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# Higher spin polynomial solutions of quantum Knizhnik–Zamolodchikov equation

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**Abstract:** We provide explicit formulae for highest-weight to highest-weight correlation functions of perfect vertex operators of $U_q(\widehat{\mathfrak{sl}(2)})$ in arbitrary integer level $\ell$. They are given in terms of certain Macdonald polynomials. We apply this construction to the computation of the ground state of higher spin vertex models, spin chains (spin $\ell/2$ XXZ) or loop models in the root of unity case $q = -e^{-i\pi/(\ell+2)}$.

## 1. Introduction

In the seminal paper [13], Frenkel and Reshetikhin showed that correlation functions of $q$-deformed vertex operators (VOs) associated to quantized affine algebras satisfy a set of holonomic $q$-difference equations which are a $q$-deformation of the Knizhnik–Zamolodchikov equation [20] ($q$KZ). The analysis is performed for generic value of the level $\ell$ of the affine algebra, but also works for positive integer level provided the obvious modifications are made. The most important one is that the paths in the Weyl chamber describing the various conformal blocks (correlation functions of vertex operators) have to be restricted to the "Weyl alcove". In all that follows we focus on the simplest algebra, $U_q(\widehat{\mathfrak{sl}(2)})$. In this case, the integrable irreducible highest weight modules are characterized by a spin $s \in \frac{1}{2}\mathbb{Z}_+$ (defined e.g. as the spin of the $U_q(\mathfrak{sl}(2))$ representation of the top degree part, where $U_q(\mathfrak{sl}(2))$ is the horizontal subalgebra of $U_q(\widehat{\mathfrak{sl}(2)})$), which in level $\ell$ only exist when $s \leq \ell/2$. Each vertex operator corresponds to a step of the path and is itself characterized by a spin $j \in \{1/2, \ldots, \ell/2\}$. The present paper is entirely dedicated to the case of so-called perfect vertex operators, that is, $j$ has its maximal value $j = \ell/2$, for which there is exactly one possible step, namely, $s \to \ell/2 - s$. This implies that various simplifications occur, and we expect this unique conformal block to be particularly simple; and indeed we provide an explicit formula for them.

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The reason to revisit this somewhat old subject is the observation made in [7] that certain solutions of $q$KZ also provide the ground state entries of integrable models at special values of the deformation parameter $q$. In fact, this idea can be traced back to [25], where Reshetikhin obtained solutions of $q$KZ using an "off-shell Bethe Ansatz" and observed that Bethe equations appeared in the semi-classical approximation, and then pursued in [26], where the Knizhnik–Zamolodchikov equation in the quasi-classical limit was found to produce eigenvectors of the Gaudin model. Here, we are only concerned with one especially simple solution of $q$KZ – with appropriate normalization, a *polynomial* solution, which produces one eigenvector of the higher spin $U_q(\widehat{\mathfrak{sl}(2)})$ (inhomogeneous, twisted) transfer matrix.

Note that in earlier work [7] (see also [6,24]), only level 1 solutions were used, whose explicit form was already known and can be found e.g. in [17]. In the present paper, we obtain a novel expression for the correlation functions of level $\ell$ perfect VOs, and show that once specialized, they provide the ground state entries of the integrable spin $\ell/2$ chain when $q = -e^{-i\pi/(\ell+2)}$. We also establish the connection with the loop model of [32] and thus prove Conj. 1 in it, which is concerned with the degree of the ground state entries as polynomials.

Also note that a *different* procedure to produce (arbitrary level) solutions of the $q$KZ equation is to compute finite temperature correlation functions of the type $\mathrm{tr}(x^d \Phi \ldots \Phi)$, see e.g. [17]. However the resulting integral formulae are of a more complicated nature than those we consider here. In particular, they only make sense for $|q| < 1$ and it is not clear how to continue them to $|q| = 1$. Here we restrict ourselves to the zero temperature correlation functions of the type $\langle 0|\Phi \ldots \Phi|0\rangle$, though $\mathrm{tr}(x^d \Phi \ldots \Phi)$ should be computable along similar lines.

The plan of the paper is as follows: in section 2 we review the construction of $U_q(\widehat{sl(2)})$ currents in terms of bosons and parafermions and then proceed to build perfect vertex operators. Section 3, the core of the paper, contains the derivation of the correlation function of these vertex operators. Section 4 then reinterprets this correlation function as the eigenvector of an integrable transfer matrix in various contexts (spin chain, loop model, supersymmetric lattice fermions). Finally, various appendices provide additional technical details.

## 2. Construction of $U_q(\widehat{sl(2)})$ perfect vertex operators

*2.1. The algebra $U_q(\widehat{sl(2)})$.* Our reference for the quantized affine algebra $U = U_q(\widehat{sl(2)})$ is the book [17] as well as the paper [16], whose conventions we follow. The Chevalley generators are denoted by $E_i, F_i, K_i$, $i = 0, 1$, to which one must add the grading operator $d$: we choose the homogeneous gradation, i.e., $E_0$ (resp. $F_0$) has degree $+1$ (resp. $-1$) and all other generators have degree 0. In particular there is a horizontal subalgebra $U_1 \subset U$ which is generated by $E_1, F_1, K_1$.

We also use in what follows Drinfeld's realization of $U$ in terms of currents [9].

Until specified otherwise we are in the regime $|q| < 1$.

We shall consider the spin $\ell/2$ evaluation representation $\rho_z$ acting on $V_z \cong \mathbb{C}^{\ell+1}$ with standard basis $v_b$, $b = 0, \ldots, \ell$. It is defined by:

$$\begin{aligned} \rho_z(F_1)v_b &= [b]v_{b-1} & \rho_z(F_0)v_b &= [\ell - b]z^{-1}q^{\ell+2}v_{b+1} \\ \rho_z(E_1)v_b &= [\ell - b]v_{b+1} & \rho_z(E_0)v_b &= [b]zq^{-\ell-2}v_{b-1} \\ \rho_z(K_1)v_b &= q^{2b-\ell}v_b & \rho_z(K_0)v_b &= q^{\ell-2b}v_b \end{aligned} \tag{2.1}$$

where $[n] := (q^n - q^{-n})/(q - q^{-1})$. In order to make $d$ act, one must consider $z$ a formal variable, in which case $d = z\frac{d}{dz} + \Delta$, where $\Delta$ is a constant to be fixed below. When restricted to $E_1, F_1, K_1$, it is an ordinary irreducible representation of $U_1 = U_q(\mathfrak{sl}(2))$ of spin $\ell/2$, and we simply denote the underlying space $V$.

We shall also need the $R$-matrix $\bar{R}(z_1/z_2)$ acting on a pair of such representations $V_{z_1} \otimes V_{z_2}$. The bar denotes the fact that we choose for now a particular normalization of the $R$-matrix where it is the identity on the highest weight vectors. Other normalizations will be considered below. Equivalently, consider $\check{\bar{R}}(z) = \mathcal{P}\bar{R}(z)$ where $\mathcal{P}$ permutes factors of the tensor product. We then have the following formula:

$$\check{\bar{R}}(z) = \sum_{j=0}^{\ell} \prod_{r=j+1}^{\ell} \frac{z - q^{2r}}{1 - q^{2r} z} P_j \tag{2.2}$$

where $P_j$ is the projector onto the spin $j$ irreducible subrepresentation of $V \otimes V$ w.r.t. the $U_1$-action.

Finally, we shall denote collectively by $\mathcal{H}_\ell$ any level $\ell$ representation space of $U$, with $\sigma : U \to \mathcal{L}(\mathcal{H}_\ell)$.

*2.2. Level $\ell$ $U_q(\widehat{sl(2)})$ currents.* The results which are summarized here are based on the work of Ding and Feigin [8] (see also [4]). For the sake of completeness and because some details do not appear in this article we give a detailed introduction to the subject in appendix A.

Consider the bosonic operators:

$$[a_n, a_m] = \delta_{m,-n} \frac{[2n][\ell n]}{n} \qquad [a_n, \beta] = \delta_{n,0} \qquad n, m \in \mathbb{Z} \tag{2.3}$$

with grading

$$[d, a_n] = n a_n \qquad [d, \beta] = -\frac{a_0}{2\ell}$$

They act on the direct sum of copies of the bosonic Fock space $\mathcal{H}_\ell^B = \bigoplus_{h \in \mathbb{Z}} \left\langle \prod_{m>0} a_{-m}^{k_m} |h\rangle_B \right\rangle$ which differ by the eigenvalue of $a_0$: $a_0 |h\rangle_B = h |h\rangle_B$. $\beta$ only appears via its exponential, which is such that $e^{\pm\beta} |h\rangle_B = |h \pm 1\rangle_B$. The grading is given by $d |h\rangle_B = -\frac{1}{4\ell} h^2 |h\rangle_B$. When $\ell = 1$, this is a realization of $U$, and therefore it is denoted by $\mathcal{H}_1$.

We build the following current operators:

$$\begin{aligned}
e(z) &= \xi^+(z) e^{2\beta} z^{a_0/\ell} e^{\sum_{n>0} q^{-\frac{n\ell}{2}} \frac{a_{-n}}{[n\ell]} z^n} e^{-\sum_{n>0} q^{-\frac{n\ell}{2}} \frac{a_n}{[n\ell]} z^{-n}} \\
f(z) &= \xi^-(z) e^{-2\beta} z^{-a_0/\ell} e^{-\sum_{n>0} q^{\frac{n\ell}{2}} \frac{a_{-n}}{[n\ell]} z^n} e^{\sum_{n>0} q^{\frac{n\ell}{2}} \frac{a_n}{[n\ell]} z^{-n}} \\
k^\pm(z) &= q^{\pm a_0} e^{\pm(q-q^{-1}) \sum_n a_{\pm n} z^{\mp n}}
\end{aligned} \tag{2.4}$$

where the $\xi^\pm(z)$ are parafermions, see appendix A. A full list of relations they satisfy is given in (A.2), as well as the relation with Chevalley generators. Here we list a few:

$$\begin{aligned}
k^+(z) f(w) &= \frac{z - q^{2+\frac{\ell}{2}} w}{q^2 z - q^{\frac{\ell}{2}} w} f(w) k^+(z) \\
e(z) e(w) \left(z - q^2 w\right) &= e(w) e(z) \left(q^2 z - w\right) \\
f(z) f(w) \left(q^2 z - w\right) &= f(w) f(z) \left(z - q^2 w\right) \\
(q - q^{-1}) z w [e(z), f(w)] &= \delta\left(q^{-\ell} \frac{z}{w}\right) k^+(q^{-\ell/2} z) - \delta\left(q^{\ell} \frac{z}{w}\right) k^-(q^{\ell/2} z)
\end{aligned} \tag{2.5}$$

where $\delta(z) = \sum_{i \in \mathbb{Z}} z^i$.

These operators act on the tensor product of $\ell$ copies of the Fock space $\mathcal{H}_1$, wich is a level $\ell$ representation of $U$. A state $|k_1\rangle_B \otimes \ldots \otimes |k_\ell\rangle_B$ is denoted by $|k_1, \ldots, k_\ell\rangle$. As we shall see in Section 3, when $k_i = 0$ or 1 for all $i$, $|k_1, \ldots, k_\ell\rangle$ is a highest weight vector, with highest weight given by the sum $k = \sum_i k_i$; for fixed $k = \sum_i k_i$, these states are interchangeable and we denote any of them by $|k\rangle$.

Remark: for an alternative construction of these vertex operators, see [5].

*2.3. Intertwining relations.* Let $\Phi(z)$ be a type I vertex operator, i.e., an intertwiner

$$\Phi(z) : \mathcal{H}_\ell \to \mathcal{H}_\ell \otimes V_z$$

whose normalization will be fixed later.

The intertwining condition writes:

$$(\sigma \otimes \rho_z)(\Delta(x))\Phi(z) = \Phi(z)\sigma(x) \tag{2.6}$$

for all $x \in U$, or explicitly for the Chevalley generators:

$$\begin{aligned}
(\sigma(E_i) \otimes 1 + \sigma(K_i) \otimes \rho_z(E_i))\Phi(z) &= \Phi(z)\sigma(E_i) \\
(\sigma(F_i) \otimes \rho_z(K_i^{-1}) + 1 \otimes \rho_z(F_i))\Phi(z) &= \Phi(z)\sigma(F_i) \\
(\sigma(K_i) \otimes \rho_z(K_i))\Phi(z) &= \Phi(z)\sigma(K_i)
\end{aligned} \tag{2.7}$$

plus the intertwining condition for $d$ which will be discussed separately below.

In what follows application of $\sigma$ will be implicit. Using the form of $\rho_z$, cf (2.1), we can expand in components: $\Phi(z) = \sum_{b=0}^{\ell} \Phi_b(z) v_b$ with

$$\begin{aligned}
\Phi_b(z)E_0 - E_0\Phi_b(z) &= [b+1] z q^{-\ell-2} K_0 \Phi_{b+1}(z) \\
\Phi_b(z)F_1 - q^{\ell-2b} F_1 \Phi_b(z) &= [b+1]\Phi_{b+1}(z) \\
\Phi_b(z)E_1 - E_1\Phi_b(z) &= [\ell-b+1] K_1 \Phi_{b-1}(z) \\
\Phi_b(z)F_0 - q^{2b-\ell} F_0 \Phi_b(z) &= [\ell-b+1] z^{-1} q^{\ell+2} \Phi_{b-1}(z) \\
q^{2b-\ell} K_1 \Phi_b(z) &= \Phi_b K_1
\end{aligned} \tag{2.8}$$

Since $\mathcal{H}_\ell$ is a level $\ell$ representation, we have $K_0 K_1 = q^\ell$.

We are mostly interested in operators $F_1$ and $E_0$; expanding the lowering current $f(z)$ in modes: $f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1}$, we have $f_0 = F_1$, $f_1 = E_0 K_1$, so that we rewrite the two first equations in (2.8):

$$\begin{aligned}
\Phi_b(z) f_0 - q^{\ell-2b} f_0 \Phi_b(z) &= [b+1]\Phi_{b+1}(z) \\
\Phi_b(z) f_1 - q^{2b-\ell} f_1 \Phi_b(z) &= q^{2b-\ell}[b+1] z \Phi_{b+1}(z)
\end{aligned} \tag{2.9}$$

This means starting from $\Phi_0(z)$, one can build the $\Phi_b$ by iterated $q$-commutator. We obtain:

$$\begin{aligned}
\Phi_b(z) &= \sum_{k=0}^{b} \frac{(-1)^k q^{k(\ell-b+1)}}{[k]![b-k]!} f_0^k \Phi_0(z) f_0^{b-k} \\
&= z^{-b} \sum_{k=0} \frac{(-1)^k q^{(b-k)(\ell-b+1)}}{[k]![b-k]!} f_1^k \Phi_0(z) f_1^{b-k}
\end{aligned} \tag{2.10}$$

where $[n]! = [n][n-1]\dots[1]$.

Finally, we write $f_0 = \oint f(w)dw$, $f_1 = \oint f(w)wdw$, so that:

$$\begin{aligned}\Phi_b(z) &= \oint \dots \oint \prod_{i=1}^{b} \frac{dw_i}{2\pi i} \sum_{k=0}^{b} \frac{(-1)^k q^{k(\ell-b+1)}}{[k]![b-k]!} f(w_1)\dots f(w_k)\Phi_0(z) f(w_{k+1})\dots f(w_b) \\ &= \oint \dots \oint \prod_{i=1}^{b} \frac{w_i dw_i}{2z^b \pi i} \sum_{k=0}^{b} \frac{(-1)^k q^{(b-k)(\ell-b+1)}}{[k]![b-k]!} f(w_1)\dots f(w_k)\Phi_0(z) f(w_{k+1})\dots f(w_b)\end{aligned} \tag{2.11}$$

The summation over $k$ will be performed in section 2.4.2.

*2.4. Explicit formulae for the perfect vertex operators.* So far we have not used the fact that twice the spin of the VO is equal to the level. For so-called perfect VOs, that is for rank 1 case precisely the VOs of maximal spin $\ell/2$, we also have the following form of the highest weight entry in terms of the bosonic field:

$$\Phi_0(z) = e^{\ell\beta}(-z)^{a_0/2} e^{\sum_n q^{\frac{n\ell}{2}} \frac{a_{-n}}{[2n]} z^n} e^{-\sum_n q^{\frac{n\ell}{2}} \frac{a_n}{[2n]} z^{-n}} \tag{2.12}$$

In order to check the consistency of this Ansatz, we shall require certain identities.

