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Author/s:

Chern, S;Li, Z;Stanton, D;Xue, T;Yee, AJ

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The Ariki–Koike algebras and Rogers–Ramanujan type partitions

Shane Chern¹ · Zhitai Li² · Dennis Stanton³ · Ting Xue⁴ · Ae Ja Yee²

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Abstract

Ariki and Mathas (Math Z 233(3):601–623, 2000) showed that the simple modules of the Ariki–Koike algebras $\mathcal{H}_{\mathbb{C}, v; Q_1, \dots, Q_m}(G(m, 1, n))$ (when the parameters are roots of unity and $v \neq 1$) are labeled by the so-called Kleshchev multipartitions. This together with Ariki’s categorification theorem enabled Ariki and Mathas to obtain the generating function for the number of Kleshchev multipartitions by making use of the Weyl–Kac character formula. In this paper, we revisit their generating function relation for the $v = -1$ case. In particular, this $v = -1$ scenario is of special interest as the corresponding Kleshchev multipartitions are closely tied with generalized Rogers–Ramanujan type partitions when $Q_1 = \dots = Q_a = -1$ and $Q_{a+1} = \dots = Q_m = 1$. Based on this connection, we provide an analytic proof of the result of Ariki and Mathas for the above choice of parameters. Our second objective is to investigate simple modules of the Ariki–Koike algebra in a fixed block, which are, as widely known, labeled by the Kleshchev multipartitions with a fixed partition residue statistic. This partition statistic is also studied in the works of Berkovich, Garvan, and Uncu. Employing their results, we provide two bivariate generating function identities for the $m = 2$ scenario.

✉ Ting Xue
ting.xue@unimelb.edu.au

Shane Chern
xh375529@dal.ca; chenxiaohang92@gmail.com

Zhitai Li
zfl5082@psu.edu

Dennis Stanton
stant001@umn.edu

Ae Ja Yee
yee@psu.edu

- ¹ Department of Mathematics and Statistics, Dalhousie University, Halifax, NS B3H 4R2, Canada
- ² Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA
- ³ School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
- ⁴ School of Mathematics and Statistics, University of Melbourne, Parkville, VIC 3010, Australia

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1 Introduction

The most fascinating identities in the theory of partitions are the *Rogers–Ramanujan identities*, which were originally proved by Rogers [36] and rediscovered by Ramanujan [35]. These identities have been a great source of research in the past several decades. In particular, they were reproved through many different approaches in the literature, and there also exist numerous similar identities and generalizations. Such identities are called *Rogers–Ramanujan type identities* or *generalized Rogers–Ramanujan identities*.

The importance of the Rogers–Ramanujan identities is also demonstrated by their mathematical relevance in other areas such as the representation theory of Lie algebras [30, 31]. In these papers, Lepowsky, Milne, and Wilson laid the groundwork for the study of the relationship between Lie algebras and Rogers–Ramanujan type identities by providing the first Lie-theoretic explanation of the Rogers–Ramanujan identities. Since then, a great deal of papers have been published in this spirit. Recently, Griffin, Ono, and Warnaar [25] provided a general framework that extends the Rogers–Ramanujan identities to doubly infinite families of q -series identities.

In this paper, we will discuss another instance, which connects these identities to the representations of the so-called *Ariki–Koike algebras*, or *cyclotomic Hecke algebras of type $G(m, 1, n)$* . The *Ariki–Koike algebras*, denoted by $\mathcal{H}_{\mathbb{C}, v; Q_1, \dots, Q_m}(G(m, 1, n))$, can be viewed as the Iwahori–Hecke algebras associated to the complex reflection groups $G(m, 1, n) \cong S_n \ltimes (\mathbb{Z}/m\mathbb{Z})^n$, where v and Q_i with $i = 1, \dots, m$ are parameters. They were introduced by Ariki and Koike [6] and independently by Broué and Malle [15], wherein Broué and Malle defined cyclotomic Hecke algebras for all complex reflection groups. In [5, 8], Ariki and Mathas showed that the simple modules of the Ariki–Koike algebras (when the parameters are roots of unity) are labeled by the so-called *Kleshchev multipartitions*. These multipartitions are in general defined recursively. A nonrecursive description of them is known when $v = -1$ by Mathas [33], and for bipartitions by Ariki, Kreiman, and Tsuchioka [7] (where in particular they recovered the result of Mathas and an unpublished result of Fayers for v a cube root of unity). We also note that in [28] Jacson gave a new characterization of Kleshchev multipartitions, which is simpler but still recursive.

Our interest here will revolve around the set $\Lambda^{a, m}(n)$ of Kleshchev multipartitions, whose exact definition will be given in Sect. 2.1.1 in the way of Mathas. Such multipartitions parametrize the simple modules of

$$\mathcal{H}_{\mathbb{C}, v; Q_1, \dots, Q_m}(G(m, 1, n)),$$

where $Q_1 = \cdots = Q_a = -1$, $Q_{a+1} = \cdots = Q_m = 1$, and $v = -1$. One of our motivation to study these special multipartitions is the appearance of the above Hecke algebras in the study of character sheaves for graded Lie algebras together with their connections to certain cyclotomic rational Cherednik algebras; see [40, 41].

The generating function for $|\Lambda^{a,m}(n)|$, the cardinality of $\Lambda^{a,m}(n)$, can be deduced as a special case of a theorem due to Ariki and Mathas.

Theorem 1.1 (Ariki–Mathas [8]) *For $m \geq 2$ and $0 \leq a \leq m$,*

$$\sum_{n \geq 0} |\Lambda^{a,m}(n)| q^n = \frac{(q^{a+1}; q^{m+2})_{\infty} (q^{m+1-a}; q^{m+2})_{\infty} (q^{m+2}; q^{m+2})_{\infty}}{(q; q)_{\infty} (q; q^2)_{\infty}}, \quad (1.1)$$

where the q -Pochhammer symbol is defined by

$$(x; q)_{\infty} := \prod_{j=0}^{\infty} (1 - xq^j).$$

The above infinite product on the right-hand side without $(q; q^2)_{\infty}$ in the denominator appears in the well-known generalized Rogers–Ramanujan identities of Andrews, Bressoud, and Gordon [1, 14, 24]. As briefly mentioned earlier, this infinite product can also be found in the works of Lepowsky, Milne, and Wilson. More specifically, in [30], Lepowsky and Milne gave a Lie-theoretic interpretation of it, and later on, Lepowsky and Wilson [31] offered a complete explanation of the whole infinite product by showing that $(q; q^2)_{\infty}$ is associated with the principal Heisenberg subalgebra of $\widehat{sl}(2)$.

In this paper, we revisit Theorem 1.1 from a partition-theoretic perspective and prove the following equivalent theorem.

Theorem 1.2 *For $m \geq 2$ and $0 \leq a \leq m$,*

$$\begin{aligned} & \sum_{\substack{N_m \geq N_{m-1} \geq \cdots \geq N_{a+1} \\ N_{a+1}+1 \geq N_a \geq \cdots \geq N_2 \geq 0}} \frac{q^{\sum_{i=2}^m \binom{N_i+1}{2}} (-q; q)_{N_2+\delta_{a+1,2}} (1 - q^{N_{a+1}+1})}{(1 - q^{N_{a+1}-N_a+1}) \prod_{i=2}^m (q; q)_{N_i - N_{i-1}}} \\ &= \frac{(q^{a+1}; q^{m+2})_{\infty} (q^{m+1-a}; q^{m+2})_{\infty} (q^{m+2}; q^{m+2})_{\infty}}{(q; q)_{\infty} (q; q^2)_{\infty}}, \quad (1.2) \end{aligned}$$

where δ is the Kronecker delta function, $N_1 = 0$, and $N_{m+1} = \infty$.

Our second objective is to study the set $\Lambda^{a,m}(n)$ under the consideration of the partition residue statistics. Integer partitions possess rich arithmetic properties such as the well-known partition congruences modulo 5, 7, and 11 due to Ramanujan [34]. In the study of such properties, combinatorial statistics play an important role. Amongst these statistics, the very first one was constructed by Dyson [17], who defined *ranks* for partitions to combinatorially interpret Ramanujan's congruences modulo 5 and 7. In the meantime, he conjectured the existence of another statistic called *crank*, which shall explain all three arithmetic relations of Ramanujan. This statistic was later

discovered by Andrews and Garvan [4]. Subsequently, Garvan, Kim, and Stanton [20] introduced a new version of the crank, which stems from core and quotient partitions, one of the fundamental concepts in the representation theory of symmetric groups. To define core and quotient partitions, residue statistics become an essential component.

The *mod 2 residue*, or *2-residue*, statistic for partitions was also investigated by Berkovich and Garvan [9]. In their work on the Andrews–Stanley srnk for integer partitions, they defined a new partition statistic, namely, the *BG-rank*, and they showed that the BG-rank gives another combinatorial account for Ramanujan’s mod 5 partition congruence. Here, the BG-rank is in essence the 2-residue statistic, as indicated in [10]. Recently, Berkovich and Uncu [11] launched a further investigation and obtained a refined generating function relation for partitions with a fixed BG-rank.

In this paper, we will consider this 2-residue statistic for multipartitions in $\Lambda^{a,2}(n)$. For a multipartition μ in $\Lambda^{a,2}(n)$, let $\omega(\mu)$ denote the 2-residue difference of μ , whose definition will be given in Definition 2.4. Our main results are the following bivariate generating function identities.

Theorem 1.3 *We have*

$$\sum_{n \geq 0} \sum_{\mu \in \Lambda^{1,2}(n)} x^{\omega(\mu)} q^n = (-q^2; q^2)_{\infty} (-xq; q^2)_{\infty} (-q/x; q^2)_{\infty} \quad (1.3)$$

and

$$\begin{aligned} & \sum_{n \geq 0} \sum_{\mu \in \Lambda^{2,2}(n)} x^{\omega(\mu)} q^n \\ &= \frac{1}{2} \left((-q; q^2)_{\infty} (-x; q^2)_{\infty} (-q^2/x; q^2)_{\infty} + (q; q^2)_{\infty} (x; q^2)_{\infty} (q^2/x; q^2)_{\infty} \right). \end{aligned} \quad (1.4)$$

Let us write $\Lambda_{\omega}^{a,2}(n)$ for the set of multipartitions in $\Lambda^{a,2}(n)$ with 2-residue difference ω . An application of Jacobi’s triple product identity to each of the right-hand sides of (1.3) and (1.4) yields the following corollary.

Corollary 1.4 *We have*

$$\sum_{n \geq 0} |\Lambda_{\omega}^{1,2}(n)| q^n = q^{\omega^2} \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \quad (1.5)$$

and

$$\sum_{n \geq 0} (|\Lambda_{\omega}^{2,2}(n)| + |\Lambda_{1-\omega}^{2,2}(n)|) q^n = q^{\omega^2 - \omega} \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (1.6)$$

It is known from [32] that the set $\Lambda_{\omega}^{a,2}(n)$ parametrizes the set of simple modules in a fixed block, labeled by ω , of the Ariki–Koike algebra $\mathcal{H}_{-1, -1, 1}(G(2, 1, n))$

(resp. $\mathcal{H}_{-1,-1,-1}(G(2, 1, n))$), when $a = 1$ (resp. 2). Note that in this case, the Ariki–Koike algebra is an Iwahori–Hecke algebra of a Weyl group of type B_n (or C_n). It is interesting to note that $|\Lambda_0^{1,2}(2n)| = |\Lambda_1^{1,2}(2n+1)|$ (resp. $|\Lambda_0^{2,2}(2n)|, |\Lambda_1^{2,2}(2n+1)|$), namely, the number of simple modules in the principal block of the corresponding Hecke algebra equals the number of nilpotent orbits in the Lie algebra of type C_n (resp. D_n, B_n).

We also note that Theorem 1.3 and Corollary 1.4 can be derived by combining results in [5, 8, 32] and the Weyl–Kac character formula computations for affine Lie algebras in [19]. However, we emphasize that our proof given in Sect. 5 is more direct and does not rely on deep results from representation theory, in particular, Ariki’s categorification theorem and the Weyl–Kac character formula. In addition, as a byproduct of our proof, we obtain the following identities.

Corollary 1.5 *We have*

$$\frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} = \sum_{r,s \geq 0} \frac{q^{r^2+s^2+r+s} (q^2; q^2)_{r+s+1}}{(q^2; q^2)_r (q^2; q^2)_r (q^2; q^2)_s (q^2; q^2)_{s+1}} \quad (1.7)$$

and

$$\begin{aligned} \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} &= \sum_{r,s \geq 0} \frac{q^{r^2+s^2+2s} (q^2; q^2)_{r+s}}{(q^2; q^2)_r (q^2; q^2)_s (q^2; q^2)_r (q^2; q^2)_s} \\ &\quad + \sum_{r,s \geq 0} \frac{q^{r^2+s^2+2r+2s+2} (q^2; q^2)_{r+s+1}}{(q^2; q^2)_r (q^2; q^2)_{r+1} (q^2; q^2)_s (q^2; q^2)_{s+1}}. \end{aligned} \quad (1.8)$$

The remainder of this paper is organized as follows. In Sect. 2, we recall some definitions, notation, and necessary results on basic hypergeometric series. In Sect. 3, we give a proof of Theorem 1.2. This proof is quite long and involved, so some necessary lemmas and their proofs are given at the end of this paper in “Appendix A.” In Sect. 4, we present some combinatorial facts about partition pairs in $\Lambda^{a,2}(n)$, and in Sect. 5, we prove Theorem 1.3 and Corollaries 1.4 and 1.5. We then conclude our paper by providing some remarks in Sect. 6.

2 Preliminaries

In this section, we review some basic definitions for integer partitions and standard notation for q -series. We also recall some necessary results on basic hypergeometric series for later use.

Fig. 1 The Young diagram
 $Y_{(5,4,4,2)}$

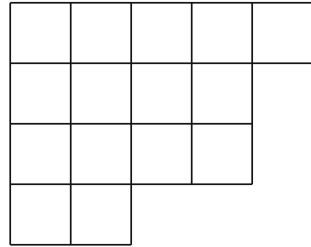
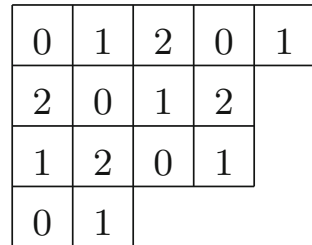


Fig. 2 Residues mod 3 for
 $Y_{(5,4,4,2)}$



2.1 Partition definitions

A *partition* of n is an integer sequence $\lambda = (\lambda_1, \dots, \lambda_\ell)$ such that $\lambda_1 \geq \dots \geq \lambda_\ell > 0$ and that $|\lambda| := \lambda_1 + \dots + \lambda_\ell = n$. We write it as $\lambda \vdash n$. The λ_i are parts of λ , and the number of parts in λ is denoted by $\ell(\lambda)$.

For $\lambda \vdash n$, its *Young diagram*, also known as the *Ferrers diagram*, is the graphical representation consisting of n boxes (or dots) placed in rows such that there are λ_i boxes (or dots) in the i th row. We denote the Young diagram of λ by Y_λ . The *conjugate partition* of λ , denoted by $\lambda^\top$, is the partition associated with the Young diagram resulting from reflecting the Young diagram of λ about the main diagonal. In Fig. 1, the Young diagram of $\lambda = (5, 4, 4, 2)$ is illustrated, and $\lambda^\top = (4, 4, 3, 3, 1)$.

