is a mapping $f : V(G) \to \{0,1,2,\ldots,k\}$ such that $|f(u)-f(v)| \geq 2$ if $uv \in E(G)$ and $f(u) \neq f(v)$ if u,v are distance two apart. The smallest positive integer k such that G admits a k-$L(2,1)$-labelling is called the λ-number of G. In this paper we study this quantity for cubic Cayley graphs (other than the prism graphs) on dihedral groups, which are called brick product graphs or honeycomb toroidal graphs. We prove that the λ-number of such a graph is between 5 and 7, and moreover we give a characterisation of such graphs with λ-number 5.

Key words: $L(2,1)$-labelling; λ-number; brick product; honeycomb toroidal graph; honeycomb torus; Cayley graph; dihedral group

AMS subject classification: 05C78

1 Introduction

Let G be a finite simple undirected graph and $h \geq k \geq 0$ be integers. An $L(h,k)$*-labelling* of G is a mapping f from the vertex set of G to the set of nonnegative integers such that $|f(u)-f(v)| \geq h$ if u and v are adjacent in G and $|f(u)-f(v)| \geq k$ if u and v are distance two apart in G. We call these requirements the $L(h,k)$*-conditions* and $f(v)$ the *label* of v under f. Without loss of generality we always assume that the minimum label used is 0. Under this assumption the *span* of f is defined to be the largest label used by f. Define $\lambda_{h,k}(G)$ to be the minimum span over all $L(h,k)$-labellings of G. In particular, $\lambda(G) = \lambda_{2,1}(G)$ is called the λ*-number* of G. An $L(2,1)$-labelling with span at most s is called an s-$L(2,1)$*-labelling*.

Motivated by the channel assignment problem [17] for radio networks, the $L(h,k)$-labelling problem [16] has received extensive attention especially in the case when $(h,k) = (2,1)$. The reader is referred to [7] for a survey on this topic. Griggs and Yeh [16] conjectured that $\lambda(G) \leq \Delta^2$ for any graph G with maximum degree $\Delta \geq 2$. This conjecture has been confirmed for several classes of graphs, including chordal graphs [28], outerplanar graphs [5, 6, 25], generalized Petersen graphs [14], Hamiltonian graphs with $\Delta \leq 3$ [20], two families of Hamming graphs [10, 33], etc. Improving earlier results [9, 21], Goncalves [15] proved that $\lambda(G) \leq \Delta^2 + \Delta - 2$ for any graph G with $\Delta \geq 2$. In [27], Molloy and Salavatipour proved that, for $h \geq k \geq 1$, $\lambda_{h,k}(G) \leq k\lceil 5\Delta/3\rceil + 18h + 77k - 18$ for any planar graph G; in particular, $\lambda(G) \leq \lceil 5\Delta/3\rceil + 77$ and thus the Griggs-Yeh conjecture is true for planar graphs with $\Delta \geq 9$. Recently, Havet, Reed and Sereni [18] proved that for any $h \geq 1$ there exists a constant $\Delta(h)$ such that every graph with maximum degree $\Delta \geq \Delta(h)$ has an $L(h,1)$-labelling with span at most Δ^2. In particular, this implies that the Griggs-Yeh conjecture is true for any graph with sufficiently large Δ.

*Email: xwli2808@yahoo.com (Li), vicky@deakin.edu.au (Mak-Hau), smzhou@ms.unimelb.edu.au (Zhou).

The results [18, 27] above suggests the need of studying the $L(2,1)$-labelling problem for graphs with small Δ. This is the first motivation of the present paper. Since the case $\Delta = 2$ is relatively easy, it would be interesting to investigate the 'smallest' nontrivial instances, namely graphs with $\Delta = 3$ and cubic graphs in particular. (A graph is cubic if all its vertices have degree 3.) There are signs indicating that this 'smallest' case might have different behaviours, as shown in [6, Section 5] when restricted to outerplanar graphs. Answering a question in [6], recently the first and third authors of the present paper proved [24] that $\lambda(G) \le 6$ for every outerplanar graph G with $\Delta = 3$. In [20], Kang proved that the λ-number of any graph with $\Delta = 3$ which contains a Hamiltonian cycle is at most 9. In [14], Georges and Mauro proved that any generalized Petersen graph (which is cubic), with the exception of the Petersen graph itself, has λ-number at most 8. They also conjectured that the Petersen graph is the only connected cubic graph whose λ-number is equal to 9. On the other hand, it is known [16] that the λ-number of any connected cubic graph is at least 5. It would be interesting to identify (or even characterise) those cubic graphs whose λ-numbers achieve this smallest possible value.

Given a group X with identity element 1 and a subset S of X such that $1 \notin S$ and $x \in S$ implies $x^{-1} \in S$, the *Cayley graph* of X with respect to S, denoted by $\mathrm{Cay}(X,S)$, is the graph with vertex set X in which $x, y \in X$ are adjacent if and only if $x^{-1}y \in S$. The second motivation of the present paper is the recent studies of the $L(h,k)$-labelling problem for Cayley graphs. This was initiated in [33], where a general approach to $L(h,k)$-labelling Cayley graphs on Abelian groups was proposed and results on $\lambda_{h,k}$ for hypercubes and some Hamming graphs were obtained. The work in [33] was continued in [32] and [11], where the focus was on a certain family of Cayley graphs containing hypercubes, and no-hole $L(2,0)$-labelling of Cayley graphs on Abelian groups, respectively. Recently, Bahls [4] proved that $\lambda_{h,1}(G) \le 2(h+n-1)$ if G is a Cayley graph on an n-generator group with a cetain kind of presentation, and equality holds if $h < 2n+1$.

With motivations above, in this paper we study the $L(2,1)$-labelling problem for a family of cubic Cayley graphs. These graphs have been studied independently in several contexts under different names by various authors. They were studied, and called *brick products*, in [1, 3] with an emphasis on Hamiltonian cycles and paths. In [26] they were studied as hexagonal embeddings on the torus in the context of molecular structures. In recent years they have also been studied, and called *generalised honeycomb tori*, as an attractive architecture for interconnection networks in parallel and distributed computing (see e.g. [12, 29, 30, 31]). (See [8] for a recent paper about $L(h,k)$-labelling some interconnection networks such as the butterfly networks, the cube-connected cycles, the trivalent Cayley networks, etc.) The term *honeycomb toroidal graphs* was suggested to name such graphs in a recent paper [2]. Yet, much earlier than all these works, in 1950 some of these graphs were studied in the seminal paper [13] by the great geometer Coxeter in the context of self-dual 1-designs. The purpose of the present paper is to study the $L(2,1)$-labelling problem for this remarkable family of Cayley graphs. We adapt the following definition of such graphs from [1, 3] (see Figure 1 for an illustration).

Definition 1.1. ([1, 3]) Let $l \ge 2$, $m \ge 1$ and $r \ge 0$ be integers such that $m+r$ is even. Let C_{2l} be a cycle of length $2l$. The (m,r)*-brick-product* of C_{2l}, denoted by $\mathrm{Br}(2l,m,r)$, is the graph with adjacency defined in two cases. For $m = 1$, $r \ge 3$ must be odd and $\mathrm{Br}(2l,1,r)$ is obtained from the cycle $C_{2l} = (v_0, v_1, v_2, \ldots, v_{2l-1}, v_0)$ by adding chords joining v_{2i} and v_{2i+r} for $i = 0, 1, \ldots, l-1$, where subscripts are taken modulo $2l$. In the general case where $m \ge 2$, $\mathrm{Br}(2l,m,r)$ is obtained by first taking the vertex-disjoint union of m copies of C_{2l}, denoted by

$$C_{2l}(i) = (v_{i,0}, v_{i,1}, \ldots, v_{i,2l-1}, v_{i,0}),\ i = 0, 1, \ldots, m-1. \tag{1}$$

Next, for each pair $(i,j) \in \{0, 1, \ldots, m-2\} \times \{0, 1, \ldots, 2l-1\}$ such that i and j have the same parity, an edge is added to join $v_{i,j}$ to $v_{i+1,j}$. Finally, for odd $j = 1, 3, \ldots, 2l-1$, an edge is added to join $v_{0,j}$ to $v_{m-1,j+r}$, where the second subscript is modulo $2l$.

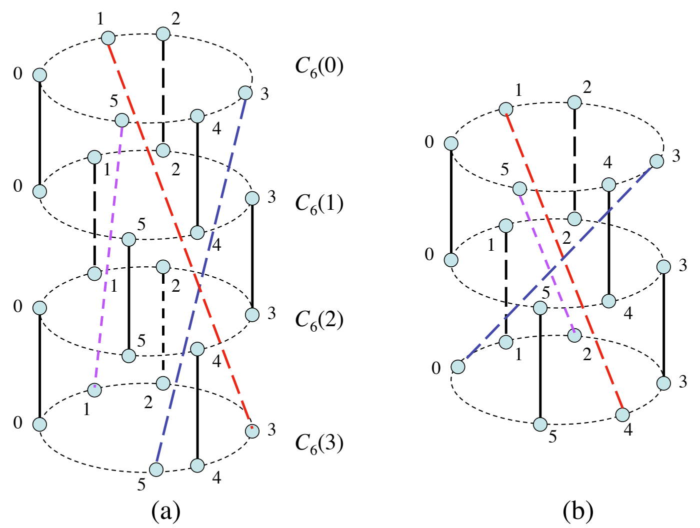

Figure 1: An illustration of brick-products: (a) Br(6, 4, 2); (b) Br(6, 3, 3).

In [2] it was proved that all brick products are Cayley graphs on generalised dihedral groups. In particular, they are all vertex-transitive.

In the present paper we focus our attention on those brick products with $m \geq 2$. (The case $m = 1$ will be dealt with in another paper [23] since it requires different techniques.) As implied in [3], these are precisely the brick products $\mathrm{Br}(2l, m, r)$ with $m \geq 2$ and $m + r \equiv 0 \pmod{2l}$. Let $D_{2n} = \langle \rho, \varepsilon \,|\, \rho^n = \varepsilon^2 = 1, \varepsilon\rho\varepsilon = \rho^{-1} \rangle$ be the dihedral group of order $2n$. It is implied in the proof of [3, Theorem 3.1] that, for any integer $n \geq 3$ and distinct integers a, b, c between 0 and $n-1$, $\mathrm{Cay}(D_{2n}, \{\rho^a\varepsilon, \rho^b\varepsilon, \rho^c\varepsilon\})$ is isomorphic to some $\mathrm{Br}(2l, m, r)$ with $m+r \equiv 0 \pmod{2l}$. (The interested reader is referred to [3] for detail.) Conversely, using essentially the same argument as in [3], one can show that any $\mathrm{Br}(2l, m, r)$ such that $m+r \equiv 0 \pmod{2l}$ is isomorphic to some $\mathrm{Cay}(D_{2n}, \{\rho^a\varepsilon, \rho^b\varepsilon, \rho^c\varepsilon\})$ with $n = lm$.