*2.4.1. Commutation relations.* In order to compute the commutation relations between the operators $e(w)$, $f(w)$ and $\Phi_0(z)$, it is convenient to define the normal ordering $:\dots:$, which pushes the negative modes to the left, that is:

$$\left.\begin{matrix} :a_{-m}a_n: \\ :a_n a_{-m}: \end{matrix}\right\} = a_{-m}a_n \qquad\qquad \left.\begin{matrix} :a_0\beta: \\ :\beta a_0: \end{matrix}\right\} = \beta a_0$$

where $n$ and $m$ are positive integers. The normal ordering has the following important properties:

**Lemma 1.** *Let $\mathcal{O}$ be a product of $N$ operators $\mathcal{O} = \mathcal{O}_1 \dots \mathcal{O}_N$. If $\mathcal{O}_i\mathcal{O}_j = g_{i,j} :\mathcal{O}_i\mathcal{O}_j:$, then $\mathcal{O} = \prod_{i<j} g_{i,j} :\mathcal{O}:$.*

**Lemma 2.** *Let $\mathcal{O}$ be an operator such that $:\mathcal{O}: = e^{g\beta} f(a_0) e^{\sum_{n>0} d_n a_{-n}} e^{\sum_{n>0} c_n a_n}$, where $g$, $c_n$ and $d_n$ are scalar constants. Then*

$${}_B\langle k| :\mathcal{O}: |k\rangle_B = f(k)$$

*if $g = 0$ and it vanishes otherwise.*

*In the more general context where $:\mathcal{O}: = \bigotimes_i e^{g_i\alpha} f_i(a_0) e^{\sum_{n>0} d_{i,n} a_{-n}} e^{\sum_{n>0} c_{i,n} a_n}$ this property is conserved:*

$$\langle k_1, \dots, k_\ell| :\mathcal{O}: |k_1, \dots, k_\ell\rangle = \prod_i f_i(k_i)$$

*if $g_i = 0$ for all $i$ and it vanishes otherwise.*

For convenience, we split the currents in bosonic and parafermionic components, that is, $e(w) = \xi^+(w)\mathfrak{e}(w)$ and $f(w) = \xi^-(w)\mathfrak{f}(w)$.

**Proposition 1.** *The relation between the products and the normal ordered products is:*

$$\Phi_0(z_1)\Phi_0(z_2) = (-z_1)^{\frac{\ell}{2}} \frac{\left(q^2\frac{z_2}{z_1};q^4\right)_\infty}{\left(q^{2\ell+2}\frac{z_2}{z_1};q^4\right)_\infty} :\Phi_0(z_1)\Phi_0(z_2):$$
$$\mathfrak{f}(w)\Phi_0(z) = \frac{1}{w-q^\ell z} :\mathfrak{f}(w)\Phi_0(z):$$
$$\Phi_0(z)\mathfrak{f}(w) = \frac{1}{q^\ell w - z} :\mathfrak{f}(w)\Phi_0(z):$$
$$\mathfrak{e}(w)\Phi_0(z) = (w-z):\mathfrak{e}(w)\Phi_0(z):$$
$$\Phi_0(z)\mathfrak{e}(w) = (w-z):\mathfrak{e}(w)\Phi_0(z):$$

*Remark 1.* In the previous computations, we perform infinite sums which are convergent under the following conditions:

$$|q^2 z_2| < |z_1| \qquad |q^\ell z| < |w| \qquad |q^\ell w| < |z| \qquad |z| < |w| \qquad |w| < |z|$$

for each expression in proposition 1, respectively.

From these expressions we find immediately:

$$\begin{aligned} \mathfrak{e}(w)\Phi_0(z) &= \mathfrak{e}(w)\Phi_0(z) \\ \mathfrak{f}(w)\Phi_0(z) &= \frac{q^\ell w - z}{w - q^\ell z}\Phi_0(z)\mathfrak{f}(w) \end{aligned} \tag{2.13}$$

As proven in appendix A, the parafermions do not interact with the bosons. Therefore, in these formulae we can replace $\mathfrak{f}(w)$ with $f(w)$.

From the first identity we conclude that $[e_k, \Phi_0(z)] = 0$. For $k = 0, 1$, this coincides with the intertwining equations in (2.8) for $E_1$ and $F_0$ at $b = 0$. We of course also have $K_1\Phi_0 = q^\ell \Phi_0 K_1$, using $K_1 = q^{a_0}$. Finally, we have:

$$x^d \Phi_0(z) x^{-d} = x^{-\ell/4}\Phi_0(x^{-1}z) \tag{2.14}$$

which coincides with the intertwining condition for $d$ on condition that one set $\Delta = \ell/4$. [1]

*2.4.2. Closed expression for the $\Phi_b(w)$.* Equation (2.10) provides us with two expressions for $\Phi_b(z)$. Performing the summation results in:

**Lemma 3.** *The following expression holds:*

$$\Phi_b(z) = \alpha_b \oint \dots \oint \prod_{i=1}^{b} \frac{w_i dw_i}{2\pi i} \frac{\Phi_0(z) f(w_1)\dots f(w_b)}{\prod_i (w_i - q^\ell z)}$$

*where $\alpha_b$ is a constant given by:*

$$\alpha_b = \frac{\prod_{i=1}^{b}(1-q^{2\ell+2-2i})}{[b]!} = (-1)^b (q-q^{-1})^b q^{b(2\ell-b+1)/2} \begin{bmatrix} \ell \\ b \end{bmatrix}$$

*with the notation* $\begin{bmatrix} \ell \\ b \end{bmatrix} = \frac{[\ell]!}{[b]![\ell-b]!}$.

[1] Note that in Conformal Field Theory (corresponding to $q \to 1$), this would be interpreted as a consequence of the conformal dimension $\ell/4$ of $\Phi_0(z)$. The $q$-deformation breaks the conformal symmetry by marking the two points 0 and $\infty$, leaving only the residual scaling symmetry $z \to xz$.

*Proof.* We shall use the first expression of $\Phi_b(z)$ in (2.11); the second expression would lead to the same result. We have a sum over several terms. We commute all $f$ to the right, so that each term becomes:

$$f(w_1)\dots f(w_k)\Phi_0(z)f(w_{k+1})\dots f(w_b) = \frac{\prod_{j\le k}(q^\ell w_j - z)}{\prod_{j\le k}(w_j - q^\ell z)}\Phi_0(z)f(w_1)\dots f(w_b)$$
$$= \frac{\prod_{j\le k}(q^\ell w_j - z)\prod_{j>k}(w_j - q^\ell z)}{\prod_j (w_j - q^\ell z)}\Phi_0(z)f(w_1)\dots f(w_b)$$

Our purpose is to compute the expectation value of an operator, which contains the operator $\Phi_0(z)f(w_1)\dots f(w_b)$. By lemmas 1 and 2, this will make appear one term per each pair of operators. From $\Phi_0(z)f(w_i)$ we get a pole $\frac{1}{q^\ell w_i - z}$. The contribution of the pairs $f(w_i)f(w_j)$ is more complicated and we need to consider the decomposition of parafermions in modes $f^k(w_i)$, which we explain in appendix A. Here we only need a couple of properties. The relations (A.2), which define $U_q(\widehat{\mathfrak{sl}(2)})$, imply that

$$f(w_1)\dots f(w_b)\prod_{i<j}\frac{q^2 w_i - w_j}{w_i - w_j}$$

is symmetric in the exchange of the $w_i$. This expression decomposes as a sum of terms of the form $f^{i_1}(w_1)\dots f^{i_b}(w_b)\prod_{i<j}($ $w_j)$, which each satisfy by equation (A.23)

$$f^{i_1}(w_1)\dots f^{i_b}(w_b)\prod_{i<j}(q^2 w_i - w_j) = P(w_1,\dots,w_b) : f^{i_1}(w_1)\dots f^{i_b}(w_b) :$$

where $P$ is some multivariate polynomial. We conclude from the definition of normal ordering that $f(w_1)\dots f(w_b)\prod_{i<j}\frac{q^2 u}{w_i}$ is a symmetric function of the $\{w_1,\dots,w_b\}$ without poles of the form $(w_i - \zeta w_j)^{-1}$.

In summary, if we write $f_b^i(z;w_1,\dots,w_b) = \prod_{j\le i}(q^\ell w_j - z)\prod_{j>i}(w_j - q^\ell z)$ (which is linear in each variable $w_j$), then the quantity that we want to compute is:

$$\Phi_b(z) = \oint\dots\oint\prod_{i=1}^b\frac{dw_i}{2\pi i}\sum_{k=0}^b\frac{(-1)^k q^{k(\ell-b+1)}}{[k]![b-k]!}f_b^k(z;w_1,\dots,w_b)\prod_{i<j}\frac{w_i - w_j}{q^2 w_i - w_j}S(z;w_1,\dots,w_b)$$

where $S(z;w_1,\dots,w_b)$ is some symmetric function of the $\{w_1,\dots,w_b\}$ without poles of the form $(w_i - \zeta w_j)^{-1}$.

We now want to prove that

$$\sum_{k=0}^b\frac{(-1)^k q^{k(\ell-b+1)}}{[k]![b-k]!}f_b^k(z;w_1,\dots,w_b) = \alpha_b\prod_{i=1}^b w_i$$

up to some terms which vanish when integrated, i.e., which are divisible by $(q^2 w_i - w_{i+1})$. They produce a zero contribution because the integral becomes skew-symmetric.

This is true when $b = 1$:

$$f_1^0(z;w_1) - q^\ell f_1^1(z;w_1) = (w_1 - q^\ell z) - q^\ell(q^\ell w_1 - z) = (1 - q^{2\ell})w_1$$

Suppose that this is true for $b-1$, then:

$$\sum_{k=0}^{b} \frac{(-1)^k q^{k(\ell-b+1)}}{[k]![b-k]!} f_b^k = \sum_{k=0}^{b-1} \frac{(-1)^k q^{k(\ell-b+2)}}{[k]![b-k-1]![b]} \prod_{j\le k}(q^\ell w_j - z)\prod_{j>k}(w_j - q^\ell z)$$
$$- q^{\ell+2-2b}\sum_{k=1}^{b} \frac{(-1)^{k-1} q^{(k-1)(\ell-b+2)}}{[k-1]![b-k]![b]} \prod_{j\le k}(q^\ell w_j - z)\prod_{j>k}(w_j - q^\ell z)$$
$$= \frac{\alpha_{b-1}}{[b]}(w_b - q^\ell z)\prod_{i=1}^{b-1} w_i - \frac{q^{\ell+2-2b}\alpha_{b-1}}{[b]}(q^\ell w_1 - z)\prod_{i=2}^{b} w_i$$
$$= \frac{\alpha_{b-1}}{[b]}(1 - q^{2\ell+2-2b})\prod_{i=1}^{b} w_i - zq^\ell \frac{\alpha_{b-1}}{[b]}(w_1 - q^{2-2b} w_b)\prod_{i=2}^{b-1} w_i$$

by the induction hypothesis. The last term can be rewritten:

$$(w_1 - q^{2-2b})w_2 \dots w_{b-1} = \sum_{i=1}^{b-1} q^{-2(i-1)} \left(\prod_{j<i} w_j\right)(w_i - q^{-2} w_{i+1})\left(\prod_{j>i+1} w_j\right)$$

which vanishes when integrated.

The constant $\alpha_b$ is given by this recursion.

*2.5. The R matrix.* Consider next the operator $\check{R}(z_1/z_2)\Phi(z_1)\Phi(z_2) : \mathcal{H}_\ell \to \mathcal{H}_\ell \otimes V_{z_2} \otimes V_{z_1}$, with the normalization of the $R$-matrix to be adjusted below. $\check{R}(z_1/z_2)\Phi(z_1)\Phi(z_2)$ is an intertwiner by the defining properties of $R$ and $\Phi$. We now use the fact that $\Phi$ is a perfect VO, i.e., that it is of (maximal) spin $\ell/2$ where $\ell$ is the level. This means that each highest weight irreducible representation (say of spin $s$) in $\mathcal{H}_\ell$, when tensored with $V_z$, is sent onto another *unique* irreducible representation (namely, of spin $\ell/2 - s$), and the same when tensored twice. Such an intertwiner is therefore unique; so that it should be proportional to $\Phi(z_2)\Phi(z_1)$. In order to fix the proportionality constant, we compute

$$r(z_1/z_2)\Phi^{(0)}(z_1)\Phi^{(0)}(z_2) = \Phi^{(0)}(z_2)\Phi^{(0)}(z_1)$$

with

$$r(z) = z^{-\ell/2}\frac{(q^2 z; q^4)_\infty (q^{2\ell+2}/z; q^4)_\infty}{(q^2/z; q^4)_\infty (q^{2\ell+2} z; q^4)_\infty}$$

and now choose (as in [16])

$$R(z) = r(z)\bar{R}(z)$$

where $\bar{R}(z)$ is the identity on the tensor product of highest weight vectors, in such a way that

$$\check{R}(z_1/z_2)\Phi(z_1)\Phi(z_2) = \Phi(z_2)\Phi(z_1) \tag{2.15}$$

## 3. Correlation functions of $U_q(\widehat{sl(2)})$ perfect vertex operators

In this section we explain how to compute the correlation function in even size $L = 2n$:

$$\Psi^{(k)}_{b_1,\dots,b_{2n}}(z_1,\dots,z_{2n}) = \langle k|\,\Phi_{b_1}(z_1)\dots\Phi_{b_{2n}}(z_{2n})\,|k\rangle$$

$\Psi^{(k)}_{b_1,\dots,b_{2n}}(z_1,\dots,z_{2n})$ is nonzero only when the neutrality condition $\sum_{i=1}^{2n} b_i = n\ell$ it satisfied. The final result will be a multiple contour integral.

*3.1. Integral formulae for correlation functions.* By lemma 3, the product of vertex operators $\Phi_{b_1}(z_1)\dots\Phi_{b_{2n}}(z_{2n})$ can be expressed as a multiple integral containing $2n$ operators $\Phi_0(z_i)$ and $\ell n$ operators $f(w_j)$. Then we can use lemmas 1 and 2 to compute the correlation function.

*3.1.1. Correlation functions of q-parafermions.* Take a product of $\ell n$ currents, multiplied as before by some appropriate prefactors, and do a mode decomposition:

$$\mathcal{F} = f(w_1)\dots f(w_{\ell n})\prod_{i<j}\frac{w_j - q^2 w_i}{w_j - w_i} = \sum_{i_1,\dots,i_{\ell n}} f^{i_1}(w_1)\dots f^{i_{\ell n}}(w_{\ell n})\prod_{i<j}\frac{w_j - q^2 w_i}{w_j - w_i} \tag{3.1}$$

Equation (A.23) implies that the correlation function $\langle 0|\,e^{2\ell n\beta}\mathcal{F}\,|0\rangle$ is a rational function. The poles which appear on (A.23) cancel with the product $\prod_{i<j}(w_j - q^2 w_i)$, thus the expression is of the form:

$$\langle 0|\,e^{2\ell n\beta}\mathcal{F}\,|0\rangle = \frac{\mathcal{P}(w_1,\dots,w_{\ell n})}{\prod_{i<j}(w_i - w_j)}$$

where $\mathcal{P}$ is some polynomial.

By equation (A.2), $\mathcal{F}$ is symmetric and therefore $\mathcal{P}$ is antisymmetric. It follows that $\langle 0|\,e^{2\ell n\beta}\mathcal{F}\,|0\rangle$ is a symmetric polynomial in the $w_i$.[2] Moreover, as a consequence of theorem 4 for its parafermionic part, it satisfies a *wheel condition*.

This wheel condition can be defined in a general setting as follows. Let $t_M$, $q_M$ be two scalars such that $q_M^{r-1} t_M^{k+1} = 1$. A symmetric polynomial is said to satisfy the $(r,k)$-wheel condition, if it vanishes whenever we set the first $k+1$ variables such that $\frac{z_{i+1}}{z_i} = t_M q_M^{s_i}$ for all $i \le k$ and $\sum_i s_i \le r-1$. Here we find that $\langle 0|\,e^{2\ell n\beta}\mathcal{F}\,|0\rangle$ satisfies the wheel condition when we set $r = 2$, $k = \ell$ and $t_M = q^{-2}$.

For each term on the decomposition (3.1) we can compute the effect of normal ordering:

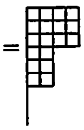
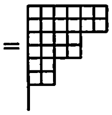
$$f^{i_1}(w_1)\dots f^{i_{\ell n}}(w_{\ell n}) = P^{i_1,\dots,i_{\ell n}}(w_1,\dots,w_{\ell n}) :f^{i_1}(w_1)\dots f^{i_{\ell n}}(w_{\ell n}):$$

Now, we ignore everything but the terms in $e^{\alpha}$. On the one hand, we have $e^{2\ell n\beta} = \bigotimes_j e^{n\alpha}$. On the other hand, each parafermionic mode $f^k$ is proportional to $\bigotimes_{j<k} 1 \otimes e^{-\alpha} \bigotimes_{j>k} 1$. Then, by lemma 2, each mode $k$ should appear exacly $n$ times.

