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A note on input-to-state stability of sampled-data nonlinear systems

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Abstract

It is shown for nonlinear systems that sampling sufficiently fast an input-to-state stabilizing (ISS) continuous time control law results in an ISS sampled-data control law. Two main features of our approach are: we show how the nonlinear sampled-data system can be modeled by a functional differential equation (FDE); we exploit a Razumikhin type theorem for ISS of FDE that was recently proved in [14] to analyze the sampled-data system.

Keywords: Functional differential equations, input-to-state stability, nonlinear systems, sampling, small gain.

1 Introduction

The prevalence of computer controlled systems motivates the investigation of discrete-time and sampled-data control systems (for related linear results see, e.g. [3, 4] and for nonlinear results see [6, 9, 1, 11, 8]). There are many ways in which the controller design for systems consisting of a continuous time plant and digital controller can be done [4]. One possible approach is to design a continuous time controller for the continuous time plant and then discretize the controller using a sampler and hold device. In the present paper we concentrate on this approach to the design of sampled-data nonlinear controllers. When the continuous time controller stabilizes the continuous time plant, stability properties of the sampled-data system depend crucially on the choice of the sampling period T .

It has been shown in [3] for linear systems that, if the continuous time controller is stabilizing and the sampling is fast enough (T is small enough), the obtained sampled-data closed loop system remains stable. Moreover, fast sampling of a continuous time controller (with an anti-aliasing filter) preserves L_p stability for

any $p \in [1, \infty]$. In the nonlinear context, some results in this spirit have been obtained for the stabilization problem in [5], the regulation problem in [1], L_2 stability of a class of nonlinear systems in [11] and output dead-beat control in [6].

We concentrate on the input-to-state stability (ISS) property introduced by Sontag in [13] and recently extended to functional differential equations by Teel in [14]. Our main tool is the small gain theorem for functional differential equations proved in [14]. The connection between functional differential equations and sampled-data systems that we establish appears to be new.

As a nonlinear counterpart of [3], we show that, if a continuous time input-to-state stabilizing controller is sampled fast enough, the obtained sampled-data nonlinear system still has the same (but in general deteriorated) properties as the continuous time system. It is of practical significance that we can compute a desired sampling rate and use it in applications where the sampling rate is a design parameter. We note, however, that the estimates obtained using the small gain arguments may be conservative.

The paper is organized as follows. In Section 2 we present notation and key results from [14] which are needed in the sequel. In Section 3 we prove and discuss the main results of the paper. In Section 4 we point out several directions in which our results can be generalized, and also show how using our results we can recover some known results from the literature. A summary of our results is given in the last section.

2 Preliminaries

We use the following notation introduced in [14]. Let t_d be a nonnegative number. Given a function $w : [-t_d, \infty[\rightarrow R^m$ and fixing $t \in [0, \infty[$, $w_d(t)(\cdot)$ represents a function from $[0, t_d]$ to R^m , defined by $w_d(t)(\tau) := w(t - \tau)$.

Consider $\gamma : R_{\geq 0} \rightarrow R_{\geq 0}$. $\gamma \in \mathcal{G}$ if it is continuous, zero at zero and nondecreasing. $\gamma \in \mathcal{K}$ if $\gamma \in \mathcal{G}$ and

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strictly increasing. $\gamma \in \mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and is unbounded. $\beta : R_{\geq 0} \times R_{\geq 0} \rightarrow R_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s > 0$.

We consider the ordinary differential equation:

$$\dot{x}(t) = f(x(t), w(t)) \quad (1)$$

and the functional differential equation

$$\dot{x}(t) = F(x_d(t), w_d(t)). \quad (2)$$

We define:

$$|x_d(t)| := \max_{t-t_d \leq s \leq t} |x(s)|, \quad \|x_d\|_{t_0} := \sup_{s \geq t_0} |x_d(s)|.$$

Given continuous functions $x : [-t_d, \infty[\rightarrow R^n$ and $V : [-t_d, \infty[\times R^n \rightarrow R_{\geq 0}$, we define $V(t, x(t)) =: V(t)$. Considering the solution x of (2), the upper right-hand derivative of V along the solution is defined as:

$$D^+V(t) = \limsup_{h \rightarrow 0^+} \frac{V(t+h) - V(t)}{h}$$

Of the two following definitions, the first is standard (see [13]), while the second was recently introduced in [14].