For a positive integer p , the *residue* of a node $(i, j) \in Y_\lambda \bmod p$ is defined to be

$$\text{Res}(i, j) := (j - i) \bmod p.$$

Figure 2 shows the residues mod 3 of $\lambda = (5, 4, 4, 2)$.

For a positive integer e , a partition λ is *e-restricted* if

$$\lambda_i - \lambda_{i+1} < e \text{ for every } 1 \leq i \leq \ell(\lambda),$$

where $\lambda_{\ell(\lambda)+1} = 0$. In other words, if λ is e -restricted, then its conjugate $\lambda^\top$ can have at most $e - 1$ parts of the same size.

We now introduce multipartitions. An m -*multipartition* of n is an m -tuple of partitions $(\lambda^{(1)}, \dots, \lambda^{(m)})$ such that

$$\sum_{i=1}^m |\lambda^{(i)}| = n.$$

By abuse of notation, we denote a multipartition as $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ and

$$|\lambda| := \sum_{i=1}^m |\lambda^{(i)}|.$$

Also, if $|\lambda| = n$, we say λ is a multipartition of n , and we still write $\lambda \vdash n$.

Definition 2.1 For a sequence $\mathbf{t} = (t_1, \dots, t_m)$, an m -multipartition λ is $(2, \mathbf{t})$ -restricted if

- (i) each $\lambda^{(i)}$ is 2-restricted;
- (ii) $\ell(\lambda^{(i)}) + t_i \leq \ell(\lambda^{(i+1)}) + t_{i+1}$ for every $1 \leq i \leq m-1$.

Definition 2.2 For a $(2, \mathbf{t})$ -restricted multipartition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$, let

$$Y_\lambda := \{(i, j, s) \mid (i, j) \in Y_{\lambda^{(s)}}, s = 1, \dots, m\},$$

where $Y_{\lambda^{(s)}}$ denotes the Young diagram of $\lambda^{(s)}$. Define the p -residue of a node $x = (i, j, s) \in Y_\lambda$ by

$$\text{Res}_p(x) := (j - i + t_s) \bmod p.$$

2.1.1 Kleshchev multipartitions

We are now in a position to give the definition of *Kleshchev multipartitions* for the simple modules of $\mathcal{H}_{\mathbb{C}, v; Q_1, \dots, Q_m}(G(m, 1, n))$ for $Q_1 = \dots = Q_a = -1$, $Q_{a+1} = \dots = Q_m = 1$, and $v = -1$.

Definition 2.3 For $m \geq 1$ and $0 \leq a \leq m$, we set

$$\mathbf{t} = (t_1, \dots, t_m),$$

where

$$t_1 = \dots = t_a = 0, \quad t_{a+1} = \dots = t_m = 1. \quad (2.1)$$

Such a $(2, \mathbf{t})$ -restricted m -multipartition is called a *Kleshchev multipartition*. We define

$$\Lambda^{a,m} := \{(2, \mathbf{t})\text{-restricted } m\text{-multipartitions}\}$$

and

$$\Lambda^{a,m}(n) := \{\pi \in \Lambda^{a,m} \text{ with } \pi \vdash n\}.$$

Remark 2.1 It easily follows from the definition of restricted multipartitions that a multipartition is $(2, \mathbf{t})$ -restricted if and only if its conjugate $\pi = (\pi^{(1)}, \dots, \pi^{(m)})$ is a multipartition such that

- (i) each $\pi^{(i)}$ is a *strict partition*, i.e., parts are all distinct;
- (ii) $\pi_1^{(i)} + t_i \leq \ell(\pi^{(i+1)}) + t_{i+1}$ for every $1 \leq i \leq m-1$.

Throughout this section and the rest of this paper, we will use this equivalent definition for Kleshchev multipartitions.

2.1.2 2-Residue statistics

Now we consider the *2-residues* for Kleshchev multipartitions.

Definition 2.4 For a Kleshchev multipartition $\pi \in \Lambda^{a,m}$, define its *2-residue difference* $\omega(\pi)$ by

$$\omega(\pi) := C_0(\pi) - C_1(\pi),$$

where

$$C_i(\pi) := |\{x \in Y_\pi \mid \text{Res}_2(x) = i\}|.$$

For a fixed integer ω , we also define

$$\Lambda_\omega^{a,m} := \{\pi \in \Lambda^{a,m} \mid \omega(\pi) = \omega\},$$

and

$$\Lambda_\omega^{a,m}(n) := \{\pi \in \Lambda^{a,m}(n) \mid \omega(\pi) = \omega\}.$$

Note that $\omega(\pi) \equiv |\pi| \pmod{2}$. Hence, if $\omega \not\equiv n \pmod{2}$, then $\Lambda_\omega^{a,m}(n)$ is the empty set.

2.2 q -Series notation

Throughout this paper, we will adopt the standard *q -Pochhammer symbol* for $n \in \mathbb{Z}$:

$$(a; q)_n := \frac{(a; q)_\infty}{(aq^n; q)_\infty}.$$

Also, for brevity, the following compact notation will be used frequently:

$$\begin{aligned} (a_1, a_2, \dots, a_r; q)_n &:= (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n, \\ (a_1, a_2, \dots, a_r; q)_\infty &:= (a_1; q)_\infty (a_2; q)_\infty \cdots (a_r; q)_\infty. \end{aligned}$$

In addition, the *q -binomial coefficients*, also known as the *Gaussian polynomials*, are given by

$$\begin{bmatrix} N \\ M \end{bmatrix} := \begin{bmatrix} N \\ M \end{bmatrix}_q := \begin{cases} \frac{(q; q)_N}{(q; q)_M (q; q)_{N-M}}, & \text{if } 0 \leq M \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

They satisfy two basic recurrences [2, p. 35, eqs. (3.3.4) and (3.3.3)]:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N-1 \\ M \end{bmatrix} + q^{N-M} \begin{bmatrix} N-1 \\ M-1 \end{bmatrix}, \quad (2.2)$$

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} N-1 \\ M-1 \end{bmatrix} + q^M \begin{bmatrix} N-1 \\ M \end{bmatrix}. \quad (2.3)$$

Also, the following trivial relation can be deduced from their definition:

$$(1 - q^M) \begin{bmatrix} N \\ M \end{bmatrix} = (1 - q^N) \begin{bmatrix} N-1 \\ M-1 \end{bmatrix}. \quad (2.4)$$

2.3 Lemmas on basic hypergeometric series

Recall that the *basic hypergeometric series* ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r \left(\begin{matrix} A_1, A_2, \dots, A_{r+1} \\ B_1, B_2, \dots, B_r \end{matrix}; q, z \right) := \sum_{n \geq 0} \frac{(A_1, A_2, \dots, A_{r+1}; q)_n z^n}{(q, B_1, B_2, \dots, B_r; q)_n}.$$

We list a set of useful identities:

► **The q -binomial theorem** [2, eq. (3.3.6)]:

$$(z; q)_N = \sum_{n \geq 0} (-1)^n z^n q^{\binom{n}{2}} \begin{bmatrix} N \\ n \end{bmatrix}. \quad (2.5)$$

► **The q -binomial theorem** [21, eq. (II.3)]:

$$\sum_{n \geq 0} \frac{(a; q)_n z^n}{(q; q)_n} = \frac{(az; q)_\infty}{(z; q)_\infty}. \quad (2.6)$$

► **Jacobi's triple product identity** [21, eq. (II.28)]:

$$(q, z, q/z; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n z^n q^{\binom{n}{2}}. \quad (2.7)$$

► **Heine's first transformation** [21, eq. (III.1)]:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} c/b, z \\ az \end{matrix}; q, b \right). \quad (2.8)$$

► **Heine's second transformation** [21, eq. (III.2)]:

$${}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, z \right) = \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} {}_2\phi_1 \left(\begin{matrix} abz/c, b \\ bz \end{matrix}; q, \frac{c}{b} \right). \quad (2.9)$$

We also establish the following evaluation of a well-poised ${}_2\phi_1$ series.

Lemma 2.1 *We have*

$${}_2\phi_1 \left(\begin{matrix} a, b \\ aq^2/b \end{matrix}; q^2, q/b \right) = \frac{(q; q^2)_\infty ((-\sqrt{a}, q\sqrt{a}/b; q)_\infty + (\sqrt{a}, -q\sqrt{a}/b; q)_\infty)}{2(aq^2/b; q^2)_\infty (q/b; q^2)_\infty}. \quad (2.10)$$

Proof Notice that

$$\begin{aligned} & {}_2\phi_1 \left(\begin{matrix} a, b \\ aq^2/b \end{matrix}; q^2, q/b \right) \\ (\text{by (2.8)}) &= \frac{(a; q^2)_\infty (q; q^2)_\infty}{(aq^2/b; q^2)_\infty (q/b; q^2)_\infty} {}_2\phi_1 \left(\begin{matrix} q^2/b, q/b \\ q \end{matrix}; q^2, a \right) \\ &= \frac{(a; q^2)_\infty (q; q^2)_\infty}{(aq^2/b; q^2)_\infty (q/b; q^2)_\infty} \sum_{k=0}^{\infty} \frac{(q/b; q)_{2k}}{(q; q)_{2k}} a^k \\ &= \frac{(a; q^2)_\infty (q; q^2)_\infty}{2(aq^2/b; q^2)_\infty (q/b; q^2)_\infty} \left(\sum_{k=0}^{\infty} \frac{(q/b; q)_k}{(q; q)_k} \sqrt{a}^k + \sum_{k=0}^{\infty} \frac{(q/b; q)_k}{(q; q)_k} (-\sqrt{a})^k \right) \\ (\text{by (2.6)}) &= \frac{(a; q^2)_\infty (q; q^2)_\infty}{2(aq^2/b; q^2)_\infty (q/b; q^2)_\infty} \left(\frac{(q\sqrt{a}/b; q)_\infty}{(\sqrt{a}; q)_\infty} + \frac{(-q\sqrt{a}/b; q)_\infty}{(-\sqrt{a}; q)_\infty} \right) \\ &= \frac{(q; q^2)_\infty ((-\sqrt{a}, q\sqrt{a}/b; q)_\infty + (\sqrt{a}, -q\sqrt{a}/b; q)_\infty)}{2(aq^2/b; q^2)_\infty (q/b; q^2)_\infty}. \end{aligned}$$

This is exactly (2.10). $\square$

Remark 2.2 Due to the convergence condition for Heine's transformation in (2.8), it is assumed that $|a| < 1$ and $|q/b| < 1$ for the identity in (2.10). However, by analytic continuation, we may drop the assumption that $|a| < 1$. Thus, setting $a = q^{-2M}$ for any positive integer M in (2.10) yields a valid terminating series, which will be used later in Sect. 5.2.

3 The generating function for general $\Lambda^{a,m}$

In this section, we discuss the generating function for $\Lambda^{a,m}$. Our main goal is to give a q -series proof of Theorem 1.2.

We start by recalling the conditions for $\pi = (\pi^{(1)}, \dots, \pi^{(m)}) \in \Lambda^{a,m}$ from Definition 2.3:

- (i) each $\pi^{(i)}$ is a strict partition;
- (ii) $\pi_1^{(i)} \leq \ell(\pi^{(i+1)})$ for $i \neq a$ and $\pi_1^{(a)} \leq \ell(\pi^{(a+1)}) + 1$. Here, if $a = m$, then $\ell(\pi^{(a+1)}) = \infty$.

Our first observation is that the generating function for m -multipartitions in $\Lambda^{a,m}$ can be represented as a multisum.

Theorem 3.1 For $m \geq 1$ and $0 \leq a \leq m$,

$$\begin{aligned} & \sum_{n \geq 0} |\Lambda^{a,m}(n)| q^n \\ &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_m}} \begin{bmatrix} N_2 + \delta_{a+1,2} \\ N_1 \end{bmatrix} \begin{bmatrix} N_3 + \delta_{a+1,3} \\ N_2 \end{bmatrix} \dots \begin{bmatrix} N_m + \delta_{a+1,m} \\ N_{m-1} \end{bmatrix} \end{aligned} \quad (3.1)$$

$$= \sum_{\substack{N_m \geq N_{m-1} \geq \dots \geq N_{a+1} \\ N_{a+1} + 1 \geq N_a \geq \dots \geq N_2 \geq 0}} \frac{q^{\sum_{i=2}^m \binom{N_i+1}{2}} (-q; q)_{N_2 + \delta_{a+1,2}} (1 - q^{N_{a+1}+1})}{(1 - q^{N_{a+1} - N_a + 1}) \prod_{i=2}^m (q; q)_{N_i - N_{i-1}}}. \quad (3.2)$$

Here, δ is the Kronecker delta function. Also, in (3.2), we take $N_1 = 0$ and $N_{m+1} = \infty$.

Proof We make use of the fact that the generating function for strict partitions with exactly M parts and largest part at most N is

$$q^{\binom{M+1}{2}} \begin{bmatrix} N \\ M \end{bmatrix}.$$

Now, assume that each $\pi^{(i)}$ has exactly N_i parts. Then, for each $1 \leq i \leq m-1$, the strict partition $\pi^{(i)}$ has N_i parts with its largest part at most $N_{i+1} + \delta_{a+1,i+1}$, and therefore the generating function is

$$q^{\binom{N_i+1}{2}} \begin{bmatrix} N_{i+1} + \delta_{a+1,i+1} \\ N_i \end{bmatrix}.$$

Furthermore, since the strict partition $\pi^{(m)}$ has exactly N_m parts, its generating function is

$$q^{\binom{N_m+1}{2}} \frac{1}{(q; q)_{N_m}}.$$

Thus, (3.1) follows. For (3.2), we calculate the inner summation in (3.1) over N_1 by (2.5). Then from (3.1),

$$\begin{aligned} \sum_{n \geq 0} |\Lambda^{a,m}(n)| q^n &= \sum_{N_m, \dots, N_2 \geq 0} \frac{q^{\sum_{i=2}^m \binom{N_i+1}{2}} (-q; q)_{N_2 + \delta_{a+1,2}}}{(q; q)_{N_m}} \\ &\quad \times \begin{bmatrix} N_3 + \delta_{a+1,3} \\ N_2 \end{bmatrix} \begin{bmatrix} N_4 + \delta_{a+1,4} \\ N_3 \end{bmatrix} \dots \begin{bmatrix} N_m + \delta_{a+1,m} \\ N_{m-1} \end{bmatrix}, \end{aligned}$$

and (3.2) follows by rewriting the q -binomial coefficients using the q -Pochhammer symbols. $\square$

Remark 3.1 Recently, Tsuchioka [39] obtained a result on a multisum formula for the generating function of $|\Lambda^{0,m}(n)|$.

For notational convenience, we reverse the order of $\pi^{(i)}$, i.e., replacing i by $m+1-i$ in (3.1), which also affects the order of t_i 's in $\mathbf{t} = (t_1, \dots, t_m)$. Namely, we have to replace a by $m+1-a$. Then the generating function in Theorem 3.1 becomes

$$\sum_{n \geq 0} |\Lambda^{a,m}(n)| q^n = \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 + \delta_{m-a,1} \\ N_2 \end{bmatrix} \\ \times \begin{bmatrix} N_2 + \delta_{m-a,2} \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} + \delta_{m-a,m-1} \\ N_m \end{bmatrix}.$$

Thus, Theorem 1.2 is equivalent to the following theorem.

Theorem 3.2 For $m \geq 1$ and $0 \leq a \leq m$,

$$\frac{(q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\ = \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 + \delta_{m-a,1} \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 + \delta_{m-a,2} \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} + \delta_{m-a,m-1} \\ N_m \end{bmatrix}. \quad (3.3)$$

Interestingly, we have a piece of evidence from an identity due to Andrews [3, Theorem 3, eq. (1.9) with $a = 2$] in combination with Kim and Yee [29, Theorem 1.4, eq. (1.2) with $a = 1$]:

$$\sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} = \frac{(q, q^{m+1}, q^{m+2}; q^{m+2})_\infty}{(q; q)_\infty (q; q^2)_\infty},$$

which corresponds to the case $a = 0$.