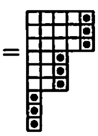
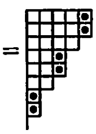
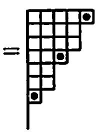
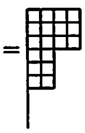
Each time we have a repeated mode (that is, $\epsilon_i = \epsilon_j$) we get a term of degree 2, more precisely $q^{2\ell - 2\epsilon_i}(w_i - q^2 w_j)(q^2 w_i - w_j)$. Otherwise, we get a term of degree zero. Therefore, the degree of $\langle 0|\,e^{2\ell n\beta}\mathcal{F}\,|0\rangle$ is $\ell n(n-1)$. The same analysis can be used to prove that $(\prod_i w_i)\,\langle k|\,e^{2\ell n\beta}\,|k\rangle$ is a symmetric polynomial of degree $\ell n(n-1) + n(\ell - k)$.

[2] It can be shown that the coefficients of the polynomial are Laurent polynomials in $q$.

We can do a more detailed analysis and obtain the dominant monomial. Let $Y_{\ell,n}$ be the staircase Young diagram, corresponding to $n$ steps of $\ell\times 2$: $Y_{\ell,n} = (2(n-1),\dots,2(n-1),2(n-2),\dots,2(n-2),2,\dots,2,0,\dots,0)$. For example:

$Y_{3,3} =$  $Y_{2,4} =$ 

More generally we define a modified staircase Young diagram $Y^{(k)}_{\ell,n}$. Which is obtained from $Y_{\ell,n}$ by adding $\ell - k$ boxes vertically at each step. For example:

$Y^{(0)}_{3,3} =$  $Y^{(1)}_{3,3} =$  $Y^{(2)}_{3,3} =$  $Y^{(3)}_{3,3} =$ 

where the doted boxes correspond to the boxes added to $Y_{\ell,n}$.

Recall that a monomial corresponding to some partition $\lambda = \{\lambda_1,\dots,\lambda_N\}$ is defined by $w^\lambda = \prod_i w_i^{\lambda_i}$.

**Proposition 2.** *The leading monomial of* $(\prod_i w_i)\,\langle k|\,e^{2\ell n\beta}\mathcal{F}\,|k\rangle$ *is:*

$$q^{\ell^2 n(n-1)+n\binom{\ell}{2}}\,([k]![\ell-k]!)^n\,w^{Y^{(k)}_{\ell,n}}$$

*In the case $k=0$, the expression simplifies:*

$$\langle 0|\,e^{2\ell n\beta}\mathcal{F}\,|0\rangle = q^{\ell^2 n(n-1)+n\binom{\ell}{2}}\,([\ell]!)^n\,w^{Y_{\ell,n}} + \textit{ lower terms.}$$

Sketch of proof: multiply $(\prod_i w_i)\,\langle k|\,e^{2\ell n\beta}\mathcal{F}\,|k\rangle$ by the Vandermonde determinant, and try to maximize the degree of $w_1$, then try to maximize the degree of $w_2$ and so on.

Call a Young diagram $(r,k)$-admissible if $\lambda_i - \lambda_{i+k} \geq r$ for all $i$. Then, it is clear that a staircase $(r,k)$-admissible Young diagram has the minimum number of boxes possible. If we set $r=2$, $k=\ell$ and we restrict ourselves to diagrams of lenght $2n$ the only diagram with $\ell n(n-1)$ boxes is exacly $Y_{\ell,n}$. For the case of modified staircases, the situation is different: there are several $(2,\ell)$-admissible Young diagrams with $\ell n(n-1)+(\ell-k)n$ boxes, the smallest of them being $Y^{(k)}_{\ell,n}$.

Macdonald polynomials [22] form a basis for the symmetric polynomials depending on two extra variables which we call $q_M$ and $t_M$. The only result related to Macdonald polynomials that we need for our purposes is the following theorem:

**Theorem 1 (Feigin, Jimbo, Miwa and Mukhin [11]).** *Let $q_M$, $t_M$ be such that $q_M^{r-1}t_M^{k+1}=1$. Let $\mathcal{V}$ be the vector space of symmetric polyynomials on $N$ variables satisfying the $(r,k)$–wheel condition. And let $\mathcal{M}$ be the vector space spanned by the Macdonald polynomials given by $(r,k)$-admissible Young diagrams and the Macdonald parameters $q_M$, $t_M$.*

*Then $\mathcal{V}=\mathcal{M}$.*

As a direct consequence we can compute the correlation function:

**Theorem 2.**
$$\langle 0| e^{2\ell n\beta} f(w_1)\dots f(w_{\ell n})|0\rangle \prod_{i<j}\frac{w_j-q^2w_i}{w_j-w_i}=\gamma_{\ell,n}P_{Y_{\ell,n}}(w_1,\dots,w_{\ell n})$$

*where $P_{Y_{\ell,n}}$ is the Macdonald polynomial with parameters $t_M=q^{-2}$ and $q_M=t_M^{-(\ell+1)}$.*

The proportionality factor is the leading coefficient, i.e.

$$\gamma_{\ell,n}=q^{\ell^2 n(n-1)+n\binom{\ell}{2}}\left([\ell]!\right)^n$$

*Proof.* We have proven above that the correlation function in the theorem is a homogeneous symmetric polynomial that satisfies the wheel condition, more precisely the $(2,\ell)$-wheel condition. Therefore it lives in a vector subspace spanned by Macdonald polynomials associated with $(2,\ell)$-admissible Young diagrams.

The polynomial depends in $2n$ variables and has a combined degree of $\ell n(n-1)$. The diagram $Y_{\ell,n}$ is the smallest one that satisfies the wheel condition (with $2n$ parts) and it has exacly $\ell n(n-1)$. Therefore, it corresponds to the only Macdonald polynomial with such conditions.

As $\langle k| e^{2\ell n\beta}\mathcal{F}|k\rangle$ also satisfies the wheel condition, we can express it as well as a sum over Macdonald polynomials: $(\prod_i w_i)\langle k| e^{2\ell n\beta}\mathcal{F}|k\rangle=\sum_Y c_{k,Y}P_Y(w_1,\dots,w_{\ell n})$ where the sum runs over all $(2,\ell)$-admissible Young diagrams with $\ell n(n-1)+n(\ell-k)$ boxes. By proposition 2, we know that the leading monomial is $w^{Y^{(k)}_{\ell,n}}$ and therefore only term appears in this sum:

$$\left(\prod_{i=1}^{\ell n} w_i\right)\langle k| e^{2\ell n\beta}\mathcal{F}|k\rangle=\gamma^{(k)}_{\ell,n}P_{Y^{(k)}_{\ell,n}}(w_1,\dots,w_{\ell n})$$

with coefficient

$$\gamma^{(k)}_{\ell,n}=q^{\ell^2 n(n-1)+n\binom{\ell}{2}}\left([k]![\ell-k]!\right)^n$$

Notice that this implies that $P_{Y^{(0)}_{\ell,n}}=(\prod_i w_i)\,P_{Y_{\ell,n}}$.

*3.1.2. Derivation of integral formulae.* We have computed all the ingredients to compute $\Psi^{(k)}_{b_1,\dots,b_{2n}}$. In order to express the result, we use the following reparameterization of the index sequence $\{b_1,\dots,b_{2n}\}$. Let $\epsilon=\{\epsilon_1,\dots,\epsilon_{\ell n}\}$ be a non-decreasing sequence, such that $1\le\epsilon_i\le 2n$, and such that, the same value can not be repeated more that $\ell$ times. Then, there is a bijection between the set of neutral spin sequences $\{b_1,\dots,b_{2n}\}$ and the non-decreasing sequences $\epsilon$, given by: $b_j=\#\{\epsilon_i \text{ such that } \epsilon_i=j\}$. To each $\epsilon_i$ equal to $j$ corresponds a current $f(w_i)$ that is used to raise the spin of $\Phi_0(z_j)$.

We then compute:

- From the product $\Phi_0(z_1)\dots\Phi_0(z_{2n})$ we get:

$$\Phi_0(z_1)\dots\Phi_0(z_{2n})=\prod_{i<j}(-z_i)^{\frac{\ell}{2}}\frac{\left(q^2\frac{z_j}{z_i};q^4\right)_\infty}{\left(q^{2\ell+2}\frac{z_j}{z_i};q^4\right)_\infty}:\Phi_0(z_1)\dots\Phi_0(z_{2n}):$$

- When $\epsilon_i<j$, we have pairs of the form $f(w_i)\Phi_0(z_j)$, and then:

$$f(w_i)\Phi_0(z_j)=\frac{1}{w_i-q^\ell z_j}:f(w_i)\Phi_0(z_j):$$

– When $\epsilon_i \geq j$, we have pairs of the form $\Phi^0(z_j)f(w_i)$, and then:

$$\Phi_0(z_j)f(w_i) = \frac{1}{q^\ell w_i - z_j} : f(w_i)\Phi_0(z_j) :$$

– Finally, we get an extra term for the case $\epsilon_i = j$:

$$\Phi_{b_j}(z_j) = \alpha_{b_j} \oint \cdots \oint \left( \prod_{i:\epsilon_i = j} \frac{w_i dw_i}{2\pi i} \frac{1}{w_i - q^\ell z_j} \right) \Phi_0(z_j) \prod_{i:\epsilon_i = j} f(w_i)$$

where the last product is in the increasing order.

We conclude using lemma 1. We first write the case $k = 0$:

$$\Psi^{(0)}_{b_1,\ldots,b_{2n}}(z_1,\ldots,z_{2n}) = \gamma_{\ell,n} \prod_{i=1}^{2n} \alpha_{b_i} \prod_{1\leq i<j\leq 2n} (-z_i)^{\frac{\ell}{2}} \frac{\left(q^2 \frac{z_j}{z_i}; q^4\right)_\infty}{\left(q^{2\ell+2} \frac{z_j}{z_i}; q^4\right)_\infty}$$
$$\times \oint \cdots \oint \prod_{i=1}^{\ell n} \frac{w_i dw_i}{2\pi i} \prod_{i<j} \frac{w_j - w_i}{w_j - q^2 w_i} \frac{P_{Y_{\ell,n}}(w_1,\ldots,w_{\ell n})}{\prod_{j\leq \epsilon_i}(q^\ell w_i - z_j) \prod_{j\geq \epsilon_i}(w_i - q^\ell z_j)} \quad (3.2)$$

Where the countours are chosen such that

$$|w_i| < |q^{-\ell} z_j| \quad \text{for all } j \leq \epsilon_i \qquad |w_i| > |q^\ell z_j| \quad \text{for all } j \geq \epsilon_i$$
$$|z_i| > |q^2 z_j| \quad \text{for all } i < j \qquad |w_i| > |q^{-2} w_j| \quad \text{for all } i < j$$

This choice is consistent with remark 1.

The same computation can be repeated for $\Psi^{(k)}_{b_1,\ldots,b_{2n}} = \langle k | \Phi_{b_1} \ldots \Phi_{b_{2n}} | k \rangle$, and we obtain:

$$\Psi^{(k)}_{b_1,\ldots,b_{2n}}(z_1,\ldots,z_{2n}) = \gamma^{(k)}_{\ell,n} \prod_{i=1}^{2n} \alpha_{b_i} (-z_i)^{\frac{k}{2}} \prod_{1\leq i<j\leq 2n} (-z_i)^{\frac{\ell}{2}} \frac{\left(q^2 \frac{z_j}{z_i}; q^4\right)_\infty}{\left(q^{2\ell+2} \frac{z_j}{z_i}; q^4\right)_\infty}$$
$$\times \oint \cdots \oint \prod_{i=1}^{\ell n} \frac{dw_i}{2\pi i} \prod_{i<j} \frac{w_j - w_i}{w_j - q^2 w_i} \frac{P_{Y^{(k)}_{\ell,n}}(w_1,\ldots,w_{\ell n})}{\prod_{j\leq \epsilon_i}(q^\ell w_i - z_j) \prod_{j\geq \epsilon_i}(w_i - q^\ell z_j)} \quad (3.3)$$

In general, we do not expect a simple formula for the Macdonald polynomial $P_{Y^{(k)}_{\ell,n}}$, except in two cases:

1. If $\ell = 1$, the parafermionic part of the current is trivial, and $P_{Y^{(k)}_{1,n}} = \prod_i w_i^{1-k} \prod_{i<j}(w_i - q^2 w_j)(w_i - q^{-2} w_j)$, and we recover the standard level 1 formulae found in e.g. [17].
2. If $\ell = 2$, the parafermionic field reduces to a ($q$-deformed) fermionic field [3], and $P_{Y^{(k)}_{2,n}} = \prod_{i<j} \frac{(w_i - q^2 w_j)(w_i - q^{-2} w_j)}{w_i - w_j} \mathrm{Pf} \Big($ where $E_m(w_i, w_j)$ is the elementary symmetric polynomial, reproducing results of [15].

In the rest of this section, we shall write $\Psi := \Psi^{(k)}$ when there is no risk of confusion.

*3.2. Quantum Knizhnik–Zamolodchikov equation.* A detailed proof of the qKZ equation can be found in [16, appendix A]). For the convenience of the reader, we provide a short graphical proof in appendix B.2.

Define

$$R_\pm(z) = \phi_\pm(z)\check{R}(z)$$

with

$$\phi_-(z) = q^{-\ell^2/2}\frac{(q^2z;q^4)_\infty^2}{(q^{2\ell+2}z;q^4)_\infty\,(q^{-2\ell+2}z;q^4)_\infty}$$
$$\phi_+(z) = \phi_-(1/z)^{-1}$$

Note that compared to the other normalization $R(z)$ we have used before, one has:

$$R_\pm(z) = \bar\phi_\pm(z)R(z) \qquad \bar\phi_-(z) = \begin{cases}(-1)^m & \ell = 2m \text{ even}\\ (-1)^m(z/q)^{1/2}\frac{(q^2z;q^4)_\infty(q^2/z;q^4)_\infty}{(z;q^4)_\infty(q^4/z;q^4)_\infty} & \ell = 2m+1 \text{ odd}\end{cases}$$

and $\bar\phi_+(z) = \bar\phi_-(1/z)^{-1}$.

The following equation is then satisfied by $\Psi(z_1,\dots,z_L) = \langle k|\,\Phi(z_1)\dots\Phi(z_L)\,|k\rangle$:

$$R_-(z_{i+1}/(sz_i))\dots R_-(z_L/(sz_i))K_1^{1+k}R_+(z_1/z_i)\dots R_+(z_{i-1}/z_i)\Psi(z_1,\dots,z_L) = \Psi(z_1,\dots,s\,z_i,\dots,z_L) \qquad i=1,\dots,L \quad (3.4)$$

where $s = q^{-2(\ell+2)}$; all the indices are suppressed and can be recovered by following the spectral parameters $z_i$, see appendix B.2 for details. This is the quantum Knizhnik–Zamolodchikov equation, first introduced in [13].

As a consequence of (2.15) for the VOs we also have the following identity (exchange relation):

$$\check{R}_{i,i+1}(z_i/z_{i+1})\Psi(z_1,\dots,z_L) = \Psi(z_1,\dots,z_{i+1},z_i,\dots,z_L) \qquad (3.5)$$

We shall come back to this equation in section 3.4.

*3.3. Recurrence relations.* Define for convenience the "dual vertex operator" $\Phi^*$, which by self-duality of evaluation representation can be expressed in terms of $\Phi$ as:

$$\Phi_b^*(z) = c_b z^{\ell/2}\Phi_{\ell-b}(q^{-2}z) \qquad (3.6)$$

where $c_b = (-1)^{\ell/2+b+k}q^{k(\ell-k)-b(\ell-b)}\begin{bmatrix}\ell\\k\end{bmatrix}/\begin{bmatrix}\ell\\b\end{bmatrix}$.

From general arguments one can prove that

$$\Phi_b(z)\Phi_{b'}^*(z) = \delta_{b,b'}g_k \qquad (3.7a)$$

$$\sum_{b=0}^{\ell}\Phi_b^*(z)\Phi_b(z) = g_k \qquad (3.7b)$$

where $g_k = q^{k(\ell-k)}\begin{bmatrix}\ell\\k\end{bmatrix}\frac{(q^{2(\ell+1)};q^4)_\infty}{(q^2;q^4)_\infty}$. cf [16, Prop. 4.1].

This implies immediately the following recurrence relations for the correlation functions:

$$\Psi_{\ldots,b,b',\ldots}(\ldots,z,q^{-2}z,\ldots)=\frac{g_k}{c_b}\delta_{b+b',\ell}z^{-\ell/2}\Psi_{\ldots}(\ldots) \tag{3.8a}$$

$$\sum_{b=0}^{\ell}c_b\Psi_{\ldots,\ell-b,b,\ldots}(\ldots,q^{-2}z,z,\ldots)=g_k z^{-\ell/2}\Psi_{\ldots}(\ldots) \tag{3.8b}$$

where $\ldots$ are omitted (arbitrary) arguments.