Definition 1 *If there exist $\alpha_i \in \mathcal{K}_\infty, i = 1, 2, 3$, smooth function $V : R^n \rightarrow R_{\geq 0}$, and $\gamma \in \mathcal{G}$ such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ and $|x| \geq \gamma(|w|) \Rightarrow \frac{\partial V}{\partial x} f(x, w) \leq -\alpha_3(|x|)$, then we say that the system (1) is ISS with gain $\tilde{\gamma} := \alpha_1^{-1} \circ \alpha_2 \circ \gamma$.*

Definition 2 *Let $\gamma \in \mathcal{G}, \nu \in R_{\geq 0}$ and $\Delta_x, \Delta_w \in R_{\geq 0} \cup \infty$. The zero solution of (2) is said to be uniformly input-to-state stable (ISS) with gain γ [and offset ν and restriction (Δ_x, Δ_w)] if $|x_d(t_0)| < \Delta_x$ and $\|w_d\|_{t_0} < \Delta_w$ imply the solutions of (2) are defined $\forall t$ and that the following properties hold uniformly in $t_0 \geq 0$:*

1. *for each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$|x_d(t_0)| \leq \delta \Rightarrow \|x_d\|_{t_0} \leq \max\{\epsilon, \gamma(\|w_d\|_{t_0}), \nu\}$$

2. *for each $\epsilon > 0, \eta_x \in]0, \Delta_x[$ and $\eta_w \in]0, \Delta_w[$ there exists $T > 0$ such that*

$$\{|x_d(t_0)| \leq \eta_x, \|w_d\|_{t_0} \leq \eta_w\} \Rightarrow$$

$$\|x_d\|_{t_0+T} \leq \max\{\epsilon, \gamma(\|w_d\|_{t_0}), \nu\}.$$

We need the results which are listed with their sources:

Lemma 1 [14, 15] *Let $\mu \geq 0$ and $\alpha \in \mathcal{K}$. If $V(t) \geq \mu$ implies $D^+V(t) \leq -\alpha(V(t))$ then there exists $\beta \in \mathcal{KL}$ (independent of μ) with $\beta(s, 0) \geq s$ such that $V(t) \leq \max\{\beta(V(t_0), t - t_0), \mu\}$.*

One of the main results in [14] is:

Theorem 1 [14] *Suppose there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuous function $V : [-t_d, \infty[\times R^n \rightarrow R_{\geq 0}$, $\gamma_x, \gamma_w \in \mathcal{G}$, $\alpha_3 \in \mathcal{K}$ and nonnegative real numbers $\delta < \Delta$ such that*

1. $\alpha_1(|x(t)|) \leq V(t) \leq \alpha_2(|x(t)|)$,
2. $|x(t)| \geq \max\{\gamma_x(|x_d(t)|), \gamma_w(|w_d(t)|)\}$ implies that $D^+V(t) \leq -\alpha_3(|x(t)|)$,
3. $\alpha_1^{-1} \circ \alpha_2 \circ \gamma_x(s) < s$, for $\delta < s < \Delta$.

Let $\beta \in \mathcal{KL}$ be as in the conclusion of Lemma 1 when $\alpha = \alpha_3 \circ \alpha_2^{-1}$. Then the origin is uniformly ISS with gain $\tilde{\gamma}_w := \alpha_1^{-1} \circ \alpha_2 \circ \gamma_w$, offset δ and restriction (Δ_x, Δ_w) such that $\max\{\alpha_1^{-1}(\beta(\alpha_2(s_1), 0), \tilde{\gamma}_w(s_2))\} < \Delta$ when $s_1 < \Delta_x, s_2 < \Delta_w$.