Remark 3.2 For other multisum representations of the infinite products in Theorem 3.2, see for instance, Chen, Sang, and Shi [16, Theorem 1.8, eq. (1.7)] and Sang and Shi [38, Theorem 1.11, eq. (1.4)]. The main combinatorial objects considered in those two papers are overpartitions.

3.1 Necessary theorems for the proof of Theorem 3.2

For the proof of Theorem 3.2, there are three main ingredients:

- (1) Theorem 3.3: A *symmetry property* that allows us to reduce (3.3) to the cases of $a \leq \frac{m}{2} + 1$;
- (2) Theorem 3.4: A *q -binomial multisum transformation formula* that allows us to rewrite the summation side of (3.3) in an alternative way;

- (3) Theorems 3.5 and 3.6: *Two identities of Andrews and Kim–Yee* that build a connection between the product side of (3.3) and the above alternative summation side.

3.1.1 A symmetry property

Notice that the product side of (3.3) has some sort of symmetry in the sense that if we replace a by $m - a$, the product stays invariant. We will show that the same symmetry property also holds true for the summation side.

Theorem 3.3 For $m \geq 1$ and $1 \leq a \leq m - 1$,

$$\begin{aligned} & \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} \begin{bmatrix} N_a+1 \\ N_{a+1} \end{bmatrix} \begin{bmatrix} N_{a+1} \\ N_{a+2} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-a-1} \\ N_{m-a} \end{bmatrix} \\ & \quad \times \begin{bmatrix} N_{m-a}+1 \\ N_{m-a+1} \end{bmatrix} \begin{bmatrix} N_{m-a+1} \\ N_{m-a+2} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix}. \end{aligned} \quad (3.4)$$

The proof of this symmetry property is based on a key lemma, which will be given in Lemma A.4 in “Appendix A” with a detailed proof.

Proof of Theorem 3.3 First, (3.4) is trivial when $a = m - a$. Now, without loss of generality, we assume that $a \geq m - a + 1$. By (2.2),

$$\begin{bmatrix} N_a+1 \\ N_{a+1} \end{bmatrix} = \begin{bmatrix} N_a \\ N_{a+1} \end{bmatrix} + q^{N_a+1-N_{a+1}} \begin{bmatrix} N_a \\ N_{a+1}-1 \end{bmatrix}.$$

Hence,

LHS(3.4)

$$\begin{aligned} &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} \begin{bmatrix} N_a+1 \\ N_{a+1} \end{bmatrix} \begin{bmatrix} N_{a+1} \\ N_{a+2} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\binom{N_1+1}{2} + \cdots + \binom{N_m+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ & \quad + \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} + (N_a+1)}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_a \\ N_{a+1} \end{bmatrix} \end{aligned}$$

$$\times \begin{bmatrix} N_{a+1} + 1 \\ N_{a+2} \end{bmatrix} \begin{bmatrix} N_{a+2} \\ N_{a+3} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix},$$

where N_{a+1} is replaced by $N_{a+1} + 1$ to get the second sum in the last line. If we apply (2.2) to $\begin{bmatrix} N_{a+1}+1 \\ N_{a+2} \end{bmatrix}$ in the second sum above, then

LHS(3.4)

$$\begin{aligned} &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\binom{N_1+1}{2} + \cdots + \binom{N_m+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &+ \sum_{N_1, \dots, N_m} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} + (N_a+1)}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &+ \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} + (N_a+1) + (N_{a+1}+1)}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a+1} \\ N_{a+2} \end{bmatrix} \\ &\quad \times \begin{bmatrix} N_{a+2} + 1 \\ N_{a+3} \end{bmatrix} \begin{bmatrix} N_{a+3} \\ N_{a+4} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix}. \end{aligned}$$

Upon repeating this process, we are eventually led to

LHS(3.4)

$$\begin{aligned} &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &+ \sum_{j=a}^{m-1} \left(\sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} + \sum_{i=a}^j (N_i+1)}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \right). \end{aligned}$$

Replacing a with $m - a$ also gives

$$\text{RHS(3.4)} = \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix}$$

$$+ \sum_{j=m-a}^{m-1} \left(\sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} + \sum_{i=m-a}^j (N_i+1)}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \right).$$

In Lemma A.4 (ii), taking $(m, a, i) \mapsto (1, a-1, m-a)$, choosing $\underline{k} = ((1)_m)$, and fixing $F(N) \equiv 1$ for all $N \geq 0$, we shall see that LHS(3.4) = RHS(3.4).

3.1.2 A q -binomial multisum transformation formula

For $1 \leq a \leq m-1$, exactly one q -binomial coefficient on the summation side of (3.10) is of the form $\begin{bmatrix} N_i+1 \\ N_{i+1} \end{bmatrix}$. Our next result allows us to get rid of the annoying “plus one” in $N_i + 1$.

Theorem 3.4 *Let $\{F(N)\}_{N \geq 0}$ be a family of series in q . For $a \geq 1$,*

$$\begin{aligned} & \sum_{N_1, \dots, N_{2a} \geq 0} \frac{q^{\sum_{i=1}^{2a} \binom{N_i+1}{2}} F(N_{2a})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} \begin{bmatrix} N_a+1 \\ N_{a+1} \end{bmatrix} \begin{bmatrix} N_{a+1} \\ N_{a+2} \end{bmatrix} \cdots \begin{bmatrix} N_{2a-1} \\ N_{2a} \end{bmatrix} \\ &= \sum_{N_1, \dots, N_{2a} \geq 0} \frac{q^{\sum_{i=1}^{2a} \binom{N_i+1}{2} - \sum_{i=1}^a N_{2i}} F(N_{2a})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{2a-1} \\ N_{2a} \end{bmatrix}. \end{aligned} \quad (3.5)$$

We still use Lemma A.4 to establish Theorem 3.4. However, the proof of this theorem is far too lengthy, so it will be given in “Appendix A.”

3.1.3 Identities of Andrews and Kim–Yee

The product side of (3.7) in connection to a certain multisum has already appeared in the works of Andrews [3], and Kim and Yee [29]. In particular, Andrews considered the m even case, while Kim and Yee studied the m odd case. We record an equivalent form of their results.

Theorem 3.5 (Andrews) *For $m \geq 1$ and $a \geq 0$ with $m \geq a$,*

$$\begin{aligned} & \frac{(q^{a+1}, q^{2m+1-a}, q^{2m+2}; q^{2m+2})_{\infty}}{(q; q)_{\infty} (q; q^2)_{\infty}} \\ &= \sum_{N_1, \dots, N_{2m} \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} - \sum_{i=1}^a N_{2i}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{2m-1} \\ N_{2m} \end{bmatrix}. \end{aligned} \quad (3.6)$$

Proof Let $N_i = n_i + n_{i+1} + \cdots + n_{k-1}$. Recall that [3, Theorem 3, eq. (1.9)] tells us that for $k, b \geq 2$ and $k \geq b-1$ with k odd and b even,

$$\frac{(-q^2; q^2)_{\infty} (q^b, q^{2k+2-b}, q^{2k+2}; q^{2k+2})_{\infty}}{(q^2; q^2)_{\infty}}$$

$$= \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{\sum_{i=1}^{k-1} N_i^2 + n_1 + n_3 + \dots + n_{b-3} + N_{b-1} + N_b + \dots + N_{k-1}}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} \dots (q^2; q^2)_{n_{k-1}}}. \quad (3.7)$$

Although Andrews only stated this result for $k \geq b$ in [3, Theorem 3], the case of $b = k + 1$ had also been shown by him by making use of [3, eq. (4.28) with $A = 2k + 2$], and then recalling [3, eqs. (4.5) and (4.27) for $x = 1$] with [3, eq. (2.8)] invoked.

Let us choose $(b, k) \mapsto (2a + 2, 2m + 1)$. Since $k \geq b - 1$, we have $m \geq a$. Now we replace q^2 with q . Noticing that $n_i = N_i - N_{i+1}$ for $1 \leq i \leq 2m - 1$ and that $n_{2m} = N_{2m}$, we have

$$\begin{aligned} & \frac{(-q; q)_\infty (q^{a+1}, q^{2m+1-a}, q^{2m+2}, q^{2m+2})_\infty}{(q; q)_\infty} \\ &= \sum_{n_1, \dots, n_{2m} \geq 0} \frac{q^{\sum_{i=1}^{2m} \frac{N_i^2}{2} + \frac{1}{2}(n_1 + n_3 + \dots + n_{2a-1} + N_{2a+1} + N_{2a+2} + \dots + N_{2m})}}{(q; q)_{n_1} (q; q)_{n_2} \dots (q; q)_{n_{2m}}} \\ &= \sum_{n_1, \dots, n_{2m} \geq 0} \frac{q^{\sum_{i=1}^{2m} \frac{N_i^2}{2} + \frac{1}{2}(N_1 - N_2 + N_3 - N_4 + \dots + N_{2a-1} - N_{2a} + N_{2a+1} + N_{2a+2} + \dots + N_{2m})}}{(q; q)_{N_1 - N_2} (q; q)_{N_2 - N_3} \dots (q; q)_{N_{2m}}} \\ &= \sum_{N_1, \dots, N_{2m} \geq 0} \frac{q^{\binom{N_1+1}{2} + \binom{N_2}{2} + \dots + \binom{N_{2a-1}+1}{2} + \binom{N_{2a}}{2} + \binom{N_{2a+1}+1}{2} + \dots + \binom{N_{2m}+1}{2}}{(q; q)_{N_1}} \\ & \quad \times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \dots \begin{bmatrix} N_{2m-1} \\ N_{2m} \end{bmatrix}, \end{aligned}$$

thereby confirming the desired result. $\square$

Theorem 3.6 (Kim–Yee) *For $m \geq 1$ and $a \geq 0$ with $m \geq a + 1$,*

$$\begin{aligned} & \frac{(q^{a+1}, q^{2m-a}, q^{2m+1}, q^{2m+1})_\infty}{(q; q)_\infty (q; q^2)_\infty} \\ &= \sum_{N_1, \dots, N_{2m-1} \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} - \sum_{i=1}^a N_{2i}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \dots \begin{bmatrix} N_{2m-2} \\ N_{2m-1} \end{bmatrix}. \quad (3.8) \end{aligned}$$

Proof Let $N_i = n_i + n_{i+1} + \dots + n_{k-1}$. We know from [29, Theorem 1.4, eq. (1.2)] that for $k \geq b \geq 1$ with k even and b odd,

$$\begin{aligned} & \frac{(-q^2; q^2)_\infty (q^{b+1}, q^{2k+1-b}, q^{2k+2}, q^{2k+2})_\infty}{(q^2; q^2)_\infty} \\ &= \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{\sum_{i=1}^{k-1} N_i^2 + n_1 + n_3 + \dots + n_{b-2} + N_b + N_{b+1} + \dots + N_{k-1}}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} \dots (q^2; q^2)_{n_{k-1}}}. \quad (3.9) \end{aligned}$$

Choosing $(b, k) \mapsto (2a + 1, 2m)$ and replacing q^2 with q , the desired result follows. $\square$

3.2 Proof of Theorem 3.2

For notational convenience, we map $a \mapsto m - a$ in (3.3) and shall prove the following equivalent identity.

Theorem 3.7 For $m \geq 1$ and $0 \leq a \leq m$,

$$\begin{aligned} & \frac{(q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2})_{\infty}}{(q; q)_{\infty}(q; q^2)_{\infty}} \\ &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 + \delta_{a,1} \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 + \delta_{a,2} \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} + \delta_{a,m-1} \\ N_m \end{bmatrix}. \end{aligned} \quad (3.10)$$

Proof We first observe that the $a = 0$ or m cases of (3.10) are exactly (3.6) and (3.8) with $a = 0$. Further, by the symmetry property in Theorem 3.3, it suffices to show (3.10) for $1 \leq a \leq \frac{m}{2}$. Now in Theorem 3.4, we choose

$$F(N_{2a}) \mapsto \sum_{N_{2a+1}, \dots, N_m \geq 0} q^{\binom{N_{2a+1}+1}{2} + \dots + \binom{N_m+1}{2}} \begin{bmatrix} N_{2a} \\ N_{2a+1} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix}.$$

Then,

$$\begin{aligned} \text{RHS}(3.10) &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} \begin{bmatrix} N_a + 1 \\ N_{a+1} \end{bmatrix} \begin{bmatrix} N_{a+1} \\ N_{a+2} \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &= \sum_{N_1, \dots, N_m \geq 0} \frac{q^{\sum_{i=1}^m \binom{N_i+1}{2} - \sum_{i=1}^a N_{2i}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{m-1} \\ N_m \end{bmatrix} \\ &= \frac{(q^{a+1}, q^{m+1-a}, q^{m+2}; q^{m+2})_{\infty}}{(q; q)_{\infty}(q; q^2)_{\infty}}, \end{aligned}$$

where the second equality follows from (3.5) and we make use of (3.6) and (3.8) for the last equality. $\square$

4 2-Kleshchev multipartitions

Recall the 2-residue difference statistic $\omega(\pi)$ of a Kleshchev multipartition π from Definition 2.4. In this section, we will investigate how the 2-residues behave for partition pairs in $\Lambda^{a,2}$ with $a = 1$ and 2 , respectively. We will first consider the $a = 2$ case and then the $a = 1$ case. However, to bring out a delicate account of the behavior

of the residues, we need a few more definitions and results from [11], which will be discussed in the next subsection.

4.1 Odd- and even-indexed parts

For a partition λ , we call $\lambda_1, \lambda_3, \lambda_5, \dots$ *odd-indexed parts*, and $\lambda_2, \lambda_4, \lambda_6, \dots$ *even-indexed parts*. Let $d_N(i, j, n)$ count the number of partitions of n into distinct parts less than or equal to N with i odd-indexed odd parts and j even-indexed odd parts. For instance, the following partition is counted by $d_{10}(2, 3, 43)$:

$$(10, 9, 7, 6, 5, 3, 2, 1).$$

For $k \in \mathbb{Z}$, define

$$B_N(k, q) := \sum_{j \geq 0} \sum_{n \geq 0} d_N(j + k, j, n) q^n.$$

Theorem 4.1 ([11], Theorem 4.1) *For $v = 0, 1$,*

$$B_{2N+v}(k, q) = q^{2k^2-k} \begin{bmatrix} 2N+v \\ N+k \end{bmatrix}_{q^2}.$$

Throughout, we write

$$f_N(x, q) := \sum_{k=-\infty}^{\infty} B_N(k, q) x^k.$$

Furthermore, for a partition λ , let

$$|\lambda|_a := \sum_{i \geq 1} (-1)^{i-1} \lambda_i,$$

which is called the *alternating sum* of λ . We also write

$$g_N(x, q) := \sum_{\lambda} x^{|\lambda|_a} q^{|\lambda|},$$

where the sum is over all partitions λ with parts less than or equal to N . We have the following nice formula for $g_N(x, q)$.