The following is the main result of this paper. We would like to emphasize that, in view of the paragraph above, this result is essentially above cubic Cayley graphs on dihedral groups.

Theorem 1.2. *Let $l, m \geq 2$ and $r \geq 0$ be integers such that $m + r \equiv 0 \pmod{2l}$. Then*

$$5 \leq \lambda(\mathrm{Br}(2l, m, r)) \leq 7. \tag{2}$$

Moreover, $\lambda(\mathrm{Br}(2l, m, r)) = 5$ if and only if one of the following holds:

(a) 3 *divides l and* 6 *divides m;*

(b) 6 *divides l and* 3 *divides m.*

Furthermore, if neither (a) nor (b) is satisfied, then $\lambda(\mathrm{Br}(2l, m, r)) = 6$ provided that $m = 2$ (and l is even or odd), or both l and m are even.

In [20] it is proved that the λ-number of any Hamiltonian graph with $\Delta = 3$ is at most 9. Since all brick product graphs are cubic and Hamiltonian [3], Theorem 1.2 can be viewed as an improvement of this bound for the family of cubic Cayley graphs on dihedral groups. Theorem 1.2 also confirms the conjecture [14] that any connected cubic graph other than the Petersen graph has λ-number at most 8, for the same family of graphs.

Theorem 1.2 gives an infinite family of cubic graphs with λ-number 5, which is smallest possible as mentioned above. As far as we know, this is the second known infinite family of cubic Cayley graphs with smallest possible λ-number, the first being the family of prisms $C_n \square K_2$ with 3 dividing n [19, 22], where $C_n \square K_2$ is the Cartesian product of the cycle C_n and the complete graph K_2 on two vertices. Interestingly, such graphs are also Cayley graphs on D_{2n}, namely $C_n \square K_2 \cong \mathrm{Cay}(D_{2n}, S)$ [3] for any $S \subset D_{2n}$ such that $1 \notin S, |S| = 3, S \cap \langle \rho \rangle \neq \emptyset$, and $x \in S$ implies $x^{-1} \in S$.

The rest of the paper is structured as follows. In Section 2 we set up notation and prove a preliminary result. In Section 3 we prove that $\lambda(\mathrm{Br}(2l, m, r)) = 5$ if and only if one of (a) and (b) in Theorem 1.2 is satisfied. In Section 4 we prove the sufficient conditions for $\lambda(\mathrm{Br}(2l, m, r)) = 6$, and in Section 5 we prove the remaining statements in Theorem 1.2.

We would like to point out that the proof of Theorem 1.2 is technical in nature, but it seems difficult to avoid such technicality.

2 Preliminaries

We will use the following lemma in the proof of Theorem 1.2.

Lemma 2.1. *([16]) The λ-number of any connected cubic graph is at least* 5.

Definition 2.2. *Given a graph G, a labelling $f : V(G) \to \{0, 1, 2, \ldots\}$ and a subset A of $\{0, 1, 2, \ldots\}$, denote*

$$f^{-1}(A) = \{u \in V(G) : f(u) \in A\}.$$

If $\lambda(G) = 5$ and f is a 5-$L(2,1)$-labelling of G, then we define

$$H_0 = \langle f^{-1}(\{0, 2, 4\}) \rangle, \; H_1 = \langle f^{-1}(\{1, 3, 5\}) \rangle$$

to be the subgraphs of G induced by $f^{-1}(\{0, 2, 4\})$, $f^{-1}(\{1, 3, 5\})$, respectively.

Lemma 2.3. *Let G be a cubic graph with $\lambda(G) = 5$ and f a 5-$L(2,1)$-labelling of G. Then the following hold.*

(a) *Each of H_0 and H_1 is a vertex-disjoint union of paths and cycles.*

(b) *The end-vertices of each path component of H_0 must be labelled 0, and that of each path component of H_1 must be labelled 5; so the length of such a path is a nonzero multiple of 3. Moreover, the vertices of any path component of H_0 are labelled (in an order of their appearance on the path) by*

$$0, 2, 4, 0, 2, 4, \ldots, 0, 2, 4, 0; \quad \text{or} \quad 0, 4, 2, 0, 4, 2, \ldots, 0, 4, 2, 0; \tag{3}$$

and the vertices of any path component of H_1 are labelled by

$$5, 3, 1, 5, 3, 1, \ldots, 5, 3, 1, 5; \quad \text{or} \quad 5, 1, 3, 5, 1, 3, \ldots, 5, 1, 3, 5. \tag{4}$$

(c) *The length of any cycle in H_0 or H_1 must be a multiple of 3. Moreover, the vertices of any cycle in H_0 are labelled (in a cyclic order) in the way described in (3), with the two end labels used on the same vertex; and the vertices of any cycle of H_1 are labelled in the way described in (4), with the two end labels used on the same vertex.*

Proof It can be verified that H_0 and H_1 have minimum degree at least one and maximum degree at most two. So each of them is a vertex-disjoint union of paths (with length at least one) and cycles.

Let $P = v_1, v_2, \ldots, v_k$ be a path component of H_0. Then $k \geq 2$ for otherwise all three neighbours of v_1 receive labels from $\{1, 3, 5\}$ but this violates the $L(2,1)$-conditions as $f(v_1) = 0, 2$ or 4. Since P is a component of H_0, the two neighbours u, w of v_1 other than v_2 receive labels from $\{1,3,5\}$. Since u and w are distance two apart or adjacent and $f(v_1) \in \{0,2,4\}$, the only possibility is that $(f(u), f(w)) = (3,5), (5,3)$ and $f(v_1) = 0$. From this one can see that the labelling of P must be in the form of (3). In the same fashion, one can prove that the end-vertices of any path component of H_1 must be labelled 5, and such a path has to be labelled as in (4). This proves the statements in (b). Part (c) can be proved similarly. □

Notation: We use $u \sim v$ and $u \simeq v$ to denote respectively that u and v are adjacent and distance two apart in the brick product under consideration. A typical argument in the following proofs goes as follows. If some vertices adjacent to a vertex u or with distance two apart from u have been labelled, then we can get the range of labels that can be assigned to u without violating the $L(2,1)$-conditions or declare that no feasible label can be assigned to u at all. For example, if $u \sim v, u \sim w$, $u \simeq x, u \simeq y$, $f(v) = 2$, $f(w) = 3$ and $f(x) = f(y) = 0$ for some 5-$L(2,1)$-labelling f, then we have to have $f(u) = 5$. This will be abbreviated to '$f(u) = 5$ $[v, w; x, y]$', and similar abbreviations will be used throughout the next two sections.

The cycles $C_{2l}(i)$ of $\mathrm{Br}(2l, m, r)$ are as defined in (1) with the understanding that the first subscript in $v_{i,j}$ is taken modulo m and the second modulo $2l$.

We assume without mentioning explicitly that $l, m \geq 2$ and $r \geq 0$ are integers such that $m + r \equiv 0 \pmod{2l}$. The last condition implies that $v_{0,2j+1} \sim v_{m-1,2j+1-m}$ for every j.

3 Brick products with λ-number 5

The plan of this section is as follows. Lemmas 3.3 and 3.4 pave the way to Lemma 3.5, which states that if $C_{2l}(j)$ is not a component of H_i, then each component of $H_i \cap C_{2l}(j)$ has exactly two vertices. This together with Lemma 3.6 leads to Lemma 3.8, which concludes that, if $C_{2l}(j)$ is not entirely in H_0 or H_1, then every component of H_0 or H_1 is in the form of a wrapped staircase. Finally, in Lemma 3.9, we will make use of all these results to prove that $\lambda(\mathrm{Br}(2l, m, r)) = 5$ if and only if one of (a) and (b) in Theorem 1.2 is satisfied.

Assumption 3.1. *In this section, before Lemma 3.9, we assume $\lambda(Br(2l, m, r)) = 5$ and we use f to denote a 5-$L(2,1)$-labelling of $Br(2l, m, r)$.*

Definition 3.2. *Define*

$$H_{0j} = \langle f^{-1}(\{0,2,4\}) \cap V(C_{2l}(j)) \rangle,\ H_{1j} = \langle f^{-1}(\{1,3,5\}) \cap V(C_{2l}(j)) \rangle,\ j = 0, 1, \ldots, m-1$$

to be the subgraphs of $\mathrm{Br}(2l, m, r)$ *induced by $V(H_0) \cap V(C_{2l}(j)), V(H_1) \cap V(C_{2l}(j))$ respectively, where H_0 and H_1 are as defined in Definition 2.2.*

Note that some H_{ij} may be empty, that is, it may have no vertices. Note also that if $C_{2l}(j)$ is not entirely in H_0 or H_1, then it is a union of path components of H_0 and H_1.

The roles of H_0 and H_1 are symmetric, and so are the roles of the H_{0j}'s and H_{1j}'s. The reason is that, for any 5-$L(2,1)$-labelling f of $\mathrm{Br}(2l, m, r)$, the labelling which assigns $5 - f(u)$ to every vertex u is also a 5-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$. This kind of symmetry will be used to simplify some proofs in the rest of this section.

Lemma 3.3. *For $i = 0, 1$ and $j = 0, 1, \ldots, m-1$, either H_{ij} is empty or every component of H_{ij} contains at least two vertices.*

Proof It suffices to prove this for H_{00} due to the symmetry of $\mathrm{Br}(2l, m, r)$ and the symmetric roles of H_0 and H_1, and we do so by way of contradiction. Suppose to the contrary that a component of H_{00} is a single vertex. Without loss of generality we may assume that this component consists of $v_{0,0}$ only, so that $v_{0,1}, v_{0,2l-1} \in V(H_1)$. This assumption together with Lemma 2.3(b) implies $v_{1,0} \in V(H_0)$. Since $v_{0,0} \sim v_{0,1}$ and $v_{0,0} \sim v_{0,2l-1}$, if $f(v_{0,0}) = 2$ or 4, then both $v_{0,1}$ and $v_{0,2l-1}$ have to be labelled 5 or 1 respectively, which is impossible since

$v_{0,1} \simeq v_{0,2l-1}$. Hence $f(v_{0,0}) = 0$ which implies $f(v_{1,0}) \in \{2,4\}$. By symmetry we may assume $f(v_{0,1}) = 3$ and $f(v_{0,2l-1}) = 5$. It follows that $f(v_{0,2}) \in \{1,5\}$ $[v_{0,1}; v_{0,0}]$.

