

Investigating connections between
fractional competence and algebraic
reasoning in the middle years

Catherine Anne Pearn
0000-0002-5007-9162

Doctor of Philosophy

January 2020

Melbourne Graduate School of Education

Submitted in total fulfilment for the degree of

Doctor of Philosophy

Abstract

To succeed in mathematics, middle-years' students must move from additive to multiplicative thinking and from arithmetic calculations to generalised algebraic reasoning. This thesis' focus on fractions gives insight into 607 middle-years students' solution strategies and the potential to prompt algebraic thinking and reasoning as they attempted two paper and pencil assessments: The Fraction Screening Test (FST) and the Algebraic Thinking Questionnaire (ATQ). Analysis of the results has revealed that students exhibit many previously reported misconceptions. For example, students ordered fractions by size of denominator or by the difference between the numerator and denominator (gap thinkers). While many students struggled with the ATQ, which focussed on multiplication and division of whole numbers, fractions and decimals, others demonstrated their understanding about equivalence which is an important component of algebraic reasoning.

A revised version of the FST included 'reverse fraction' tasks, so named because they require students to find the number of objects in the whole collection when given the number of objects representing a particular fraction. Analysis of student responses to both tests, in conjunction with these additional items, provided new insights revealing evidence of emerging algebraic reasoning.

A semi-structured interview was constructed using reverse fraction tasks similar to those in the FST. The interview questions included the same three fractions but with variations. These variations included changes of the number of objects, the more generalised situation with 'any number' of objects; and if students were successful with 'any number' concluded with the generalised question of 'any fraction' and 'any number' of objects. All 43 students interviewed had successfully solved at least two of the three reverse fraction tasks on the written version of the revised FST. Descriptions of nine case study students are included to highlight the range of strategies students used in their written tests and those elaborated during their interview.

Detailed analysis of the strategies used to solve the three reverse fractions has allowed the development of two new frameworks for classifying student strategies. The first, the *Framework for Reverse Fraction Tasks*, assists the classification of students' solution strategies for the reverse fraction tasks for both the written assessment and the interview questions as being: diagram dependent, additive, or partially, fully or advanced multiplicative. This framework highlights the importance of presenting fraction tasks where students need to calculate the whole from the given fractional part, length or number of objects as well as finding parts of a whole length, area or collection. The second, the *Emerging Algebraic Reasoning Framework*, monitors the extent to which students exhibit emerging algebraic reasoning when opportunities for generalised responses are prompted, as for the questions in the Structured Interview. These two frameworks highlight the connection between middle-years students' fractional competence and reasoning and emerging evidence of algebraic reasoning. The frameworks enable teachers to identify the strategies students use to solve reverse fraction tasks, and then monitor their students' progress by giving clear suggestions about what is needed to move to the next level. The new test items and frameworks add to mathematics teachers' tools for formative assessment.

Declaration

This is to certify that:

- the thesis comprises of only my original work towards the PhD except where indicated in the preface;
- due acknowledgement has been made in the text to all other material used;
- the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices

Catherine Anne Pearn

Preface

As the thesis research developed and pilot studies were conducted, preliminary results were published in peer-reviewed national and international conference proceedings. The process of peer review of these conference papers and comments from conference attendees contributed significantly to the focus and consequent depth of the thesis. The individual contributions of the publications to the overall thesis is described below.

1. Pearn, C., & Stephens, M. (2015a). Strategies for solving fraction tasks and their link to algebraic thinking. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics Education in the Margins*. (Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 493-500). Sunshine Coast, Australia: MERGA
This peer reviewed conference paper discusses the links to the Lee and Hackenberg interview tasks (2014) which is discussed in Section 2.3 and introduces Pilot Study 1 where 67 students trialled the revised Fraction Screening Test to be used in the main study. Pilot Study 1 is discussed in detail in Section 3.3.

The first-named author contributed approximately 70% and the second-named author contributed approximately 30% towards the preparation of this 2015 MERGA conference paper.

2. Pearn, C., & Stephens, M. (2015b). The importance of fractional thinking as a bridge to algebraic reasoning. In N. Davis, K. Manuel & T. Spencer (Eds.), *Mathematics: learn, lead, link*. (Proceedings of the 25th Biennial Conference of the Australian Association of Mathematics Teachers Inc., pp. 103-111). Adelaide, Australia: AAMT
This peer reviewed conference paper introduces Pilot Study 2 where 18 students trialled the three reverse fraction tasks (Tasks C5, C6 and C7) from the Fraction Screening Test and either the multiplication or division parts of the Algebraic Thinking Questionnaire to be used in the main study. Pilot Study 2 is discussed in detail in Section 3.4.

The first-named author contributed approximately 70% and the second-named author contributed approximately 30% towards the preparation of this 2015 AAMT conference paper.

3. Pearn, C., & Stephens, M. (2016). Competence with fractions in fifth or sixth grade as a unique predictor of algebraic thinking? In B. White, M. Chinnappan, & S. Trenholm (Eds.), *Opening up mathematics education research*. (Proceedings of the 39th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 519-526). Adelaide, Australia: MERGA.

This peer reviewed conference paper is a summary of the results from 162 Years 5 and 6 students from three schools. These are a subset of students from the main study. Overall scores are compared for both the Fraction Screening Test, and the Algebraic Thinking Questionnaire, but this form of analysis was not used for the whole student cohort in the main study.

The first-named author contributed approximately 65% and the second-named author contributed approximately 35% towards the preparation of this 2016 MERGA conference paper.

4. Pearn, C., & Stephens, M. (2017). Generalising fraction structures as a means for engaging in algebraic thinking. In A. Downton, S. Livy & J. Hall (Eds.), *40 years on: We are still learning*. (Proceedings of the 40th Annual Conference of Mathematics Education Research Group of Australasia, pp. 405-412). Adelaide, Australia: MERGA.

This peer reviewed conference paper discusses the results of 45 Year 5 and Year 6 students from one primary school and includes an initial classification for the three reverse fraction tasks (C5, C6 and C7) shown in Figure 3.7. For the final classification for the reverse fraction tasks see Section 4.3.2. This paper introduces the interview protocol discussed in Section 4.5 and the initial Scoring Framework discussed in Section 4.5.5.

The first-named author contributed approximately 60% and the second-named author contributed approximately 40% towards the preparation of this 2017 MERGA conference paper.

5. Pearn, C., Pierce, R., & Stephens, M. (2017a). Questions to reveal and guide fractional thinking. In V. Barker (Ed.), *Capital Maths*. (Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers Inc., pp. 125-322). Adelaide, Australia: AAMT.

This peer reviewed conference paper discusses the results of a subset of 570 students from the main study (607 students) for a subset of five fraction tasks from the Fraction Screening Test. The results for all Fraction Screening Test tasks are discussed in detail in Section 5.1.

The first-named author contributed approximately 60% and the second- and third-named authors contributed approximately 20% each towards the preparation of this 2017 AAMT conference paper.

6. Pearn, C., Pierce, R., & Stephens, M. (2017b). Reverse fraction tasks reveal algebraic thinking. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*. (Vol 4 pp. 1-8) Singapore: PME.

This peer reviewed conference paper discusses the 117 Year 8 students' responses to the three reverse fraction tasks with a second revised solution method framework for these tasks which was

further revised and shown in Section 4.3.2. The final results for the three reverse fraction tasks are included in Section 5.1.4.

The first-named author contributed approximately 60% and the second- and third-named authors contributed approximately 20% each towards the preparation of this 2017 PME conference paper.

7. Pearn, C., & Stephens, M. (2018). Generalizing fractional structures: A critical precursor to Algebraic Thinking. In C. Kieran (Ed.), *Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice*. (pp. 237-260). Cham, Switzerland: Springer.

This peer reviewed chapter summarises several sections in the thesis. These include:

1.1 The Australian Context

2.3 Links between fraction competence and algebra

3.4 Pilot Study 2

3.6 Consultation with some experts

4.5 The Structured Interview

The first-named author contributed approximately 65% and the second-named author contributed approximately 35% towards the preparation of this 2018 chapter.

8. Pearn, C., Stephens, M., & Pierce, R. (2018a). Diagram effective or diagram dependent? In J. Hunter, L. Darragh, & P. Perfer, (Eds.), *Making Waves, Opening Spaces*. (Proceedings of the 41st Annual Conference of the Mathematics Education Research Group of Australasia. pp. 631-638). Auckland, NZ: MERGA.

This peer reviewed conference paper focuses on two students who depended on diagrammatic representations in both the Fraction Screening Test and the Structured Interview. A more detailed description for each of these students is given in case study Sections 7.1 and 7.2 in Chapter 7.

The first-named author contributed approximately 60% and the second-named author contributed approximately 30% and the third-named author contributed 10% towards the preparation of this 2018 MERGA conference paper.

9. Pearn, C., Stephens, M., & Pierce, R. (2018b). Interviews reveal Year 8 students' struggle to generalise solutions to reverse fraction tasks. In F. J. Hsieh (Ed). *ICMI-EARCOME 8 Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education*. (pp. 215-226). Taipei, Taiwan: ICME.

This peer reviewed conference paper includes the Year 8 students' results for the three reverse fraction tasks using the final classification (See Section 5.1.4). The results of the first three interview tasks for 26 Year 8 students are classified according to the Classification Framework

for Reverse Fraction Tasks given in Table 5.15. One Year 8 student's responses to the three reverse fraction tasks and the first five questions of the Structured Interview are discussed. This student's results are discussed in more detail in Section 7.1.

The first-named author contributed approximately 65% and the second-named author contributed approximately 25% and the third-named author contributed 10% towards the preparation of this 2018 ICMI-EARCOME 8 conference paper.

10. Pearn, C., Stephens, M., & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley & A. Cooke (Eds.), *Mathematics Education Research Impacting Practice*. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA.

This peer reviewed paper summarises the two paper and pencil instruments (Sections 4.3 and 4.4) and the Structured Interview (Section 4.5). The relationship between the results for the two paper and pencil instruments for all students in the main study are graphed. The two important frameworks are presented: The Framework for Reverse Fraction Strategies (Table 5.15) and The Emerging Algebraic Reasoning Framework (Table 6.12) and discussed in more detail in Section 8.7.

The first-named author contributed approximately 70% and the second-named author contributed approximately 20% and the third-named author contributed 10% towards the preparation of this 2019 MERGA conference paper.

11. Pearn, C., Stephens, M., Zhang, Q., & Pierce, R. (2019). Identifying emerging algebraic reasoning. In M. Graven, H. Venkat, A. Essien, & P. Vale, (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education*. (Vol 2, pp. 201-208). Pretoria, South Africa: PME.

This peer reviewed conference paper compared the results of the 17 Australian and 12 Chinese primary students who completed the three reverse fraction tasks and were interviewed. Results from this exploratory study confirmed that the The Framework for Reverse Fraction Strategies (Table 5.15) could be used to analyse the results from the three reverse fraction tasks and each of the questions in the Structured Interview and that both Chinese and Australian students could be placed on a level of The Emerging Algebraic Reasoning Framework (Table 6.11.)

The first-named author contributed approximately 60% and the second-named author contributed approximately 20% while the other two authors contributed 10% each towards the preparation of this 2019 PME conference paper.

The publications that informed and reshaped the individual chapters of this thesis are provided in the following Table.

Chapter	Relevant publication(s)
1. Introduction	Pearn, C., & Stephens, M. (2018). Generalizing fractional structures: A critical precursor to Algebraic Thinking. In C. Kieran (Ed.), <i>Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice</i> . (pp. 237-260). Cham, Switzerland: Springer.
2. Literature review	Pearn, C., & Stephens, M. (2015a). Strategies for solving fraction tasks and their link to algebraic thinking. In M. Marshman, V. Geiger, & A. Bennison (Eds.), <i>Mathematics Education in the Margins</i>. (Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 493-500). Sunshine Coast, Australia: MERGA Pearn, C., & Stephens, M. (2018). Generalizing fractional structures: A critical precursor to Algebraic Thinking. In C. Kieran (Ed.), <i>Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice</i>. (pp. 237-260). Cham, Switzerland: Springer.
3. Methodology and trial of instruments	Pearn, C., & Stephens, M. (2015a). Strategies for solving fraction tasks and their link to algebraic thinking. In M. Marshman, V. Geiger, & A. Bennison (Eds.), <i>Mathematics Education in the Margins</i>. (Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 493-500). Sunshine Coast, Australia: MERGA Pearn, C., & Stephens, M. (2015b). The importance of fractional thinking as a bridge to algebraic reasoning. In N. Davis, K. Manuel & T. Spencer (Eds.), <i>Mathematics: learn, lead, link</i>. (Proceedings of the 25th Biennial Conference of the Australian Association of Mathematics Teachers Inc., pp. 103-111). Adelaide, Australia: AAMT Pearn, C., & Stephens, M. (2018). Generalizing fractional structures: A critical precursor to Algebraic Thinking. In C. Kieran (Ed.), <i>Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice</i>. (pp. 237-260). Cham, Switzerland: Springer.
4. Main study method	Pearn, C., & Stephens, M. (2017). Generalising fraction structures as a means for engaging in algebraic thinking. In A. Downton, S. Livy & J. Hall (Eds.), <i>40 years on: We are still learning</i> . (Proceedings of the 40 th Annual Conference of Mathematics Education Research Group of Australasia, pp. 405-412). Adelaide, Australia: MERGA.

	Pearn, C., Pierce, R., & Stephens, M. (2017b). Reverse fraction tasks reveal algebraic thinking. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), <i>Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education</i>. (Vol 4 pp. 1-8) Singapore: PME. Pearn, C., & Stephens, M. (2018). Generalizing fractional structures: A critical precursor to Algebraic Thinking. In C. Kieran (Ed.), <i>Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice</i>. (pp. 237-260). Cham, Switzerland: Springer. Pearn, C., Stephens, M., & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley, & A. Cooke (Eds.), <i>Mathematics Education Research Impacting Practice</i>. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA.
5. Exploratory analysis of paper and pencil assessment results	Pearn, C., & Stephens, M. (2016). Competence with fractions in fifth or sixth grade as a unique predictor of algebraic thinking? In B. White, M. Chinnappan, & S. Trenholm (Eds.), <i>Opening up mathematics education research</i>. (Proceedings of the 39th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 519-526). Adelaide, Australia: MERGA. Pearn, C., Pierce, R., & Stephens, M. (2017a). Questions to reveal and guide fractional thinking. In V. Barker (Ed.), <i>Capital Maths</i>. (Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers Inc., pp. 125-322). Adelaide, Australia: AAMT. Pearn, C., Stephens, M., & Pierce, R. (2018b). Interviews reveal Year 8 students' struggle to generalise solutions to reverse fraction tasks. In F. J. Hsieh (Ed.), <i>ICMI-EARCOME 8 Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education</i>. (pp. 215-226). Taipei, Taiwan: ICME. Pearn, C., Stephens, M., & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley, & A. Cooke (Eds.), <i>Mathematics Education Research Impacting Practice</i>. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA. Pearn, C., Stephens, M., Zhang, Q., & Pierce, R. (2019). Identifying emerging algebraic reasoning. In M. Graven, H. Venkat, A. Essien, & P. Vale, (Eds.), <i>Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education</i>. (Vol 2, pp. 201-208). Pretoria, South Africa: PME.

6. Results from the Structured Interview	Pearn, C., Stephens, M., & Pierce, R. (2018a). Diagram effective or diagram dependent? In J. Hunter, L. Darragh, & P. Perfer, (Eds.), <i>Making Waves, Opening Spaces</i>. (Proceedings of the 41st Annual Conference of the Mathematics Education Research Group of Australasia. pp. 631-638). Auckland, NZ: MERGA. Pearn, C., Stephens, M., & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley, & A. Cooke (Eds.), <i>Mathematics Education Research Impacting Practice</i>. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA. Pearn, C., Stephens, M., Zhang, Q. & Pierce, R. (2019). Identifying emerging algebraic reasoning. In M. Graven, H. Venkat, A. Essien, & P. Vale, (Eds.), <i>Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education</i>. (Vol 2, pp. 201-208). Pretoria, South Africa: PME.
7. Case Studies	Pearn, C., Stephens, M., & Pierce, R. (2018a). Diagram effective or diagram dependent? In J. Hunter, L. Darragh, & P. Perfer, (Eds.), <i>Making Waves, Opening Spaces</i>. (Proceedings of the 41st Annual Conference of the Mathematics Education Research Group of Australasia. pp. 631-638). Auckland, NZ: MERGA. Pearn, C., Stephens, M., & Pierce, R. (2018b). Interviews reveal Year 8 students' struggle to generalise solutions to reverse fraction tasks. In F. J. Hsieh (Ed). <i>ICMI-EARCOME 8 Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education</i>. (pp. 215-226). Taipei, Taiwan: ICME.
8. Discussing the research questions	Pearn, C., Stephens, M., & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley, & A. Cooke (Eds.), <i>Mathematics Education Research Impacting Practice</i>. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA.
9. Conclusion and implications	Pearn, C., Stephens, M. & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley, & A. Cooke (Eds.), <i>Mathematics Education Research Impacting Practice</i>. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA.

The research documented in this thesis has been supported by the Australian Government Research Training Program Fees Offset Scholarship.

Acknowledgements

This thesis was completed under the supervision of Associate Professor Robyn Pierce and Dr Max Stephens and would not have been possible without their ongoing guidance, unwavering encouragement and support. Their belief that my work was important and worthwhile ensured the timely completion of my thesis. I am eternally grateful for their assistance and support.

In addition to the support provided by Robyn and Max, I am grateful to Associate Professor Wee Tiong Seah for chairing my panel and providing ongoing encouragement and support. Thanks also to Brian Doig for his suggestions and support in the early stages of this research, Tony Jones for reading the penultimate draft of my thesis while on holiday, Vicki Steinle for assisting with the data analysis, and colleagues Kate Copping, Carmel Mesiti and Scott Cameron for technical assistance during the final stages of my writing.

My sincere thanks to Robert Hunting and Gary Davis who introduced me to the world of fraction research when I was privileged to be their research assistant in their projects. This thesis builds on the findings of their earlier fraction research.

Many of my colleagues in the Mathematic Education group at the Melbourne Graduate School of Education have provided ongoing support and encouragement during the writing of the thesis. These include Duncan Symons, Lynda Ball, Beth Price, Natasha Ziebell, Derek Holton, Meredith Begg and John Dowsey. Thank you also to Professor Jan van Driel for his support during the final stages of this thesis. Thanks also to my ACER colleagues who also supported me during this time.

This research would not have been possible without my participants – thank you to all the students who completed the paper and pencil assessments with special thanks to the students who participated in the Structured Interview. Your responses to the fraction tasks gave a great insight into your ways of thinking, understanding and reasoning about fraction tasks. Thank you also to the teachers who changed their programs to allow me access to their students.

My friends and family have played an important role in supporting me throughout my candidature. I would like to thank my children: Jen, Suze, Ben and Michael; their partners: Guy, Shaun, Jen and Sarah; and my wonderful grandchildren: Jackson, Angus, Scarlett, Daisy, Pepper, Julian and Rafael. Special thanks to Jackson and Angus who assisted with the data entry.

This thesis is dedicated to my father, Walter Murray Lonie (19.12.1920 - 19.11.2018) who actively supported my research but unfortunately did not live long enough to see this thesis finished.

CHAPTER 1: INTRODUCTION	1
1.1 THE AUSTRALIAN CONTEXT.....	2
1.2 RESEARCH QUESTIONS	7
CHAPTER 2: LITERATURE REVIEW	9
2.1 FRACTIONAL COMPETENCE	9
2.2 ALGEBRAIC THINKING.....	15
2.2.1 <i>Transition from arithmetic to algebra</i>	16
2.2.2 <i>Relational Thinking</i>	17
2.3 LINKS BETWEEN FRACTION COMPETENCE AND ALGEBRA.....	18
2.4 FOCUS OF THIS RESEARCH	21
CHAPTER 3: METHODOLOGY AND TRIAL OF INSTRUMENTS.....	22
3.1 PLANNING THE MAIN STUDY.....	23
3.2 THE FRACTION SCREENING TEST	25
3.2.1 <i>Background to the Fraction Screening Test</i>	25
3.2.2 <i>Connections to Kieren's interpretations of fractions (1980)</i>	29
3.3 PILOT STUDY 1: TRIALING THE REVISED FRACTION SCREENING TEST.....	30
3.3.1 <i>Scoring the Fraction Screening Test</i>	30
3.3.2 <i>Results for Fraction Screening Test: Part A</i>	31
3.3.3 <i>Results for Fraction Screening Test: Part B</i>	32
3.3.4 <i>Results for Fraction Screening Test: Part C</i>	33
3.3.5 <i>Elaborating written responses to the reverse fraction tasks</i>	34
3.3.6 <i>Internal consistency for the Fraction Screening Test</i>	37
3.3.7 <i>Finding evidence of algebraic thinking</i>	39
3.3.8 <i>Conclusion for Pilot Study 1</i>	42
3.4 PILOT STUDY 2: TRIALING THE ALGEBRAIC THINKING QUESTIONNAIRE	42
3.4.1 <i>Background to the Algebraic Thinking Questionnaire</i>	42
3.4.2 <i>Second trial of the three Reverse Fraction Tasks</i>	44
3.4.3 <i>Algebraic Thinking Questionnaire: Part M</i>	46
3.4.4 <i>Algebraic Thinking Questionnaire Part D</i>	48
3.4.5 <i>Conclusion for Pilot Study 2</i>	50
3.5 SIGNIFICANCE OF PILOT STUDIES.....	50
3.6 CONSULTATION WITH SOME EXPERTS	50

3.7	IMPLICATIONS FOR THE MAIN STUDY	53
CHAPTER 4:	MAIN STUDY METHOD	54
4.1	PARTICIPANTS.....	55
4.2	DATA COLLECTION: INSTRUMENTS	56
4.3	THE FRACTION SCREENING TEST	57
4.3.1	<i>Scoring the Fraction Screening Test</i>	<i>57</i>
4.3.2	<i>Analysing strategies used to solve individual Part C reverse fraction tasks</i>	<i>59</i>
4.3.3	<i>Dominant strategies used to solve the set of Part C reverse fraction tasks</i>	<i>61</i>
4.4	THE ALGEBRAIC THINKING QUESTIONNAIRE	63
4.4.1	<i>Scoring the Algebraic Thinking Questionnaire</i>	<i>63</i>
4.5	THE STRUCTURED INTERVIEW	64
4.5.1	<i>Development of the Structured Interview</i>	<i>65</i>
4.5.2	<i>Comparison with Lee and Hackenberg Interview Tasks</i>	<i>67</i>
4.5.3	<i>Trialling the Structured Interview</i>	<i>70</i>
4.5.4	<i>Administration of the Structured Interview</i>	<i>75</i>
4.5.5	<i>Scoring the Structured Interview</i>	<i>77</i>
CHAPTER 5:	EXPLORATORY ANALYSIS OF PAPER AND PENCIL ASSESSMENT RESULTS	79
5.1	FRACTION SCREENING TEST.....	79
5.1.1	<i>Fraction Screening Test Part A task results.....</i>	<i>80</i>
5.1.2	<i>Fraction Screening Test Part B task results.....</i>	<i>87</i>
5.1.3	<i>Fraction Screening Test Part C task results.....</i>	<i>95</i>
5.1.4	<i>Analysing the written responses for Tasks C5, C6 and C7</i>	<i>99</i>
5.2	RELATIONSHIP BETWEEN THE REVERSE FRACTION TASKS AND OTHER FST TASKS .	107
5.2.1	<i>Relationship between FST and reverse fraction tasks scores</i>	<i>108</i>
5.2.2	<i>Relationships between parts of FST and reverse fraction tasks.....</i>	<i>109</i>
5.2.3	<i>Relationship between reverse fraction and individual tasks</i>	<i>112</i>
5.3	ALGEBRAIC THINKING QUESTIONNAIRE	114
5.3.1	<i>Algebraic Thinking Questionnaire Part M Question 1 results.....</i>	<i>115</i>
5.3.2	<i>Algebraic Thinking Questionnaire Part M Question 2 results.....</i>	<i>125</i>
5.3.3	<i>Algebraic Thinking Questionnaire Part D Question 1 results</i>	<i>131</i>
5.3.4	<i>Algebraic Thinking Questionnaire Part D Question 2 results</i>	<i>139</i>
5.3.5	<i>Comparing results from Part M and Part D (ATQ)</i>	<i>145</i>
5.4	COMPARISON OF RESULTS FOR FST AND ATQ	148

5.4.1	<i>Comparing the ATQ total scores and FST total scores</i>	148
5.4.2	<i>Comparison of ATQ and Part A (FST) total scores</i>	149
5.4.3	<i>Comparison of ATQ and Part B (FST) total scores</i>	151
5.4.4	<i>Comparison of ATQ and Part C (FST) total scores</i>	153
5.5	INTERNAL CONSISTENCY FOR THE TWO PAPER AND PENCIL ASSESSMENTS	154
5.6	SUMMARISING THE RESULTS FOR THE PAPER AND PENCIL ASSESSMENTS	155
CHAPTER 6: RESULTS FROM THE STRUCTURED INTERVIEWS.....		157
6.1	PRIMARY INTERVIEWS.....	157
6.1.1	<i>Variation 1: Change of number 1</i>	158
6.1.2	<i>Variation 2: Change of number 2 and ‘any number’</i>	161
6.1.3	<i>Variation 3: ‘Any fraction’ and ‘any number’</i>	162
6.2	YEAR 8 INTERVIEWS	166
6.2.1	<i>Variation 1: Change of number 1</i>	167
6.2.2	<i>Variation 2: Change of number 2 and ‘any number’</i>	168
6.2.3	<i>Variation 3: ‘Any fraction’ and ‘any number’</i>	169
6.3	SCORING THE STRUCTURED INTERVIEWS	171
6.3.1	<i>Comparison of primary students’ written and interview responses</i>	172
6.3.2	<i>Comparison of Year 8 students’ written and interview responses</i>	173
6.4	IMPLICATIONS FROM THE ANALYSIS OF THE STRUCTURED INTERVIEW	174
CHAPTER 7: CASE STUDIES		176
7.1	CASE STUDY 1: VIOLET	181
7.1.1	<i>Fraction Screening Test</i>	181
7.1.2	<i>Reverse Fraction Tasks</i>	182
7.1.3	<i>Structured Interview</i>	184
7.1.4	<i>Algebraic Thinking Questionnaire</i>	186
7.2	CASE STUDY 2: GLORIA.....	188
7.2.1	<i>Fraction Screening Test</i>	188
7.2.2	<i>Reverse Fraction Tasks</i>	189
7.2.3	<i>Structured Interview</i>	191
7.2.4	<i>Algebraic Thinking Questionnaire</i>	194
7.3	CASE STUDY: KATE	197
7.3.1	<i>Fraction Screening Test</i>	198
7.3.2	<i>Reverse Fraction Tasks</i>	199
7.3.3	<i>Structured Interview</i>	200

7.3.4	<i>Algebraic Thinking Questionnaire</i>	201
7.4	CASE STUDY 4: DANIEL	203
7.4.1	<i>Fraction Screening Test</i>	203
7.4.2	<i>Reverse Fraction Tasks</i>	204
7.4.3	<i>Structured Interview</i>	205
7.4.4	<i>Algebraic Thinking Questionnaire</i>	207
7.5	CASE STUDY 5: ANDREW	210
7.5.1	<i>Fraction Screening Test</i>	211
7.5.2	<i>Reverse Fraction Tasks</i>	212
7.5.3	<i>Structured Interview</i>	213
7.5.4	<i>Algebraic Thinking Questionnaire</i>	214
7.6	CASE STUDY 6: ALEX	216
7.6.1	<i>Fraction Screening Test</i>	216
7.6.2	<i>Reverse Fraction Tasks</i>	218
7.6.3	<i>Structured Interview</i>	218
7.6.4	<i>Algebraic Thinking Questionnaire</i>	219
7.7	CASE STUDY 7: TANYA	224
7.7.1	<i>Fraction Screening Test</i>	224
7.7.2	<i>Reverse Fraction Tasks</i>	225
7.7.3	<i>Structured Interview</i>	226
7.7.4	<i>Algebraic Thinking Questionnaire</i>	227
7.8	CASE STUDY 8: LEWIS	230
7.8.1	<i>Fraction Screening Test</i>	231
7.8.2	<i>Reverse Fraction Tasks</i>	232
7.8.3	<i>Structured Interview</i>	232
7.8.4	<i>Algebraic Thinking Questionnaire</i>	233
7.9	CASE STUDY 9: JACK	237
7.9.1	<i>Fraction Screening Test</i>	237
7.9.2	<i>Reverse Fraction Tasks</i>	239
7.9.3	<i>Structured Interview</i>	239
7.9.4	<i>Algebraic Thinking Questionnaire</i>	240
7.10	SUMMARISING THE CASE STUDIES	244
CHAPTER 8: DISCUSSING THE RESEARCH QUESTIONS.....		246
8.1	RESEARCH QUESTION 1	247

8.2	RESEARCH QUESTION 2	248
8.2.1	<i>Relationship between reverse fraction tasks and Part A tasks</i>	248
8.2.2	<i>Relationship between reverse fraction tasks and Part B tasks</i>	249
8.2.3	<i>Relationship between reverse fraction and other Part C tasks</i>	250
8.2.4	<i>Relationship between reverse fraction tasks and Parts A, B, C (FST)</i>	251
8.3	RESEARCH QUESTION 3	252
8.3.1	<i>The written responses to the three reverse fraction tasks (FST)</i>	252
8.3.2	<i>The Structured Interview</i>	253
8.4	RESEARCH QUESTION 4	254
8.5	RESEARCH QUESTION 5	255
8.5.1	<i>Three reverse fraction tasks</i>	255
8.5.2	<i>The Algebraic Thinking Questionnaire</i>	257
8.5.3	<i>The Structured Interview</i>	257
8.6	RESEARCH QUESTION 6	258
8.7	RESEARCH QUESTION 7	261
CHAPTER 9: CONCLUSION AND IMPLICATIONS		262
9.1	KEY FINDINGS FROM THIS RESEARCH.....	263
9.1.1	<i>The three reverse fraction tasks</i>	263
9.1.2	<i>The Structured Interview</i>	264
9.1.3	<i>The two Frameworks</i>	264
9.2	IMPLICATIONS FOR TEACHING.....	265
9.2.1	<i>Assessments and frameworks monitor progress towards algebraic reasoning.</i>	266
9.2.2	<i>Broader interpretation of the curriculum</i>	267
9.2.3	<i>Appropriate sequencing – going beyond the numbers</i>	267
9.2.4	<i>Use transferable models</i>	268
9.2.5	<i>Value students’ strategies</i>	269
9.2.6	<i>Awareness of common misconceptions and difficulties</i>	269
9.3	LIMITATIONS OF THIS STUDY	269
9.4	IMPLICATIONS FOR FURTHER RESEARCH	270
9.5	THE IMPORTANCE OF THIS STUDY	271
REFERENCES		272
APPENDIX A: FRACTION SCREENING TEST		283
APPENDIX B: ALGEBRAIC THINKING QUESTIONNAIRE		289

List of Tables

Table 1-1: The sub-strands for <i>Number and Algebra</i> Strand (ACARA, 2018).....	3
Table 1-2: Descriptions of Proficiencies (Australian Curriculum: Mathematics, ACARA, 2018)3	
Table 1-3: Examples of proficiencies for Years 5, 6 and 8 (ACARA, 2018)	4
Table 1-4: Examples of Australian Curriculum: Mathematics sub-strands (ACARA, 2018).....	5
Table 1-5: Comparison of the Australian and Victorian Content Descriptions for Year 6	7
Table 2-1: Rational number interpretations compared to Content Descriptions (ACARA, 2018)	10
Table 2-2: Fractional Operations (See McCloskey & Norton, 2009, p.46).....	13
Table 2-3: Fractional schemes with descriptions of operations and sample tasks	14
Table 2-4: Examples of Lee & Hackenberg (2014) tasks	19
Table 3-1: Trialling the assessment instruments	24
Table 3-2: The structure of the two assessment instruments.....	25
Table 3-3: Examples from the FST for Keiren's five interpretations of fractions (1980)	30
Table 3-4: Scoring for Reverse Fraction Task 1 (Task C5, FST).....	31
Table 3-5: Percentage of students achieving each result for Part A tasks, FST (n = 67)	32
Table 3-6: Percentage of students achieving each result for Part B tasks, FST (n = 67)	33
Table 3-7: Percentage of students achieving each score for Part C tasks, FST (n = 67).....	33
Table 3-8: FST Tasks with Corrected Item-Total values of less than 0.3	38
Table 3-9: Correlations between results for Parts A, B and C (FST).....	39
Table 3-10: Examples of Question 1 tasks from Parts M and D (ATQ)	43
Table 3-11: Examples of Question 2 tasks from Parts M and D (ATQ)	43
Table 3-12: Mathematics Content Descriptors with corresponding ATQ tasks.....	44
Table 3-13: Scores for Tasks C5 - C7, FST (n = 18)	44
Table 3-14 Scores for Question 1, Part M (ATQ) (n = 10)	46
Table 3-15: Scores for Question 2, Part M (ATQ) (n = 10)	47
Table 3-16: Correlations between Tasks C5 – C7 (FST) and Part M tasks (ATQ).....	48
Table 3-17: Scores for Question 1 Part D ATQ (n = 8)	48
Table 3-18: Scores for Question 2, Part D, ATQ (n = 8)	49
Table 3-19: Correlation of scores for reverse fraction tasks and Part D (ATQ) (n = 8)	49
Table 3-20: Green group	51
Table 3-21: Blue Group.....	52

Table 4-1: Phases of the sequential explanatory mixed method	54
Table 4-2: Number and year level of participants by school (n = 607).....	56
Table 4-3: Example of the scoring scheme used for Task A1 (FST)	58
Table 4-4: Example of the marking scheme for Task M1a (ATQ)	64
Table 4-5: Comparison of Structured Interview questions with Tasks C5 - C7 (FST).....	65
Table 4-6: Examples of Type A fraction tasks	68
Table 4-7: Examples of Type B fraction tasks	68
Table 4-8: Examples of Type C fraction tasks	69
Table 4-9: Example of Type D fraction task	70
Table 4-10: Gender and year level of interviewed students (n = 43)	76
Table 4-11: Dominant methods used by students for the three reverse fraction tasks (n = 43) ..	76
Table 4-12 : The Structured Interview Scoring Framework.....	77
Table 5-1: Percentage of correct responses by year level for Tasks A1 – A4 (FST)	81
Table 5-2: Percentage of correct responses by year level for Tasks A5 – A8 (FST)	83
Table 5-3: Content Descriptions related to counting by fractions (ACARA, 2018).....	85
Table 5-4: Percentage of correct responses by year level for Tasks A9 – A12 (FST)	86
Table 5-5: Percentage of correct responses by year level for Tasks B1a – B2d (FST).....	90
Table 5-6: Percentage of correct responses by year level for Tasks B3 – B5b (FST).....	93
Table 5-7: Percentage of correct responses by year level for Tasks C1 and C2 (FST).....	97
Table 5-8: Percentage of correct responses by year level for Tasks C5 – C7 (FST).....	98
Table 5-9: Classification of strategies used to solve Task C5 (FST)	100
Table 5-10: Percentage of students by year level for each response type for Task C5 (FST) ..	101
Table 5-11: Classification of strategies used to solve Task C6 (FST)	102
Table 5-12: Percentage of students by year level for each response type for Task C6 (FST) ..	103
Table 5-13: Classification of strategies used to solve Task C7 (FST)	104
Table 5-14: Percentage of students by year level for each response type for Task C7 (FST)) .	105
Table 5-15: Classification Framework for Reverse Fraction Tasks	106
Table 5-16: Relationship between Tasks C5 - C7 and Parts (FST).....	110
Table 5-17: Relationship between Tasks C5 – C7 and Part A tasks (FST).....	112
Table 5-18: Relationship between Tasks C5 – C7 scores and Part B scores (FST).....	113
Table 5-19: Relationship between Tasks C5 – C7 scores and Tasks C1 – C4 scores (FST)	114
Table 5-20: Percentage of correct responses by year level for Task M1a (ATQ).....	116
Table 5-21: Percentage of responses by year level for explanation for Task M1a (ATQ).....	117
Table 5-22: Percentage of responses by year level for Task M1b (ATQ).....	118

Table 5-23: Percentage of responses by year level for explanation for Task M1b (ATQ)	119
Table 5-24: Percentage of responses by year level for Task M1c (ATQ).....	120
Table 5-25: Percentage of responses by year level for explanation for Task M1c (ATQ).....	121
Table 5-26: Percentage of responses by year level for Task M1d (ATQ).....	123
Table 5-27: Percentage of responses by year level for explanation for Task M1d (ATQ).....	124
Table 5-28: Percentage of responses by year level for Task M2a (ATQ).....	126
Table 5-29: Percentage of responses by year level for Task M2b (ATQ).....	126
Table 5-30: Percentage of responses by year level for Task M2c (ATQ).....	127
Table 5-31: Percentage of responses by year level for explanation of Task M2c (ATQ).....	128
Table 5-32: Percentage of responses by year level for Task M2d (ATQ).....	129
Table 5-33: Percentage of responses by year level for Task M2e (ATQ).....	130
Table 5-34: Percentage of responses by year level for Task D1a (ATQ).....	132
Table 5-35: Percentage of responses by year level for explanation for Task D1a (ATQ)	132
Table 5-36: Percentage of responses by year level for Task D1b (ATQ)	134
Table 5-37: Percentage of responses by year level for explanation for Task D1b (ATQ).....	134
Table 5-38: Percentage of responses by year level for Task D1c (ATQ).....	136
Table 5-39: Percentage of responses by year level for explanation for Task D1c (ATQ)	136
Table 5-40: Percentage of responses by year level for Task D1d (ATQ)	137
Table 5-41: Percentage of responses by year level for explanation for Task D1d (ATQ).....	138
Table 5-42: Percentage of responses by year level for Task D2a (ATQ).....	139
Table 5-43: Percentage of responses by year level for Task D2b (ATQ)	140
Table 5-44: Percentage of responses by year level for Task D2c (ATQ).....	142
Table 5-45: Percentage of responses by year level for explanation of Task D2c (n = 607).....	142
Table 5-46: Percentage of responses by year level for explanation of Task D2d (FST).....	143
Table 5-47: Percentage of responses by year level for explanation of Task D2e (ATQ).....	144
Table 6-1: Number of correct primary responses by year level for SI questions (n = 17).....	158
Table 6-2: Primary students' strategies by year level for first set of SI questions (n = 17).....	159
Table 6-3: Primary students' strategies by year level for second set of SI questions (n = 17) .	161
Table 6-4: Primary students' strategies for final SI question (n = 17)	165
Table 6-5: Strategies used by the primary students for the SI Questions (n = 17).....	166
Table 6-6: Number of correct Year 8 responses for SI questions (n = 26)	167
Table 6-7: Analysis of Year 8 students' strategies for first set of SI questions (n = 26)	168
Table 6-8: Analysis of Year 8 students' strategies for second set of SI questions (n = 26).....	168
Table 6-9: Year 8 student responses to Question 7 (Structured Interview).....	169

Table 6-10: Strategies used by Year 8 students for the SI Questions (n = 26).....	170
Table 6-11: The Emerging Algebraic Reasoning Framework.....	171
Table 6-12: Classification of students' interview responses.....	172
Table 6-13: Comparison of primary students' written (FST) and interview responses (SI).....	172
Table 6-14: Comparison of Year 8 students' written (FST) and interview responses (SI).....	173
Table 7-1: Structured Interview results for case study students (n = 9).....	177
Table 7-2: Results from the three assessment protocols for the case study students	178
Table 7-3: Comparison of scores for case study students for FST and ATQ.....	179

List of Figures

Figure 3-1: Task A9 (FST)	26
Figure 3-2: Part A reverse thinking fraction tasks A4 and A12 (FST).....	26
Figure 3-3: Task B3 (FST)	27
Figure 3-4: Task B4 (FST)	27
Figure 3-5: Task B5 (FST)	28
Figure 3-6: Task C1 and C2 (FST).....	28
Figure 3-7: Tasks C5, C6 & C7 (FST)	29
Figure 3-8: Student KL scales up fractions and whole numbers in parallel for Task C5 (FST) .	35
Figure 3-9: Student JL uses a pro-numeral to solve Task C6 (FST)	35
Figure 3-10: Student BM scales up fractions and whole numbers in parallel for Task C6 (FST)	36
Figure 3-11: WA's initial and elaborated verbal response for Task C7 (FST)	36
Figure 3-12: Student JL's example of Response Type 1 for Task C5 (FST)	40
Figure 3-13: Student KL's example of Response Type 3 for Task 5C (FST).....	41
Figure 3-14: Examples of Response Type 5 for Tasks C5 & C7 (FST).....	41
Figure 3-15: Student AA's response to Task C5 (FST)	45
Figure 3-16: Student AP's multiplicative response to Task C7 (FST).....	45
Figure 3-17: Student ZB's response to Question 2a and 2b Part M (ATQ)	47
Figure 4-1: Student BM's uses a <i>Diagram Dependent Strategy</i> to solve Task C5 (FST).....	59
Figure 4-2: Student RH uses an <i>Additive Strategy</i> to solve Task C6 (FST).....	60
Figure 4-3: Student JB uses a <i>Partially Multiplicative Strategy</i> to solve Task C5 (FST).....	60
Figure 4-4: Student DC uses a <i>Fully Multiplicative Strategy</i> to solve Task C6 (FST).	61
Figure 4-5: Student JL uses <i>Advanced Multiplicative</i> strategies to solve Task C5 and C6 (FST)	61
Figure 4-6 : Student MB used <i>Partially Multiplicative</i> strategies for Tasks C5 - C7 (FST).....	62
Figure 4-7: Student DC used <i>Fully Multiplicative</i> strategies for Tasks C5 - C7 (FST)	62
Figure 4-8: Student JP used an <i>Advanced Multiplicative</i> strategy for Tasks C5 - C7 (FST)	63
Figure 4-9: Question 7 (SI).....	66
Figure 4-10: Student AM's solution for Task C5 (FST).....	70
Figure 4-11: Student AM's attempted solution for Task C7 (FST)	71
Figure 4-12: Student AM's <i>Fully Multiplicative</i> methods for Tasks C5 and C6 (FST)	71
Figure 4-13: Student AM's <i>Partially Multiplicative</i> solution to Task C7 (FST).....	72
Figure 4-14: Student AM's <i>Fully Multiplicative</i> solutions for Questions 1 and 2 (SI)	72

Figure 4-15: Student AM's <i>Partially Multiplicative</i> response to Question 3 (SI).....	72
Figure 4-16: Student AM's response to Question 4a (SI).....	73
Figure 4-17: Student AM's response to Question 4b (SI).....	73
Figure 4-18: Student AM's response to Question 5a (SI)	73
Figure 4-19: Student AM's response to Question 5b (SI).....	74
Figure 4-20: Student AM's response to Question 6a (SI)	74
Figure 4-21: Student AM's response to Question 6b (SI).....	75
Figure 4-22: Student AM's response to Question 7 (SI).....	75
Figure 5-1: Percentage of correct responses by year level for Tasks A1 – A4 (FST).....	82
Figure 5-2: Percentage of correct responses by year level for Tasks A5 – A8 (FST).....	84
Figure 5-3: Percentage of correct responses by year level for Tasks A9 – A12 (FST).....	87
Figure 5-4: Student RC's response to Task B1 (FST)	88
Figure 5-5: Student JM's response to Task B2 (FST).....	88
Figure 5-6: Student RC's incorrect responses for Tasks B2a and B2b (FST).....	89
Figure 5-7: Student RC's response to Task B2d (FST).....	90
Figure 5-8: Percentage of correct responses by year level for Tasks B1a, B2a and B2c (FST) .	91
Figure 5-9: Examples of two students' correct responses for Task B3 (FST)	91
Figure 5-10: Student KH's incorrect response to Task B3 (FST)	92
Figure 5-11: Examples of two correct responses for Task B4 (FST).....	92
Figure 5-12: Examples of two correct responses for Task B5 (FST).....	94
Figure 5-13: Student KH's incorrect response for Task B5 (FST)	94
Figure 5-14: Percentage of correct responses by year level for Tasks B3, B4 and B5a (FST)...	95
Figure 5-15: Examples of two students' responses for Task C1 (FST)	95
Figure 5-16: Student KH uses incorrect 'gap thinking' for Task C1 (FST).....	96
Figure 5-17: Examples of two students' responses for Task C4 (FST)	97
Figure 5-18: Percentage of correct responses by year level for Tasks C1 – C4 (FST)	97
Figure 5-19: Percentage of correct responses by year level for Tasks C5 – C7 (FST)	99
Figure 5-20: Comparison of strategies by year level for Task C5 (FST).....	102
Figure 5-21: Comparison of strategies by year level for Task C6 (FST).....	104
Figure 5-22: Comparison of strategies by year level for Task C7 (FST).....	106
Figure 5-23: Comparison of the three reverse fraction task and FST scores (n = 607)	108
Figure 5-24: Comparison of the three reverse fraction task and total for all other FST tasks. .	109
Figure 5-25: Comparison of scores for Tasks C5 – C7 and Part A scores (FST)	110
Figure 5-26: Comparison of scores for Tasks C5 – C7 and Part B scores (FST).....	111