*3.4. Polynomiality, cyclicity.* It is convenient to redefine

$$\Psi^{(k)}(z_1,\ldots,z_L)=\prod_{i=1}^{L}(-z_i)^{(L-1)\ell/4+k/2}\prod_{1\le i<j\le L}F(z_j/z_i)\Psi^{(k)}(z_1,\ldots,z_L) \tag{3.9}$$

where

$$F(z)=z^{-\ell/4}\frac{(q^4z;q^4)_\infty}{(q^{2(\ell+2)}z;q^4)_\infty}$$

Note that the prefactor $\prod_{i=1}^{L}z_i^{(L-1)\ell/4+k/2}$ combined with equation (2.14) implies that $\Psi(xz_1,\ldots,xz_L)=x^{\ell n(n-1)+kn}\Psi(z_1,\ldots,z_L)$ (where $L=2n$); in fact, rewriting the integral formula (3.2) as:

$$\begin{aligned}\Psi^{(k)}_{b_1,\ldots,b_{2n}}(z_1,\ldots,z_{2n})=\gamma^{(k)}_{\ell,n}\prod_{i=1}^{2n}\alpha_{b_i}z_i^k\prod_{1\le i<j\le 2n}\prod_{r=1}^{\ell}(q^{2r}z_j-z_i)\oint\cdots\oint\prod_{i=1}^{\ell n}\frac{dw_i}{2\pi i}\\ \times\prod_{i<j}\frac{w_j-w_i}{w_j-q^2w_i}\frac{P_{Y^{(k)}_{\ell,n}}(w_1,\ldots,w_{\ell n})}{\prod_{j\le\epsilon_i}(q^\ell w_i-z_j)\prod_{j\ge\epsilon_i}(w_i-q^\ell z_j)}\end{aligned} \tag{3.10}$$

(where we recall that $\epsilon=(\epsilon_1,\ldots,\epsilon_{\ell n})$ is the sequence such that $b_j=\#\{i\text{ such that }\epsilon_i=j\}$) one can show (see appendix D) that $\Psi^{(k)}$ is a (vector-valued) *polynomial* in the variables $z_1,\ldots,z_L$ of degree $\ell n(n-1)+kn$. Furthermore, its coefficients are rational functions of $q$ whose denominators are products of $1-q^{2j}$, $j=1,\ldots,\ell$.

For example, when $\ell=2$ and $n=2$ there are nineteen components, among which:

$$\begin{aligned}\Psi^{(0)}_{1,2,1,0}(z_1,z_2,z_3,z_4)&=(q+q^3)(z_1-q^2z_2)(z_2-q^2z_3)(z_3-q^2z_4)(z_1-q^6z_4)\\ \Psi^{(0)}_{1,2,0,1}(z_1,z_2,z_3,z_4)&=-(q+q^3)(z_1-q^2z_2)(z_3-q^2z_4)\\ &\quad\times(q^2z_1z_2+z_1z_3-q^4z_1z_3-q^4z_2z_3-q^4z_1z_4+q^{10}z_3z_4)\end{aligned}$$

For generic $n$ and $\ell$ the simplest component is the one with $b_i=0$ for all $i\le n$ and $b_i=\ell$ for the remainings $i>\ell$:

$$\Psi^{(k)}_{0,\ldots,0,\ell,\ldots,\ell}=(-1)^{\ell\frac{n(n+1)}{2}}q^{(\ell-k)n+\ell n\frac{n-\ell}{2}}\prod_{i=1}^{n}z_i^k\prod_{1\le i<j\le n}\prod_{r=1}^{\ell}(q^{2r}z_j-z_i)(q^{2r}z_{n+j}-z_{n+i}) \tag{3.11}$$

and all its rotations.

Now define yet another normalization of the $R$-matrix, namely $\mathbf{R}(z) = \frac{F(z)}{F(1/z)} R(z)$, which is adapted to $\Psi$ from (3.9). Explicitly,

$$\check{\mathbf{R}}(z) = \sum_{j=0}^{\ell} \prod_{r=1}^{j} \frac{1-q^{2r}z}{z-q^{2r}} P_j \tag{3.12}$$

The exchange relation (3.5) becomes

$$\check{\mathbf{R}}_{i,i+1}(z_i/z_{i+1})\Psi(z_1,\ldots,z_L) = \Psi(z_1,\ldots,z_{i+1},z_i,\ldots,z_L) \tag{3.13}$$

Starting from (3.4) at say $i=1$, switching to $\Psi$, applying repeatedly (3.13) and using $\bar\phi_+(1/z)F(sz)/F(1/z) = (-1)^\ell$, we find the cyclicity relation:

$$(-1)^\ell s^{(n-1)\ell/2} q^{(1+k)a_{0,L}} S\Psi(z_L,z_1,\ldots,z_{L-1}) = \Psi(z_1,z_2,\ldots,sz_L) \tag{3.14}$$

where $S: V_{z_L}\otimes V_{z_1}\otimes\cdots\otimes V_{z_{L-1}} \to V_{z_1}\otimes\cdots\otimes V_{sz_L}$ rotates cyclically the tensor product.

The system of equations (3.13–3.14) is similar to the one that appeared first in [28] in the study of form factors. Note that the power of $s$ is a reflection of the homogeneity of $\Psi(z_1,\ldots,z_L)$. In components, (3.14) writes:

$$(-1)^\ell s^{(n-1)\ell/2} q^{(1+k)(2b_L-\ell)} \Psi_{b_L,b_1,\ldots,b_{L-1}}(z_L,z_1,\ldots,z_{L-1}) = \Psi_{b_1,\ldots,b_L}(z_1,z_2,\ldots,sz_L)$$

Other formulae can be rewritten in terms of $\Psi$ as well; in particular, the recurrence relations (3.8) become:

$$\Psi_{\ldots,b,b',\ldots}(\ldots,z_i,z_{i+1}=q^{-2}z_i,\ldots) = (-1)^{k-\ell/2}\frac{1}{c_b}\delta_{b+b',\ell}\begin{bmatrix}\ell\\k\end{bmatrix} q^{k(\ell-k-1)}$$
$$z_i^k \prod_{j=1}^{i-1}\prod_{r=1}^{\ell}(z_j-q^{2r}z_i)\prod_{j=i+2}^{L}\prod_{r=1}^{\ell}(z_j-q^{-2(r+1)}z_i)\Psi_{\ldots}(\ldots) \tag{3.15a}$$

$$\sum_{b=0}^{\ell} c_b \Psi_{\ldots,\ell-b,b,\ldots}(\ldots,z_i=q^{-2}z_{i+1},z_{i+1},\ldots) = (-1)^{k-\ell/2}\begin{bmatrix}\ell\\k\end{bmatrix} q^{k(\ell-k+1)}$$
$$z_i^k \prod_{j=1}^{i-1}\prod_{r=1}^{\ell}(z_j-q^{2r}z_{i+1})\prod_{j=i+2}^{L}\prod_{r=1}^{\ell}(z_j-q^{-2(r+1)}z_{i+1})\Psi_{\ldots}(\ldots) \tag{3.15b}$$

*3.5. Wheel condition.* Finally, we point out that $\Psi$ satisfies a wheel condition of a different nature than that of the Macdonald polynomials of sect. 3.1.1. Namely, we have the following theorem:

**Theorem 3.** *Assume that three parameters $z_{i_1}$, $z_{i_2}$, $z_{i_3}$, $i_1 < i_2 < i_3$, form a "wheel": $z_i = q^{2(a+b)}z$, $z_j = q^{2b}z$, $z_k = z$ with $a, b$ integers such tht $a, b > 0$ and $a+b < \ell+2$. Then*

$$\Psi(\ldots,q^{2(a+b)}z,\ldots,q^{2b}z,\ldots,z,\ldots) = 0 \tag{3.16}$$

The proof is similar to the one given in [32, theorem 3] and is reproduced in appendix C.3.

Note furthermore that the entries of $\Psi$ are *nonsymmetric* polynomials, hence the ordering condition for its arguments in the wheel condition. This wheel condition is not a special case of the one considered in [18] for nonsymmetric Macdonald polynomials.

## 4. Application to integrable spin chains

In all the discussion that precedes concerning vertex operators, it was assumed that $|q| < 1$. However, we have seen in section 3.4 that one can get rid of all infinite products in explicit formulae by a redefinition of the solution $\Psi$ of $q$KZ and of the $R$-matrix $\mathbf{R}$. The former becomes a polynomial of $z_1, \dots, z_L$ (with coefficients of the form $P(q)/\prod_{i=1}^{\ell}(1-q^{2i})^{k_i}$) whereas the latter becomes a rational function of them. As a result, we can now relax the constraint $|q| < 1$ and in particular consider the case where $q^2$ is a primitive $(\ell+2)^{\text{th}}$ root of unity, so that $s = q^{-2(\ell+2)} = 1$. It is the purpose of this section to show that $\Psi$ then becomes an eigenvector of an integrable transfer matrix.

Note that since $\Psi \sim \langle k|\Phi \dots \Phi|k\rangle$, our procedure is similar to a "matrix product Ansatz", where the role of the matrix algebra is played here by the Zamolodchikov–Faddeev algebra [30,10,21]. In fact, a similar procedure has already been proposed in [1,2,19]. However, there is a crucial difference between our situation and theirs. In the aforementioned papers, in order to impose periodic boundary conditions, a trace is taken, which would correspond to computing $\text{tr}(\Phi \dots \Phi)$. Unfortunately such a trace is divergent in our setting. Also observe that such an Ansatz would produce eigenvectors of integrable models for an arbitrary value of the quantum parameter $q$, a claim which we do not make here. Instead we take a vacuum expectation value of VOs; the price to pay is that rotational invariance (periodic boundary conditions) is only restored at special roots of unity, namely, $q^{2(\ell+2)} = 1$. One can also regularize the trace by adding a $x^d$: $\text{tr}(x^d \Phi \dots \Phi)$, producing finite temperature correlation functions, as already mentioned in the introduction; but that would also spoil the rotational invariance, and is therefore not directly relevant in the present context.

*4.1. The model.* Define the inhomogeneous monodromy matrix to be an operator on $V_z \otimes V_{z_1} \otimes \cdots \otimes V_{z_L}$ of the form:

$$\mathcal{T}(z) = \mathbf{R}_{10}(z_1/z) \dots \mathbf{R}_{L0}(z_L/z)$$

where the 0 index corresponds to $V_z$ and non zero indices to $V_{z_i}$, and the $R$-matrix is given as before by (3.12). with $\check{\mathbf{R}} = \mathcal{P}\mathbf{R}$.

The twisted transfer matrix is then defined by taking the trace over the auxiliary space $V_z$, with a twist (in which it is convenient to absorb the sign $(-1)^\ell$):

$$\mathbf{T}(z) = \text{tr}_0((-q^{1+k})^{a_0} \mathcal{T}(z)) \tag{4.1}$$

The Yang–Baxter equation implies that transfer matrices commute for different spectral parameters:

$$[\mathbf{T}(z), \mathbf{T}(z')] = 0$$

The diagonalization of $\mathbf{T}(z)$ is usually performed using Bethe Ansatz, which produces eigenvectors defined in terms of "Bethe roots", i.e., solutions of certain algebraic equations. We do not pursue this route here.

A local Hamiltonian can be extracted from the transfer matrix in the homogeneous limit where all the $z_i$ coincide. Assume that $z_i = 1$. According to (3.12), $R(1) = \mathcal{P}$, and therefore

$$\mathbf{T}(1) = (-q^{1+k})^{a_{0,L}} S \tag{4.2}$$

where we recall that $S$ is cyclic permutation of factors of the tensor product. $\mathbf{T}(1)$ is a discrete analogue of the momentum operator. Expanding to next order, we obtain the Hamiltonian:

$$\mathbf{H} = \frac{1}{i} \mathbf{T}(1)^{-1} \frac{d\mathbf{T}}{dz}|_{z=1} = \sum_{j=1}^{L} h_{j,j+1}$$

where $h_{j,j+1} = \frac{1}{i}\frac{d}{dz}\check{\mathbf{R}}_{j,j+1}(z)|_{z=1}$, $i < L$, and $h_{L,1} = \frac{1}{i}(-q^{1+k})^{-a_{0,L}}\frac{d}{dz}\check{\mathbf{R}}_{L,1}(z)|_{z=1}(-q^{1+k})^{a_{0,L}}$ (twisted periodic boundary conditions). The factor of $1/i$ is introduced for convenience, see below.

*4.2. The simple eigenvalue.* Assume that $q^2$ is primitive $(\ell+2)^{\text{th}}$ root of unity (primitiveness being necessary for $\Psi$ to be well-defined). Consider $\Psi$ given by (3.10). An important remark is that the values of $q_M = q^{2(\ell+1)}$ and $t_M = q^{-2}$ coincide, so that the Macdonald polynomial appearing in this expression becomes simply the corresponding *Schur polynomial.* Therefore,

$$\begin{aligned}\Psi^{(k)}_{b_1,\dots,b_{2n}}(z_1,\dots,z_{2n}) = \gamma^{(k)}_{\ell,n}\prod_{i=1}^{2n}\alpha_{b_i} z_i^k \prod_{1\le i<j\le 2n}\prod_{r=1}^{\ell}(q^{2r}z_j - z_i)\oint\dots\oint\prod_{i=1}^{\ell n}\frac{dw_i}{2\pi i}\\ \times\frac{\det(w_i^{h_j-1})_{i,j=1,\dots,2n}}{\prod_{i<j}(w_j-q^2w_i)\prod_{j\le\epsilon_i}(q^\ell w_i - z_j)\prod_{j\ge\epsilon_i}(w_i-q^\ell z_j)}\end{aligned} \tag{4.3}$$

where $h_j = \lfloor\frac{j}{\ell}\rfloor + \lfloor\frac{j+\ell-k}{\ell}\rfloor + j$.

Next, note that (3.14) can be simplified if one assumes $s = q^{-2(\ell+2)} = 1$ to

$$(-q)^{(1+k)a_{0,L}} S\Psi(z_2,\dots,z_L,z_1) = \Psi(z_1,\dots,z_L) \tag{4.4}$$

Now consider the effect of the transfer matrix on the correlation function $\Psi(z_1,\dots,z_L)$. Writing the transfer matrix $\mathbf{T}(z)$ as the trace of the monodromy matrix $\mathcal{T}(z)$ and using (3.15a), we have[3]

$$\mathbf{T}(z)\Psi(z_1,\dots,z_L) = \Pi^{-1}\sum_{b,b'} c_b((-q^{1+k})^{a_0})_{b,b}\mathcal{T}_{b,b'}(z)\Psi_{b,\ell-b',\dots}(z,q^{-2}z,z_1,\dots,z_L) \tag{4.5}$$

where $\Pi = \begin{bmatrix}\ell\\k\end{bmatrix} z_i^k\prod_{j=1}^{i-1}\prod_{r=1}^{\ell}(z_j-q^{2r}z_i)\prod_{j=i+2}^{L}\prod_{r=1}^{\ell}(z_j-q^{-2(r+1)}z_i)$ is the factor appearing in (3.15a), and the … in subscript mean that only the first and the last index are fixed, the rest forming a vector in $V_{z_1}\otimes\dots\otimes V_{z_L}$.

Using (4.4), we can rewrite this as

$$\mathbf{T}(z)\Psi(z_1,\dots,z_L) = \Pi^{-1}\sum_{b,b'} c_b\mathcal{T}_{b,b'}(z)\Psi_{\ell-b',\dots,b}(q^{-2}z,z_1,\dots,z_L,z)$$

Finally, writing $\mathcal{T}(z)$ as a product of $\mathbf{R}$-matrices and applying the exchange relation (3.13) repeatedly, we find:

$$\begin{aligned}\mathbf{T}(z)\Psi(z_1,\dots,z_L) &= \Pi^{-1}\sum_b c_b\Psi_{\ell-b,b,\dots}(q^{-2}z,z,z_1,\dots,z_L)\\ &= \Psi(z_1,\dots,z_L)\end{aligned}$$

where the last equality follows from (3.15b).

Noting that $\Psi\neq 0$ because of (3.11), we conclude that if $q^2$ is a primitive $(\ell+2)^{\text{th}}$ root of unity, $\Psi(z_1,\dots,z_L)$ is an eigenvector of the (inhomogeneous, twisted) transfer matrix $\mathbf{T}(z)$, with a trivial eigenvalue. A graphical interpretation of this proof can be found in appendix B.3.

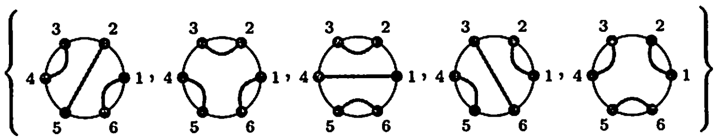
The twist is $-q^{1+k}$, but note that only its square is meaningful. As $k$ varies from 0 to $\ell$, $q^{2(1+k)}$ spans all $\ell+1$ nontrivial $(\ell+2)^{\text{th}}$ roots of unity.