Theorem 4.2 ([11, Theorem 5.2] or [27]) *For $v = 0, 1$,*

$$g_{2N+v}(x, q) = \frac{1}{(q^2; q^2)_{2N+v}} \sum_{i=0}^N \begin{bmatrix} N \\ i \end{bmatrix}_{q^4} x^{2i} q^{2i} (-xq; q^4)_{N-i+v} (-x^{-1}q; q^4)_i$$

$$= \frac{1}{(q^2; q^2)_{2N+v}} \sum_{j=0}^{2N+v} x^j q^j \begin{bmatrix} 2N+v \\ j \end{bmatrix}_{q^2}.$$

Proof The last equality follows from the identities for the Rogers–Szegő polynomial $H_n(t, q)$:

$$H_n(t, q) = \sum_{j=0}^n t^j \begin{bmatrix} n \\ j \end{bmatrix} = \sum_{r=0}^{\lfloor n/2 \rfloor} t^{2r} (-q/t; q^2)_r (-t; q^2)_{\lfloor n/2 \rfloor - r} \begin{bmatrix} \lfloor n/2 \rfloor \\ r \end{bmatrix}_{q^2}.$$

See [12, eq. (8.3)]. □

4.2 The $a = 2$ case

For a strict partition π , let

$$\begin{aligned} \delta(\pi) &:= (\ell(\pi), \ell(\pi) - 1, \dots, 1), \\ \sigma(\pi) &:= (\pi_1 - \ell(\pi), \pi_2 - \ell(\pi) + 1, \dots, \pi_{\ell(\pi)} - 1)^\top. \end{aligned}$$

Then, it is clear that

$$\pi = \delta(\pi) + \sigma(\pi)^\top.$$

By the definition of residues with $t_1 = t_2 = 0$, we see that

$$\omega(\pi) = (-1)^{\ell(\pi)-1} \left(\left\lceil \frac{\ell(\pi)}{2} \right\rceil - |\sigma(\pi)|_a \right).$$

Example 4.1 We consider $\pi = (9, 7, 6, 3)$. Then

$$\begin{aligned} \delta(\pi) &= (4, 3, 2, 1), \\ \sigma(\pi) &= (5, 4, 4, 2)^\top = (4, 4, 3, 3, 1), \end{aligned}$$

and

$$\begin{aligned} \omega(\pi) &= 12 - 13 = -1 = (-1)^{4-1} (2 - (4 - 4 + 3 - 3 + 1)) \\ &= (-1)^{\ell(\pi)-1} \left(\left\lceil \frac{\ell(\pi)}{2} \right\rceil - |\sigma(\pi)|_a \right). \end{aligned}$$

In Fig. 3, the shifted Young diagram of π with residues is given. The thick vertical line separates $\delta(\pi)$ and $\sigma(\pi)^\top$.

Fig. 3 The strict partition $\pi = (9, 7, 6, 3)$

0	1	0	1	0	1	0	1	0
	1	0	1	0	1	0	1	
		0	1	0	1	0	1	
			1	0	1			

For $\pi = (\pi^{(1)}, \pi^{(2)}) \in \Lambda^{2,2}$, we apply the above analysis to $\pi^{(2)}$ and obtain that

$$\omega(\pi) = \omega(\pi^{(1)}) + \omega(\pi^{(2)}) = \omega(\pi^{(1)}) + (-1)^{\ell(\pi^{(2)})-1} \left(\left\lceil \frac{\ell(\pi^{(2)})}{2} \right\rceil - |\sigma(\pi^{(2)})|_a \right).$$

Hence,

$$\begin{aligned} G_2(x, q) &:= \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{2,2}(n)| x^{\omega} q^n \\ &= \sum_{(\pi^{(1)}, \pi^{(2)}) \in \Lambda^{2,2}} x^{\omega(\pi^{(1)}) + \omega(\pi^{(2)})} q^{|\pi^{(1)}| + |\pi^{(2)}|} \\ &= \sum_{n \geq 0} \sum_{(\sigma, \nu)} x^{(-1)^{n-1} \left(\lceil n/2 \rceil - |\sigma|_a \right) + \omega(\nu)} q^{n(n+1)/2 + |\sigma| + |\nu|}, \end{aligned} \quad (4.1)$$

where the inner sum in the last line runs over all (σ, ν) with σ an ordinary partition into parts $\leq n$ and ν a strict partition into parts $\leq n$. It turns out that this generating function $G_2(x, q)$ can be expressed in the following way.

Theorem 4.3 *We have*

$$\begin{aligned} &\sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{2,2}(n)| x^{\omega} q^n \\ &= \sum_{n \geq 0} \frac{x^{-n} q^{n(2n+1)}}{(q^2; q^2)_{2n}} \sum_{j=0}^{2n} x^j q^j \begin{bmatrix} 2n \\ j \end{bmatrix}_{q^2} \sum_{k=-n}^n x^k q^{2k^2-k} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} \\ &\quad + \sum_{n \geq 0} \frac{x^{n+1} q^{(n+1)(2n+1)}}{(q^2; q^2)_{2n+1}} \sum_{j=0}^{2n+1} x^{-j} q^j \begin{bmatrix} 2n+1 \\ j \end{bmatrix}_{q^2} \sum_{k=-n}^{n+1} x^k q^{2k^2-k} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^2}. \end{aligned} \quad (4.2)$$

Proof Note that by (4.1),

$$\begin{aligned} \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{2,2}(n)| x^{\omega} q^n &= \sum_{n \geq 0} x^{-n} q^{n(2n+1)} g_{2n}(x, q) f_{2n}(x, q) \\ &\quad + \sum_{n \geq 0} x^{n+1} q^{(n+1)(2n+1)} g_{2n+1}(x^{-1}, q) f_{2n+1}(x, q). \end{aligned}$$

Making use of Theorems 4.1 and 4.2 gives the desired result. $\square$

We compute the coefficient of x^{ω} in $G_2(x, q)$:

$$\begin{aligned} [x^{\omega}]G_2(x, q) &= \sum_{n \geq 0} \sum_{k=-\infty}^{\infty} \frac{q^{2(n^2+n+k^2-k)+\omega}}{(q^2; q^2)_{2n}} \begin{bmatrix} 2n \\ n-k+\omega \end{bmatrix}_{q^2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} \\ &\quad + \sum_{n \geq 0} \sum_{k=-\infty}^{\infty} \frac{q^{2((n+1)^2+k^2)-\omega}}{(q^2; q^2)_{2n+1}} \begin{bmatrix} 2n+1 \\ n-k+\omega \end{bmatrix}_{q^2} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^2}. \end{aligned} \quad (4.3)$$

4.3 The $a = 1$ case

For $\pi = (\pi^{(1)}, \pi^{(2)}) \in \Lambda^{1,2}$, since $\mathbf{t} = (t_1, t_2) = (0, 1)$, the analysis on the residue of $\pi^{(1)}$ from the previous section on $\Lambda^{2,2}$ is still valid. However, the residue of each node of $\pi^{(2)}$ has changed by 1 mod 2 due to $t_2 = 1$. Thus, for the weighted generating function for $\pi^{(2)}$, we have to replace the variable x by x^{-1} . It follows that

$$\begin{aligned} G_1(x, q) &:= \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{1,2}(n)| x^{\omega} q^n \\ &= \sum_{(\pi^{(1)}, \pi^{(2)}) \in \Lambda^{1,2}} x^{\omega(\pi^{(1)}) + \omega(\pi^{(2)})} q^{|\pi^{(1)}| + |\pi^{(2)}|} \\ &= \sum_{n \geq 0} \sum_{(\sigma, \nu)} x^{(-1)^n (\lceil n/2 \rceil - |\sigma|_a) + \omega(\nu)} q^{n(n+1)/2 + |\sigma| + |\nu|}, \end{aligned} \quad (4.4)$$

where the inner sum in the last line runs over all (σ, ν) with σ an ordinary partition into parts $\leq n$ and ν a strict partition into parts $\leq n+1$. Thus, this generating function $G_1(x, q)$ can be expressed as follows.

Theorem 4.4 *We have*

$$\begin{aligned} \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{1,2}(n)| x^{\omega} q^n \\ = \sum_{n \geq 0} \frac{x^n q^{n(2n+1)}}{(q^2; q^2)_{2n}} \sum_{j=0}^{2n} x^{-j} q^j \begin{bmatrix} 2n \\ j \end{bmatrix}_{q^2} \sum_{k=-n}^{n+1} x^k q^{2k^2-k} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^2} \end{aligned}$$

$$+ \sum_{n \geq 0} \frac{x^{-(n+1)} q^{(n+1)(2n+1)}}{(q^2; q^2)_{2n+1}} \sum_{j=0}^{2n+1} x^j q^j \begin{bmatrix} 2n+1 \\ j \end{bmatrix}_{q^2} \sum_{k=-n-1}^{n+1} x^k q^{2k^2-k} \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_{q^2}. \quad (4.5)$$

Proof Note that by (4.4),

$$\begin{aligned} \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{1,2}(n)| x^{\omega} q^n &= \sum_{n \geq 0} x^n q^{n(2n+1)} g_{2n}(x^{-1}, q) f_{2n+1}(x, q) \\ &\quad + \sum_{n \geq 0} x^{-(n+1)} q^{(n+1)(2n+1)} g_{2n+1}(x, q) f_{2n+2}(x, q). \end{aligned}$$

Making use of Theorems 4.1 and 4.2 gives the desired result. $\square$

We now compute the coefficient of x^{ω} in $G_1(x, q)$:

$$\begin{aligned} [x^{\omega}] G_1(x, q) &= \sum_{n \geq 0} \sum_{k=-\infty}^{\infty} \frac{q^{2(n^2+n+k^2)-\omega}}{(q^2; q^2)_{2n}} \begin{bmatrix} 2n \\ n+k-\omega \end{bmatrix}_{q^2} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^2} \\ &\quad + \sum_{n \geq 0} \sum_{k=-\infty}^{\infty} \frac{q^{2((n+1)^2+k^2-k)+\omega}}{(q^2; q^2)_{2n+1}} \begin{bmatrix} 2n+1 \\ n+k-\omega \end{bmatrix}_{q^2} \begin{bmatrix} 2n+2 \\ n+1+k \end{bmatrix}_{q^2}. \end{aligned} \quad (4.6)$$

5 The bivariate generating functions for $\Lambda^{1,2}$ and $\Lambda^{2,2}$

In Theorems 4.3 and 4.4, we have expressed the bivariate generating functions for $\Lambda^{1,2}$ and $\Lambda^{2,2}$ as triple q -summations. In this section, we will simplify these triple summations and hence provide a proof of Theorem 1.3. In particular, our proof relies on some basic hypergeometric series manipulations of the coefficients in (4.3) and (4.6).

5.1 Proof of (1.3)

Recall that

$$G_1(x) := G_1(x, q) = \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{1,2}(n)| x^{\omega} q^n.$$

In (4.6), letting $r = n + k$ and $s = n - k$ in the first double summation and letting $r = n - k + 1$ and $s = n + k$ in the second double summation, we have

$$[x^{\omega}] G_1(x) = \sum_{\substack{r, s=-\infty \\ r \equiv s \pmod{2}}}^{\infty} \frac{q^{r^2+r+s^2+s-\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s+1 \\ r \end{bmatrix}_{q^2}$$

$$+ \sum_{\substack{r,s=-\infty \\ r \not\equiv s \pmod{2}}}^{\infty} \frac{q^{r^2+r+s^2+s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r+\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s+1 \\ r \end{bmatrix}_{q^2}. \quad (5.1)$$

Now we first deduce from (5.1) that

$$\begin{aligned} G_1(x) + G_1(x^{-1}) \\ = \sum_{\omega=-\infty}^{\infty} (x^{\omega} + x^{-\omega}) \sum_{r,s=-\infty}^{\infty} \frac{q^{r^2+r+s^2+s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r+\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s+1 \\ r \end{bmatrix}_{q^2}. \end{aligned} \quad (5.2)$$

Throughout, we introduce a family of auxiliary series

$$H_{\omega}^{+} = H_{\omega}^{+}(q) := \sum_{r,s=-\infty}^{\infty} \frac{q^{r^2+r+s^2+s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r+\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s+1 \\ r \end{bmatrix}_{q^2}. \quad (5.3)$$

Then,

$$\begin{aligned} H_{\omega}^{+} &= \sum_{s=-\infty}^{\infty} \frac{q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (q^2; q^2)_{s+1}} \sum_{r \geq 0} \frac{q^{r^2+r} (q^2; q^2)_{r+s+1}}{(q^2; q^2)_{r+\omega} (q^2; q^2)_r} \\ &= \lim_{u \rightarrow 1} \sum_{s=-\infty}^{\infty} \frac{q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (uq^2; q^2)_{\omega}} \sum_{r \geq 0} q^{2 \binom{r}{2} + 2r} \frac{(q^{4+2s}; q^2)_r}{(uq^{2+2\omega}; q^2)_r (q^2; q^2)_r} \\ &= \lim_{u \rightarrow 1} \sum_{s=-\infty}^{\infty} \frac{q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (uq^2; q^2)_{\omega}} \\ &\quad \times \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^{4+2s}, 1/\tau \\ uq^{2+2\omega} \end{matrix}; q^2, -q^2\tau \right) \\ &= \lim_{u \rightarrow 1} \sum_{s=-\infty}^{\infty} \frac{q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (uq^2; q^2)_{\omega}} \lim_{\tau \rightarrow 0} \frac{(uq^{2+2\omega}\tau, -q^2; q^2)_{\infty}}{(uq^{2+2\omega}, -q^2\tau; q^2)_{\infty}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} -q^{4+2s-2\omega}/u, 1/\tau \\ -q^2 \end{matrix}; q^2, uq^{2+2\omega}\tau \right) \\ &= \sum_{s=-\infty}^{\infty} \frac{q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega}} \lim_{\tau \rightarrow 0} \frac{(q^{2+2\omega}\tau, -q^2; q^2)_{\infty}}{(q^2, -q^2\tau; q^2)_{\infty}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} -q^{4+2s-2\omega}, 1/\tau \\ -q^2 \end{matrix}; q^2, q^{2+2\omega}\tau \right) \\ &= \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{r,s=-\infty}^{\infty} \frac{q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega}} \frac{(-1)^r q^{r^2+r+2r\omega} (-q^{4+2s-2\omega}; q^2)_r}{(q^4; q^4)_r}, \end{aligned}$$

where we have applied (2.9) to get the ${}_2\phi_1$ series in the fourth line. Notice that we add the limit $u \rightarrow 1$ to make our computation more rigorous by avoiding the zero denominator in $(q^{2+2\omega}; q^2)_r$ when $\omega < 0$. However, since this step is more or less straightforward, it will be omitted in our subsequent calculations. Now, letting $s \mapsto s + \omega - 1$ in the above yields

$$\begin{aligned} H_\omega^+ &= \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{r,s \geq 0} \frac{(-1)^r q^{r^2+r+s^2-s+2r\omega+2s\omega} (1-q^{2s})(-q^2; q^2)_{r+s}}{(q^4; q^4)_r (q^4; q^4)_s} \\ &= \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{M \geq 1} \frac{q^{2M\omega}}{(q^2; q^2)_M} \sum_{\substack{r,s \geq 0 \\ r+s=M}} (-1)^r q^{r^2+r+s^2-s} (1-q^{2s}) \begin{bmatrix} M \\ r \end{bmatrix}_{q^4}. \end{aligned}$$