Figure 5-27: Comparison of scores from Tasks C1 – C4 and Tasks C5 - C7 (FST).....	111
Figure 5-28: Examples of two students' correct responses to Task M1a (ATQ).....	116
Figure 5-29: Examples of two students' responses to Task M1b (ATQ)	119
Figure 5-30: Examples of two students' correct responses to Task M1c (ATQ).....	121
Figure 5-31: Examples of three students' responses to Task M1d (ATQ).....	123
Figure 5-32: Comparison of results by year level for Question 1 Part M (ATQ)	125
Figure 5-33: Examples of two students' correct responses to Task M2a (ATQ).....	125
Figure 5-34: Student GD's correct response to Task M2b (ATQ).....	127
Figure 5-35: Examples of two students' responses to Task M2c (ATQ).....	128
Figure 5-36: Student JM's response for Task M2d (ATQ).....	129
Figure 5-37: Examples of students' responses to Task M2e (ATQ).....	130
Figure 5-38: Comparison of results by year level for Question 2 tasks, Part M (ATQ)	131
Figure 5-39: Examples of two students' responses to Task D1a (ATQ).....	133
Figure 5-40: Examples of students' responses to Task D1b (ATQ)	135
Figure 5-41: Examples of students' responses to Task D1c (ATQ)	136
Figure 5-42: Student RK's response for Task D1d (ATQ)	138
Figure 5-43: Comparison of results by year level for Part D Question 1 tasks (ATQ).....	138
Figure 5-44: Examples of two students' responses for Task D2a.....	140
Figure 5-45: Examples of two students' responses to Task D2a (ATQ)	140
Figure 5-46: Examples of two students' correct responses to Task D2b.....	141
Figure 5-47: MB's response to Task D2b (ATQ)	141
Figure 5-48: Examples of correct student responses for Task D2c (ATQ).....	142
Figure 5-49: Examples of two students' responses for Task D2d (ATQ).....	143
Figure 5-50: Student MB's use of incorrect strategy for D2d (ATQ).....	144
Figure 5-51: MB's incorrect response for Task D2e (ATQ).....	145
Figure 5-52: Three students' responses to Task D2e (ATQ)	145
Figure 5-53: Comparison of results by year level for Question 2 Part D (ATQ)	145
Figure 5-54: Comparison of Tasks M1a and D1a and Tasks M1b And D1b results (ATQ).....	146
Figure 5-55: Comparison of Tasks M1c and D1c and Tasks M1d and D1d results (ATQ).....	146
Figure 5-56: Comparison of Task M2a and Task D2a (ATQ) results.....	147
Figure 5-57: Comparison of Tasks M2b and D2b and Tasks M2c and D2c results (ATQ).....	147
Figure 5-58: Comparison of Tasks M2d and D2d and Tasks M2e and D2e (ATQ).....	148
Figure 5-59: Comparison of total scores for ATQ and FST.....	149
Figure 5-60: Comparison of ATQ and Part A (FST) total scores	150

Figure 5-61: Comparison of Part M (ATQ) and Part A (FST) total scores.....	150
Figure 5-62: Comparison of total scores for Part D (ATQ) and Part A (FST).....	151
Figure 5-63: Comparison of scores for ATQ and Part B (FST).	152
Figure 5-64: Comparison of Part M (ATQ) and Part B (FST) scores	152
Figure 5-65: Comparison of total scores for Part D (ATQ) and Part B (FST).....	153
Figure 5-66: Comparison of ATQ and Part C (FST) scores.....	153
Figure 5-67: Comparison of Part M (ATQ) and Part C (FST) scores	154
Figure 5-68: Comparison of Part D (ATQ) and Part C (FST) scores.....	154
Figure 6-1: Student BM's responses to Question 1 (SI).....	159
Figure 6-2: Comparison of BM's responses to written task C6 (FST) and Question 2 (SI).....	160
Figure 6-3: Student CZ uses same strategies for Task C7 (FST) and Question 3 (SI).....	160
Figure 6-4: Student LG's response to Question 3 (SI).....	161
Figure 6-5: Student LG's additive response to Question 7 (SI).....	163
Figure 6-6: Student CE's responses to Question 6b and 7 (SI).....	164
Figure 6-7: Student PK's responses to Question 6b and Question 7 (SI)	164
Figure 6-8: Student MJ's symbolic multiplicative response to Question 7 (SI)	164
Figure 7-1: Comparison of ATQ and FST total scores for case study students	180
Figure 7-2: Violet's response to Task C5 (FST).....	182
Figure 7-3: Violet's response to Task C6 (FST)	183
Figure 7-4: Violet's response to Task C7 (FST)	183
Figure 7-5: Violet's responses to Question 1 (SI).....	184
Figure 7-6: Violet's responses to Question 2 (SI).....	185
Figure 7-7: Violet's responses to Question 3 (SI).....	185
Figure 7-8: Violet's responses to Question 4 (SI)	185
Figure 7-9: Violet's responses to Question 5 (SI)	186
Figure 7-10: Violet's response to Task M1c (ATQ)	187
Figure 7-11: Violet's response to Tasks M2a and M2b (ATQ)	187
Figure 7-12: Violet's response to Task M2c (ATQ)	187
Figure 7-13: Gloria's solution to Task C5 (FST)	190
Figure 7-14: Gloria's solution to Task C6 (FST)	190
Figure 7-15: Gloria's solution to C7 (FST).....	190
Figure 7-16: Gloria's response to Question 2 (SI).....	191
Figure 7-17: Gloria's response to Question 3 (SI).....	192
Figure 7-18: Gloria's response to Question 5 (SI)	192

Figure 7-19: Gloria's response to Question 6 (SI)	193
Figure 7-20: Gloria's response to Question 7 (SI)	193
Figure 7-21: Gloria's response to Task M1a (ATQ)	195
Figure 7-22: Gloria's incorrect response for Question 1b, Part M (ATQ)	195
Figure 7-23: Gloria's incorrect additive response for Task M1d (ATQ)	195
Figure 7-24: Gloria's response to Task M2a (ATQ)	196
Figure 7-25: Gloria's response to Tasks M2b and M2c (ATQ)	196
Figure 7-26: Gloria's response to Tasks D1a and D1b (ATQ)	196
Figure 7-27: Gloria's response to Task D1d (ATQ)	196
Figure 7-28: Gloria's response to Tasks D2a and D2b (ATQ)	197
Figure 7-29: Kate's responses to the Tasks C5 and C6 (FST)	199
Figure 7-30: Kate's responses to Task C7 (FST)	199
Figure 7-31: Kate's responses to Questions 6a and 6b (SI)	200
Figure 7-32: Kate's response to Question 7 (SI)	201
Figure 7-33: Kate's response to Question M1b (ATQ)	201
Figure 7-34: Kate's response to Task D1a (ATQ)	202
Figure 7-35: Kate's response to Task D1b (ATQ)	202
Figure 7-36: Kate's response to Task D1c (ATQ)	202
Figure 7-37: Daniel's responses to Tasks A11 and A12 (FST)	204
Figure 7-38: Daniel's responses to Task C5 and Task C6 (FST)	204
Figure 7-39: Daniel's responses to Task C7 (FST)	205
Figure 7-40: Daniel's response to Questions 1 and 2 (SI)	206
Figure 7-41: Daniel's response to Question 3 (SI)	206
Figure 7-42: Daniel's response to Question 4 (SI)	207
Figure 7-43: Daniel's response to Questions 5a and 5b (SI)	207
Figure 7-44: Daniel's response to Tasks M1a and M1b (ATQ)	208
Figure 7-45: Daniel's response to Tasks M1c and M1d (ATQ)	208
Figure 7-46: Daniel's responses to Tasks M2a and M2b (ATQ)	209
Figure 7-47: Daniel's response to Task M2c (ATQ)	209
Figure 7-48: Daniel's responses to Tasks M2d and M2e (ATQ)	209
Figure 7-49: Daniel's response to Tasks D1a and D1b (ATQ)	210
Figure 7-50: Daniel's responses to Tasks D1c and D1d (ATQ)	210
Figure 7-51: Daniel's response to Tasks D2a and D2b (ATQ)	210
Figure 7-52: Andrew's response to Task B3 (FST)	212

Figure 7-53: Andrew's 'gap thinking' response to Task C1 (FST)	212
Figure 7-54: Andrew's responses for Tasks C3 and C4 (FST)	212
Figure 7-55: Andrew's partially multiplicative response to Task C5 (FST)	213
Figure 7-56: Andrew's solution for Task C6	213
Figure 7-57: Andrew's fully multiplicative response to Task C7 (FST)	213
Figure 7-58: Andrew's correct responses for Questions 1 and 2 (SI)	214
Figure 7-59: Andrew's response for Question 4b	214
Figure 7-60: Andrew's responses for tasks M1a and M1b (ATQ)	215
Figure 7-61: Andrew's two correct responses for Task M2a (ATQ)	215
Figure 7-62: Andrew's incorrect response for Task M2e (ATQ)	215
Figure 7-63: Andrew's correct responses to Task M2a (ATQ)	216
Figure 7-64: Alex's response to Task B5 (FST)	217
Figure 7-65: Alex's response to Task C2 (FST)	218
Figure 7-66: Alex's strategies for the Tasks C5, C6 and C7 (FST)	218
Figure 7-67: Alex's response to Question 1 (SI)	219
Figure 7-68: Alex's response to Question 7 (SI)	219
Figure 7-69: Alex's response to Task M1a (ATQ)	220
Figure 7-70: Alex's response to Task M1b (ATQ)	220
Figure 7-71: Alex's response to Task M1c (ATQ)	220
Figure 7-72: Alex's response to Task M2a (ATQ)	221
Figure 7-73: Alex's responses to Tasks M2b and M2c (ATQ)	221
Figure 7-74: Alex's response to Task M2d (ATQ)	221
Figure 7-75: Alex's correct response to M2e (ATQ)	222
Figure 7-76: Alex's responses to Tasks D1a and D1b (ATQ)	222
Figure 7-77: Alex's incorrect responses to Tasks D1c and D1d (ATQ)	222
Figure 7-78: Alex's responses to Task D2a (ATQ)	223
Figure 7-79: Alex's correct response to Task D2b (ATQ)	223
Figure 7-80: Alex's response to Task D2c (ATQ)	223
Figure 7-81: Alex's responses to Tasks D2d and D2e (ATQ)	223
Figure 7-82: Tanya's responses to Task C5 and Task C6 (FST)	225
Figure 7-83: Tanya's responses to the three reverse fraction tasks (Tasks C5 - C7)	226
Figure 7-84: Tanya's response to Questions 1 and 2 (SI)	226
Figure 7-85: Tanya's responses to Questions 4a and 4b (SI)	227
Figure 7-86: Tanya's response to Question 7 (SI)	227

Figure 7-87: Tanya's response to Task M1a (ATQ).....	228
Figure 7-88: Tanya's responses to Tasks M1b and M1c (ATQ)	228
Figure 7-89: Tanya's response to Task M2a (ATQ).....	228
Figure 7-90: Tanya's response to Tasks 2b and 2c (ATQ).....	228
Figure 7-91: Tanya's responses to Tasks 2d and 2e (ATQ).....	229
Figure 7-92: Tanya's response to Tasks D1a and D1b (ATQ).....	229
Figure 7-93: Tanya's response to Tasks D1c and D1d, Part A (ATQ)	229
Figure 7-94: Tanya's response to Task M2a (ATQ)	229
Figure 7-95: Tanya's responses to Tasks D2b and D2c (ATQ).....	230
Figure 7-96: Tanya's responses to Tasks D2d and D2e (ATQ)	230
Figure 7-97: Lewis's responses to Task C5 and Task C6 (FST).....	232
Figure 7-98: Lewis's responses to Tasks C5 – C7 (FST)	232
Figure 7-99: Lewis's response to Question 1 (SI)	233
Figure 7-100: Lewis's responses to Tasks M1a and M1b (ATQ).....	233
Figure 7-101: Lewis's response to Task M1c (ATQ)	234
Figure 7-102: Lewis's response to Task M1c ATQ.....	234
Figure 7-103: Lewis's solution to Tasks M2a, M2b and M2c (ATQ)	234
Figure 7-104: Lewis's solution to Task M2d (ATQ)	235
Figure 7-105: Lewis's response to Task M2e (ATQ).....	235
Figure 7-106: Lewis's response to Tasks D1b and D1c (ATQ).....	235
Figure 7-107: Lewis's response to Task D2a (ATQ).....	236
Figure 7-108: Lewis responses to Tasks D2b and D2c (ATQ)	236
Figure 7-109: Lewis's response to Task D2e (ATQ).....	237
Figure 7-110: Jack's responses to Task C5 and Task C6 (FST).....	239
Figure 7-111: Jack's response to Task C7 (FST).....	239
Figure 7-112: Jack's response to Question 1 (SI)	240
Figure 7-113: Jack's response to Question 7 (SI)	240
Figure 7-114: Jack's response to task M1a (ATQ)	241
Figure 7-115: Jack's responses to Task M1b (ATQ)	241
Figure 7-116: Jack's responses to Tasks M1c and M1d (ATQ)	241
Figure 7-117: Jack's responses to Task M2a (ATQ)	242
Figure 7-118: Jack's response to Task M2d (ATQ).....	242
Figure 7-119: Jack's response to M2e (ATQ).....	242
Figure 7-120: Jack's responses to Tasks D1a and D1b (ATQ).....	243

Figure 7-121: Jack's response to Task D1c (ATQ)	243
Figure 7-122: Jack's response to D1d (ATQ)	243
Figure 7-123: Jack's response to Task D2a (ATQ)	244
Figure 7-124: Jack's response to D2e (ATQ)	244

Chapter 1: Introduction

To succeed in mathematics, middle-years students must move from additive to multiplicative thinking and from arithmetic calculations to generalised algebraic reasoning. Jacobs, Franke, Carpenter, Levi, and Battey (2007) emphasise the need to “facilitate students’ transition to the formal study of algebra in the later grades (of the elementary school) so that no distinct boundary exists between arithmetic and algebra” (p.261). The National Mathematics Advisory Panel (NMAP, 2008) in the USA stated that the conceptual understanding of fractions, and fluency in solving fraction problems, are central goals of students’ mathematical development and are the critical foundations for algebraic learning. Monitoring and prompting students’ progress along this learning trajectory is one of the challenges for middle years’ mathematics teaching.

Student solutions to fractional tasks can take many forms. Solutions can be diagrammatic e.g. when asked to demonstrate what one-half looks like students may draw a shape with one of two equal parts shaded or draw a collection of small shapes and circle one-half of these shapes. Subsequently some students learn that they can use multiplicative operations to calculate a fractional part of a given whole e.g. to find two-thirds of 12 objects they may find one-third of 12 that is 4 and then double it to get 8 (See for example, Stephens & Pearn, 2003).

However, many students appear to find it more difficult to find the number of objects in the whole group if given the number of objects in a specified part (Pearn & Stephens, 2007). In simple cases, such as where the known part of objects represents a unit fraction, repeated addition or multiplication can be used to find the unknown whole using either concrete materials, diagrams or mental calculations. For example, if students were told that there were four apples left in a box and this was one-half of the apples that were in box at the beginning of the day many would double four and state correctly that there were eight apples at the beginning of the day. However, these procedures become more difficult to undertake when the known part represents a proper or improper fraction of the unknown whole. This research will focus on the types of fraction problems that require more sophisticated multiplicative thinking (usually involving both multiplication and division) to find the unknown whole.

Some of the methods used by students to solve these latter problems seem to anticipate or mimic those algebraic methods that are required to solve equations of the form $\frac{a}{b}(x) = c$. Algebraic reasoning in this study is not evidenced by students setting up and solving algebraic equations of the type mentioned previously. However, student thinking can be described as algebraic or emerging algebraic, if, when presented with a specific problem, they reason in one or more of the following ways. First, for example, when told that the known part is $\frac{2}{7}$ of the unknown whole, they reason that: “I need to find $\frac{1}{7}$ and I can do this by dividing the known part by two, and then

to find $\frac{7}{7}$ I need to multiply this new part by 7 to find how many make up the whole.” Some students will complete this task using a sequence of division and multiplication operations; and sometimes these may be expressed in idiosyncratic ways, but the algebraic thinking is clear. Alternatively, some students argue that if the known part is $\frac{2}{7}$ of an unknown whole, the quantity representing the known part must be multiplied by $\frac{7}{2}$ to find the whole. These kinds of numerical representations closely mirror what students need to do to solve $\frac{2}{7}(x) = 16$ or some given number. In this study these problems will be referred to as “*reverse fraction problems*” to distinguish them from those more familiar problems where students are asked to find fractional parts of a known whole. Reverse fraction problems cannot be solved only by using additive methods.

For the study described in this thesis algebraic thinking or reasoning is defined in terms of students’ capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then to operate multiplicatively on both in order to find the whole. This definition has been derived from an investigation of the literature as discussed in Chapter 2 and in consultation with current subject experts which is discussed in Chapter 3.

1.1 The Australian Context

The Australian Curriculum: Mathematics (ACARA, 2018) includes three content strands which describe what is to be taught: *Number and Algebra*, *Measurement and Geometry*, and *Statistics and Probability*. According to the rationale for the curriculum document on the ACARA website:

The Australian Curriculum: Mathematics provides students with essential mathematical skills and knowledge in number and algebra, measurement and geometry, and statistics and probability. It develops the numeracy capabilities that all students need in their personal, work and civic life, and provides the fundamentals on which mathematical specialties and professional applications of mathematics are built.

Each content stand has several sub-strands. Under each sub-strand are numerous content descriptions which are included to illustrate the sequence of concept development across year levels. Table 1.1 includes the sub-strands for the *Number and Algebra* strand, the year levels for which they are designed, and one example of a Year 6 content description for that sub-strand. There are no content descriptions for two of the sub-strands, *Real numbers* and *Linear and non-linear relationships*, as these topics are only introduced at Year 7.

Table 1-1: The sub-strands for *Number and Algebra* Strand (ACARA, 2018)

Sub-strands	Example of Year 6 Content Descriptions for each sub-strand
Number and Place Value (F - 8)	Select and apply efficient mental and written strategies and appropriate digital technologies to solve problems involving all four operations with whole numbers (ACMNA123)
Fractions and decimals (1- 6)	Compare fractions with related denominators and locate and represent them on a number line (ACMNA)
Real numbers (7 - 10)	n/a
Money and financial mathematics (1 - 10)	Investigate and calculate percentage discounts of 10%, 25% and 50% on sale items, with and without digital technologies (ACMNA132)
Patterns and algebra (F - 10)	Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA 133)
Linear and non-linear relationships (7 - 10)	n/a

In addition to the three content strands the Australian Curriculum: Mathematics (ACARA, 2018) includes four proficiency strands that describe how the mathematical content is developed and the actions in which students can engage when learning and using the content of the Australian Curriculum: Mathematics. The four proficiencies: *Understanding*, *Fluency*, *Problem-solving* and *Reasoning* provide the language to build in the developmental aspects of the learning of mathematics. The proficiencies reinforce the significance of working mathematically within the content and describe how the content is explored or developed. Table 1.2 documents the descriptions of the four proficiencies as given in the *Key Ideas* document from the Australian Curriculum: Mathematics (ACARA, 2018 retrieved 18th October 2019 from <https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/key-ideas/>).

Table 1-2: Descriptions of Proficiencies (Australian Curriculum: Mathematics, ACARA, 2018)

Proficiency	Description as given in <i>Key Ideas</i> (ACARA, 2018)
Understanding	Students build a robust knowledge of adaptable and transferable mathematical concepts. They make connections between related concepts and progressively apply the familiar to develop new ideas. They develop an understanding of the relationship between the ‘why’ and the ‘how’ of mathematics. Students build understanding when they connect related ideas, when they represent concepts in different ways, when they identify commonalities and differences between aspects of content, when they describe their thinking mathematically and when they interpret mathematical information.
Fluency	Students develop skills in choosing appropriate procedures; carrying out procedures flexibly, accurately, efficiently and appropriately; and recalling factual knowledge and concepts readily. Students are fluent when they calculate answers efficiently, when they recognise robust ways of answering questions, when they choose appropriate methods and approximations, when they recall definitions and regularly use facts, and when they can manipulate expressions and equations to find solutions.

Problem Solving	Students develop the ability to make choices, interpret, formulate, model and investigate problem situations, and communicate solutions effectively. Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are reasonable.
Reasoning	Students develop an increasingly sophisticated capacity for logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying and generalising. Students are reasoning mathematically when they explain their thinking, when they deduce and justify strategies used and conclusions reached, when they adapt the known to the unknown, when they transfer learning from one context to another, when they prove that something is true or false, and when they compare and contrast related ideas and explain their choices.

Table 1.3 documents the proficiencies for Years 5, 6, and 8 that relate to the topics of fractions and algebra. Year 7 proficiencies have not been included in Table 1.3 as no Year 7 students participated in this research. Year 9 is not included in this table as none of the Year 9 elaborations of the proficiencies are specifically focused on either fractions or algebraic reasoning.

Table 1-3: Examples of proficiencies for Years 5, 6 and 8 (ACARA, 2018)

Proficiencies	Year 5	Year 6	Year 8
Understanding	includes making connections between representations of numbers comparing and ordering fractions and decimals and representing them in various ways	includes representing fractions and decimals in various ways and describing connections between them	includes identifying commonalities between operations with algebra and arithmetic
Fluency	includes using estimation to check the reasonableness of answers to calculations	includes representing integers on a number line converting between fractions and decimals using operations with fractions, decimals and percentages	recognising equivalence of common decimals and fractions factorising and simplifying basic algebraic expressions
Problem Solving	includes formulating and solving authentic problems using whole numbers	includes formulating and solving authentic problems using fractions, decimals, percentages	includes formulating and modelling practical situations involving ratios
Reasoning	includes investigating strategies to perform calculations efficiently continuing patterns involving fractions and decimals	includes explaining mental strategies for performing calculations describing results for continuing number sequences	includes justifying the result of a calculation or estimation as reasonable

The specific Content Descriptions for the sub-strands *Fractions and Decimals* and *Patterns and Algebra*, from the *Number and Algebra* strand of the Australian Curriculum: Mathematics (ACARA, 2016) are shown in Table 1.4. These are the Content Descriptions for both sub-strands that should have been implemented by teachers at the time this research was conducted. The content descriptions in the left-hand column of Table 1.4 have also been reported in Pearn, Pierce and Stephens (2017a) and a subset of both columns given in Pearn and Stephens (2017; 2018).

Table 1-4: Examples of Australian Curriculum: Mathematics sub-strands (ACARA, 2018)

Year	Fractions and Decimals	Pattern and Algebra
	Compare and order common unit fractions and locate and represent them on a number line (ACMNA102)	Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction (ACMNA107)
5	Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator (ACMNA103)	Find unknown quantities in number sentences involving multiplication and division and identify equivalent number sentences involving multiplication and division (ACMNA121)
6	Compare fractions with related denominators and locate and represent them on a number line (ACMNA125) Solve problems involving addition and subtraction of fractions with the same or related denominators (ACMNA126) Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)	Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133) Explore the use of brackets and order of operations to write number sentences (ACMNA134)
7	Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line (ACMNA152) Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153) Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154) Express one quantity as a fraction of another, with and without the use of digital technologies (ACMNA155) Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157) Recognise and solve problems involving simple ratios (ACMNA173)	Introduce the concept of variables as a way of representing numbers using letters (ACMNA175) Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176) Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)

8	Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188)	Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190) Factorise algebraic expressions by identifying numerical factors (ACMNA191) Simplify algebraic expressions involving the four operations (ACMNA192)
---	--	--

As shown in the content descriptions given in Table 1.4 fractions is seen as an important topic in mathematics across Years 5 – 8. The focus at Year 6 is on finding fractional parts of a known whole but students are not expected to find the unknown whole if given a known fractional part. This may be addressed in Year 7 when students are asked to solve problems involving addition and subtraction of fractions, but the curriculum document does not specifically state the focus, purpose or content of these problems. However, there is no mention of a focus on multiplicative solutions to fraction problems especially those involving finding an unknown whole.

The *Content Descriptions* contained in Table 1.4 appear to imply that the link between fractions and algebra is limited to number patterns and sequences involving fractions. In Year 7 students are expected to be introduced to the concept of variables and to use letters to represent numbers. But any bridge between fractional knowledge and algebraic thinking is left unstated or at best implicit. The focus of this research is to highlight the importance of linking fractional competence, and, in particular demonstrating understanding, using and generalising fractional structure.

At the time when this research was undertaken the designated mathematics curriculum for Victorian schools was the Australian Curriculum: Mathematics (ACARA, 2018). Since that time the new Victorian Curriculum: Mathematics (VCAA, 2018) has been introduced. Table 1.5 compares Content Descriptions for Year 6 students from the Australian Curriculum (ACARA, 2018) and the current Victorian Curriculum (VCAA, 2018).

The only difference between these two sets of Content Descriptions appears to be the codes. This means that although there has been a change in mathematics curriculum documents for Victorian schools the purpose or intention of the mathematics curriculum has not changed in Victoria since this study was conducted.

Table 1-5: Comparison of the Australian and Victorian Content Descriptions for Year 6

Victorian Curriculum: Mathematics Content Descriptions (VCAA, 2018)	Australian Curriculum: Mathematics Content Descriptions (ACARA, 2018)
Compare fractions with related denominators and locate and represent them on a number line (<u>VCMNA211</u>)	Compare fractions with related denominators and locate and represent them on a number line (ACMNA 125)
Solve problems involving addition and subtraction of fractions with the same or related denominators (<u>VCMNA212</u>)	Solve problems involving addition and subtraction of fractions with the same or related denominators (ACMNA 126)
Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (<u>VCMNA213</u>)	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)

While there has been some research in the USA there have not been any Australian studies that investigate the links between fractional competence and algebraic thinking. Previous Australian research has focused on the links between whole numbers and rational numbers (Hunting, Davis and Pearn, 1996), multiplicative thinking and fraction knowledge (Siemon, 2003; Siemon, Breed, & Virgona, 2005), fraction misconceptions (Clarke, Roche, Mitchell & Sukenik, 2006; Mitchell & Clarke, 2004; Roche, 2005). Previous research will be discussed in more detail in Chapter 2.

1.2 Research Questions

The overall research question for this study is: How does middle-years students' fractional competence and reasoning show evidence of (anticipate) algebraic reasoning? To address the key question of this study the following research sub-questions will be investigated:

RQ1. Is overall fraction ability as measured by a broad fraction screening test a predictor of success on reverse fraction tasks?

RQ2. How are performances on specific fraction tasks linked to success on the reverse fraction tasks?

RQ3. To what degree do proficient multiplicative thinkers for fraction tasks (i.e. students who demonstrate relational thinking, represent fractions in various ways and use reverse or reciprocal thinking when solving rational number tasks) use solution strategies that demonstrate emerging or established generalised algebraic reasoning?

RQ4. How does middle-years students' fractional competence and reasoning show evidence of non-symbolic algebraic thinking and its progressive transition towards more explicit forms of symbolic thinking as expected in lower secondary classes?

RQ5. How is competence with reverse fraction tasks associated with equivalence, transformation using equivalence, and generalisation anticipated in algebraic reasoning?

RQ6. What is the relationship between verbal fluency in describing mathematical reasoning in the context of fraction tasks and demonstrated generalisations that are anticipated in algebraic reasoning?

RQ7. How can students' levels of emerging algebraic reasoning be described and identified so that teachers may monitor and prompt students' progress?

In Chapter 2 the research literature is discussed with a focus on the relevant research related to both fractional competence and algebraic reasoning. Chapter 3 describes the methodology and the trialling of the assessment instruments while Chapter 4 focuses on the specific methods used for this research. Chapter 5 focuses on the results from the two paper and pencil assessment instruments while Chapter 6 focuses on the results from the semi-structured interviews used with one group of primary-aged students and one group of Year 8 students. Chapter 7 includes case studies of nine students who used a range of strategies to respond to the questions from the Structured Interview. Chapter 8 synthesises results in order to answer the research sub-questions and address how these results support the main question being explored in this thesis: "How does middle-years students' fractional competence and reasoning show evidence of algebraic reasoning?" Chapter 9 is the concluding chapter and includes implications for both teaching and further research.

Chapter 2: Literature Review

Many researchers believe that much of the basis for algebraic thinking rests on a clear understanding of rational number concepts (Kieren, 1980; Lamon, 1999; Wu, 2001) and the ability to manipulate common fractions. According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). Kieran (2004) described algebraic thinking as: analysing relationships between quantities, noticing structure, studying change, generalising, problem solving, modelling, justifying, proving, and predicting. There is also research documenting the link between multiplicative thinking and rational number ideas (Harel & Confrey, 1994; Baturo, 1997).

The National Mathematics Advisory Panel (NMAP, 2008) from the USA stated that the conceptual understanding of fractions, and fluency in using procedures to solve fraction problems, are central goals of students' mathematical development and are the critical foundations for algebra learning. Siegler et al.'s (2012) analysis of longitudinal data from both the United States and United Kingdom showed that competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of their attainment in algebra and overall mathematics performance five or six years later when other factors were controlled. They controlled for factors such as whole number arithmetic, intelligence, working memory, and family background.

Empson, Levi and Carpenter (2010) also argue that the key to learning algebra meaningfully is to help students: "to see the continuities among whole numbers, fractions and algebra" (p. 411). They suggest that students should develop and use computational procedures using relational thinking to integrate their learning of whole numbers and fractions. Kieren (1980) suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions, and language to precede the formal work with fractions, decimals, and rational algebraic forms.

In this chapter the focus will be on the research literature that looks at fractional competence, algebraic thinking and reasoning and finally the research that links fractional competence with success in algebraic reasoning.

2.1 Fractional Competence

Over the past 20 years, research about rational number learning has focused on development of basic fraction concepts, including partitioning of a whole into fractional parts, naming of

fractional parts, and order and equivalence (see for example, Behr, Wachsmuth, Post, & Lesh, 1984; Kieren, 1983; Pothier & Sawada, 1983; Streefland, 1984; 1991; Vergnaud, 1983). Kieren (1976) distinguished seven interpretations of rational number which were necessary to enable the learner to acquire sound rational number knowledge, but subsequently (Kieren, 1980; 1988) condensed these into five interpretations: whole-part relations, measures, operators, quotients and ratios. Examples are given in Table 2.1 to explain each of Kieren's interpretations (1988) with links to the Content Descriptors from the Australian Curriculum: Mathematics (ACARA, 2018).

Table 2-1: Rational number interpretations compared to Content Descriptions (ACARA, 2018)

Rational number interpretations (Kieren, 1988)		Australian Curriculum: Mathematics Content Descriptors (ACARA, 2018)
Interpretation	Example	
Whole-part relations	A continuous quantity can be partitioned or divided (e.g. length, area, and volume models) or a set of discrete objects can be shared into equal-sized parts or groups.	Compare and order common unit fractions (ACMNA102, Year 5) Compare fractions with related denominators (ACMNA125, Year 6)
Measures	A fraction can represent a measure of a quantity relative to one unit of that quantity. The number of equal parts in a unit can vary depending on how many times it is divided or partitioned e.g. number lines	Compare and order common unit fractions and locate and represent them on a number line (ACMNA102, Year 5) Compare fractions with related denominators and locate and represent them on a number line (ACMNA125, Year 6) Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line (ACMNA152, Year 7)
Operators	A fraction can be used as an operator to scale a number up or down e.g. $\frac{2}{3}$ of 12 = 8 or $\frac{7}{6}$ x 12 = 14	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127, Year 6) Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154, Year 7)
Quotients	A fraction ($\frac{a}{b}$) may represent the operation of division or the result of a division e.g. $4 \div 5 = \frac{4}{5}$. For example, this could represent the amount of cake each student would receive if four cakes were shared equally between five students.	Express one quantity as a fraction of another, with and without the use of digital technologies (ACMNA155, Year 7)
Ratios	Fractions can be used to compare the size of two sets or two measurements e.g. The ratio of blue counters to red counters can be expressed as 2:3. If there were 10 counters 4 counters would be blue and 6 counters would be red. This means that $\frac{2}{5}$ of the set of 10 counters are blue and $\frac{3}{5}$ are red.	Compare fractions using equivalence ... (ACMNA152, Year 7) Recognise and solve problems involving simple ratios (ACMNA173, Year 7) Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188, Year 8)

Pothier and Sawada (1983) and Kieren (1976) have shown that students have a rich store of informal knowledge of partitioning and equivalence of fractions. Gunderson and Gunderson (1957) and Mack (1990) concluded that students not only possess informal knowledge, which Mack defined as "applied, real-life circumstantial knowledge constructed by an individual student in response to problems posed in the context of real-life situations familiar to him or her" (p. 17), but that they are often able to connect this knowledge with formal symbols and procedures. Mack also found that students focused on symbolic manipulations when they possessed knowledge of rote procedures. The findings from Mack's study suggested that if partitioning was used to develop a strand of rational number using the students' informal knowledge, then other strands of rational number could be developed more easily.

Saenz-Ludlow (1994) maintained that students needed to conceptualise fractions as quantities before being introduced to standard fractional symbolic computational algorithms. Streefland (1984; 1991) discussed the importance of students developing their understanding of fractions by constructing the procedures of the operations, rules and language of fractions.

Lamon (1993) noted that student's modelling and counting strategies demonstrated conceptually based understanding and an adept ability to use whole number strategies. Students were more likely to be able to solve ratio and proportion tasks when they combined counting and matching strategies. To determine the whole number strategies that might facilitate or impede children's progress towards rational number related concepts, researchers need to determine the extent of the children's whole number knowledge and the type and quality of the strategies used.

Several researchers have noted how children's whole number schemes can interfere with their efforts to learn fractions (Behr, Wachsmuth, Post, & Lesh, 1984; Bezuk, 1988; Hunting, 1986; Hunting & Davis, 1991; Streefland, 1984; Van Hoof, Lijnen, Verschaffel & Van Dooren, 2013). Behr, Wachsmuth, Post and Lesh (1984) referred to this focus on whole number thinking as "whole number dominance" which they defined as "making separate comparisons of numerators and denominators using the ordering of whole numbers" (p. 332). Hart (1981) noted that: "a fraction of course involves two whole numbers which have to be dealt with as if they were irrevocably linked" (p.69). According to Hart (1981) the "irrevocable" link is the ratio between numerator and denominator.

Behr and Post (1988) stated that students needed to be competent in the four operations using whole numbers, along with an understanding of measurement, to enable them to understand rational numbers. They noted that rational numbers are the first set of numbers experienced by children that are not dependent on a counting algorithm and the required shift of thinking caused

difficulty for many students. Siegler, Fazio, Bailey and Zhou (2013) suggested that “understanding fractions requires learning that many principles that apply to whole numbers do not apply to fractions” (p.15).

An Australian research project was designed to investigate the extent to which children's thinking processes might be associated with qualitative differences in their whole number knowledge when solving rational number tasks (Davis, Hunting, & Pearn, 1993a, 1993b; Hunting, Davis, & Pearn, 1996). This research highlighted the vast difference in the children's mathematical knowledge and the types of whole number strategies they used when solving mathematical tasks. The most successful students in solving whole number tasks were more successful, and used superior strategies, when solving rational number tasks. Students who relied on rules and procedures when solving whole number tasks were less successful with rational number tasks. These students experienced some success with partitioning and ratio tasks but little or no success with fraction tasks set in other contexts.

Pearn, Stephens, and Lewis (2002) discussed the differences in students' responses to a set of fraction tasks. In particular, the difficulties with fraction tasks experienced by students struggling with mathematics, were elaborated. In this study it was found that many secondary students classified by their teachers to be good at mathematics also had many misconceptions about fractions. These misconceptions had remained hidden as the students could apply the correct rules and procedures to successfully complete conventional classroom mathematics tasks and tests. The misconceptions about fractions were only revealed when students are asked to apply their fractional knowledge in unfamiliar contexts. For example, many Year 8 students were able to identify or complete equivalent fractions but when asked to compare two fractions many appeared to use a range of inappropriate procedures including looking at the differences between the numerator and denominator.

Steffe and Olive (1990, 1993) showed that students used their natural language in their construction of fraction knowledge, concepts and operations. The goal of their project was to identify accommodations which children made to their counting sequences that would yield fraction schemes. Two distinct fraction schemes emerged from their research. In the iterative scheme, children established a unit fraction as part of a continuous but segmented unit. From this, children developed their own fraction knowledge by iterating unit fractions.

Several researchers have noted the limitations of part-whole reasoning (e.g. Behr, Harel, Post, & Lesh, 1992; Mack, 2001; Streefland, 1993). Part-whole reasoning allows students to recognise and use proper fractions as they show that the proper fraction $\frac{a}{b}$ (where a is less than b) has a

parts out of b equal parts altogether in the whole. For example, students using a part-whole scheme who are given the fraction $\frac{4}{5}$ will most likely partition a continuous bar or rectangular shape into five equal parts and indicate four of the parts as the number required. However, part-whole reasoning is very limited, especially in situations involving improper fractions, as showing $\frac{7}{5}$ as seven parts out of five equal parts makes little sense (Tzur, 1999). This approach does not promote generalised reasoning.

Building on the work of Olive (1999) and Steffe (2002) Norton and McCloskey (2008) identified four key fractional operations (mental actions) that students use to understand fractions: Unitizing, Partitioning, Disembedding and Iterating. A year later McCloskey and Norton (2009) added Splitting to their list of fractional operations. Table 2-2 replicates Table 1 given in McCloskey and Norton (2009, p.46) and includes the definitions for the five fractional operation terms.

Table 2-2: Fractional Operations (See McCloskey & Norton, 2009, p.46)

Operation	Description	Example of using the operation
Unitizing	Treating an object or collection of objects as a unit or whole	Treating two hexagons as a whole (as with pattern blocks)
Partitioning	Separating a unit/whole into equal parts	Sharing a pizza among four people
Disembedding	Imaginatively pulling out a fraction from the whole while keeping the whole intact and unaltered	After the pizza has been sliced in fourths, imagining what three-fourths of the pizza would look like
Iterating	Repeating a part to produce identical copies of it	Use a one-fifth piece to identify a three-fifths piece (as with fraction rods)
Splitting	The simultaneous composition of partitioning and iterating	“This bar is five times as long as another bar. Draw the other bar.”

McCloskey and Norton (2009) discuss Steffe’s fraction schemes used to describe the operations or mental actions that students use to solve fraction tasks but emphasise that schemes are different to strategies because: “schemes describe ways of operating that usually occur outside the student’s awareness. Also, schemes are activated at once, rather than in a sequential manner” (p. 46).

Norton and McCloskey (2008) described schemes as teacher constructs because: “The teacher attributes the scheme to the student only because it is useful in explaining and predicting the student’s actions.” (p. 49). McCloskey and Norton (2009) included researchers as well as teachers stating that: “Schemes are constructs used by teachers and researchers to model students’ cognitive structures” (p. 46). Table 2-3 includes a summary of Steffe’s fraction schemes used by

teachers and researchers to characterise student thinking with examples of tasks designed to encourage students to use these schemes (See Table 2 in McCloskey and Norton, 2009, p.47).

Table 2-3: Fractional schemes with descriptions of operations and sample tasks

Scheme	Operations	Sample Task
Simultaneous partitioning scheme	Unitizing the whole, partitioning the continuous whole using a composite unit as a template	Share this candy bar equally among you and two friends
Part-whole scheme	Unitizing, partitioning, disembedding a part from the partitioned whole	Show me two-thirds of the candy bar
Equi-partitioning scheme	Unitizing, partitioning, iterating any part to determine its identity with the other parts	If you share this candy bar equally among you and two friends, show me what your piece would look like.
Partitive unit fractional scheme	Iterating a given unit fraction to produce a continuous partitioned unitized whole.	If I give you this much (show a one-third piece and an unpartitioned whole), what fraction of the candy bar would you have?
Partitive fractional scheme	Unitizing, disembedding a proper fraction from the whole, hypothetically partitioning the proper fraction to produce a unit fraction, iterating the unit fraction to produce the proper fraction and the whole, coordinating unit fractions within a composite fraction (units coordinating at two levels)	If I give you this much (show an unpartitioned two-thirds piece and unpartitioned whole), what fraction of the candy bar would you have?
Reversible partitive fractional scheme	Splitting (that is, partitioning and then iterating) an unpartitioned piece of a larger whole to re-create the whole.	If the bar is four-fifths as long as your candy bar [show an unpartitioned piece], draw what your candy bar would look like.
Iterative fractional scheme	Splitting (that is, partitioning and then iterating) an unpartitioned piece of a smaller whole to re-create the whole	If the bar is five-fourths as long as your candy bar [show an unpartitioned piece], draw what your candy bar would look like.

More recently, Steffe and Olive (2010) suggested a possible learning trajectory for students' progress from less sophisticated schemes, such as the part-whole scheme, toward more sophisticated schemes that account for fractional sizes relative to the whole.

Stephens and Pearn (2003) identified Year 8 proficient fractional thinkers as students who demonstrated a capacity to represent fractions in various ways, and to use reverse thinking with fractions to solve problems. Their research also suggested that effective reverse thinking depends on a capacity to apply multiplicative operations to transform a known fraction to the whole.

Siegler, Fazio, Bailey and Zhou (2013) suggest that accurate representation of fraction magnitudes is crucial both to conceptual understanding of fractions and computations with fractions. Fuchs et al. (2013) showed that when students received instruction that emphasised the use of number lines and other representations that focussed on fraction magnitudes rather than an emphasis on part-whole understanding and fraction arithmetic procedures made greater improvement in conceptual understanding of fractions and proficiency with fraction arithmetic. Thus, it would seem that improving understanding of fraction magnitudes should be an important goal to ensure improvement of fraction knowledge more generally.

An examination of the literature would indicate that fraction competence includes: understanding fraction size and relationships, demonstrating understanding of fraction concepts and basic arithmetic competence with simple fractions.

2.2 Algebraic Thinking

The terms algebraic thinking and algebraic reasoning appear to be used interchangeably in much of the mathematics education research literature. Jacobs, Franke, Carpenter, Levi and Battey (2007) and Stephens and Ribeiro (2012) define algebraic thinking as students' understanding of equivalence, transformation using equivalence, and the use of generalisable methods. Kaput (2008) suggested that algebraic thinking consisted of two important aspects. The first aspect is making and expressing generalisations in increasingly formal and conventional symbol systems. The second is the ability to reason with, and manipulate, symbolic forms. Similarly, Kieran (1989) stated that a necessary component of algebraic generalisation "is the use of algebraic symbolism to reason about and to express that generalization" (p. 165). Radford and Pierce (2006) suggested that algebraic thinking is a particular form of reflecting mathematically. Radford noted that generalising a pattern algebraically depends on students noticing the similarity about the terms in that particular pattern, being aware that the similarity is common for all the terms of the pattern, and then being able to use this similarity to articulate the pattern.

Kaput (1988) suggested that algebraic reasoning involves students' capacity to build, justify and express conjectures about mathematical structure and relationships. Blanton and Kaput (2011) suggest that one form of algebraic reasoning involves functional thinking that they describe as: "building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right" (p.8). More recently Blanton et al (2018) stated that primary students

can engage in algebraic thinking when they generalise, represent, justify and reason with a variety of mathematical structures and relationships.

Blanton and Kaput (2011, p. 8) used the work of Smith (2008) to elaborate the different types of functional thinking which include:

- Recursive patterning that includes finding variation within a sequence of values.
- Co-variational thinking that includes the analysis of two quantities and how they vary simultaneously.
- Correspondence relationship that is based on identifying the relationship between variables.

Blanton and Kaput (2011) suggest that:

... varying a problem parameter enables students to generate a set of data that has a mathematical relationship, and using sufficiently large quantities for that parameter leads to the algebraic use of number (p.17)

2.2.1 Transition from arithmetic to algebra

Kieran (2004) suggests that students need to make many adjustments in the way they think when making the transition from arithmetic to algebra. To develop an algebraic way of thinking, students need to focus on:

- the relationships and not just on the calculation
- the inverses of the operations and the related idea of doing/undoing
- both the representation and the solution of a problem rather than just the solution
- focus on both numbers and letters/symbol
- refocus on the meaning of the equal sign

Tall (1991) stated that:

A procedural view of arithmetic may lead to the interpretation of an algebraic expression as a process which cannot be carried out and cause a considerable conceptual obstacle, which, even if overcome, may lead to a procedural view of algebra. On the other hand, a flexible, proceptual, view of arithmetic, manipulating symbols so that known relationships can be used to deduce derived facts is more appropriate to form a foundation for meaningful manipulation of algebraic symbolism (p. 18).

According to Mason, Graham, Pimm and Gower (1985) thinking algebraically relies on the recognition and description of patterns used to solve specific questions but does not require the

use of symbols. However Kaput, Blanton and Moreno (2008) link algebra to the use of conventional algebraic symbols while Carraher and Schliemann (2018) extend the notion of algebraic thinking to a variety of representations including linguistic, tabular, graphical and diagrammatic.

... reasoning that expresses itself as statements or other representations denoting relations among sets of elements, typically numbers or quantities. We have found it useful to include, as representations of algebraic thinking, various forms (linguistic, tabular, graphical, diagrammatic etc.) in addition to algebraic notation because algebraic thinking may be cultivated before algebraic notation is introduced (p. 131)

2.2.2 Relational Thinking

Symbolic representations and problem structure may hinder students' ability to solve problems (Carpenter, Hiebert, & Moser, 1981). Many students misunderstand the equals sign (see for example, Kieran, 1981; Falkner, Levi & Carpenter, 1981). Many students do not understand that the equal sign is relational, that is, shows that a relationship exists between the numbers or expressions on each side of the equal sign (Jacobs, Franke, Carpenter, Levi and Battey, 2007). The number or expression on one side of the equal sign should have the same value as the number or expression on the other. Many students see the equal sign as an operational symbol, that is, the equal sign indicates that they must give an answer. Misinterpretation of the equals sign may develop from a misunderstanding of symbols or may develop when students are used to solving standard equations which requires a calculation such as $5 + 3 = _$. If this task was rewritten as $5 + 3 = _ + 5$ a common incorrect response would be 8.

Powell (2012) suggests that understanding the equal sign in a relational way is important once students start solving equations with missing information and when the equal sign is not in the standard position e.g. $9 = _ + 6$. Herscovics and Kieran (1980) and Powell and Fuchs (2010) noted that students need to understand the relational nature of the equal sign to solve algebraic equations rather than the operational use. Carpenter, Franke, and Levi (2003) suggested that if primary/elementary students struggle to solve equations, and do not learn the correct interpretation of the equal sign, solving mathematical calculations will become more difficult as they progress through school.

Many primary students tend to think of the equals sign as indicating that they need to give an answer or record the result of a calculation. Jones, Inglis, Gilmore and Evans (2013) discussed three different conceptions of the equals sign: operational conception, sameness-relational

conception, and substitutive-relational conception. The operational conception of the equals sign is described as the expectation that the equals sign indicates that the student needs to ‘give an answer’ (Jones et al., 2013, p. 36). Researchers such as Baroody and Ginsburg (1983) and Rittle-Johnson et al. (2011) suggest that students’ reliance on the operational conception of the equals sign is reinforced by the way arithmetical equations are presented on worksheets and in textbooks for primary students. Researchers such as Knuth et al. (2008) believe that students’ dependence on the operational conception of the equals sign hinders both arithmetic and algebraic calculations while McNeil (2007) suggests that the operational conception will prove to be increasingly unsuccessful for secondary students trying to solve symbolic algebra.

The sameness-relational conception of the equals sign involves seeing the equals symbol or sign as meaning ‘is the same as’ (Jones et al., 2013, p. 34) which encourages students to see the sameness of the expressions on both sides of the equals sign thus seeing the equivalence when comparing each expression. The substitutive-relational conception involves students thinking that the equals sign also means ‘can be substituted for’ (Jones et al., 2013, p. 35) and enables students to use arithmetic rules, such as commutativity, to change the arithmetic expressions on either side of the equals sign but retain the equality. In 2008 Jones investigated whether the substitutive-relational conception was distinct from the sameness-relational conception using a computer-based arithmetic notating task called Sum Puzzles. In 2013 Jones and his colleagues attempted to address the limitations of the 2008 study and found that:

...the tasks designed to engage children with the structure of arithmetic statements in novel ways impact on conceptions of the equals sign, with a shift towards a more relational understanding (p. 52).

substitutive-relational conceptions of the equals sign are important for algebraic thinking. Gagnon and Maccini (2001) suggested that understanding the equal sign is foundational to algebraic competence.