We conjecture that if $q = -e^{\pm i\pi/(\ell+2)}$ (for all twists $-q^{1+k}$), the eigenvector we have just constructed corresponds to the largest eigenvalue of $\mathbf{T}(z)$ for $z_i$ of modulus 1 and sufficiently close to 1. In particular, for $z_i = 1$, we conjecture that $\Psi$ is the *ground state* eigenvector of the Hamiltonian $\mathbf{H}$ (with zero ground state energy). For a discussion of the latter statement, see sect. 4.4. Note that at $q = \pm e^{\pm i\pi/(\ell+2)}$, the spectrum of $\mathbf{H}$ is real.

[3] The argument that follows was suggested to us by R. Weston, to whom we are indebted.

*4.3. Relation to loop model.* In [32], a "higher spin" loop model was built by fusing the standard Temperley–Lieb loop model. Since the latter is related to the spin 1/2 representation of $U = U_q(\widehat{\mathfrak{sl}(2)})$ in the sense that it is equivalent to the XXZ spin chain/6-vertex model, the fused loop model must be related to higher spin integrable Hamiltonians/transfer matrices. The only nontrivial issue is that of boundary conditions. Let us discuss this here, first in the spin 1/2 case, then in the fused case.

*4.3.1. Equivalence to spin model and twisted boundary conditions.* We first define a link pattern of size $L$ to be a planar pairing of $\{1,\ldots,L\}$ viewed as boundary points of a disk, e.g.,

$\mathcal{L}_6 =$ 


We shall identify a link pattern with the fixed-point-free involution which sends paired points to each other. Note that $L$ has to be even for $\mathcal{L}_L$ to be nonempty.

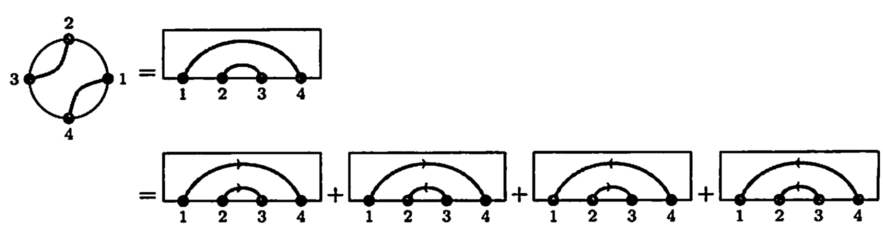
The span of link patterns, $\mathbb{C}[\mathcal{L}_L]$, can be identified with a subspace of $(\mathbb{C}^2)^{\otimes L}$; indeed, there is a linear map $\varphi$ which to each $\pi \in \mathcal{L}_L$ associates a vector in $(\mathbb{C}^2)^{\otimes L}$, which is given by the following explicit formula:

$$\varphi(\pi) = \sum_{\epsilon\in\{0,1\}^L} \prod_{1\le i<\pi(i)\le L} \delta_{\epsilon_i+\epsilon_{\pi(i)},1}(-q)^{-\epsilon_i+\frac{1}{2}} v_\epsilon \tag{4.6}$$

where on the r.h.s. the $v_\epsilon$ are the standard basis of $(\mathbb{C}^2)^{\otimes L}$. In appendix C, we show that $\varphi$ is injective, and that for generic $q$ it is in fact an isomorphism from $\mathbb{C}[\mathcal{L}_L]$ to the $U_1$-invariant subspace of $(\mathbb{C}^2)^{\otimes L}$ (recall that $U_1$ is the horizontal subalgebra of $U$).

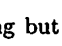
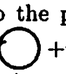
The change of basis described by (4.6) has the following diagrammatic interpretation: one sums over all possible orientations of the arcs, assigns a weight which is equal to $(-q)^{\frac{1}{2\pi}\text{total angle spanned by the arcs}}$, and then reconstructs the spin state by recording the orientations at the endpoints of the arcs, identifying $v_1 = v_\uparrow, v_0 = v_\downarrow$.

The actual shape of the disk and the positioning of the points on its boundary are irrelevant and result in gauge transformations (i.e., moving around the twist). The choice above corresponds to all points aligned on a straight region of the boundary, e.g.,




$= -q^{-1}v_{\uparrow\uparrow\downarrow\downarrow} + v_{\uparrow\downarrow\uparrow\downarrow} + v_{\downarrow\uparrow\downarrow\uparrow} - qv_{\downarrow\downarrow\uparrow\uparrow}$

The simplest way to compute the twist is to consider the rotation one-step to the left of link patterns, $\tilde{S}$, and map it to the spin space. We find that $\tilde{S} = (-q)^{a_{0,L}} S$ where $S$ is (untwisted) spin rotation, compare with (4.2) at $k = 0$.

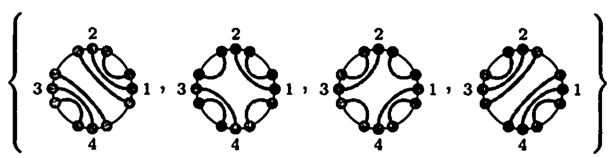
For the sake of completeness we also indicate how to map covectors (and therefore, operators). It is convenient to think of covectors in the loop picture as planar pairings *outside* the disk.[4] The pairing between vectors and covectors is to paste them together, and assign them $\tau$ to the power the number of closed loops thus formed, where $\tau = -(q+q^{-1})$. Note that $\tau$ is nothing but  + , where we have used the same rule as before that the weight of a line is equal to $(-q)^{\frac{1}{2\pi}\text{total angle spanned by the arcs}}$. Therefore, the exact same graphical rule must be used for covectors: sum over all orientations, give a weight of $(-q)^{\frac{1}{2\pi}\text{total angle spanned by the arcs}}$ and build the dual spin state from the orientations at the endpoints of the arcs, with $v_1^* = v_\uparrow^*, v_0^* = v_\downarrow^*$, $v_i^* v_j = \delta_{ij}$.

*4.3.2. Fusion.* We shall not reproduce here the fusion procedure, i.e., consider representation theory of $U$, but rather consider only the non-affine part $U_1$. We therefore view the $U_1$-module $V = \mathbb{C}^{\ell+1}$ as a submodule of $(\mathbb{C}^2)^{\otimes \ell}$, as well as the projection $p : (\mathbb{C}^2)^{\otimes \ell} \to V$ (see C.2 for its explicit definition).

The $\ell$-fused link pattern of size $L$ of [32] is then obtained from ordinary link patterns of size $\ell L$ by grouping together vertices into groups indexed by $r_i = 1 + \lfloor \frac{i-1}{\ell} \rfloor$ and by restricting to link patterns with no connections within a group, i.e.,

$$\mathcal{L}_{\ell,L} = \{\pi \in \mathcal{L}_{\ell L} : \pi(i) = j \ \Rightarrow r_i \neq r_j\}$$

For example,

$\mathcal{L}_{3,4} =$ 


In the fused case, to a $\pi \in \mathcal{L}_{\ell,L}$ is then associated the corresponding vector *after projection*:

$$\varphi_\ell(\pi) = \prod_{i=1}^{L} p_i \ \varphi(\pi) \tag{4.7}$$

where $p_i$ is $p$ acting on the factors $\ell i + 1, \ldots, \ell(i+1)$ of the tensor product $(\mathbb{C}^2)^{\otimes \ell L}$.

Clearly, twist and fusion commute, so that we have the same twist $(-1)^\ell K_1 = (-q)^{a_0}$ in the fused case.

In appendix C, we show that the map $\varphi_\ell$ is well-defined and injective provided $q^2$ is not a $r^{\text{th}}$ root of unity, $2 \leq r \leq \ell$, and that for generic $q$ it is in fact an isomorphism from $\mathbb{C}[\mathcal{L}_{\ell;L}]$ to the $U_1$-invariant subspace of $(\mathbb{C}^2)^{\otimes L}$.

The integrable inhomogeneous transfer matrix can be formulated directly in the language of loops, as is done in [32]. Once converted to the spin space, it becomes exactly the twisted transfer matrix $\mathbf{T}(z)$ discussed in 4.1, with twist given by $k = 0$.

[4] Note the asymmetry: the region outside the disk is homeomorphic to an interval times a circle ("identified connectivities"), not a disk ("unidentified connectivities"). It would be tempting to add the point at infinity to restore the symmetry, but that is not allowed because we want the winding angle of any closed curve to be $2\pi$.

*4.3.3. Some results.* Consider now $\Psi^{(0)}(z_1,\ldots,z_L)$. It is proportional to $\Psi^{(0)}(z_1,\ldots,z_L) = \langle 0|\,\Phi(z_1)\ldots\Phi(z_L)\,|0\rangle$, and therefore, from the intertwining property of $\Phi(z)$, is $U_1$-invariant. We conclude that it is in the image of $\varphi_\ell$ for generic $q$, and by continuity it is so for $q^2$ $(\ell+2)^{\text{th}}$ root of unity. We can therefore consider its preimage $\varphi_\ell^{-1}(\Psi^{(0)})$. By construction it is an eigenvector of the transfer matrix of the loop model, with eigenvalue 1. And it is a polynomial of degree $\ell n(n-1)$ in the $z_i$. At the particular value $q = -e^{-i\pi/(\ell+2)}$, which is the only one considered in [32], $\Psi^{(0)}$ coincides with the eigenvector studied in that paper, and its degree is the content of Conjecture 1 of [32], which is thus proved.

*4.4. Relation to supersymmetric fermions with exclusion.* In [12], a supersymmetric model of fermions on the one-dimensional lattice was introduced. It has the exclusion rule that at most $\ell$ consecutive sites may be occupied. In the case $\ell = 1$, it was found to be equivalent to the XXZ spin chain at $\Delta = -1/2$ [29]. More generally, in a recent paper [14], Hagendorf conjectured that for any $\ell$, this fermionic model was equivalent to the integrable spin $\ell/2$ chain at $q = e^{i\pi/(\ell+2)}$. Note that models related by $q \to -q$ and $q \to 1/q$ are closely connected (in particular the choice $q \to -q$ depends on the sign convention for the Hamiltonian). It was also stated that the admissible twists are of the form $e^{2\pi i m/(\ell+2)a_0}$, where $m = 0,1,\ldots,p$, $p = \ell+1$ for $\ell$ odd and $p = \ell/2$ for $\ell$ even. However, it seems difficult to compare his twists with ours because the equivalence is only conjectural and not explicit in the general case, and may modify the twist. See also the computation of the Witten index in [14, sect. 3.5].

Adapting this discussion to our setting, we conclude (conjecturally) that the Hamiltonian H at $q = -e^{\pm i\pi/(\ell+2)}$ should be supersymmetric for all twists of the form $\pm q^n$. Furthermore, it should have a supersymmetric ground state exactly when $n \neq 0 \pmod{\ell+2}$. This would imply that our state $\Psi$, which has zero energy, is indeed the (supersymmetric) ground state of H.

## A. Construction of $q$-deformed parafermions

Here we follow the procedure of Ding and Feigin [8], we intend to construct a level $\ell$ Drinfeld realization of $U_q(\widehat{\mathfrak{sl}(2)})$, and to rewrite the current in terms of bosonic and parafermionic operators (these being appropriate $q$-deformations of Conformal Field Theory operators, cf the parafermionic fields of [31]).

Let

$$\begin{aligned} e(z) &= \sum_{i\in\mathbf{Z}} e_i z^{-i-1} \qquad f(z) = \sum_{i\in\mathbf{Z}} f_i z^{-i-1} \\ k^+(z) &= \sum_{i\in\mathbf{Z}_0^+} k_i^+ z^{-i} \quad k^-(z) = \sum_{i\in\mathbf{Z}_0^-} k_i^- z^{-i} \end{aligned} \tag{A.1}$$

We want to construct operators that satisfy the following relations:

$$\begin{aligned} k_0^+ k_0^- &= 1 = k_0^- k_0^+ \\ k^\pm(z)k^\pm(w) &= k^\pm(w)k^\pm(z) \\ k^-(z)k^+(w) &= \frac{g\,(q^{-c}z/w)}{g\,(q^{c}z/w)} k^+(w)k^-(z) \\ k^\pm(z)e(w) &= g\left(q^{-c/2}\,(w/z)^{\pm1}\right)^{\mp1} e(w)k^\pm(z) \\ k^\pm(z)f(w) &= g\left(q^{c/2}\,(w/z)^{\pm1}\right)^{\pm1} f(w)k^\pm(z) \end{aligned} \tag{A.2}$$

$$
\begin{aligned}
e(z)e(w)\left(z-q^2w\right) &= e(w)e(z)\left(q^2z-w\right)\\
f(z)f(w)\left(q^2z-w\right) &= f(w)f(z)\left(z-q^2w\right)\\
(q-q^{-1})zw[e(z),f(w)] &= \delta\left(q^{-c}z/w\right)k^+(q^{-c/2}z)-\delta\left(q^{c}z/w\right)k^-(q^{c/2}z)
\end{aligned}
$$

where $c$ is a central element. The functions $g(z)$ and $\delta(z)$ are defined by:

$$
g(z)=\frac{1-q^2z}{q^2-z} \qquad\qquad \delta(z)=\sum_{i\in\mathbf{Z}} z^i
$$

*Remark 2.* The Chevalley generators are given by:

$$
\begin{aligned}
K_0 &= q^c k_0^- & E_0 &= f_1(k_0^+)^{-1} & F_0 &= k_0^+ e_{-1}\\
K_1 &= k_0^+ & E_1 &= e_0 & F_1 &= f_0
\end{aligned}
$$

A Hopf algebra structure is obtained as follows. We define the co-multiplication:

$$
\begin{aligned}
\tilde{\Delta}(q^h) &= q^h\otimes q^h\\
\tilde{\Delta}(e(z)) &= e(z)\otimes 1 + k^-(q^{c_1/2}z)\otimes e(q^{c_1}z)\\
\tilde{\Delta}(f(z)) &= f(q^{c_2}z)\otimes k^+(q^{c_2/2}z) + 1\otimes f(z)\\
\tilde{\Delta}(k^\pm(z)) &= k^\pm(q^{\pm c_2/2}z)\otimes k^\pm(q^{\mp c_1/2}z)
\end{aligned}
\tag{A.3}
$$

where $c_1 = c\otimes 1$ and $c_2 = 1\otimes c$; the co-unit:

$$
\tilde{\epsilon}(q^h) = 1 = \tilde{\epsilon}(k^\pm(z)) \qquad\qquad \tilde{\epsilon}(e(z)) = 0 = \tilde{\epsilon}(f(z)) \tag{A.4}
$$

and the antipode:

$$
\begin{aligned}
\tilde{a}(q^h) &= q^{-h}\\
\tilde{a}(e(z)) &= -k^-(q^{-c/2}z)^{-1}e(q^{-c}z)\\
\tilde{a}(f(z)) &= -f(q^{-c})k^+(q^{-c/2}z)^{-1}\\
\tilde{a}(k^\pm(z)) &= k^\pm(z)^{-1}
\end{aligned}
\tag{A.5}
$$

*A.1. Highest weight modules.* We define a highest weight module as the unique module $V(\Lambda) := U_q(\widehat{\mathfrak{sl}(2)})v_\Lambda$, where $\Lambda = n_0\Lambda_0 + n_1\Lambda_1$ is the highest weight, and $v_\Lambda$ is the highest weight vector, which satisfies:

$$
E_i v_\Lambda = 0 \qquad\qquad K_i v_\Lambda = q^{(\Lambda,\alpha_i)}v_\Lambda \qquad\qquad F_i^{(\Lambda,\alpha_i)+1}v_\Lambda = 0
$$

the inner product being defined by $(\Lambda_i,\alpha_j) = \delta_{i,j}$. Except in the trivial case $\Lambda = 0$, the modules thus generated are irreducible and infinite-dimensional.

We will give a more explicit way of constructing these modules depending on the eigenvalue $\ell = n_0 + n_1$ of the central element $c$.

*A.2. Level* 1 *realization.* We first build a realization of the $U_q(\widehat{\mathfrak{sl}(2)})$ algebra for the case $c = \ell = 1$ in terms of bosonic currents. We introduce a new set of bosonic operators, which satisfy the relations:

$$[a_n, a_m] = \delta_{n+m,0}\frac{[2n][n]}{n} \qquad\qquad [a_n, \alpha] = 2\delta_{n,0}$$

Then we can build the currents by:

$$\begin{aligned} e(z) &= e^{\alpha} z^{a_0} e^{\sum q^{-n/2}\frac{a_{-n}}{[n]}z^n} e^{-\sum q^{-n/2}\frac{a_n}{[n]}z^{-n}} \\ f(z) &= e^{-\alpha} z^{-a_0} e^{-\sum q^{n/2}\frac{a_{-n}}{[n]}z^n} e^{\sum q^{n/2}\frac{a_n}{[n]}z^{-n}} \\ k^+(z) &= q^{a_0} e^{(q-q^{-1})\sum a_n z^{-n}} \\ k^-(z) &= q^{-a_0} e^{-(q-q^{-1})\sum a_{-n} z^n} \end{aligned} \tag{A.6}$$

these satisfy all the above relations. We omit such computations.