Lemma 2 [2] *Consider the system:*

$$\dot{x} = \Phi(x, u_1, u_2) \quad (3)$$

If the system (3) with $u_2(t) \equiv 0$ is ISS with gain γ_{u_1} , then there exists an $m \times m$ matrix $G(x, u_1)$ of smooth functions, invertible for all $x \in R^n, u_1 \in R^s$ that satisfies $G(x, u_1) \equiv I_{m \times m}$ in a neighborhood of the origin, and a function γ_{u_2} of class $\mathcal{K}_\infty$ such that the system

$$\dot{x} = \Phi(x, u_1, G(x, u_1)u_2)$$

is ISS with gain $(\gamma_{u_1}, \gamma_{u_2})$. More precisely, there exist $\alpha_i, i = 1, 2, 3$ functions of class $\mathcal{K}$ such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ and following holds:

$$|x(t)| \geq \max\{\gamma_{u_1}(|u_1(t)|), \gamma_{u_2}(|u_2(t)|)\} \Rightarrow \frac{\partial V}{\partial x} \Phi(x, u_1, G(x, u_1)u_2) \leq -0.5\alpha_3(|x(t)|) \quad (4)$$

Finally, if γ_{u_1} is class $\mathcal{K}_\infty$, then the matrix G can be chosen to be independent of u_1 .

3 ISS of sampled-data nonlinear systems

We show in this section how one of the main results developed in [14], Theorem 1 above, can be used to analyze ISS of nonlinear sampled-data systems. Consider the nonlinear system

$$\dot{x} = f(x, u, w) \quad (5)$$

for which an ISS controller

$$u = u_c(x) \quad (6)$$

has been designed. Vectors x, u , and w denote respectively the state, control input and disturbance. It is assumed that f and u_c are locally Lipschitz. If the controller (6) is implemented using the sampler and zero order hold, that is,

$$u_T(x, t) = u_c(x(kT)), t \in [kT, (k+1)T[, k = 0, 1, 2, \dots \quad (7)$$

we obtain the closed loop system in the form:

$$\dot{x} = f(x, u_T(x, t), w) \quad (8)$$

To stress the presence of the time delay due to sampling we rewrite (8) as

$$\dot{x}(t) = f(x(t), u_c(x(t - \tau(t))), w) \quad (9)$$

where $\tau(t) = t - kT, t \in [kT, (k+1)T[$ is a time-varying time delay. The digital control law (7) and the function $\tau(t)$ depend on the sampling period T and an important question is: *If the system (5), (6) is ISS, does there exist a positive number $T^* > 0$ such that for any sampling interval $T \in]0, T^*[$, the system (9) is uniformly ISS?* The answer to this question is of practical significance in cases when the sampling rate can be chosen by the designer.

Theorem 2 *Suppose that the origin of the closed loop (5, 6) is ISS with gain $\tilde{\gamma}$. Then, given arbitrary numbers $0 < \delta < \Delta$, there exists T^* such that, for all $T \in]0, T^*[$, the origin of (9) is uniformly ISS with gain $\tilde{\gamma}$, nonzero offset δ and nonzero restriction Δ .*

Proof: Since the continuous time system:

$$\dot{x} = f(x, u_c(x), w)$$

is ISS, recall from Definition 1 that there exists an ISS Lyapunov function $V(x)$ such that $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ and :

$$|x| \geq \gamma(|w|) \Rightarrow \frac{\partial V}{\partial x} f(x, u_c(x), w) \leq -\alpha_3(|x|).$$

In this case the corresponding gain is $\tilde{\gamma} := \alpha_1^{-1} \circ \alpha_2 \circ \gamma$.