Notice that for $M \geq 1$,

$$\begin{aligned} &\sum_{\substack{r,s \geq 0 \\ r+s=M}} (-1)^r q^{r^2+r+s^2-s} (1-q^{2s}) \begin{bmatrix} M \\ r \end{bmatrix}_{q^4} \\ &= q^{M^2-M} \sum_{r \geq 0} (-1)^r q^{4\binom{r}{2}+(4-2M)r} \begin{bmatrix} M \\ r \end{bmatrix}_{q^4} \\ &\quad - q^{M^2+M} \sum_{r \geq 0} (-1)^r q^{4\binom{r}{2}+(2-2M)r} \begin{bmatrix} M \\ r \end{bmatrix}_{q^4} \\ &(\text{by (2.5)}) = q^{M^2-M} (q^{4-2M}; q^4)_M - q^{M^2+M} (q^{2-2M}; q^4)_M \\ &= \begin{cases} -(-1)^N q^{2N^2+2N} (q^2; q^4)_N^2, & \text{if } M = 2N, \\ (-1)^N q^{2N^2+2N} (q^2; q^4)_N (q^2; q^4)_{N+1}, & \text{if } M = 2N + 1. \end{cases} \end{aligned}$$

Thus,

$$H_\omega^+ = \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 - (1 - q^{2\omega}) \sum_{N \geq 0} \frac{(-1)^N q^{2N^2+2N+4N\omega} (q^2; q^4)_N}{(q^4; q^4)_N} \right).$$

Now, we rewrite the summation over N as follows,

$$\begin{aligned} \sum_{N \geq 0} \frac{(-1)^N q^{2N^2+2N+4N\omega} (q^2; q^4)_N}{(q^4; q^4)_N} &= \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^2, 1/\tau \\ \tau \end{matrix}; q^4, q^{4+4\omega}\tau \right) \\ &(\text{by (2.9)}) = \lim_{\tau \rightarrow 0} \frac{(\tau^2, q^{4+4\omega}; q^4)_\infty}{(\tau, q^{4+4\omega}\tau; q^4)_\infty} {}_2\phi_1 \left(\begin{matrix} q^{6+4\omega}/\tau, 1/\tau \\ q^{4+4\omega} \end{matrix}; q^4, \tau^2 \right) \\ &= (q^{4+4\omega}; q^4)_\infty \sum_{N=-\infty}^{\infty} \frac{q^{4N^2+2N+4N\omega}}{(q^4; q^4)_N (q^{4+4\omega}; q^4)_N} \end{aligned}$$

$$= (q^4; q^4)_\infty \sum_{N=-\infty}^{\infty} \frac{q^{4N^2+2N+4N\omega}}{(q^4; q^4)_N (q^4; q^4)_{N+\omega}}.$$

Thus,

$$H_\omega^+ = \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 - (q^4; q^4)_\infty (1 - q^{2\omega}) \sum_{N=-\infty}^{\infty} \frac{q^{4N^2+2N+4N\omega}}{(q^4; q^4)_N (q^4; q^4)_{N+\omega}} \right).$$

Also,

$$\begin{aligned} H_{-\omega}^+ &= \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 - (q^4; q^4)_\infty (1 - q^{-2\omega}) \sum_{N=-\infty}^{\infty} \frac{q^{4N^2+2N-4N\omega}}{(q^4; q^4)_N (q^4; q^4)_{N-\omega}} \right) \\ &= \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 - (q^4; q^4)_\infty (1 - q^{-2\omega}) \sum_{N=-\infty}^{\infty} \frac{q^{4(N+\omega)^2+2(N+\omega)-4(N+\omega)\omega}}{(q^4; q^4)_{N+\omega} (q^4; q^4)_N} \right) \\ &= \frac{q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \left(1 - (q^4; q^4)_\infty (q^{2\omega} - 1) \sum_{N=-\infty}^{\infty} \frac{q^{4N^2+2N+4N\omega}}{(q^4; q^4)_N (q^4; q^4)_{N+\omega}} \right). \end{aligned}$$

We conclude that for any integer ω ,

$$H_\omega^+ + H_{-\omega}^+ = \frac{2q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty}. \quad (5.4)$$

Therefore,

$$\begin{aligned} G_1(x) + G_1(x^{-1}) &= \sum_{\omega=-\infty}^{\infty} (x^\omega + x^{-\omega}) H_\omega^+ = \sum_{\omega=-\infty}^{\infty} x^\omega (H_\omega^+ + H_{-\omega}^+) \\ &\stackrel{\text{(by (5.4))}}{=} \sum_{\omega=-\infty}^{\infty} \frac{2x^\omega q^{\omega^2}(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \\ &\stackrel{\text{(by (2.7))}}{=} \frac{2(-q^2; q^2)_\infty (-xq, -x^{-1}q, q^2; q^2)_\infty}{(q^2; q^2)_\infty}. \end{aligned} \quad (5.5)$$

Next, we may also use (5.1) to get

$$\begin{aligned} G_1(x) - G_1(x^{-1}) &= - \sum_{\omega=-\infty}^{\infty} (x^\omega - x^{-\omega}) \sum_{r,s=-\infty}^{\infty} \frac{(-1)^{r+s} q^{r^2+r+s^2+s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r+\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s+1 \\ r \end{bmatrix}_{q^2}. \end{aligned} \quad (5.6)$$

Now, we define

$$H_{\omega}^{-} = H_{\omega}^{-}(q) := \sum_{r,s=-\infty}^{\infty} \frac{(-1)^{r+s} q^{r^2+r+s^2+s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r+\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s+1 \\ r \end{bmatrix}_{q^2}. \quad (5.7)$$

Then,

$$\begin{aligned} H_{\omega}^{-} &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (q^2; q^2)_{s+1}} \sum_{r \geq 0} \frac{(-1)^r q^{r^2+r} (q^2; q^2)_{r+s+1}}{(q^2; q^2)_{r+\omega} (q^2; q^2)_r} \\ &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (q^2; q^2)_{\omega}} \sum_{r \geq 0} \frac{(-1)^r q^{\binom{r}{2}+2r} (q^{4+2s}; q^2)_r}{(q^{2+2\omega}; q^2)_r (q^2; q^2)_r} \\ &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (q^2; q^2)_{\omega}} \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^{4+2s}, 1/\tau \\ q^{2+2\omega} \end{matrix}; q^2, q^2 \tau \right) \\ (\text{by (2.9)}) &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+s+\omega}}{(q^2; q^2)_{s-\omega} (q^2; q^2)_{\omega}} \lim_{\tau \rightarrow 0} \frac{(q^{2+2\omega} \tau, q^2; q^2)_{\infty}}{(q^{2+2\omega}, q^2 \tau; q^2)_{\infty}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} q^{4+2s-2\omega}, 1/\tau \\ q^2 \end{matrix}; q^2, q^{2+2\omega} \tau \right) \\ &= \sum_{r,s=-\infty}^{\infty} \frac{(-1)^{r+s} q^{r^2+r+s^2+s+\omega}}{(q^2; q^2)_{s-\omega}} \frac{q^{r^2+r+2r\omega} (q^{4+2s-2\omega}; q^2)_r}{(q^2; q^2)_r^2} \\ (s \mapsto s + \omega - 1) &= -(-1)^{\omega} q^{\omega^2} \sum_{r,s \geq 0} \frac{(-1)^{r+s} q^{r^2+r+s^2-s+2r\omega+2s\omega} (1 - q^{2s}) (q^2; q^2)_{r+s}}{(q^2; q^2)_r^2 (q^2; q^2)_s^2}. \end{aligned}$$

We also have

$$\begin{aligned} H_{-\omega}^{-} &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2+r-\omega}}{(q^2; q^2)_{r-\omega} (q^2; q^2)_r} \sum_{s \geq -1} \frac{(-1)^s q^{s^2+s} (q^2; q^2)_{r+s+1}}{(q^2; q^2)_{s+\omega} (q^2; q^2)_{s+1}} \\ &= - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2+r-\omega}}{(q^2; q^2)_{r-\omega} (q^2; q^2)_r} \sum_{s \geq 0} \frac{(-1)^s q^{s^2-s} (q^2; q^2)_{r+s}}{(q^2; q^2)_{s+\omega-1} (q^2; q^2)_s} \\ &= - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2+r-\omega}}{(q^2; q^2)_{r-\omega} (q^2; q^2)_{\omega-1}} \lim_{u \rightarrow 1} \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^{2+2r}, u/\tau \\ q^{2\omega} \end{matrix}; q^2, \tau \right) \\ (\text{by (2.9)}) &= - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2+r-\omega}}{(q^2; q^2)_{r-\omega} (q^2; q^2)_{\omega-1}} \lim_{u \rightarrow 1} \lim_{\tau \rightarrow 0} \frac{(q^{2\omega} \tau, u; q^2)_{\infty}}{(q^{2\omega}, \tau; q^2)_{\infty}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} u q^{2+2r-2\omega}, \frac{u}{\tau} \\ u \end{matrix}; q^2, \frac{q^{2\omega} \tau}{u} \right) \end{aligned}$$

$$\begin{aligned}
&= - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2+r-\omega}}{(q^2; q^2)_{r-\omega} (q^2; q^2)_{\omega-1}} \lim_{u \rightarrow 1} \frac{(u; q^2)_{\infty}}{(q^{2\omega}; q^2)_{\infty}} \\
&\quad \times \sum_{s \geq 0} \frac{(-1)^s q^{s^2-s+2s\omega} (uq^{2+2r-2\omega}; q^2)_s}{(q^2; q^2)_s (u; q^2)_s} \\
&= - \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2+r-\omega}}{(q^2; q^2)_{r-\omega}} \sum_{s \geq 0} \frac{(-1)^s q^{s^2-s+2s\omega} (1-q^{2s}) (q^{2+2r-2\omega}; q^2)_s}{(q^2; q^2)_s (q^2; q^2)_s} \\
(r \mapsto r + \omega) &= -(-1)^{\omega} q^{\omega^2} \sum_{r,s \geq 0} \frac{(-1)^{r+s} q^{r^2+r+s^2-s+2r\omega+2s\omega} (1-q^{2s}) (q^2; q^2)_{r+s}}{(q^2; q^2)_r^2 (q^2; q^2)_s^2}.
\end{aligned}$$

Thus, for any integer ω ,

$$H_{\omega}^{-} = H_{-\omega}^{-}. \quad (5.8)$$

We conclude that

$$\begin{aligned}
G_1(x) - G_1(x^{-1}) &= - \sum_{\omega=-\infty}^{\infty} (x^{\omega} - x^{-\omega}) H_{\omega}^{-} \\
&= - \sum_{\omega=-\infty}^{\infty} x^{\omega} (H_{\omega}^{-} - H_{-\omega}^{-}) \\
&\quad \text{(by (5.8))} = 0.
\end{aligned} \quad (5.9)$$

Finally, summing (5.5) and (5.9) yields (1.3).

5.2 Proof of (1.4)

Recall that

$$G_2(x) := G_2(x, q) = \sum_{\omega=-\infty}^{\infty} \sum_{n \geq 0} |\Lambda_{\omega}^{2,2}(n)| x^{\omega} q^n.$$

In (4.3), letting $r = n + k$ and $s = n - k$ in the first double summation and letting $r = n - k + 1$ and $s = n + k$ in the second double summation, we have

$$\begin{aligned}
[x^{\omega}] G_2(x) &= \sum_{\substack{r,s=-\infty \\ r \equiv s \pmod{2}}}^{\infty} \frac{q^{r^2+s^2+2s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^2} \\
&\quad + \sum_{\substack{r,s=-\infty \\ r \not\equiv s \pmod{2}}}^{\infty} \frac{q^{r^2+s^2+2s+1-\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-1+\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^2}.
\end{aligned} \quad (5.10)$$

We start with the difference of $G_2(x)$ and $xG_2(x^{-1})$. It follows from (5.10) that

$$\begin{aligned} G_2(x) - xG_2(x^{-1}) \\ = \sum_{\omega=-\infty}^{\infty} (x^{\omega} - x^{1-\omega}) \sum_{r,s=-\infty}^{\infty} \frac{(-1)^{r+s} q^{r^2+s^2+2s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^2}. \end{aligned} \quad (5.11)$$

We define

$$I_{\omega}^{-} = I_{\omega}^{-}(q) := \sum_{r,s=-\infty}^{\infty} \frac{(-1)^{r+s} q^{r^2+s^2+2s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^2}. \quad (5.12)$$

Then,

$$\begin{aligned} I_{\omega}^{-} &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+2s+\omega}}{(q^2; q^2)_{s+\omega} (q^2; q^2)_s} \sum_{r \geq 0} \frac{(-1)^r q^{r^2} (q^2; q^2)_{r+s}}{(q^2; q^2)_{r-\omega} (q^2; q^2)_r} \\ &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+2s+\omega}}{(q^2; q^2)_{s+\omega} (q^2; q^2)_{-\omega}} \sum_{r \geq 0} \frac{(-1)^r q^{2\binom{r}{2}+r} (q^{2+2s}; q^2)_r}{(q^{2-2\omega}; q^2)_r (q^2; q^2)_r} \\ &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+2s+\omega}}{(q^2; q^2)_{s+\omega} (q^2; q^2)_{-\omega}} \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^{2+2s}, 1/\tau \\ q^{2-2\omega} \end{matrix}; q^2, q\tau \right) \\ (\text{by (2.9)}) &= \sum_{s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+2s+\omega}}{(q^2; q^2)_{s+\omega} (q^2; q^2)_{-\omega}} \lim_{\tau \rightarrow 0} \frac{(q^{2-2\omega}\tau, q; q^2)_{\infty}}{(q^{2-2\omega}, q\tau; q^2)_{\infty}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} q^{1+2s+2\omega}, 1/\tau \\ q \end{matrix}; q^2, q^{2-2\omega}\tau \right) \\ &= \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{r,s=-\infty}^{\infty} \frac{(-1)^s q^{s^2+2s+\omega}}{(q^2; q^2)_{s+\omega}} \frac{(-1)^r q^{r^2+r-2r\omega} (q^{1+2s+2\omega}; q^2)_r}{(q^2; q^2)_r (q; q^2)_r}. \end{aligned}$$

Letting $s \mapsto s - \omega$ in the above yields

$$\begin{aligned} I_{\omega}^{-} &= \frac{(-1)^{\omega} q^{\omega^2-\omega} (q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{r,s \geq 0} \frac{(-1)^{r+s} q^{r^2+r+s^2+2s-2r\omega-2s\omega} (q; q^2)_{r+s}}{(q^2; q^2)_r (q; q^2)_r (q^2; q^2)_s (q^2; q^2)_s} \\ &= \frac{(-1)^{\omega} q^{\omega^2-\omega} (q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} + \frac{(-1)^{\omega} q^{\omega^2-\omega} (q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \\ &\quad \times \sum_{M \geq 1} (-1)^M q^{-2M\omega} (q; q^2)_M S_{1,M}, \end{aligned}$$

where

$$S_{1,M} := \sum_{\substack{r,s \geq 0 \\ r+s=M}} \frac{q^{r^2+r+s^2+2s}}{(q^2; q^2)_r (q; q^2)_r (q^2; q^2)_s (q; q^2)_s}.$$