Case 1: $f(v_{0,2}) = 1$. In this case we have $f(v_{1,2}) \in \{4,5\}$ $[v_{0,2}; v_{0,1}]$. If $f(v_{1,2}) = 4$, then $f(v_{1,0}) = 2$ because $f(v_{1,0}) \in \{2,4\}$ as shown above. So there is no label available for $v_{1,1}$ without violating the $L(2,1)$-conditions. This contradiction together with $f(v_{1,2}) \in \{4,5\}$ implies $f(v_{1,2}) = 5$. Thus $f(v_{1,0}) = 4$ for otherwise no label would be available for $v_{1,1}$. Consequently, $f(v_{1,1}) = 2$ and $f(v_{1,2l-1}) = 1$. We then have $f(v_{0,2l-2}) \in \{2,3\}$ $[v_{0,0}, v_{0,2l-1}; v_{1,2l-1}]$ and $f(v_{1,2l-2}) = 3$ $[v_{1,2l-1}; v_{0,2l-1}, v_{1,0}]$. Since $v_{0,2l-2} \sim v_{1,2l-2}$, this cannot happen.

Case 2: $f(v_{0,2}) = 5$. Since $f(v_{0,0}) = 0$ and $f(v_{1,0}) \in \{2,4\}$ as shown above, we have $f(v_{1,1}) \neq 0,3,5$. Moreover, if $f(v_{1,1}) = 1$, then no label is available for $v_{1,2}$ $[v_{1,1}, v_{0,2}; v_{0,1}]$. Thus $f(v_{1,1}) \in \{2,4\}$ and so $(f(v_{1,0}), f(v_{1,1})) = (2,4)$ or $(4,2)$. Hence $v_{0,0}, v_{1,0}, v_{1,1}$ is a path in H_0. Since $v_{1,0} \sim v_{1,2l-1}$, we have $v_{1,2l-1} \in V(H_1)$ by Lemma 2.3. If $(f(v_{1,0}), f(v_{1,1})) = (4,2)$, then $f(v_{1,2l-1}) = 1$ $[v_{1,0}; v_{0,0}, v_{1,1}]$ and consequently $f(v_{1,2l-2}) = 3$ $[v_{1,2l-1}; v_{1,0}, v_{0,2l-1}]$. It follows that no label is available for $v_{0,2l-2}$ $[v_{0,2l-1}, v_{1,2l-2}; v_{0,0}, v_{1,2l-1}]$. Thus, $(f(v_{1,0}), f(v_{1,1})) = (2,4)$, which implies $f(v_{1,2l-1}) = 5$ and $f(v_{1,2}) \in \{0,1\}$.

Note that $m + r \equiv 0 \pmod{2l}$ by our assumption. Thus, if $m = 2$, then $v_{0,1} \sim v_{1,2l-1}$. So $v_{0,2} \simeq v_{1,2l-1}$, but this contradicts the fact $f(v_{0,2}) = f(v_{1,2l-1}) = 5$.

Henceforth we assume $m \geq 3$. Then $f(v_{2,1}) \in \{0,1\}$ $[v_{1,1}; v_{1,0}]$ and so $(f(v_{1,2}), f(v_{2,1})) = (0,1)$ or $(1,0)$ as $v_{1,2} \simeq v_{2,1}$.

Subcase 2.1: $(f(v_{1,2}), f(v_{2,1})) = (0,1)$. We have $f(v_{1,3}) \in \{2,3\}$ $[v_{1,2}; v_{1,1}, v_{0,2}]$. If $f(v_{1,3}) = 2$, then $f(v_{2,3}) \in \{4,5\}$ $[v_{1,3}; v_{1,2}, v_{2,1}]$ and $f(v_{2,2}) \in \{3,5\}$ $[v_{2,1}; v_{1,1}]$. Since $v_{2,2} \sim v_{2,3}$, the only possibility is $f(v_{2,2}, v_{2,3}) = (3,5)$. We then have $f(v_{2,0}) = 5$ $[v_{2,1}; v_{2,2}, v_{1,1}]$. However, this contradicts the fact that $v_{2,0} \simeq v_{1,2l-1}$ and $f(v_{1,2l-1}) = 5$. Thus $f(v_{1,3}) \neq 2$ and so $f(v_{1,3}) = 3$. Hence $f(v_{2,3}) = 5$ $[v_{1,3}; v_{1,2}, v_{2,1}]$ and no label is available for $v_{2,2}$ $[v_{2,1}, v_{2,3}; v_{1,1}, v_{1,3}]$.

Subcase 2.2: $(f(v_{1,2}), f(v_{2,1})) = (1,0)$. Similar to Subcase 2.1, we have $f(v_{1,3}) = 3$, $f(v_{2,3}) = 5$ and $f(v_{2,2}) = 2$. We then obtain $f(v_{1,4}) = 0$ $[v_{1,3}; v_{1,2}, v_{2,3}]$, $f(v_{0,3}) = 2$ $[v_{0,2}; v_{0,1}, v_{1,2}, v_{1,4}]$ and $f(v_{0,4}) = 4$ $[v_{0,3}, v_{1,4}; v_{1,3}, v_{0,2}]$ in succession.

Since $m + r \equiv 0 \pmod{2l}$, we have $v_{m-1,1-m} \sim v_{0,1}$ and $v_{m-1,-1-m} \sim v_{0,2l-1}$. Hence $f(v_{m-1,1-m}) = 1$ $[v_{0,0}, v_{0,2}]$, $f(v_{m-1,-m}) = 4$ $[v_{m-1,1-m}; v_{0,1}, v_{0,2l-1}]$, and $f(v_{m-1,2-m}) = 5$ $[v_{m-1,1-m}; v_{m-1,-m}, v_{0,1}]$. We have $f(v_{m-1,-1-m}) = 2$ $[v_{m-1,-m}; v_{m-1,1-m}, v_{0,0}]$, $f(v_{m-2,-m}) = 0$ $[v_{m-1,-m}; v_{m-1,-1-m}, v_{m-1,1-m}]$, and $f(v_{m-2,1-m}) \in \{2,3\}$ $[v_{m-2,-m}; v_{m-1,-m}, v_{m-1,2-m}]$. Finally, we have $f(v_{m-2,2-m}) \in \{2,3\}$ $[v_{m-1,2-m}; v_{m-2,-m}, v_{m-1,1-m}]$. This is a contradiction because $v_{m-2,1-m} \sim v_{m-2,2-m}$ and hence their labels should differ by at least two.

In summary, it is impossible to have any component of H_{00} with only one vertex. By symmetry the same statement holds for other H_{ij}. □

We will use Lemma 3.3 and the following lemma to prove that each component of H_{ij} is isomorphic to K_2 or $C_{2l}(j)$ (see Lemma 3.5).

Lemma 3.4. *If $C_{2l}(i)$ is not a component of H_0, $0 \leq i \leq m-1$, then for any path component of H_{0i} with length at least two, say, $v_{i,j}, v_{i,j+1}, v_{i,j+2}, \ldots$, we have $f(v_{i,j+1}) \neq 4$ and $f(v_{i,j+2}) \neq 4$.*

Due to the symmetry of the 5-labellings $f(u)$ and $5 - f(u)$, the result in Lemma 3.4 is equivalent to the following: If $C_{2l}(i)$ is not a component of H_1, then for any path component $v_{i,j}, v_{i,j+1}, v_{i,j+2}, \ldots$ of H_{1i} with length at least two, we have $f(v_{i,j+1}) \neq 1$ and $f(v_{i,j+2}) \neq 1$.

Proof By symmetry it suffices to prove the result for $(i,j) = (0,0)$. Let $v_{0,0}, v_{0,1}, \ldots, v_{0,k-1}$ be a path component of H_{00} with length $k - 1 \geq 2$. Since $C_{2l}(0)$ is not a component of H_0, we have $H_{00} \neq C_{2l}(0)$, $3 \leq k < 2l$, and $v_{0,k}, v_{0,2l-1} \in V(H_1)$.

Claim 1: $f(v_{0,1}) \neq 4$.

Proof: Suppose otherwise. Then $f(v_{0,0}) \in \{0,2\}$. If $f(v_{0,0}) = 2$, then $(f(v_{0,2}), f(v_{m-1,1-m})) = (0,1)$ $[v_{0,1}; v_{0,0}, v_{0,2}]$. Since $v_{0,2l-1} \sim v_{0,0}$ and $f(v_{0,2l-1}) \in \{1,3,5\}$ by noting $H_{00} \neq C_{2l}(0)$, we have $f(v_{0,2l-1}) = 5$ and so $f(v_{m-1,-1-m}) \in \{0,3\}$ $[v_{0,2l-1}; v_{0,0}, v_{m-1,1-m}]$. If $f(v_{m-1,-1-m}) = 3$, then no feasible label is available for $v_{m-1,-m}$ $[v_{m-1,-1-m}, v_{m-1,1-m}; v_{0,2l-1}]$, a contradiction. Thus $f(v_{m-1,-1-m}) = 0$ and so $f(v_{m-1,-m}) = 3$ $[v_{0,1}]$, $f(v_{m-1,2-m}) = 5$ $[v_{m-1,1-m}; v_{m-1,-m}, v_{0,1}]$ and $f(v_{m-2,-m}) = 5$. We then have $f(v_{0,3}) \in \{2,3\}$ $[v_{0,2}; v_{0,1}, v_{m-1,2-m}]$, and $f(v_{0,3}) = 3$ occurs only when $k = 3$. Since $v_{0,3} \sim v_{m-1,3-m}$, no label is available for $v_{m-1,3-m}$, a contradiction.

If $f(v_{0,0}) = 0$, then $(f(v_{0,2}), f(v_{m-1,1-m})) = (2,1)$ and $f(v_{0,2l-1}) \in \{3,5\}$. If $f(v_{0,2l-1}) = 3$, then $f(v_{m-1,-1-m}) = 5$ $[v_{0,2l-1}; v_{m-1,1-m}, v_{0,0}]$ and so no feasible label is available for $v_{m-1,-m}$. So $f(v_{0,2l-1}) = 5$. If $v_{m-1,-1-m} \in V(H_1)$, then $f(v_{m-1,-1-m}) = 3$ and no feasible label can be assigned to $v_{m-1,-m}$. Thus, $v_{m-1,-1-m} \in V(H_0)$ and so $f(v_{m-1,-1-m}) = 2$ $[v_{0,2l-1}; v_{0,0}]$, implying that no feasible label is available for $v_{m-1,-m}$ $[v_{m-1,-1-m}, v_{m-1,1-m}; v_{0,1}, v_{0,2l-1}]$.

Claim 2 $f(v_{0,2}) \neq 4$.

Proof: Suppose otherwise. Then $f(v_{0,3}) \in \{0,1,2\}$. If $f(v_{0,3}) = 2$, then by Lemma 2.3(b) and noting $v_{0,2l-1} \in V(H_1)$, we have $(f(v_{0,1}), f(v_{0,0}), f(v_{1,0})) = (0,2,4)$. By Lemma 2.3(a) we then have $f(v_{1,2}) = 1$ and so no label is available for $v_{1,1}$. It remains to consider the following two cases.