Consider now (9), (7) and write it as follows

$$\dot{x}(t) = f(x(t), u_c(x(t)) + \xi(t), w(t)) \quad (10)$$

$$\xi(t) = u_c(x(t - \tau(t))) - u_c(x(t))$$

Introduce the following $|x_d(t)| = \max_{s \in [t-2T, t]} |x(s)|$. Since u_c is locally Lipschitz there exists a continuous nondecreasing function $L : R_{\geq 0} \rightarrow R_{\geq 0}$, such that:

$$\begin{aligned} |\xi(t)| &\leq L(|x_d(t)|) |x(t - \tau(t)) - x(t)| \\ &\leq L(|x_d(t)|) \left| \int_{t-\tau(t)}^t f(x(s), x(s - \tau(s)), w) ds \right| \end{aligned} \quad (11)$$

Since f is locally Lipschitz, $|\tau(t)| < T, \forall t$ and $f(0, u_c(0), 0) = 0$, there exist $\gamma_1^*, \gamma_2^* \in \mathcal{K}$ such that

$$\left| \int_{t-\tau(t)}^t f(x(s), x(s - \tau(s)), w) ds \right| \leq$$

$$T \max\{\gamma_1^*(|x_d(t)|), \gamma_2^*(|w_d(t)|)\}$$

and hence we can write

$$|\xi(t)| \leq TL(|x_d(t)|) \max\{\gamma_1^*(|x_d(t)|), \gamma_2^*(|w_d(t)|)\}. \quad (12)$$

Let us introduce a function $\rho \in \mathcal{K}_\infty$. Then if $|x_d(t)| \geq \rho(|w_d(t)|)$ from (12) it follows that

$$\begin{aligned} |\xi(t)| &\leq TL(|x_d(t)|) \max\{\gamma_1^*(|x_d(t)|), \gamma_2^*(\rho^{-1}(|x_d(t)|))\} \\ &=: T \gamma'_1(|x_d(t)|) \end{aligned}$$

Otherwise, when $|w_d(t)| \geq \rho^{-1}(|x_d(t)|)$, it follows from (12) that

$$\begin{aligned} |\xi(t)| &\leq TL(\rho(|w_d(t)|)) \max\{\gamma_1^*(\rho(|w_d(t)|)), \gamma_2^*(|w_d(t)|)\} \\ &=: T \gamma'_2(|w_d(t)|) \end{aligned}$$

Hence, we can write

$$|\xi(t)| \leq T \max\{\gamma'_1(|x_d(t)|), \gamma'_2(|w_d(t)|)\} \quad (13)$$

where $\gamma'_1, \gamma'_2 \in \mathcal{K}$. Since the system (5), (6) is ISS, from Lemma 2 it follows that there exists a smooth, invertible $m \times m$ matrix G and $\gamma_\theta \in \mathcal{K}_\infty$ such that the system

$$\dot{x} = f(x, u_c(x) + G(x, w)\theta, w) \quad (14)$$

satisfies

$$\begin{aligned} |x(t)| &\geq \max\{\gamma_\theta(|\theta(t)|), \gamma(|w(t)|)\} \\ &\Rightarrow \dot{V} \leq -0.5\alpha_3(|x(t)|). \end{aligned} \quad (15)$$

The original system (9) can be written in form (14) with $\theta = G(x, w)^{-1}\xi$, and this together with (13) implies that there exist γ_1 and γ_2 such that

$$|\theta(t)| \leq T \max\{\gamma_1(|x_d(t)|), \gamma_2(|w_d(t)|)\}. \quad (16)$$

The functions γ_1, γ_2 are arrived at using the same type of argument used to construct γ'_1, γ'_2 . We define $\hat{\gamma} \in \mathcal{G}$ as $\hat{\gamma}(s) := \max\{\gamma_\theta(T\gamma_2(s)), \gamma(s)\}$ and write

$$\begin{aligned} |x(t)| &\geq \max\{\gamma_\theta(T\gamma_1(|x_d(t)|)), \hat{\gamma}(|w_d(t)|)\} \\ &\Rightarrow \dot{V} \leq -0.5\alpha_3(|x(t)|) \end{aligned} \quad (17)$$

Since $\gamma_\theta(0) = 0$, for any positive δ, Δ , there exists $\hat{T} > 0$ such that for any $T \in]0, \hat{T}[$ we have

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(T\gamma_1(s)) < s, \forall s \in]\delta, \Delta[\quad (18)$$