2.3 Links between fraction competence and algebra

This current study builds on the research of Hackenberg (2005), Lee (2012), Lee and Hackenberg (2014), and Hackenberg and Lee (2015), who investigated students’ quantitative reasoning with fractions and algebraic reasoning in writing and solving equations. Hackenberg (2005) tracked the development of four Year 6 students’ multiplicative reasoning with fractional quantities when solving problems using basic linear equations. Hackenberg identified several areas for further research that included students’ construction of fractions as operators and the use of reciprocity.

Research conducted by Lee (2012), Lee and Hackenberg (2014) and Hackenberg and Lee (2015) with 18 middle school and high school students showed that fractional knowledge appeared to be closely related to establishing algebra knowledge in the domains of writing and solving linear equations. While their research is about a total of 18 students the articles about this study each report a subset of the 18 students: Lee (2012) focused on two seventh grade students, Lee and Hackenberg (2014) wrote about one of the two students, and Hackenberg and Lee (2015) focused on 12 of the larger group of 18 middle and high school students.

Their research used both a Fraction based interview and an Algebra based interview. The two interview protocols were designed so that the reasoning involved in the Fraction based interview provided a foundation for solving problems in the Algebra interview. In both interviews, students were asked to draw a picture as part of the solution. For the Fraction tasks they were also asked to find the answer whereas in the Algebra tasks they were asked to write an appropriate equation but not solve it.

In the Lee and Hackenberg research, students were given a fractional relationship between two collections of objects. The actual size of the collections was initially given but in later questions was unstated and so needed to be represented by “unknowns”. Students were first asked to “Draw a picture of the situation”, and then asked to write an algebraic equation to represent this relationship. An example of the first kind of problem used by Lee and Hackenberg (2014) is the *Tanya-David Money Problem* first presented and discussed in Pearn and Stephens (2015a) and shown in Table 2-4.

Table 2-4: Examples of Lee & Hackenberg (2014) tasks

Fraction Task	Algebra Task
Tanya has \$84, which is $\frac{4}{7}$ of David’s money. Could you draw a picture of this situation? How much does David have?	Theo has a stack of CDs some number of cm tall. Sam’s stack is $\frac{2}{5}$ of that height. Can you draw a picture of this situation? Can you write an equation?

Neither the Fraction Task nor the Algebra Task from the Lee and Hackenberg study (2014) shown in Table 2.4 can be solved additively, for example, by saying: “I have to add another three-sevenths” for the Fraction Task or “I need another three-fifths” for the Algebra Task unless they have already found a representation or measure for the remaining three-sevenths or three-fifths. The thinking required to solve these types of fraction tasks is similar to the kind of thinking required to “solve for x ” in a corresponding algebraic equation. In the Algebra task students are

not required to solve the algebra equation $S = \frac{2}{5} T$ (where S and T represent the number of CDs that Sam and Theo have).

In the Fraction Task shown in the left-hand column of Table 2-4 the fraction $\frac{4}{7}$ represents an amount of \$84. Students may attempt to represent Tanya's money by drawing a rectangular shape and dividing into four equal parts, while representing David's money by a rectangular shape divided into seven equal parts. With an appropriate pictorial representation, this problem could be solved, without writing an algebraic equation, relying on the fact that the \$84 can be split into four equal parts consisting of \$21 and so the seven equal parts would be presented by $\$21 \times 7$, that is \$147. The unknown three-sevenths can be represented by \$63 which can then be added to the \$84 giving \$147 which is described as a partially multiplicative method.

However, in the Algebra Task shown in the right-hand side of Table 2.4 the corresponding numerical quantity is unstated. Lee and Hackenberg (2014) presented this second problem as an example of reciprocal reasoning. Having drawn a picture of a rectangle that was partitioned into five equal parts, Willa, the student who was the focus of the Lee and Hackenberg study, then drew another rectangle that spanned two of those five parts. Willa used the letter t to represent the height of Theo's stack and the letter s to represent the height of Sam's stack. Willa then wrote a correct equation $s = (\frac{2}{5}) t$. A correct pictorial representation of the fractional relationship allowed Willa to show the relationship between Sam's smaller CD stack and Theo's larger CD stack regardless of the specific number of CDs making up the respective collections. Willa was also able to write the equation $t = (\frac{5}{2}) s$, explaining that she "got 10 pieces by taking Sam's two-part stack height five times and the divided by two to find Theo's five-part stack height". This use of a reciprocal solution can be described as demonstrating "reverse fractional thinking". It is interesting to note that Willa obtained the second "reciprocal" relationship by manipulating her pictorial representations, as distinct from transposing the first equation algebraically.

The research by Lee (2012), Lee and Hackenberg (2014), and Hackenberg and Lee (2015) shows that fractional knowledge is closely related to establishing algebra knowledge in the domains of writing and solving linear equations. After analysing the data, Lee (2012) constructed models to determine the fraction schemes used by students and their reasoning about unknowns and writing equations.

While these studies point to some connections between fractional competence and algebraic thinking they are based on small samples and do not include tasks requiring a generalised solution. For the Hackenberg study (2005) it was four Grade 6 students, for the Lee and Hackenberg study (2014) it was 18 students. Lee (2012) concluded: "Teaching fractions and equation writing

together can create synergy in developing students' fractional knowledge and algebra ideas" (p. 9). Willa and the other students interviewed in their study needed a clear understanding of fractional structures in order to write correct algebraic equations in those cases where the physical amounts being represented by a fraction were left unstated.

2.4 Focus of this research

Many researchers believe that much of the basis for algebraic thinking rests on a clear understanding of rational number concepts (Kieran, 1980; Lamon, 1999; Wu, 2001), the ability to manipulate common fractions and the importance of fraction/rational number knowledge. The National Mathematics Advisory Panel (NMAP, 2008) stated that the conceptual understanding of fractions and fluency in solving fraction problems are central goals of students' mathematical development and are critical foundations for algebraic learning. It is currently not clear which specific fraction or rational number knowledge is needed to make the transition to successful algebraic learning as stated by the National Mathematics Advisory Panel.

The focus of the Lee and Hackenberg research was on whether students were able to write specific algebraic equations using appropriate algebraic notation, namely, being able to write and solve an algebraic equation to represent a multiplicative relationship between two unknown quantities. However, the research study described here includes students in the final two years of primary school who have met fractions but are not yet expected to use and write algebraic notation as well as secondary students where algebraic notation has been introduced. While it would be reasonable to expect younger students to attempt to solve problems, similar to the *Tanya-David Money* problem, it would be less reasonable to present problems like the *Sam-Theo CD* problem, unless students can be transitioned into solving these more abstract problems by careful scaffolding. With younger students, the focus is the methods they use to approach problems, similar to the *Tanya-David Money* problem, including contexts where they are provided with diagrammatic representations of the quantities involved and other contexts where no diagram is supplied.

Three distinct aspects of algebraic thinking identified by Jacobs et al. (2007) and by Stephens and Ribeiro (2012) are important for this study. They are students' understanding of equivalence, transformation using equivalence, and the use of generalizable methods. In this current research algebraic thinking is defined in terms of students' capacity to identify an equivalence relationship between a given collection of objects and the fraction this collection represents of an unknown whole, and then to operate multiplicatively on both in order to find the whole.

Chapter 3: Methodology and trial of instruments

This study has been designed to investigate whether the conceptual understanding of fractions, and fluency with fraction calculations are linked to algebraic reasoning, as advocated by researchers and mathematics educators such as Kieran (1980), Lamon (1999), and Wu (2001). The study used a mixed method approach (Creswell & Plano Clark, 2011) employing both quantitative and qualitative methods. Students' understanding and competence with both fractions and algebra were assessed using paper and pencil assessments and analysed using quantitative approaches. A subset of students was then interviewed using a semi-structured interview protocol analysed using qualitative methods to gather further insights into students' thinking and solutions to fraction tasks. The two data collections were then integrated to answer the overarching research question: "How does middle-years students' fractional competence and reasoning show evidence of algebraic reasoning?"

Mixed methods research involves collecting, analysing, and interpreting quantitative and qualitative data in a single study or series of studies that investigate the same underlying phenomenon (Creswell & Plano Clark, 2011; Leech & Onwuegbuzie, 2008; Onwuegbuzie & Combs, 2010). Mixed methods use a combination of quantitative and qualitative approaches that are believed to provide a better understanding of the research being investigated as it includes the strengths of both (Creswell, 2003; Creswell & Plano Clark, 2011). While quantitative methodologies address questions about causality, generalisability, or effect size, qualitative research methodologies are used to explore why or how a phenomenon occurs, to develop a theory, or describe the nature of an individual's experience (Fetters, Curry, & Creswell, 2013).

However, using both qualitative and quantitative data collection methods in a single study is not sufficient enough to categorise a study as 'mixed methods'. Researchers have stated that mixed methods research depends on the integration or linking of the two strands of data. Integration can happen at multiple levels of a study – design-level, methods-level, or interpretation-level – and can happen in a variety of different ways – connecting, building, merging, or embedding (Fetters, Curry, & Creswell 2013; Creswell & Plano Clark 2011). Researchers have suggested that it is the integration of quantitative and qualitative data that enhances the value of mixed methods research (Bryman, 2006; Creswell & Plano Clark, 2011; Teddlie and Tashakkori, 2009).

According to Yin (2006) a mixed methods study must address a set of research questions that deliberately requires complementary qualitative and quantitative evidence and methods. Both the

quantitative and qualitative data would be analysed and interpreted together, before arriving at a study's main conclusion(s).

This mixed method approach has been used extensively in mathematics education research studies where students complete a paper-and-pencil assessment or task-based interviews and their responses are analysed to determine the strategies that they used to complete the particular mathematical tasks (see for example, Siemon, Virgona, & Corneille; 2001).

3.1 Planning the main study

The initial intention of this research was to collect data from middle-years students' responses to fraction and algebra assessment tasks, analyse the students' responses and then draw conclusions as to whether or not there was a link between the students' fractional competence and their algebraic thinking. In particular, the aim was to determine whether or not there was evidence that middle-years students were thinking algebraically as they elaborated their responses to a variety of fraction tasks.

Initially the use of semi-structured task-based interviews was considered (Goldin, 2000). Interviews such as those used in the Australian Fraction Research Project (See for example, Davis, Hunting & Pearn, 1993a; 1993b; Hunting, Davis and Pearn, 1996) or the Early Numeracy Research project (see for example, Clarke, Sullivan, Cheeseman, Clarke, 2000). However, if students were to be interviewed this would limit the number of students in the study. To be able to extend the study to include a larger number of students it was decided that the students could initially complete paper and pencil assessments and then several could be interviewed to allow them to elaborate their written responses used in the written assessments. This was a method used to determine the levels of middle-years students' multiplicative thinking in the Middle Years Research Project (Siemon, Virgona, & Corneille, 2001).

Previously, the researcher had used a set of fraction assessment instruments extensively with middle-years' students from Years 5 - 9. These included a paper and pencil test: The Fraction Screening Test, and two interview protocols: The Fraction Interview and The Probing Fraction Interview. Results from these assessments have been reported and discussed previously at national and international conferences. (see for example, Pearn & Stephens, 2007; Stephens & Pearn, 2003).

A colleague had also previously used an algebraic assessment instrument with middle-years students: The Algebraic Thinking Questionnaire (Stephens, 2006). The results from earlier

versions of this assessment instrument have also been presented nationally and internationally. (Stephens & Ribeiro, 2012; Mason, Stephens, & Watson, 2009).

An examination of the two paper and pencil assessment instruments revealed that both needed to be modified to ensure there were a number of opportunities for students to demonstrate their algebraic thinking as they completed both the Fraction Screening Test (FST) and the Algebraic Thinking Questionnaire (ATQ).

This chapter will focus on two pilot studies that were designed to trial the two paper and pencil assessment instruments which had been adapted specifically for this research: The FST and the ATQ. Initially, the development and composition of each of these paper and pencil assessment instruments will be outlined, then the results of the trials in the pilot studies will be presented and discussed. Pilot Study 1 trialled the adapted FST designed to be used in the main study. Pilot Study 2 trialled the adapted ATQ plus the three reverse fraction tasks from Part C (Tasks C5, C6, C7) as these reverse fraction tasks had not had the extensive testing like most of the other items from the FST. Table 3-1 gives details of the trialling of the two paper and pencil assessments.

Table 3-1: Trialling the assessment instruments

Pilot Study 1: School A 67 Year 6 students	Pilot Study 2: School B 18 Year 6 students
• trial of revised FST tasks • results of FST tasks analysed • 19 students asked to give more detailed response for one of the three reverse fraction tasks • more detailed responses analysed • reverse fraction tasks analysed to determine whether there was evidence of algebraic reasoning in written responses	• additional trial of 3 reverse fraction tasks (FST) • initial trial of revised ATQ tasks • analysis of results from 3 reverse fraction tasks to determine whether there was evidence of algebraic thinking • analysis of the ATQ tasks

Table 3-2 shows the structure of the two assessment instruments and the details of the changes made to earlier versions. The FST is made up of three parts with 24 tasks overall and the ATQ is split into two main parts with two types of questions in each part and 18 tasks altogether. The FST is detailed in Section 3.2 and the ATQ is discussed in Section 3.3.

Table 3-2: The structure of the two assessment instruments

Assessment Instruments		Focus	Number of tasks
FST	Part A	Simple geometric representations of fractions	12
		Counting by fractions	
		Equivalent fractions	
		Addition of fractions with related denominators	
	Part B	Number lines	5
	Part C	Equivalence of fractions, decimals and percentages	4
		Ordering fractions	3
		Reverse fraction tasks	
ATQ	Part M: Question 1	Focus on multiplication and the relationship between known numbers	4
	Part M: Question 2	Focus on multiplication and the relationship between unknown numbers	5
	Part D: Question 1	Focus on division and the relationship between known numbers	4
	Part D: Question 2	Focus on division and the relationship between unknown numbers	5

3.2 The Fraction Screening Test

The background to the FST is discussed in Section 3.2.1 and the tasks are linked to Kieren's five interpretations of fractions (1980) in Section 3.2.2. In Section 3.3 the trialling of the revised FST in Pilot Study 1 is discussed in detail.

3.2.1 Background to the Fraction Screening Test

The FST (Pearn & Stephens, 2014) intended for the main research study was revised and adapted from an earlier version used extensively with students from Years 5 – 8 (Pearn & Stephens, 2007). The original FST contained 20 tasks and four additional reverse thinking fraction tasks were included in the trial version, making 24 tasks altogether. The additional tasks include Task A12 in Part A (Figure 3-2) and Tasks C5 - C7 in Part C (Figure 3-7). Another structural change was the division of the FST into three parts: Part A, Part B and Part C. A copy of the revised FST can be found in Appendix A.

Part A of the revised FST contains 12 routine fraction tasks, with 11 tasks trialled previously (see for example, Pearn & Stephens, 2007). Part A tasks include pictorial representations of the fraction three-quarters, counting backwards by a given fraction, equivalent fractions and the addition of fractions with related denominators. The tasks in Part A would be similar to those

fraction tasks students would be expected to complete in class. For example, in Task A9 (Figure 3-1) students are asked to write a number in the box to complete the equivalent fraction, a task commonly found in most school texts. Analysis of the results for Part A tasks will be used to determine students’ fractional competence with routine fraction tasks and discussed in response to RQ1 and RQ2.

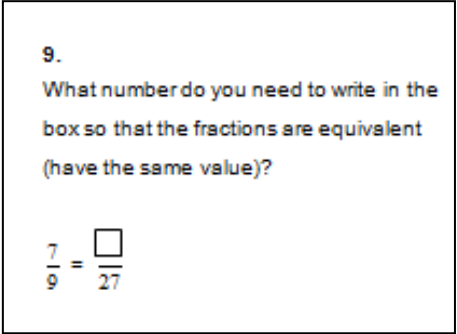


Figure 3-1: Task A9 (FST)

Figure 3-2 shows two Part A reverse fraction tasks. These tasks are called reverse tasks as each task presents a quantity of objects that represents a given fraction and students are asked to determine the number of objects that would represent the whole collection. For example, in Task A4 (left-hand side of Figure 3-2) students are asked to determine how many lollies were in the whole group if the four lollies given in the diagram represented one-half of the whole group. In Task A12 (right-hand side of Figure 3-2) students are asked to find the amount of money saved given that \$4 represented one-seventh of the amount saved. However, Task A12 does not include a picture or diagram that shows the number of objects representing the given fraction, but students could draw a diagram if required.

4. This is one -half of the lollies I started with.  How many lollies did I start with?	12. To buy a new workbook I spent \$4. This is $\frac{1}{7}$ of what I saved up. How much did I save up? _____
--	--

Figure 3-2: Part A reverse thinking fraction tasks A4 and A12 (FST)

Researchers such as Wong (2009) and Mitchell and Horne (2008) have suggested that assessment for fractional competence should include number-line tasks as many students have difficulty with positioning fractions correctly on number lines. Many students will find the fractional part of the line rather than the correct position of the fraction on a marked number line. Part B includes five

number-line tasks with four used previously (see for example, Pearn & Stephens, 2007). Task B3, shown in Figure 3-3, was included to see whether students placed the number three-fifths accurately on the line between the points indicating zero and one or whether they marked three-fifths of the entire line (Pearn & Stephens, 2007). Part B tasks have been included to determine students' fractional competence with the measurement interpretation of fractions (Kieren, 1980) shown in Table 3-3 and are needed to respond to research questions RQ1 and RQ2.

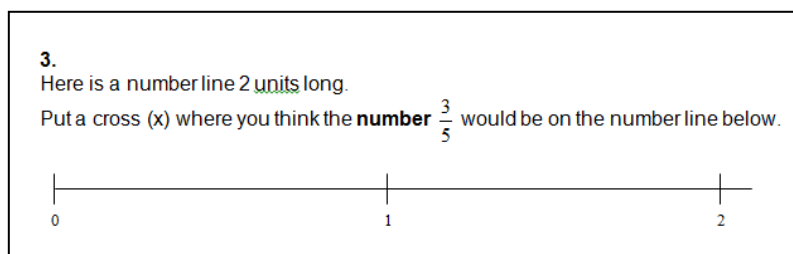


Figure 3-3: Task B3 (FST)

Task B4, shown in Figure 3-4, is a task that requires students to work in reverse and could be considered another example of a reverse fraction task. Typically, students in middle-years' classrooms would be given number-line tasks similar to Task B3, where they would be given the two end points and asked to position a given fraction on the number-line. In Task B4 students are asked to mark the position of the number one (the whole) having been given the part (one-third). This means that instead of dividing a line into parts, as with Task B3, students need to replicate or iterate the length from zero to one-third twice more. Students who have difficulties with this task typically place the mark for the number one at the end of the number line while some students place their response for the number one between zero and one-third.

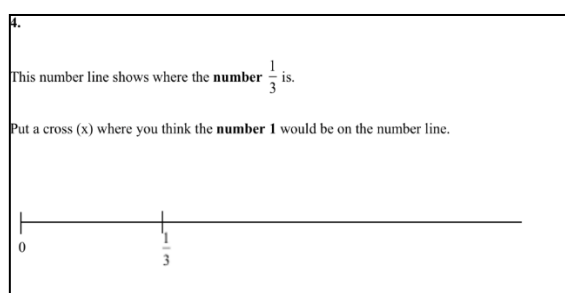


Figure 3-4: Task B4 (FST)

Task B5 in Figure 3-5 is a new task for the revised version of the FST. This task was included, as previously many students placed the fraction three-quarters on a number line very close to the number one and justified their response by stating: "three-quarters is nearly one". Students are expected to choose the correct response and then explain or justify why the other two responses were not appropriate.

5.

On this number line 0 and the number $\frac{3}{4}$ are shown.

a. Circle the letter that represents where the **number 1** would be.

b. How did you decide the other two choices were incorrect?

Figure 3-5: Task B5 (FST)

Part C includes fraction tasks that assess whether or not students can use their understanding of the size of fractions to order a set of fractions as in Task C1 (left-hand side of Figure 3-6) and whether they can match fractions with their decimal equivalents as in Task C2 (right-hand side of Figure 3-6).

1. Place these fractions in order from largest to smallest. $\frac{1}{2}$ $\frac{1}{4}$ $\frac{3}{5}$ $\frac{1}{10}$ _____	2. Draw a line from each fraction to the equivalent decimal. <table> <tbody> <tr> <td>$\frac{1}{2}$</td> <td>0.4</td> </tr> <tr> <td>$\frac{1}{4}$</td> <td>0.1</td> </tr> <tr> <td>$\frac{1}{10}$</td> <td>0.25</td> </tr> <tr> <td>$\frac{2}{5}$</td> <td>0.5</td> </tr> </tbody> </table>	$\frac{1}{2}$	0.4	$\frac{1}{4}$	0.1	$\frac{1}{10}$	0.25	$\frac{2}{5}$	0.5
$\frac{1}{2}$	0.4								
$\frac{1}{4}$	0.1								
$\frac{1}{10}$	0.25								
$\frac{2}{5}$	0.5								

Figure 3-6: Task C1 and C2 (FST)

For Tasks C3 and C4 students are asked to ‘choose the one that does not belong’ when given a choice of different symbolic representations including fractions, decimals and percentages. All four tasks have been used in previous studies and found to be challenging for most students (see for example, Pearn & Stephens, 2007). Both ordering fractions and being able to relate fractions to their decimal and percentage equivalents would be considered important for fractional competence and thus needed to respond to RQ1 and RQ2.

In addition to the four tasks described previously Part C includes three new tasks which require students to apply reverse thinking to less familiar fraction tasks (Figure 3-7). These tasks also offer students opportunities to use more explicit algebraic thinking which was not needed to solve Task A4 relating to one-half (left-hand side of Figure 3.2). These types of tasks are not usually included in Australian or Victorian mathematics textbooks.

These three reverse fraction tasks specifically require students to use reverse or reciprocal thinking to find the number of objects in the whole collection when given the number of objects

representing a fractional part of that collection. Two of the three tasks expected students to include a written statement in response to: Explain how you decided that your answer is correct” while the other task asked students to: “Show all your working”. Two tasks included diagrams or pictures showing the number of objects representing the fractional part. While the diagram in Task C5 (left-hand side of Figure 3-7) is assumed to be helpful the diagram in Task C7 (right-hand side of Figure 3-7) was not.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____ Show all your working.	7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct.
--	--	--

Figure 3-7: Tasks C5, C6 & C7 (FST)

3.2.2 Connections to Kieren’s interpretations of fractions (1980)

According to Kieren (1980) there are five interconnected sub-constructs or interpretations of fractions that are both mathematically and psychologically dependent (see Table 3-3). Researchers have suggested that conceptual understanding of fractions includes the ability to make connections between these different interpretations, including the similarities and differences of the various interpretations (see for example, Cathcart, Pothier, Vance & Bezuk, 2011; Kieren, 1980; Lesh, Landau, & Hamilton, 1983). Wong (2009) suggests that students with conceptual understanding of fraction equivalence have an integrated knowledge of fractions and are able to display and articulate the five interpretations as given by Kieren (1980).

The five interpretations by Kieren (1980) are shown in the left-hand column of Table 3-3. Examples of these interpretations as given by Lamon (2001) for the fraction three-quarters are shown in the middle column of Table 3-3. The right-hand column of Table 3-3 includes items from the FST that address each of the five interpretations. While there are no specific ratio tasks similar to the example in Table 3.3 there are tasks related to equivalent fractions which are deemed to be ratio tasks by researchers such as Charalambous and Pitta-Pantazi (2005) who suggested that: “the ratio subconstruct is considered as the most “natural” to promote the concept of equivalence, and, subsequently, the process of finding equivalent fractions” (p. 234).

Table 3-3: Examples from the FST for Keiren's five interpretations of fractions (1980)

Fraction Interpretation	Example given for three-quarters (Lamon, 2001)	Examples from the FST
Part/whole	3 out of 4 equal parts of a whole or set of objects	Part A Tasks: A1, A2, A4, A5
Measure	Three times the distance from zero to one-quarter starting at zero on the number line	Part B Tasks: B1, B2, B3, B4, B5
Operator	$\frac{3}{4}$ of something, stretching or shrinking	Part A Task: A8
Quotient	$\frac{3}{4}$ is the amount each person receives	Part A Tasks: A3, A6, A12
Ratio	3 parts cement to 4 parts sand	Part A Tasks: A9, A10

3.3 Pilot Study 1: Trialing the revised Fraction Screening Test

Since, as detailed above, new tasks had been added to the original FST, the revised FST was trialled with the whole cohort of 67 Year 6 students from an eastern metropolitan school in Melbourne (Pearn & Stephens, 2015a). Most students completed the FST in approximately 30 minutes. To decide whether the revised FST was appropriate for the main study students' responses were scored and analysed. Section 3.3.1 details the scoring used for the tasks used in Pilot Study 1 and Section 3.3.2 details results for Part A, Section 3.3.3 for Part B, Section 3.3.4 for Part C results and Section 3.3.5 focuses on the written responses for Tasks C5, C6 and C7. Section 3.3.6 gives details about the internal consistency of the tasks in the FST and Section 3.3.7 describes the attempt to elaborate algebraic reasoning as seen in the written responses for the three, Part C reverse fraction tasks i.e. Tasks C5, C6 and C7. This was done to ensure the revised FST was appropriate for the main study and would provide the level of detail needed to answer the research questions.

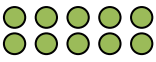
3.3.1 Scoring the Fraction Screening Test

For Pilot Study 1 marks were allotted to each FST task according to the detail included in the students' responses. Task A1, where students had to choose the two correct pictorial representations of the fraction three-quarters, was scored out of three, Task A7, the counting backwards task, was scored one for a correct response and zero for an incorrect response. All other Part A tasks were scored out of two while all Part B tasks were given a score out of three.

Tasks C3 and C4 were scored one for a correct response and zero for an incorrect response. The other five Part C tasks were scored out of three. For example, each of the three reverse thinking fraction questions (Figure 3-7) is marked out of three. One mark was given for a correct response with no explanation, or if there was some evidence of a correct diagram, or an initial

representation, which indicates that students had a starting point but were not able to proceed. Two marks were given for a correct answer with limited explanation and three marks were given for a correct answer with an appropriate explanation. A score of zero was given if the task was not attempted or an incorrect response was given. Table 3-4 gives an example of this scoring for Task C5.

Table 3-4: Scoring for Reverse Fraction Task 1 (Task C5, FST)

Task C5	Score = 1 mark	Score = 2 marks	Score = 3 marks
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided that your answer is correct.	Correct response but no explanation or some evidence of the correct use of diagram (e.g. drew an extra row of counters) but did not give a correct answer	Correct response given but with a limited explanation or incorrect mathematical statements used in the explanation e.g. $3 \div 15$ instead of $15 \div 3$	Correct response with appropriate explanation. For example, one-third is 5 thus three-thirds is 15 or uses reciprocal thinking $10 \times \frac{3}{2} = 15$

A similar scoring rubric was applied to the other two reverse fraction tasks (Tasks C6 and C7) shown in Figure 3.7.

3.3.2 Results for Fraction Screening Test: Part A

Table 3-5 shows the percentage of Year 6 students who scored zero, one, two or three marks for each Part A task. Seventy-two percent of the students chose the two correct representations of three-quarters for Task A1 with an additional 24% choosing only one of the two correct representations. Eighty-eight percent of the students circled one-third of the six beetles in Task A2 while 97% could share 20 fish equally between four friends in Task A3. Eighty-seven percent of the students were successful with Task A4 determining the total number of lollies when given that four lollies represented one-half of the group. Ninety percent of the students were able to shade one-fifth of the rectangle given in Task A5.

Only 30% of students were successful with Task A6 where they were expected to imagine sharing 3 pizzas between four people and indicate how much pizza each person would get. For Task A7 81% of the students completed the counting backwards by two-fifths pattern and 88% shaded three-quarters of the ribbon in Task A8.

Table 3-5: Percentage of students achieving each result for Part A tasks, FST (n = 67)

Tasks	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11	A12
Score 0	0	12	1	9	6	63	19	4	18	6	31	9
Score 1	24	0	1	4	4	7	81	7	6	4	22	10
Score 2	4	88	97	87	90	30	0	88	76	90	46	66
Score 3	72											
Mean Score*	2.48	1.76	1.96	1.78	1.84	0.67	0.81	1.84	1.58	1.84	1.15	2.27

* Possible Scores: A1: 0, 1, 2, 3; A2 – A6 & A8 – A12: 0, 1, 2; A7: 0, 1

While 76% of the students gave the correct numerator for the equivalent fraction for seven-ninths (Task A9) 90% were able to simplify the fraction eight-tenths (Task A10) to give the correct numerator of the equivalent fraction. Forty-six percent of the students were successful adding two fractions with related denominators in Task A11 and simplified the answer while another 22% correctly added the fractions but did not simplify the answer. That is, 68% could add three-tenths and one-fifth. In Task A12 66% could calculate the amount of money that had been saved.

As the Part A tasks are the easiest of the FST these results indicate that these tasks appear to be appropriate for students from Year 5 to Year 8. The two tasks where students did not perform well have been designed to highlight possible misconceptions. The results for these tasks, from the main study, will be used in the discussion about RQ1 and RQ2.

3.3.3 Results for Fraction Screening Test: Part B

The five Part B number-line tasks were each scored out of three. Task B2 was divided into two parts and each part was also scored out of three. Table 3-6 shows the percentage of students who scored zero, one, two or three marks for each Part B number-line task. Overall 87% of the students correctly wrote the numbers on the four interim marks on the number line marked zero to 100 (Task B1). More than 70% of the students correctly marked the position of the numbers 75 (Task B2a) and 5 (Task B2b) on a number line marked 0 and 25. However, only 27% of the students could place the number three-fifths accurately on a number line marked zero, one and two with an additional 13% of students marking three-fifths of the line rather than the correct position of three-fifths.

Table 3-6: Percentage of students achieving each result for Part B tasks, FST (n = 67)

Part B Tasks	B1	B2a	B2b	B3	B4	B5
Score 0	7	0	3	27	24	3
Score 1	3	13	7	12	13	22
Score 2	3	12	16	34	33	7
Score 3	87	75	73	27	30	67
Mean Score*	2.69	2.61	2.60	1.61	1.69	2.37

* Possible Scores for all Part B tasks: 0, 1, 2, 3

Many researchers (See for example, Wong, 2013a, 2013b; Michell & Horne, 2008) have suggested that students need to mark and interpret number lines and that these are important for fraction competence.

3.3.4 Results for Fraction Screening Test: Part C

The Part C tasks have been designed to test students' understanding of fraction concepts and thus deemed to be more difficult than those presented in either Part A or Part B and includes three new reverse thinking fraction tasks shown in Figure 3-7. Tasks C3 and C4 were marked either correct or incorrect and scored one for correct and zero for an incorrect response. Responses for the other five Part C tasks were given a score out of three. Table 3-7 includes the percentage of students who scored zero, one, two or three marks for each Part C task.

Table 3-7: Percentage of students achieving each score for Part C tasks, FST (n = 67)

Part C Tasks	C1	C2	C3	C4	C5	C6	C7
0	12	3	12	57	30	37	43
1	18	4	88	43	1	6	9
2	31	6			4	12	7
3	39	87			64	45	40
Mean Score*	1.97	2.76	0.88	0.43	2.03	1.64	1.45

* Possible scores: C1, C2 & C5 – C7: 0, 1, 2, 3; C3 & C4: 0, 1

As shown in Table 3.7 70% of the students ordered the four fractions correctly with 39% ordering them from largest to smallest as requested in Task C1 while 31% ordered from smallest to largest. This means 70% of students could order the set of fractions correctly although they may not have given the correct response for the task. For Task C2 87% were able to match fractions with their decimal equivalents. While 88% identified that 0.4 did not belong to the group that were

equivalent representations of one-quarter (Task C3) only 43% of the students correctly identified that 12% was not an equivalent representation for the improper fraction six-fifths (Task C4).

While 64% of the students correctly answered Task C5, 45% correctly answered Task C6 and 40% correctly answered Task C7 the percentage of incorrect or non-responses increased over the three tasks: C5, C6 and C7. An initial analysis of the responses to the three reverse fraction tasks appeared to provide an important lens into the students' strategies and possible algebraic reasoning but this was not easily captured in the scoring used for Pilot Study 1. The students' responses for the three, Part C reverse fraction tasks are discussed in more detail in the next section (Section 3.3.5).

3.3.5 Elaborating written responses to the reverse fraction tasks

When students' written responses to the three reverse fraction tasks (Tasks C5, C6, C7 shown in Figure 3-7) were initially analysed it was noted that some responses were very brief and not elaborated as requested and consequently the student's thinking was unclear. Nineteen students, who had correctly solved all three reverse fraction tasks and provided adequate but brief explanations of their thinking, were asked to provide a more detailed written explanation of their solution to one question only. It was hoped that their more detailed explanation would give a greater insight into their thinking. This request was relayed from the researcher to the students by their teacher.

The left-hand side of Figure 3-8 shows Student KL's initial response to Task C5 while the right-hand side shows his elaborated response along with an alternative solution. In his initial response it is unclear why he divided three by two and then multiplied the answer by two. His explanation in the top part of the right-hand column clearly shows an alternative version of his thinking for the solution of this task. The second part in the right-hand column is a more elaborated version of his initial solution. Note that he scaled up the fraction, and the whole number that represents the fraction, in parallel positions in the separate number sentences. In the top part of the right-hand column Student KL wrote: "If 10 counters = $\frac{2}{3}$ where the equals sign is used to convey 'is equivalent to' rather than 'equals'. He abbreviated this statement in the second part when he wrote $10 = \frac{2}{3}$ again indicating that ten is equivalent to two-thirds of the whole.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? <u>15</u> b. Explain how you decided that your answer is correct. $3 \div 2 = 1\frac{1}{2}$ $2 \times 1\frac{1}{2} = 3$ $10 \times 1\frac{1}{2} = 15$	1. Please explain briefly why you answered this question the way that you did. If 10 counters = $\frac{2}{3}$ then 5 counters = $\frac{1}{3}$ because $\frac{2}{3} \div 2 = \frac{1}{3}$ and $10 \div 2 = 5$. → The answer = 15 because $\frac{1}{3} \times 3 = 1$ and $5 \times 3 = 15$. 1 = the answer. 2. Can you think of another way in which you could have answered this question? Please describe briefly what you might do. Thank you for your help. → $\frac{2}{3} \times 1\frac{1}{2} = 1$, $10 = \frac{2}{3}$ $10 \times 1\frac{1}{2} = 15$ 1 = the answer
---	--

Figure 3-8: Student KL scales up fractions and whole numbers in parallel for Task C5 (FST)

Figure 3-9 shows Student JL's two responses to Task C6, which unlike Task C5 has no diagram. Student JL's initial response (left-hand column) used a pro-numeral and it was unclear whether he had used a rote-learned rule or understood the underpinning mathematics for his calculations.

6. $\frac{4}{7}$ Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. Susie = 12 Kay = x $12 = \frac{4}{7}x$ $12 \div \frac{4}{7} = 3 \times 7 = 21$ $x = 21$	1. Please explain briefly why you answered this question the way that you did. Susie has 12 CDs. We do not know Kay's number of CDs, so Kay is x (unknown number). 12 is $\frac{4}{7}$ of x, meaning $x = 12 \div \frac{4}{7}$. The ($\div$) is changed to ($\times$) by flipping ($\frac{4}{7}$) to ($\frac{7}{4}$). The equation is now $x = 12 \times \frac{7}{4}$. The equation is simplified to $x = 3 \times 7$, therefore $x = 3 \times 7$. $x = 21$ 2. Can you think of another way in which you could have answered this question? Please describe briefly what you might do. Thank you for your help. Susie = $\frac{4}{7}$ Kay = 12 $12 \div 4 = 3$ $\frac{1}{7} = 3$ $\frac{1}{7} \times 7 = 3 \times 7 = 21$
--	--

Figure 3-9: Student JL uses a pro-numeral to solve Task C6 (FST)

Student JL's response given in the right-hand column of Figure 3.9 clearly shows that he has learnt an appropriate method to solve Task C6 and is able to verbally describe his method. This is not a method he was taught at school. His alternative method in the right-hand side of Figure 3-9 is the method used by many Year 6 students. He has found the number of objects representing the one-seventh and then multiplied by seven to find the number in the whole group.

Student BM's initial response to Task C6 in the left-hand side of Figure 3-10 is unconventional as she scales up both whole numbers and fractions in parallel. BM's second response given in the right-hand side of Figure 3-10, although also unconventional, has elaborated the thinking she used for her initial response. She has written the fractional number sentence on the top line and the whole number equivalent underneath. Although unclear in Figure 3.10 she starts with four-sevenths divided by four equals one-seventh before multiplying by seven.

Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. $\boxed{12}$ $12 \div \frac{4}{7} = \frac{3}{7}$ $3 \times 7 = 21$	$\frac{\boxed{4}}{\boxed{7}} \div 4 = \frac{1}{7} \times 7 = \frac{7}{7} = 1$ $12 \div 4 = \boxed{3}$ $3 \times 7 = \boxed{21}$
---	---

Figure 3-10: Student BM scales up fractions and whole numbers in parallel for Task C6 (FST)

Student WA's initial response is shown in the left-hand column of Figure 3-11 and his more elaborated written response to Reverse Fraction Task 3 is shown in the right-hand column of Figure 3-11. He finds one-seventh of the 14 counters then multiplies by six to determine the number in the whole collection. His recording '2 equals $\frac{1}{6}$ ' is his way of saying that '2 counters is equivalent to one-sixth of the whole' as seen with Student KL.

7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with. $\frac{1}{6} = 2$  a. How many counters did I start with? b. Explain how you decided that your answer is correct. I just divide 7 into 14 then times 6	1. Please explain briefly why you answered this question the way that you did. Because 14 counters is $\frac{7}{6}$. I divided 7 into 14 and got the answer 2. So 2 equals $\frac{1}{6}$, $\frac{2}{6}$ to make the fraction 1. 1 times 2 by 6 which equal 12.
--	--

Figure 3-11: WA's initial and elaborated verbal response for Task C7 (FST)

The four students shown above used a range of multiplicative strategies to solve these three reverse fraction tasks (Tasks C5 – C7). Despite pro-numerals not being introduced until Year 7 one Year 6 student used a pro-numeral to solve these tasks but the other three used strategies that demonstrated either emerging or established algebraic reasoning. Analysis of these responses confirmed that these three tasks are appropriate for use in the main study. Results for the three reverse fraction tasks (Tasks C5, C6 and C7) are important for responses to the research questions: RQ5 and RQ6.

3.3.6 Internal consistency for the Fraction Screening Test

To determine the internal consistency of the revised FST the students' results from Pilot Study 1 were entered into SPSS (IBM SPSS Statistics, 2017). The scores for each individual task were adjusted so that each task on the FST was scored out of two marks where 0 was given for no response or an incorrect response, 1 for a partially correct response and 2 for a correct response.

Cronbach's Alpha was calculated using the scores of the 67 students for the 25 tasks of the revised FST. Cronbach's Alpha is a measure of internal consistency, that is, how closely related a set of items is as a group and is a standard measure of scale reliability. An analysis of the Inter-item Correlation Matrix showed that all Inter-item correlations were positive, so all items appear to be measuring the same underlying characteristic. While values of Cronbach's Alpha above 0.7 are considered acceptable, values above 0.8 are preferable so the Cronbach's Alpha (.898) given in the Reliability Statistics table for this sample suggests very good internal consistency for the scale from the FST.

An analysis of the *Corrected Item-Total Correlation* values shown in the *Item-Total Statistics* table indicates the degree to which each item correlates with the total score. Values less than 0.3 indicate that the item is measuring something different from the scale as a whole. The three fraction tasks that had values of less than 0.3 are shown in Table 3-8.

These three fraction tasks (A3, A4 and A8) were found by this group of Year 6 students to be the easiest on the FST and are typical of the types of routine fraction tasks students would have completed in class at an earlier year level. These items had a high success rate: 97%, 87% and 88% respectively. Since the overall *Cronbach Alpha* for the FST is more than 0.7 these items do not need to be removed.

Table 3-8: FST Tasks with Corrected Item-Total values of less than 0.3

FST Tasks	Corrected item -total correlation	Cronbach's Alpha if item deleted	Percentage correct	Mean Score
3. I want to share 20 fish equally between 4 friends. How many fish would each friend get?	.201	.899	97	1.96
4. This is one-half of the lollies I started with.  How many lollies did I start with? _____	.290	.898	87	1.78
8. Here is a picture of a ribbon. Shade $\frac{3}{4}$ of the ribbon. 	.233	.899	88	1.84

The results were then checked to see if any of the *Alpha If Item Deleted* values was higher than the final Alpha value of .898. Two of the three tasks have a higher Alpha value, while the third has the same value. If any value is higher than the final Alpha value, it is suggested that the items be removed. The decision was made to retain these tasks as weaker students are more comfortable with these tasks than some of the other tasks on the FST. Thus, it was decided to retain these tasks in an endeavour to reduce possible test anxiety.

While a high value for Cronbach's alpha indicates good internal consistency of the items in the scale for the FST, it does not mean that the scale is unidimensional. To determine the dimensionality of the scale a factor analysis could have been used but is beyond the scope of this research which is limited by the constraints of the samples.

The FST was chosen for the main study as a test of fraction competence, so it was important that the tasks appear to be related as a group despite the variety of fraction topics. There is a strong relationship between each of the individual parts of the FST as shown in Table 3-9. While the FST tasks have been separated into three parts this has only been done as way of discussing the tasks. Part A tasks are those routine type tasks most commonly found in texts and exercises for middle-years students. Part B tasks are those that focus on number line tasks, the importance of

which have been highlighted in research and Part C tasks include tasks that test understanding and reasoning in a variety of contexts.

Table 3-9: Correlations between results for Parts A, B and C (FST)

	Part A	Part B	Part C
Part A	1.00	0.71	0.67
Part B		1.00	0.69
Part C			1.00

3.3.7 Finding evidence of algebraic thinking

Nineteen students were asked to elaborate their initial response to one of the reverse fraction tasks. In many cases students had given a very brief response in their initial attempt at the FST tasks making an analysis of their strategies difficult to interpret. Section 3.3.5 includes four examples of students' initial responses and their more elaborated responses. The four examples given in Section 3.3.5 demonstrated a range of multiplicative strategies.

In this section the responses of all students in Pilot Study 1 to the three reverse fraction tasks (Figure 3-7) were examined to attempt to identify features that could be confidently taken to indicate evidence of algebraic reasoning, namely, understanding of equivalence, transformation using equivalence, and the use of generalisable methods.

This initial analysis of the responses to the Part C reverse fraction tasks suggested that students who were capable reverse thinkers were able to step back from a visual representation and could relate the fraction to the numerical quantity it represented without a visual cue. These students were able to scale down to find the number of objects representing the unit fraction and then scale up to find the number representing the whole. They scaled down, then up, both fractions and the quantities they represented to determine the number of objects needed to represent the whole. Scaling up and scaling down is a reliable two-step procedure for finding the whole that may even be compacted into one-step. These students are not dependent on using additive strategies which may be appropriate for simple fraction problems like the one-half fraction task in A4 (left-hand side of Figure 3-2). These students appeared to be using algebraic reasoning in their solutions to these Part C reverse fraction tasks.

Five different types of algebraic reasoning responses were evident in the initial analysis of the reverse fraction tasks shown in Figure 3-7. The first four types exhibit explicit and competent use

of multiplicative thinking while the fifth shows an effective mix of multiplicative and additive thinking. This initial analysis focussed on the types of algebraic reasoning responses for the three reverse fraction tasks. When the results for the main study were analysed a more comprehensive framework was developed to include the range of all types of strategies used not just algebraic reasoning strategies.

3.3.7.1 Response Type 1: Fully Multiplicative Response

Response Type 1 is an example of a fully multiplicative response where the student finds the number of objects representing the unit fraction by dividing and then multiplies that number by the number needed to find the whole. Student JL, whose elaborated response to Task C5 (left-hand column of Figure 3-7) is shown in Figure 3-12, started by writing: “If I have 10 counters now and that is two parts out of three ...”. He then divided by two to find the number of dots in one-third then multiplied by three to find three-thirds or the whole.

In some cases, the connection between the number of objects and the fractional parts of the problem was implied rather than clearly stated. For example, when initially solving the same reverse fraction task, Student JL wrote $10 = \frac{2}{3}$. While this is mathematically incorrect, it is clear that the student means that the ten dots represented two-thirds of the whole group.

This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.



a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct.
 If I have 10 counters now and that is two parts out of three, then
 when I divided 2 parts by two I will get 1 part ($10 \div 2 = 5$) 1 part = 5
 1 part $\times$ (multiplied) by 3 = 3 parts ($5 \times 3 = 15$) You started with 15 counters.

Figure 3-12: Student JL’s example of Response Type 1 for Task C5 (FST)

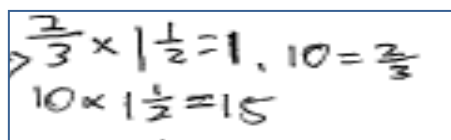
3.3.7.2 Response Type 2: Using fractions and quantities multiplicatively in parallel

Some students employed equivalent operations using fractions and whole number quantities in parallel. See for example Student BM’s response to Task C6 in the right-hand column of Figure 3-10 where she wrote: $\frac{4}{7} \div 4 = \frac{1}{7} \times 7 = \frac{7}{7} = 1$ on one line and $12 \div 4 = 3, 3 \times 7 = 21$ on the line underneath thus tracking both fractional and whole number computation in parallel. This response can be directly compared to a two-step solution for: $\frac{4}{7}x = 12$. The equal sign is used idiosyncratically to connect the steps as in the first line of her response. However, this student

clearly understands the need for equivalent operations to relate the two parallel lines of her solution.