Similarly,

$$\begin{aligned} I_{1-\omega}^- &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2-\omega+1}}{(q^2; q^2)_{r+\omega-1} (q^2; q^2)_r} \sum_{s \geq 0} \frac{(-1)^s q^{s^2+2s} (q^2; q^2)_{r+s}}{(q^2; q^2)_{s-\omega+1} (q^2; q^2)_s} \\ &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2-\omega+1}}{(q^2; q^2)_{r+\omega-1} (q^2; q^2)_{1-\omega}} \sum_{s \geq 0} \frac{(-1)^s q^{2\binom{s}{2}+3s} (q^{2+2r}; q^2)_s}{(q^{4-2\omega}; q^2)_s (q^2; q^2)_s} \\ &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2-\omega+1}}{(q^2; q^2)_{r+\omega-1} (q^2; q^2)_{1-\omega}} \lim_{\tau \rightarrow 0} {}_2\phi_1 \left(\begin{matrix} q^{2+2r}, 1/\tau \\ q^{4-2\omega} \end{matrix}; q^2, q^3 \tau \right) \\ (\text{by (2.9)}) &= \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r^2-\omega+1}}{(q^2; q^2)_{r+\omega-1} (q^2; q^2)_{1-\omega}} \lim_{\tau \rightarrow 0} \frac{(q^{4-2\omega} \tau, q^3; q^2)_{\infty}}{(q^{4-2\omega}, q^2; q^2)_{\infty}} \\ &\quad \times {}_2\phi_1 \left(\begin{matrix} q^{1+2r+2\omega}, 1/\tau \\ q^3 \end{matrix}; q^2, q^{4-2\omega} \tau \right) \\ &= \frac{(q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{r,s=-\infty}^{\infty} \frac{(-1)^r q^{r^2-\omega+1}}{(q^2; q^2)_{r+\omega-1}} \frac{(-1)^s q^{s^2+3s-2s\omega} (q^{1+2r+2\omega}; q^2)_s}{(q^2; q^2)_s (q^3; q^2)_s} \\ &= \frac{(q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{r,s=-\infty}^{\infty} \frac{(-1)^r q^{r^2-\omega+1}}{(q^2; q^2)_{r+\omega-1}} \frac{(-1)^s q^{s^2+3s-2s\omega} (q^{1+2r+2\omega}; q^2)_s}{(q^2; q^2)_s (q; q^2)_{s+1}}. \end{aligned}$$

Letting $r \mapsto r - \omega + 1$ and $s \mapsto s - 1$ in the above yields

$$\begin{aligned} I_{1-\omega}^- &= \frac{(-1)^{\omega} q^{\omega^2-\omega} (q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \\ &\quad \times \sum_{r,s \geq 0} \frac{(-1)^{r+s} q^{r^2+2r+s^2+s-2r\omega-2s\omega} (q; q^2)_{r+s}}{(q^2; q^2)_r (q; q^2)_{r+1} (q^2; q^2)_{s-1} (q; q^2)_s} \\ &= \frac{(-1)^{\omega} q^{\omega^2-\omega} (q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{M \geq 1} (-1)^M q^{-2M\omega} (q; q^2)_M S_{2,M}, \end{aligned}$$

where

$$S_{2,M} := \sum_{\substack{r,s \geq 0 \\ r+s=M}} \frac{q^{r^2+2r+s^2+s}}{(q^2; q^2)_r (q; q^2)_{r+1} (q^2; q^2)_{s-1} (q; q^2)_s}.$$

We claim that for $M \geq 1$,

$$S_{1,M} = S_{2,M} = \frac{q^{M(M+3)/2}(-q; q)_{M-1}}{(q; q^2)_M(q; q)_M}. \quad (5.13)$$

To see this, we have

$$\begin{aligned} S_{1,M} &= \sum_{\substack{r, s \geq 0 \\ r+s=M}} \frac{q^{r^2+r+s^2+2s}}{(q^2; q^2)_r(q; q^2)_r(q^2; q^2)_s(q; q^2)_s} \\ &= \frac{q^{M^2+2M}}{(q; q^2)_M(q^2; q^2)_M} {}_2\phi_1 \left(\begin{matrix} q^{-2M}, q^{1-2M} \\ q \end{matrix}; q^2, q^{2M} \right) \\ (\text{by (2.10)}) &= \frac{q^{M^2+2M}}{(q; q^2)_M(q^2; q^2)_M} \frac{(-q^{-M}, q^M; q)_\infty}{2(q^{2M}; q^2)_\infty} \\ &= \frac{q^{M(M+3)/2}(-q; q)_{M-1}}{(q; q^2)_M(q; q)_M}, \end{aligned}$$

and

$$\begin{aligned} S_{2,M} &= \sum_{\substack{r, s \geq 0 \\ r+s=M}} \frac{q^{r^2+2r+s^2+s}}{(q^2; q^2)_r(q; q^2)_{r+1}(q^2; q^2)_{s-1}(q; q^2)_s} \\ &= \lim_{u \rightarrow 1} \frac{1-u}{1-q} \sum_{\substack{r, s \geq 0 \\ r+s=M}} \frac{q^{r^2+2r+s^2+s}}{(q^2; q^2)_r(q^3; q^2)_r(u; q^2)_s(q; q^2)_s} \\ &= \frac{q^{M^2+M}}{(1-q)(q; q^2)_M(q^2; q^2)_{M-1}} {}_2\phi_1 \left(\begin{matrix} q^{-2M}, q^{1-2M} \\ q^3 \end{matrix}; q^2, q^{2M} \right) \\ (\text{by (2.10)}) &= \frac{q^{M^2+M}}{(q; q^2)_M(q^2; q^2)_{M-1}} \frac{(-q^{1-M}, q^{1+M}; q)_\infty}{2(q^{2M}; q^2)_\infty} \\ &= \frac{q^{M(M+3)/2}(-q; q)_{M-1}}{(q; q^2)_M(q; q)_M}, \end{aligned}$$

thereby confirming the claim. It follows that for any integer ω ,

$$I_\omega^- - I_{1-\omega}^- = \frac{(-1)^\omega q^{\omega^2-\omega}(q; q^2)_\infty}{(q^2; q^2)_\infty}. \quad (5.14)$$

We conclude that

$$G_2(x) - xG_2(x^{-1}) = \sum_{\omega=-\infty}^{\infty} (x^\omega - x^{1-\omega}) I_\omega^-$$

$$\begin{aligned}
&= \sum_{\omega=-\infty}^{\infty} x^{\omega} (I_{\omega}^{-} - I_{1-\omega}^{-}) \\
&\text{(by (5.14))} = \sum_{\omega=-\infty}^{\infty} \frac{(-1)^{\omega} x^{\omega} q^{\omega^2-\omega} (q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \\
&\text{(by (2.7))} = \frac{(q; q^2)_{\infty} (x, x^{-1} q^2, q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (5.15)
\end{aligned}$$

Next, we deduce from (5.10) that

$$\begin{aligned}
&G_2(x) + xG_2(x^{-1}) \\
&= \sum_{\omega=-\infty}^{\infty} (x^{\omega} + x^{1-\omega}) \sum_{r,s=-\infty}^{\infty} \frac{q^{r^2+s^2+2s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^2}. \quad (5.16)
\end{aligned}$$

Define

$$I_{\omega}^{+} = I_{\omega}^{+}(q) := \sum_{r,s=-\infty}^{\infty} \frac{q^{r^2+s^2+2s+\omega}}{(q^2; q^2)_{r+s}} \begin{bmatrix} r+s \\ r-\omega \end{bmatrix}_{q^2} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^2}. \quad (5.17)$$

Note that

$$I_{\omega}^{+}(q) = (-1)^{\omega} I_{\omega}^{-}(-q).$$

It follows that for any integer ω ,

$$\begin{aligned}
I_{\omega}^{+} + I_{1-\omega}^{+} &= (-1)^{\omega} I_{\omega}^{-}(-q) + (-1)^{1-\omega} I_{1-\omega}^{-}(-q) \\
&= (-1)^{\omega} (I_{\omega}^{-}(-q) - I_{1-\omega}^{-}(-q)) \\
&\text{(by (5.14))} = \frac{q^{\omega^2-\omega} (-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (5.18)
\end{aligned}$$

We conclude that

$$\begin{aligned}
G_2(x) + xG_2(x^{-1}) &= \sum_{\omega=-\infty}^{\infty} (x^{\omega} + x^{1-\omega}) I_{\omega}^{+} \\
&= \sum_{\omega=-\infty}^{\infty} x^{\omega} (I_{\omega}^{+} + I_{1-\omega}^{+}) \\
&\text{(by (5.18))} = \sum_{\omega=-\infty}^{\infty} \frac{x^{\omega} q^{\omega^2-\omega} (-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \\
&\text{(by (2.7))} = \frac{(-q; q^2)_{\infty} (-x, -x^{-1} q^2, q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}}. \quad (5.19)
\end{aligned}$$

Finally, summing (5.15) and (5.19) yields (1.4).

5.3 Proof of Corollary 1.4

If we expand the right-hand side of (1.3) using Jacobi's triple product identity (2.7), it becomes

$$\frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{\omega=-\infty}^{\infty} x^\omega q^{\omega^2}.$$

Equating the coefficient of x^ω , we deduce (1.5).

Similarly, (1.6) follows from (1.4) with an application of Jacobi's triple product identity (2.7) to its right-hand side and after some cancellations. We omit the details.

5.4 Proof of Corollary 1.5

The identity in (1.7) corresponds to the coefficient of x^0 in (1.3). For the summation side, we use (5.1) and rewrite the q -binomial notation as q -Pochhammer symbols. For the product side, we apply Jacobi's triple product identity in (2.7).

Similarly, the identity in (1.8) follows by summing the coefficients of x^0 and x^1 in (1.4). Note that for the summation side, we make use of (5.10) and also rewrite the q -binomial coefficients as q -Pochhammer symbols. For the product side, we apply Jacobi's triple product identity in (2.7).

5.5 Further refined results on $\Lambda^{a,2}$

It immediately follows from (1.3) and (1.4) that

$$\sum_{n \geq 0} |\Lambda^{1,2}(n)| q^n = (-q; q)_\infty (-q; q^2)_\infty \quad (5.20)$$

and

$$\sum_{n \geq 0} |\Lambda^{2,2}(n)| q^n = (-q; q)_\infty (-q^2; q^2)_\infty. \quad (5.21)$$

Also, since $\omega(\pi) \equiv n \pmod{2}$, if we sum over $\omega \equiv n \pmod{2}$, we can obtain a 2-dissection for each of the generating functions.

▷ **The $a = 1$ case:**

▷ If $n \equiv 0 \pmod{2}$, then

$$|\Lambda^{1,2}(n)| = \sum_{\omega \equiv 0 \pmod{2}} |\Lambda_\omega^{1,2}(n)|.$$

Hence, by considering the specializations with $x = 1$ and $x = -1$ in (1.3), we get

$$\begin{aligned} \sum_{\substack{n \geq 0 \\ n \equiv 0 \pmod{2}}} |\Lambda^{1,2}(n)| q^n &= \frac{1}{2} \left((-q; q^2)_\infty^2 (-q^2; q^2)_\infty + (q; q^2)_\infty^2 (-q^2; q^2)_\infty \right) \\ &= \frac{1}{2} \left((-q; q^2)_\infty^2 + (q; q^2)_\infty^2 \right) (-q^2; q^2)_\infty \\ &= \frac{(-q^2; q^2)_\infty (-q^4; q^8)_\infty^2 (q^8; q^8)_\infty}{(q^2; q^2)_\infty}. \end{aligned} \quad (5.22)$$

▷ If $n \equiv 1 \pmod{2}$, then

$$|\Lambda^{1,2}(n)| = \sum_{\omega \equiv 1 \pmod{2}} |\Lambda_\omega^{1,2}(n)|.$$

Again, by considering the specializations with $x = 1$ and $x = -1$ in (1.3), we get

$$\begin{aligned} \sum_{\substack{n \geq 0 \\ n \equiv 1 \pmod{2}}} |\Lambda^{1,2}(n)| q^n &= \frac{1}{2} \left((-q; q^2)_\infty^2 (-q^2; q^2)_\infty - (q; q^2)_\infty^2 (-q^2; q^2)_\infty \right) \\ &= \frac{1}{2} \left((-q; q^2)_\infty^2 - (q; q^2)_\infty^2 \right) (-q^2; q^2)_\infty \\ &= \frac{2q(-q^2; q^2)_\infty (q^{16}; q^{16})_\infty^2}{(q^2; q^2)_\infty (q^8; q^8)_\infty}. \end{aligned} \quad (5.23)$$

By combining (5.20) with (5.22) and (5.23), we obtain the following result.

Corollary 5.1 *We have*

$$\begin{aligned} &(-q; q)_\infty (-q; q^2)_\infty \\ &= \frac{(-q^2; q^2)_\infty}{(q^2; q^2)_\infty} \left((-q^4; q^8)_\infty^2 (q^8; q^8)_\infty + 2q(-q^8; q^8)_\infty (q^{16}; q^{16})_\infty \right). \end{aligned}$$

Remark 5.1 The last equality in each of the above generating functions can be derived as follows. We first note that

$$(q; q^2)_\infty^2 = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty^2}.$$

Therefore, we may evaluate $(-q; q^2)_\infty^2 \pm (q; q^2)_\infty^2$ by recalling the 2-dissection formula of Ramanujan's classical theta function $\phi(-q)$ [13, p. 49, Entry 25]:

$$\phi(-q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q; q)_\infty^2}{(q^2; q^2)_\infty} = \frac{(q^8; q^8)_\infty^5}{(q^4; q^4)_\infty^2 (q^{16}; q^{16})_\infty^2} - \frac{2q(q^{16}; q^{16})_\infty}{(q^8; q^8)_\infty}.$$

▷ **The $a = 2$ case:**

▷ If $n \equiv 0 \pmod{2}$, then

$$|\Lambda^{2,2}(n)| = \sum_{\omega \equiv 0 \pmod{2}} |\Lambda_{\omega}^{2,2}(n)|.$$

Hence,

$$\begin{aligned} \sum_{\substack{n \geq 0 \\ n \equiv 0 \pmod{2}}} |\Lambda^{2,2}(n)| q^n &= \frac{1}{2} \left((-q; q^2)_{\infty} (-q^2; q^2)_{\infty}^2 + (q; q^2)_{\infty} (-q^2; q^2)_{\infty}^2 \right) \\ &= \frac{1}{2} \left((-q; q^2)_{\infty} + (q; q^2)_{\infty} \right) (-q^2; q^2)_{\infty}^2 \\ &= \frac{(-q^2; q^2)_{\infty} (-q^6, -q^{10}, q^{16}; q^{16})_{\infty}}{(q^2; q^2)_{\infty}}. \end{aligned} \quad (5.24)$$

▷ If $n \equiv 1 \pmod{2}$, then

$$|\Lambda^{2,2}(n)| = \sum_{\omega \equiv 1 \pmod{2}} |\Lambda_{\omega}^{2,2}(n)|.$$

Hence,

$$\begin{aligned} \sum_{\substack{n \geq 0 \\ n \equiv 1 \pmod{2}}} |\Lambda^{2,2}(n)| q^n &= \frac{1}{2} \left((-q; q^2)_{\infty} (-q^2; q^2)_{\infty}^2 - (q; q^2)_{\infty} (-q^2; q^2)_{\infty}^2 \right) \\ &= \frac{1}{2} \left((-q; q^2)_{\infty} - (q; q^2)_{\infty} \right) (-q^2; q^2)_{\infty}^2 \\ &= \frac{q(-q^2; q^2)_{\infty} (-q^2, -q^{14}, q^{16}; q^{16})_{\infty}}{(q^2; q^2)_{\infty}}. \end{aligned} \quad (5.25)$$

By combining (5.21) with (5.24) and (5.25), we obtain the following result.