Case 1: $f(v_{0,3}) = 1$. Assume $f(v_{0,0}) = 0$ first. Then $(f(v_{0,1}), f(v_{m-1,1-m})) = (2,5)$ and so $(f(v_{m-1,3-m}), f(v_{m-1,2-m})) = (3,0)$. Hence $f(v_{1,2}) = 0$ $[v_{0,2}; v_{0,1}, v_{0,3}]$ and $f(v_{0,4}) = 5$ $[v_{0,3}; v_{0,2}, v_{m-1,3-m}]$. If $v_{1,3} \in V(H_0)$, then $f(v_{1,3}) = 2$ by Lemma 2.3(b) and so there is no feasible label for $v_{1,4}$ $[v_{1,3}, v_{0,4}; v_{1,2}]$. Thus $v_{1,3} \in V(H_1)$ and so $f(v_{1,3}) = 3$. Hence no feasible label is available for $v_{1,4}$ $[v_{1,3}, v_{0,4}; v_{1,2}, v_{0,3}]$.

Next assume $f(v_{0,0}) = 2$ so that $f(v_{0,1}) = 0$. Since $v_{0,2l-1} \sim v_{0,0}$ and $v_{0,2l-1} \in V(H_1)$ by $k < 2l$, we have $f(v_{0,2l-1}) = 5$. Also $f(v_{0,4}) \in \{3,5\}$ $[v_{0,3}; v_{0,2}]$ and $f(v_{1,2}) = 2$ by Lemma 2.3(b). If $v_{1,3} \in V(H_1)$, then $f(v_{1,3}) = 5$, $(f(v_{0,4}), f(v_{1,4})) = (3,0)$, $f(v_{m-1,3-m}) = 5$ $[v_{0,3}; v_{0,2}, v_{0,4}]$ and $f(v_{m-1,1-m}) = 3$ $[v_{0,1}; v_{0,0}, v_{0,2}, v_{m-1,3-m}]$, resulting in no feasible label for $v_{m-1,2-m}$ $[v_{m-1,1-m}, v_{m-1,3-m}; v_{0,1}, v_{0,3}]$. Hence $v_{1,3} \in V(H_0)$. In view of the path $v_{0,0}, v_{0,1}, v_{0,2}$ of H_0 and by Lemma 2.3(b), we then have $(f(v_{1,2}), f(v_{1,3})) = (2,0)$. By Lemma 2.3(a), we have $v_{1,1} \in H_1$ and hence $f(v_{1,1}) = 5$ $[v_{1,2}; v_{1,3}, v_{0,2}]$, resulting in no feasible label for $v_{1,0}$ $[v_{1,1}, v_{0,0}; v_{1,2}, v_{0,1}]$.

Case 2: $f(v_{0,3}) = 0$. In this case $(f(v_{0,0}), f(v_{0,1})) = (0,2)$ by Lemma 2.3(b). So $f(v_{1,2}) = 1$ $[v_{0,2}; v_{0,1}, v_{0,3}]$. If $v_{1,0} \in V(H_0)$, then $f(v_{1,0}) = 4$ and so no feasible label can be assigned to $v_{1,1}$. Thus $v_{1,0} \in V(H_1)$ and $f(v_{1,0}) \in \{3,5\}$ as $v_{1,0} \simeq v_{1,2}$.

Subcase 2.1: $f(v_{1,0}) = 5$. We have $f(v_{1,1}) = 3$ $[v_{1,0}, v_{1,2}]$ and $f(v_{0,2l-1}) = 3$ $[v_{0,0}; v_{1,0}]$ by noting $v_{0,2l-1} \in V(H_1)$. We also have $f(v_{m-1,1-m}) = 5$ $[v_{0,1}; v_{0,0}, v_{0,2}]$ and $f(v_{m-1,-1-m}) = 1$ $[v_{0,2l-1}; v_{0,0}, v_{m-1,1-m}]$. So no feasible label is available for $v_{m-1,-m}$ $[v_{m-1,-1-m}, v_{m-1,1-m}; v_{0,2l-1}]$.

Subcase 2.2: $f(v_{1,0}) = 3$. We have $f(v_{1,1}) = f(v_{0,2l-1}) = f(v_{m-1,1-m}) = 5$ and $f(v_{1,2l-1}) = 1$. Hence $f(v_{m-1,-1-m}) \in \{1,2,3\}$ $[v_{0,2l-1}; v_{0,0}]$.

If $f(v_{m-1,-1-m}) = 2$, then $f(v_{0,2l-2}) = 3$ and there is no feasible label for $v_{1,2l-2}$.

If $f(v_{m-1,-1-m}) = 1$, then $(f(v_{m-1,-m}), f(v_{m-1,-2-m})) = (3,4)$ and $f(v_{m-1,-3-m}) \in \{0,2\}$. If $f(v_{m-1,-3-m}) = 2$, then $(f(v_{0,2l-2}), f(v_{0,2l-3})) = (3,0)$ and no label is available for $v_{1,2l-2}$. If $f(v_{m-1,-3-m}) = 0$, then $f(v_{0,2l-3}) \in \{2,3\}$ and no label is available for $v_{0,2l-2}$.

If $f(v_{m-1,-1-m}) = 3$, then $(f(v_{0,2l-2}), f(v_{1,2l-2}), f(v_{0,2l-3})) = (2,4,0)$ and $f(v_{m-1,-m}) \in \{0,1\}$. If $f(v_{m-1,-m}) \neq 1$, then no feasible label is available for $v_{m-1,-2-m}$ $[v_{m-1,-1-m}; v_{m-1,-m}, v_{0,2l-3}, v_{0,2l-1}]$. Thus, $f(v_{m-1,-m}) = 0$, leading to $f(v_{m-1,-2-m}) = 1$ and $f(v_{m-2,-2-m}) \in \{4,5\}$. If $f(v_{m-2,-2-m}) = 4$, then $f(v_{m-2,-m}) = 2$ and no label is available for $v_{m-2,-1-m}$. If $f(v_{m-2,-2-m}) = 5$, then $(f(v_{m-1,-3-m}), f(v_{m-2,-1-m})) = (4,2)$ or $(4,3)$. In the latter case, no feasible label is available for $v_{m-2,-m}$. In the former case, $(f(v_{m-2,-m}), f(v_{m-2,1-m})) = (4,1)$

and $f(v_{m-1,2-m}) = 3$ since $v_{m-1,2-m} \in V(H_1)$ by Lemma 2.3(b); consequently, no label is available for $v_{m-2,2-m}$. □

Lemma 3.5. *For $i = 0, 1$ and $j = 0, 1, \ldots, m-1$, if $C_{2l}(j)$ is not a component of H_i, then every component of H_{ij} is isomorphic to K_2.*

Proof Consider the case $i = 0$ first. It suffices to prove the statement for $j = 0$ due to the symmetry. Suppose $C_{2l}(0)$ is not a component of H_0. Then each component of H_{00} is a path with at least two vertices (Lemma 3.3). Suppose a path component of H_{00}, say, $v_{0,0}, v_{0,1}, \ldots, v_{0,k-1}$, contains three or more vertices, that is, $3 \leq k < 2l - 1$. Then $v_{0,k}, v_{0,2l-1} \in V(H_1)$ and $(f(v_{0,1}), f(v_{0,2})) = (0, 2)$ or $(2, 0)$ by Lemma 3.4. In either case we have $f(v_{0,0}) = 4$. So $f(v_{0,2l-1}) = 1$ as $v_{0,2l-1} \in V(H_1)$ and $(f(v_{0,2l-2}), f(v_{m-1,-1-m})) = (3, 5)$ or $(5, 3)$.

In the case when $(f(v_{0,1}), f(v_{0,2})) = (0, 2)$, we have $f(v_{m-1,1-m}) \in \{3, 5\}$ and so $(f(v_{0,2l-2}), f(v_{m-1,-1-m}), f(v_{m-1,1-m})) = (3, 5, 3)$ or $(5, 3, 5)$. In either case, no feasible label is available for $v_{m-1,-m}$ $[v_{m-1,-1-m}, v_{m-1,1-m}; v_{0,1}, v_{0,2l-1}]$, a contradiction.

In the case when $(f(v_{0,1}), f(v_{0,2})) = (2, 0)$, we have $f(v_{m-1,1-m}) = 5$. Hence $(f(v_{m-1,-1-m}), f(v_{0,2l-2}), f(v_{m-1,-m})) = (3, 5, 0)$. Thus $v_{m-1,-m}$ is in H_0 but the two neighbours of it on $C_{2l}(m-1)$ are in H_1, contradicting Lemma 3.3.

So far we have proved that each path component of H_{00} has exactly two vertices and so is isomorphic to K_2, completing the proof for $i = 0$. In the case $i = 1$, the same result holds due to the symmetry between f and $5 - f$. □

Lemma 3.6. *The following hold for $i = 0, 1$, $j = 0, 1, 2, \ldots, m-1$ and any integer k.*

(a) *If j is even, then H_i contains neither the path $v_{j,2k-1}, v_{j,2k-2}, v_{j+1,2k-2}, v_{j+1,2k-1}$ nor the path $v_{j,2k-1}, v_{j,2k}, v_{j+1,2k}, v_{j+1,2k-1}$;*

(b) *If j is odd, then H_i contains neither the path $v_{j,2k}, v_{j,2k-1}, v_{j+1,2k-1}, v_{j+1,2k}$ nor the path $v_{j,2k}, v_{j,2k+1}, v_{j+1,2k+1}, v_{j+1,2k}$.*

Proof It suffices to prove that H_0 does not contain path $P(j, k) : v_{j,2k-1}, v_{j,2k-2}, v_{j+1,2k-2}, v_{j+1,2k-1}$ when j is even. The other three statements are consequences of this because of the symmetry of $\mathrm{Br}(2l, m, r)$ and the symmetric roles of H_0 and H_1.