From Theorem 1 it follows that, given arbitrary offset δ and arbitrary restriction (Δ_x, Δ_w) , there exists $\hat{T}$ such that if $T \in]0, \hat{T}[$, then the origin of (9) is uniformly ISS

with gain $\alpha_1^{-1} \circ \alpha_2 \circ \gamma$. Moreover, a possibly smaller T can be found so that for any $T \in]0, T^*[$ we have

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(T\gamma_2(\Delta_w)) \leq \delta$$

and we can replace $\alpha_1^{-1} \circ \alpha_2 \circ \hat{\gamma}$ with $\alpha_1^{-1} \circ \alpha_2 \circ \gamma = \tilde{\gamma}$, which proves Theorem 2. Q.E.D.

Comment 1 *If condition (18) is satisfied for all $s > 0$, then the origin of (9) is globally uniformly ISS. This can happen if the system (5) is globally exponentially stable and f and u_c are globally Lipschitz. Then we can find T^* so that the system (9) is globally uniformly ISS for any $T \in]0, T^*[$, but the gain may have to change near the origin and at infinity. Also, if $\alpha_1, \alpha_2, \gamma_\theta$ and γ_1 in (18) are locally sector bounded gains, we can render the offset equal to zero by choosing small T , whereas the restriction may still be finite.*

The result of Theorem 2 can be applied when instead of a static state feedback controller we design a dynamic controller, and implement it digitally using some form of approximation or numerical scheme. Suppose that the continuous time plant:

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ y &= h(x) \end{aligned} \quad (19)$$

has been rendered ISS with the use of the controller

$$\begin{aligned} \dot{z} &= g(z, y) \\ u &= u_c(z, y) \end{aligned} \quad (20)$$

To illustrate our approach we assume that the digital controller is implemented as follows:

1. $y(kT), k = 0, 1, 2, \dots$ is measured,
2. controller dynamics are approximated by the Euler discretization scheme and a zero order hold:

$$\begin{aligned} z(k+1) &= z(k) + Tg(z(k), y(k)), \\ u_T(z, y, t) &= u_c(z(k), y(k)), \quad t \in [kT, (k+1)T[\end{aligned} \quad (21)$$

We have a result analogous to Theorem 2:

Theorem 3 *If the closed loop system (19), (20) is ISS with a gain $\tilde{\gamma}$, then given arbitrary numbers $0 < \delta < \Delta$, there exists a positive number $T^* > 0$ such that for any sampling period $T \in]0, T^*[$ the closed loop system (19), (21) is uniformly ISS with gain $\tilde{\gamma}$, nonzero offset δ and restriction Δ .*

Proof of Theorem 3: We introduce the notation $\eta = (x^T \ z^T)^T$, $F(\eta, u, w) = (f^T(x, u, w) \ g^T(\eta))^T$ and with

$$\alpha_1(|\eta|) \leq V(\eta) \leq \alpha_2(|\eta|) \quad (22)$$

$$|\eta| \geq \gamma(|w|) \Rightarrow \frac{\partial V}{\partial \eta} F(\eta, u_c(\eta), w) \leq -\alpha_3(|\eta|)$$

where the gain is $\tilde{\gamma} = \alpha_1^{-1} \circ \alpha_2 \circ \gamma$.

The closed loop system (19), (21) is first written in the form:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_c(z(t - \tau(t)), y(t - \tau(t))), w(t)) \\ \dot{z}(t) &= g(z(t - \tau(t)), y(t - \tau(t))) \end{aligned} \quad (23)$$

where $\tau(t) = t - kT, t \in [kT, (k+1)T[$ is a time varying time delay. The first equation in (23) is obtained by noting that $x(t - \tau(t)) = x(kT), t \in [kT, (k+1)T[$ and by integrating the second equation in (23) over one sampling interval we obtain (21). The equation can be rewritten as:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u_c(z(t), y(t)) + \xi_1(t), w(t)) \\ \dot{z}(t) &= g(z(t), y(t)) + \xi_2(t) \end{aligned} \quad (24)$$