They act on the usual bosonic Fock space $\mathcal{H}_1 = \bigoplus_{h\in\mathbb{Z}} \left\langle (\prod_{m>0} a_{-m}^{k_m} |h\rangle_B \right\rangle$ by the following rules:

$$\begin{aligned} a_i f\,|h\rangle_B &= a_i f\,|h\rangle_B \quad \text{if } i<0 & \qquad e^{\alpha} f\,|h\rangle_B &= f\,|h+2\rangle_B \\ a_i f\,|h\rangle_B &= [a_i, f]\,|h\rangle_B \quad \text{if } i>0 & \qquad a_0 f\,|h\rangle_B &= f h\,|h\rangle_B \end{aligned}$$

where $f \in \mathbb{C}[a_{-1}, a_{-2}, \ldots]$.

This produce two highest weight modules, depending on the parity of $h$, corresponding to the two highest weight vectors

$$v_{\Lambda_0} = |0\rangle_B \qquad\qquad v_{\Lambda_1} = |1\rangle_B$$

*A.3. Level $\ell$ realization.* Let us denote $\sigma_1 : U \to \mathcal{L}(\mathcal{H}_1)$ the level 1 realization. We produce, using the coproduct $\tilde{\Delta}$, a level $\ell$ representation as the tensor product of $\ell$ copies of $V$, i.e., if we define $\tilde{\Delta}^\ell = (1\otimes\ldots\otimes\tilde{\Delta})\circ\ldots\circ(1\otimes\tilde{\Delta})\circ\tilde{\Delta}$, then $\sigma = \tilde{\Delta}^\ell(\sigma_1\otimes\cdots\otimes\sigma_1)$ is the representation on $\mathcal{H}_\ell = \mathcal{H}_1^{\otimes\ell}$.

We also define currents in $\mathcal{H}_\ell$ in the obvious way:

$$\begin{aligned} e(z) &= \tilde{\Delta}^\ell(e(z)) \qquad f(z) = \tilde{\Delta}^\ell(f(z)) \\ k^{\pm}(z) &= \tilde{\Delta}^\ell(k^{\pm}(z)) \qquad q^c = \tilde{\Delta}^\ell(q^c) = q^c\otimes\ldots\otimes q^c \end{aligned} \tag{A.7}$$

As $\tilde{\Delta}$ is a co-multiplication, this will satisfy automatically the relations (A.2), with $c = \ell$.

In that way we can compute $e(z)$ for any $\ell$. For example, if $\ell = 2$ we get:

$$e(z) = e(z)\otimes 1 + k^-(q^{c_1/2}z)\otimes e(q^{c_1}z) = e^1(z) + e^2(z). \tag{A.8}$$

where $c_i$ is the value of $c$ in $V_i$.

We perform a mode expansion of $e(z)$, for generic $\ell$:

$$e(z) = \sum_{i\le\ell} e^i(z) \tag{A.9}$$

where (using $c_i = 1$)

$$e^i(z) = \bigotimes_{j<i} k^-(q^{j-1/2}z) \otimes e(q^{i-1}z) \bigotimes_{j>i} 1 \tag{A.10}$$

where $j$ indicates the position on the tensor product.

We repeat the process for $f(z)$:

$$f(z) = \sum_{i \leq \ell} f^i(z) \tag{A.11}$$

where (using $c_i = 1$)

$$f^i(z) = \bigotimes_{j<i} 1 \otimes f(q^{\ell-i}z) \bigotimes_{j>i} k^+(q^{\ell-j+1/2}z) \tag{A.12}$$

The same goes for $k^\pm(z)$:

$$k^\pm(z) = \bigotimes_j k^\pm(q^{\pm\frac{\ell-2j+1}{2}}z) \tag{A.13}$$

Let us denote $|k_1, \ldots, k_\ell\rangle = |k_1\rangle_B \otimes \ldots \otimes |k_\ell\rangle_B$, where $k_1 = 0,\ 1$. Let $k = \sum_i k_i$. $|k_1, \ldots, k_\ell\rangle$, as a tensor product of highest weight vectors, is a highest weight vector with highest weight $\Lambda = (\ell - k)\Lambda_0 + k\Lambda_1$. Note that for a given $\Lambda$, except for $k = 0$ or $\ell$, this construction is not unique. For our purposes, any such highest weight vector provides a realization of the highest weight module, and we will use the notation $|k\rangle$ for any of them (our results do not depend on the choice, i.e., when we permute $k_i \leftrightarrow k_j$). Moreover, these vectors are normalized such that $\langle k|k\rangle = 1$.

*A.4. Parafermions.* The computation of commutation relations between $e^i(z)$ (or $f^i(z)$) and $k^\pm(w)$ it is not complicated, although it is long. The result for $e^i(z)$ is:

$$\begin{aligned}
e^i(z)k^+(w) &=: e^i(z)k^+(w) : \\
k^+(w)e^i(z) &= \frac{q^2w - q^{-\ell/2}z}{w - q^{2-\ell/2}z} : e^i(z)k^+(w) : \\
e^i(z)k^-(w) &= \frac{z - q^{-2-\ell/2}w}{z - q^{2-\ell/2}w} : e^i(z)k^-(w) : \\
k^-(w)e^i(z) &= q^{-2} : e^i(z)k^-(w) :
\end{aligned} \tag{A.14}$$

and for $f^i(z)$ is:

$$\begin{aligned}
f^i(z)k^+(w) &=: f^i(z)k^+(w) : \\
k^+(w)f^i(z) &= \frac{w - q^{2+\ell/2}z}{q^2w - q^{\ell/2}z} : f^i(z)k^+(w) : \\
f^i(z)k^-(w) &= \frac{z - q^{2+\ell/2}w}{z - q^{-2+\ell/2}w} : f^i(z)k^-(w) : \\
k^-(w)f^i(z) &= q^{-2} : f^i(z)k^-(w) :
\end{aligned} \tag{A.15}$$

Surprisingly, these results do not depend on $i$. This fact, they tell us that we can factor $e^i(z)$ and $f^i(z)$ into two parts: one (parafermionic part) that commutes with $k^\pm(w)$ and depends on $i$ and a second part (bosonic part) that does not depend on $i$ and does not commute with $k^\pm(w)$.

Following Ding and Feigin, let us build the bosonic part. Define (let $n$ be a non-negative integer):

$$\begin{aligned} a_{\pm n} &= \sum_i a^i_{\pm n} \\ a^i_{\pm n} &= \bigotimes_{j<i} 1 \otimes q^{-n\frac{\ell-2i+1}{2}} a_{\pm n} \bigotimes_{j>i} 1 \end{aligned} \tag{A.16}$$

(correcting a sign misprint in [8]).

We could think that $a_n$ is only the result of applying the co-multiplication to $a_n$. But this is false. In fact, there is no Hopf algebra for these operators.

These new operators satisfy the following commutation realation:

$$[a_n, a_m] = \delta_{n+m,0} \frac{[2n][\ell n]}{n} \tag{A.17}$$

It can be easily proved.

The main reason for this definition is the fact that:

$$k^{\pm}(z) = q^{\pm a_0} e^{\pm(q-q^{-1})\sum_n a_{\pm n} z^{\mp n}} \tag{A.18}$$

where, for example, $q^{a_0}$ means $q^{a_0} \otimes \ldots \otimes q^{a_0}$. Notice that this is the natural generalization of $k^{\pm}(z)$.

Now the idea is to factor $e^i(z)$ and $f^i(z)$ using these new bosonic operators. That is, we want to write them as $e^i(z) = \mathfrak{e}(z)\xi^{+i}(z)$ and $f^i(z) = \mathfrak{f}(z)\xi^{-i}(z)$, where $\mathfrak{e}(z)$ and $\mathfrak{f}(z)$ are expressed in terms of $a_{\pm n}$ and $e^{\alpha} = \bigotimes_j e^{\alpha}$ (compared to the notations of the main text, we have renormalized the zero mode as $\alpha = 2\ell\beta$).

The following Ansatz is made:

$$\begin{aligned} \mathfrak{e}(z) &= e^{\alpha/\ell} z^{a_0/\ell} e^{\sum_n q^{-\frac{n\ell}{2}} \frac{a_{-n}}{[n\ell]} z^n} e^{-\sum_n q^{-\frac{n\ell}{2}} \frac{a_n}{[n\ell]} z^{-n}} \\ \mathfrak{f}(z) &= e^{-\alpha/\ell} z^{-a_0/\ell} e^{-\sum_n q^{\frac{n\ell}{2}} \frac{a_{-n}}{[n\ell]} z^n} e^{\sum_n q^{\frac{n\ell}{2}} \frac{a_n}{[n\ell]} z^{-n}} \end{aligned} \tag{A.19}$$

In this way, the commutation relations of $\mathfrak{e}(z)$ (or $\mathfrak{f}(z)$) and $k^{\pm}(w)$ have the form of (A.14) and (A.15), i.e.,

$$\begin{aligned} 0 &= [\xi^{+i}(z), k^{\pm}(w)] \\ 0 &= [\xi^{-i}(z), k^{\pm}(w)] \end{aligned} \tag{A.20}$$

for all $i$.

In fact, one can prove that

$$[a_n, \xi^{\pm i}(w)] = 0 \qquad\qquad [\alpha, \xi^{\pm i}(w)] = 0 \tag{A.21}$$

for any integer $n$ and $i \in \{1, 2, \ldots, \ell\}$.

The operators $\xi^+(z) = \sum_i \xi^{+i}(z)$ and $\xi^-(z) = \sum_i \xi^{-i}(z)$ are the so-called parafermionic operators.

We can split a vector $|k\rangle$ into a bosonic component and a parafermionic component. The first being an average $|k\rangle_B = |k/\ell\rangle_B \otimes \ldots \otimes |k/\ell\rangle_B$. This requires that we enlarge the Fock space to $\mathcal{H}_1 = \bigoplus_{h \in \mathbb{Z}/\ell} \left\langle (\prod_{m>0} a_{-m}^{k_m} \, |h\rangle_B \right\rangle$.

*A.5. The wheel condition.* These operators satisfy an important relation.

**Theorem 4.** *$e(z), f(z)$ satisfy the wheel condition:*

$$e(z)e(q^2z)\dots e(q^{2\ell}z)=0$$
$$f(q^{2\ell}z)\dots f(q^2z)f(z)=0$$

This is theorem 2.5 in [8].

In order to prove this fact, we need the folowing relations for $e^i(z)$ (let $j>i$):

$$\begin{aligned} e^i(z)e^j(w) &= \frac{z-q^{-2}w}{z-q^2w} : e^i(z)e^j(w) : \\ e^i(z)e^i(w) &= q^{2i-2}(z-w)(z-q^{-2}w) : e^i(z)e^i(w) : \\ e^j(z)e^i(w) &= q^{-2} : e^j(z)e^i(w) : \end{aligned} \tag{A.22}$$

And the equivalent for $f^i(z)$:

$$\begin{aligned} f^i(z)f^j(w) &= \frac{z-q^2w}{q^2z-w} : f^i(z)f^j(w) : \\ f^i(z)f^i(w) &= q^{2\ell-2i}(z-w)(z-q^2w) : f^i(z)f^i(w) : \\ f^j(z)f^i(w) &=: f^j(z)f^i(w) : \end{aligned} \tag{A.23}$$

Note that when $w=q^2z$ (or $w=q^{-2}z$ for the $f^i(z)$ case) these expressions simplify. This is our main tool to prove theorem 4:

*Proof.* The product $e(z_1)\dots e(z_n)$ can be written as the sum over all possible decompositions:

$$e(z_1)\dots e(z_n)=\sum_{\epsilon_1,\dots,\epsilon_n} e^{\epsilon_1}(z_1)\dots e^{\epsilon_n}(z_n)$$

The regular part, that is the $:\dots:$ part, has no poles (aside 0 and $\infty$). Thus, setting $w=q^2z$, we see that only $e^j(z)e^i(q^2z)$ survives (we are assuming $j>i$).

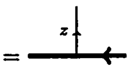
$$e(z)\dots e(q^{2n-2}z)=\sum_{\epsilon_1>\dots>\epsilon_n} e^{\epsilon_1}(z)\dots e^{\epsilon_n}(q^{2n-2}z)$$

but such sequence $\epsilon_1>\dots>\epsilon_n$ is impossible if $n=\ell+1$, considering that $\epsilon_i\in\{1,\dots,\ell\}$. The result follows. The proof for $f^i(z)$ is identical.

## B. Graphical calculus

We describe here the graphical calculus to represent $U_q(\widehat{\mathfrak{sl}(2)})$ invariants. It is a convenient way to derive various relations while keeping track of the spaces involved, i.e., it acts in a similar way as a "type checking" tool for programming.

*B.1. Basics.* Vector spaces are represented by oriented lines, such that tensor products correspond to stacking lines from left to right if looking in the direction of the orientation. Highest weight representations $\mathcal{H}_\ell$ are denoted by thick lines, evaluation representations $V_z$ by thin lines with a label $z$. Thus, a (type I) vertex operator $\Phi(z)$ becomes:

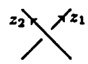
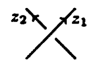
$\Phi(z) =$ 

and its (vacuum to vacuum) correlation function is:

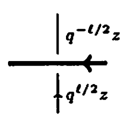
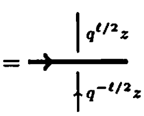
$\Psi(z_1,\dots,z_L) = \langle 0|\,\Phi(z_1)\dots\Phi(z_L)\,|0\rangle =$ 

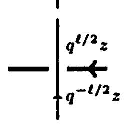
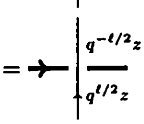
where the white dots on the left (resp. right) correspond to $\langle 0|$ (resp. $|0\rangle$).

We need two more objects obtained by applying representations to the universal $R$-matrix $\mathcal{R}$ (or to $\mathcal{P}(\mathcal{R}^{-1})$): the $R$-matrix

$R_+(z_1/z_2) =$  $R_-(z_1/z_2) =$ 

where the argument is the ratio of spectral parameters of left and right incoming lines (note that $R_+$ and $R_-$ only differ by a normalization); and the $L$-matrix

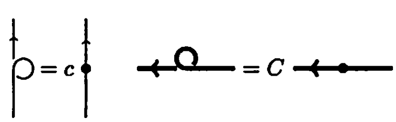
$L_+(z) =$  $a(L_+(z)) =$ 

$L_-(z) =$  $a(L_-(z)) =$ 

Note that thin lines pick up a factor of $q^{\pm\ell}$ when they cross thick lines.

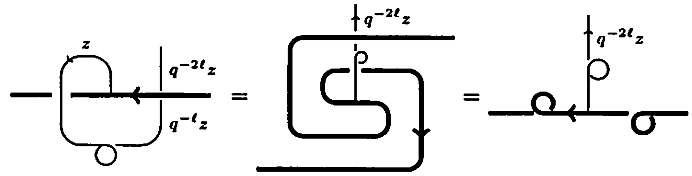
Lines can also be slid across intersections of other lines (Reidemeister move III) due to the Yang–Baxter equation, or across the trivalent vertex of a VO ($L$/VO commutation).

Finally, it is easy to check on generators that the square of the antipode satisfies $a^2(x) = q^{-a_0+4d}xq^{a_0-4d}$ for all $x$ in $U$. This implies that any line that does a full $\pm 2\pi$ rotation can be replaced with a straight line with an insertion of $q^{\pm(a_0-4d)}$ times a central element, e.g.,

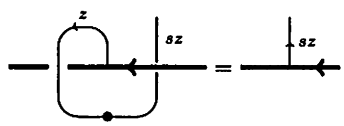
 (B.1)

where the dot represents insertion of $q^{a_0-4d}$. We shall not need the values $c, C$ of the central element in what follows. The $q^{-4d}$ acts as a multiplicative shift of $q^{-4}$ of the spectral parameter, which propagates along the lines.

*B.2. Proof of qKZ equation.* We begin with a local relation: consider the following equality, obvious diagrammatically:

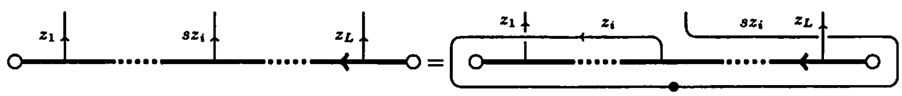



If we use (B.1), recognizing conjugation by $q^{a_0-4d}$ on the thick line, the coproduct $\Delta(q^{a_0-4d}) = q^{a_0-4d} \otimes q^{a_0-4d}$ and finally the intertwining property of the VO, we can simplify this to:




where $s = q^{-2(\ell+2)}$, and we recall that the dot represents insertion of $K_1 = q^{a_0}$ accompanied by a shift of the spectral parameter $z \to q^{-4}z$, which we pulled out of the thin line.