Suppose to the contrary that H_0 contains $P(j, k)$ where j is even. Then the four vertices on $P(j, k)$ receive labels from $\{0, 2, 4\}$ and moreover $f(v_{j,2k-1}) = f(v_{j+1,2k-1})$ by Lemma 2.3(b). By Lemma 3.5, we have $v_{j,2k}, v_{j+1,2k} \in V(H_1)$. Since $v_{j,2k} \sim v_{j+1,2k}$, we have $(f(v_{j,2k}), f(v_{j+1,2k})) = (1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3)$. However, since $v_{j,2k-1} \sim v_{j,2k}$ and $v_{j+1,2k-1} \sim v_{j+1,2k}$, in each case we cannot have $f(v_{j,2k-1}) = f(v_{j+1,2k-1}) = 2$ or 4. That is, $f(v_{j,2k-1}) = f(v_{j+1,2k-1}) = 0$, and hence $(f(v_{j,2k-2}), f(v_{j+1,2k-2})) = (2, 4)$ or $(4, 2)$. By Lemma 3.5, we have $v_{j,2k-3}, v_{j,2k-4}, v_{j+1,2k-3}, v_{j+1,2k-4} \in V(H_1)$. Thus $f(v_{j,2k-3}) = f(v_{j+1,2k-3}) \in \{1, 3, 5\}$ by Lemma 2.3(b). Since $(f(v_{j,2k-2}), f(v_{j+1,2k-2})) = (2, 4)$ or $(4, 2)$, and since $v_{j,2k-3} \sim v_{j,2k-2}$ and $v_{j+1,2k-3} \sim v_{j+1,2k-2}$, there exists no single label in $\{1, 3, 5\}$ that can be assigned to both $v_{j,2k-3}$ and $v_{j+1,2k-3}$ without violating the $L(2, 1)$-conditions, a contradiction. □

Definition 3.7. *For $k = 1, 2, \ldots, l$, denote*

$$L(k) = (v_{0,2k-1}, v_{0,2k-2}, v_{1,2k-2}, v_{1,2k-3}, \ldots, v_{m-2,2k-m}, v_{m-1,2k-m}, v_{m-1,2k-m-1}, v_{0,2k-1}) \quad (5)$$

$$R(k) = (v_{0,2k-1}, v_{0,2k}, v_{1,2k}, v_{1,2k+1}, \ldots, v_{m-2,2k-m-2}, v_{m-1,2k-m-2}, v_{m-1,2k-m-1}, v_{0,2k-1}). \quad (6)$$

These are cycles of $\mathrm{Br}(2l, m, r)$ *with length* $2m$.

Based on Lemmas 3.5 and 3.6 we now prove the following lemma.

Lemma 3.8. *Suppose $m \geq 3$. For $i = 0, 1$, if there exists $0 \leq j \leq m-1$ such that $C_{2l}(j)$ is not a component of H_i, then every component of H_i is $L(k)$ or $R(k)$ for some k.*

Proof It suffices to prove the result for $(i, j) = (0, 0)$ due to the symmetry. Suppose $C_{2l}(0)$ is not a component of H_0. Then by Lemma 3.5 each component of H_{00} is isomorphic to K_2. Suppose to the contrary that a path component P of H_0 is not equal to $L(k)$ or $R(k)$ for any k. By symmetry, and by Lemmas 3.5-3.6 we may assume without loss of generality that P is in one of the following forms for some $1 \leq q < m-1$:

$$v_{0,2k-1}, v_{0,2k-2}, v_{1,2k-2}, v_{1,2k-3}, \ldots, v_{q-1,2k-q}, v_{q-1,2k-q-1}, v_{q,2k-q-1}; \tag{7}$$

$$v_{0,2k-1}, v_{0,2k-2}, v_{1,2k-2}, v_{1,2k-3}, \ldots, v_{q-1,2k-q}, v_{q-1,2k-q-1}, v_{q,2k-q-1}, v_{q,2k-q-2}. \tag{8}$$

By Lemma 2.3(b), that the length of P is a multiple of 3, it is clear that $q \equiv 0 \pmod 3$ in (7) and $q \equiv 1 \pmod 3$ in (8).

Case 1: P is given by (7). Then $v_{q-1,2k-q-2}, v_{q-1,2k-q-3}, v_{q,2k-q-3}, v_{q,2k-q-2}$ is a path in H_1 by Lemma 3.5. However, this contradicts the results in Lemma 3.6 no matter q is odd or even.

Case 2: P is given by (8). By Lemma 2.3(b), $f(v_{0,2k-1}) = f(v_{q,2k-q-2}) = 0$. By Lemma 3.5, H_1 contains the edges $v_{q-1,2k-q-2}v_{q-1,2k-q-3}$, $v_{q,2k-q-3}v_{q,2k-q-4}$ and $v_{q,2k-q}v_{q,2k-q+1}$. Since P is a component of H_0, $v_{q+1,2k-q-2} \in V(H_1)$ and so by Lemma 3.5, H_1 contains either $v_{q+1,2k-q-2}v_{q+1,2k-q-3}$ or $v_{q+1,2k-q-2}v_{q+1,2k-q-1}$.

Assume first that H_1 contains $v_{q+1,2k-q-2}v_{q+1,2k-q-3}$. Then, by Lemma 3.5, H_0 contains $v_{q+1,2k-q-4}, v_{q+1,2k-q-5}$ and $v_{q+1,2k-q-1}v_{q+1,2k-q}$. Since $v_{q+1,2k-q-2}$ and $v_{q,2k-q}$ are end-vertices of H_1, by Lemma 2.3(b) we have $f(v_{q+1,2k-q-2}) = f(v_{q,2k-q}) = 5$. Similarly, $f(v_{q+1,2k-q-4}) = 0$ as $v_{q+1,2k-q-4}$ is an end-vertex of H_0. We then have $f(v_{q,2k-q-3}) = 3$ $[v_{q,2k-q-2}; v_{q+1,2k-q-2}]$ and therefore $f(v_{q,2k-q-4}) = 5$. It follows that $f(v_{q-1,2k-q-3}) = 1$ $[v_{q-1,2k-q-2}, v_{q,2k-q-3}; v_{q,2k-q-4}]$ and hence $f(v_{q-1,2k-q-2}) = 5$. As $v_{q-1,2k-q-1} \in V(H_0)$, we then have $f(v_{q-1,2k-q-1}) = 2$ $[v_{q-1,2k-q-2}; v_{q,2k-q-2}]$. Thus, by Lemma 2.3(b), it must be that $f(v_{q,2k-q-1}) = 4$. This, however, contradicts the fact that $f(v_{q,2k-q}) = 5$ and $v_{q,2k-q-1} \sim v_{q,2k-q}$.

Now we assume that H_1 contains $v_{q+1,2k-q-2}v_{q+1,2k-q-1}$. By Lemma 3.5(b), H_0 contains $v_{q+1,2k-q-3}, v_{q+1,2k-q-4}$, $v_{q+1,2k-q}v_{q+1,2k-q+1}$, and $v_{q,2k-q-5}v_{q,2k-q-6}$. Note that $v_{q+1,2k-q}$ is an end-vertex of H_0, and both $v_{q+1,2k-q-2}$ and $v_{q,2k-q}$ are end-vertices of H_1. Thus, by Lemma 2.3(b), $f(v_{q+1,2k-q}) = 0$ and $f(v_{q+1,2k-q-2}) = f(v_{q,2k-q}) = 5$. Hence, $f(v_{q+1,2k-q-1}) = 3$, $f(v_{q+1,2k-q-3}) = 2$, $f(v_{q,2k-q-1}) = 2$, and $f(v_{q,2k-q-3}) = 3$. Applying Lemma 2.3(b) to P yields $f(v_{q-1,2k-q-1}) = 4$. Since $v_{q+1,2k-q-4}$ is an end-vertex of H_0, by Lemma 2.3, $f(v_{q+1,2k-q-4}) = 0$. So $f(v_{q,2k-q-4}) = 5$ and $f(v_{q-1,2k-q-3}) = 1$. Since $v_{q-1,2k-q-2}v_{q-1,2k-q-3}$ is an edge of H_1, we then have $f(v_{q-1,2k-q-2}) = 5$. This, however, is impossible as $v_{q-1,2k-q-2} \sim v_{q-1,2k-q-1}$ and $f(v_{q-1,2k-q-1}) = 4$. □

Equipped with the results above, we are now ready to prove the following lemma which forms part of the proof of Theorem 1.2.

Lemma 3.9. *Suppose $m \geq 3$. Then $\lambda(\mathrm{Br}(2l, m, r)) = 5$ if and only if one of the following holds:*

(a) 3 *divides* l *and* 6 *divides* m;

(b) 6 *divides* l *and* 3 *divides* m.

Proof Suppose $\lambda(\mathrm{Br}(2l, m, r)) = 5$. We will prove that one of (a) and (b) holds.

Case 1: There exists at least one pair (i, j) such that $H_{ij} = C_{2l}(j)$. In this case, by Lemma 3.8, for all $0 \leq j \leq m-1$, $C_{2l}(j)$ is a component of H_0 or H_1. Thus, by Lemma 2.3, 3 must be a divisor of $2l$ and hence a divisor of l. Using Lemma 2.3 one can verify that, if $C_{2l}(j)$ is a component of H_0 (H_1 respectively) for some j, then both $C_{2l}(j-1)$ and $C_{2l}(j+1)$ are components of H_1 (H_0 respectively), where $j \pm 1$ is taken modulo m. Hence m is even. Because of the symmetry of $\mathrm{Br}(2l, m, r)$ we may assume that $C_{2l}(0)$ is a component of H_1. Then, by Lemma 2.3, there are only three possible patterns for the labels of $v_{0,0}, \ldots, v_{0,2l-1}$:

(i) 3, 5, 1, …, 3, 5, 1; (ii) 1, 3, 5, …, 1, 3, 5; (iii) 3, 1, 5, …, 3, 1, 5.

(Three other equivalent patterns are obtained by reversing these sequences.) One can verify that, for each of these possibilities, the labels for all other vertices are uniquely determined. In fact, in Case (i), for odd j, $v_{j,0}, \ldots, v_{j,2l-1}$ must be labelled $0, 2, 4, \ldots, 0, 2, 4$; and for even j, $v_{j,0}, \ldots, v_{j,2l-1}$ must be labelled $3, 5, 1, \ldots, 3, 5, 1$. In Case (ii), for odd j, $v_{j,0}, \ldots, v_{j,2l-1}$ must be labelled $4, 0, 2, \ldots, 4, 0, 2$; and for even j, $v_{j,0}, \ldots, v_{j,2l-1}$ must be labelled $1, 3, 5, \ldots, 1, 3, 5$. In Case (iii), for odd j, $v_{j,0}, \ldots, v_{j,2l-1}$ must be labelled $0, 4, 2, \ldots, 0, 4, 2$; and for even j, $v_{j,0}, \ldots, v_{j,2l-1}$ must be labelled $3, 1, 5, \ldots, 3, 1, 5$. Note that $v_{m-1,1} \sim v_{0,m+1}$ since m is even and $m + r \equiv 0 \pmod{2l}$. In Case (i) ((iii) respectively), since $v_{m-1,1}$ is labelled 2 (4 respectively), $v_{0,m+1}$ must be labelled 5 (1 respectively). Hence m is a multiple of 3. Since m is even, we see that 6 divides m and therefore condition (a) is satisfied. In Case (ii), since $v_{m-1,1}$ is labelled 0, $v_{0,m+1}$ cannot be labelled 1. If $v_{0,m+1}$ is labelled 5, then $v_{0,m+3}$ is labelled 3, which contradicts the fact that $v_{0,m+3} \sim v_{m-1,3}$ and $v_{m-1,3}$ is labelled 4. Thus, $v_{0,m+1}$ must be labelled 3 in Case (ii), and similar to Cases (i) and (iii), we see that (a) is satisfied.