where

$$\begin{aligned} \xi_1(t) &= u_c(z(t - \tau(t)), y(t - \tau(t))) - u_c(z(t), y(t)) \\ \xi_2(t) &= g(z(t - \tau(t)), y(t - \tau(t))) - g(z(t), y(t)) \end{aligned} \quad (25)$$

and if we introduce the new variable $\xi = (\xi_1^T \ \xi_2^T)^T$, the equation (24) can be written in the form

$$\dot{\eta}(t) = \Phi(\eta(t), \xi(t), w(t)). \quad (26)$$

From Lemma (2) and the fact that the continuous time closed loop system (19), (20) is ISS, i.e., (26) with $\xi(t) \equiv 0$ is ISS, we know that there exists a smooth invertible $G(\eta, w)$ and $\gamma_\xi \in \mathcal{K}_\infty$ such that the system

$$\dot{\eta}(t) = \Phi(\eta(t), G(\eta(t), w(t))\xi(t), w(t)) \quad (27)$$

satisfies

$$\begin{aligned} |\eta| &\geq \max\{\gamma_\xi(|\xi(t)|), \gamma(|w(t)|)\} \\ &\Rightarrow \dot{V} \leq -0.5\alpha_3(|\eta(t)|) \end{aligned} \quad (28)$$

Using (25) and (28) and following the steps of the proof of Theorem 2, the proof is completed. Q.E.D.

4 Further generalizations and related results

Results similar to Theorems 2 and 3 can be proved for more general situations:

1. The controller discrete-time approximation is not Euler as in Theorem 3 but some higher order approximation of controller dynamics (see for instance [10] for such formulas for a class of nonlinear systems) or obtained using some other numerical schemes, such as Runge-Kutta.

2. Other higher order hold functions are used instead of the zero order hold, such as

$$u_d(t) = u_c(x(kT)) + \frac{t - kT}{T}(u_c(x(kT)) - u_c(x((k-1)T))) \quad (29)$$

We sketch below a general approach to all such problems. The first step is to write a functional differential equation with time varying delay whose solution coincides with the solution of the sampled-data nonlinear system (for example in Theorem 3 it is the equation (23)). The form of the equation depends on the approximation scheme we use for the controller dynamics and the type of the hold device which we use. In general, however, we obtain the closed loop system of the form (26) where η includes both plant and controller states and ξ is the vector of “new” disturbances. Since we know that the continuous time system $\dot{\eta} = \Phi(\eta, 0, w)$ is ISS, using Lemma 2 it follows that the system (27) is uniformly ISS with respect to w and ξ . Also, the bound of the form $|\xi| \leq T \max\{\gamma_1(|x_d(t)|), \gamma_2(|w_d(t)|)\}$ can be obtained, where T is sampling period and $|x_d(t)| = \max_{s \in [t, t-nT]} |x(s)|$, $|w_d(t)| = \max_{s \in [t, t-nT]} |w(s)|$, and the integer n depends directly on the type of hold function and the approximation scheme we use for the controller. Finally, Theorem 1 is invoked to show that given arbitrary $0 < \delta < \Delta$, there exists $T^* > 0$ such that for any sampling period $T \in]0, T^*[$, the system (26) is uniformly ISS with the same gain, offset δ and restriction Δ .

In Theorems 2 and 3 we have not addressed the rate of convergence of the state of the sampled system, although this information can be obtained in certain situations. Consider, for instance, the case of convergence to the origin, i.e., $w \equiv 0$, and suppose we can choose the sampling period $T > 0$ such that, there exist $1 > \lambda > 0, K > 0, \alpha > 0$ such that for all $t_0 \geq 0$ the following holds

$$|x_d(t)| \leq \max\{K e^{-\alpha(t-t_0)} |x_d(t_0)|, \lambda \|x_d\|_{t_0}\}, t \geq t_0. \quad (30)$$

at least for $|x_d(t)|$ sufficiently small.