Now apply this relation to the correlation function $\Psi(z_1, \dots, z_L)$ described graphically above, and use the highest/lowest weight property of $|0\rangle$ and $\langle 0|$, i.e., $L_+(z)\,|0\rangle = 1$ and $\langle 0|\,L_-(z) = 1$:

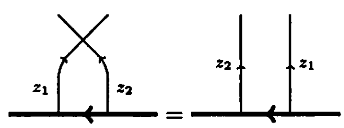



(B.2)

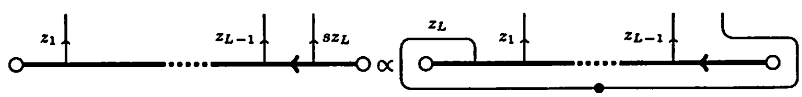
which is exactly the quantum Knizhnik–Zamolodchikov equation (3.4).

In case $\langle 0|\dots|0\rangle$ is replaced with $\langle k|\dots|k\rangle$, $k = 0, \dots, \ell$, there is an additional contribution coming from the zero modes of $L_\pm(z) \sim q^{a_0 \otimes a_0/2}$, so that the twist in the dot becomes $q^{(1+k)a_0}$.

The $q$KZ equation is valid for the correlation function of a product of arbitrary VOs; however, in the special case of perfect VOs, we have the additional relation (2.15):




where the flat crossing corresponds to another normalization of the $R$-matrix, denoted by $R(z)$ in the text. This in turn implies the exchange relation (3.5) for correlation functions, and, combined with the $q$KZ equation at $i = L$, produces the cyclicity property:

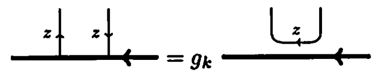



The change from $\Psi$ to $\Psi$ removes the proportionality factor, which leads to (3.14).

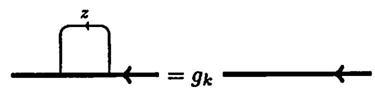
*B.3. Proof of eigenvector property.* We provide here a graphical proof of the eigenvector property of section 4.2. What follows is not rigorous because the VO construction becomes divergent when $|q| = 1$; however, the proper proof, given in the text, is along the exact same lines as this graphical proof, except with diverging prefactors removed. We shall therefore ignore prefactors in what follows.

Recall the dual vertex operator (3.6). The properties (3.7) it satisfies can be depicted as:

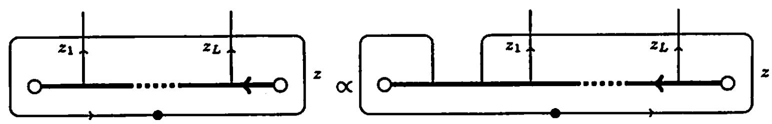
$$\Phi_b(z)\Phi^*_{b'}(z) = \delta_{b,b'} g_k \qquad \text{(B.3a)}$$



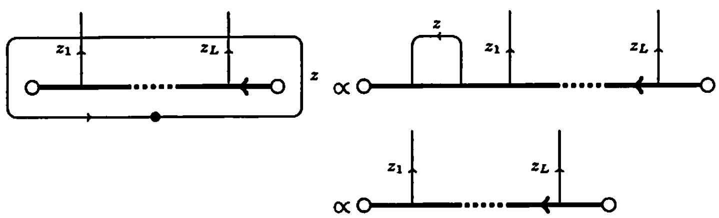

$$\sum_{b=0}^{\ell} \Phi^*_b(z)\Phi_b(z) = g_k \qquad \text{(B.3b)}$$




Now use the first relation to rewrite the action of the transfer matrix $\mathbf{T}(z)$ on $\Psi(z)$ in a similar way as in (4.5):




Applying the $q$KZ equation (B.2) at $i = L$ and ignoring the spectral parameter shift since $s = 1$, we find:




which is the desired eigenvector property.

## C. Properties of the spin-loop mapping

We discuss here in more detail the linear map $\varphi_\ell$ from the loop space to the spin space that is used in section 4.3.

*C.1. Case $\ell = 1$.* This case is well-known, and we only review it briefly. $\varphi_1 = \varphi : \mathbb{C}[\mathcal{L}_L] \to (\mathbb{C}^2)^{\otimes L}$ is defined explicitly in 4.3.1. Both spaces come equipped with standard bases, indexed by $\mathcal{L}_L$ and $\{0,1\}^L$ respectively, and its matrix is given by (4.6); its entries are powers of $-q$, and therefore $\varphi$ is well-defined for any $q \in \mathbb{C}^\times$. Up to an overall power of $q$, these are nothing but maximal parabolic Kazhdan–Lusztig polynomials, see e.g. [27] for a discussion in a related context.

There is a partial order on $\{0,1\}^L$: $\epsilon \le \epsilon'$ iff $\sum_{j=1}^{i}(\epsilon_j - \epsilon'_j) \le 0$ for all $i = 1,\dots,L$. Denote $\epsilon_0 = \{1,0,\dots,1,0\}$ and $\tilde{\mathcal{L}}_L = \{\epsilon \in \{0,1\}^L : \epsilon \ge \epsilon_0\}$ (Dyck words). There is a natural bijection between $\mathcal{L}_L$ and

$\tilde{\mathcal{L}}_L$ which to $\pi \in \mathcal{L}_L$ associates $\epsilon$ such that

$$\epsilon_i = \begin{cases} 1 & \text{if } \pi(i) > i \\ 0 & \text{if } \pi(i) < i \end{cases} \qquad i = 1, \ldots, L$$

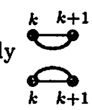
The claim is that the submatrix of $\phi$ where indices are restricted to $\tilde{\mathcal{L}}_L \times \mathcal{L}_L$, is upper triangular w.r.t. the partial order above (modulo the bijection). This is fairly obvious in the graphical interpretation where one orients the arcs: the diagonal elements of the matrix correspond to all arcs pointing right, and each subsequent change of orientation of an arc clearly increases the partial sums $\sum_{j\leq i} \epsilon_i$. Furthermore, the diagonal elements are all $(-q)^{-L/2}$. Therefore, the matrix has maximal rank for all $q \in \mathbb{C}^\times$, and $\varphi$ is an isomorphism of $\mathbb{C}[\mathcal{L}_L]$ onto its image.

For $L = 2$, the unique link pattern is sent onto the state $(-q)^{-1/2}v_{10} + (-q)^{1/2}v_{01}$, which is checked explicitly to be $U_1$-invariant. One can then proceed inductively by noting that any link pattern possesses an arc connecting neighbors, applying the $L = 2$ observation to it and then removing it. Therefore all link patterns are sent into the $U_1$-invariant subspace $((\mathbb{C}^2)^{\otimes L})_{inv}$ of $(\mathbb{C}^2)^{\otimes L}$.

For generic $q$, the dimension of the invariant subspace is the same as for $\mathfrak{sl}(2)$ and is well-known to be the Catalan number $L!/((L/2)!(L/2+1)!)$ ($L$ even), which enumerates Dyck paths or link patterns. Therefore, for generic $q$, $\varphi$ is an isomorphism of $\mathbb{C}[\mathcal{L}_L]$ onto $((\mathbb{C}^2)^{\otimes L})_{inv}$.

*C.2. Fused case.* Let us first define the projection map $p = p^{(\ell)} : (\mathbb{C}^2)^{\otimes \ell} \to V \cong \mathbb{C}^{\ell+1}$. We use induction on $\ell$: $p^{(1)} = 1$ and

$$p^{(k+1)} = p^{(k)} \left(1 + \frac{q^k - q^{-k}}{q^{k+1} - q^{-(k+1)}} e_k\right) p^{(k)}$$

where $e_k = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & 1 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{k,k+1}$ ; in the loop language, $e_k$ is simply 

. From its definition, it commutes with the $U_1$-action, and in fact the claim (see e.g. [23]) is that it is exactly the projection onto the maximal spin $\ell/2$ irreducible subrepresentation.

Now define $\varphi_\ell$ from $\varphi$ by projection, as in (4.7). It is well-defined when the projection is, that is, when $q^{2r} \neq 1$, $r = 2, \ldots, \ell$. By definition, the $\varphi_\ell(\pi)$ live in the $U_1$-invariant subspace $(V^{\otimes L})_{inv}$ of $V^{\otimes L}$. Furthermore, borrowing an argument from [32], we note that linear independence of the $\varphi_\ell(\pi)$ is equivalent to linear independence of the $\varphi(\pi)$, because $\varphi_\ell(\pi) = \varphi(\pi) + \sum_{\rho \in \mathcal{L}_{\ell L} - \mathcal{L}_{\ell;L}} c_{\pi,\rho}\varphi(\rho)$. But the latter was proved in the previous section.

It is not hard to show once again that $\dim(V^{\otimes L})_{inv} = \#\mathcal{L}_{\ell,L}$ for generic $q$ (a short proof is to use a bijection – generalizing the one of the previous section – between fused link patterns and Bratteli diagrams for spin $\ell/2$, the increments of the paths being now given as the difference of numbers of opening and closing arcs in a given group); equivalently, using the crystal limit $q \to 0$ one can conclude directly that the map above is an isomorphism between $\mathbb{C}[\mathcal{L}_{\ell,L}]$ and $(V^{\otimes L})_{inv}$ for generic $q$. We conjecture that it remains an isomorphism for $q = -e^{-i\pi/(\ell+2)}$, though we do not have a rigorous proof of that.

*C.3. Wheel condition.* We prove here the wheel condition of Theorem 3.

We work inside $(\mathbb{C}^2)^{\otimes \ell L}$. We fix a subset of consecutive $2\ell$ indices, say $\ell(i-1)+1, \ldots, \ell(i+1)$, and consider the projector $P_{j;i,i+1}$ ($j = 0, \ldots, \ell$) onto the spin $j$ subrepresentation of the action of $U_q(\mathfrak{sl}(2))$ acting on these $2\ell$ spaces. Also, given a link pattern $\pi \in \mathcal{L}_{L,\ell}$. denote $2c_{i,i+1}(\pi)$ the number of connections between the sites $\ell(i-1)+1,\dots,\ell(i+1)$ and the outside.

Then we have the following lemma: $j > c_{i,i+1}(\pi)$ implies $P_{j;i,i+1}\varphi_1(\pi) = 0$ (and the $P_{j;i,i+1}\varphi_1(\pi)$, where $\pi$ runs over link patterns with $c_{i,i+1} = j$, are linearly independent). The equality is easy proved by inductively removing the "little arches" connecting sites $\ell(i-1)+1,\dots,\ell(i+1)$. Explicitly, suppose $\pi(k) = k+1$, $\ell(i-1)+1 \le k < k+1 \le \ell(i+1)$, and consider the vector space

$$V = (\mathbb{C}^2)_{\ell(i-1)+1} \otimes \cdots \otimes (\mathbb{C}^2)_{k-1} \otimes ((-q)^{-1/2}\,|\uparrow\downarrow\rangle - (-q)^{1/2}\,|\downarrow\uparrow\rangle)_{k,k+1} \otimes (\mathbb{C}^2)_{k+2} \otimes \cdots \otimes (\mathbb{C}^2)_{\ell(i+1)}$$

Note that $\varphi_1(\pi) \in W_{\text{left}} \otimes V \otimes W_{\text{right}}$, where $W_{\text{left}} \cong (\mathbb{C}^2)^{\otimes \ell(i-1)}$ and $W_{\text{right}} \cong (\mathbb{C}^2)^{\otimes \ell(L-i-1)}$ correspond to the sites outside $[\ell(i-1)+1, \ell(i+1)]$. Since the vector at sites $k, k+1$ is a singlet of $U_q(\mathfrak{sl}(2))$, $V$ is isomorphic to $(\mathbb{C}^2)^{\otimes 2(\ell-1)}$ as a representation space of $U_q(\mathfrak{sl}(2))$. Iterating the process, we find that $\varphi_1(\pi) \in W_{\text{left}} \otimes V \otimes W_{\text{right}}$ as above, where $V$ is a subspace of $(\mathbb{C}^2)^{\otimes 2\ell}$ which is isomorphic to $(\mathbb{C}^2)^{\otimes 2c_{i,i+1}(\pi)}$. But we know explicitly the decomposition of the latter into irreducible representations, and in particular that the highest possible spin is $c_{i,i+1}(\pi)$, hence the equality of the lemma. For the linear independence, consider the unique spin state obtained from $\pi$ which has all outside connections of the form $|\uparrow\rangle$, i.e., by orienting all lines to the outside "outwards".

Next, we note that since the projection $\prod_{i=1}^{L} p_i$ *cannot increase* the number of connections to the outside, then the same statement can be made about the "fused" vector $\varphi_\ell(\pi)$, $\pi \in \mathcal{L}_{L,\ell}$ (noting that in the case of fused link patterns, the connections to the outside are shared equally between groups $i$ and $i+1$). So,

$$P_{j;i,i+1}\varphi_\ell(\pi) = 0 \qquad \pi \in \mathcal{L}_{L;\ell},\ j > c_{i,i+1}(\pi)$$

(and the $P_{j;i,i+1}\varphi_1(\pi)$, where $\pi$ runs over fused link patterns with $c_{i,i+1} = j$, are linearly independent).

From the coproduct of $U_q(\mathfrak{sl}(2))$ it not hard to see that $P_{j;i,i+1}$ commutes with its action *on the whole of* $(\mathbb{C}^{\ell+1})^{\otimes L}$, and in particular leaves invariant the image of $\varphi_\ell$. This implies that $P_{j;i,i+1}\varphi_\ell(\pi)$ is a linear combination of $\varphi_\ell(\pi')$.

Now writing $P_{\ell;i,i+1}P_{j;i,i+1}\varphi_\ell(\pi) = 0$ for $j < \ell$, and expanding $P_{j;i,i+1}\varphi_\ell(\pi)$ in $\varphi_\ell(\pi')$ as above, we find that no link patterns $\pi'$ with $c_{i,i+1}(\pi') = \ell$ can appear in the expansion. Repeating the argument with $P_{\ell-1;i,i+1}P_{j;i,i+1}\varphi_\ell(\pi) = 0$, etc, until $P_{j+1;i,i+1}P_{j;i,i+1}\varphi_\ell(\pi) = 0$, we conclude that

$$\operatorname{Im} P_{j;i,i+1} \circ \varphi_\ell \subseteq \langle \varphi_\ell(\pi) : c_{i,i+1}(\pi) \le j \rangle$$

Now consider $\Psi := \Psi^{(0)}(\dots, z_i, z_{i+1}, z_{i+2}, \dots)$ (the case $k > 0$ will be discussed separately) with $z_i = q^{2(a+b)}z$, $z_{i+1} = q^{2b}z$, $z_{i+2} = z$, $a, b > 0$, $a + b < \ell + 2$. Plugging these equalities into (3.13) and using the form (3.12) of the $R$-matrix, we conclude that $\Psi$ is a linear combination of $P_{j;i,i+1}\Psi$, $j < a$, and a linear combination of $P_{j;i+1,i+2}\Psi$, $j < b$.

As mentioned in sect. 4.3, $\Psi$ is $U_1$-invariant. We can therefore apply the reasoning above and conclude that

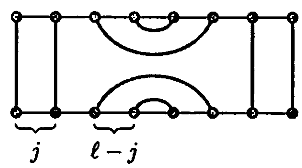
$$\Psi \in \langle \varphi_\ell(\pi) : c_{i,i+1}(\pi) < a,\ c_{i+1,i+2}(\pi) < b \rangle$$

For any such $\pi$ the group of sites $i+1$ has at least $\ell+1-a$ connections to the sites $i$, and $\ell+1-b$ connections to the sites $i+2$; but $\ell+1-a+\ell+1-b > \ell$ which is contradictory. Therefore $\Psi = 0$.

This is the special case $i_1 = i$, $i_2 = i+1$, $i_3 = i+2$ of (3.16). In order to obtain the general case, one simply needs to apply the exchange relation (3.13) repeatedly, to move $(i, i+1, i+2) \to (i_1, i_2, i_3)$; assuming that all other arguments of $\Psi$ are generic, all $R$-matrices involved in this process have nonzero denominator and are therefore well-defined. We have thus proved Theorem 3.

The case of $\Psi^{(k)}$, $k > 0$, can be treated similarly by noting that the reasoning above is purely local and cannot depend on the twist. Explicitly, one can build a $U_1$-invariant element of $\mathbb{C}^{k+1} \otimes (\mathbb{C}^2)^{\otimes \ell L} \otimes \mathbb{C}^{k+1}$ given by $\langle x | \Phi(z_1) \dots \Phi(z_{2n}) | y \rangle$ where $\langle x|$ and $|y\rangle$ run over the $U_1$-representation generated by $\langle k|$ and $|k\rangle$, each isomorphic to $\mathbb{C}^{k+1}$. It satisfies the wheel condition, and therefore $\Psi^{(k)}$ does.