Case 2: $H_{ij} \neq C_{2l}(j)$ for all (i, j). In this case, by Lemma 3.8, for $i = 0, 1$ each component of H_i is of the form $L(k)$ or $R(k)$. Since the roles of $L(k)$'s and $R(k)$'s are symmetric, without loss of generality we may assume that the components of H_0 and H_1 are of the form $L(k)$. By Lemma 3.5, $L(1), L(2), \ldots, L(l)$ must be in H_0, H_1 alternatively. Hence l is even. Because of the symmetry we may assume that H_0 consists of $L(2), L(4), \ldots, L(l)$ and H_1 consists of $L(1), L(3), \ldots, L(l-1)$. By Lemma 2.3(c), the length $2m$ of these cycles is a multiple of 3. So 3 is a divisor of m. We take each $L(k)$ as oriented in (5) and treat $v_{0,2k-1}$ as the first vertex of $L(k)$ in the following labelling. By symmetry, without loss of generality, we may assume that the vertices of $L(1)$ are labelled $5, 3, 1, \ldots, 5, 3, 1$. Then other cycles must be labelled as follows: $L(2)$: $4, 2, 0, \ldots, 4, 2, 0$; $L(3)$: $3, 1, 5, \ldots, 3, 1, 5$; $L(4)$: $2, 0, 4, \ldots, 2, 0, 4$; $L(5)$: $1, 5, 3, \ldots, 1, 5, 3$; $L(6)$: $0, 4, 2, \ldots, 0, 4, 2$; $L(7)$: $5, 3, 1, \ldots, 5, 3, 1$; etc. Since $L(7)$ has to be labelled in the same way as $L(1)$, this pattern is repeated in a cyclic manner modulo 6. Thus 6 divides l and condition (b) is satisfied.

Now we prove sufficiency. Suppose that one of (a) and (b) is satisfied. We now show that $\lambda(\mathrm{Br}(2l, m, r)) = 5$. Since by Lemma 2.1, $\lambda(\mathrm{Br}(2l, m, r)) \geq 5$, it suffices to prove the existence of a 5-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$.

Case 3: 3 divides l and 6 divides m. In this case, for each even j, we label the vertices $v_{j,0}, v_{j,1}, \ldots, v_{j,2l-1}$ of $C_{2l}(j)$ by $0, 4, 2, \ldots, 0, 4, 2$; and for each odd j, we label $v_{j,0}, v_{j,1}, \ldots, v_{j,2l-1}$ by $3, 1, 5, \ldots, 3, 1, 5$. Under this labelling any two vertices distance two apart receive distinct labels since they are either on the same $C_{2l}(j)$ or on consecutive $C_{2l}(j)$ and $C_{2l}(j+1)$ respectively. It remains to verify that any two adjacent vertices receive labels differing by at least 2. By the symmetry of $\mathrm{Br}(2l, m, r)$ we may assume without loss of generality that one vertex is on $C_{2l}(0)$ and the other one is on $C_{2l}(m-1)$. Since m is a multiple of 6 and $m + r \equiv 0 \pmod{2l}$, $v_{0,m+2j-1}$ is adjacent to $v_{m-1,2j-1}$. If $v_{m-1,2j-1}$ is assigned 3, then $2j - 1 \equiv 0 \pmod 3$ and hence $m + 2j - 1 \equiv 0 \pmod 3$. Thus $v_{0,m+2j-1}$ is labelled 0, and hence the labels of $v_{m-1,2j-1}$ and $v_{0,m+2j-1}$ differ by at least 2. Similarly, a 5-labelled vertex on $C_{2l}(m-1)$ can be adjacent to a 2-labelled vertex on $C_{2l}(0)$ only, and a 1-labelled vertex on $C_{2l}(m-1)$ can be adjacent to a 4-labelled vertex on $C_{2l}(0)$ only. Therefore the above-defined labelling is a 5-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$.

Case 4: 6 divides l and 3 divides m. Since 3 divides m, the length $2m$ of $L(k)$ is a multiple of 6. Label these cycles $L(k)$ in the way as described in Case 2 above. One can verify that this is a 5-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$. □

4 Brick products with λ-number 6

In this section we prove two sufficient conditions for $\lambda(\mathrm{Br}(2l, m, r)) = 6$ as stated in Theorem 1.2.

Lemma 4.1. $\lambda(\mathrm{Br}(2l, 2, r)) = 6$ *for any* $l \geq 2$.

Proof Since $m = 2$ and $2+r \equiv 0 \pmod{2l}$ by our assumption, $v_{0,2j+1} \sim v_{1,2j-1}$ for all j. We will first prove $\lambda(\mathrm{Br}(2l, 2, r)) \geq 6$ by way of contradiction. We will then present a 6-$L(2,1)$-labelling of $\mathrm{Br}(2l, 2, r)$ and thus complete the proof.

Suppose to the contrary that $\lambda(\mathrm{Br}(2l, 2, r)) \leq 5$. Let f be a 5-$L(2,1)$-labelling of $\mathrm{Br}(2l, 2, r)$, and let H_0, H_1 and H_{ij} be as defined in Definitions 2.2 and 3.2 respectively. It can be verified that, for $i = 0, 1$ and $j = 0, 1$, $C_{2l}(j)$ is not entirely in H_i. Thus, from Lemma 3.5, we have:

Claim 1. For $i = 0, 1$ and $j = 0, 1$, each path component of H_{ij} is isomorphic to K_2.

This implies that $C_{2l}(j) \cap (H_0 \cup H_1)$ has alternating pairs of vertices in H_0 and H_1. In particular, l must be even.

Claim 2. For $i = 0, 1$, H_i does not contain any edge $v_{0,j}v_{0,j+1}$ where j is even, or any edge $v_{1,j}v_{1,j+1}$ where j is odd.

Suppose to the contrary that H_0 contains an edge $v_{0,j}v_{0,j+1}$ where j is even. Since j is even and $m = 2$, $v_{0,j} \sim v_{1,j}$ and $v_{0,j+1} \sim v_{1,j-1}$ for every j. Thus, since by Claim 1, $v_{0,j-1}, v_{0,j+2} \in V(H_1)$, at least one of $v_{1,j}$ and $v_{1,j-1}$ is in H_0 by Lemma 2.3. Suppose first that $v_{1,j} \in V(H_0)$. Then $v_{1,j-1} \in V(H_1)$ for otherwise $(v_{0,j}, v_{0,j+1}, v_{1,j-1}, v_{1,j}, v_{0,j})$ is a 4-cycle in H_0, which is impossible. Now that $f(v_{1,j-1}) \in \{1, 3, 5\}$, by Lemma 2.3 we must have $(f(v_{0,j+1}), f(v_{0,j}), f(v_{1,j})) = (0, 4, 2)$ and $f(v_{1,j-1}) = 5$. By Claim 1, we have $f(v_{1,j-2}) \in \{1, 3\}$. Since $v_{0,j-1} \in V(H_1)$ but $f(v_{0,j}) = 4$, we have $f(v_{0,j-1}) = 1$ and consequently $f(v_{0,j-2}) = 3$ $[v_{1,j-1}]$. However, we now have no feasible label from $\{1, 3\}$ for $v_{1,j-2}$ $[v_{0,j-2}; v_{0,j-1}]$, a contradiction. The case $v_{1,j-1} \in V(H_0)$ can be dealt with similarly. This proves Claim 2 when $i = 0$ and j is even.

The case when $i = 0$ and j is odd is reduced to the case when $i = 0$ and j is even by swapping $C_{2l}(0)$ and $C_{2l}(1)$ and reversing their directions. This proves Claim 2 for H_0 and therefore for H_1 as well due to the symmetry between H_0 and H_1 (see the comments before Lemma 3.3).

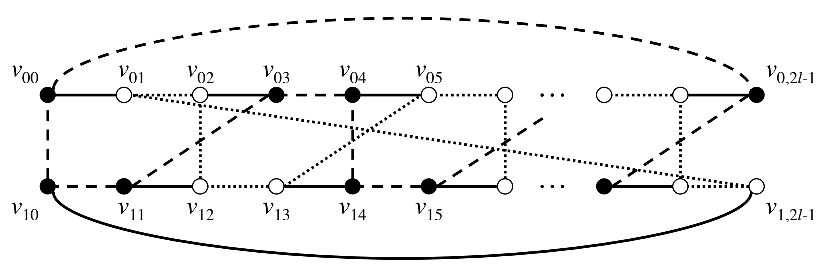

Figure 2: Proof of Lemma 4.1.

Combining Claims 1-2 with Lemma 2.3 and noting that l is even as mentioned above, we conclude that H_0 and H_1 must be cycles. (See Figure 2, where H_0 consists of dashed lines and H_1 dotted lines.) Without loss of generality we may assume

$$H_0 = (v_{0,0}, v_{1,0}, v_{1,1}, v_{0,3}, v_{0,4}, v_{1,4}, \ldots, v_{1,2l-3}, v_{0,2l-1}, v_{0,0})$$

$$H_1 = (v_{0,1}, v_{0,2}, v_{1,2}, v_{1,3}, v_{0,5}, v_{0,6}, \ldots, v_{0,2l-3}, v_{0,2l-2}, v_{1,2l-2}, v_{1,2l-1}, v_{0,1}).$$

Since $v_{0,0}, v_{1,0}, v_{1,1}, v_{0,3}$ is a path in H_0, by Lemma 2.3(b) we have $f(v_{0,0}) = f(v_{0,3}) \in \{0, 2, 4\}$. However, since $f(v_{0,1}), f(v_{0,2}) \in \{1, 3, 5\}$, if $f(v_{0,0}) = f(v_{0,3}) = 2$ or 4, then the only label for both $v_{0,1}$ and $v_{0,2}$ is 5 or 1 respectively. Since this contradicts the fact $v_{0,1} \sim v_{0,2}$, we must have $f(v_{0,0}) = f(v_{0,3}) = 0$. Thus, $(f(v_{0,1}), f(v_{0,2})) = (3, 5)$ or $(5, 3)$, and $f(v_{1,1}) = f(v_{1,4}) \in \{2, 4\}$. However, since $v_{1,2}, v_{1,3} \in V(H_1)$, if $f(v_{1,1}) = f(v_{1,4}) = 2$, then the only label for both $v_{1,2}$ and $v_{1,3}$ is 5. Similarly, if $f(v_{1,1}) = f(v_{1,4}) = 2$, then the only label for both $v_{1,2}$ and $v_{1,3}$ is 1. Since $v_{1,2} \sim v_{1,3}$, in both cases we have a contradiction. Therefore, $\lambda(\mathrm{Br}(2l, 2, r)) \geq 6$.