3.3.7.3 Response Type 3: Two-step reverse operation

Sometimes a two-step reverse operation is compacted into one step as shown in Figure 3-13. Student KL's response mirrors very closely the kind of transformational thinking needed to solve the algebraic equation: $\frac{2}{3}x = 10 \rightarrow x = 10 \times 1\frac{1}{2}$



$$\frac{2}{3} \times 1\frac{1}{2} = 1, 10 = \frac{2}{3}$$

$$10 \times 1\frac{1}{2} = 15$$

Figure 3-13: Student KL's example of Response Type 3 for Task 5C (FST)

3.3.7.4 Response Type 4: Using a pro-numeral

Some Year 6 students set up the appropriate equation using a pro-numeral. See for example Student JL's response given in the left-hand side of Figure 3-9 where he started with the equation $12 = \frac{4}{7}x$. While this method appears to be a dramatic change it is really just a symbolic change. The reasoning here is no different from that shown in the previous three response types. Using an equation such as this is more economical and uses recognisable algebraic notation, but mathematically it is identical in its reasoning. Student JL used Response Type 1 in his more elaborated explanation shown in the right-hand side of Figure 3-9.

3.3.7.5 Response Type 5: Mixed method approach

Sometimes students use a mixed method approach that uses both multiplicative and additive strategies. Figure 3-14 shows responses where two students initially found the unit fraction and then added on the amount needed for Task C5 or subtracted the amount for Task C7.

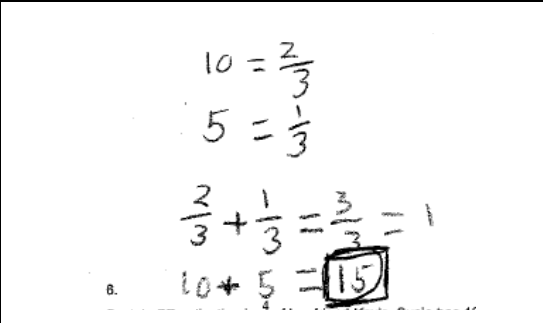
 $10 = \frac{2}{3}$ $5 = \frac{1}{3}$ $\frac{2}{3} + \frac{1}{3} = \frac{3}{3} = 1$ $10 + 5 = 15$	a. How many counters did I start with? 12 b. Explain how you decided that your answer is correct. $\frac{7}{6} = 1\frac{1}{6}, \frac{1}{6} = 2, 14 - 2 = 12$ $\frac{7}{6} - \frac{1}{6} = \text{Answer}$ 
--	--

Figure 3-14: Examples of Response Type 5 for Tasks C5 & C7 (FST)

Student JB (left-hand side of Figure 3-14) initially found that five counters represented one-third of the whole group so added this onto the initial 10 counters that represented two-thirds of the

whole group. As for Student BM, Student JB recorded the two processes in parallel. Student AX (right-hand side of Figure 3-14) found one-sixth of the whole group and subtracted that amount from the number of counters representing seven-sixths.

3.3.8 Conclusion for Pilot Study 1

After examining the results from the analysis of the FST tasks it appears that the FST is appropriate as a test of fractional competence. The reverse fraction tasks in Part C allowed students to demonstrate a range of algebraic reasoning strategies as shown in Section 3.3.7.

Students in Pilot Study 2 trialled the ATQ tasks as well as a second trial of the three reverse fraction tasks from Part C (FST). These results will be discussed in Section 3.4.

3.4 Pilot Study 2: Trialing the Algebraic Thinking Questionnaire

A second pilot study was conducted with 18 Year 6 students from another eastern suburban metropolitan school in Melbourne (Pearn & Stephens, 2015b; 2018). There were two reasons for this second pilot study. The first was a second trial of the three reverse fraction tasks (Tasks C5 – C7) and the second was the initial trial of the revised ATQ. Due to time restraints the students participating in Pilot Study 2 completed a paper and pencil assessment that included Tasks C5, C6 and C7 (FST) and either Part M or Part D of the ATQ. The ATQ can be found in Appendix B. This was not the same school used in Pilot Study 1 where the revised FST was initially trialled.

The 18 Year 6 students were chosen as they were deemed by their teachers to be highly successful in mathematics. All 18 students were given the three reverse fraction tasks while ten students were given Part M (the multiplication tasks) and eight were given Part D (the division tasks) of the ATQ. As there were so few results the Cronbach's Alpha was not calculated. In the main study all students completed both sections (Part M and Part D), that is, both the multiplication and division sections of the ATQ.

3.4.1 Background to the Algebraic Thinking Questionnaire

The ATQ trialled in Pilot Study 2 and used in the main study was developed based on items from an assessment instrument that had already been used extensively in seven countries (Stephens, 2006). One consideration in designing questions for the ATQ was to provide tasks that would allow students to either use arithmetic calculations or relational thinking that could be interpreted as demonstrating algebraic reasoning. The focus for the ATQ tasks was on multiplication and division of whole numbers, fractions and decimals.

The purpose of the ATQ was to ensure that a purely operational understanding of the equals sign (Jones et al, 2013), while suitable for some questions on the FST, would not enable students to successfully solve all ATQ tasks. The ATQ questions were constructed so that students had the opportunity to demonstrate either of the two different relational conceptions of the equals sign: sameness-relational conception or the substitutive-relational conception which Jones and colleagues (2013) indicated was important for algebraic thinking. The aim was to determine whether using relational aspects of equivalence led to evidence of generalisation.

Part M focuses on multiplication tasks and Part D focuses on division tasks. Examples of the types of tasks involving both whole numbers and fractions from Question 1 of both parts of the ATQ are shown in Table 3-10.

Table 3-10: Examples of Question 1 tasks from Parts M and D (ATQ)

Question 1 (ATQ)	Part M: Multiplication focus	Part D: Division focus
Task 1	$36 \times 25 = 9 \times \square$	$3 \div 4 = 15 \div \square$
Task 4	$\frac{2}{5} \times - = 1$	$\frac{7}{6} \div - = 1$

Question 2 of both Part M and Part D of the ATQ focuses on students' understanding of equivalence relationships with two unknown numbers represented by Box A and Box B, and by symbolic representations of two unknowns (see Table 3-11). Students are also expected to explain the relationships between the respective unknown numbers or the given symbolic representations (c and d or a and b).

Table 3-11: Examples of Question 2 tasks from Parts M and D (ATQ)

Question 2 (ATQ)	Part M	Part D
Task 2: When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?	$5 \times \square_{\text{Box A}} = 10 \times \square_{\text{Box B}}$	$3 \div \square_{\text{Box A}} = 15 \div \square_{\text{Box B}}$
Task 4: What can you say about c and d in this mathematical sentence?	$c \times 2 = d \times 14$	$c \div 8 = d \div 24$
Task 5: What can you say about a and b in this mathematical sentence?	$a \times \frac{3}{4} = b \times 1\frac{1}{2}$	$a \div \frac{2}{7} = b \div \frac{3}{7}$

Pattern and Algebra is one sub-strand in the Australian Curriculum: Mathematics. Table 3-12 shows the two Content Descriptions for Pattern and Algebra that are relevant for this study along with the Year Levels and the related ATQ tasks.

Table 3-12: Mathematics Content Descriptors with corresponding ATQ tasks

Content Descriptions (ACARA, 2018)	Year Level	ATQ tasks
Find unknown quantities in number sentences involving multiplication and division and identify equivalent number sentences involving multiplication and division (ACMNA121)	5	Question 1 Tasks 1 - 4
Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)	7	Question 2 Tasks a – e

3.4.2 Second trial of the three Reverse Fraction Tasks

The 18 Year 6 students in Pilot Study 2 from a second primary school completed Tasks C5 - C7 (FST) and the responses were scored as in Pilot Study 1 and described in Section 3.2.1. That is, one mark was given for a correct response with no explanation or if there was some evidence of a correct diagram or an initial representation which the student did not take further (starting point). Two marks were given for a correct answer with limited explanation and three marks given for a correct answer with appropriate explanation. Zero was given if no response or an incorrect response was given. For an example of the results using the scoring for Task C5 see Table 3-13.

Table 3-13: Scores for Tasks C5 - C7, FST (n = 18)

Scores	Task C5 (FST)	Task C6 (FST)	Task C7 (FST)
Score 0	0	4	4
Score 1	0	0	0
Score 2	4	4	5
Score 3	14	10	9
Mean Score*	2.78	2.11	2.05

* Possible Scores: Tasks C5 - C7: 0, 1, 2, 3

All students correctly answered Task C5. Fourteen students gave a correct response with an explanation such as $\frac{1}{3}$ is 5 thus $\frac{3}{3}$ is 15. Four students gave a limited written explanation or used incorrect mathematical language e.g. one student wrote $3 \div 15$ instead of the required $15 \div 3$. Student AA's correct additive response is given in Figure 3-15.

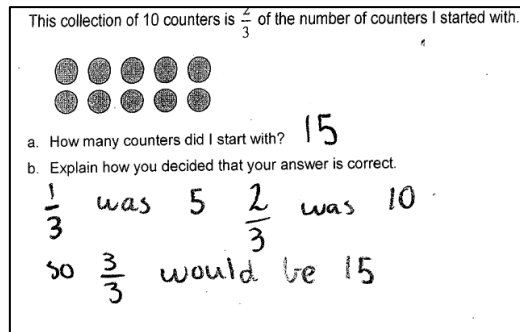


Figure 3-15: Student AA's response to Task C5 (FST)

Fourteen students answered Task C6 correctly. Ten of these students gave explanations for their answers while four students gave the correct answer with either a limited explanation or used incorrect mathematical statements in their explanations. Four students either gave an incorrect response or did not attempt the task.

Fourteen students answered Task C7 correctly. Nine students explained their answers while five students gave the correct answer with either a limited explanation or used incorrect mathematical statements in their explanations. Four students either did not attempt the task or gave an incorrect response. Student AP's correct response is given in Figure 3-16. While Student AP has written equations which are mathematically incorrect such as $14 = \frac{7}{6}$, $2 = \frac{1}{6}$, $2 \times 6 = \frac{6}{6}$ and $12 = \frac{6}{6}$ it is assumed that this is their shorthand way of writing that 14 represents seven-sixths of the whole collection thus two dots represent one-sixth and 12 dots represents the whole collection.

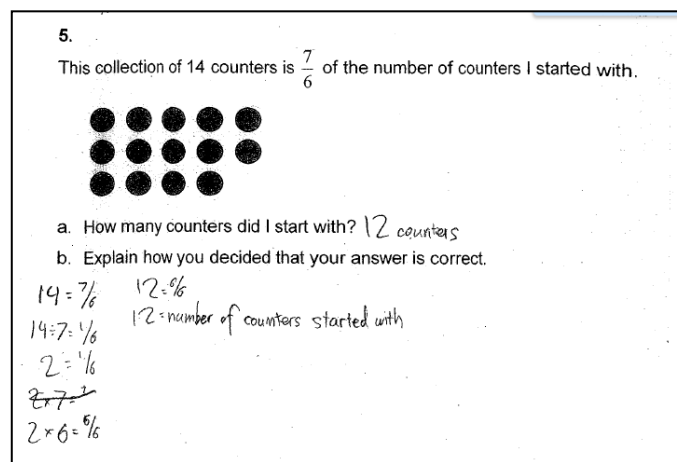


Figure 3-16: Student AP's multiplicative response to Task C7 (FST)

The total scores for the three reverse thinking fraction questions ranged from three to nine. The average score was 6.94. Twelve of the 18 students scored higher than the average for the three reverse fraction tasks i.e. seven or more. Seven students answered all three reverse fraction tasks correctly and gave a comprehensive explanation for all three reverse fraction tasks.

Year 6 students from Pilot Study 2 also demonstrated that these reverse fraction tasks from Part C of the FST allowed them to demonstrate algebraic reasoning when completing these tasks. These tasks are deemed to be appropriate for use in the main study and will assist in the discussion about RQ5 and RQ6.

3.4.3 Algebraic Thinking Questionnaire: Part M

Ten of the 18 students in Pilot Study 2 completed Part M of the ATQ. There is a total of nine multiplication-based tasks. Question 1 consists of four tasks and Question 2 has five tasks. In Question 1, students who did not attempt a task or gave an incorrect response scored zero. A score of one was given for a response that was partially correct, a score of two for a response that was a correct arithmetical calculation and three for a response that used relational thinking. Table 3-14 shows the number of students scoring from zero to three for each of the four tasks from Question 1 of Part M (ATQ).

Table 3-14 Scores for Question 1, Part M (ATQ) (n = 10)

Part M Question 1	Task M1a	Task M1b	Task M1c	Task M1d
Score 0	0	1	2	3
Score 1	0	0	1	1
Score 2	5	5	3	3
Score 3	5	4	4	3
Mean	2.5	2.2	1.9	1.6

* Possible scores: Tasks M1a – M1d: 0, 1, 2, 3

Similar scoring was used for Question 2 of Part M (ATQ). Students who did not attempt the task or gave an incorrect response scored zero. A score of one was given for a response that was partially correct, a score of two for a response that was a correct calculation and three for a response that used relational thinking. For an example of a correct response with an appropriate explanation for Question 2a and 2b see Student ZB's response in Figure 3.17. This response would score one for Task M2a and three for Task M2b.

2. Can you think about the following mathematical sentence:

$$5 \times \boxed{20} = 10 \times \boxed{10}$$

Box A Box B

a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$$5 \times \boxed{35} = 10 \times \boxed{15}$$

Box A Box B

$$5 \times \boxed{15} = 10 \times \boxed{7.5}$$

Box A Box B

b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

Box B is half of Box A

Figure 3-17: Student ZB's response to Question 2a and 2b Part M (ATQ)

Table 3-15 shows the number of students scoring from zero to three for each of the five tasks from Question 2 of Part M.

Table 3-15: Scores for Question 2, Part M (ATQ) (n = 10)

Part M Question 2	Task M2a	Task M2b	Task M2c	Task M2d	Task M2e
Score 0	0	1	1	1	3
Score 1	1	0	2	3	1
Score 2	9	0	0	1	0
Score 3	0	9	7	5	6
Mean Score*	2.0	2.7	2.3	2.0	1.9

* Possible scores Task M2a: 0, 1, 2; Tasks M2b – M2e: 0, 1, 2, 3

Students' total scores ranged from 7 to 26 for Part M of the ATQ with a mean score of 19. While most students were able to answer the nine questions relating to multiplication only five students showed consistent evidence of relational thinking to solve all multiplication tasks. One student started off using calculations for the easier tasks but was able to switch to relational thinking for the more abstract tasks.

The correlation between the total results of the reverse fraction tasks and the total for Part M (ATQ) for these students was 0.79. Table 3-16 shows the correlations between the individual tasks from both instruments used for Pilot Study 2. That is, Tasks C5 - C7 (FST) and the multiplication tasks (Part M) from the ATQ.

The correlations varied from very weak relationships between a reverse fraction task and an ATQ task to strong positive relationships between the tasks. For example, the correlation between Task C5 and Task M2d was zero while the correlation between Task C7 and Task M1b was -0.09 which is very close to zero. However, there were strong relationships between Task C5 and Tasks M2c, Task C6 and Tasks M1a – M1d and Task C7 and Tasks M1d and M2d.

Table 3-16: Correlations between Tasks C5 – C7 (FST) and Part M tasks (ATQ)

FST Tasks	Part M (ATQ)								
	M1a	M1b	M1c	M1d	M2a	M2b	M2c	M2d	M2e
Task C5	0.33	0.08	0.56	0.44	1.0	1.0	0.70	0	0.46
Task C6	0.74	0.68	0.72	0.68	0.50	0.50	0.54	0.48	0.49
Task C7	0.30	- 0.09	0.53	0.68	0.50	0.50	0.54	0.68	0.49

3.4.4 Algebraic Thinking Questionnaire Part D

Eight students completed Part D of the ATQ. Part D focuses on division tasks. There are nine division-based tasks in Part D with four tasks in Question 1 and five tasks in Question 2. Similar scoring to that used in Part M was used for Question 1 of Part D. Students who did not attempt the task or gave an incorrect response scored zero. A score of one was given for a response that was partially correct, a score of two for a response that was a correct calculation and three for a response that used relational thinking. Table 3-17 shows the number of students achieving each of the scores for the four Question 1 tasks of Part D.

Table 3-17: Scores for Question 1 Part D ATQ (n = 8)

Scores	Task D1a	Task D1b	Task D1c	Task D1d
Score 0	1	0	3	0
Score 1	1	1	0	5
Score 2	2	1	1	0
Score 3	5	6	4	3
Mean Score	2.3	2.6	1.8	1.8

* Possible Scores: Tasks D1a – D1e: 0, 1, 2, 3

Similar scoring was used for Question 2 of Part D (ATQ). Students who did not attempt the task or gave an incorrect response scored zero. A score of one was given for a response that was partially correct, a score of two for a response that was a correct calculation and three for a response that used relational thinking. Table 3-18 shows the number of students for each score for each task of Question 2 of Part D.

Table 3-18: Scores for Question 2, Part D, ATQ (n = 8)

Scores	Task D2a	Task D2b	Task D2c	Task D2d	Task D2e
Score 0	1	2	3	2	3
Score 1	1	3	2	1	1
Score 2	6	1	1	2	3
Score 3	0	2	2	3	1
Mean Score*	1.6	1.4	1.3	1.8	1.3

* Possible Scores: Task D2a: 0, 1, 2 Task D2b – D2e: 0, 1, 2, 3

Two students demonstrated consistent evidence of relational thinking for all division tasks. One other student was able to give the correct answers to all division tasks – at times elaborating responses. For an example of a correct response with explanation for Part D (ATQ) see Student RT's response in Figure 3-19.

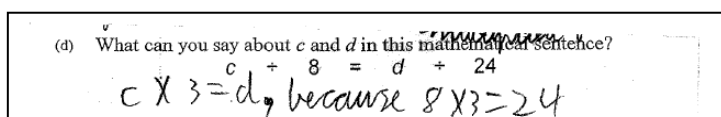


Figure 3-19: Student RT's response to Question 2d, Part D (ATQ)

Scores for Part D of the ATQ ranged from 6 to 26 with a mean score for the division tasks of 15.6. Table 3-19 shows the correlation between the three reverse fraction tasks (FST) and the nine division tasks on Part D (ATQ). The correlation between the total score for the fraction tasks and the total score for the division tasks for these students was 0.64.

Table 3-19: Correlation of scores for reverse fraction tasks and Part D (ATQ) (n = 8)

FST Tasks	Part D Tasks (ATQ)								
	D1a	D1b	D1c	D1d	D2a	D2b	D2c	D2d	D2e
Task C5	-0.30	-0.42	-0.51	-0.47	0.70	0.03	-0.05	-0.16	-0.30
Task C6	0.89	0.51	0.51	0.33	0.14	0.61	0.59	0.27	0.53
Task C7	0.83	0.46	0.61	0.17	-0.12	0.44	0.53	0.48	0.34

The correlations between the reverse fraction tasks and the division tasks from the ATQ ranged from moderate negative relationships to strong positive relationships between the tasks. For example, the correlation between Task C5 and the first four division tasks were moderately negative, a strong positive relationship with Task D2a but weaker negative correlations for the last four division tasks. There was a strong positive relationship between Task C6 and Task C7 with Task D1a with weaker relationships between the next three division tasks.

3.4.5 Conclusion for Pilot Study 2

Students in Pilot Study 2 trialled the ATQ tasks as well as the three reverse fraction tasks from Part C (FST). After examining the results from the three reverse fraction tasks (FST) in Pilot Study 2 it appears that these three tasks allowed the Year 6 students to demonstrate a range of algebraic reasoning when completing these tasks as shown in Section 3.3.7. These tasks are deemed to be appropriate for use in the main study and will assist in the discussion about RQ5 and RQ6.

Ten of the 18 students in Pilot Study 2 completed the nine multiplication-based tasks of Part M (ATQ) while eight completed the nine division-based tasks of Part D (ATQ). While there were some students who did not complete some of these ATQ tasks many students successfully completed these tasks using either correct arithmetical computations or demonstrated their use of relational thinking in their solutions. These ATQ tasks are deemed to be appropriate for use in the main study and will assist in the discussion about RQ3, RQ4, RQ5 and RQ6.

3.5 Significance of Pilot Studies

These two pilot studies were designed to trial the paper and pencil assessment instruments before implementing the main research study. The results from these studies draw attention to three quite specific aspects of fractional operations that are not sufficiently emphasised in earlier studies. The first is being able to transform (operate on) a given fraction in order to find the whole, regardless of whether the fraction is expressed in proper or improper form. The second is students' understanding of equivalence, meaning that the operations that are required to restore a fraction to a whole need to be applied to the corresponding numerical quantities represented by the fraction. The third is to utilise efficient and generalisable multiplicative methods to achieve this goal; in contrast to other methods, usually additive, which may work only with simple fractions. All three aspects are essential for the subsequent solution of algebraic equations. This study will allow us to assist teachers to identify these multiplicative methods and thus they will be able to encourage their students to use these efficient and generalizable strategies to solve fraction tasks.

3.6 Consultation with some experts

As part of this research, an international group of 15 mathematicians, mathematics educators, and practicing teachers were asked to contrast responses of the Green Group of students shown in Table 3-20 with responses from students in the Blue Group shown in Table 3-21. This consultation has been discussed briefly in Pearn and Stephens (2018).

The respondents were asked to comment on the following two questions:

- What in your opinion distinguishes the first group of responses (green) from the second (blue)? You may wish to comment on particular students, but we would like you to identify any general features of the second blue group that distinguish it from the first.
- What, in your opinion, in the second group of responses (blue) could be seen as anticipating algebra in the sense of algebra as these students will typically encounter it in secondary school?

As shown in Table 3-20 the Green group of students had obtained correct answers by predominantly using additive or pictorial methods as well as some multiplicative methods. For example, Student DW counted by fives to solve Task C5 while student used tracking of the fractions while counting by threes for Task C6 and twos for Task C7. Both these students used additive methods for one of the reverse fraction tasks. Student CE gave several incorrect responses. She incorrectly stated: "If you do $3 \div 15$ it equals 5" but used the result to find the correct answer for Task C5 (FST). Both her other attempts were incorrect.

Table 3-20: Green group

Reverse Fraction Tasks	Student DW	Student JD	Student CE
C5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with. How many counters did I start with? Explain how you decided that your answer is correct.	Correct response If ten is two-thirds you count by 5 so 5, 10, 15	Correct response There are 10 counters. So I halved 10 and that equals 5. I multiplied 5×3 and that equalled 15.	Correct response If you do $3 \div 15$ it equals 5 and since it is $\frac{2}{3}$ double 5 will be 10 and an extra 5 is left over.
C6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? Show all your working.	Correct response $12 \div 4 = 3$ so $3 \times 7 = 21$	Correct response $3 \times 4 = 12$ Tracking used $\frac{4}{7} \quad \frac{5}{7} \quad \frac{6}{7} \quad \frac{7}{7}$ 12 15 18 21	Incorrect response Divided 7 into 12 then 7 into 48 then scribbled both out.
C7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided that your answer is correct.	Correct response I found the closest multiple of 6 to 14 which is 12. I divided by six which equals 2 so $\frac{1}{6} = 2$	Correct response Each $\frac{1}{6}$ is 2 then I kept adding 2 until $\frac{6}{6}$ and that number was 12.	Incorrect response Rewrote $\frac{7}{6}$ as $\frac{11}{6}$ and added $28 + 7$ to get 35.

The three students in the Blue Group use a range of multiplicative methods to correctly solve all three reverse fraction tasks. Note some of the idiosyncratic recording of these multiplicative

methods. They all use the equals sign as shorthand. For example, Student AM writes that $10 = \frac{2}{3}$ but really means that the group of 10 counters represents two-thirds of the whole group. These students would not have been introduced to formal algebraic notation.

Table 3-21: Blue Group

Reverse Fraction Tasks	Student AP	Student WL	Student EJ
C5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided that your answer is correct.	Correct response $10 = \frac{2}{3}$ $10 \div 2 = \frac{1}{3}$ $5 = \frac{1}{3}$ $5 \times 3 = \frac{3}{3}$ $15 = \frac{3}{3}$ $15 =$ the number of counters started with	Correct response 10 is 2 parts, $= 5 \text{ each} \times 3 = 15$	Correct response $10 = \frac{2}{3}$ therefore $(10 \div 2) = \frac{1}{3}$ $5 (\frac{1}{3}) \times 3 = 15$
C6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? — Show all your working.	Correct response $12 = \frac{4}{7}$ $12 \div 4 = \frac{1}{7}$ $3 = \frac{1}{7}$ $3 \times 7 = \frac{7}{7}$ $21 = \frac{7}{7}$ $21 =$ Kay's CDs	Correct response $12 \div 4 = 3 \times 7 = 21$	Correct response $12 = \frac{4}{7}$ therefore $(12 \div 4) = \frac{1}{7}$ $3 (\frac{1}{7}) \times 7 = 21$
C7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with. How many counters did I start with? Explain how you decided that your answer is correct.	Correct response $14 = \frac{7}{6}$ $14 \div 7 = \frac{1}{6}$ $2 = \frac{1}{6}$ $2 \times 6 = \frac{6}{6}$ $12 = \frac{6}{6}$ $12 =$ the number of counters started with	Correct response $14 \div 7 = 2 \times 6 = 12$ 14 divided by 7 units as that is the total, then times it by 6 as it is the actual amount of units.	Correct response $14 = \frac{7}{6}$ therefore $(14 \div 7) = \frac{1}{6}$ $2 (\frac{1}{6}) \times 6 = 12$

Three responses to the sets of students' work samples shown are quoted here. One Professor of Mathematics expressed concern: "These students may have difficulty with algebra because they do not understand the meaning of the equals sign". By contrast, a Professor of Mathematics Education, referring to Student AP's response to Task C5 (Table 3.21) said that $10 = \frac{2}{3}$ is treated like $\frac{2}{3}x = 10$. She went on to say that this latter algebraic expression is solved by dividing by 2 and multiplying by 3, or by multiplying by $1\frac{1}{2}$ as the student had done. A third Professor of Mathematics, from outside Australia, argued that the students were using: 'tracking arithmetic': not allowing parameters (numbers) in the problem to be calculated, so that at the end their role and influence can be tracked, and generalisation consists of treating each original number as place holder, *so treating it algebraically*.

Practicing teachers who were consulted generally supported the view that the kind of thinking shown in the three examples from the Blue Group was algebraic through its use of equivalence and generalisability. While there was no uniform agreement about what exactly anticipated algebra in the work samples presented to this group, the majority of the respondents drew attention to features that indicated a shift away from arithmetical thinking. The proposed study seeks to develop a stronger conceptual framework for the terms ‘anticipating algebra’ and ‘algebraic reasoning’.

3.7 Implications for the Main Study

The results from the pilot studies informed the development of the instruments used in the main study. The assessment instruments were proven to be appropriate for the purpose of providing opportunities for students to demonstrate their algebraic thinking as they solved fractional tasks especially the three reverse fraction tasks from Part C of the FST. The ATQ was challenging for the Year 6 students in Pilot Study 2 but the tasks appeared to address the issues of equivalence, transformation using equivalence, and generalisation anticipated in algebraic reasoning.

As the number of students in the main study are much higher than the numbers of students in the pilot studies consideration was given to the scoring of the instruments to ensure that the details from the students’ responses are not lost. The details contained in the student responses will be used in the exploratory analysis of the two written assessments in Chapter 5, the case studies in Chapter 7 and responses to the research sub-questions in Chapter 8.

Chapter 4: Main Study Method

As discussed in Chapter 3 this is a mixed methods research study which uses the integration of the data from both quantitative and qualitative approaches to provide a better understanding of the research being investigated (Bryman, 2006; Creswell & Plano Clark, 2011; Teddlie & Tashakkori, 2009; Yin, 2006). This main research study uses a sequential explanatory approach (Creswell, 2003) characterised by the collection and analysis of quantitative data using descriptive statistics followed by the collection and analysis of the qualitative data. Thus, the results from the qualitative analysis in Chapter 6 help to explain and interpret the quantitative data from Chapter 5. The results of the data collections have been exemplified in the case studies in Chapter 7 and integrated to address the research questions in Chapter 8.

Ethics approval for this study was obtained from the Melbourne Graduate School of Education's Human Research Ethics Committee (1544444) and permission to conduct research in schools was obtained from the Department of Education and Training (2015_002714) and the Catholic Education Office (Project #2016 Pearn). Principals of all schools, and parents of all students, participating in this research completed the permission letters approved by the ethics committee.

Details of the phases of this sequential exploratory mixed method study are shown in Table 4.1. This main study uses descriptive statistics in the quantitative phase as the relationships or correlations between the results from the two paper and pencil assessments have been examined to determine how one may be associated with the other (Kervin, Vialle, Herrington and Okely, 2006, p.62). This study examines the relationship between students' responses to a range of fraction tasks to determine their fractional competence and their responses to both algebraic tasks and reverse fraction tasks to find evidence of emerging or developed algebraic reasoning.

Table 4-1: Phases of the sequential explanatory mixed method

Phase 1: Quantitative (n = 607)	Phase 2: Qualitative (n = 45)	Integration and Interpretation
- Students attempt FST and ATQ tasks - FST and ATQ results analysed and correlated using descriptive statistics - Development of <i>Framework for Reverse Fraction Task Strategies</i>	- Students interviewed using the Structured Interview - Results of individual tasks from the Structured Interview analysed using the <i>Framework for Reverse Fraction Task Strategies</i>	- Case studies illustrate the integration of the results - Analysis of all data for responses to main research question and sub-questions. - Development of <i>Emerging Algebraic Reasoning Framework</i>

In this study, a sequential explanatory mixed method research design was selected in order to investigate connections between fractional competence and algebraic reasoning. In the first phase of the main study, quantitative data was collected from 607 students from 11 schools and the subsequent data was analysed using SPSS using descriptive statistical methods. The quantitative data collection and subsequent analysis has been described in Chapter 5. The qualitative data collection included semi-structured interviews with 43 students which have been described in Chapter 6.

The first linking or integrating of data occurred when the quantitative data from the two paper and pencil assessments was analysed to inform the construction of the semi-structured interviews. The second integration of data occurred when the quantitative data from Phase 1 was integrated with the qualitative data from the semi-structured interviews used in Phase 2. This integration is discussed in the case studies elaborated to “draw out new insights beyond the information gained from the separate quantitative and qualitative results” (Fetters, Curry & Creswell, 2013, p. 2143). The third integration of the data occurred when analyses of all data sources were synthesised to respond to the research question and sub-questions.

In this research the qualitative and quantitative methods were linked and integrated to address the overarching research question looking at the connections between fractional competence and algebraic reasoning. The analysis of the quantitative data, for example the correlation between fractional competence and algebraic reasoning as indicated by the scores on the two assessments, did not give enough information as to why students were choosing to use particular strategies. Conversely, only having the qualitative data from the Structured Interview would not have identified the relationship between fractional competence and algebraic reasoning for the large numbers of students in the quantitative study. Data from both the qualitative and quantitative phases were analysed in the final phase to form an overall understanding developed through the integration of all the data strands (Teddle and Tashakkori, 2009). The synthesis of all data sources and responses to the main and sub research questions are included in the discussion in Chapter 8.

4.1 Participants

The students participating in this study were a convenience sample. Their teachers volunteered to participate either when the research proposal was presented at the annual conference of the Mathematical Association of Victoria or after attending professional learning sessions presented by the researcher. There were 607 middle-years students from 11 Victorian schools with nine schools from metropolitan Melbourne and two from regional Victoria. There were nine primary

schools and two secondary schools. Table 4.2 shows the number of students by year level from each school. There were no Year 7 students in this research, as their teachers did not volunteer to participate. Altogether there were 190 Year 5 students, 269 Years 6 students, 122 Year 8 students, 15 Year 9 students and 11 primary students who did not identify whether they were in Year 5 or Year 6.

Table 4-2: Number and year level of participants by school (n = 607)

	School										
	1	2	3	4	5	6	7	8	9	10	11
ICSEA	1144	1134	1013	1135	1104	1162	1180	1156	1150	1160	1181
Year 5 (n = 190)	29	-	27	-	-	9	27	51	20	27	-
Year 6 (n = 269)	36	40	52	-	31	15	20	39	17	19	-
Year 8 (n = 122)	-	-	-	120	-	-	-	-	-	-	2
Year 9 (n = 15)	-	-	-	-	-	-	-	-	-	-	15
Unknown * (n = 11)	4	-	-	-	-	6	-	1	-	-	-
Total	69	40	79	120	31	30	47	91	37	46	17

* These primary students did not indicate whether they were in Year 5 or Year 6.

Table 4.2 also shows the Index of Community Socio-Educational Advantage [ICSEA]. For each school this measure takes into account parents' occupations, parents' education, geographical location and the proportion of indigenous students. ICSEA values are calculated on a scale which has a median of 1000 and a standard deviation of 100 and typically range from a value of about 500 (representing extremely educationally disadvantaged backgrounds) to about 1300 (representing schools with students with very educationally advantaged backgrounds) [ACARA, 2018]. As shown in Row 1 of Table 4.2, at the time of this study, the participating schools had ICSEA values that ranged from 1013 to 1181 and thus would be deemed to be educationally advantaged.

4.2 Data collection: Instruments

Unlike the Lee and Hackenberg study (2014), which utilised two interviews with small groups of students participating in both a Fraction Interview and an Algebra interview, this study uses two paper and pencil assessments: The FST (Pearn & Stephens, 2014) and the ATQ (Stephens, 2015) with 607 students. The contents, and details of the trials, of these two assessment instruments are discussed in Chapter 3. The FST is in Appendix A and the ATQ is in Appendix B. Solutions from

the paper and pencil assessments were examined for any evidence of relationships between students' fractional competence and algebraic reasoning.

After a preliminary analysis of the three, Part C reverse fraction tasks (FST) scripted one-on-one interviews were conducted with 45 students who had successfully completed two of the three reverse fraction tasks from Part C. These students were purposefully selected on the basis of their responses to the reverse fraction tasks so that a variety of strategies were represented. These students were prompted to explain their thinking and to suggest alternative solution strategies to the interview tasks.

The three instruments used in this main study are described briefly in both Pearn and Stephens (2018) and Pearn, Stephens and Pierce (2019) while the Structured Interview is described in Pearn, Stephens, Zhang and Pierce (2019).

4.3 The Fraction Screening Test

The FST (Pearn & Stephens, 2014) used for this main study was developed and trialled in Pilot Study 1 and discussed in Chapter 3. This version is divided into three parts. Part A includes 12 routine fraction tasks such as equivalent fractions, ordering fractions and recognising simple representations of fractions. Part B includes five number-line tasks. Part C includes fraction tasks that assess whether or not students can use their understanding of the size of fractions to order a set of fractions, match fractions with their decimal equivalents or 'choose the one that does not belong' when given a choice of different symbolic representations including fractions, decimals and percentages. Part C also includes the three reverse fraction tasks that require students to determine the number of objects representing given the number of objects representing a fractional part. Results from the FST will be used to respond to research questions RQ1 and RQ2 which focus on fractional competence while results from the three, Part C reverse fractions tasks will be specifically used to respond to RQ3, RQ5 and RQ6 which have a greater focus on algebraic reasoning

4.3.1 Scoring the Fraction Screening Test

The students in Pilot Study 1 trialled the revised FST. When analysing the Year 6 students' responses to the FST the number of marks allocated varied according to the amount of detail expected to be included in these responses. This scoring is described in detail in Section 3.3.1. For example, each of the three, Part C reverse fraction questions (Figure 3-7) was marked out of three. One mark was given for a correct response with no explanation or if there was some

evidence of a correct diagram or an initial representation that the student did not complete. Two marks were given for a correct answer with limited explanation and three marks given for a correct answer with adequate explanation. Zero was allocated if an incorrect response was given or there was no response.

After careful consideration of the data it became clear that the Pilot Study scoring guide for the FST was not sufficiently comprehensive and details of students' responses were not being adequately recorded. As a result, the scoring scheme was revised to capture more of the detail available in the data. Consequently, an initial alphabetical coding was used to record the information and then alphabetical scores were converted to numerical scores and some categories were collapsed. In Table 4.3 is an example of the initial coding and the subsequent scoring scheme for Task A1 (FST) where students were asked to choose the correct representations of three-quarters.

Table 4-3: Example of the scoring scheme used for Task A1 (FST)

Task A1 (FST)	a	b	c	d	e
Numerical score	0	1	1	1	2
	No correct shapes chosen	One correct shape and one incorrect shape chosen	Two correct shapes and one incorrect shape chosen	One correct shape and no incorrect shapes chosen	Two correct shapes chosen

In the initial analysis of the students' responses in Pilot Study 1 to Part C reverse fraction tasks (Tasks C5, C6 and C7) shown in Figure 3-7 five different types of algebraic reasoning strategies were identified. This initial analysis, shown in Section 3.3.7, focussed on the types of algebraic reasoning responses students used when responding to the three reverse fraction tasks. The algebraic reasoning type responses included:

- Response Type 1: Fully multiplicative
- Response Type 2: Using fractions and quantities multiplicatively in parallel
- Response Type 3: Two-step reverse operation
- Response Type 4: Using a pro-numeral
- Response Type 5: Mixed method approach

So that all student responses could be included this initial coding needed to be enlarged to include all the strategies students used to solve the three reverse fraction tasks not just those that demonstrated algebraic reasoning. The range of strategies will be analysed in the next section (Section 4.3.2).

4.3.2 Analysing strategies used to solve individual Part C reverse fraction tasks

Written responses for each of the three reverse fraction tasks (Figure 3-7) were analysed to determine the types of strategies students used to solve each task. Six categories of strategies were established using the thematic analysis approach suggested by Braun & Clarke (2006). For this initial attempt at the analysis of the results from the students' responses for the main study these categories of strategies were classified as: *Not Clear Strategy*, *Diagram Dependent Strategy*, *Additive/Subtractive Strategy*, *Partially Multiplicative Strategy*, *Fully Multiplicative Strategy*, and *Advanced Multiplicative Strategy* (Pearn, Pierce, & Stephens, 2017b).

Written responses that were either incomplete or not attempted were classified as *Not Clear*. When students explicitly partitioned the given diagrams and drew extra objects their response was classified as demonstrating the *Diagram Dependent Strategy*. For example, in Student BM's response shown in Figure 4.1 she divided the drawing of the ten counters into two groups to determine one-third and then drew another five counters to determine that three-thirds was 15.

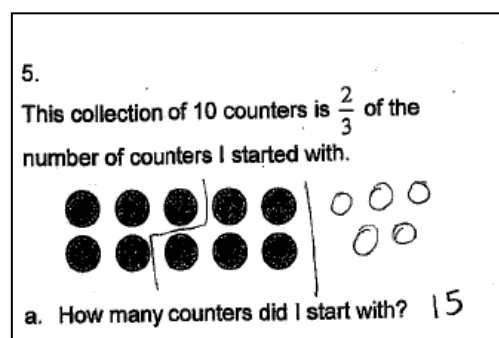


Figure 4-1: Student BM's uses a *Diagram Dependent Strategy* to solve Task C5 (FST)

The *Additive/Subtractive Strategy* refers to the use of additive or subtractive methods without explicit partitioning of the given diagram or creating a new diagram. Students using the *Additive Strategy* find the number of objects needed to represent the unit fraction and then use counting or repeated addition to find the number of objects needed to represent the whole. Student RH has determined that one-seventh of the group is three counters and keeps track of the fractional parts by adding an extra three objects for each seventh (Figure 4.2). Although the recording is unconventional it is apparent that this student's expression $5 = 15$ is used as shorthand meaning that five-sevenths of the original group would be 15 counters, similarly $6 = 18$ means that six-

sevenths is represented by 18 counters so seven-sevenths or the whole group would be 21 counters.

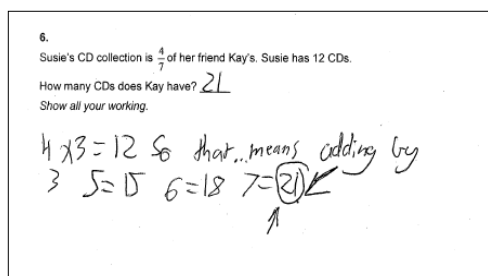


Figure 4-2: Student RH uses an *Additive Strategy* to solve Task C6 (FST)

The *Partially Multiplicative Strategy* refers to the use of both multiplicative and additive methods. In Figure 4.3 Student JB found the number of objects representing one-third of the group then added these to the two-thirds to find the appropriate number of objects needed to make the whole. While the written recording is unconventional it is clear to see that the student is recording his method step-by-step. Note the parallel tracking in Line 3 and Line 4 in Figure 4.3. In Line 3 Student JB has demonstrated that he can add two-thirds and one-third to get three-thirds or the whole. In Line 4 the number of counters in each of the fractional parts is recorded under each fraction in Line 3.

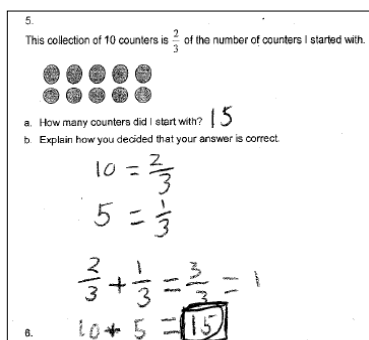


Figure 4-3: Student JB uses a *Partially Multiplicative Strategy* to solve Task C5 (FST)

The *Fully Multiplicative Strategy* refers to the use of only multiplicative methods to solve each reverse fraction task. Students find the quantity represented by the unit fraction using division and then multiply the quantity representing the unit fraction to find the number of objects in the whole. Although Student DC in Figure 4.4 uses no written explanations, and while the recording is unconventional, the solution method is clear. The fraction four-sevenths is not stated explicitly but is implied in the division by four. In the second line Student DC states the equivalence between the number of objects and the fraction one-seventh while the third line shows the transformation needed to go from one-seventh to a whole without needing to refer to the fraction. This strategy anticipates how students would solve $\frac{4}{7}x = 12$

Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.

How many CDs does Kay have? 21

Show all your working.

$$12 \div 4 = 3$$

$$3 = \frac{1}{7}$$

$$3 \times 7 = 21$$

Figure 4-4: Student DC uses a *Fully Multiplicative Strategy* to solve Task C6 (FST).

The *Advanced Multiplicative Strategy* describes the more advanced multiplicative strategies students use to solve Tasks C5 – C7. Student JL (Figure 4.5) has used a one-step strategy to find the whole by dividing the given quantity by the known fraction (left-hand side) and the correct use of algebraic notation to find the whole (right-hand side).

$10 \div \frac{2}{3} = 15$ $= 5 \times 3$ $= 15$	$\text{Susie} = 12$ $\text{Kay} = x$ $12 = \frac{4}{7}x$ $12 \div \frac{4}{7} = 3 \times 7 = 21$ $x = 21$
--	---

Figure 4-5: Student JL uses *Advanced Multiplicative* strategies to solve Task C5 and C6 (FST)

4.3.3 Dominant strategies used to solve the set of Part C reverse fraction tasks

Students from one primary school and one secondary school were interviewed using the Structured Interview. Students were to be chosen to be interviewed so that they represented a range of strategies used to solve Tasks C5 - C7. Written responses were analysed to determine whether students consistently used the same strategies across the three reverse fraction tasks. That is, the written responses were investigated to determine whether students had a dominant strategy that they used for at least two of the three reverse fraction tasks. Just as for the strategies the students used for each of the individual reverse fraction tasks six categories of dominant methods were identified for the set of three reverse fraction tasks: *Not Clear Strategy*, *Diagram Dependent Strategy*, *Additive/Subtractive Strategy*, *Partially Multiplicative Strategy*, *Fully Multiplicative Strategy*, and *Advanced Multiplicative Strategy*. These classifications are used to categorise the students' responses to the set of the three reverse fraction tasks not the individual tasks.

As shown in Figure 4.6 Student MB consistently used *Partially Multiplicative* strategies for all three reverse fraction tasks. He calculated the number of objects needed to represent the unit fraction and then added or subtracted the appropriate number of objects needed to make the whole. In Task C5 (left-hand column of Figure 4.6) Student MB found that five counters represented one-third so added that onto the ten representing two-thirds to determine the total of the whole

group was 15 counters. In Task C6 (middle-column of Figure 4.6) he found three-sevenths and added that to the quantity representing four-sevenths while he subtracted the number representing one-sixth from the number representing seven-sixths to calculate the whole for Task C7 (right-hand column of Figure 4.6).

1 divided ten by two and got five. Since it is $\frac{2}{3}$ of the number I added another five and got 15.	$12 \div 4 = 3$ $3 \times 3 = 9$ $12 + 9 = 21$	$7 \times 2 = 14$ so $6 \times 2 = 12$ $14 - 2 = 12$
---	--	--

Figure 4-6 : Student MB used *Partially Multiplicative* strategies for Tasks C5 - C7 (FST)

Student DC uses *Fully Multiplicative Strategies* for all three of the reverse fraction tasks as shown in Figure 4.7. Regardless of the fraction or the quantity representing that fraction he consistently found the quantity represented by the unit fraction and then scaled up find the whole as shown. Note that in all three solutions he has used the same idiosyncratic recording that a fraction was equal to a whole number. For example, in the left-hand column he has written $5 = \frac{1}{3}$ to indicate that five counters represent or is equivalent to one-third of the whole collection.

$10 \div 2 = 5$ $5 = \frac{1}{3}$ $5 \times 3 = 15$	$12 \div 4 = 3$ $3 = \frac{1}{7}$ $3 \times 7 = 21$	$14 \div 7 = 2$ $2 = \frac{1}{6}$ $2 \times 6 = 12$
---	---	---

Figure 4-7: Student DC used *Fully Multiplicative* strategies for Tasks C5 - C7 (FST)

Student JP, shown in Figure 4.8, consistently used a one-step method to find the whole by dividing the given quantity by the known fraction for all three reverse fraction tasks and is deemed to have used an *Advanced Multiplicative Strategy*. Note that Student JP has used the same idiosyncratic recording as Student DC in Figure 4.6 ' $12 = \frac{4}{7}$ ' to indicate that 12 counters represent or is equivalent to, four-sevenths of the whole group.

$$10 \div \frac{2}{3} = 10 \times \frac{3}{2} = \frac{30}{2} = 15 \quad 12 = \frac{4}{7} \quad 12 \div \frac{4}{7} = 3 \times 2 \times \frac{7}{4} = 21$$

$$14 \div \frac{2}{6} = 14 \times \frac{6}{2} = 12$$

Figure 4-8: Student JP used an *Advanced Multiplicative* strategy for Tasks C5 - C7 (FST)

This preliminary coding of the strategies used for the individual reverse fraction tasks and for the set of three reverse fraction tasks will be discussed in more detail in Chapter 5 which focuses on the analysis of the results for the two assessment instruments: The FST and the ATQ. In Chapter 5 the development of a more advanced and generalizable framework for classifying the strategies used to solve the reverse fraction tasks will be discussed. This new framework will be important in the discussion of research questions RQ3, RQ5 and RQ6.