Corollary 5.2 *We have*

$$\begin{aligned} &(-q; q)_{\infty} (-q^2; q^2)_{\infty} \\ &= \frac{(-q^2; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left((-q^6, -q^{10}, q^{16}; q^{16})_{\infty} + q(-q^2, -q^{14}, q^{16}; q^{16})_{\infty} \right). \end{aligned}$$

Remark 5.2 The last equality in each of the above generating functions can be derived in the following way. First, note that

$$(q; q^2)_{\infty} = \frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}}.$$

Therefore, we may evaluate $(-q; q^2)_\infty \pm (q; q^2)_\infty$ by recalling the 2-dissection formula of $(q; q)_\infty$ [26, p. 332, eq. (34.10.2)]:

$$(q; q)_\infty = (q^{12}, q^{20}; q^{32})_\infty (q^2, q^{14}, q^{16}; q^{16})_\infty \\ - q(q^4, q^{28}; q^{32})_\infty (q^6, q^{10}, q^{16}; q^{16})_\infty.$$

6 Concluding remarks

Let us write $W_n = G(2, 1, n)$ for the Weyl group of type B_n (or C_n). Let $\mathcal{H}_{c_1, c_2}^{rat}(W_n)$ denote the rational Cherednik algebra of W_n , with parameter c_1 (resp. c_2) assigned to the reflections associated with hyperplanes $z_i = z_j$ (resp. $z_i = 0$), as defined by Etingof and Ginzburg [18]. Let $\Omega^{c_1, c_2}(n)$ denote the number of finite-dimensional simple modules of $\mathcal{H}_{c_1, c_2}^{rat}(W_n)$. It is known [23] (private communication with E. Norton) that

$$\sum_{n \geq 0} \Omega^{\frac{1}{2}, 1}(n) x^n = \prod_{n \geq 1} \frac{1 + x^{2n}}{1 - x^{2n}}$$

and

$$\sum_{n \geq 0} \Omega^{\frac{1}{2}, \frac{1}{2}}(n) x^n = \prod_{n \geq 1} \frac{1 + x^{2n-1}}{1 - x^{2n}}.$$

Therefore, our results imply that the number of simple modules in the block of $\mathcal{H}_{-1; -1, 1}(W_n)$ (resp. $\mathcal{H}_{-1; -1, -1}(W_n)$), labeled by ω , equals the number of finite-dimensional simple modules of $\mathcal{H}_{\frac{1}{2}, 1}^{rat}(W_{n-\omega^2})$ (resp. $\mathcal{H}_{\frac{1}{2}, \frac{1}{2}}^{rat}(W_{n-\omega^2+\omega})$). This provides evidence (in our very special case) for the anticipation of Lie theory experts (private communication with P. Shan) that maximal support and minimal support simple modules in dual blocks of category $\mathcal{O}$'s of cyclotomic rational double affine Hecke algebras correspond to each other under the so-called level-rank duality (see, for example, [37]). It is likely that this can be deduced by combining results in [22] and [37].

It is also shown in [16] that the infinite product in Theorem 1.1 is an overpartition analog of the generalized Rogers–Ramanujan identities. Thus, Theorem 1.1 tells us a connection of the multipartitions in $\Lambda^{a, m}(n)$ to the overpartition analogs of the generalized Rogers–Ramanujan type partitions. It would be interesting to find a direct combinatorial explanation of this connection and perhaps to investigate whether this connection can be extended to Kleshchev multipartitions and overpartitions in general.

Appendix A: Results on q -binomial multisums

A.1 A decomposition formula for q -binomial multisums

Let us start with a decomposition formula that lays the foundation for our key lemmas to be established in the next subsection of this Appendix.

Lemma A.1 *Let $\{F(N)\}_{N \geq 0}$ be a family of series in q . Given an arbitrary integer $a \geq 2$, we have that, for any nonnegative integers $k_1, \dots, k_{a-1}$,*

$$\begin{aligned} & \sum_{N_1, \dots, N_a \geq 0} \frac{q^{\sum_{i=1}^{a-2} \binom{N_i + k_i}{2} + \binom{N_{a-1} + k_{a-1}}{2} + \binom{N_a + k_{a-1} + 1}{2}} F(N_a)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} \\ &= \sum_{N_1, \dots, N_a \geq 0} \frac{q^{\sum_{i=1}^{a-2} \binom{N_i + k_i}{2} + \binom{N_{a-1} + k_{a-1}}{2} + \binom{N_a + k_{a-1}}{2}} F(N_a)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} \\ &+ \sum_{N_1, \dots, N_a \geq 0} \frac{q^{\sum_{i=1}^{a-2} \binom{N_i + k_i + 1}{2} + \binom{N_{a-1} + k_{a-1} + 1}{2} + \binom{N_a + k_{a-1} + 1}{2}} F(N_a)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix}. \end{aligned} \quad (\text{A.1})$$

Proof First, by replacing N_i with $N_i - 1$ for $1 \leq i \leq a - 1$, we can rewrite the second sum on the right-hand side of (A.1) as follows:

$$\begin{aligned} & \sum_{\substack{N_1, \dots, N_{a-1} \geq 1 \\ N_a \geq 0}} \frac{q^{\sum_{i=1}^{a-2} \binom{N_i + k_i}{2} + \binom{N_{a-1} + k_{a-1}}{2} + \binom{N_a + k_{a-1} + 1}{2}} F(N_a)}{(q; q)_{N_1}} \\ & \times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a-2} \\ N_{a-1} \end{bmatrix} \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} (1 - q^{N_{a-1} - N_a}) \\ &= \sum_{N_1, \dots, N_a \geq 0} \frac{q^{\sum_{i=1}^{a-2} \binom{N_i + k_i}{2} + \binom{N_{a-1} + k_{a-1}}{2} + \binom{N_a + k_{a-1} + 1}{2}} F(N_a)}{(q; q)_{N_1}} \\ & \times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{a-2} \\ N_{a-1} \end{bmatrix} \begin{bmatrix} N_{a-1} - 1 \\ N_a \end{bmatrix} (1 - q^{N_{a-1}}), \end{aligned}$$

where the equality follows from (2.4). Thus, to prove (A.1), it is sufficient to prove that the coefficients of $F(N_a)$ on both sides match. That is, for fixed N_{a-1} and N_a , we need to show

$$q^{N_a} \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} = q^{N_{a-1}} \begin{bmatrix} N_{a-1} \\ N_a \end{bmatrix} + q^{N_a} \begin{bmatrix} N_{a-1} - 1 \\ N_a \end{bmatrix} (1 - q^{N_{a-1}}). \quad (\text{A.2})$$

Finally, the above relation follows directly from the definition of the q -binomial coefficients. $\square$

A.2 Key lemmas

For notational convenience, we write for $(k_1, k_2, \dots, k_b)$ a finite list of nonnegative integers and $\{F(N)\}_{N \geq 0}$ a family of series in q ,

$$\mathfrak{S} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix} := \mathfrak{S}_F \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix} := \sum_{N_1, \dots, N_b \geq 0} \frac{q^{\sum_{i=1}^b \binom{N_i + k_i}{2}} F(N_b)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{b-1} \\ N_b \end{bmatrix}. \quad (\text{A.3})$$

For each k_i , we *always* maintain the subscript i to keep track of its corresponding summation index, i.e., the $\binom{N_i + k_i}{2}$ term in the power of q in (A.3). In particular, we write $k_i = (m)_i$ if m is a specific value. For instance, on the left-hand side of (A.4), we have the entry $(k_{a-1} + 1)_a$ wherein $k_{a-1} + 1$ is a specific value which corresponds to the summation index a so that there is an associated term $\binom{N_a + k_{a-1} + 1}{2}$ in the power of q .

Notice that in Lemma A.1, $\{F(N)\}_{N \geq 0}$ is an arbitrary family of series. So, by abuse of notation, if we take

$$F(N_a) \mapsto \sum_{N_{a+1}, \dots, N_b \geq 0} q^{\sum_{i=a+1}^b \binom{N_i + k_i}{2}} F(N_b) \begin{bmatrix} N_a \\ N_{a+1} \end{bmatrix} \cdots \begin{bmatrix} N_{b-1} \\ N_b \end{bmatrix},$$

then Lemma A.1 implies the following result.

Corollary A.2 For $b \geq a \geq 2$,

$$\mathfrak{S} \begin{pmatrix} k_1 \\ \vdots \\ k_{a-2} \\ k_{a-1} \\ (k_{a-1} + 1)_a \\ k_{a+1} \\ \vdots \\ k_b \end{pmatrix} = \mathfrak{S} \begin{pmatrix} k_1 \\ \vdots \\ k_{a-2} \\ (k_{a-1} + 1)_{a-1} \\ (k_{a-1})_a \\ k_{a+1} \\ \vdots \\ k_b \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (k_1 + 1)_1 \\ \vdots \\ (k_{a-2} + 1)_{a-2} \\ (k_{a-1} + 1)_{a-1} \\ (k_{a-1} + 1)_a \\ k_{a+1} \\ \vdots \\ k_b \end{pmatrix}. \quad (\text{A.4})$$

The above relation is a key to the next two lemmas.

Lemma A.3 For $a \geq 1$, $m \geq 0$ and $\underline{k}$ a finite list of nonnegative integers,

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{a-1} \\ (m)_a \\ (m+1)_{a+1} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{a-1} \\ (m+1)_a \\ (m)_{a+1} \\ \underline{k} \end{pmatrix} = \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{a-1} \\ (m+1)_a \\ (m+1)_{a+1} \\ \underline{k} \end{pmatrix}. \quad (\text{A.5})$$

Proof This is a direct consequence of (A.4) by taking $k_1 = \dots = k_{a-1} = m$ and $(k_{a+1}, \dots, k_b) = \underline{k}$, and then replacing a by $a+1$. $\square$

Lemma A.4 Let $\underline{k}$ be a finite list of nonnegative integers. The following relations are true.

(i) For $a \geq i \geq 2$ and $m \geq 0$,

$$\begin{aligned} & \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_a \\ (m+1)_{a+1} \\ \vdots \\ (m+1)_{a+i-1} \\ (m+1)_{a+i} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{a-1} \\ (m+1)_a \\ \vdots \\ (m+1)_{a+i-1} \\ (m)_{a+i} \\ \underline{k} \end{pmatrix} \\ &= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ \vdots \\ (m+1)_{a+i-1} \\ (m+1)_{a+i} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-2} \\ (m+1)_{i-1} \\ \vdots \\ (m+1)_{a+i-1} \\ (m)_{a+i} \\ \underline{k} \end{pmatrix}. \end{aligned} \quad (\text{A.6})$$

(ii) For $a \geq i \geq 1$ and $m \geq 0$,

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_a \\ (m+1)_{a+1} \\ (m)_{a+2} \\ \vdots \\ (m)_{a+i} \\ \underline{k} \end{pmatrix} + \dots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_a \\ (m+1)_{a+1} \\ (m+1)_{a+2} \\ \vdots \\ (m+1)_{a+i} \\ \underline{k} \end{pmatrix}$$

$$= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ (m)_{i+1} \\ \vdots \\ (m)_{a+i} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ (m+1)_{i+1} \\ \vdots \\ (m+1)_{a+i} \\ \underline{k} \end{pmatrix}. \quad (\text{A.7})$$

Proof We prove this lemma by induction on a . The basic idea can be outlined as follows:

- (1) We start with the base case, $a = 1$, of Part (ii).
- (2) Next, assuming that Part (ii) is true for some $a = b \geq 1$, we will show that Part (i) is true for $a = b + 1$.
- (3) Finally, we use Part (i) for $a = b + 1$ and Part (ii) for $a = b$ to show Part (ii) for $a = b + 1$ and therefore complete the proof.

Step (1). The $a = 1$ (and thus $i = 1$) case of Part (ii) is

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ (m+1)_2 \\ \underline{k} \end{pmatrix} = \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ (m)_2 \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ (m+1)_2 \\ \underline{k} \end{pmatrix},$$

which is a direct consequence of (A.4).

Step (2). Assume that Part (ii) is true for some $a = b \geq 1$. We will show that for any j with $2 \leq j \leq b + 1$,

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m)_{b+1} \\ (m+1)_{b+2} \\ \vdots \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m+1)_{b+2} \\ \vdots \\ (m+1)_{b+j} \\ (m)_{b+j+1} \\ \underline{k} \end{pmatrix}$$

$$= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{j-2} \\ (m)_{j-1} \\ (m+1)_j \\ \vdots \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{j-2} \\ (m+1)_{j-1} \\ (m+1)_j \\ \vdots \\ (m+1)_{b+j} \\ (m)_{b+j+1} \\ \underline{k} \end{pmatrix}. \quad (\text{A.8})$$

We repeatedly make use of (A.4) to obtain

$$\begin{aligned} \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} &= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m)_{b+2} \\ (m+1)_{b+3} \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} \\ &= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ (m+1)_{b+4} \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_b \\ (m+2)_{b+1} \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ (m+1)_{b+4} \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} \\ &= \dots \\ &= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ \vdots \\ (m+1)_{b+j-1} \\ (m+1)_{b+j} \\ (m)_{b+j+1} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_b \\ (m+2)_{b+1} \\ \vdots \\ (m+2)_{b+j-1} \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} \end{aligned}$$

$$+ \cdots + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_b \\ (m+2)_{b+1} \\ (m+1)_{b+2} \\ \vdots \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix}.$$

Thus,

$$\begin{aligned} \text{LHS(A.8)} = & \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_b \\ (m+2)_{b+1} \\ \vdots \\ (m+2)_{b+j-1} \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_b \\ (m+2)_{b+1} \\ (m+1)_{b+2} \\ \vdots \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} \\ & + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix}. \end{aligned}$$

A similar application of (A.4) to the right-hand side of (A.7) gives

$$\begin{aligned} \text{RHS(A.8)} = & \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{j-2} \\ (m+2)_{j-1} \\ \vdots \\ (m+2)_{b+j-1} \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{j-2} \\ (m+2)_{j-1} \\ (m+1)_j \\ \vdots \\ (m+1)_{b+j} \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix} \\ & + \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_{b+j+1} \\ \underline{k} \end{pmatrix}. \end{aligned}$$

Now, in (A.7), we first take $(m, a, i) \mapsto (m + 1, b, j - 1)$ and then with abuse of notation choose $\underline{k} \mapsto ((m + 1)_{b+j}, (m + 1)_{b+j+1}, \underline{k})$. It follows that $\text{LHS}(\text{A.8}) = \text{RHS}(\text{A.8})$, and therefore (A.8) holds true.