To complete the proof, we give the following labelling for each $l \geq 2$, where the labels in bold font comprise the pattern of repeating sequence. When labelling $C_{2l}(i)$ $(i = 1, 2)$ we label its vertices according to the order $v_{i,0}, v_{i,1}, \ldots, v_{i,2l-1}$ beginning with $v_{i,0}$.

Case 1: l is even.

$$\begin{array}{rl} C_{2l}(0): & \mathbf{2,6,0,4,2,6,0,4},\ldots,\mathbf{2,6,0,4} \\ C_{2l}(1): & \mathbf{0,6,2,4,0,6,2,4},\ldots,\mathbf{0,6,2,4} \end{array}$$

Case 2: l is odd and $l \equiv 0 \pmod 3$.

$$\begin{array}{rl} C_{2l}(0): & \mathbf{0,6,4,0,6,2,0,6,4,0,6,2},\ldots,\mathbf{0,6,4,0,6,2} \\ C_{2l}(1): & \mathbf{5,3,1,5,3,1,5,3,1,5,3,1},\ldots,\mathbf{5,3,1,5,3,1} \end{array}$$

Case 3: $l = 7$.

$$\begin{array}{rl} C_{2l}(0): & 0,2,6,4,2,5,3,0,5,2,0,6,3,5 \\ C_{2l}(1): & 6,1,3,0,6,4,1,6,3,1,4,2,0,4 \end{array}$$

Case 4: $l \geq 13$ is odd and $l \equiv 1 \pmod 3$.

$$\begin{array}{rl} C_{2l}(0): & 0,5,3,6,4,1,6,4,2,6,4,\mathbf{1,5,3,1,5,3,6,1,4,2,5,3},\ldots,1,6,2 \\ C_{2l}(1): & 4,2,0,5,2,0,3,1,5,3,0,\mathbf{6,2,0,4,2,0,6,3,0,6,4,0},\ldots,5,3,1 \end{array}$$

Case 5: l is odd and $l \equiv 2 \pmod 3$.

$$\begin{array}{rl} C_{2l}(0): & \mathbf{3,5,0,3,5,0,3,5,0,3,5,0},\ldots,\mathbf{3,5,0,3,5,0},2,4,6,1 \\ C_{2l}(1): & \mathbf{6,1,4,6,2,0,6,1,4,6,2,0},\ldots,\mathbf{6,1,4,6,2,0},6,4,0,2 \end{array}$$

In each case the above is a 6-$L(2,1)$-labelling of $\mathrm{Br}(2l, 2, r)$. Therefore, $\lambda(\mathrm{Br}(2l, 2, r)) = 6$. □

In what follows m' denotes the unique integer between 1 and $2l$ such that $m \equiv m' \pmod{2l}$. One may simply assume $m = m'$ in the following proofs. In fact, if $m > 2l$, then we simply insert $m - m'$ rows between $C_{2l}(0)$ and $C_{2l}(1)$ and label them by using the patterns of $C_{2l}(1)$ and $C_{2l}(2)$ alternatively.

Lemma 4.2. *If both $m \geq 4$ and $l \geq 2$ are even, then $\lambda(\mathrm{Br}(2l, m, r)) = 5$ or 6.*

Proof By Lemma 2.1 it suffices to prove the existence of a 6-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$.

Since l is even, we are able to cyclically label each $C_{2l}(i)$ by a group of four labels: beginning with $v_{i,0}$ we label the vertices of $C_{2l}(i)$ by $6, 0, 4, 2, \ldots, 6, 0, 4, 2$ when i is even and $4, 2, 6, 0, \ldots, 4, 2, 6, 0$ when $1 \leq i \leq m - 2$ is odd. We label the vertices of $C_{2l}(m - 1)$ by $4, 2, 6, 0, \ldots, 4, 2, 6, 0$ if $m' \equiv 0 \pmod 4$, and by $4, 0, 6, 2, \ldots, 4, 0, 6, 2$ if $m' \equiv 2 \pmod 4$. This labelling is shown in the following table:

$$\begin{array}{rl} C_{2l}(0): & 6,0,4,2,6,0,4,2,\ldots,6,0,4,2 \\ C_{2l}(1): & 4,2,6,0,4,2,6,0,\ldots,4,2,6,0 \\ & \vdots \\ C_{2l}(m-4): & 6,0,4,2,6,0,4,2,\ldots,6,0,4,2 \\ C_{2l}(m-3): & 4,2,6,0,4,2,6,0,\ldots,4,2,6,0 \\ C_{2l}(m-2): & 6,0,4,2,6,0,4,2,\ldots,6,0,4,2 \\ C_{2l}(m-1)\ (m' \equiv 0 \bmod 4): & 4,2,6,0,4,2,6,0,\ldots,4,2,6,0 \\ C_{2l}(m-1)\ (m' \equiv 2 \bmod 4): & 4,0,6,2,4,0,6,2,\ldots,4,0,6,2 \end{array}$$

Since m is even, $v_{0,m+2j-1} \sim v_{m-1,2j-1}$ for every j. Thus, when $m' \equiv 0 \bmod 4$, the vertices of $C_{2l}(m-1)$ labelled 2 or 0 are adjacent to the vertices of $C_{2l}(0)$ labelled 0 and 2 respectively. Similarly, when $m' \equiv 2 \bmod 4$, the vertices of $C_{2l}(m-1)$ labelled 0 or 2 are adjacent to the vertices of $C_{2l}(0)$ labelled 2 and 0 respectively. One can verify that the labelling above is indeed a 6-$L(2,1)$-labelling of $\mathrm{Br}(2l,m,r)$. □

5 Brick products with λ-number 6 or 7

In this section we prove the rest of the statements in Theorem 1.2. As before, when labelling $C_{2l}(i)$ we label its vertices according to the order $v_{i,0}, v_{i,1}, \dots, v_{i,2l-1}$ beginning with $v_{i,0}$. Denote by m' the unique integer between 1 and $2l$ such that $m \equiv m' \pmod{2l}$. (See the comments before Lemma 4.2.) We assume $m \geq 3$ since the case $m = 2$ was handled in Lemma 4.1 already.

Lemma 5.1. *If $m \geq 4$ is even and $l \geq 3$ is odd, then $\lambda(\mathrm{Br}(2l,m,r)) \leq 7$.*

Proof It suffices to prove the existence of a 7-$L(2,1)$-labelling of $\mathrm{Br}(2l,m,r)$. Our assumption on m, l and r implies $2l \equiv 2 \pmod 4$ and $v_{0,m+2j-1} \sim v_{m-1,2j-1}$ for every j.

In the case when $m' \equiv 0 \pmod 4$, we label the vertices of the cycles $C_{2l}(j)$ as follows:

$$\begin{array}{rl}
C_{2l}(0): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,5,3 \\
C_{2l}(1): & 4,2,6,0,4,2,6,0,\dots,4,2,6,0,7,1 \\
C_{2l}(2): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,5,3 \\
\vdots & \\
C_{2l}(m-3): & 4,2,6,0,4,2,6,0,\dots,4,2,6,0,7,1 \\
C_{2l}(m-2): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,5,3
\end{array}$$

$$C_{2l}(m-1): \overbrace{4,2,6,0,\dots,4,2,6,0}^{2l-m'-2},4,1,6,2,\overbrace{4,0,6,2,\dots,4,0,6,2}^{m'-4},7,0$$

The assumption $m' \equiv 0 \pmod 4$ implies that, among the first $2l - m' - 2$ (≥ 4) vertices of $C_{2l}(m-1)$, those with label 2 or 0 are adjacent to the vertices of $C_{2l}(0)$ between $v_{0,m+1}$ and $v_{0,2l-3}$ with label 0 or 2 respectively. Among the next four vertices of $C_{2l}(m-1)$, $v_{m-1,2l-m-1}$ (with label 1) and $v_{m-1,2l-m+1}$ (with label 2) are adjacent to $v_{0,2l-1}$ (with label 3) and $v_{0,1}$ (with label 0) respectively, and so on. Based on these observations one can verify that the labelling above is a 7-$L(2,1)$-labelling of $\mathrm{Br}(2l,m,r)$.

In the case when $m' \equiv 2 \pmod 4$, we label $C_{2l}(j)$, $0 \leq j \leq m-2$, as follows:

$$\begin{array}{rl}
C_{2l}(0): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,5,3 \\
C_{2l}(1): & 4,2,6,0,4,2,6,0,\dots,4,2,6,0,7,1 \\
C_{2l}(2): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,5,3 \\
\vdots & \\
C_{2l}(m-3): & 4,2,6,0,4,2,6,0,\dots,4,2,6,0,7,1 \\
C_{2l}(m-2): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,5,3
\end{array}$$

In addition, if $m' = 2l$, then we label $C_{2l}(m-1)$ by:

$$C_{2l}(m-1): 4,2,6,0,4,2,6,0,\dots,4,2,6,0,7,1;$$

otherwise, $6 \leq m' \leq 2l-4$ and we label $C_{2l}(m-1)$ by:

$$C_{2l}(m-1): \overbrace{4,0,6,2,\dots,4,0,6,2}^{2l-m'-4},4,0,6,1,\overbrace{4,2,6,0,\dots,4,2,6,0}^{m'-2},7,2.$$

Similar to the case $m' \equiv 0 \pmod 4$, one can verify that this is a 7-$L(2,1)$-labelling of $\mathrm{Br}(2l,m,r)$.
□

Lemma 5.2. *If $m \geq 3$ is odd and $l \geq 2$ is even, then $\lambda(\mathrm{Br}(2l, m, r)) \leq 7$.*

Proof Since m is odd, $v_{0,m+2j} \sim v_{m-1,2j}$ for every j. Using this and the fact that $2l$ is a multiple of 4, one can verify that the following is a 7-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$:

$$\begin{array}{rl}
C_{2l}(0): & 0,4,2,5,0,4,2,5,\dots,0,4,2,5\\
C_{2l}(1): & 2,6,0,4,2,6,0,4,\dots,2,6,0,4\\
C_{2l}(2): & 0,4,2,6,0,4,2,6,\dots,0,4,2,6\\
\vdots & \\
C_{2l}(m-4): & 2,6,0,4,2,6,0,4,\dots,2,6,0,4\\
C_{2l}(m-3): & 0,4,2,6,0,4,2,6,\dots,0,4,2,6\\
C_{2l}(m-2): & 2,5,0,4,2,5,0,4,\dots,2,5,0,4\\
C_{2l}(m-1)\ (m' \equiv 1 \bmod 4): & 1,7,3,6,1,7,3,6,\dots,1,7,3,6\\
C_{2l}(m-1)\ (m' \equiv 3 \bmod 4): & 3,7,1,6,3,7,1,6,\dots,3,7,1,6
\end{array}$$

This completes the proof. □

Lemma 5.3. *If both $m \geq 3$ and $l \geq 3$ are odd, then $\lambda(\mathrm{Br}(2l, m, r)) \leq 7$.*

Proof Since m is odd, $v_{0,m+2j} \sim v_{m-1,2j}$ for every j. Since l is odd, $2l \equiv 2 \pmod 4$.