4.4 The Algebraic Thinking Questionnaire

The ATQ includes tasks that test students' understanding of the relationship between numbers. Part M of the ATQ includes multiplication tasks using whole numbers and fractions while Part D includes tasks that focus on division. The ATQ was trialled in Pilot Study 2 and discussed in Section 3.4 with Part M described in Section 3.4.3 and Part D in Section 3.4.4. Examples of Question 1 tasks are shown in Table 3-10 and examples for Question 2 are given in Table 3-11. The ATQ can be found in Appendix B.

4.4.1 Scoring the Algebraic Thinking Questionnaire

For the main study it was felt that the scoring used for the ATQ in Pilot Study 2 was not adequate as some of the details of the students' responses were not being recorded. In Chapter 5 the analysis includes percentages of students who responded correctly to the ATQ tasks but also includes details of misconceptions, arithmetical responses as well as responses that demonstrate relational thinking. To enable the relationships between the ATQ tasks and the FST tasks be examined in order to answer the sub research questions it is important that none of this detail be ignored. An alphabetic system was devised similar to that used to code students' responses for the FST. Table 4.4 shows an example of the initial alphabetic scoring scheme and the numerical marking scheme used to code responses for Task M1a.

Table 4-4: Example of the marking scheme for Task M1a (ATQ)

Task M1a (ATQ)	a	b	c	d
Numerical score	0	0	1	1
M1a. Write a number in the box to make a true statement. $36 \times 25 = 9 \times \square$	No attempt	Attempted a response but incorrect	Correct arithmetical calculations	Correct relational thinking

The results for the ATQ tasks are discussed in detail in Chapter 5. These results are used to inform the discussion about the main research question and research questions RQ3 and RQ4. In the next section the development and trial of the Structured Interview will be described. This semi-structured interview was designed especially for the Main Study.

4.5 The Structured Interview

The Structured Interview protocol was developed in response to suggestions made by Carolyn Kieran (personal communication, July 2016, at ICME-13). She suggested that the quantities associated with the three reverse fraction tasks be varied, and that non-specific quantities be included. These suggestions are in line with the research of Marton, Runesson, and Tsui (2004) which demonstrated that varying numbers in mathematical tasks can foster a generalisable pattern. It was anticipated that there would be stronger evidence of algebraic thinking or reasoning if students consistently used multiplicative strategies, or progressed to using multiplicative strategies, when responding to reverse fraction questions where the quantities were changed but the fractions remained the same as in the first three interview tasks (middle column of Table 4.5).

The Structured Interview conducted with 17 primary students was initially described in Pearn and Stephens (2017; 2018) and included a brief discussion about the administration of the interview, the initial scoring of the interview (see Table 4.12) and gave some illustrative examples. Other papers describing the Structured Interview include: Pearn, Stephens, Pierce (2018a; 2018b), and Pearn, Stephens, Zhang and Pierce (2019).

Section 4.5.1 describes the development of the Structured Interview, Section 4.5.2 compares the interview tasks with the Lee and Hackenberg tasks, Section 4.5.3 describes the trialling of the Structured Interview with a Year 5 student, Section 4.5.4 details the administration of the Structured Interview while Section 4.5.5 details the initial attempt at scoring the interview. Chapter 6 discusses the results from both the primary and Year 8 students for the Structured Interview.

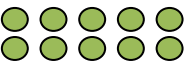
4.5.1 Development of the Structured Interview

The Structured Interview protocol includes reverse fraction tasks similar to those shown in Figure 3-7 but with progressive levels of abstraction, starting from particular instances and becoming progressively more generalised as shown in Table 4.5.

In the left-hand column are the reverse fraction tasks from the FST. The initial variation in the Structured Interview is a change in the given number of counters (*Change of number 1*). The middle column of Table 4.5 shows Questions 1 - 3 of the Structured Interview, with the same fraction as used in the FST, but a different number of counters.

The second variation used for Question 4 - 6 (right-hand column of Table 4.5) is another change in the given number of counters (*Change of number 2*) followed by another change with an unknown number of counters (*any number*).

Table 4-5: Comparison of Structured Interview questions with Tasks C5 - C7 (FST)

Tasks C5 - C7 (FST)	Structured Interview	
	First variation <i>Change of number 1</i>	Second variation: <i>Change of number 2</i> followed by 'any number'
Task C5 This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided that your answer is correct.	1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	4a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with? 4b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?
Task C6 Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____ Show all your working.	2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. How many CDs does Kay have? _____ Explain your thinking.	5a. If Susie had 20 CDs, which was $\frac{4}{7}$ of her friend Kay's CDs, how would you find the number of CDs Kay has? 5b. If it was any number of CDs that Susie had, and this was still $\frac{4}{7}$ of the number CDs Kay had, what would you need to do to find the number of CDs Kay had?

Task C7 This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  How many counters did I start with? Explain how you decided that your answer is correct.	3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	6a. If I gave you 70 counters, which was $\frac{7}{6}$ of the number of counters I started with, how would you find the number of counters I started with? 6b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?
---	--	---

While Questions 4a, 5a and 6a (right-hand column of Table 4.5) use the same fractions as those used in the FST (Tasks C5, C6 and C7) the number of counters is changed. Questions 4b, 5b, 6b also use the same three fractions but were designed to make additive and/or subtractive strategies less attractive and less easy to use. In these questions, students were asked how to find an unknown whole if they had any number of counters which represented a specific fraction of the whole.

If students could respond to these six questions, they were presented with Question 7 (Figure 4.9) which required them to come up with a completely generalisable method as they were given neither the fraction nor the number of objects representing the fraction. The aim was to ascertain whether students could generalise their solution method, for example, by dividing the unknown quantity by the numerator and then multiplying by the denominator. This question also provided an opportunity for more capable multiplicative thinkers to use algebraic notation to represent the unknown quantity and its accompanying fraction.

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?

Can you tell me what you would do?

Please write your explanation in your own words.

Figure 4-9: Question 7 (SI)

Where the original three reverse fractions (left-hand column of Table 4.5) were used with different known quantities (middle column of Table 4.5), the aim was to see whether students used the same strategies for the interview questions as they had used on the paper and pencil assessments. Further rationale for the structure of these items was to see whether the change of quantities induced students to change from additive/subtractive methods to generalisable multiplicative methods. In particular, were students who had relied on additive or subtractive methods, with or

without a diagram, able to use multiplicative methods once diagrams were no longer provided? Interview questions involving ‘*any number*’ were designed to make additive and subtractive strategies less attractive and encourage students to focus on the fractional structures.

4.5.2 Comparison with Lee and Hackenberg Interview Tasks

This study builds on the work of Lee and Hackenberg (Hackenberg & Lee, 2015; Lee & Hackenberg, 2014) in an attempt to confirm the link between fractional competence and algebraic reasoning in the middle years. While they worked with a small number of older students this study was designed to be undertaken with larger numbers of students with a subset of those students being interviewed.

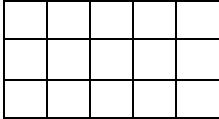
Lee and Hackenberg (2014) interviewed students from middle and high school years who had already been introduced to formal symbolic notation and were therefore able to write an appropriate equation to represent both the fraction and algebra tasks. However, since this study involves younger students from Year 5 and 6 who have not typically been introduced to formal algebraic notation, the Structured Interview tasks needed to allow students to record their responses in whichever way was appropriate for them. Thus, the Structured Interview aimed to allow students to demonstrate that they had identified the structural properties of fractions and were able to make robust generalisations with or without formal symbolic notation.

Four types of fraction tasks are given in the left-hand columns of Table 4.6, 4.7, 4.8 and Table 4.9. Type A tasks include those where students are asked to find the number of objects in a fractional part of a whole collection (Table 4.6). In Type B tasks students are given the equivalent quantity representing the fractional part and asked to find the number in the whole collection (Table 4.7). In Type C tasks the fraction is stated but the equivalent quantity remains unstated and students are asked to show how they would work out the number in the whole collection (Table 4.8) and in Type D tasks both the fraction and the equivalent quantity remain unstated and students are asked how they would determine the number in the whole collection (Table 4.9).

In Type A fraction tasks students are given the number of objects (or size) in a whole collection and asked to find a fractional part of that collection. Type A tasks were used in the Hackenberg and Lee interviews (Hackenberg & Lee, 2015; Lee & Hackenberg, 2014) and included in the FST tasks but not in the Structured Interview questions. Examples of the tasks from both research studies are shown in Table 4.6 where both the whole and the fractional part to be found are specified. In the Lee and Hackenberg example students need to translate that ‘5 times the height of Roberto’s stack’ means finding ‘one-fifth of Sara’s stack’. Once this translation has been made

it becomes equivalent to one-fifth of 65. Both tasks in Table 4.6 assumes students will use the operational meaning for the equals sign (Jones et al. 2013) to give an answer to the worded tasks.

Table 4-6: Examples of Type A fraction tasks

Lee & Hackenberg task		Fraction Screening Test	
Sara has a stack of CDs that is 65 cm tall. That's 5 times the height of Roberto's stack. Draw a picture of this situation. How tall is Roberto's stack of CDs? How did you solve this problem?	Whole (<i>Given</i>) Sara has a stack of CDs that is 65cms tall.	A5 (FST) Shade one-fifth of this rectangle. 	Whole (<i>Given</i>) One large rectangle divided into 15 small rectangles
	Fractional part (<i>Find</i>) Find the height of Roberto's stack given that Sara's stack is 5 times Roberto's stack. (This implies that Roberto's stack is one-fifth the height of Sara's stack.)		Fractional part (<i>Find</i>) Find one-fifth of 15 small rectangles.

Both studies used Type B tasks and examples are shown in Table 4.7. Students are given the equivalent quantity for the fractional part and asked to find the size or number in the whole collection. The Lee and Hackenberg task states that four-sevenths represents \$84 and the FST task relates two-thirds and 18 counters. In both cases students have been given both the fraction and the amount representing the fractional. In this case students need to use the 'sameness' meaning of the equals sign in order to balance each side of the equals sign (Jones et al., 2013).

Table 4-7: Examples of Type B fraction tasks

Lee & Hackenberg task		Structured Interview Task	
Tanya has \$84 which is $\frac{4}{7}$ the size of David's amount of money. Draw a picture of this situation. How much does David have?	Whole (<i>Find</i>): David's amount of money	Q.1 (SI) If I gave you 18 counters, which is $\frac{2}{3}$ of the counters I started with, how would you find the number of counters I started with?	Whole (<i>Find</i>): The number of counters I started with.
	Fractional part (<i>Given</i>) Tanya's amount of money (\$84) is four-sevenths the amount of David's money		Fractional part (<i>Given</i>) Two-thirds of the number of counters I started with is 18 counters.

Students will get a whole number result if they use the operational meaning of the equals sign for both tasks shown in Table 4.7. That is, find the fractional part of the given number/quantity rather than equate the fractional part with the given number/quantity. For example, four-sevenths of \$84

is \$48 and two-thirds of 18 is twelve. Students would have no perception that their response is incorrect.

Type C fraction tasks are used in both studies; students are given the fraction, but the equivalent quantity remains unstated and students are required to show how they would find the whole. Hackenberg and Lee (2015) and Lee and Hackenberg (2014) refer to Type C tasks as Algebra Tasks and students were expected to draw a picture, then write a symbolic equation and test the given equation using particular numbers. In the Structured Interview initially the numbers in the Type B tasks were varied before moving onto Type C tasks. This allowed interviewers to ask students what they would need to do to find the whole given a particular fraction and an unstated number of objects represented by that fraction. Examples from both studies are shown in Table 4.8.

Table 4-8: Examples of Type C fraction tasks

Lee & Hackenberg task		Structured Interview Task	
Theo has a stack of CDs some number of cm tall. Sam's stack is two-fifths of that height. a. Draw a picture to represent this situation. b. Can you write an expression for how tall the height of Sam's stack is?	Whole (<i>Find</i>): Represent the two stacks of CDs - Theo's stack and Sam's stack. Express the height of Sam's stack in terms of the height of Theo's stack.	Q4a. (SI) If I gave you any number of counters, which is $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?	Whole (<i>Find</i>): Find the number of counters I started with.
	Fractional Part (<i>Given</i>): Sam's stack is two-fifths of Theo's stack. Unstated ("some"): Theo has a stack of CDs <i>some number</i> of cm tall.		Fractional Part (<i>Given</i>): $\frac{2}{3}$ of the number of counters I started with Unstated ("some"): If I gave you <i>any number</i> of counters

Type D tasks were not included in any of the Hackenberg and Lee research reports. They concluded from students' responses to their Type C Algebra Tasks that students' algebraic thinking was evident if they were able to draw a picture, write an algebraic equation and trial specific numbers. Three out of 12 secondary students in their 2015 study having written, for example, $y = (\frac{2}{5})x$ were able to successfully transform the first equation into $x = (\frac{5}{2})y$.

Type D tasks were vital to the current study to demonstrate that students were able to give a fully generalisable strategy for finding the whole where both the fraction and the equivalent quantity

remain unstated as the primary students had not yet been introduced to symbolic algebraic notation and the Year 8 students appeared reluctant to use this notation unless specifically required to do so. The example of the Type D task from the Structured Interview is included in Table 4.9.

Table 4-9: Example of Type D fraction task

Structured Interview Task		
7. What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?	Whole (<i>Find</i>): The number of counters I started with. Unstated (“some”): ... any number of counters The number of counters I started with.	Fractional Part (<i>Find</i>) Unstated (“some”): ... any fraction of the number of counters I started with

This comparison was completed after the trialling of the Structured Interview. However, the results shown in Tables 4.6 – 4.9 validate the tasks used for the Structured Interview in the main study.

4.5.3 Trialling the Structured Interview

The Structured Interview was initially trialled with a Year 5 student who had attempted to complete the three reverse fraction tasks when in Year 4 with limited success. In solving Task C5 in Year 4, Student AM halved the 10 counters to determine that five counters represented one-third (“I found out 5 equals one-third”) thus the whole or three-thirds was 15 (Figure 4.10).

3.
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.



a. How many counters did I start with? 15

b. Explain how you decided that your answer is correct. I found out five = 1/3

Figure 4-10: Student AM's solution for Task C5 (FST)

When attempting to solve Task C7 Student AM wrote several responses and his confusion is demonstrated in his use of both tally marks and question marks (14? 12?) in Figure 4.11.

5.
This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with. 7



5 + 11

a. How many counters did I start with? 14? 12?

b. Explain how you decided that your answer is correct.

Figure 4-11: Student AM's attempted solution for Task C7 (FST)

Before trialling the Structured Interview in Year 5 Student AM was asked to complete the three reverse fractions. For Task C5 in the left-hand column of Figure 4.12 He used a multiplicative strategy, but his recording looks as though he is counting by fives to determine the whole if ten counters represented two-thirds. This recording was made as he explained his method to the interviewer. In Task C6 Student AM used and recorded his multiplicative strategy. As shown in the right-hand side of Figure 4.12 he found the number of objects (3) representing the unit fraction (one-seventh) and then multiplied by seven to find the whole (seven-sevenths).

1. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? 4 15 b. Explain how you decided that your answer is correct. $\frac{1}{3} = 5$ counters $\frac{2}{3} = 10$ counters so $\frac{3}{3} = 15$ counters	2. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? 21 Show all your working. $\frac{1}{7} = 3$ cds $3 \times 7 = 21$
--	--

Figure 4-12: Student AM's *Fully Multiplicative* methods for Tasks C5 and C6 (FST)

For Task C7 Student AM used a partially multiplicative strategy as shown in Figure 4.13. He subtracted the number representing the unit fraction (2) from the quantity representing seven-sixths (14) to get the number representing the six-sixths or the whole (12). He has used the unconventional recording of $\frac{1}{6} = 2$ being shorthand for 'one-sixth of the group of 14 counters is two counters'.

3.
This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.



a. How many counters did I start with? 12
b. Explain how you decided that your answer is correct.

$\frac{1}{6} = 2$ $\frac{7}{6}$ is improper so ~~14~~ minus 2

Figure 4-13: Student AM's *Partially Multiplicative* solution to Task C7 (FST)

After completing the three reverse fraction tasks Student AM was interviewed using the Structured Interview. The first three questions used the same three fractions but with different numbers of objects representing these fractions and there were no diagrams for any of the interview tasks. Student AM used multiplicative strategies for these three interview questions. Student AM used fully multiplicative solution strategies for the first two questions of the Structured Interview as shown in Figure 4.14.

12 counters is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? 18 b. Explain your thinking. $6 \text{ counters} = \frac{1}{3}$ $6 \times 3 = 18$	Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. a. How many CDs does Kay have? 14 b. Explain your thinking. 8 CDs $2 \text{ CDs} = \frac{1}{7}$ $2 \times 7 = 14$
--	---

Figure 4-14: Student AM's *Fully Multiplicative* solutions for Questions 1 and 2 (SI)

Figure 4.15 shows Student AM's partially multiplicative response to Question 3 when he subtracted three from 21 to find the number of counters in the whole collection. This is the same strategy he used for Task C7 from the FST as shown in Figure 4.13.

21 counters is $\frac{7}{6}$ of the number of counters I started with.

a. How many counters did I start with? 18
b. Explain your thinking.

$3 \times 7 = 21$ $21 - 3 = 18$

Figure 4-15: Student AM's *Partially Multiplicative* response to Question 3 (SI)

When Student AM was presented with Question 4a he was unsure what was meant by the expression: "If it was another number, say 16, which was two-thirds of the number I started with".

After clarification he used a fully multiplicative strategy as shown in his response in Figure 4.16 where he found ‘half of 16’ then multiplied by three (the denominator).

16 counters is $\frac{2}{3}$
a. If it was another number, say 16, which was two-thirds of the number I started with, how would you find the number of counters I started with?
half of 16 is 8 $8 \times 3 = 24$

Figure 4-16: Student AM's response to Question 4a (SI)

When asked about “another number” representing two-thirds in Question 4b Student AM again used a fully multiplicative method. He stated that the ‘any number’ needed to be divided by the numerator (2) when he halved it and multiplied by the denominator (“times it by 3”). He also provided an example of his own. He chose six counters to represent two-thirds and stated that the whole group would be nine after halving six and multiplying by three. Note his use of an arrow in his solution given in Figure 4.17 to indicate that he is using the result from the first part of Question 4b in the second part.

b. If it was another number and it was still two-thirds of the number you started with what would you need to do to find the number of counters I started with?
half it times it by 3
(eg) 6 halved = 3 $\rightarrow 3 \times 3 = 9$

Figure 4-17: Student AM's response to Question 4b (SI)

When presented with Question 5a Student AM used the fully multiplicative method (Figure 4.18) he had used for Questions 1 and 2 (Figure 4.14) and both parts of Question 4 (Figure 4.15 and Figure 4.16). Neither the change in the fraction, nor the change in the number of CDs represented by the fraction, appeared to influence Student AM’s choice of strategy. As shown in Figure 4.18 he again used an arrow to indicate that he was using the number of CDs represented by the unit fraction (5) and multiplying it by the number of parts (7) needed to find the number in the whole collection (35).

Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's.
a. If Susie had 20 CDs, which was four-sevenths of her friend Kay's CDs, how would you find the number of CDs Kay has?
 $20 \div 4 = 5 \rightarrow 5 \times 7 = 35$

Figure 4-18: Student AM’s response to Question 5a (SI)

To respond to Question 5b (Figure 4.19) Student AM used his *Fully Multiplicative* strategy and suggested that ‘any number’ representing the fraction four-sevenths would need to be divided by the numerator (4) and multiplied by the denominator (7). Without any prompting from the interviewer he gave his own example where he doubled the number given in Question 5a to determine that if 16 CDs represented four-sevenths of the whole collection then the whole collection would contain 28 CDs.

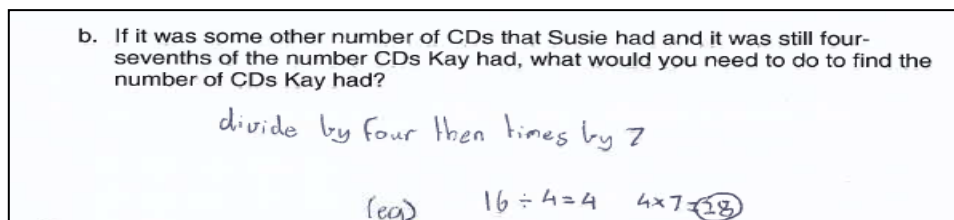


Figure 4-19: Student AM’s response to Question 5b (SI)

Question 6a, uses the same fraction ($\frac{7}{6}$) as Task C7 (FST) and Question 3 (SI). For these tasks Student AM used a *Partially Multiplicative* strategy where he calculated the number of counters representing one-sixth then subtracted that amount from the total given. See Figure 4.15 for his response to Question 3 (SI). For Question 6a Student AM changed from the *Partially Multiplicative* strategy that he had previously used with the tasks using the improper fraction seven-sixths to the *Fully Multiplicative* strategy that he had used for the proper fractions of two-thirds and four-sevenths. Figure 4.20 shows this fully multiplicative method for Question 6a.

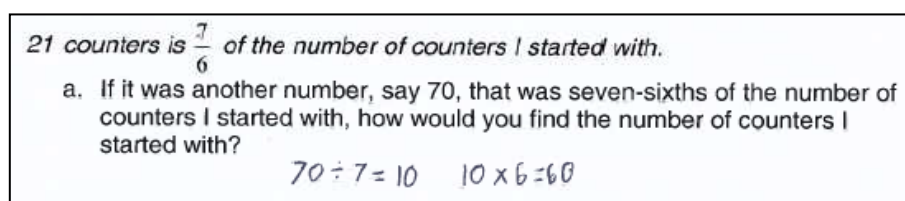


Figure 4-20: Student AM’s response to Question 6a (SI)

When responding to Question 6b Student AM verbally stated that for ‘any number’ representing seven-sixths: “you need to divide by seven then multiply the answer by six. Then you will have the number of counters you started with.” Once again, he included his own example. As for Question 5b, he doubled the number representing the fraction seven-sixths in Question 6a and calculated that if 140 counters represented seven-sixths of the whole collection then the whole collection would be 120 (Figure 4.21).

b. If it was another number and it was still seven-sixths of the number you started with what would you need to do to find the number of counters I started with?

$$140 \div 7 = 20 \quad 20 \times 6 = 120$$

Figure 4-21: Student AM's response to Question 6b (SI)

While Question 7 had not been explicitly planned before the trialling of the Structured Interview it seemed to be a logical next step. When the interviewer asked Student AM: "What if you had any number of counters, and they represented any fraction, how would you find the number of counters in the whole collection?" Student AM responded immediately: "You would divide by the top number and multiply by the bottom number". The interviewer then asked him if he knew the terms for 'top number' and 'bottom number' and he responded with the correct terms of numerator and denominator. As shown in Figure 4.22 he wrote both expressions on the response sheet.

divide by top number then multiply by bottom number
(numerator) (denominator)

Figure 4-22: Student AM's response to Question 7 (SI)

While Question 7 was not initially planned to be part of the Structured Interview it has proven to be an important component determining whether or not students can generalise their solutions to reverse fraction tasks.

4.5.4 Administration of the Structured Interview

Interviews were conducted with 17 primary students and 26 Year 8 students purposefully selected on the basis of their responses to the three reverse fraction tasks from the FST. In order to be interviewed students had to have successfully solved at least two of the three reverse fraction tasks and were chosen to represent the range of dominant solution methods for the set of three reverse fraction tasks as described in Section 4.3.3. In the final sample shown in Table 4.10 there were 20 girls and 23 boys with nine students in Year 5, eight students in Year 6 and 26 students in Year 8.

Table 4-10: Gender and year level of interviewed students (n = 43)

Year Level	Boys	Girls	Total
Year 5	5	4	9
Year 6	5	3	8
Year 8	13	13	26
Total	23	20	43

The 17 primary students from the same school were selected to represent the six categories of dominant methods for the set of three reverse fraction tasks (Task C5, Task C6, Task C7) of the written FST. For at least two of the three reverse fraction tasks two primary students used *Diagram Dependent Strategies*, one used *Additive/Subtractive Strategies*, seven used *Partially Multiplicative Strategies*, six used *Fully Multiplicative Strategies* and one used *Advanced Multiplicative Strategies*. In the Year 8 group of students one used *Diagram Dependent Strategies* two used *Additive/Subtractive Strategies*, 19 used *Partially Multiplicative Strategies* and 17 used *Fully Multiplicative Strategies* for at least two of the three reverse fraction tasks. No Year 8 student used *Advanced Multiplicative Strategies* for at least two of the three reverse fraction tasks for the written tasks on the FST. The number of students in each year level using the dominant methods are shown in Table 4.11.

Table 4-11: Dominant methods used by students for the three reverse fraction tasks (n = 43)

Year Level	Diagram Dependent	Additive/Subtractive	Partially Multiplicative	Fully Multiplicative	Advanced Multiplicative	Total
5	1	1	4	3	0	9
6	1	0	3	3	1	8
8	1	2	12	11	0	26
Total	3	3	19	17	1	43

The one-to-one Structured Interview was conducted at both schools by this researcher and three other experienced interviewers briefed by this researcher. Multiple interviewers were used to minimise any possible disruptions to the schools and enabled the interviews to be conducted on the same day. All interviewers took copious notes and one recorded the interviews on her phone.

The record of interview consisted of a three-page document, which included the questions and space for students to record their answer and to explain their thinking. This record, plus the instructions for interviewers, is provided in Appendix C. While there were multiple interviewers

to minimise interruptions to the schools, the analysis of these interviews was performed by this researcher.

At the start of their interview students were shown a copy of their responses to the three original reverse fraction tasks completed on a previous occasion. This was then left on the table for students to refer to if required. Students were encouraged to think about, and articulate, their response before writing anything on paper. Students who were unable to answer Questions 4b, 5b, or 6b, were not pressed to answer Question 7 (Figure 4.9).

Each interview took approximately 15 minutes per student. Students were free to correct their written responses to any of the interview questions. Interviewers encouraged students to first verbalise and then record their thinking. Students were able to leave the interview at any point. Some students terminated early because the tasks became too difficult.

4.5.5 Scoring the Structured Interview

Initially the written records from the interviews were analysed by two researchers independently using the Structured Interview Scoring Framework shown in Table 4.12 (see Pearn and Stephens, 2017; 2018). While this scoring meant students could be placed on a level within The Structured Interview Scoring Framework, more information was required in order to answer the main research question about the connections between fractional competence and algebraic reasoning.

Table 4-12 : The Structured Interview Scoring Framework

Level	Description
0	Not able to successfully complete any questions
1	Completed some or all of Questions 1 – 3 with known fraction and given quantity
2	Completed all questions with known fractions and a given quantity (Questions 1 -3, 4a, 5a and 6a). Relied on additive methods to solve Questions 4b, 5b, 6b. Could not give a generalizable response to Question 7.
3	Completed Questions 1 – 6 using multiplicative and/or mixed methods. Gave an appropriate non-symbolic generalizable response to Question 7
4	Completed Questions 1 – 6 using consistent multiplicative methods. Used suitable algebraic notation to give a generalizable response to Question 7

A second analysis of the results from the Structured Interview was subsequently conducted and the solutions to each of the seven interview questions were individually analysed and scored, using the same criteria as that used for the three reverse fraction tasks. The classification criteria

for the individual Part C reverse fraction tasks is described in detail in Chapter 5, as part of the *Framework for Reverse Fraction Task Strategies* (See Table 5.15). While responses to each Structured Interview question could be classified individually according to the *Framework for Reverse Fraction Task Strategies* the overall results for each interview also needed to be classified in terms of the development of algebraic reasoning.

Students' responses to the Structured Interview questions were analysed using a thematic analysis approach (Braun & Clarke, 2006) and varied from a reliance on computational methods to fully generalised responses which indicate algebraic reasoning. The new framework for the analysis of the Structured Interview is discussed in more detail in Chapter 6 and the *Emerging Algebraic Reasoning Framework* is shown in Table 6.12. Results from the Structured Interview are important for research questions RQ3, RQ4, RQ5 and RQ7 discussed in Chapter 8.

Chapter 5: Exploratory Analysis of Paper and Pencil Assessment Results

In this chapter results from the two paper and pencil assessments are analysed to help answer the sub-questions given in Section 1.2 and discussed in Chapter 8. Student work samples will be included to strengthen the analysis and, where appropriate, to illustrate both strategies and misconceptions revealed by students in their responses to the written tasks. Section 5.1 includes the descriptive analysis of results for the individual tasks of the FST while Section 5.2 describes the relationships between the three reverse fraction tasks and the results from the individual tasks of the FST. Both sections are needed to respond to research questions RQ1 and RQ2. Section 5.3 describes the relationships between the results for tasks from both the FST and the ATQ while Section 5.4 describes the relationship between the three reverse fraction tasks and the ATQ. Both Sections 5.3 and 5.4 will be used to respond to the research questions RQ3, RQ4, RQ5 and RQ6.

Pearn and Stephens (2016) performed a preliminary analysis of 162 Years 5 and 6 students' responses to the FST and ATQ instruments. These students were from three schools and were a subset of students from the main study. Their results have been incorporated into the results described in this chapter. The analysis conducted and described in the paper were not replicated in the main study as it was not deemed appropriate for the larger number of students in the main study. Pearn, Pierce and Stephens (2017a) discussed the results for a subset of 570 students for a set of six fraction tasks from the Fraction Screening Test. The six tasks include: Task A4 (Table 5.1), Task A12 (Table 5.4), Task B2a (Table 5.5), Tasks B4 and B5 (Table 5.6), and Task C6 (Table 5.8). The results for these six tasks have been included in the overall results recorded for the whole cohort.

The analysis of the data from the Structured Interviews can be found in Chapter 6 while Chapter 7 contains case studies to exemplify the findings from Chapters 5 and 6. The integration of the research findings for the research sub-questions are elaborated in Chapter 8.

5.1 Fraction Screening Test

The results for the three individual parts of the FST will be discussed in this section. Section 5.1.1 will focus on Part A results, Section 5.1.2 on Part B and 5.1.3 on Part C results. The results for the Part A tasks are shown in Tables 5.1, 5.2 and 5.3. The results for the Part B tasks are shown in Tables 5.4 and 5.5 while the results for the Part C tasks are shown in Tables 5.6 and 5.7. The strategies that students used to solve the three reverse fraction tasks (C5, C6 and C7) are

elaborated in Tables 5.8, 5.9 and 5.10. The data from all three parts of the FST will contribute to the responses to the research questions: RQ1 and RQ2 in Chapter 8.

5.1.1 Fraction Screening Test Part A task results

Part A tasks (A1 – A4) are shown in the left-hand column of Table 5.1 and focus on the identification of correct pictorial representations of three-quarters (Task A1), the selection of one-third of a group of discrete items (Task A2), the equal sharing of 20 discrete items (Task A3) and the calculation of the number of lollies in the whole group if one-half of the group is four lollies (Task A4). These initial four tasks are deliberately easy in an attempt to prevent or at least alleviate any possible test anxiety experienced by any student in this study but particularly those students in Year 5.

In Task A1, 87% of all middle-years students identified the two correct pictorial representations of three-quarters. Three students chose one correct representation and one incorrect representation, 51 students chose one correct representation with no incorrect representations, and 16 students chose the two correct representations and one incorrect representation. Many students correctly identified the prototypical square with the horizontal base showing three of four equal parts shaded but were unable to recognise that the square, rotated 45 degrees from the vertical, also has three of four equal parts shaded. Row 1 of Task 5.1 shows the similarity of the percentage of correct responses for each year level. Eighty-four percent of Year 5, 88% of Year 6, 89% of Year 8 and 100% of Year 9 students correctly identified the two representations of three-quarters.

In Task A2, 92% of all middle-years' students correctly identified one-third of the group of six beetles. The second row of Table 5.1 shows the percentage of correct responses for each year level for Task A2, with 88% of Year 5, 92% of Year 6, 97% of Year 8 and 100% of Year 9 students correctly identifying one-third of a group of six beetles.

Ten students incorrectly circled one beetle while 19 students incorrectly circled three of the beetles shown in Task A2. These students are demonstrating one of two common misconceptions related to their beliefs about whole number operations (Siegler, Thompson & Schneider, 2011) or 'whole number bias' (Ni & Zhou, 2005) where they use whole number knowledge to make inferences about fractions. For example, students who focus on the two whole numbers comprising any fraction ($\frac{a}{b}$) will focus on either the numerator (a), or the denominator (b), rather than the actual fraction ($\frac{a}{b}$) or the number of objects in the collection. Hence for these students one-third of any collection of objects would be stated as one object if the student's focus is on the numerator or three objects if their focus is on the denominator. They would use similar logic if

finding one-fifth of any sized group. That is, their response would be either one or five regardless of the number of objects in the group.

In Task A3, 96% of all students correctly calculated the number of fish each of four friends would receive if they shared 20 fish equally. The third row of Table 5.1 shows the similarity in the percentages of correct responses for each year level with 96% of both Year 5 and Year 6 students and 98% of Year 8 students able to divide 20 by four. Ten students gave an incorrect answer while 14 students did not complete the task. While most students just wrote a correct response in the space provided others used tally marks or a division algorithm to complete the task.

Task A4 is the first of the fraction tasks that requires students to use the number of objects representing a fractional part to find the whole rather than finding a fractional part of a whole. Ninety-four percent of all students calculated that eight lollies would be the number in the whole group when four lollies represented one-half of the group. Row 4 shows the percentage of correct responses for Task A4 for each year level. Ninety-two percent of Year 5, 94% of Year 6, 96% of Year 8 and 100% of Year 9 students successfully solved this task. Overall, 1% of students did not complete the task while 5% gave incorrect responses. Seven students stated there would be six lollies which is one-half more than the four depicted and three students stated there would be 16 lollies which is four times the number depicted.

Table 5-1: Percentage of correct responses by year level for Tasks A1 – A4 (FST)

Part A Tasks (FST)	Year 5 (n = 195)	Year 6 (n = 275)	Year 8 (n = 122)	Year 9* (n = 15)
A1. Circle the picture(s) that represent(s) three-quarters ($\frac{3}{4}$)				
	84	88	89	100
A2. Draw a circle around $\frac{1}{3}$ of these beetles.				
	88	92	97	100
A3. I want to share 20 fish equally between 4 friends. How many fish would each friend get?	96	96	98	100
A4. This is one-half of the lollies I started with.				
	92	94	96	100
How many did I start with?				

The graph in Figure 5.1 compares the percentage of correct responses for the four, Part A tasks (A1 – A4). More students were successful with Task A3 (sharing task) than the other three tasks (A1, A2 and A4). The primary students' results followed a similar pattern in the responses for these four tasks with Year 6 students being more successful than Year 5 students. The percentage of correct responses for Task A3 was very similar for both Year 5 and Year 6 primary students. Year 8 students were more successful than the primary students and were more successful with Task A2 than Task A4 which is the reverse of the primary results. As expected all Year 9 students correctly answered all four easy fraction tasks.

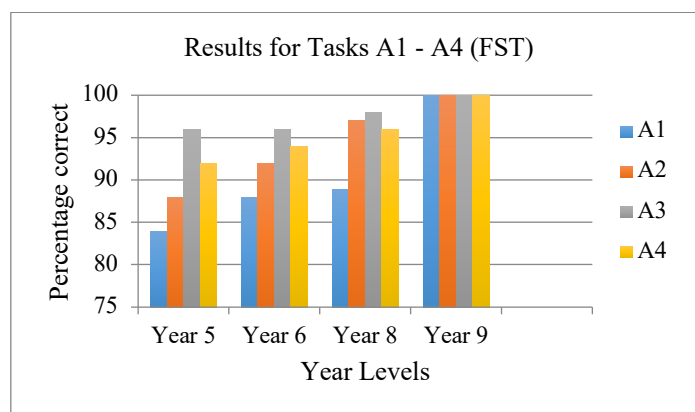


Figure 5-1: Percentage of correct responses by year level for Tasks A1 – A4 (FST)

The Part A tasks (A5 - A8) shown in the left-hand side of Table 5.2 focus on shading one-fifth of a rectangle marked into 15 squares (Task A5), dividing three pizzas into four equal pieces and stating the size of each share (Task A6), counting backwards by two-fifths (Task A7), and partitioning and shading three-quarters of a length (Task A8).

Task A5 requires students to shade in one-fifth of a rectangle divided equally into three rows of five small rectangles. Row 1 of Table 5.2 shows the percentage of correct responses for each year level. Eighty-five percent of Year 5, 89% of Year 6 and 93% of the Year 8 students correctly shaded one-fifth of a rectangle divided into 15 equal pieces. While 89% of all middle-years students shaded the correct number of small rectangles (3), seven students shaded one small rectangle and 43 shaded five small rectangles i.e. 7% of the whole group shaded five small rectangles to show one-fifth of the rectangle divided into 15 small rectangles. These responses reveal similar misconceptions to those shown in Task A2. Many students who had chosen the number of objects based on the size of the numerator or denominator in Task A2 also used similar thinking in Task A5. However, more students shaded in five small rectangles to show one-fifth of 15 than the number who circled three beetles to show one-third of six beetles. As the diagram

given for Task A5 uses an area representation rather than discrete objects this may have increased the difficulty for some students.

Task A6 requires students to calculate the fraction of a pizza that each person would receive if three pizzas equally between four friends. Row 2 shows the percentage of correct responses for each year level. Forty-eight percent of Year 5, 58% of Year 6 and 60% of Year 8 students successfully solved this task. While 5% of all students did not attempt this task, 57% gave a correct response and 17% gave the incorrect response of one-quarter. One-half of the students who stated that each friend would receive one-quarter of a pizza divided each of the three pictures representing pizzas into four equal parts.

Task A7 requires students to recognise, and continue, the counting backwards by two-fifths pattern. Row 3 of Table 5.2 shows the percentage of correct responses for each year level. Sixty-eight percent of both Year 5 and Year 6, and 75% of Year 8 students successfully solved this task. Seventy percent of all middle-years students correctly identified that the missing number in the counting backwards sequence was two-fifths while seven percent did not attempt this counting task. Six students stated that the next number in the counting sequence was six-fifths, which showed they recognised the two-fifths counting sequence, but they counted forwards rather than backwards. Twenty-seven students wrote that the next number was one-fifth.

Table 5-2: Percentage of correct responses by year level for Tasks A5 – A8 (FST)

Part A Tasks	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
A5. Shade one-fifth ($\frac{1}{5}$) of this rectangle. 	85	89	93	100
A6. Four friends shared three pizzas equally. What fraction of a pizza would each friend get? 	48	58	60	87
A7. Here is a counting pattern: $1\frac{3}{5}$ $1\frac{1}{5}$ $\frac{4}{5}$ What is the next term?	68	68	75	87
A8. Here is a picture of a ribbon.  Shade $\frac{3}{4}$ of the ribbon.	91	92	95	93

Task A8 requires students to mark, and shade, three-quarters of a length of ribbon. Row 4 of shows the percentage of correct responses for each year level. Ninety-one percent of Year 5, 92% of Year 6, 95% of Year 8 and 93% of Year 9 students successfully marked and shaded three-

quarters of a length of ribbon. Overall, 90% of middle-years students accurately placed a mark indicating three-quarters of the ribbon. Of these, 60% used accurate interim marks indicating one-quarter and one-half of the ribbon while 40% accurately placed the final mark for three-quarters without using any interim marks. Twenty-four students were unable to mark or shade three-quarters of the ribbon accurately. Overall, 92% of all students correctly shaded three-quarters of the ribbon.

The graph in Figure 5.2 compares the percentage of correct responses by year level for Tasks A5, A6, A7, and A8. Students found shading one-fifth of an area easier (Task A5) than sharing a pizza (Task A6) where many responses indicated the amount per pizza rather than the amount of pizza each person would receive altogether. Some students divided the ribbon into three parts (Task A8) but the parts were not equal.

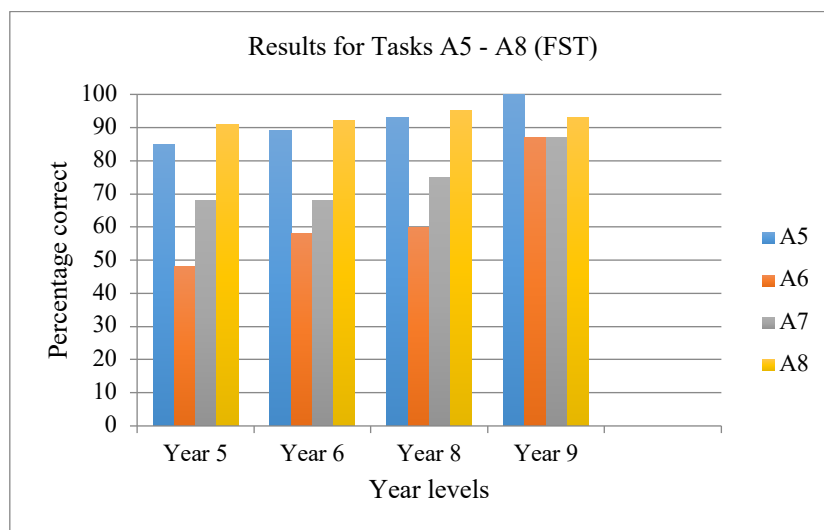


Figure 5-2: Percentage of correct responses by year level for Tasks A5 – A8 (FST)

Students found Task A7 (counting backwards by the fraction two-fifths) to be one of the more difficult questions in Part A. An investigation of the content descriptions from the Australian Curriculum: Mathematics (ACARA, 2016) as shown in Table 5.3 reveals that counting by fractions is only included at Year 4 when students are expected to count by quarters, halves and thirds. While there is no mention of counting by fractions at Year 5 or Year 6 students are expected to order fractions on number lines at both these levels. While unit fractions are compared and ordered at Year 5, fractions with related denominators are compared at Year 6. As counting backwards by two-fifths could be deemed to be similar to completing a number-line then this question is appropriate for inclusion in a middle-years assessment of fractions.

Table 5-3: Content Descriptions related to counting by fractions (ACARA, 2018)

Year 4	Year 5	Year 6
Count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line (ACMNA078)	Compare and order common unit fractions and locate and represent them on a number line (ACMNA102)	Compare fractions with related denominators and locate and represent them on a number line (ACMNA125)

The Part A tasks shown in the left-hand column of Table 5.4 focus on equivalence of fractions (A9, A10), addition of fractions with related denominators (A11) and finding the total amount of money given the amount representing a fractional part (A12). Table 5.4 shows the percentage of correct responses by year level for the tasks A9 – A12.

Two Part A tasks require students to find the value of the numerator of a fraction so that two fractions are equivalent. Task A9 requires students to complete an equivalent fraction for seven-ninths by writing the missing numerator, given the denominator (27). Row 1 of Table 5.4 shows the percentage of correct responses for each year level for Task A9. Sixty-two percent of Year 5, 78% of Year 6, 91% of Year 8 and 100% of Year 9 students successfully wrote the correct numerator for Task A9. While 76% of all middle-years students wrote the correct numerator (21), 17 students repeated the numerator of the given fraction (7), 18 students wrote 25, the number that is two less than the denominator, which suggests that some students are focusing on the difference or gap between the numerator and denominator. Twenty-two students gave the numerator as three.

Task A10 requires students to simplify the fraction eight-tenths to the equivalent number of fifths. Students generally found this task easier than Task A9. Row 2 of Table 5.4 shows the percentage of correct responses for each year level for Task A10. Seventy-six Year 5, 89% Year 6, 94% Year 8 and 100% of Year 9 students were able simplify eight-tenths to four-fifths. However, 23 students wrote the number that is two less than the given denominator (3) that implies they are focusing on the difference or gap between the numerator and denominator.

Task A11 requires students to add two fractions with related denominators (fifths and tenths) and simplify the answer if appropriate. Row 3 of Table 5.4 shows the percentage of correct responses for each year level. Fifteen percent of Year 5, 29% of Year 6, 25% of Year 8 and 73% of Year 9 students added the two related fractions and simplified the answer. A further 25% of Year 5, 29% Year 6, 27% Year 8 and 20% of Year 9 students correctly added the two fractions but did not simplify the answer.

Altogether 52% of all middle-years students correctly added three-tenths and one-fifth. Forty-eight percent of the successful students simplified the answer of five-tenths to one-half. This means that 25% of all students correctly added the two fractions and simplified the final answer. However, 23% of all students added numerators and denominators and stated the answer as four-fifteenths while approximately 7% gave a response of four-tenths, where they added the numerators and chose the larger denominator.

Table 5-4: Percentage of correct responses by year level for Tasks A9 – A12 (FST)

Part A Tasks (FST)	Year 5 (n = 195)	Year 6 (n = 275)	Year 8 (n = 122)	Year 9* (n = 15)
A9. What number do you need to write in the box so that the fractions are equivalent (have the same value)? $\frac{7}{9} = \frac{\square}{27}$	62	78	91	100
A10. What number do you need to write in the box so that the fractions are equivalent? $\frac{8}{10} = \frac{\square}{5}$	76	89	94	100
A11. Complete $\frac{3}{10} + \frac{1}{5} = \square$	15	29	25	73
A12. To buy a new workbook I spent \$4. This is $\frac{1}{7}$ of what I saved up. How much did I save up?	59	68	71	93

Task A12 is another reverse thinking fraction task where students need to use the amount of money representing the fraction one-seventh to calculate the total of the whole amount. This is a relatively easy reverse thinking task as the value representing the unit fraction has been given and to find the total amount students need to multiply by seven. Row 4 of Table 5.4 shows the percentage of correct responses for each year level. Fifty-nine percent of Year 5, 68% of Year 6, 71% of Year 8 and 93% of Year 9 students successfully calculated the whole amount of money given the amount for one-seventh and included the dollar sign as required by the context. While 67% of all students calculated that the whole amount would be \$28, another 16% wrote the number 28 without the dollar sign. This means that 83% of all students performed the correct calculation but 16% ignored the context. Eight percent of all students did not attempt this task. Nine students wrote \$4, 13 wrote \$24, two wrote \$20 and two wrote \$21. If these students had been interviewed, it may have been possible to determine the reason for these answers but as this was a written test it is not clear why these responses were given.

The graph in Figure 5.3 compares the percentage of correct responses by year level for Part A Tasks A9, A10, A11 and A12. Students were more successful with the equivalent fraction tasks than the addition of fractions with related denominators or the reverse fraction style task. More students were able to simplify eight-tenths to four-fifths than determine that $\frac{21}{27}$ was equivalent to seven-ninths. The pattern of responses for Tasks A9, A10 and A12 was similar for Years 5, 6 and 8 with an increasing percentage of responses across the year levels. The percentage of students at Year 6 successfully responding to the addition of two fractions with related denominators (Task A11) was higher than either the percentage of Year 5 students (15%) or the Year 8 students (25%).