*Remark*: the connection between the projectors $P_j$ and link patterns can be made more explicit by introducing generalized Temperley–Lieb operators $E_j$ (denoted $e^{(\ell-j)}$ in [32]), which are best described graphically as

$E_j = p_1 p_2$ 


Note that the $E_j$ are (up to normalization) a family of non-orthogonal projectors (as opposed to the $P_j$ which are orthogonal to each other).

Comparing the expression of the $R$-matrix (3.12) with the following expression (correcting a sign mistake in Eq. (2.8) of [32]):

$$\check{\mathrm{R}}(z) = (-1)^\ell \sum_{j=0}^{\ell} \left( \prod_{r=1}^{j} \frac{q^{\ell-r+1} - q^{-(\ell-r+1)}}{q^r - q^{-r}} \right) \left( \prod_{r=0}^{j} \frac{q^r z - q^{-r} w}{q^{\ell-r} w - q^{r-\ell} z} \right) E_j$$

we see that there is a triangular change of basis between the $P_j$ and the $E_j$: $P_j = \sum_{j' \le j} \alpha_{j,j'} E_{j'}$ with $\alpha_{j,j} \neq 0$.

## D. Polynomiality

In this appendix we prove that the expression (3.10) is a polynomial. As a bonus we get that the coefficients are Laurent polynomials in $q$ up to some known factor. Recall the expression:

$$\begin{aligned} \Psi^{(k)}_{b_1,\dots,b_{2n}}(z_1,\dots,z_{2n}) = \gamma^{(k)}_{\ell,n} \prod_{i=1}^{2n} \alpha_{b_i} z_i^k \prod_{1 \le i < j \le 2n} \prod_{r=1}^{\ell} (q^{2r} z_j - z_i) \oint \dots \oint \prod_{i=1}^{\ell n} \frac{dw_i}{2\pi i} \\ \times \prod_{i<j} \frac{w_j - w_i}{w_j - q^2 w_i} \frac{P_{Y^{(k)}_{\ell,n}}(w_1,\dots,w_{\ell n})}{\prod_{j \le \epsilon_i} (q^\ell w_i - z_j) \prod_{j \ge \epsilon_i} (w_i - q^\ell z_j)} \end{aligned} \tag{D.1}$$

where the contour integration is made in the following way: we start by integrating $w_{\ell n}$ arround the points $q^{\ell-2k} z_s$, for all $1 \le i \le 2n$ and $0 \le k < \ell$; we proceed by doing the same with $w_{\ell n-1}$, $w_{\ell n-2}$, $\dots$, up to $w_1$. By Cauchy's residue theorem this is the same as replacing $\{w_1,\dots,w_{\ell n}\} \to \{q^{\ell-2k_1} z_{s_1}, \dots, q^{\ell-2k_{\ell n}} z_{s_{\ell n}}\}$, removing the correspondant poles $(w_i - q^{\ell-2k_i} z_{s_i})^{-1}$ and summing over all possible combinations. Notice that, due to the presense of the Vandermonde polynomial we can not repeat a pole. Also, $q^{\ell-2k} z_s$ is only possible, for $k \ge 1$ if $q^{\ell-2k+2} z_s$ appears to the right.

We want to prove that there is neither any pole of the kind $(z_s - q^a z_r)^{-1}$ nor of the kind $z_r^{-1}$. In order to prove it we will look in the details of the computation. We can ignore all integrations except the ones that concern $z_r$ and $z_s$, with $r < s$. There is a subset of the integration variables $\{w_{j_1},\dots,w_{j_m}\}$ which is replaced by $q^{\ell-2k} z_r$ and $q^{\ell-2k} z_s$, the others being replaced by other variables $z_t$.

Fix $m$. We will represent it diagramatically, representing all poles in $z_s$ by a blue circle and the ones in $z_r$ by a red square. For example, we represent

$$\{w_1, w_2, w_3, w_4\} \to \{q^{\ell-4} z_s, q^{\ell-2} z_s, q^{\ell} z_r, q^{\ell} z_s\}$$

by 

where we simplified the indices of the integration variables.

Let $\psi_{d,\epsilon}$ be the relevant part of the integral, where $d$ stands for an arbitrary diagram with $m$ circles and squares,

$$\begin{aligned}\psi_{d,\epsilon}(z_r, z_s) = \oint \cdots \oint \prod_{i=1}^{m} \frac{dw_i}{2\pi i} \prod_{i<j} \frac{w_i - w_j}{w_i - q^{-2} w_j} \prod_{i} \frac{1}{(w_i - q^{\ell} z_r)(w_i - q^{\ell} z_s)} \\ \times P_{Y^{(k)}_{\ell,n}} z_r^k z_s^k \prod_{i=1}^{m} \frac{\prod_{\epsilon_i > r}(w_i - q^{\ell} z_r) \prod_{\epsilon_i > s}(w_i - q^{\ell} z_s)}{\prod_{\epsilon_i \geq r}(q^{\ell} w_i - z_r) \prod_{\epsilon_i \geq s}(q^{\ell} w_i - z_s)} \prod_{t=1}^{\ell} (q^{2t} z_s - z_r)\end{aligned} \tag{D.2}$$

where the integration is performed according to the diagram $d$. We restrict $\epsilon$ to the relevant indices: $\{\epsilon_{j_1}, \dots \epsilon_{j_m}\}$.

The following statement is necessary to the proof:

**Lemma 4.** *The Macdonald polynomial $P_{Y^{(k)}_{\ell,n}}$ satisfies:*

$$P_{Y^{(k)}_{\ell,n}}(z, q^2 z, \dots, q^{2j-2} z, w_{j+1}, \dots) = z^{\max\{j-k,0\}} \times (\dots)$$

*where $j \leq \ell$.*

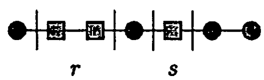
Recall that $P^{(k)}_{Y_{\ell,n}}$ is obtained from $(\prod_i w_i) \langle k_1, \dots, k_\ell | e^{2\ell n \beta} \mathcal{F} | k_1, \dots, k_\ell \rangle$, which can be seen as a sum over all parafermionic decompositions. Chosing $j$ variables $\{z, q^2 z, \dots, q^{2j-2} z\}$ forces the parafermions to be in distinct modes, according to (A.23). Each parafermion $f^i(z)$ generates a factor $z^{-k_i}$. The result follows from minimizing the power of $z$ on the expression.

We split formula (D.2) into two contributions, the first row and the second row. All the poles of the second row are outside of the contour integral.

**Proposition 3.** *For any diagram $d$, the second row of formula* (D.2) *is a polynomial in $z_r$ and $z_s$. Moreover, if multiplied by $\frac{[\ell]!}{[\ell-m_r]!} \frac{[\ell]!}{[\ell-m_s]!}$, where $m_r$ (and $m_s$) is the number of $\epsilon_i$ equal to $r$ (respectivelly $s$), it is a Laurent polynomial in $q$.*

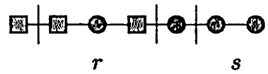
Notice that $m_r \leq b_r$ (and $m_s \leq b_s$), the equality being only possible if we chose all $w_i$ such that $\epsilon_i = r$ (or $s$) to belong to our subset of variables.

*Proof.* By choice $r < s$. Pick a random diagram, for example




where the vertical lines split the variables $w_i$ in five regions $\epsilon_i < r$, $\epsilon_i = r$, $r < \epsilon_i < s$, $\epsilon_i = s$ and $\epsilon_i > s$. The product $(w_i - q^{\ell} z_r)(w_i - q^{\ell} z_s)$ vanishes whenever there is a blue circle on the region $\epsilon_i > s$; and it vanishes also if there is a red square on the regions $\epsilon_i > r$. Then this example will vanish.

We pick now a non vanishing example:




The term $(q^\ell w_i - z_s)^{-1}$ only appears in the region $\epsilon_i = s$, exacly $m_s$ times, when we replace $w_i$ by the sucessive $q^{\ell-2p} z_s$ we get $(q^{2\ell}-1)^{-1} \dots (q^{2\ell-2m_s+2}-1)^{-1} z_s^{-m_s}$. There are at least $m_s$ variables $z_s$ in $P_{Y^{(k)}_{\ell,n}}$ and we can apply lemma 4, therefore the power in $z_s$ is at least $z_s^k z_s^{\max\{m_s-k,0\}} z_s^{-m_s} = z_s^{\max\{0,k-m_s\}}$. The term $(q^\ell w_i - z_r)^{-1}$ appears in all regions such $\epsilon_i \geq r$. Let $m'_r$ be the number of red squares in the region $\epsilon_i = r$. Then we can do the same analysis for the squares. The circles will generate a term $(q^{2\ell} z_s - z_r) \dots (q^{2\ell-2m'+2} z_s - z_r)$, where $m'$ is the number of circles in the regions $\epsilon_i \geq r$. This term cancels with the product $\prod_p (q^{2p} z_s - z_r)$.

In general, the first row of the formula (D.2) will give a rational function.

**Proposition 4.** *The sum over all possible diagrams with m nodes of $\psi_{d,\epsilon}(z_r, z_s)$ is a polynomial in $z_r$ and $z_s$.*

*Proof.* From proposition 3 we know that

$$\psi_d(z_r, z_s) = \oint \dots \oint \prod_{i=1}^{m} \frac{dw_i}{2\pi i} \prod_{i<j} \frac{w_j - w_i}{w_j - q^2 w_i} \prod_i \frac{Q_d(z_r, z_s)}{(w_i - q^\ell z_r)(w_i - q^\ell z_s)}$$

where $Q_d(z_r, z_s)$ is a polynomial depending on the diagram $d$. We ignore the dependence in $\epsilon$. For convenience we allow diagrams with more than $\ell$ circles (or squares), in which case we set $Q_d$ to zero. This is naturally done by using the Macdonald Polynomial $P_{Y_{\ell,n}}$.

On the other hand, $Q_d(z_r, z_s)$ can be seen as the evaluation of a function, i.e. $Q_d(z_r, z_s) = Q(z_r, z_s; z^{(1)}, \dots, z^{(m)})$, where $z^{(i)}$ is given by the diagram $d$. We will omit the explicit dependence on $z_r$ and $z_s$, and for the analysis we will consider that the $z^{(i)}$ are independent variables. Then, this is a rational function of the form

$$Q(z^{(1)}, \dots, z^{(m)}) = \frac{R(z^{(1)}, \dots, z^{(m)})}{\prod_i \prod_{j_i} (z^{(i)} - \zeta_{j_i})} \tag{D.3}$$

where the polynomial $R$ and $\zeta_{j_i}$ depend on $z_r$ and $z_s$.

The diagram  corresponds to the following expression:

$$\psi_d(z_r, z_s) = \frac{Q(q^{\ell-4} z_s, q^\ell z_r, q^{\ell-2} z_s, q^\ell z_s)}{(q^{\ell-4} z_s - q^{\ell-2} z_r)(q^\ell z_r - q^{\ell-4} z_s)(q^{\ell-2} z_s - q^\ell z_r)(q^\ell z_s - q^\ell z_r)}$$

The denominator can be constructed by a simple rule: we read the diagram $d$ from right to left, at each node we can put either a circle or a square (corresponding to some $q^a z_s$ or $q^b z_r$); if we choose a circle (resp. a square) we add the pole $(q^a z_s - q^b z_r)^{-1}$ (resp. $(q^b z_r - q^a z_s)^{-1}$).

Now we sum over all $2^m$ possible diagrams. Let $d_1$ and $d_2$ be two diagrams that only differ in the first element, i.e. $d_1 = \square d^{(1)}$ and $d_2 = \bullet d^{(1)}$, where $d^{(1)}$ is some diagram with $m-1$ nodes. Then:

$$\psi_{\square d^{(1)}}(z_r, z_s) + \psi_{\bullet d^{(1)}}(z_r, z_s) = \frac{1}{(\dots)} \frac{Q_{\square d^{(1)}} - Q_{\bullet d^{(1)}}}{q^a z_r - q^b z_s}$$

where the three dots are replacing the commun $m-1$ poles. Using the formulation (D.3) and setting $q^a z_r = z^{(1)}$ and $q^b z_s = z'^{(1)}$, the numerator is given by

$$\frac{1}{\prod_{i\geq 2}\prod_{j_i}(z^{(i)}-\zeta_{j_i})}\left(\frac{R(z^{(1)},\ldots)}{\prod_{j_1}(z^{(1)}-\zeta_{j_1})}-\frac{R(z'^{(1)},\ldots)}{\prod_{j_1}(z'^{(1)}-\zeta_{j_1})}\right)$$

which vanishes if $z^{(1)} = z'^{(1)}$. Then the division by $(z^{(1)} - z'^{(1)})$ is well defined and we can define the polynomial:

$$Q^{(1)}_{\circ d^{(1)}} := \frac{Q_{\square d^{(1)}} - Q_{\bullet d^{(1)}}}{q^a z_s - q^b z_r}$$

where the open circle indicates that we already performed the sum. This can be written in the rational function perpesctive:

$$Q^{(1)}(z^{(1)}, z'^{(1)}; z^{(2)} \ldots, z^{(m)}) = \frac{R^{(1)}(z^{(1)}, z'^{(1)}; z^{(2)} \ldots, z^{(m)})}{\prod_k (z^{(1)}-\zeta_k)(z'^{(1)}-\zeta_k)\prod_{i\geq 2}\prod_{j_i}(z^{(i)}-\zeta_{j_i})}$$

Notice that this is a symmetric function on $z^{(1)}$ and $z'^{(1)}$.

Now we sum over the second node, let $d_1 = \circ\square d^{(2)}$ and $d_2 = \circ\bullet d^{(2)}$, where the open circle indicates the first node where we already performed the sum. Then:

$$\psi_{\circ\square d^{(2)}}(z_r, z_s) + \psi_{\circ\bullet d^{(2)}}(z_r, z_s) = \frac{1}{(\ldots)}\frac{Q^{(1)}_{\circ\square d^{(2)}} - Q^{(1)}_{\circ\bullet d^{(2)}}}{q^a z_s - q^b z_r}$$

We consider now that $z^{(1)}$ and $z'^{(1)}$ depend on the two variables $z^{(2)}$ and $z'^{(2)}$, then the numerator is given by:

$$\frac{1}{\prod_{i\geq 3}\prod_{j_i}(z^{(i)}-\zeta_{j_i})}\left(\frac{R^{(1)}(q^{-2}z^{(2)}, z'^{(2)}; z^{(2)},\ldots)}{\prod_k(q^{-2}z^{(2)}-\zeta_k)(z'^{(2)}-\zeta_k)\prod_{j_2}(z^{(2)}-\zeta_{j_2})}\right.$$
$$\left. - \frac{R^{(1)}(z^{(2)}, q^{-2}z'^{(2)}; z'^{(2)},\ldots)}{\prod_k(z^{(2)}-\zeta_k)(q^{-2}z'^{(2)}-\zeta_k)\prod_{j_2}(z'^{(2)}-\zeta_{j_2})}\right)$$

Once again this vanishes if $z^{(2)} = z'^{(2)}$, then the division by $(z^{(2)} - z'^{(2)})$ is well defined and we can define a new polynomial:

$$Q^{(2)}_{\circ\circ d^{(2)}} := \frac{Q^{(1)}_{\circ\square d^{(2)}} - Q^{(1)}_{\circ\bullet d^{(2)}}}{q^a z_s - q^b z_r}$$

or in the rational function perspective:

$$Q^{(2)}(z^{(2)}, z'^{(2)}; z^{(3)} \ldots, z^{(m)}) = \frac{R^{(2)}(z^{(2)}, z'^{(2)}; z^{(3)} \ldots, z^{(m)})}{\prod_k (z^{(2)}-\zeta_k)(z'^{(2)}-\zeta_k)\prod_{i\geq 2}\prod_{j_i}(z^{(i)}-\zeta_{j_i})}$$

which is once again symmetric in $z^{(2)}$ and $z'^{(2)}$.

Repeating the process $m$ times we obtain:

$$\psi(z_r, z_s) = Q^{(m)}(q^\ell z_r, q^\ell z_s)$$

which is a polynomial.

This concludes the proof. Note that if $Q_d$ is a Laurent polynomial in $q$ for all $d$, then $\sum_d \psi_d$ will be also a Laurent polynomial in $q$. Then:

**Corollary 1.** *The coefficients of* $([\ell]!)^{-n}\prod_i [b_i]!\Psi^{(k)}_{b_1,\ldots,b_{2n}}(z_1,\ldots,z_{2n})$ *are Laurent polynomials in* $q$.

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