In the case when $m' \equiv 1 \pmod 4$, we label $C_{2l}(i)$ for odd i between 1 and $m-4$ using the same pattern, and label $C_{2l}(i)$ for even i between 2 and $m-3$ using the same pattern. We label $C_{2l}(m-2)$ in the same way as $C_{2l}(1)$ except that $v_{m-2,2l-m'}$ is labelled 7 instead of 6. Moreover explicitly, we label $\mathrm{Br}(2l, m, r)$ as follows.

$$\begin{array}{rl}
C_{2l}(0): & 0,7,2,5,0,7,2,5,\dots,0,7,2,5,1,4\\
C_{2l}(1): & 2,6,0,4,2,6,0,4,\dots,2,6,0,4,7,5\\
C_{2l}(2): & 0,4,2,6,0,4,2,6,\dots,0,4,2,6,1,3\\
\vdots & \\
C_{2l}(m-4): & 2,6,0,4,2,6,0,4,\dots,2,6,0,4,7,5\\
C_{2l}(m-3): & 0,4,2,6,0,4,2,6,\dots,0,4,2,6,1,3
\end{array}$$

$$C_{2l}(m-2): 2,6,0,4,2,6,0,4,\dots,2,\overbrace{7}^{v_{m-2,2l-m'}},0,4,\dots,2,6,0,4,7,5$$

$$C_{2l}(m-1): \overbrace{1,4,7,1,5,3,7,1,\dots,5,3,7,1}^{2l-m'-1},6,3,5,1,\overbrace{7,3,5,1,\dots,7,3,5,1}^{m'-5},7,3$$

In the case where $m' \equiv 3 \pmod 4$, we give the following labelling:

$$\begin{array}{rl}
C_{2l}(0): & 6,1,4,2,6,1,4,2,\dots,6,1,4,2,7,3\\
C_{2l}(1): & 4,2,6,0,4,2,6,0,\dots,4,2,6,0,5,1\\
C_{2l}(2): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,7,3\\
\vdots & \\
C_{2l}(m-4): & 4,2,6,0,4,2,6,0,\dots,4,2,6,0,5,1\\
C_{2l}(m-3): & 6,0,4,2,6,0,4,2,\dots,6,0,4,2,7,3
\end{array}$$

$$C_{2l}(m-2): \overbrace{4,2,6,1,\dots,4,2,6,1}^{2l-m'-3},\overbrace{4,2,6,0,4,2,6,0,\dots,4,2,6,0}^{m'+1},5,1$$

$$C_{2l}(m-1): \overbrace{0,7,3,5,\dots,0,7,3,5}^{2l-m'-3},0,7,5,2,\overbrace{7,0,5,3,\dots,7,0,5,3}^{m'-7},7,0,5,7,2,6$$

In each case one can verify that the above gives a 7-$L(2,1)$-labelling of $\mathrm{Br}(2l, m, r)$. □

Proof of Theorem 1.2 This follows from Lemmas 3.9, 4.1, 4.2 and 5.1-5.3 immediately. □

6 Concluding remarks

In this paper we studied the $L(2,1)$-labelling problem for the family of cubic Cayley graphs (other than the prism graphs) on dihedral groups. Such graphs have been studied extensively in several contexts, including Hamiltonicity of Cayley graphs, computer architecture and interconnection networks, chemical structures, and combinatorics and geometry. They are called 'brick products', 'generalised honeycomb tori' and 'honeycomb toroidal graphs' in the literature. We proved that in the case when $m \geq 2$ and $m + r \equiv 0 \pmod{2l}$ the λ-number of the brick product $\mathrm{Br}(2l, m, r)$ (see Definition 1.1) is equal to $5, 6$ or 7, and moreover we give a characterisation of such brick products with λ-number 5. This result confirms a conjecture of Georges and Mauro [14] in the special case of cubic Cayley graphs on dihedral groups. It also gives an infinite family of cubic graphs with smallest possible λ-number. The case when $m = 1$ will be dealt with in a separate paper [23] because it requires different techniques.

At present we do not know any $\mathrm{Br}(2l, m, r)$, $m \geq 2$, $m+r \equiv 0 \pmod{2l}$, with λ-number 7. We did extensive computation by using the computer package CPLEX. All instances implemented output a λ-number 5 or 6. This prompts us to propose the following conjecture.

Conjecture 6.1. *$\lambda(\mathrm{Br}(2l, m, r)) = 5$ or 6 for all brick products $\mathrm{Br}(2l, m, r)$ with $m \geq 2$ and $m + r \equiv 0 \pmod{2l}$.*

Acknowledgements We appreciate the referees for their helpful comments. The work was supported by a Discovery Project Grant (DP0558677) of the Australia Research Council. Li was supported by a grant (11171129) of the National Natural Science Foundation of China. Zhou was supported by a Future Fellowship (FT110100629) and a Discovery Project Grant (DP120101081) of the Australian Research Council.

References

[1] B. Alspach, C. C. Chen and K. Mcavanney, On a class of Hamiltonian laceable 3-regular graphs, *Discrete Math.* **151** (1996), 19–38.

[2] B. Alspach and M. Dean, Honeycomb toroidal graphs are Cayley graphs, *Inform. Process. Lett.* **109** (2009), 705–708.

[3] B. Alspach and C. Q. Zhang, Hamilton cycles in cubic graphs on dihedral groups, *Ars Combinatoria* **28** (1989), 101–108.

[4] P. Bahls, Channel assignment on Cayley graphs, *J. Graph Theory* **67** (3) (2011), 169–177.

[5] H. L. Bodlaender, T. Kloks, R. B. Tan and J. van Leeuwen, Approximations for λ-coloring of graphs, *The Computer Journal* **47** (2004), 193–204.

[6] T. Calamoneri and R. Petreschi, L(h,1)-labeling subclasses of planar graphs, *Journal on Parallel and Distributed Computing*, **64** (3) (2004), 414–426.

[7] T. Calamoneri, The $L(h, k)$-labelling problem: An updated survey and annotated bibliography, *The Computer Journal* **54** (2011), 1344–1371.

[8] T. Calamoneri, S. Caminiti and R. Petreschi, A general approach to $L(h, k)$- label interconnection networks, *Journal of Computer Science & Technology* **23** (4) (2008), 652–659.

[9] G. J. Chang and D. Kuo, The $L(2,1)$-labelling problem on graphs, *SIAM J. Discrete Math.* **9** (1996), 309–316.

[10] G. J. Chang, C. H. Lu and S. Zhou, Distance-two labellings of Hamming graphs, *Discrete Appl. Math.* **157** (2009), 1896–1904.

[11] G. J. Chang, C. H. Lu and S. Zhou, No-hole 2-distant colorings for Cayley graphs on finitely generated Abelian groups, *Discrete Math.* **307** (2007), 1808–1817.

[12] H. Cho and L. Hsu, Generalized honeycomb torus, *Inform. Process. Lett.* **86** (2003), 185–190.

[13] H. S. M. Coxeter, Self dual configurations and regular graphs, *Bull. Amer. Math. Soc.* **56** (1950), 413–455.

[14] J. P. Georges and D. W. Mauro, On generalized Petersen graphs labelled with a condition at distance two, *Discrete Math.* **259** (2002), 311–318.

[15] D. Goncalves, On the $L(p,1)$-labelling of graphs, *Discrete Math.* **308** (2008), 1405–1414.

[16] J. R. Griggs and R. K. Yeh, Labelling graphs with a condition at distance 2, *SIAM J. Discrete Math.* **5** (1992), 586–595.

[17] W. K. Hale, Frequency assignment: Theory and applications, *Proc. IEEE* **68** (1980), 1497–1514.

[18] F. Havet, B. Reed and J.-S. Sereni, Griggs and Yeh's Conjecture and $L(p,1)$-labelings, *SIAM J. Discrete Math.* **26** (2012), 145–168.

[19] P. K. Jha, A. Narayanan, P. Sood, K. Sundaram and V. Sunder, On $L(2;1)$-labeling of the Cartesian product of a cycle and a path, *Ars Combin.* **55** (2000), 81–89.

[20] J-H. Kang, $L(2,1)$-labelling of Hamiltonian graphs with maximum degree 3, *SIAM J. Discrete Math.* **22** (2008), 213–230.

[21] D. Král' and R. Škrekovski, A theorem about the chennel assignment problem, *SIAM J. Discrete Math.* **16** (2003), 426–437.

[22] D. Kuo and J.-H Yan, On $L(2,1)$-labellings of Cartesian products of paths and cycles, *Discrete Math.* **283** (2004), 137–144.

[23] X. Li, V. Mak-Hau and S. Zhou, Distance-two labellings for chordal rings of degree three, in preparation.

[24] X. Li and S. Zhou, Labeling outerplanar graphs with maximum degree three, preprint.

[25] D. D. Liu and X. Zhu, Circular distance two labelling and the λ-number for outerplanar graphs, *SIAM J. Discrete Math.* **19** (2005), 281–291.

[26] D. Marušič and T. Pisanski, Symmetries of hexagonal molecular graphs on the torus, *Croatica Chem.* **73** (2000) 969–981.

[27] M. Molloy and M. R. Salavatipour, A bound on the chromatic number of the square of a planar graph, *J. Combin. Theory (B)* **94** (2005), 189–213.

[28] D. Sakai, Labelling chordal graphs: Distance two condition, *SIAM J. Discrete Math.* **7** (1994), 133–140.

[29] I. Stojmenovic, Honeycomb networks: Topological properties and communication algorithms, *IEEE Trans. Parallel Distrib. Systems* **8** (1997), 1036–1042.

[30] W. Xiao and B. Parhami, Further mathematical properties of Cayley digraphs applied to hexagonal and honeycomb meshes, *Discrete Appl. Math.* **155** (2007), 1752–1760.

[31] X. Yang, D. J. Evans, H. Lai and G. M. Megson, Generalized honeycomb torus is hamiltonian, *Inform. Process. Lett.* **92** (2004), 31–37.

[32] S. Zhou, A channel assignment problem for optical networks modelled by Cayley graphs, *Theoretical Computer Science* **310** (2004), 501–511.

[33] S. Zhou, Labelling Cayley graphs on Abelian groups, *SIAM J. Discrete Math.* **19** (2006), 985–1003.