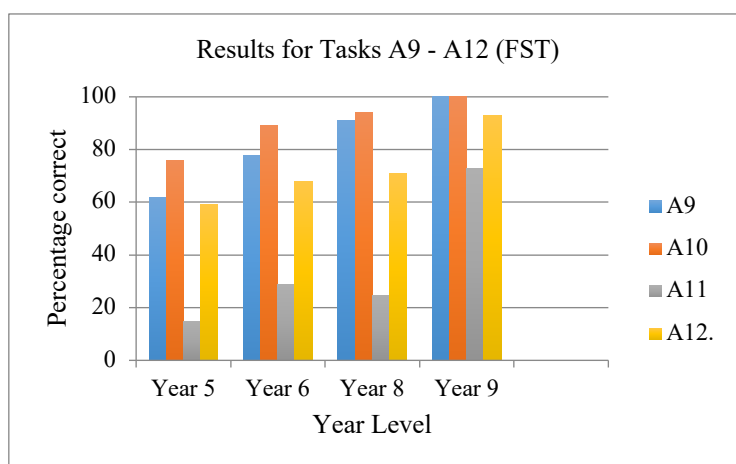


Figure 5-3: Percentage of correct responses by year level for Tasks A9 – A12 (FST)

As expected, students from all year levels found most of the Part A tasks easy as demonstrated by the results shown in Tables 5.1, Table 5.2 and Table 5.3 and shown in Figure 5.1, Figure 5.2 and Figure 5.3. This data will be used to inform the overall research question and in particular research questions RQ1 and RQ2. The results for Part A tasks will also be compared with the results for the three reverse fraction tasks (C5 – C7) in Table 5.17 and the ATQ in Table 5.64.

5.1.2 Fraction Screening Test Part B task results

Part B tasks focus on number lines, including completing the numbers required for a number line with equally spaced marks, dividing number lines into fractional parts and marking number lines given the length of a fractional part. Table 5.5 shows the results for the first two questions of Part B (Task B1 and Task B2) and Table 5.6 shows the results for three questions (B3 – B5). The analysis of this data will be used to respond to the research questions RQ1 and RQ2.

Task B1 has a number line divided into five equal parts, with the two endpoints given as zero and 100. Students are required to write the correct numbers on four equally spaced interim marks.

Row 1 of Table 5.5 shows the percentage of correct responses for each year level. Eighty-four percent of Year 5, 92% of Year 6, 98% of Year 8 and 100% of Year 9 correctly wrote the missing numbers on the number line marked zero to 100. Thirty-four percent of Year 5, 45% of Year 6, 66% of Year 8 and 73% of Year 9 students included an appropriate explanation that referred to the partitioning of the number line. While 91% of all students wrote the correct numbers on the marks on the number line (20, 40, 60, 80) only 47% gave an appropriate explanation based on the equal division of the number line, 22% based their explanation on counting by 20 and another 22% gave explanations that either did not make sense or were incomplete. Student RC in Figure 5.4 explains his method of dividing 100 by five to determine the correct response for Task B1.

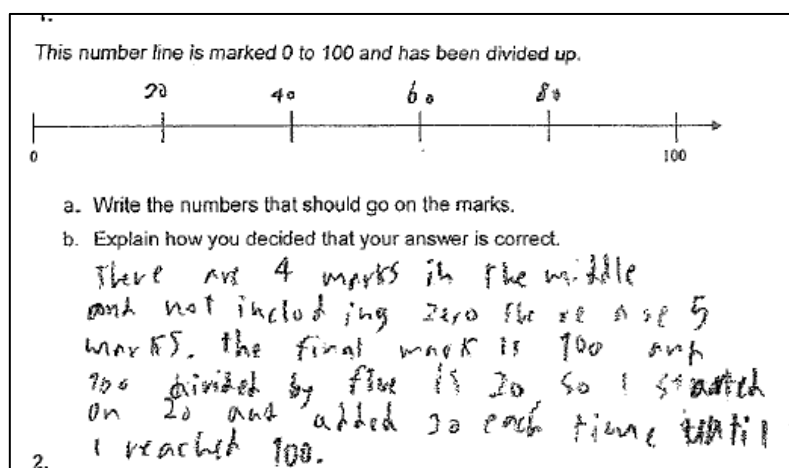


Figure 5-4: Student RC's response to Task B1 (FST)

The number-line in Task B2 is marked with the numbers zero and 25. Students are initially asked to mark the point on the line that indicates the position of the number 75 as accurately as they can (Task B2a) and subsequently where the number 5 would be on the same line (Task B2c). Student JM's response in Figure 5.5 indicates that he iterated the length from 0 to 25 two more times to determine the position of 75 on the number-line.

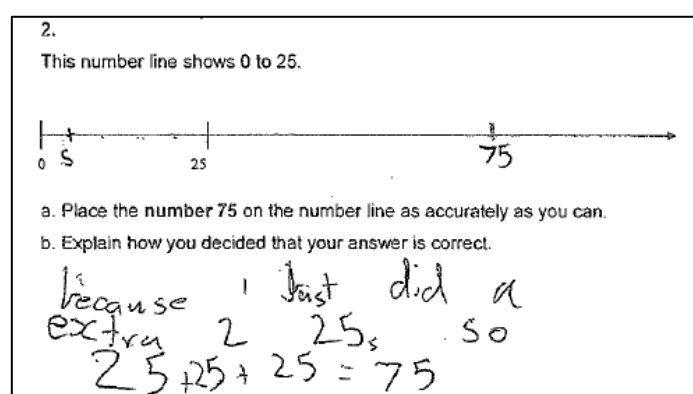


Figure 5-5: Student JM's response to Task B2 (FST)

Rows 3 and 4 of Table 5.5 shows the percentage of correct responses for each year level for this task. Fifty-two percent of Year 5, 55% of Year 6, 64% of Year 8 and 93% of Year 9 students were able to accurately place the number 75 on a number line using the given length for zero to 25. However, only 28% percent of Year 5, 38% of Year 6, 57% of Year 8 and 87% of Year 9 students justified their choice of position for the number 75 by stating that the distance needed to be three times the distance from zero to 25. Forty-four percent of students drew the interim mark for 50 before marking the position of 75. Another 13% accurately marked the position for 75 without any interim marks. This means that 57% of all students accurately marked where the number 75 should be positioned given the length from zero to 25. Forty percent of all students explained that the number 75 was placed at a position that was three times the length from zero to 25.

Many students assumed that the line represented a number line from zero to 100. This misconception was revealed when 33% of all students placed the number 75 three-quarters along the given line and justified this when explaining their response for Task B2. Student RC's response in Figure 5.6 illustrates his belief that the end of the number-line is 100 and describes how he decided the position for 75 based on this faulty assumption.

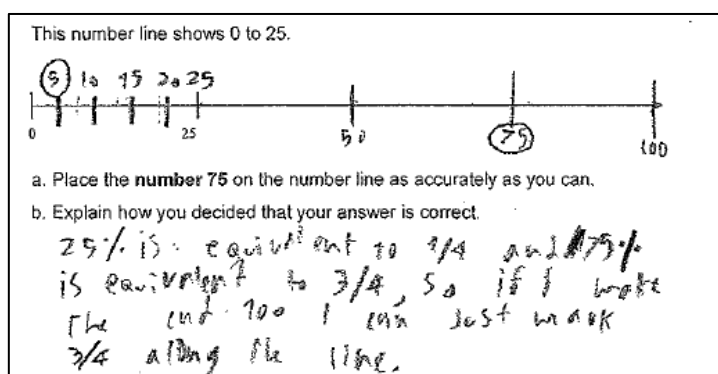


Figure 5-6: Student RC's incorrect responses for Tasks B2a and B2b (FST)

Students found Task B2c much easier than Task B2a as they partitioned the length from zero to 25 into five equal parts. For example, Student RC was unsuccessful with Task B2a as shown in Figure 5.6 as he believed the number-line ranged from zero to 100 but he was able to partition the given length from zero to 25 into five equal parts for Task B2c. His explanation shown in Figure 5.7 indicates that he is able to divide the length into five equal parts.

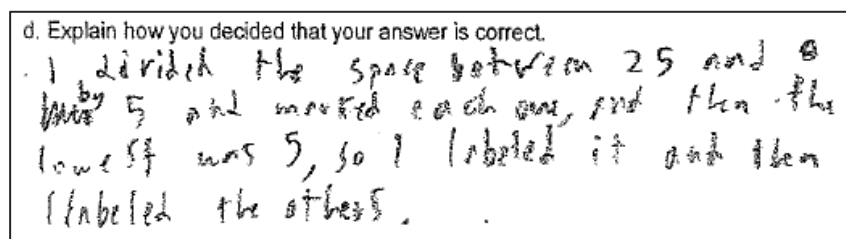
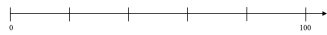


Figure 5-7: Student RC's response to Task B2d (FST)

Rows 5 and 6 of Table 5.5 shows the percentage of correct responses for each year level for Task B2c and Task B2d. Sixty-nine percent of Year 5, 74% of Year 6, 85% of Year 8 and 100 of Year 9 students accurately marked one-fifth of the distance from zero to 25. Fifty-seven percent of all students used interim marks to divide the length from zero to 25 into five equal parts while another 18% accurately placed one-fifth without interim marks. Thus 75% of all students successfully marked one-fifth of the distance from zero to 25. Forty-nine percent of all students explained that they knew their answer was correct “as five was one-fifth of 25” while 36% gave explanations that were incomplete or did not make sense.

Table 5-5: Percentage of correct responses by year level for Tasks B1a – B2d (FST)

Part B Tasks (FST)	Year 5 (n = 195)	Year 6 (n = 275)	Year 8 (n = 122)	Year 9* (n = 15)
B1a. This number line is marked 0 to 100 and has been divided up.  Write the numbers that should go on the marks.	84	92	98	100
B1b. Explain how you decided that your answer is correct.	34	45	66	73
B2a. This number line shows 0 to 25. Place the number 75 on the number line as accurately as you can. 	52	55	64	93
B2b. Explain how you decided that your answer is correct.	28	38	57	87
B2c. Mark the number 5 on the number line as accurately as you can.	69	74	85	100
B2d. Explain how you decided that your answer is correct.	39	43	75	80

The graph in Figure 5.8 shows the percentage of correct responses by year level for the number-line tasks that required students to write the correct numbers on the marks between zero and 100 (Task B1a), marking the number line to show the number 75 given the distance representing zero to 25 (Task B2a) and dividing the number line marked zero to 25 to show where the number 5 should be placed (Task B2c). Students found it more difficult to determine where the number 75 should be placed (Task B2a) than dividing a section of a number line to show where the number

5 should be placed (Task B2c). The percentage of correct responses for all three tasks increased across the year levels.

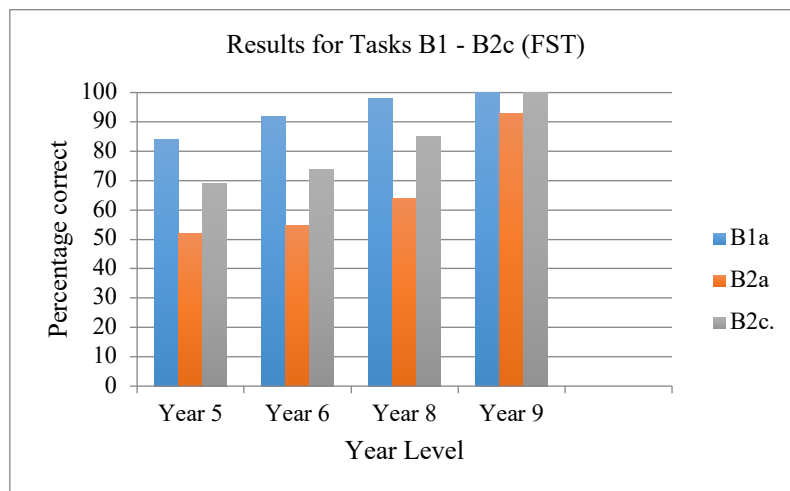


Figure 5-8: Percentage of correct responses by year level for Tasks B1a, B2a and B2c (FST)

The Part B tasks shown in Table 5.6 has one task where the scale has been changed, and two tasks which rely on students replicating, or iterating, the length that represents a given fractional part to then establish where the number one would be.

In Task B3 students were given a number line marked zero, one and two and asked to mark the position for the number three-fifths. The two students shown in Figure 5.9 have correctly marked the fraction three-fifths on the number line. Student RC (top) has marked in all the fifths between zero and one while Student BE (bottom) has given the marks for the fifths between zero and one and named three-fifths.

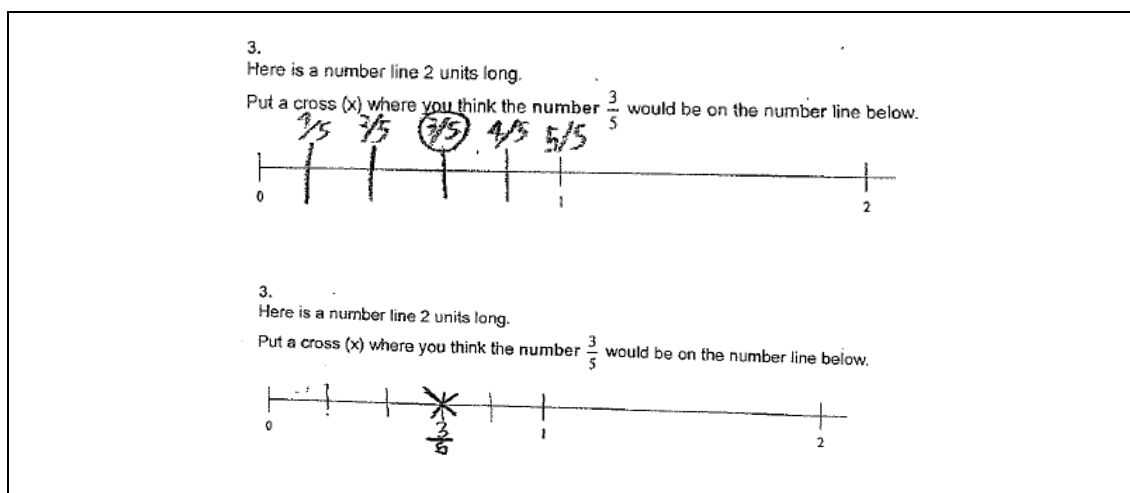


Figure 5-9: Examples of two students' correct responses for Task B3 (FST)

Row 1 of Table 5.6 shows the percentage of correct responses for each year level. Forty-four Year 5, 51% Year 6, 64% Year 8 and 87% Year 9 students correctly marked where the number three-fifths would be placed on a number-line marked zero to three. While 31% of all students accurately placed the number three-fifths using appropriate interim marks, an additional 21% of the students accurately positioned the number three-fifths without reference to any interim marks. This means that a total of 52% of all students correctly placed the number three-fifths on a number line marked zero to two. However, 14% of all students placed the number three-fifths on a mark that was three-fifths of the distance of the whole line marked zero to two rather than three-fifths of the distance between zero and one. Student KH has placed the mark about three-quarters of the length of the line (Figure 5.10).

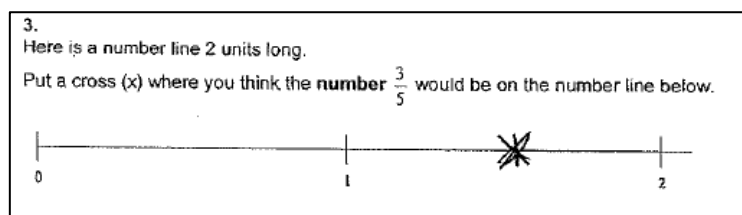


Figure 5-10: Student KH's incorrect response to Task B3 (FST)

In Task B4 students are given a number line marked with the numbers zero and one-third and asked to put a cross “where the number one would be on the number line”. Student RK (left-hand side of Figure 5.11) has iterated the length from zero to one-third twice and then indicated with a cross where the number one would be positioned. Student JM (right-hand side of Figure 5.11) has estimated the two iterations and although the marks are not accurately placed the intention is clear.

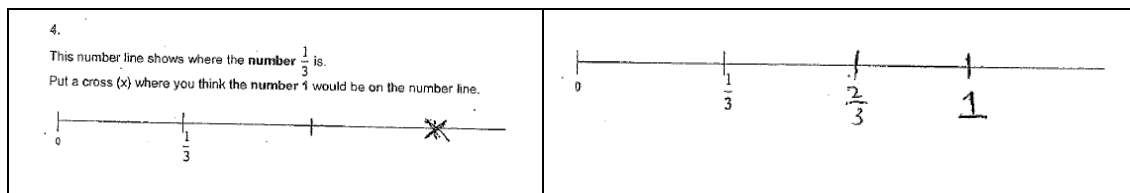
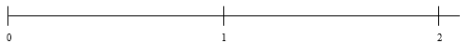
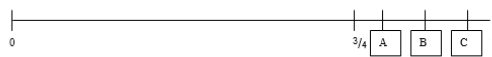


Figure 5-11: Examples of two correct responses for Task B4 (FST)

Row 2 of Table 5.6 shows the percentage of correct responses for each year level. Fifty-four percent of Year 5, 62% Year 6, 75% Year 8 and 87% Year 9 students successfully marked the position of the number one given the positions of zero and one-third. Thirty-eight percent of all students used appropriate interim marks to accurately mark the position of the number. Another 25% accurately positioned the number one without using any interim marks, giving a total of 63% of all students who could use the distance from one to one-third to position the number one. However, 15% of all students placed the number one at the end of the given number line.

Row 3 of Table 5.6 shows the percentage of correct responses for each year level. Sixty-nine percent of Year 5, 75% Year 6, 84% Year 8 and 87% Year 9 students successfully chose the correct alternative to indicate where the number one would be on the number-line marked zero to three-quarters. Altogether 18% of all students chose either of the two incorrect alternatives with the same number of students choosing each alternative. However, while 76% chose the correct alternative, only 59% gave an appropriate explanation for their choice. Row 4 shows the results for Task B5 which asks students to explain how they decided on the correct answer for Task B5.

Table 5-6: Percentage of correct responses by year level for Tasks B3 – B5b (FST)

Part B Tasks (FST)	Year 5 (n = 195)	Year 6 (n = 275)	Year 8 (n = 122)	Year 9* (n = 15)
3. Here is a number line 2 units long. Put a cross (x) where you think the number $\frac{3}{5}$ would be on the number line below. 	44	51	64	87
4. This number line shows where the number $\frac{1}{3}$ is. Put a cross (x) where you think the number 1 would be on the number line. 	54	62	75	87
5. On this number line 0 and the number $\frac{3}{4}$ are shown.  a. Circle the letter that represents where the number 1 would be.	69	75	84	87
B5b. How did you decide the other two choices were incorrect?	54	54	73	80

In Task B5 students were asked to decide which of three alternatives was in the correct position for the number one and to write an explanation about how they knew that the other two alternatives were incorrect. The two students in Figure 5.12 have chosen the correct alternative.

Student JM (left-hand side of Figure 5.12) gave an appropriate response and while the second half of Student RC's explanation is hard to interpret (right-hand side of Figure 5.12), he marked in the approximate positions of one-quarter and one-half proving that he knew C was the correct position for the number one.

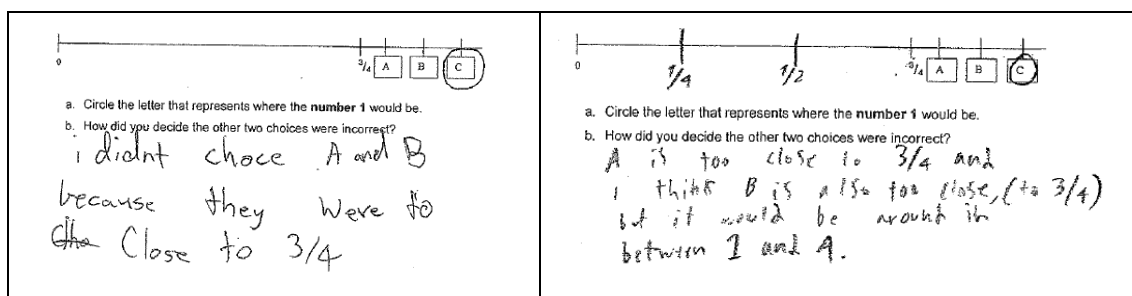


Figure 5-12: Examples of two correct responses for Task B5 (FST)

It is unclear how Student KH (Figure 5.13) decided that the response was A. He had obviously chosen B before crossing it out to choose A. Many students believe that ‘three-quarters is nearly one’ which might explain his response that B and C were too far from three-quarters to be the number one.

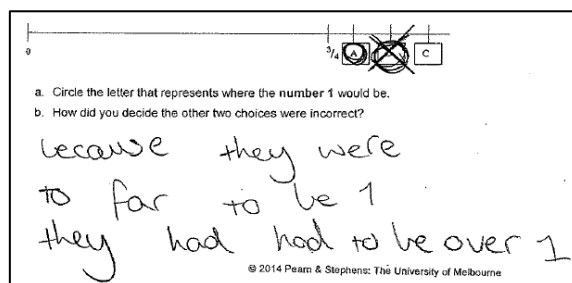


Figure 5-13: Student KH's incorrect response for Task B5 (FST)

The graph in Figure 5.14 shows the percentage of correct responses by year level for Tasks B3 – B5a. Students at Years 5, 6 and 8 found these tasks progressively easier as demonstrated by the increasing percentage of correct responses at each level. The pattern across these year levels was similar and shows that more students were successful with the successive tasks at the higher level. The hardest task was accurately marking the position of the number three-fifths on the number-line marked zero to two.

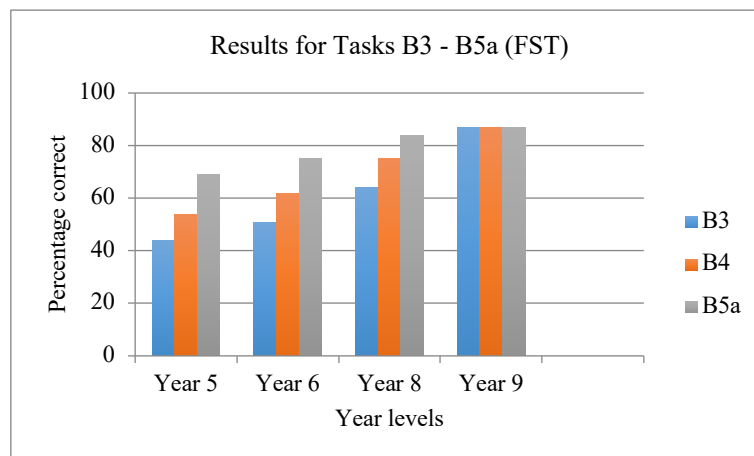


Figure 5-14: Percentage of correct responses by year level for Tasks B3, B4 and B5a (FST)

Students found these number line tasks more difficult than the Part A tasks. This data will be used to inform the overall research question and in particular the responses to Research Questions RQ1 and RQ2.

5.1.3 Fraction Screening Test Part C task results

Part C tasks are the most difficult in the FST. Part C tasks C1 to C4 have been designed to assess whether students can order four fractions (C1), match fractions to their equivalent decimal (C2) and decide which of the four alternatives does not belong in a group of fractions, decimals and percentages (C3 and C4). The results for Tasks C1 – C4 are shown in Table 5.7. As for the data discussed in the previous sections, this data will be used to inform the overall research question and in particular to Research Questions RQ1 and RQ2.

In Task C1 40% of all students ordered the four fractions from largest to smallest. Another 23% ordered the fractions correctly but ordered them from smallest to largest. The two students in Figure 5.15 ordered the fractions correctly. Student RK (left-hand side) converted all the fractions to equivalent tenths before writing the fractions from largest to smallest as directed. Student JM (right-hand side) ordered the fractions correctly but wrote them from smallest to largest rather than largest to smallest as required.

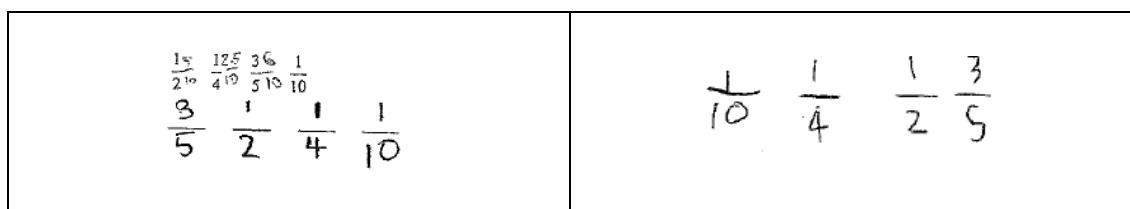


Figure 5-15: Examples of two students' responses for Task C1 (FST)

Row 1 of Table 5.7 shows the percentage of correct responses for each year level. Thirty-five percent of Year 5, 37% of Year 6, 54% of Year 8 and 60% of Year 9 students successfully ordered the fractions from largest to smallest.

Misconceptions were revealed when the incorrect responses for Task C1 were analysed. Incorrect responses demonstrated whole number thinking defined by Behr, Wachsmuth, Post, and Lesh (1984) as thinking which involves: “making separate comparisons of numerators and denominators using the ordering of whole numbers” (p. 332). Four percent of all students ordered the fractions by denominator i.e. they ordered the fractions from smallest to largest denominators, 3% ordered the fractions from largest to smallest denominators, 3% ordered initially by numerators then denominators and 18% ordered according to the size of the difference between the numerator and denominator. An example is shown in Figure 5.16 where Student KH has incorrectly ordered the fractions according to the difference between the numerator and the denominator. This is an example of ‘gap thinking’ (Pearn & Stephens, 2004). Students using gap thinking believe that the smaller the gap the larger the fraction.

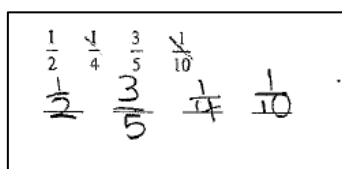


Figure 5-16: Student KH uses incorrect 'gap thinking' for Task C1 (FST)

In Task C2 82% of all students matched the four pairs of fractions and decimals with 14% matching two pairs correctly and a further 3% matching one pair e.g. $\frac{1}{2}$ and 0.5. Row 2 of Table 5.7 shows the percentage of correct responses for each level. Sixty-five percent of Year 5, 74% of Year 6, 84% of Year 8 and 93% of Year 9 matched all four pairs of fractions and their equivalent decimals.

In Task C3 students were asked to choose the representation that does not belong to the following set of numbers: 25% 0.4 0.25 $\frac{1}{4}$. Row 3 of Table 5.7 shows the percentages of correct responses for each year level. Seventy-one percent of Year 5, 81% of Year 6, 80% of Year 8 and 93% of Year 9 students correctly recognised that 0.4 did not belong in the group representing one-quarter. Seventy-eight percent of all middle-years students correctly chose 0.4 as not belonging to the group that are equivalent representations for one-quarter.

In Task C4 students need to select the representation that does not belong to the following set of numbers: 1.2 $\frac{6}{5}$ 12% $1\frac{6}{30}$. In Figure 5.17 Student JM (left-hand side) circled the correct response while Student KH (right-hand side) circled one of the incorrect responses.

4. Circle the one that does not belong: 1.2 $\frac{6}{5}$ <u>12%</u> $1\frac{6}{30}$	4. Circle the one that does not belong: <u>1.2</u> $\frac{6}{5}$ 12% $1\frac{6}{30}$
--	--

Figure 5-17: Examples of two students' responses for Task C4 (FST)

Row 4 of Table 5.7 shows the percentages of correct responses for each year level. Results were very similar for the primary students with 39% of Year 5, 40% of Year 6, 50% of Year 8 and 80% of Year 9 students correctly recognised that 12% did not belong in the group representing the number 1.2. Forty-three percent of all students correctly chose 12% as not belonging to the group that were equivalent representations for 1.2.

Table 5-7: Percentage of correct responses by year level for Tasks C1 and C2 (FST)

Part C tasks (FST)	Year 5 (n = 195)	Year 6 (n = 275)	Year 8 (n = 122)	Year 9* (n = 15)
C1. Place these fractions in order from largest to smallest : $\frac{1}{2}$ $\frac{1}{4}$ $\frac{3}{5}$ $\frac{1}{10}$	35	37	54	60
C2. Draw a line from each fraction to the equivalent decimal: $\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{10}$ $\frac{2}{5}$ 0.5 0.4 0.1 0.25	65	74	84	93
C3. Circle the one that does not belong: 25% 0.4 0.25 $\frac{1}{4}$	71	81	80	93
C4. Circle the one that does not belong: 1.2 $\frac{6}{5}$ 12% $\frac{16}{30}$	39	40	50	80

The graph in Figure 5.18 shows the percentage of correct responses by year level for Tasks C1, C2, C3 and C4. Primary students were more successful with Task C3 than Task C2 while Year 8 students were more successful with Task C2 than Task C3. Primary students were more successful with Task C4 than Task C1 while Year 8 students were more successful with Task C1 than Task C4.

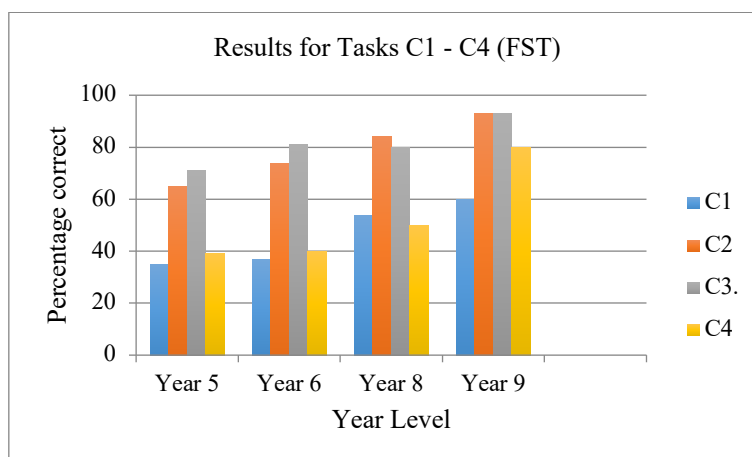


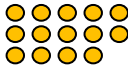
Figure 5-18: Percentage of correct responses by year level for Tasks C1 – C4 (FST)

Table 5.8 shows the percentage of students who successfully solved the three reverse fraction tasks C5 to C7. In the pilot studies discussed in Chapter 3 these three reverse fraction tasks allowed students the opportunity to reveal their algebraic thinking as they solved them. The results from these tasks are integral to all the research questions: RQ3, RQ4, RQ5 and RQ6. Table 5.7 shows the percentage of correct responses for each of these three tasks.

Most students found Task C5 the easiest of the three reverse tasks as shown in Row 1 of Table 5.8. Overall 81% of all students successfully solved this task with very similar results in the two primary years with 77% in Year 5 and 81% in Year 6. There were similar results for the secondary levels with 88% of Year 8 and 93% of Year 9 students correctly determining that there were 15 counters in the whole collection.

There is no diagram given for Task C6 and the fraction four-sevenths is less familiar than the fraction two-thirds used in Task C5 which appears to impact on the results for Task C6 shown in Row 2 of Table 5.8. Forty-nine percent of all students were successfully solved Task C6 with very similar results for Years 5, 6 and 8 with 44% Year 5, 48% Year 6 and 53% of Year 8 successfully determining the number of CDs that Kay had. Only one Year 9 student was unable to solve this task.

Table 5-8: Percentage of correct responses by year level for Tasks C5 – C7 (FST)

Part C tasks (FST)	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
C5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  How many counters did I start with?	77	81	88	93
C6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____	44	48	53	93
C7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  How many counters did I start with?	41	44	70	93

Unlike the diagram given for Task C5 the diagram for Task C7 was not designed or intended to provide a visual representation that assisted students to solve the task just by looking at the diagram. The improper fraction used also appeared to add to the complexity of the task. Many students used the number of counters in the diagram in a ‘trial and error’ method to solve this task. This method may have contributed to the higher percentage of correct responses for the Year

8 students. As shown in Row 3 of Table 5.8 the results for the primary students were very similar with 41% Year 5 and 44% Year 6 students solving the task successfully. Seventy percent of Year 8 and 93% of Year 9 students were able to calculate the numbers needed for the whole group. Overall only 50% of all middle-years students successfully solved this task.

The graph in Figure 5.19 shows the percentage of correct responses by year level for the three reverse thinking tasks C5, C6, and C7. For each task the percentage of correct responses increased as the year levels increased. Both primary and secondary students were most successful on the first reverse fraction task (Task C5). While the results for the remaining two reverse fraction tasks were similar for the primary students, Year 8 students were more successful with Task C7 than Task C6. Task C7 included a diagram which many secondary students used to assist with solving the task.

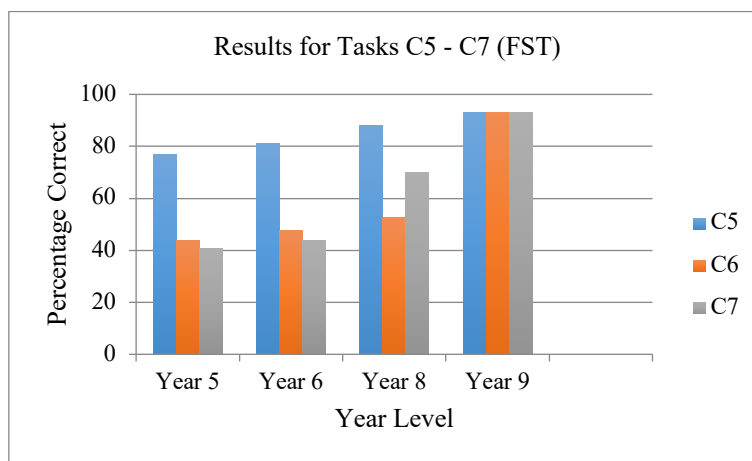


Figure 5-19: Percentage of correct responses by year level for Tasks C5 – C7 (FST)

In Section 5.1.3 the focus has been on the percentage of correct responses for all Part C tasks. This data will be used to inform the overall research question particularly the research questions RQ1 and RQ2. In the next section 5.1.4 the focus will be on the strategies the students used to solve the three reverse fraction tasks (Tasks C5, C6, and C7) from Part C.

5.1.4 Analysing the written responses for Tasks C5, C6 and C7

In this section the focus is on the analysis of students' written responses for each of the three reverse fraction tasks: Task C5, Task C6 and C7. Students were asked to: "Explain how you decided that your answer is correct." for Tasks C5 and C7 while for Task C6 students were asked to: "Show all your working". While most students gave a numerical response for each of the reverse fraction tasks many did not respond to either of these questions: "Explain how you decided that your answer is correct." or "Show all your working".

The students' responses for Task C5 were classified as *Not Attempted* if there was no attempt to respond to the task. Explanations or responses that were incomplete were classified as *Not Clear*. The other responses were classified using the five strategies: *Diagram Dependent*, *Additive/Subtractive*, *Partially Multiplicative*, *Fully Multiplicative*, and *Advanced Multiplicative*. Table 5.9 includes an explanation and example for each strategy used for Task C5.

Table 5-9: Classification of strategies used to solve Task C5 (FST)

Classification	Explanation	Example
Diagram dependent	Students use explicit partitioning of diagrams before using additive strategies	a. How many counters did I start with? 15 b. Explain how you decided that your answer is correct. because if you have $\frac{2}{3}$ and you have $\frac{1}{3}$ 00000 00000 10 Show you just add 00000 1 more 5
Additive	Students use additive methods without explicit partitioning of a diagram. Students find the number of objects needed to represent one-third and then use counting or addition e.g. "add another 5" to find the number of objects that represents the whole.	a. How many counters did I start with? 15 b. Explain how you decided that your answer is correct. one third is 5 2 thirds is 10 and 3 thirds would be 15
Partially multiplicative	Students use both multiplicative and additive methods. They calculate the missing fractional part (one-third) and then add it onto the original quantity (two-thirds).	a. How many counters did I start with? 15 b. Explain how you decided that your answer is correct. because there's 10 2 lots of 5 counters, so add another 5.
Fully multiplicative	Students use fully multiplicative methods. Students find the quantity representing one-third by dividing the quantity representing two-thirds by two. They then multiply the quantity representing one-third (5) by three to find the whole (15).	$10 \div 2 = 5$ $5 \times 3 = 15$ I divided the 10 counters in half so it's now $\frac{1}{2}$ then I timesed it by 3 to get the full amount
Advanced multiplicative	Students use more advanced multiplicative methods to solve the reverse fraction questions. These include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity (10) by two-thirds.	$10 \div \frac{2}{3} = 10 \times \frac{3}{2} = \frac{30}{2} = 15$

Table 5.10 includes the percentage of each type of response by year level for Task C5. Rows 1 and 2 show that as the year levels increase the number of students not attempting to answer the question or giving a response that is classified as *Not Clear* decreases. As shown in Row 3 very few students use either *Diagram Dependent* or *Additive* strategies for this task despite the inclusion of a picture of two rows of dots. About 40% of students from Years 5, 6 and 8 used partially multiplicative strategies for this task. This means they found the number of counters that represented one-third and then added this number onto the original number of counters representing two-thirds. The diagram may have assisted students with this strategy.

Table 5-10: Percentage of students by year level for each response type for Task C5 (FST)

Response Type	Year 5 (n=195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
Not attempted	13	12	6	0
Not clear	23	13	11	7
Diagram dependent	6	7	3	0
Additive/subtractive	3	7	7	7
Partially multiplicative	39	41	43	20
Fully Multiplicative	15	19	30	60
Advanced Multiplicative	1	3	2	7

The graph in Figure 5.20 highlights the types of strategies used successfully by students at each year level for Task C5. Most Year 5, 6 and 8 students used either *partially multiplicative* or *fully multiplicative* strategies to solve Task C5. Most Year 9 students used *fully multiplicative* or *advanced multiplicative* strategies to successfully answer Task C5. As the year levels increased so did the percentage of students who used a *fully multiplicative* strategy. Very few students from any year level used *advanced multiplicative* strategies for this task.

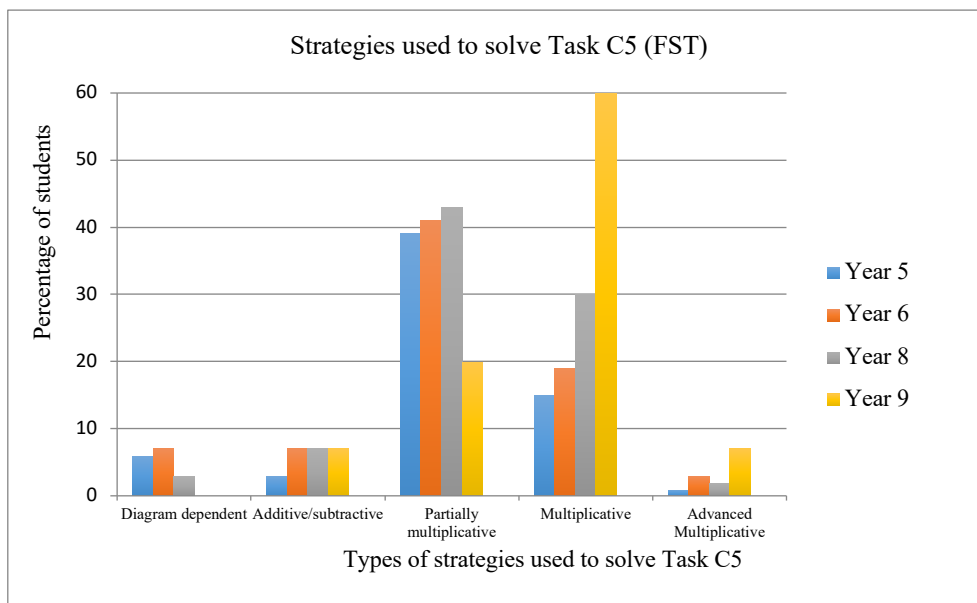


Figure 5-20: Comparison of strategies by year level for Task C5 (FST)

The students' responses for Task C6 were classified using the same criteria as that used for Task C5. Table 5.11 includes an explanation and example for Task C6 for each strategy.

Table 5-11: Classification of strategies used to solve Task C6 (FST)

Classification	Explanation	Example
Diagram dependent	Students use explicit partitioning of diagrams before using additive strategies	Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working.
Additive	Students use additive methods without explicit partitioning of a diagram. Students find the number of objects needed to represent the one-seventh and then use counting or repeated addition to find the number of objects needed to represent the whole.	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. $4 \times 3 = 12$ So that... means adding by 3 $3 \times 3 = 9$ $6 \times 3 = 18$ $7 \times 3 = 21$
Partially multiplicative	Students use both multiplicative and additive methods. In the example, they calculate three-sevenths by dividing 12 by 4 then multiplying by 3 and then add the amount for three-sevenths onto the original quantity for four-sevenths to determine the whole.	4. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. $12 \div 4 = 3$ $3 \times 3 = 9$ $9 + 12 = 21$

Fully multiplicative Students use fully multiplicative methods. Students find the quantity represented by one-seventh by dividing 12 by four and then multiply the quantity representing one-seventh by seven to find the whole.

Advanced multiplicative Students use more advanced multiplicative methods to solve the reverse fraction questions. These include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity by the known fraction.

6.
Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.
How many CDs does Kay have? 21
Show all your working.

$$12 \div 4 = 3 \quad 3 \times 7 = 21 \text{ (CDs)}$$

$$\begin{aligned} \text{Susie} &= 12 \\ \text{Kay} &= x \\ 12 &= \frac{4}{7}x \\ 12 \div \frac{4}{7} &= 3 \times 7 = 21 \\ x &= 21 \end{aligned}$$

In Table 5.12 the percentage of students are given by year level for Task C6. Overall 50% of all students were unable to explain their strategy or show their working out for Task C6. This task had no diagram and included the unfamiliar fraction of four-sevenths. Twenty-seven percent of all students made no attempt to respond and 23% gave responses that were *Not Clear*. Twenty-six percent of students successfully used a *Fully Multiplicative* strategy and the remaining 25% used a range of strategies with similar percentages relying on *Diagram Dependent*, *Additive* and *Partially Multiplicative* strategies. Students using *Fully Multiplicative* strategies worked out the number representing one-seventh then multiplied by seven to determine the number representing the whole. Only 3% overall used *Advanced Multiplicative* strategies.

Table 5-12: Percentage of students by year level for each response type for Task C6 (FST)

Response Type	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
Not attempted	32	26	22	0
Not clear	22	26	21	7
Diagram dependent	8	5	7	0
Additive/subtractive	9	7	4	0
Partially multiplicative	9	10	5	7
Fully Multiplicative	18	22	37	73
Advanced Multiplicative	1	3	3	13

The graph in Figure 5.21 shows that for Task C6 few students were successful using *Diagram Dependent*, *Additive* or *Partially Multiplicative* strategies. As the year levels increased then the percentage of students using both the *Fully Multiplicative* and the *Advanced Multiplicative* strategies also increased.

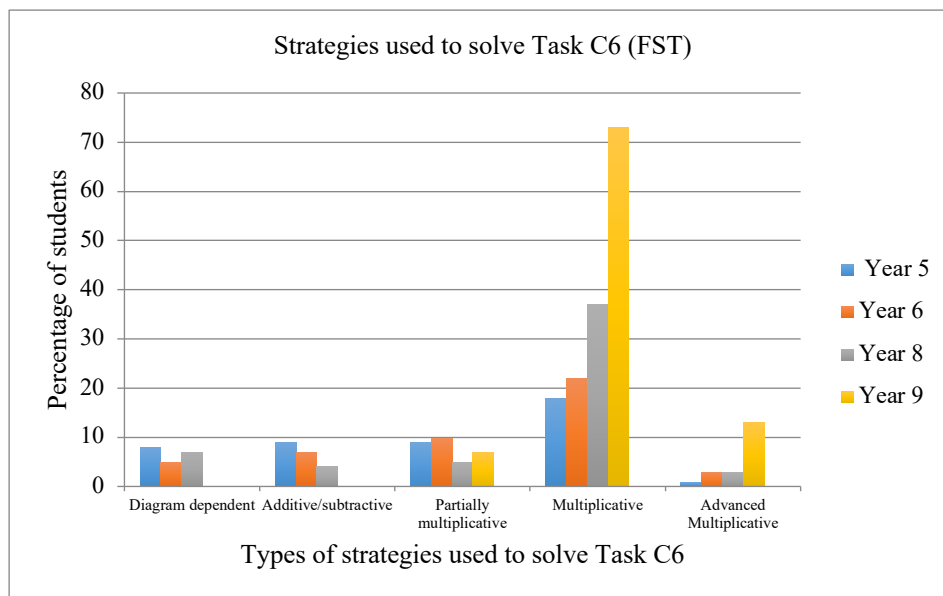


Figure 5-21: Comparison of strategies by year level for Task C6 (FST)

The students' responses for Task C7 were classified using the same criteria as that used for Task C5 and Task C6. Table 5.13 includes an explanation and example for Task C7 from students for each strategy.

Table 5-13: Classification of strategies used to solve Task C7 (FST)

Classification	Explanation	Example
Diagram dependent	Students use explicit partitioning of diagrams before using subtractive strategies.	7. This collection of 14 counters is $\frac{2}{6}$ of the number of counters I started with.  12 counters a. How many counters did I start with? b. Explain how you decided that your answer is correct. I found out that 14 is in the 2s times tables and so I split 14 into 2s and 7 tables so I knew that if I took one 2s away I would have 6 so I took it away and got twelve.
Additive/ Subtractive	Students use additive/subtractive methods without explicit partitioning of a diagram. Students find the number of objects needed to represent one-sixth and then use counting, adding or subtracting the amount representing one-sixth to find the number of objects needed to represent the whole.	$\frac{1}{6} = 2$ $\frac{2}{6} = 4$ $\frac{3}{6} = 6$ $\frac{4}{6} = 8$ $\frac{5}{6} = 10$ $1 = 12$ $\frac{7}{6} = 14$

Partially multiplicative Students use both multiplicative and subtractive methods. They calculate the unit fraction by dividing the quantity representing the unit fraction by seven (2) and then subtract this amount from the original quantity (14) to give the whole (12).

$$14 \div 7 = 2$$

$$14 - 2 = 12$$

Fully multiplicative Students use fully multiplicative methods. Students find the quantity represented by one-sixth by dividing by seven and then multiply this amount by six to find the whole.

a. How many counters did I start with? 12
b. Explain how you decided that your answer is correct.
Because I divided 14 by 7 and then timesed the answer by 6

Advanced multiplicative Students use more advanced multiplicative methods to solve the reverse fraction questions. These include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity by the known fraction.

a. How many counters did I start with? 12
b. Explain how you decided that your answer is correct.
I divided 14 by $\frac{1}{7}$

$$14 \times \frac{6}{7} = 12$$

In Table 5.14 the percentage of students for each of the strategies is given by year level. Nearly 60% of all students were unable to show their workings for Task C7. Thirty-two percent of all students made no attempt to respond and 25% gave responses that were *Not Clear*. Less than 10% of students at Year 5, Year 6 and Year 8 relied on a *Diagram Dependent* strategy while only two Year 5 students used the *Additive/Subtractive* strategy. Twenty percent of all students successfully used a *Partially Multiplicative* strategy with another 11% using a *Fully Multiplicative* strategy.