Step (3). We want to show Part (ii) for $a = b + 1$. That is, for $1 \leq i \leq b + 1$,

$$\begin{aligned} & \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \vdots \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \vdots \\ (m+1)_{b+i+1} \\ \underline{k} \end{pmatrix} \\ &= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ (m)_{i+1} \\ \vdots \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ (m+1)_{i+1} \\ \vdots \\ (m+1)_{b+i+1} \\ \underline{k} \end{pmatrix}. \end{aligned} \quad (\text{A.9})$$

First, in (A.5), we take $a \mapsto b + 1$, and then with abuse of notation choose $\underline{k} = ((m)_{b+3}, \underline{k})$. Thus,

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix} = \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ (m+1)_b \\ (m+1)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix}.$$

Adding this with the $j = 2$ case of (A.8), i.e.,

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix} = \mathfrak{S} \begin{pmatrix} (m)_1 \\ (m+1)_2 \\ \vdots \\ \vdots \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \underline{k} \end{pmatrix} - \mathfrak{S} \begin{pmatrix} (m+1)_1 \\ \vdots \\ \vdots \\ (m+1)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix},$$

we have

$$\begin{aligned} & \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \underline{k} \end{pmatrix} \\ &= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m)_1 \\ (m+1)_2 \\ \vdots \\ \vdots \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \underline{k} \end{pmatrix}. \end{aligned}$$

We then choose $\underline{k} = ((m)_{b+4}, \underline{k})$ in the above and add it with the $j = 3$ case of (A.8). Further, if we repeat this procedure until adding the $j = i$ case of (A.8), it follows that, for $1 \leq i \leq b+1$,

$$\begin{aligned}
& \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m)_{b+3} \\ \vdots \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{b+1} \\ (m+1)_{b+2} \\ (m+1)_{b+3} \\ \vdots \\ (m+1)_{b+i+1} \\ \underline{k} \end{pmatrix} \\
&= \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m)_{b+2} \\ \vdots \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ \vdots \\ (m+1)_{b+i} \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ \vdots \\ (m+1)_{b+i} \\ (m+1)_{b+i+1} \\ \underline{k} \end{pmatrix}.
\end{aligned}$$

We point out that the $i = 1$ case of the above is simply (A.5) with $a \mapsto b + 1$. Finally, according to the inductive assumption, we make use of the $a = b$ case of (A.7) with $\underline{k} \mapsto ((m)_{b+i+1}, \underline{k})$ to obtain

$$\mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ (m)_{b+2} \\ \vdots \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_b \\ (m+1)_{b+1} \\ \vdots \\ (m+1)_{b+i} \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} = \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ (m)_{i+1} \\ \vdots \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix} + \cdots + \mathfrak{S} \begin{pmatrix} (m)_1 \\ \vdots \\ (m)_{i-1} \\ (m+1)_i \\ \vdots \\ (m+1)_{b+i} \\ (m)_{b+i+1} \\ \underline{k} \end{pmatrix}.$$

Combining this with the aligned identity in the above confirms (A.9). $\square$

We record a corollary of Lemma A.4, which plays a central role in the proof of Theorem 3.4.

Corollary A.5 *Let $\{F(N)\}_{N \geq 0}$ be a family of series in q . For $b \geq 2$,*

$$\begin{aligned}
& \sum_{j=b}^{2b-2} \left(\sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b}^j (N_i+1)} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \right) \\
&= \sum_{j=b-1}^{2b-2} \left(\sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b-1}^j (N_i+1)} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \right).
\end{aligned} \tag{A.10}$$

Proof In Lemma A.4 (ii), we take $(m, a, i) \mapsto (1, b-1, b-1)$, and then choose $\underline{k} = ((1)_{2b-1}, (1)_{2b})$. The desired result follows. $\square$

A.3 Proof of Theorem 3.4

Our proof of Theorem 3.4 is based on induction on a . For convenience, we write the two sides of (3.5) as LHS(3.5) $_a$ and RHS(3.5) $_a$, respectively.

Base Case. We first prove the $a = 1$ case of (3.5). That is,

$$\sum_{N_1, N_2 \geq 0} \frac{q^{(N_1+1)+\binom{N_2+1}{2}} F(N_2)}{(q; q)_{N_1}} \begin{bmatrix} N_1+1 \\ N_2 \end{bmatrix} = \sum_{N_1, N_2 \geq 0} \frac{q^{(N_1+1)+\binom{N_2}{2}} F(N_2)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

Next, we equate the coefficient of $F(N_2)$. Then

$$\sum_{N_1 \geq 0} \frac{q^{N_1(N_1+1)/2 + N_2(N_2+1)/2}}{(q; q)_{N_1}} \begin{bmatrix} N_1+1 \\ N_2 \end{bmatrix} = \sum_{N_1 \geq 0} \frac{q^{N_1(N_1+1)/2 + N_2(N_2-1)/2}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

Using (2.5) with $N = \infty$, we can deduce that both sums become

$$\frac{q^{N_2^2} (-q^{N_2+1}; q)_{\infty}}{(q; q)_{N_2}}.$$

Inductive Step. Assume that (3.5) is true for some $a = b-1 \geq 1$. We will show (3.5) for $a = b$. First, by a similar argument to how we expand LHS(3.4) in the proof of Theorem 3.3, we have

$$\begin{aligned} & \text{LHS(3.5)}_b \\ &= \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2}} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \\ &+ \sum_{j=b}^{2b-2} \left(\sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b}^j \binom{N_i+1}{2}} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \right) \\ &+ \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b}^{2b-1} \binom{N_i+1}{2}} F(N_{2b}+1)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \end{aligned}$$

Let us define an auxiliary series

$$\text{Aux} := \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2}} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{b-2} \\ N_{b-1} \end{bmatrix}$$

$$\times \begin{bmatrix} N_{b-1} + 1 \\ N_b \end{bmatrix} \begin{bmatrix} N_b \\ N_{b+1} \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \quad (\text{A.11})$$

We may also expand this series as

$$\begin{aligned} \text{Aux} = & \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2}} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \\ & + \sum_{j=b-1}^{2b-2} \left(\sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b-1}^j (N_i+1)} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \right) \\ & + \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b-1}^{2b-1} (N_i+1)} F(N_{2b} + 1)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \end{aligned}$$

With recourse to Corollary A.5, we find that

$$\begin{aligned} & \text{LHS(3.5)}_b - \text{Aux} \\ &= \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b}^{2b-1} (N_i+1)} F(N_{2b} + 1)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \\ & - \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b-1}^{2b-1} (N_i+1)} F(N_{2b} + 1)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \end{aligned} \quad (\text{A.12})$$

Meanwhile, we rewrite RHS(3.5)_b as

$$\begin{aligned} & \text{RHS(3.5)}_b \\ &= \sum_{N_1, \dots, N_{2b-2} \geq 0} \frac{q^{\sum_{i=1}^{2b-2} \binom{N_i+1}{2} - \sum_{i=1}^{b-1} N_{2i}}}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-3} \\ N_{2b-2} \end{bmatrix} \\ & \times \sum_{N_{2b-1}, N_{2b} \geq 0} q^{\binom{N_{2b-1}+1}{2} + \binom{N_{2b}}{2}} F(N_{2b}) \begin{bmatrix} N_{2b-2} \\ N_{2b-1} \end{bmatrix} \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \end{aligned}$$

According to the inductive assumption, we make use of (3.5) with $a = b - 1$ where we choose

$$F(N_{2b-2}) \mapsto \sum_{N_{2b-1}, N_{2b} \geq 0} q^{\binom{N_{2b-1}+1}{2} + \binom{N_{2b}}{2}} F(N_{2b}) \begin{bmatrix} N_{2b-2} \\ N_{2b-1} \end{bmatrix} \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \text{RHS(3.5)}_b = & \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} - N_{2b}} F(N_{2b})}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{b-2} \\ N_{b-1} \end{bmatrix} \\ & \times \begin{bmatrix} N_{b-1}+1 \\ N_b \end{bmatrix} \begin{bmatrix} N_b \\ N_{b+1} \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \end{aligned}$$

Recalling (A.11), we further have

$$\begin{aligned} & \text{RHS(3.5)}_b - \text{Aux} \\ = & \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} - N_{2b}} F(N_{2b})}{(q; q)_{N_1}} \\ & \times (1 - q^{N_{2b}}) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{b-2} \\ N_{b-1} \end{bmatrix} \begin{bmatrix} N_{b-1}+1 \\ N_b \end{bmatrix} \begin{bmatrix} N_b \\ N_{b+1} \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \\ = & \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} - N_{2b}} F(N_{2b})}{(q; q)_{N_1}} \\ & \times (1 - q^{N_{b-1}+1}) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{b-2} \\ N_{b-1} \end{bmatrix} \begin{bmatrix} N_{b-1} \\ N_b-1 \end{bmatrix} \begin{bmatrix} N_b-1 \\ N_{b+1}-1 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1}-1 \\ N_{2b}-1 \end{bmatrix} \\ = & \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b}^{2b-1} (N_i+1)} F(N_{2b}+1)}{(q; q)_{N_1}} \\ & \times (1 - q^{N_{b-1}+1}) \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{b-2} \\ N_{b-1} \end{bmatrix} \begin{bmatrix} N_{b-1} \\ N_b \end{bmatrix} \begin{bmatrix} N_b \\ N_{b+1} \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}, \end{aligned}$$

which tells us that

$$\begin{aligned} & \text{RHS(3.5)}_b - \text{Aux} \\ = & \sum_{N_1, \dots, N_{2b} \geq 0} \frac{q^{\sum_{i=1}^{2b} \binom{N_i+1}{2} + \sum_{i=b}^{2b-1} (N_i+1)} F(N_{2b}+1)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix} \end{aligned}$$

$$- \sum_{N_1, \dots, N_{2b} \geq 0} q^{\sum_{i=1}^m \binom{N_i+1}{2} + \sum_{i=b-1}^{2b-1} (N_i+1)} \frac{F(N_{2b}+1)}{(q; q)_{N_1}} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cdots \begin{bmatrix} N_{2b-1} \\ N_{2b} \end{bmatrix}. \quad (\text{A.13})$$

Combining (A.12) and (A.13) implies that $\text{LHS}(3.5)_b = \text{RHS}(3.5)_b$, which is exactly our desired result. $\square$

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Declarations

Conflict of interest There is no conflict of interest to declare.

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References

1. Andrews, G.E.: An analytic proof of the Rogers–Ramanujan–Gordon identities. *Amer. J. Math.* **88**, 844–846 (1966)
2. Andrews, G.E.: *The Theory of Partitions*. Cambridge University Press, Cambridge (1998)
3. Andrews, G.E.: Parity in partition identities. *Ramanujan J.* **23**(1–3), 45–90 (2010)
4. Andrews, G.E., Garvan, F.G.: Dyson's crank of a partition. *Bull. Amer. Math. Soc. (N.S.)* **18**(2), 167–171 (1988)
5. Ariki, S.: On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$. *J. Math. Kyoto Univ.* **36**(4), 789–808 (1996)
6. Ariki, S., Koike, K.: A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and construction of irreducible representations. *Adv. Math.* **106**(2), 216–243 (1994)
7. Ariki, S., Kreiman, V., Tsuchioka, S.: On the tensor product of two basic representations of $U_v(\widehat{\mathfrak{sl}}_e)$. *Adv. Math.* **218**(1), 28–86 (2008)
8. Ariki, S., Mathas, A.: The number of simple modules of the Hecke algebras of type $G(r, 1, n)$. *Math. Z.* **233**(3), 601–623 (2000)
9. Berkovich, A., Garvan, F.G.: On the Andrews–Stanley refinement of Ramanujan's congruence modulo 5 and generalization. *Trans. Amer. Math. Soc.* **358**(2), 703–726 (2006)
10. Berkovich, A., Garvan, F.G.: The BG-rank of a partition and its applications. *Adv. in Appl. Math.* **40**(3), 377–400 (2008)

11. Berkovich, A., Uncu, A.K.: On partitions with fixed number of even-indexed and odd-indexed odd parts. *J. Number Theory* **167**, 7–30 (2016)
12. Berkovich, A., Warnaar, S.O.: Positivity preserving transformations for q -binomial coefficients. *Trans. Amer. Math. Soc.* **357**(6), 2291–2351 (2005)
13. Berndt, B.C.: *Ramanujan's Notebooks. Part III*. Springer, New York (1991)
14. Bressoud, D.M.: A generalization of the Rogers–Ramanujan identities for all moduli. *J. Combin. Theory Ser. A* **27**(1), 64–68 (1979)
15. Broué, M., Malle, G.: Zyklotomische Heckealgebren. *Astérisque* **212**, 119–189 (1993)
16. Chen, W.Y.C., Sang, D.D.M., Shi, D.Y.H.: The Rogers–Ramanujan–Gordon theorem for overpartitions. *Proc. Lond. Math. Soc. (3)* **106**(6), 1371–1393 (2013)
17. Dyson, F.J.: Some guesses in the theory of partitions. *Eureka* **8**, 10–15 (1944)
18. Etingof, P., Ginzburg, V.: Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism. *Invent. Math.* **147**(2), 243–348 (2002)
19. Feingold, A.J., Lepowsky, J.: The Weyl–Kac character formula and power series identities. *Adv. Math.* **29**(3), 271–309 (1978)
20. Garvan, F., Kim, D., Stanton, D.: Cranks and t -cores. *Invent. Math.* **101**(1), 1–17 (1990)
21. Gasper, G., Rahman, M.: *Basic Hypergeometric Series*, 2nd edn. Cambridge University Press, Cambridge (2004)
22. Gerber, T.: Triple crystal action in Fock spaces. *Adv. Math.* **329**, 916–954 (2018)
23. Gerber, T., Norton, E.: The $\mathfrak{sl}_\infty$ -crystal combinatorics of higher level Fock spaces. *J. Comb. Algebra* **2**(2), 103–145 (2018)
24. Gordon, B.: A combinatorial generalization of the Rogers–Ramanujan identities. *Amer. J. Math.* **83**, 393–399 (1961)
25. Griffin, M.J., Ono, K., Warnaar, S.O.: A framework of Rogers–Ramanujan identities and their arithmetic properties. *Duke Math. J.* **165**(8), 1475–1527 (2016)
26. Hirschhorn, M.D.: *The Power of q . A Personal Journey*. Springer, Cham (2017)
27. Ishikawa, M., Zeng, J.: The Andrews–Stanley partition function and Al-Salam–Chihara polynomials. *Discrete Math.* **309**(1), 151–175 (2009)
28. Jacon, N.: Kleshchev multipartitions and extended Young diagrams. *Adv. Math.* **339**, 367–403 (2018)
29. Kim, S., Yee, A.J.: The Rogers–Ramanujan–Gordon identities, the generalized Göllnitz–Gordon identities, and parity questions. *J. Combin. Theory Ser. A* **120**(5), 1038–1056 (2013)
30. Lepowsky, J., Milne, S.: Lie algebraic approaches to classical partition identities. *Adv. Math.* **29**(1), 15–59 (1978)
31. Lepowsky, J., Wilson, R.L.: A Lie theoretic interpretation and proof of the Rogers–Ramanujan identities. *Adv. Math.* **45**(1), 21–72 (1982)
32. Lyle, S., Mathas, A.: Blocks of cyclotomic Hecke algebras. *Adv. Math.* **216**(2), 854–878 (2007)
33. Mathas, A.: Simple modules of Ariki–Koike algebras. In: *Group Representations: Cohomology, Group Actions and Topology* (Seattle, WA, 1996), *Proceedings of Symposia in Pure Mathematics*, vol. 63, pp. 383–396. American Mathematical Society, Providence, RI (1998)
34. Ramanujan, S.: Some properties of $p(n)$, the number of partitions of n . *Proc. Cambridge Philos. Soc.* **19**, 207–210 (1919)
35. Ramanujan, S.: Proof of certain identities in combinatory analysis. *Proc. Cambridge Philos. Soc.* **19**, 214–216 (1919)
36. Rogers, L.J.: Second memoir on the expansion of certain infinite products. *Proc. Lond. Math. Soc.* **25**, 318–343 (1894)
37. Rouquier, R., Shan, P., Varagnolo, M., Vasserot, E.: Categorifications and cyclotomic rational double affine Hecke algebras. *Invent. Math.* **204**(3), 671–786 (2016)
38. Sang, D.D.M., Shi, D.Y.H.: An Andrews–Gordon type identity for overpartitions. *Ramanujan J.* **37**(3), 653–679 (2015)
39. Tsuchioka, S.: A proof of the second Rogers–Ramanujan identity via Kleshchev multipartitions. *Proc. Japan Acad. Ser. A Math. Sci.* **99**(3), 23–26 (2023)
40. Vilonen, K., Xue, T.: Invariant systems and character sheaves for graded Lie algebras. Preprint (2021). [arXiv:2111.08403](https://arxiv.org/abs/2111.08403)
41. Vilonen, K., Xue, T.: Character sheaves for graded Lie algebras: stable gradings. *Adv. Math.* **417**, 108935 (2023)

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