This would be the case if the origin of the continuous time closed loop system (when $T = 0$) is locally exponentially stable and then $T > 0$ is chosen sufficiently small for the sampled system. It immediately follows from the proof technique in [14] that

$$\|x_d\|_{t_0} \leq K |x_d(t_0)|,$$

i.e., we have uniform stability. For an estimate of the rate of convergence, let t_1 be such that $K e^{-\alpha t_1} = \lambda$. Using (30) for $t = \bar{t} + t_1$ we obtain that

$$|x_d(\bar{t} + t_1)| \leq \max\{K e^{-\alpha(\bar{t} + t_1 - t_0)} |x_d(t_0)|, \lambda \|x_d\|_{t_0}\}, \bar{t} \geq t_0.$$

Therefore, we have

$$\begin{aligned} \sup_{\bar{t} \geq t_0} |x_d(\bar{t} + t_1)| &= \sup_{s \geq t_1 + t_0} |x_d(s)| = \\ \|x_d\|_{t_0 + t_1} &\leq \max\{\lambda |x_d(t_0)|, \lambda \|x_d\|_{t_0}\} \\ &= \lambda \|x_d\|_{t_0} \end{aligned} \quad (31)$$

and so we can write

$$\|x_d\|_{t_0 + nt_1} \leq \lambda^n \|x_d\|_{t_0}$$

for each positive integer n . Define $b = -\frac{1}{t_1} \ln \lambda$. Let t be given and let n be the largest integer such that $t_0 + nt_1 \leq t$, that is $t_0 + (n+1)t_1 \geq t$. The following standard argument is similar to that used in [7, pp. 194-195]:

$$\begin{aligned} |x_d(t)| &\leq \|x_d\|_{t_0 + nt_1} \leq \lambda^n \|x_d\|_{t_0} \\ &= \lambda^{-1} \lambda^{n+1} \|x_d\|_{t_0} \\ &\leq \lambda^{-1} \lambda^{\frac{t-t_0}{t_1}} \|x_d\|_{t_0} \\ &= \lambda^{-1} e^{\ln \lambda \frac{t-t_0}{t_1}} \|x_d\|_{t_0} \\ &= \lambda^{-1} e^{\frac{t-t_0}{t_1} \ln \lambda} \|x_d\|_{t_0} \\ &\leq \lambda^{-1} e^{-b(t-t_0)} K |x_d(t_0)|. \end{aligned} \quad (32)$$

This proves exponential convergence of the states of the sampled system. A similar statement can be proved for the ISS stability but then we have exponential convergence to a ball whose diameter depends on the L_∞ norm of disturbances.

Using Theorem 2 for disturbance-free case $w \equiv 0$ and the above given arguments we can recover, for example, Corollary 3 in [3] which was given for the following set-up. A linear plant P_c with minimal realization (A_p, b_p, c_p) is controlled using an output dynamic linear controller K_c , whose minimal realization is (A_k, b_k, c_k) . The resulting system is given by:

$$\begin{aligned} \dot{x} &= A_p x + b_p (c_k z) \\ \dot{z} &= A_k z + b_k (c_p x) \\ u_c &= c_k z \end{aligned} \quad (33)$$

and in the implementation the control signal $c_p x$ is sampled whereas the signal $c_k z$ is implemented using a zero order hold device (it is piecewise constant). The following result is stated in [3], and now it can be derived as a consequence of our results:

Corollary 1 *If (P_c, K_c) is stable, there exists a positive number T^* such that the state of the sampled-data system converges exponentially for every $T \in]0, T^*[$.*

5 Summary

Theorem 3 is very important for applications since in the nonlinear context the controller design is usually

carried out in continuous time and the digital implementation of a nonlinear controller usually requires some form of approximate discretization. The fact that for fast sampling and even with the simplest Euler approximation of the controller dynamics we have semiglobal practical stability of the sampled-data system motivates implementation of such algorithms in cases when sampling rate is a design parameter. We remark there are cases when Euler discretization of the dynamic controller may yield global stability, as proved for the special case of a current-fed induction motor in [12].

Finally, if the bounds on the sampling rate are to be found from our method, the obtained estimates may be conservative. Obtaining less conservative estimates for the domain of attraction of the sampled-data system and/or the sampling rate appear to be important issues which should be addressed in future research.

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