Table 5-14: Percentage of students by year level for each response type for Task C7 (FST))

Response Type	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
Not attempted	42	32	24	0
Not clear	29	28	16	7
Diagram dependent	8	9	7	0
Additive/subtractive	1	0	0	0
Partially multiplicative	14	20	32	7
Fully Multiplicative	5	8	18	73
Advanced Multiplicative	2	3	3	13

The graph in Figure 5.22 shows that for Task C7 few students were successful using *Diagram Dependent*, *Additive/subtractive* or *Partially Multiplicative* strategies. As the year levels increased then the percentage of students using both the *Fully Multiplicative* and the *Advanced*

Multiplicative strategies also increased. For this task, more students used the *Partial Multiplicative* strategy rather than the *Multiplicative* strategy they used for the previous task (C6). That is, they calculated the number of objects representing one-sixth and then subtracted that from the number representing seven-sixths.

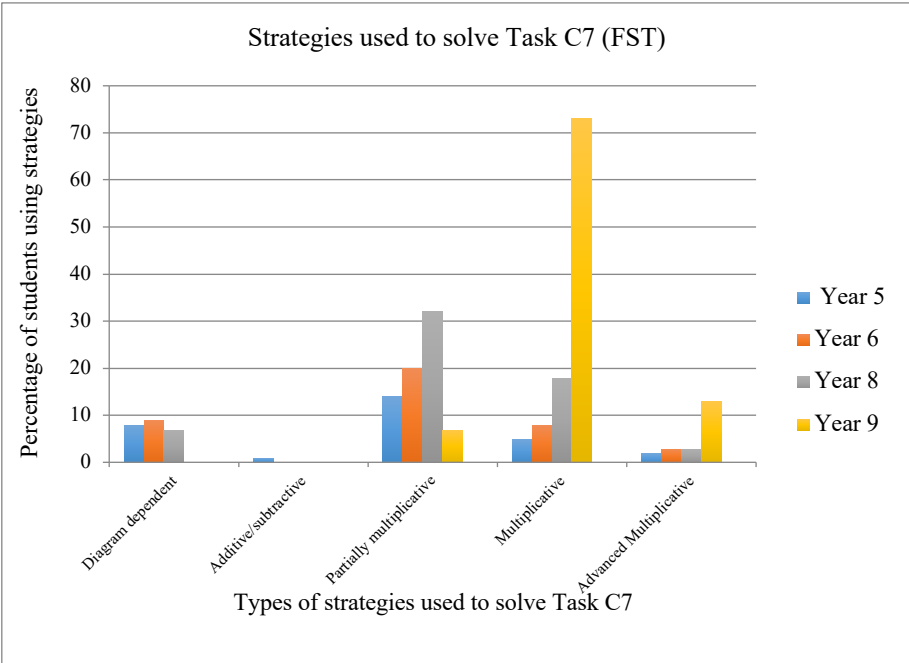


Figure 5-22: Comparison of strategies by year level for Task C7 (FST)

A further interrogation of the data has shown that students’ written responses for all three reverse fraction tasks (C5, C6 and C7) could be classified using the same strategies. (See Tables 5.10, 5.12 and 5.14). Each classification, with the elaborated but generic description, is given in Table 5.15 and is now labelled as the *Classification Framework for Reverse Fraction Tasks*. This classification framework will be used to analyse students’ responses for each question used in the Structured Interview.

Table 5-15: Classification Framework for Reverse Fraction Tasks

Classification	Explanation
Diagram dependent	Students use explicit partitioning of diagrams before using additive or subtractive strategies to find the measure or quantity representing the whole
Additive / subtractive	Students use additive or subtractive methods without explicit partitioning of a diagram. Students find the measure or quantity needed to represent the unit fraction and then use counting or repeated addition to find the measure or quantity needed to represent the whole.

Partially multiplicative	Students use both multiplicative and additive methods. Students calculate the measure or quantity representing the missing fractional part and either add this amount (proper fractions) to the original quantity or subtract this amount from the original measure or quantity (improper fractions).
Fully multiplicative	Students use fully multiplicative methods. Students find the measure or quantity represented by the unit fraction using division by the numerator of the given fraction and then multiply the measure or quantity representing the unit fraction by the denominator to find the measure or quantity representing the whole.
Advanced multiplicative	Students use more advanced multiplicative methods to solve the reverse fraction questions. These include the correct use of appropriate algebraic notation to find the whole, or a one-step method to find the whole by either dividing the given quantity by the known fraction.

Analysing the students' written responses to the reverse fraction tasks in this main study has confirmed the findings from the pilot studies that these tasks allow students to demonstrate their ability to use relational thinking, equivalence and algebraic reasoning. This data will be used to inform the overall research question and in particular for the research questions RQ3 and RQ4.

5.2 Relationship between the reverse fraction tasks and other FST tasks

The three, Part C reverse fraction tasks allow students to demonstrate their level of algebraic reasoning, and thus are important for responding to the research questions. In Section 5.2.1 the relationship between the total scores for these tasks (Tasks C5, C6 and C7) and the total score for the FST (Figure 5.11) will be investigated, before the comparison of the total score of the FST without the reverse fraction tasks (Figure 5.12). In Section 5.2.2 the total scores for the reverse fraction tasks (C5, C6 and C7) will be compared to the total score for each part of the FST. In Section 5.2.3 the relationship between the scores for individual tasks for Part A and the three reverse fraction tasks will be given. The data from this section will inform the main research question but, in particular, the data will be used to respond to research questions: RQ1, RQ2, RQ4 and RQ5.

In this section descriptive statistics using Pearson product-moment correlation coefficient are calculated using the Statistical Package for the Social Sciences (SPSS) and reported following the guidelines given by Pallant (2011). That is, small correlations (weak relationships) are values between 0.1 and 0.29, medium correlations (moderate relationships) are values between 0.3 and 0.49 while large correlations (strong relationships) are values between 0.5 and 1.0 (Pallant, 2011,

p.134). Correlations and explained variation are reported here as descriptive statistics and not to claim formal statistical inference.

5.2.1 Relationship between FST and reverse fraction tasks scores

The scores for the three reverse fraction tasks ranged from a minimum of zero to a maximum of nine with a mean of 3.50 and SD of 2.54. Total scores for the FST ranged from a minimum of zero to a maximum of 35, with a mean of 21.91 and SD of 7.14. Figure 5.23 shows the scatterplot comparing the scores for the three reverse fraction tasks (C5, C6 and C7) with the total scores for FST. The three reverse fraction tasks are a subset of the FST, and as expected, there is a strong positive correlation (0.84) between these two scores. The relationship is modelled in Figure 5.23 by a positive linear trend-line, indicating that nearly 70% of the variation, of the three reverse fraction tasks scores, may be explained by the total scores for the FST ($R^2=0.67$).

Figure 5.23 shows that students who scored less than 12 marks on the FST scored three or less out of a possible nine marks for the three reverse fraction tasks. Students who scored more than 30 on the FST scored more than six for the three reverse fraction tasks. However, as shown in Figure 5.11 some students who performed well overall on the FST were unable to complete any, or all, of the three reverse fraction tasks (C5 – C7). These students tended to use less sophisticated strategies such as drawings and diagrams or additive methods rather than multiplicative ones to solve these three reverse fraction tasks.

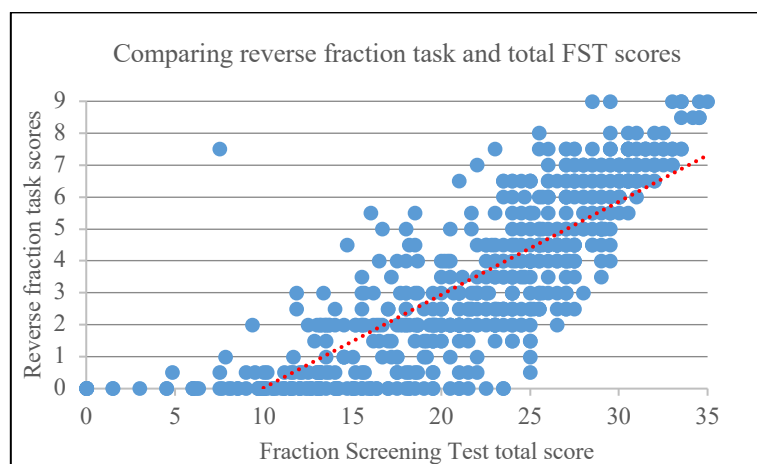


Figure 5-23: Comparison of the three reverse fraction task and FST scores (n = 607)

Figure 5.24 shows a scatterplot of the students' scores for the three reverse fraction tasks plotted against their FST scores for all tasks except the three reverse fraction tasks. That is, their scores have been added together for Part A, Part B and the first four Part C tasks, Tasks C1 – C4. Scores for the FST, without the three reverse fraction tasks, ranged from a minimum of zero to a

maximum of 26, with a mean of 18.4 and SD of 5.3. The relationship is modelled in Figure 5.24 by a positive linear trend-line, indicating that nearly 40% of the variation for the three reverse fraction scores, may be explained by the total scores for the remaining FST tasks ($R^2=0.39$).

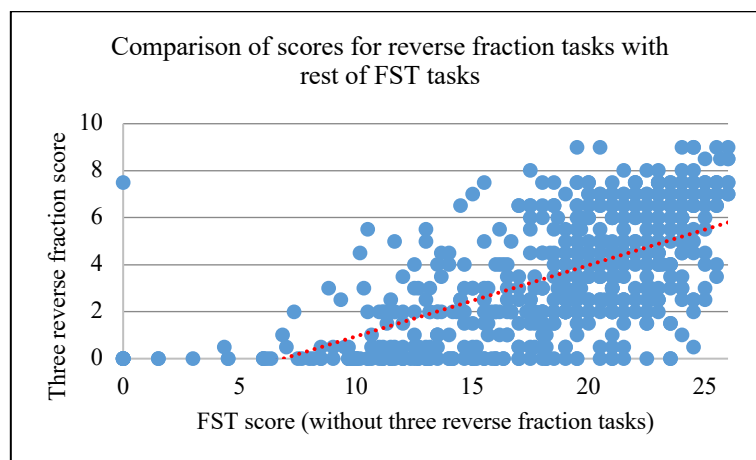


Figure 5-24: Comparison of the three reverse fraction task and total for all other FST tasks.

Figure 5.24 shows that the students who scored low marks on the FST also scored low marks for the reverse fraction tasks. Students who scored less than 10 on the FST scored less than three out of a possible nine marks on the reverse fraction tasks while students who scored between 10 and 14 on the FST scored less than six marks on the three reverse fraction tasks. However, students who scored between 14 and 26 on the FST scored the full range of marks between zero and nine for the reverse fraction tasks. All but one student who scored between six and nine marks for the three reverse fraction tasks all scored between 14 and 26 for the FST. The student who scored 7.5 for the three reverse fraction tasks did not attempt any of the other tasks on the FST.

There is a strong positive correlation (0.63) between the results for the FST (without the three reverse fraction tasks) and the three reverse fraction tasks. Students whose scores were above the mean for the FST scored the full range of scores between zero and nine for the reverse fraction tasks. However, students who scored more than six on the reverse fraction tasks scored more than 14 on the FST. The level of success on the reverse fraction tasks appears to be associated with the level of success demonstrated by the total scores for the FST.

5.2.2 Relationships between parts of FST and reverse fraction tasks

To determine whether the scores for any specific part of the FST was more strongly correlated to the results for the three reverse fraction tasks scatterplots correlations were calculated and scatterplots drawn. Table 5.16 shows the relationships between the total score for the three reverse fraction tasks and the total scores for Part A tasks, Part B tasks, and Part C tasks: C1 – C4.

Table 5-16: Relationship between Tasks C5 - C7 and Parts (FST)

	Part A	Part B	Part C (C1 – C4)
Tasks C5 - C7 (FST)	0.55	0.56	0.53

Scatterplots have been drawn to illustrate the relationships between these three parts of the FST. Figure 5.25 shows the relationship between the scores for the three reverse fraction tasks and the total scores for the Part A tasks, Figure 5.26 shows the relationship between the reverse fraction tasks and the total scores for the Part B tasks while Figure 5.27 shows the relationship between the three reverse tasks and the remaining Part C tasks (C1 – C4).

Figure 5.25 shows the relationship between students' scores for the three reverse fraction tasks and their total scores for Part A (FST) on a scatterplot. Scores for Part A ranged from a minimum of zero to a maximum of 12, with a mean of 9.63 and SD of 2.22. The relationship is modelled in Figure 5.25 by a positive linear trend-line, indicating that nearly 30% of the variation for the three reverse fraction scores, may be explained by the scores for Part A of the FST tasks ($R^2 = 0.29$).

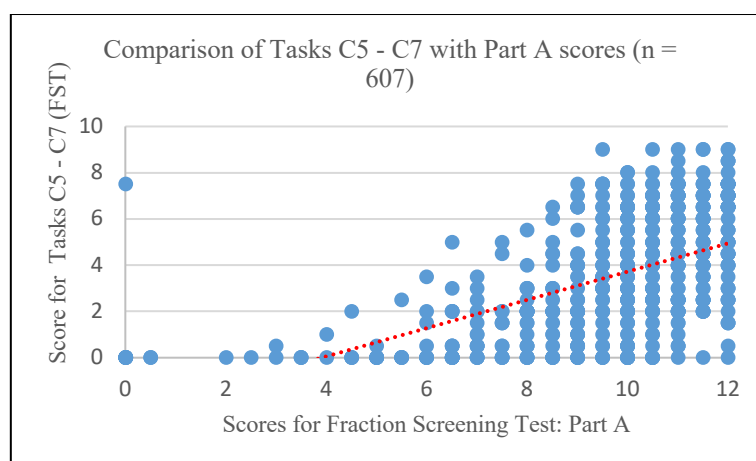


Figure 5-25: Comparison of scores for Tasks C5 – C7 and Part A scores (FST)

Part B includes number line tasks that require students to either find fractional parts of a given number line or the find the position for the number one when fractions were marked on a number line. Figure 5.26 shows a scatterplot of the students' scores for the three reverse fraction tasks (Figure 1) plotted against their Part B (FST) scores. The minimum score for the three reverse fraction tasks ranged from a minimum of zero to a maximum of nine with a mean of 3.50 and SD of 2.54. Scores for Part B ranged from a minimum of zero to a maximum of 10, with a mean of

6.21 and SD of 2.67. The relationship is modelled in Figure 5.26 by a positive linear trend-line, indicating that nearly one-third of the variation of the three reverse fraction scores (C5-C7) scores, may be explained by the scores for the FST: Part B tasks ($R^2 = 0.32$).

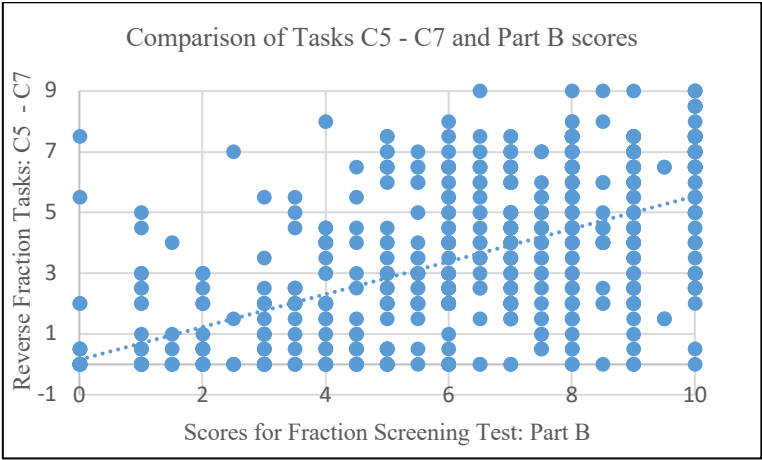


Figure 5-26: Comparison of scores for Tasks C5 – C7 and Part B scores (FST)

Figure 5.27 compares the scores from the first four tasks of Part C of the FST (C1 – C4) with the scores from the reverse fraction tasks (C5 – C7). The minimum score for the three reverse fraction tasks ranged from a minimum of zero to a maximum of nine with a mean of 3.50 and SD of 2.54. Scores for Part C tasks (C1 - C4) ranged from a minimum of zero to a maximum of four, with a mean of 2.58 and SD of 1.15. The relationship is modelled in Figure 5.27 by a positive linear trend-line, indicating that nearly 30% of the variation of the scores for the three reverse fraction tasks, may be explained by the scores for the first four Part C tasks ($R^2 = 0.28$).

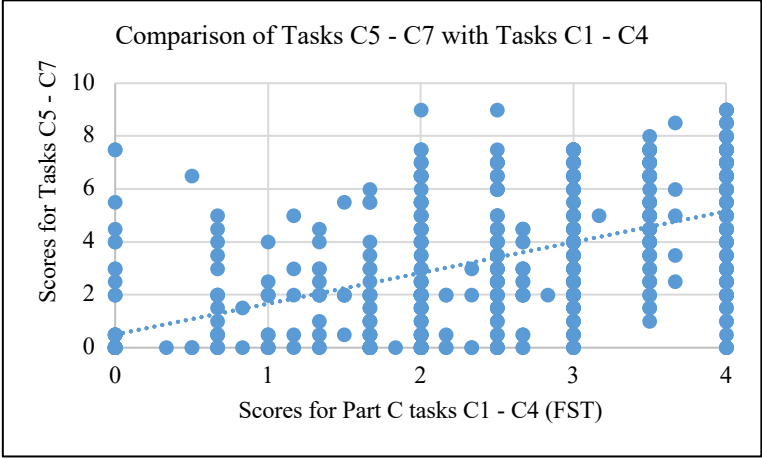


Figure 5-27: Comparison of scores from Tasks C1 – C4 and Tasks C5 - C7 (FST)

There is a similar strong positive correlation between the scores for part of the FST and scores for the three reverse fraction tasks. While the results for the relationships between the scores for Part A and the scores for the three reverse fractions (Figure 5.13) and the scores for Part B and the three reverse fraction tasks (Figure 5.14) are very similar, the graphs are noticeably different. The graph representing the relationship between the scores for Part B and the three reverse fraction tasks (Figure 5.14) is similar to the graph representing the relationship between the scores for Part C and the three reverse fraction tasks (Figure 5.15). These are both quite different to the graph representing the relationship between the scores for Part A and the three reverse fraction tasks in Figure 5.13.

5.2.3 Relationship between reverse fraction and individual tasks

To determine the links between performances on specific fraction tasks and success on the reverse fraction tasks the total scores for the three reverse fraction tasks were compared with the scores for each individual task of the FST. Table 5.17 shows the correlations between the total scores for the three reverse fraction tasks and the individual scores for individual Part A tasks, Table 5.18 the correlations between the three reverse fraction tasks total score and individual Part B task scores while Table 5.19 shows the correlations between the three reverse fraction tasks total score and the remaining Part C tasks (i.e. Tasks C1 – C4).

5.2.3.1 Relationship between Part A tasks and reverse fraction tasks

As shown in Table 5.17 the relationship between the reverse fraction tasks and the individual Part A tasks ranged from little correlation (0.09), a small correlation or weak relationship (0.14 - 0.26) and medium correlation or moderate relationship (0.33 – 0.39) with no results for the Part A tasks demonstrating a strong relationship (> 0.5) with the scores for the three reverse fraction tasks.

Table 5-17: Relationship between Tasks C5 – C7 and Part A tasks (FST)

Part A	A1	A2	A3	A4	A5	A6	A7	A8a	A8b	A9	A10	A11	A12
C5 – C7	0.15	0.21	0.09	0.14	0.26	0.33	0.24	0.14	0.15	0.36	0.25	0.39	0.38

Table 5.17 shows that there is a weak positive relationship for the total scores for the three reverse fraction tasks and eight of the Part A tasks (A1, A2, A4, A5, A7, A8a, A8b & A10). These tasks are shown in Tables 5.1, 5.2 and 5.3. These correlations ranged from 0.14 for finding the number of objects in the whole group if given the number in one-half of the group (A4 shown in Row 4 of Table 5.1), to 0.26 for shading a fractional part of a grid (A5 shown in Row 1 Table 5.2). Since

the correlation coefficient is below 0.1 Task A3 would not have even a weak relationship with the scores from the three reverse fraction tasks.

Four Part A tasks (A6, A9, A11 & A12), shown in Tables 5.2 and 5.3 have a moderate positive relationship with the three reverse fraction tasks. Task A6 has a correlation of 0.33 with the reverse fraction tasks while Task A9 has a correlation of 0.36 with the reverse fraction tasks. Task A12 (Row 4 of Table 5.3) has a correlation of 0.38 and Task A11 (Row 3, Table 5.3) has a correlation of 0.39 with the total score for the three reverse fraction tasks: Tasks C5, C6 and C7.

5.2.3.2 Relationship between Part B tasks and reverse fraction tasks

As shown in Table 5.18 the reverse fraction tasks and the individual Part B number line tasks had either a small correlation or weak relationship (0.15 - 0.27) or a medium correlation or moderate relationship (0.30 – 0.39) with no results for the Part B tasks showing a strong relationship (> 0.5) with the three reverse fraction tasks.

There is a weak positive relationship between the three reverse fraction tasks and four of the Part B tasks (B1a, B2a, B2b, B2c) ranging from a small correlation or weak relationship of 0.15 for the three reverse fractions and Task B2a (placing 75 on the number line marked zero to 25), 0.19 for Task B1a (placing numbers on the 4 marks between zero and 100), 0.25 for an explanation of the students' responses to Task B2a (i.e. B2b) and 0.27 for marking where the number 5 was on the number line marked 0 – 25 (B2c).

Table 5-18: Relationship between Tasks C5 – C7 scores and Part B scores (FST)

Part B Tasks	B1a	B1b	B2a	B2b	B2c	B2d	B3	B4	B5a	B5b
Tasks C5 – C7	0.19	0.41	0.15	0.25	0.27	0.41	0.30	0.37	0.37	0.44

Three of the six Part B tasks (B1b, B2d, B5b), which required students to explain their responses to number-line tasks involving whole numbers, had moderate positive relationships. Two tasks, B1b and B2d, which required students to: “Explain how you decided that your answer is correct” after they had completed the respective number line tasks, had the same correlation of 0.41. There was a similar medium correlation or moderate relationship, 0.44, for the three reverse fraction tasks and Task B5b when students were asked: “How did you decide the other two choices were incorrect?”

There were moderate positive relationships between the three reverse fraction task scores and the three number line questions involving fractions. These were 0.30 (Task B3) and 0.37 for Task B4

and Task B5a respectively. Task B3 required students to place three-fifths on a number line marked zero to two (Row 1 of Figure 5.6) while Task B4 required students to place the number 1 on a line marked zero to one-third (Row 2 of Figure 5.6). Task B5a (Row 3 of Figure 5.6) requires students to choose the position of number 1 from three alternatives when given the distance from zero to three-quarters. There was a moderate positive correlation (0.37) between the three fraction tasks and Task B5a.

5.2.3.3 Relationship between reverse fraction tasks and other Part C tasks

As shown in Table 5.19 there is a moderate positive correlation between the total scores for the three reverse fraction tasks (C5 – C7) and the scores for each of the other Part C tasks (C1 – C4). The relationships ranged from a correlation of 0.32 for the comparison of the reverse fraction scores and Task C4 (choosing the symbol which is not equivalent to 1.2), 0.33 for Task C3 (choosing the symbol that is not equivalent to one-quarter), 0.36 for Task C2 (matching equivalent fractions and decimals) and 0.39 for ordering fractions (Task C1).

Table 5-19: Relationship between Tasks C5 – C7 scores and Tasks C1 – C4 scores (FST)

Tasks C1 – C4	C1	C2	C3	C4
Tasks C5 – C7	0.39	0.36	0.33	0.32

In this section the focus has been on the comparison of the scores for reverse fraction tasks and their relationship the FST scores. This was followed by an investigation of the relationship between the scores for the three reverse fraction tasks and the individual parts of the FST. Finally, the relationship between the reverse fraction task scores and the individual task scores was also investigated. The data obtained from these three investigations will inform the overall research question and the research questions: RQ1, RQ2, RQ3 and RQ4 and RQ5.

5.3 Algebraic Thinking Questionnaire

The ATQ was the second paper and pencil assessment and is described in detail in Chapter 3. Students found this assessment much more difficult than the FST as evidenced by the number of non-responses for each task particularly those tasks requiring a detailed written response or explanation. As the lack of responses impacts on the research the percentages of non-responses and incorrect responses are included in the analysis of the ATQ tasks. The data from this section will address the overall research question and will inform the responses to research questions RQ3, RQ4, RQ5 and RQ6 to be discussed in Chapter 8.

The ATQ is divided into two parts. Part M focuses on multiplication while Part D focuses on division of whole numbers, fractions and decimals. Each part contains two questions divided into sub-questions. Tasks have not been specifically designed to test instant recall of multiplication facts but to encourage students to use their knowledge about the relationship between sets of numbers i.e. relational thinking.

For example, in Task M1aii (Table 5.21), students who successfully use relational thinking (Row 4) have recognised and understood that the equals sign means more than ‘give an answer’ but have also recognised and used both the sameness-relational and the substitutive-relational understanding of the equals sign (Jones et al., 2013). The sameness-relational understanding of the equals sign is demonstrated for this task when students recognise that both sides of the number sentence must balance or have the same product i.e. 36×25 must have the same value as 9 times whatever they write in the empty box. The substitutive-relational understanding of the equals sign is demonstrated when students recognise that:

- 9 is a quarter of 36 or that 36 divided by 4 is 9
- to maintain equality of the expressions then 25 must be multiplied by 4
- 25 multiplied by 4 is 100

However, if students do not recognise the relationship between the sets of numbers in the number sentence, they can still solve the tasks successfully using arithmetical calculations. Calculations usually take longer to complete than using relational thinking and there are more chances of calculation errors.

5.3.1 Algebraic Thinking Questionnaire Part M Question 1 results

Question 1 contains four sub-questions or tasks where students are expected to find the missing number that will make a correct number sentence and then to explain their working.

5.3.1.1 Task M1a

Task M1a, shown in the left-hand column of Table 5.20, focuses on the relationship between two sets of whole numbers. Row 1 of Table 5.20 shows that 18% Year 5, 11% Year 6 and 7% Year 8 students did not respond to this task while all Year 9 students attempted Task M1a. Row 2 shows the percentage of incorrect responses. Thirty-two percent of all students attempted the task but gave an incorrect response including 37% Year 5, 26% Year 6, 38% Year 8 and 13% Year 9 students.

Fifty-six percent of all students successfully completed Task M1a by writing 100 in the box on the right-hand side of the equation. As shown in Row 3 there was an increase in the percentage of the successful responses from 45% at Year 5 to 63% at Year 6. Fewer Year 8 students were successful than those at Year 6, with 55% at Year 8, while 87% Year 9 students correctly answered Task M1a using either arithmetical calculations or relational thinking. See Figure 5.28 for examples of an arithmetical solution (left-hand side) and a relational thinking response (right-hand side).

Table 5-20: Percentage of correct responses by year level for Task M1a (ATQ)

Task M1a (ATQ)		Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1ai. Write a number in the box to make a true statement. $36 \times 25 = 9 \times \square$	Not attempted	18	11	7	0
	Incorrect	37	26	38	13
	Correct	45	63	55	87

Figure 5.28 shows two correct responses for Task M1a. Student RK's correct arithmetical calculations (left-hand column) demonstrate his 'sameness-relational' understanding of the equals sign (Jones et al., 2013). He initially multiplied 36 by 25 successfully before dividing the answer by 9 to get the correct answer. Student MB's correct relational thinking response (right-hand column) also shows he understands the sameness-relational understanding of the equals sign. In order to achieve this equality, he uses the substitutive-relational meaning of the equals sign (Jones et al., 2013). He has recognised that 36 can be rewritten as 9×4 and that $25 \times 4 = 100$ and demonstrates this by arrows in his response.

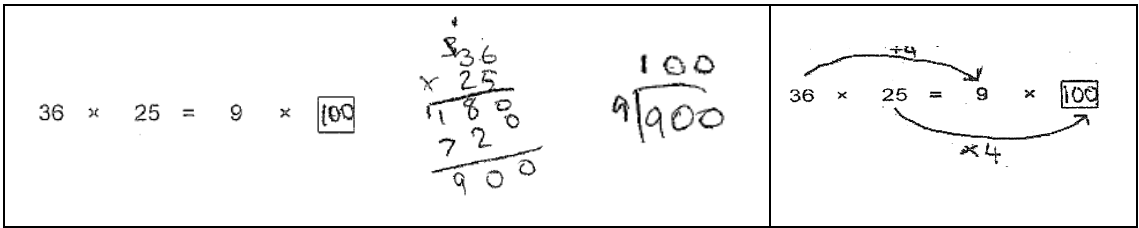


Figure 5-28: Examples of two students' correct responses to Task M1a (ATQ)

Table 5.21 shows the percentage of students by year level who did not give an appropriate response for "Explain your working briefly". Students' correct responses were categorised as either arithmetical or relational. While some students gave the correct response, many did not explain their working so these responses could not be categorised. While 56% of all students correctly wrote 100 in the empty box on the right-hand side of the equation only 45% explained

their solution method. Forty-one percent of all students used calculations to solve the task while 4% used relational thinking for the whole number multiplication task.

As shown in Row 1 (Table 5.21) 33% of all students did not respond when asked to: “explain your working briefly”. This included 41% Year 5, 33% Year 6, and 23% Year 8 students who did not provide an explanation or written calculation for this task. An additional 22% of all students attempted a response that was incorrect. This included 25% Year 5, 17% Year 6, 32% Year 8 and 13% Year 9 students (see Row 2).

Table 5-21: Percentage of responses by year level for explanation for Task M1a (ATQ)

Task M1a	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1aii. Write a number in the box to make a true statement. $36 \times 25 = 9 \times \square$ Explain your working briefly.	Not attempted	41	33	23	0
	Incorrect response	25	17	32	13
	Correct arithmetical calculation	33	48	37	40
	Correct relational thinking	2	2	8	47

The percentages of correct responses for the explanation for responses to Task M1a are given in Rows 3 and 4 of Table 5.21. Forty-five percent of all students gave a correct response with 41% of all students using an arithmetical calculation and 4% giving a response that could be described as demonstrating relational thinking. That is, 246 of the 273 students who responded correctly to Task M1ai used arithmetical calculations while 27 (slightly less than 10%) used relational thinking similar to that shown in the right-hand side of Figure 5.29.

Although 65% of all Year 5 students either did not attempt Task M1a, or gave an incorrect response, 35% gave a correct response with 33% completing arithmetical calculations and a further 2% successfully used relational thinking. Thus, 94% of the Year 5 students who successfully responded to this task, used arithmetical calculations and 6% used relational thinking similar to that shown in Figure 5.28.

Fifty percent of all Year 6 students either did not attempt the task or gave an incorrect response. However, 50% correctly demonstrated their solution method with 48% using arithmetical calculations and 2% used relational thinking. That is, 96% of the Year 6 students who successfully responded to this task, used arithmetical calculations and 4% used relational thinking.

The results for Years 5 and 6 are very similar for this task. Overall, 56% of the primary students either did not attempt this question or gave an incorrect response. Of the 44% of primary students

who successfully completed this task, 95% used arithmetical calculations and 5% used relational thinking. While 55% of all Year 8 students either did not attempt the task, or gave an incorrect response, 45% correctly demonstrated their solution method with 37% using arithmetical calculations while 8% used relational thinking. That is, 82% of the Year 8 students who successfully responded to this task, that is, 45 of the 55 Year 8 students, used arithmetical calculations and 10 students (18%) used relational thinking.

Two of the 15 Year 9 students (13%) did not attempt this question. While 87% of Year 9 students demonstrated their solution method more used relational thinking than arithmetical calculations. Forty percent of the Year 9 students used arithmetical calculations while 47% used relational thinking. That is, six of the 13 Year 9 students who responded to this task used arithmetical calculations and seven (54%) used relational thinking.

5.3.1.2 Task M1b

The left-hand side of the equation in Task M1bi, shown in Table 5.22 is a multiplication of a decimal by a two-digit number with the missing number placed directly after the equals sign and before the operation of multiplying by 10.

Row 1 of Table 5.22 shows the percentage of students who did not attempt Task M1b. Nineteen percent of all students did not attempt this question including 25% Year 5, 19% Year 6 and 11% Year 8 students while all Year 9 students attempted the task. Row 2 shows the percentage of incorrect responses. Thirty-five percent of all students attempted the task but gave an incorrect response including 42% Year 5, 33% Year 6, 34% Year 8 and 7% Year 9 students.

Forty-six percent of all students successfully completed Task M1b by writing 12 in the box on the right-hand side of the equation. As shown in Table 5.22 there was an increase in the percentage of the successful responses across the year levels with 34% Year 5, 48% Year 6, 55% Year 8 and 93% Year 9* students who correctly answered Task M1b using either arithmetical calculations or relational thinking.

Table 5-22: Percentage of responses by year level for Task M1b (ATQ)

Task M1b	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1bi. Write a number in the box to make a true statement. $48 \times 2.5 = \square \times 10$	Not attempted	25	19	11	0
	Incorrect	42	33	34	7
	Correct	34	48	55	93

Table 5.23 shows the percentage of students by year level who did not give an appropriate response to Task M1b for “Explain your working briefly”. The explanations were categorised as either arithmetical or relational. While some students gave the correct response, many did not explain their solution methods which meant these responses could not be categorised.

The left-hand side of Figure 5.29 shows Student RK’s arithmetical solution where she has performed calculations for both sides of the equation demonstrating her understanding of the sameness–relational meaning of the equal sign. Student MB’s use of arrows (right-hand side of Figure 5.30) shows the relationship between the expressions on both sides of the equals sign and demonstrates his understanding of the substitutive-relational meaning of the equals sign (Jones et al., 2013).

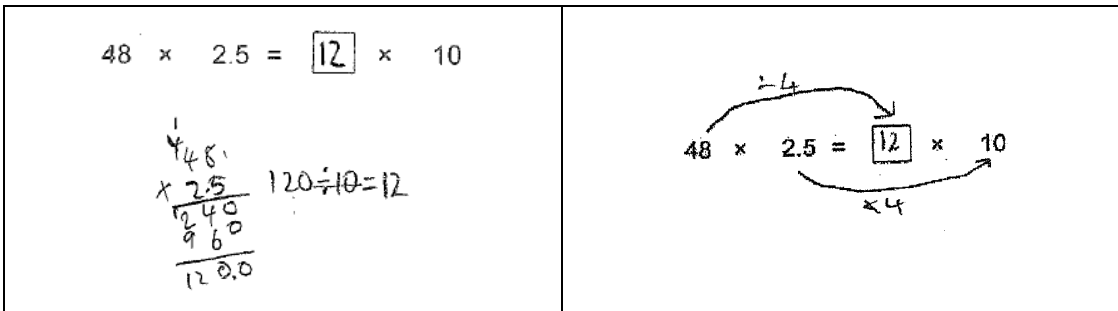


Figure 5-29: Examples of two students' responses to Task M1b (ATQ)

As shown in Row 1 of Table 5.23 52% Year 5, 43% Year 6, and 29% Year 8 students did not provide either a written calculation or explanation when asked to: “explain your working briefly” for Task M1b. While 42% did not provide an explanation or written calculation for Task M1b an additional 21% of all students gave an incorrect response. This includes 24% Year 5, 19% Year 6, 24% Year 8 and 7% Year 9 students (see Row 2).

Students’ explanations for their solutions for Task M1b have been classified as being either arithmetical or relational and these percentages are given in Rows 3 and 4 of Table 5.23.

Table 5-23: Percentage of responses by year level for explanation for Task M1b (ATQ)

Task M1b	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1bii. Write a number in the box to make a true statement. $48 \times 2.5 = \square \times 10$ Explain your working briefly.	Not attempted	52	43	29	0
	Incorrect response	24	19	24	7
	Correct arithmetical calculation	23	36	39	27
	Correct relational thinking	2	1	8	67

Thirty-seven percent of all students gave a correct response with 87% using arithmetical calculations and 13% giving a response demonstrating relational thinking similar to the solution shown in the right-hand side of Figure 5.30.

Twenty-five percent of all Year 5 and 37% of all Year 6 students stated their solution method for Task M1b. Of these students 93% of Year 5 and 96% of Year 6 students used arithmetical calculations while 7% of Year 5 and 4% of Year 6 used relational thinking similar to that shown in Figure 5.29. Forty-seven percent of all Year 8 students correctly demonstrated their solution method with 83% of these students using arithmetical calculations and 17% used relational thinking. While 87% of Year 9 students demonstrated their solution method, more used relational thinking (71%) than arithmetical calculations (29%).

5.3.1.3 Task M1c

Task M1c, shown in Table 5.24, is a reverse fraction question where students need to find the whole given that 18 represents two-thirds of that whole. This is a very similar question to Task C5 (FST) but is presented symbolically rather than as a worded problem with a diagram. Table 5.24 shows the percentage of students by year level who gave correct responses for the multiplication of a whole number by a fraction as given for Task M1c. It also records the percentage of students who either gave an incorrect response or no response in the box provided.

Table 5-24: Percentage of responses by year level for Task M1c (ATQ)

Task M1c	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1ci. Write a number in the box to make a true statement. $\frac{2}{3} \times \square = 18$	Not attempted	41	30	28	0
	Incorrect	46	50	43	13
	Correct	13	20	30	87

Row 1 of Table 5.24 shows that 41% Year 5, 30% Year 6 and 28% Year 8 students did not attempt Task M1c, but all Year 9 students attempted the task. This means that about one-third of all students did not attempt Task M1c. Row 2 shows that 46% Year 5, 50% Year 6, 43% Year 8 and 13% Year 9 students gave an incorrect response for Task M1c. That is, nearly one-half of all students attempted the task but gave an incorrect response. Some students gave the incorrect response of 12 which is two-thirds of 18. Another incorrect response was 24 which is one-third more than the original 18.

As shown in the left hand-side of Figure 5.30 Student RK used a trial and error approach with arithmetical calculations. She initially tried using 30, then tried 28 which she erased before trying the correct response of 27.

Figure 5-30: Examples of two students' correct responses to Task M1c (ATQ)

While Student RK (Figure 5.30) recognised that the expressions on both sides of the equals sign needed to be equivalent, she used the operational understanding of the equals sign by calculating the left-hand side to get the answer on the right-hand side of the equals sign. Student MB's correct response is shown in the right-hand side of Figure 5.30 but there is no indication of how he decided on this correct response.

As shown in Table 5.25 the percentage of the successful explanations and responses increased across the year levels with 13% Year 5, 20% Year 6, 30% Year 8 and 87% Year 9 students giving correct responses for Task M1c using either additive or multiplicative strategies. Overall about 20% of all students successfully completed this task by writing 27 in the box on the left-hand side of the equation given in Task M1c.

As shown in Row 1 of Table 5.25 76% Year 5, 75% Year 6, and 58% Year 8 students did not provide an explanation or written calculation for task M1c. This means 70% of students overall did not provide a response when asked to: “explain your working briefly” for the multiplication of a whole number by a fraction task.

Table 5-25: Percentage of responses by year level for explanation for Task M1c (ATQ)

Task M1c	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1cii. Write a number in the box to make a true statement. $\frac{2}{3} \times \square = 18$ Explain your working briefly.	Not attempted	76	75	58	0
	Incorrect	15	14	20	33
	Correct additive strategy	7	9	18	33
	Correct multiplicative strategy	2	2	3	33

As shown in Row 2 15% Year 5, 14% Year 6, 20% Year 8 and 33% Year 9 students attempted a response that was incorrect. This means an additional 16% of all students could not explain their solutions for Task M1c.

The percentages of correct responses for the explanation for Task M1c are given in Rows 3 and 4. Nine percent of all Year 5 students (16) gave a correct response with 13 using additive strategies and three using multiplicative strategies using either a one-step or a two-step method. The one-step method includes dividing 18 by the fraction two-thirds while the two-step method includes dividing by the numerator then multiplying by the denominator. Overall, 11% of Year 6 students correctly responded to Task M1c with 25 students using additive strategies and five using multiplicative strategies. Twenty-one percent of Year 8 students, 22 students using additive strategies and four students using multiplicative strategies to correctly solve Task M1c. Of the 10 Year 9 students who gave a correct response for Task M1c five used additive methods and five used multiplicative methods.

Overall, 14% of all students gave a correct response with 11% of all students using an additive strategy and 3% giving a response that could be described as multiplicative. That is, 65 students used additive strategies to respond correctly to Task M1c while 17 used multiplicative strategies.

5.3.1.4 Task M1d

Task M1d, shown in Table 5.26, assesses whether students understand that when a fraction is multiplied by its' reciprocal then the product is one e.g. $\frac{a}{b} \times \frac{b}{a} = 1$.

Row 1 of Table 5.26 shows that 44% Year 5, 26% Year 6 and 26% Year 8 students did not attempt Task M1d. Overall, more than 30% of all students did not attempt this question. Row 2 shows that 46% Year 5, 48% Year 6, 49% Year 8 and 20% Year 9 students gave an incorrect response. That is nearly one-half of the students who attempted the task gave an incorrect response. Many students wrote $\frac{3}{5}$ appearing to add three-fifths and two-fifths while others wrote $\frac{2}{5}$ which may indicate that they confused this multiplication task with division as two-fifths divided by two-fifths equals one.

In the left hand-side of Figure 5.32 is Student GD's incorrect response of three-fifths, in the centre is Student KH's incorrect response of two-fifths and in the right-hand side is Student MB's correct response to Task M1d. It is difficult to know how Student MB worked out his answer to this task as there are not enough details given in his solution.

$\frac{2}{5} \times \frac{\boxed{3}}{\boxed{8}} = 1$	$\frac{2}{5} \times \frac{\boxed{2}}{\boxed{5}} = 1$	$\frac{2}{5} \times \frac{\boxed{5}}{\boxed{2}} = 1$ $\frac{2}{5} \times \frac{5}{2} = 1$
--	--	---

Figure 5-31: Examples of three students' responses to Task M1d (ATQ)

As shown in Row 3 of Table 5.26 there was an increase in the percentage of successful responses from Year 5 to Year 6 but Year 6 and Year 8 had a similar percentage of correct responses with 10% Year 5, 26% Year 6, 25% Year 8 and 80% Year 9 students correctly answering Task M1d. Overall, 22% of all students successfully completed this task by writing $\frac{5}{2}$ in the box on the left-hand side of the equation given in Task M1d.

Table 5-26: Percentage of responses by year level for Task M1d (ATQ)

Task M1d	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1di. Write a number in the box to make a true statement $\frac{2}{5} \times \frac{\boxed{}}{\boxed{}} = 1$	Not attempted	44	26	26	0
	Incorrect	46	48	49	20
	Correct	10	26	25	80

Table 5.27 shows the percentage of students by year level who did not give an appropriate response for the second part of Task M1d for “Explain your working briefly” (Task M1d) as well as correct responses which were categorised as either arithmetical or relational based on the evidence in the working out space given for the task. While some students gave a correct response in the box provided many students did not explain their calculations, so these responses could not be categorised.

Many students struggled to explain their response to Task M2d. As shown in Row 1 of Table 5.27 when asked to: “explain your working briefly” 85% Year 5, 82% Year 6, 75% Year 8 and 7% of Year 9 students did not provide an explanation or written calculation for task M1d.

As shown in Row 2 many students attempted a response but were incorrect including 11% Year 5, 10% Year 6, 13% Year 8 and 27% Year 9 students. Thus while 80% of all students did not provide a response an additional 11% attempted a response that was incorrect so that more than 90% of students were unable to successfully explain their response to Task M1d.

Table 5-27: Percentage of responses by year level for explanation for Task M1d (ATQ)

Task M1d	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M1dii. Write a number in the box to make a true statement $\frac{2}{5} \times \frac{\boxed{}}{\boxed{}} = 1$	Not attempted	85	82	75	7
	Incorrect response	11	10	13	27
	Correct arithmetical calculation	3	6	6	33
Explain your working briefly.	Correct relational thinking	1	3	6	33

Only eight Year 5 students were able to give a written explanation or calculation for Task M1d with six students using arithmetical calculations and two students used relational thinking. Fifteen Year 6 students used arithmetical calculations and seven students used relational thinking while seven Year 8 students used arithmetical calculations and seven students used relational thinking. Five Year 9 students used arithmetical calculations and five used relational thinking. Overall, 34 students used arithmetical calculations and 21 used relational thinking to describe their solution to Task M1d.

5.3.1.5 Comparison of results by year for Question 1, Part M (ATQ)

The graph in Figure 5.32 compares the results across the year levels for Tasks M1a, M1b, M1c and M1d. The percentage of successful students increases with the year levels for Task M1b (multiplication of a whole number by a decimal) and Task M1c (a reverse fraction task). However, a larger percentage of Year 6 students than Year 8 students were successful with Task M1a (multiplication of two-digit whole numbers) while a similar percentage of Year 6 and Year 8 students successfully answered Task M1d (multiplication of two fractions).

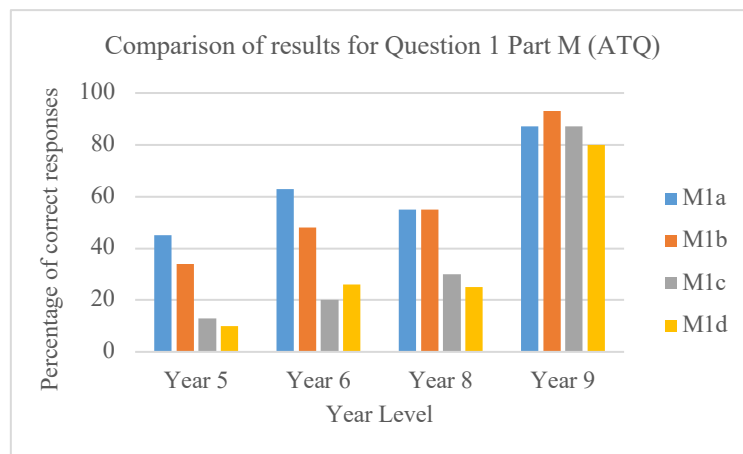


Figure 5-32: Comparison of results by year level for Question 1 Part M (ATQ)

5.3.2 Algebraic Thinking Questionnaire Part M Question 2 results

The tasks in Question 2 of Part M (ATQ) are more open-ended and allows students the opportunity to provide a range of numerical answers or state the generalised solution.

5.3.2.1 Task M2a

Task M2a asks students to provide two pairs of numbers for the equation $5 \times _ = 10 \times _$. Table 5.28 shows the percentage of students by year level who wrote either one or two pairs of correct responses, gave an incorrect response or did not respond. Student GD's two correct responses are in the left-hand side and Student MB's two correct responses for Task M2a are in the right-hand side of Figure 5.33. Note that Student MB has used arrows to show the relationship between the pairs of numbers thus highlighting his use of relational thinking. It is unclear how Student GD decided on his responses as there is no detail given.

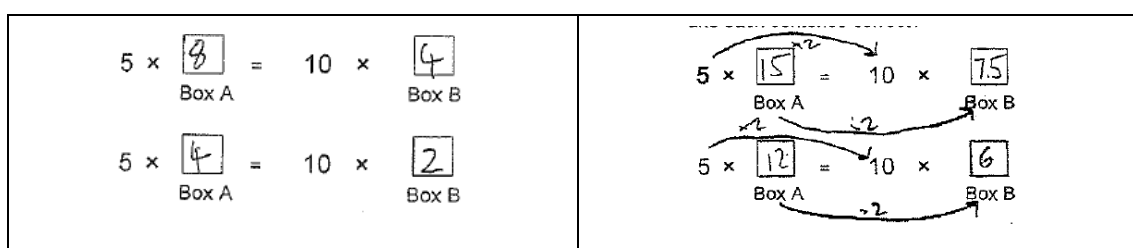


Figure 5-33: Examples of two students' correct responses to Task M2a (ATQ)

Row 1 of Table 5.28 shows the percentage of students who did not respond for each year level. While 19% Year 5, 9% Year 6 and 6% Year 8 students did not respond to this task this meant that overall 11% of all students did not respond. Row 2 shows that there were a similar percentage of students in Years 5, 6 and 8 who gave an incorrect response with 8% of all students giving an incorrect response for this task. All Year 9 students attempted Task M2a, but none gave an

incorrect response. In Task M2a, 65% of all students were able to give two correct pairs of numbers with an additional 16% giving only one correct pair of numbers.

Table 5-28: Percentage of responses by year level for Task M2a (ATQ).

Task M2a	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M2a. Put numbers in Box A and Box B to make each sentence correct. $5 \times \square = 10 \times \square$ Box A Box B	Not attempted	19	9	6	0
	Incorrect	8	9	8	0
	One correct response	19	16	13	7
	Two correct responses	54	67	73	93

As shown in Row 4 there were 54% Year 5, 67% Year 6, 73% Year 8 and 93% of Year 9 students who correctly wrote two appropriate pairs of numbers for the equation $5 \times _ = 10 \times _$. Another 19% Year 5, 10% Year 6, 13% Year 8 and 7% Year 9 students correctly wrote one pair of numbers. That is, 64% of all students were able to give two correct pairs of numbers and another 16% of all students gave one pair of correct numbers.

5.3.2.2 Task M2b

Task M2b requires students to explain the relationship between the missing numbers for the equation $5 \times _ = 10 \times _$ (left-hand side of Table 5.29). While 39% of all students were able to clearly explain the relationship between the numbers in the two boxes i.e. Box A is two times Box B or Box B is one half of Box A, 24% of all students did not attempt a response for this task with a further 37% giving incorrect responses.

Table 5-29: Percentage of responses by year level for Task M2b (ATQ)

Task M2b	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M2b. When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?	Not attempted	32	25	9	0
	Incorrect	36	39	36	27
	Correct	32	36	55	73

Row 1 of Table 5.29 shows that 32% Year 5, 25% Year 6 and 9% of Year 8 students did not respond to this task. Row 2 shows that 36% Year 5, 39% Year 6, 36% Year 8 and 27% of Year 9 students attempted to solve the task but gave incorrect responses. As shown in Row 3 32% Year 5, 36% Year 6, 55% Year 8 and 73% Year 9 students were able to explain the relationship between

the two numbers. Altogether 39% of all students explained the relationship between the two numbers for Task M2b. Student GD's correct response is shown in Figure 5.35.

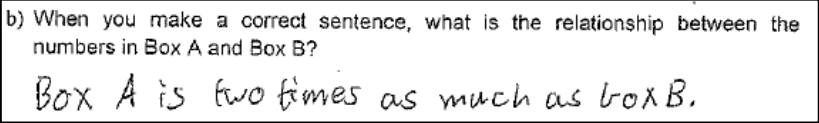


Figure 5-34: Student GD's correct response to Task M2b (ATQ)

5.3.2.3 Task M2c

Task M2c requires students to agree or disagree with the following statement: “If you put any number in Box A in the equation $5 \times _ = 10 \times _$ can you still make a correct sentence?” (left-hand column of Table 5.30)

Row 1 of Table 5.30 shows that 40% Year 5 students, 25% Year 6 and 15% Year 8 students did not respond. Row 2 shows that 35% Year 5, 38% Year 6, 30% Year 8 and one Year 9 student disagreed with the statement. That is, approximately one-third of all students (34%) disagreed with the statement while 25% Year 5, 38% Year 6, 55% Year 8 and 93% Year 9 students agreed with the statement (Row 3) showing that they could move beyond using specific numbers to generalising. While 39% of all students agreed with the statement, 27% did not attempt a response and a further 34% of all students disagreed with the statement.

Table 5-30: Percentage of responses by year level for Task M2c (ATQ)

Task M2c	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M2ci. If you put any number in Box A, can you still make a correct sentence?	Not attempted	40	25	15	0
	Incorrect	35	38	30	7
	Correct	25	38	55	93

The second part of Task M2c, shown in Table 5.31, requires students to explain their reasons for agreeing or disagreeing with the following statement: “If you put any number in Box A in the equation $5 \times _ = 10 \times _$ can you still make a correct sentence?” Only 12% of all students gave a correct mathematical generalisation having agreed with the statement. See for example, JM's correct response shown in the right-hand column of Figure 5.35. However, 8% of the students who agreed with the statement gave statements that focused on achieving a whole number answer e.g. must use even numbers. While 27% of all students did not attempt to explain their response a further 48% of all students who disagreed with the statement justified their response as if they

expected to end up with a whole number. See for example, Student GD's response in the left-hand side of Figure 5.35 where he was concerned about getting a whole number response.

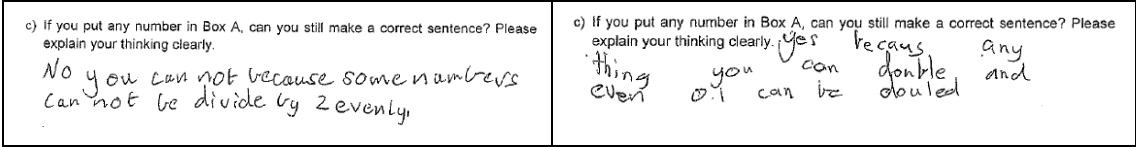


Figure 5-35: Examples of two students' responses to Task M2c (ATQ)

Row 1 of Table 5.31 shows the percentage of students who did not respond to Task M2cii for each year level. There were 40% Year 5 students, 25% Year 6 and 16% Year 8 students who did not give an explanation for their choice of yes or no. Row 2 shows that 28% of all students gave an explanation related to their 'no' response with 41% of those students whose explanation showed that they believed that any numbers could be used as long as the result was a whole number. Row 3 shows that 26% of all students responded with a correct mathematical generalisation expressed in words that stated that any number can be used in Box A as long as what is written in Box B retains the same relationship as that recorded for Task M2ci. That is 13% of all Year 5 students, 23% of Year 6, 43% of all Year 8 students and 73% of all Year 9 students were able to explain their thinking for Task M2c.

Table 5-31: Percentage of responses by year level for explanation of Task M2c (ATQ)

Task M2c	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M2cii. If you put any number in Box A, can you still make a correct sentence. Please explain your thinking clearly.	Not attempted	40	25	16	0
	Related to 'no' response	47	52	41	27
	Correct mathematical generalisation	13	23	43	73

5.3.2.4 Task M2d

Task M2d (Table 5.32) uses letters to denote unknowns on either side of the whole number multiplication equation rather than boxes as given in Task M2c (Table 5.30). Row 1 of Table 5.32 shows that 43% Year 5, 34% Year 6 and 22% Year 8 students who did not write anything when asked to explain or describe the relationship between the two unknowns, *c* and *d*. Overall 33% of all students did not respond to this task.

Row 2 (Table 5.32) shows that 26% Year 5, 29% Year 6, 37% Year 8 and 40% Year 9 students either wrote the inverse relationship for the two unknowns or their explanation was not quite

correct e.g. c is 7 more than d . Overall 30% of all students gave an incorrect relationship between c and d .

Table 5-32: Percentage of responses by year level for Task M2d (ATQ)

Task M2d	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M2d. What can you say about c and d in this mathematical sentence? $c \times 2 = d \times 14$	Not attempted	43	34	22	0
	Incorrect	26	29	37	40
	One correct response	22	28	23	7
	Correct relationship	9	10	18	53

Row 3 (Table 5.32) shows 22% Year 5, 28% Year 6, 23% Year 8 and one Year 9 student gave specific values for the letters c and d given in Task M2d. This is 24% of the whole group. Row 4 shows that 9% Year 5, 10% Year 6, 18% Year 8 and eight Year 9 (53%) students recognised and described the relationship between the two letters in the equation focused on multiplication by whole numbers. Thus, 13% of all students could express in words the relationship between the letters c and d as given in Task M2d. In Figure 5.36 Student JM has given a generalised response that focuses on the relationship between the two letters.

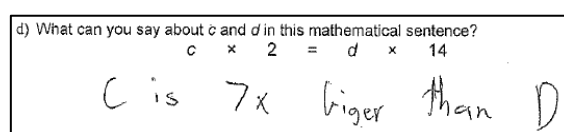


Figure 5-36: Student JM's response for Task M2d (ATQ)

5.3.2.5 Task M2e

Task M2e (left-hand column of Table 5.33) uses letters to denote unknowns on either side of the fraction multiplication equation rather than the boxes used in earlier tasks.

Row 1 of Table 5.33 shows that 66% Year 5 students, 55% Year 6 and 43% Year 8 students did not write anything when asked to explain or describe the relationship between the two unknowns, a and b in the fraction multiplication task. Overall, 54% of students did not respond to Task M2e, which is more than in the previous task (M2d) where 33% did not attempt to answer the task. Row 2 shows that 22% Year 5, 24% Year 6, 27% Year 8 and 53% Year 9 students gave incorrect responses for Task M2e. Overall, 25% of all students either wrote the inverse relationship for the two unknowns e.g. $b = 2a$ or their explanation was not quite correct e.g. a is 2 more than b or b is smaller than a .

Table 5-33: Percentage of responses by year level for Task M2e (ATQ)

Task M2e	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
M2e. What can you say about a and b in this mathematical sentence? $a \times \frac{3}{4} = b \times 1\frac{1}{2}$	Not attempted	66	55	43	0
	Incorrect	22	24	27	53
	One pair correct	7	11	7	0
	Correct relationship given	6	10	23	47

Row 3 of Table 5.33 shows that 7% Year 5, 11% Year 6, and 7% of Year 8 students responded with one pair of correct values for a and b which is 9% of all students. Row 4 shows that 6% Year 5, 10% Year 6, 23% Year 8 and 47% Year 9 students recognised and described the relationship between the two letters in the equation focused on multiplication by fractions. Thus 13% of all students could express in words the relationship between the letters a and b as given in Task M2e.

Student RK's response in the left-hand side of Figure 5.37 shows her calculations which enabled her to give specific values for the letters a and b while Student MB's generalised response in the right-hand side states that the letters can represent any number as long as the relationship is maintained.

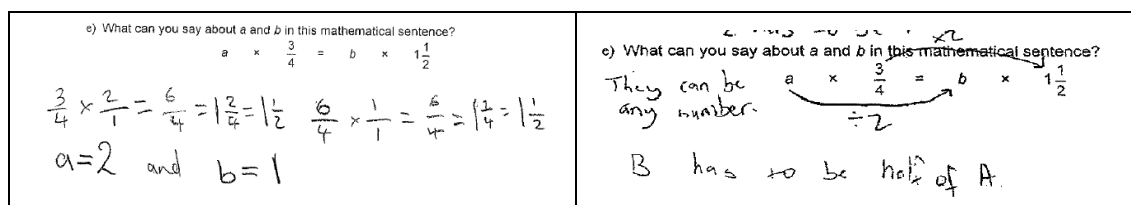


Figure 5-37: Examples of students' responses to Task M2e (ATQ)

5.3.2.6 Comparison of results by year for Question 2, Part M (ATQ)

The graph in Figure 5.38 compares the results across the year levels for the five tasks in Question 2 (ATQ). The percentage of correct responses for these Question 2 tasks increase across the year levels.

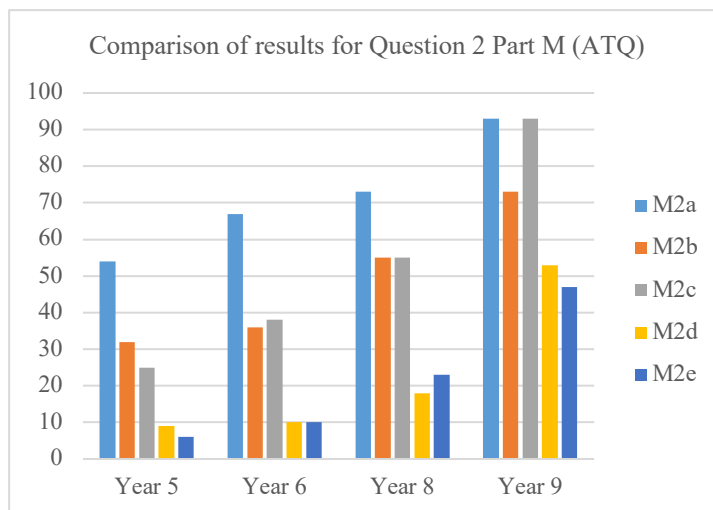


Figure 5-38: Comparison of results by year level for Question 2 tasks, Part M (ATQ)

The number of blank spaces, question marks and comments such as: “I don’t know” for Part M tasks (ATQ) were unexpected. A review of the curriculum has shown that these tasks were appropriate for students at these levels. Students at Year 5 are expected to multiply larger numbers by one or two-digit numbers (Task M1a), students at Year 6 are expected to multiply decimals by whole numbers (Task M1b) while students at Year 7 are expected to multiply fractions (Tasks M1c and M1d). Students at Year 5 are expected to find unknown quantities in multiplication sentences as given in Task M2a. While students in primary schools have not been introduced to pronumerals Year 8 students should have been able to respond to all tasks in Question 2 of Part M (ATQ).

5.3.3 Algebraic Thinking Questionnaire Part D Question 1 results

In Question 1 of Part D (ATQ) students are asked to “write a number in the box to make a true statement”. Working space is given so that students could: “Explain your working briefly”. Tasks D1a and D1b focus on division of whole numbers while Tasks D1c and D1d include division by fractions.

5.3.3.1 Task D1a

Table 5.34 shows the percentage of students by year level who wrote correct responses and the percentage of students who either gave an incorrect response or no response for Task D1a.

Row 1 of Table 5.34 shows that 41% Year 5, 29% Year 6 and 16% Year 8 students did not attempt Task D1a which means that nearly 30% of all students did not attempt this task. Row 2 shows that 39% Year 5, 36% Year 6 and 16% Year 8 gave incorrect responses. Thirty-two percent of all

students attempted the task but gave an incorrect response. This means that nearly 60% of all students were unable to give an appropriate response for Task D1a. As shown in Row 3 there was an increase in the percentage of successful responses from 20% at Year 5, 35% at Year 6, 69% at Year 8, while 100% Year 9 students correctly answered Task D1a using either arithmetical calculations or relational thinking as indicated by the use of arrows. Nearly 40% of all students successfully completed Task D1a by writing 20 in the box on the right-hand side of the equation i.e. $3 \div 4 = 15 \div 20$.

Table 5-34: Percentage of responses by year level for Task D1a (ATQ)

Task D1a	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1ai. Write a number in the box to make a true statement. $3 \div 4 = 15 \div \square$	Not attempted	41	29	16	0
	Incorrect	39	36	16	0
	Correct	20	35	69	100

As shown in Table 5.35 the percentage of non-responses decreased over the year levels with 84% Year 5, 73% Year 6, 48% Year 8 and 20% Year 9 students not able to provide an explanation for their response to the previous task (D1a). That is, nearly 70% of all students did not provide adequate working or an explanation for Task D1a. However, 7% of students from Years 5, 6 and 8 attempted the task but gave an incorrect response. Three percent of all students successfully solved the task using an arithmetical calculation including 4% of primary students (Years 5 and 6) and 2% of Year 8 students. Five percent of Year 5 students, 16% Year 6, 43% Year 8 and 80% of Year 9 students used relational thinking to explain the solution for Task D1a.

Table 5-35: Percentage of responses by year level for explanation for Task D1a (ATQ)

Task D1a	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1aii. Write a number in the box to make a true statement. $3 \div 4 = 15 \div \square$ Explain your working briefly	Not attempted	84	73	48	20
	Incorrect response	7	7	7	0
	Correct arithmetical calculation	4	4	2	0
	Correct relational thinking	5	16	43	80

While 39% of all students correctly wrote 20 in the empty box on the right-hand side of the equation only 30% explained their solution method. Three percent of all students used calculations to solve the task while 19% used relational thinking for the whole number division task with some rewriting the task as an equivalent fraction task. That is $\frac{3}{4} = \frac{15}{20}$.

In the left-hand side of Figure 5.39 is RK's correct response while in the right hand-side is MB's incorrect response. However, it is difficult to know whether Student RK used these calculations to determine the missing number or to check her solution. However, Student MB's written calculations highlight his faulty method as he attempts to perform relational thinking with division. He used the same rules for the relational thinking with division as he had used very successfully for the multiplication tasks. If he had checked the value of the expressions given either side of the equals sign, he would have realised that they were not equal as $3 \div 4 \neq 15 \div 0.8$. Each number on the left-hand side of the equals sign needed to be multiplied by five to calculate the numbers on the right-hand side to retain the relationship: $3 \div 4 = 15 \div 20$.

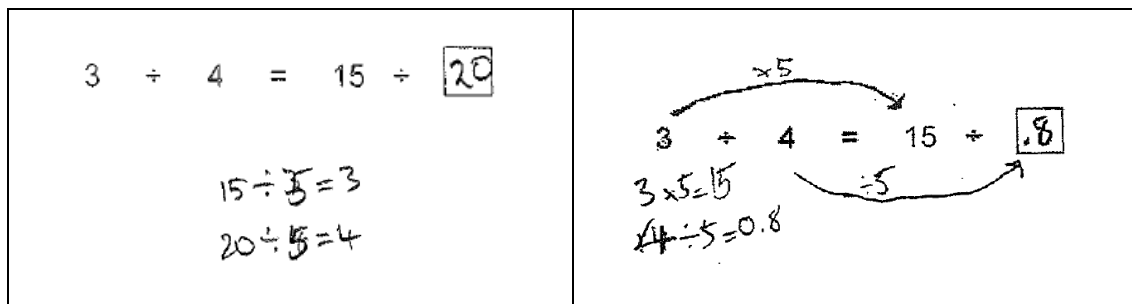


Figure 5-39: Examples of two students' responses to Task D1a (ATQ)

5.3.3.2 Task D1b

Thirty-five percent of all students successfully solved Task D1b that required them to write the missing number at the start of the following division task: $_ \div 15 = 16 \div 30$. Row 1 of Table 5.36 shows the percentage of all students who did not respond to the task. Row 2 shows the number of students who attempted the task but gave an incorrect response while Row 3 shows the percentage of students who gave a correct response by writing 8 in the empty box so writing the equation as: $8 \div 15 = 16 \div 30$.

As shown in Table 5.36 54% Year 5, 38% Year 6 and 19% Year 8 students did not attempt Task D1b. The percentage on non-responses decreased across the year levels. Many students attempted the Task D1b, but their responses were incorrect. This included 26% Year 5, 32% Year 6 and 20% Year 8 students. While 38% of all students did not attempt the task another 27% attempted the task but were incorrect thus about 65% of all students were not able to give a correct response to this task. Nineteen percent of Year 5 students, 30% Year 6, 61% Year 8 and 100% of Year 9

students correctly solved this task and wrote the number 8 in the empty box at the start of the equation. The percentages of correct responses increased across the year levels

Table 5-36: Percentage of responses by year level for Task D1b (ATQ)

Task D1b	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1bi. Write a number in the box to make a true statement. $\square \div 15 = 16 \div 30$	Not attempted	54	38	19	0
	Incorrect	26	32	20	0
	Correct	19	30	61	100

In Table 5.37 the results are shown for the written explanations for Task D1b for the statement: “Explain your working briefly”. Row 1 of Table 5.37 shows that the percentage of non-responses decreased over the year levels: 89% Year 5, 75% Year 6, 53% Year 8 and 20% Year 9 students.

Table 5-37: Percentage of responses by year level for explanation for Task D1b (ATQ)

Task D1b	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1bii. Write a number in the box to make a true statement. $\square \div 15 = 16 \div 30$ Explain your working briefly.	Not attempted	89	75	53	20
	Incorrect response	3	6	4	0
	Correct arithmetical calculation	3	1	2	0
	Correct relational thinking	5	18	41	80

Overall, 74% of students did not explain how they solved the previous task, D1b. Four percent of all students attempted a response but were incorrect. Twenty-two percent of all students gave a correct response with 2% using an arithmetical calculation and 20% using a response deemed to be relational. Arithmetical calculations include those where each side of the equation is calculated separately while relational is demonstrated where the relationship is demonstrated between numbers on each side of the equals sign. See for example Student RK’s solution shown in the left-hand side of Figure 5.40.

In the left-hand side of Figure 5.40 is RK’s correct response for Task D1b while in the right hand-side is MB’s incorrect response. However, it is difficult to know whether Student RK used these calculations to determine the missing number or to check her solution. Student MB’s written calculations highlight his faulty thinking. He has not maintained the equivalence on both sides of the equals sign: $32 \div 15$ is not equal to $16 \div 30$. He needed to divide 16 by 2 to find the missing number not divide the missing number by two to maintain the ‘sameness-relationship’ of both

sides (Jones et al., 2013) thus giving $8 \div 15$ which is equal to $16 \div 30$. Both students used the same methods for Task D1a (Figure 5.39) and Task D1b (Figure 5.40).

$\boxed{8} \div 15 = 16 \div 30$ $30 \div 2 = 15$ $16 \div 2 = 8$	$\boxed{32} \div 15 = 16 \div 30$ $\xrightarrow{\times 2}$ $\xrightarrow{\div 2}$ $15 \times 2 = 30$ $32 \div 2 = 16$
---	---

Figure 5-40: Examples of students' responses to Task D1b (ATQ)

The percentage of students using relational thinking increased across the year levels with 5% Year 5, 18% Year 6, 41% Year 8 and 80% Year 9 students using a strategy deemed to exhibit relational thinking usually indicated by the use of arrows demonstrating the relationship.

5.3.3.3 Task D1c

Task D1c focuses on the division of a whole number by a fraction: $18 \div \boxed{} = 6 \div \frac{2}{3}$. Table 5.38 shows the percentage of students by year level who wrote correct responses in the box and the percentage of students who either gave an incorrect response or no response in the box provided.

Fewer students were successful with this task with a total of 13% giving a correct response compared to 35% in the previous task. Only 10% of all students could explain their strategy. Row 1 of Table 5.38 shows the percentage of students who did not attempt the task, Row 2 shows the percentage of students who attempted the task but gave an incorrect response and Row 3 highlights the percentage of students who successfully solved the task by writing 2 in the empty box so that: $18 \div 2 = 6 \div \frac{2}{3}$.

The percentage of students not attempting the tasks lessened across the year levels with 60% Year 5, 47% Year 6, 34% Year 8 and 7% Year 9 students leaving the box blank. However, the percentage of incorrect responses increased from Year 5 to Year 8 with 33% Year 5, 42% Year 6 and 49% Year 8 students attempting to solve the task. The percentage of correct responses (Table 5.38) increased over the year levels with 7% year 5, 12% Year 6, 16% Year 8 and 73% Year 9 students giving a correct response using either an arithmetical calculation or using relational thinking.

Table 5-38: Percentage of responses by year level for Task D1c (ATQ)

Task D1c	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1ci. Write a number in the box to make a true statement.	Not attempted	60	47	34	7
$18 \div \square = 6 \div \frac{2}{3}$	Incorrect	33	42	49	20
	Correct	7	12	16	73

Overall 81% of all students did not explain their solution method for Task D1c. Row 1 in Table 5.39 shows that this percentage decreased across the year levels with 94% Year 5, 81% year 6, 68% Year 8 and 20% Year 9 students not attempting to describe their solution method. The percentage of students attempting to respond but their response was incorrect increased across the year levels from Year 5 to Year 8 including 3% Year 5, 10% Year 6 and 16% Year 8 who gave an incorrect response.

Table 5-39: Percentage of responses by year level for explanation for Task D1c (ATQ)

Task D1cii	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1c. Write a number in the box to make a true statement.	Not attempted	94	81	68	20
$18 \div \square = 6 \div \frac{2}{3}$	Incorrect response	3	10	16	13
	Correct arithmetical calculation	2	1	0	0
Explain your working briefly.	Correct relational thinking	2	7	16	67

As shown in Table 5.39 the percentage of students using relational thinking increased across the year levels with 2% at Year 5, 7% Year 6, 16% Year 8 and 67% Year 9 students correctly describing a strategy deemed to be relational thinking.

$$18 \div \boxed{2} = 6 \div \frac{2}{3}$$

$$\frac{6}{1} \div \frac{2}{3} = \frac{18}{2} = 9$$

$$\frac{6}{1} \times \frac{3}{2} = \frac{18}{2}$$

$$18 \div 9 = 2$$

$$18 \div \boxed{9} = 6 \div \frac{2}{3}$$

$\xrightarrow{\div 3}$
 $\xrightarrow{\times 3}$

Figure 5-41: Examples of students' responses to Task D1c (ATQ)

While 10% of all students correctly responded to this task only 1% used an arithmetical calculation while 9% demonstrated relational thinking in their responses. In the left-hand side of Figure 5.41 is RK’s correct response for Task D1c while in the right hand-side is MB’s incorrect response. Both students used the same methods for the first three tasks of Part D: Task D1a (Figure 5.39), Task D1b (Figure 5.40) and Task D1c (Figure 5.41).

5.3.3.4 Task D1d

Task D1d shown in the left-hand column of Table 5.40 tests students’ understanding that division of a number by the same number will give the answer one. Table 5.40 shows the percentage of students by year level who wrote correct responses in the box and the percentage of students who either gave an incorrect response or no response. The percentage of students who did not attempt the task decreased across the year levels, with 65% Year 5, 49% Year 6, 42% Year 8 and 7% Year 9 students leaving the fraction blank.

Table 5-40: Percentage of responses by year level for Task D1d (ATQ)

Task D1d	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1di. Write a number in the box to make a true statement $\frac{7}{6} \div \frac{\square}{\square} = 1$	Not attempted	65	49	42	7
	Incorrect	27	33	34	13
	Correct	8	18	25	80

While 51% of all students did not attempt this task, thirty percent of all students attempted this task but gave incorrect responses. Twenty-five percent of students who responded incorrectly gave the response of one-sixth while 19% responded with answer of six-sevenths. However, 18% of all students correctly answered this task with the percentage of correct responses increasing from 8% at Year 5, 18% at Year 6 to 25% at Year 8 and 80% at Year 9.

Many students did not explain their thinking for Task D1d. Eighty-nine percent of students did not explain their solution nor show any working out. The percentage of non-responses started at 98% for Year 5 and decreased to 88% Year 6, 81% at Year 8 and 33% at Year 9 as shown in Row 1 of Table 5.41. The results in Row 2 show that 3% of all students attempted the task but their responses were incorrect.

Table 5-41: Percentage of responses by year level for explanation for Task D1d (ATQ)

Task D1d (ATQ)	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D1dii. Write a number in the box to make a true statement $\frac{7}{6} \div \frac{\boxed{7}}{\boxed{6}} = 1$	Not attempted	98	88	81	33
	Incorrect response	0	4	8	7
	Correct calculation	1	3	0	0
Explain your working briefly	Correct relational thinking	1	5	11	60

A total of 8% of all students used either an arithmetical calculation (2%) or a strategy deemed to demonstrate relational thinking (6%). As shown in Figure 5.42 Student RK used trial and error to finally work out the answer to Task D1d. As this was a written response rather than an interview it is difficult to know why she chose the fractions she did.

$\frac{7}{6} \div \frac{\boxed{7}}{\boxed{6}} = 1$	$\frac{7}{6} \div \frac{1}{6} =$ $\frac{7}{6} \times \frac{6}{1} = \frac{42}{6} = 7$	$\frac{7}{6} \div \frac{1}{2} =$ $\frac{7}{6} \times \frac{2}{1} = \frac{14}{6}$	$\frac{7}{6} \div \frac{7}{6} = \frac{42}{42} = 1$ $\frac{7}{6} \times \frac{6}{7} = \frac{42}{42}$
--	---	---	--

Figure 5-42: Student RK's response for Task D1d (ATQ)

5.3.3.5 Comparison of the results for Part D, Question 1

As shown in the graph in Figure 5.43 students across all the year levels were more successful with the whole number division tasks than the fraction division tasks in Question 1.

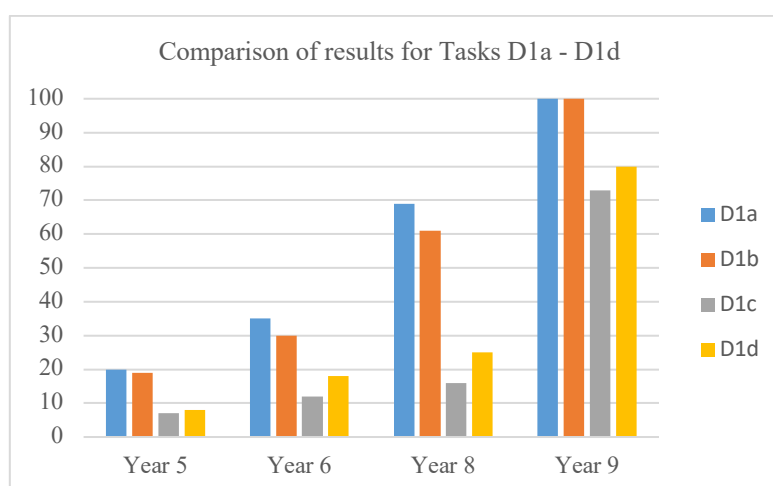


Figure 5-43: Comparison of results by year level for Part D Question 1 tasks (ATQ)

5.3.4 Algebraic Thinking Questionnaire Part D Question 2 results

The tasks in Question 2, Part D, are more open-ended division tasks than those in Question 1 where students can provide a range of numerical answers or state the generalised solution. Tasks D2b (Table 5.43), and D2c (Table 5.44 and Table 5.45) are all related to Task D2a that is shown in the left-hand column of Table 5.42. These division tasks have been designed to test students' ability to generalise and provide the data for RQ3, RQ4, RQ5 and RQ6.

5.3.4.1 Task D2a

Row 1 of Table 5.42 shows that 46% Year 5, 33% Year 6, and 43% Year 8 students did not attempt to respond to Task D2a. Row 2 shows that there were a similar percentage of students in Years 5 and 6 (31%) who gave an incorrect response, 19% of Year 8 students, and 28% of all students giving an incorrect response for this task. All Year 9 students attempted this task, but none gave an incorrect response. Nearly 70% of all students were unable to give a correct response for Task D2a with 38% not attempting a response and a further 28% giving incorrect responses. As shown in Row 3 13% Year 5, 25% Year 6, 32% Year 8 and all Year 9 students correctly wrote two appropriate pairs of numbers for the equation $3 \div \square = 15 \div \square$. An additional 11% of Year 5 and Year 6, and 7% of Year 8 students correctly gave one pair of numbers. While 24% of all students were able to give two correct pairs of numbers there were an additional 10% giving only one correct pair of numbers.

Table 5-42: Percentage of responses by year level for Task D2a (ATQ)

Task D2a	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D2a. Put numbers in Box A and Box B to make each sentence correct. $3 \div \square = 15 \div \square$ Box A Box B $3 \div \square = 15 \div \square$ Box A Box B	Not attempted	46	33	43	0
	Incorrect	31	31	19	0
	One pair correct	11	11	7	0
	Two pairs correct	13	25	32	100

Student RC in the left-hand side of Figure 5.44 has given two correct solutions for Task D2a while Student GD in the right-hand side has given one correct solution ($3 \div 1 = 15 \div 5$) and one incorrect response ($3 \div 4 = 15 \div 16$) for Task D2a.

$3 \div \boxed{4} = 15 \div \boxed{20}$ Box A Box B $3 \div \boxed{7} = 15 \div \boxed{35}$ Box A Box B	$3 \div \boxed{4} = 15 \div \boxed{16}$ Box A Box B $3 \div \boxed{1} = 15 \div \boxed{5}$ Box A Box B
--	---

Figure 5-44: Examples of two students' responses for Task D2a

In Figure 5.45 Student JM has self-corrected his initial responses as shown by the arrows given in the diagram. One of the disadvantages of not doing these tasks as interview questions is that it is not clear why Student MB chose the particular numbers that he wrote in the right-hand column of Figure 5.45.

make each sentence correct: $3 \div \boxed{10} = 15 \div \boxed{2}$ Box A Box B $3 \div \boxed{15} = 15 \div \boxed{3}$ Box A Box B	$3 \div \boxed{25} = 15 \div \boxed{5}$ Box A Box B $3 \div \boxed{100} = 15 \div \boxed{20}$ Box A Box B
--	--

Figure 5-45: Examples of two students' responses to Task D2a (ATQ)

5.3.4.2 Task D2b

Task D2b requires students to explain the relationship between the missing numbers for the equation $3 \div _ = 15 \div _$. Row 1 of Table 5.43 shows that 59% Year 5, 48% Year 6 and 36% of Year 8 students did not attempt Task D2b. Row 2 shows that 31% Year 5, 33% Year 6, 39% Year 8 attempted the task but gave an incorrect response. As shown in Row 3 9% Year 5, 19% Year 6, 25% Year 8 and 93% Year 9 students stated the correct relationship symbolically or verbally using multiplication e.g. Box B is five times Box A or division e.g. Box A is one-fifth of Box B or Box A equals Box B divided by five.

Table 5-43: Percentage of responses by year level for Task D2b (ATQ)

Task D2b	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?	Not attempted	59	48	36	0
	Incorrect	31	33	39	7
	Correct	9	29	25	93

In Figure 5.46 are responses from two students. Student JM correctly wrote the relationship between the two missing numbers as “B is $5 \times A$ ” which is assumed to mean that the number in Box B is five times the number in Box A. Student RC used a worded explanation in which the second part contradicts the initial statement which is correct. The initial part correctly describes the relationship: “Box A is Box B divided by five”. However, the second part states: “What you put in Box A is equivalent to five times that in Box B” which is the inverse of the first part of the statement.

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? box A is Box B divided by five, so what they gave put in box A is equivalent to 5x that in box B	B is $5 \times A$
---	-------------------------------------

Figure 5-46: Examples of two students' correct responses to Task D2b

In Figure 5.47 is MB's response for Task D2b that describes the relationship for one of his incorrect equations given for Task D2a given in Figure 5.45 ($3 \div 25 = 15 \div 5$) where he wrote 25 in Box A and 5 in Box B. This would not have been appropriate for his other equation where he wrote 10 in Box A and 20 in Box B.

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? Box B is $\frac{1}{5}$ of Box A.
--

Figure 5-47: MB's response to Task D2b (ATQ)

While 19% of all students were able to clearly explain the relationship between the numbers in the two boxes e.g. Box B is five times Box A or Box A is one-fifth of Box A, 48% of all students did not attempt a response for this task a further 33% of all students gave incorrect responses.

5.3.4.3 Task D2c

Task D2c requires students to agree or disagree with the following statement: “If you put any number in Box A in the equation $3 \div _ = 15 \div _$ can you still make a correct sentence?” Row 1 of Table 5.44 shows the percentage of students who did not respond to this task for each year level. There were 64% Year 5 students, 47% Year 6 and 38% Year 8 students who did not respond.

Row 2 of Table 5.44 shows that 23% Year 5, 26% Year 6, 19% Year 8 and one Year 9 student. As shown in Row 3, 14% Year 5, 26% Year 6, 43% Year 8 and all Year 9 students agreed with the statement. While 28% of all students agreed with the statement, 49% did not attempt a response and a further 23% of all students disagreed with the statement.

Table 5-44: Percentage of responses by year level for Task D2c (ATQ)

Task D2c	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D2ci. If you put any number in Box A, can you still make a correct sentence?	Not attempted	64	47	38	0
	Incorrect	23	26	19	0
	Correct	14	26	43	100

The second part of Task D2c requires students to explain their reasons for agreeing or disagreeing with the following statement: “If you put any number in Box A in the equation $3 \div _ = 15 \div _$ can you still make a correct sentence?” Only 14% of all students, gave a correct mathematical generalisation showing clearly that any number can be used in Box A while more than one-half of all students (54%) did not attempt a response. Figure 5.48 shows two examples of correct responses from Student RC (left-hand side) and Student JM in the right-hand side.

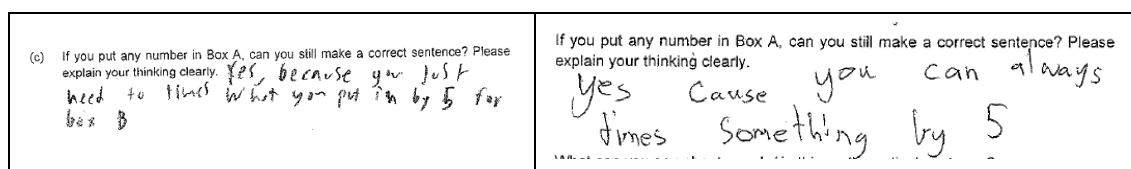


Figure 5-48: Examples of correct student responses for Task D2c (ATQ)

Row 1 of Table 5.45 shows that 70% Year 5 students, 52% Year 6 and 42% Year 8 students did not give an explanation for Task D2c. Row 2 shows that 26% Year 5, 36% Year 6, and 34% Year 8 students gave incorrect responses. Of the students who gave an incorrect response 45 stated that the answers needed to be whole numbers.

Row 3 shows that 4% Year 5, 12% Year 6 24% Year 8 and all Year 9 students with 14% of all students who successfully completed this task. These students responded with a correct mathematical generalisation expressed in words that stated that any number can be used in Box A as long as what was written in Box B retains the same relationship as in D2c.

Table 5-45: Percentage of responses by year level for explanation of Task D2c (n = 607)

Task D2c	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
Not attempted	70	52	42	0
Attempted response but incorrect	26	36	34	0
Correct	4	12	24	100

5.3.4.4 Task D2d

Task D2d (Table 5.46) uses letters to denote unknowns on either side of the whole number division equation rather than boxes as given in Task D2a (Table 5.42). Row 1 shows that 68% Year 5 students, 52% Year 6 and 39% Year 8 students did not write anything when asked to explain or describe the relationship between the two unknowns, c and d . Overall more than 50% of all students did not respond to this task.

Row 2 (Table 5.46) shows that 20% Year 5, 30% Year 6, 34% Year 8 and 7% Year 9 students attempted Task D2d. Row 3 shows that 5% Year 5, 10% Year 6, 5% Year 8 and one Year 9 student gave specific values for the letters c and d given in Task D2d. Row 4 shows that 7% Year 5, 8% Year 6, 22% Year 8 and 87% Year 9 students recognised and described the relationship between the two letters in the equation focused on division by whole numbers. Overall 13% of all students could express in words the relationship between the letters c and d as given in Task D2d, 7% of all students responded with specific and correct values c and d , while 27% of all students attempted this task but were incorrect.

Table 5-46: Percentage of responses by year level for explanation of Task D2d (FST)

Task D2d	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D2d. What can you say about c and d in this mathematical sentence? $c \div 8 = d \div 24$	Not attempted	68	52	39	0
	Incorrect	20	30	34	7
	Specific values for c and d	5	10	5	7
	Correct relationship	7	8	22	87

In Figure 5.49 Student RC gave specific values for the letters c and d (left-hand side) while Student JM expresses the relationship between the two letters (right-hand side).

(d) What can you say about c and d in this mathematical sentence? $c \div 8 = d \div 24$ $c = 3 \quad d = 9$	Dis $3 \times c$
---	------------------------------------

Figure 5-49: Examples of two students' responses for Task D2d (ATQ)

In Figure 5.50 Student MB has shown that the letters can represent “any number” but has used an incorrect relationship as shown on the arrows. Pronumeral d needs to be three times the value of c not one-third of the value of c as 24 is three times the value of 8. He has used the relationship between the numbers and written on the arrows as though it was multiplication and not division.

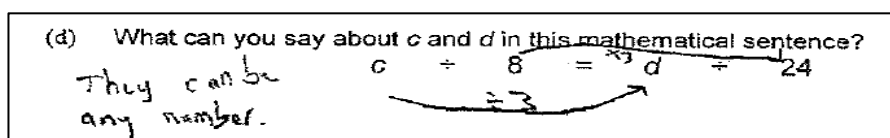


Figure 5-50: Student MB's use of incorrect strategy for D2d (ATQ)

5.3.4.5 Task D2e

Task D2e (left-hand column of Table 5.47) uses letters to denote unknowns on either side of the fraction division equation. Row 1 shows that fewer students responded to this task (D2e) than the previous task (D2d). Overall, 65% of students did not respond to Task D2e with 79% Year 5 students, 65% Year 6 and 48% Year 8 students who did not write anything when asked to explain or describe the relationship between the two unknowns, a and b in the fraction division task. Row 2 of Table 5.45 shows that 31% of all students attempted this task but their responses were incorrect. This included 17% Year 5, 31% Year 6, 51% Year 8 and 60% Year 9 students. Row 3 shows that 9% of all students responded with one set of correct values for a and b , which includes 2% Year 5, 2% Year 6, and 1% Year 8 and 13% Year 9 students.

Row 4 shows that 2% of all students could express in words the relationship between the letters a and b as given in Task D2e. This includes 3 Year 5 (2%), 4 Year 6 (1%), no Year 8 (0%) and 4 Year 9 (27%) students who recognised and described the relationship between the two letters in the equation focused on division by fractions.

Table 5-47: Percentage of responses by year level for explanation of Task D2e (ATQ)

Task D2e	Responses	Year 5 (n = 195)	Year 6 (n = 175)	Year 8 (n = 122)	Year 9* (n = 15)
D2e. What can you say about a and b in this mathematical sentence? $a \div \frac{2}{7} = b \div \frac{3}{7}$	Not attempted	79	65	48	0
	Incorrect	17	31	51	60
	Specific values given for a and b	2	2	1	13
	Correct relationship	2	1	0	27

Student MB's response in Figure 5.51 is an example of an incorrect additive response for Task D2e. This was a response given by students from both primary and Year 8 classes.

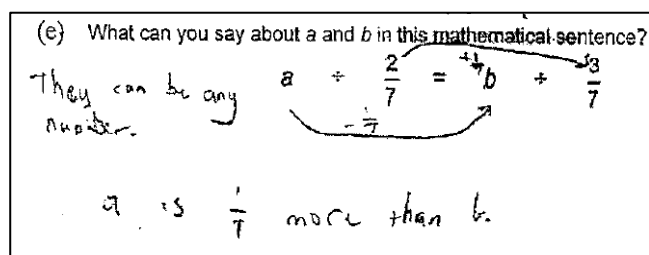


Figure 5-51: MB's incorrect response for Task D2e (ATQ)

In Figure 5.52 are three students' responses to Task D2e. Student BE (left-hand column) Student RC (middle column) and Student RK (right-hand column).

$a=1?$ $B=2?$	$a=1$ $b=1.5?$	$a=6$ $b=9$
---------------	----------------	-------------

Figure 5-52: Three students' responses to Task D2e (ATQ)

5.3.4.6 Comparison of Question 2 Part D Tasks (ATQ)

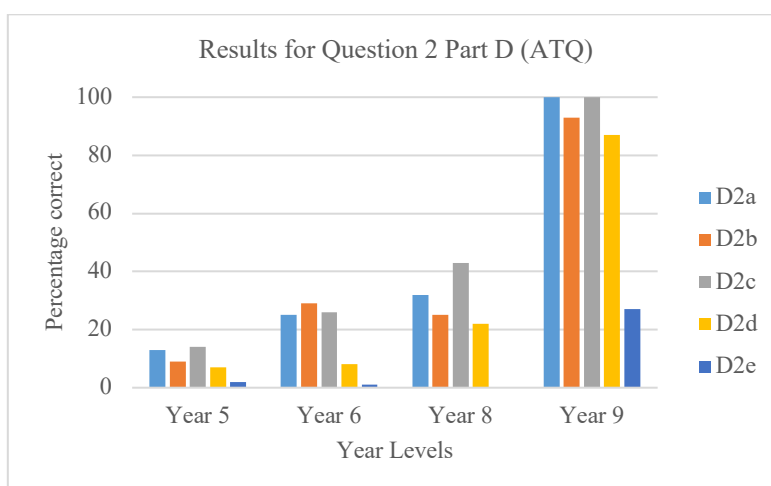


Figure 5-53: Comparison of results by year level for Question 2 Part D (ATQ)

As for Part M (ATQ) the number of non-responses, question marks and comments such as: "I don't know" or "My teacher didn't teach me this" were unexpected. As discussed earlier for the multiplication tasks these tasks were appropriate for these year levels.

5.3.5 Comparing results from Part M and Part D (ATQ)

In this section the two parts of the ATQ are compared. That is, Part M tasks with a focus on multiplication on whole numbers and fractions are compared to Part D tasks with a focus on division for both whole numbers and fractions.

The graphs in Figure 5.54 shows that more primary students were successful with the multiplication tasks than the division tasks while and more secondary students were successful with the division tasks than the multiplication tasks. Primary students were more successful with the multiplication of two-digit whole numbers (Task M1a) than multiplication of a decimal by a two-digit number (Task M1b) with similar results for the two tasks requiring division of a whole number by a whole number (Task D1a and D1b).

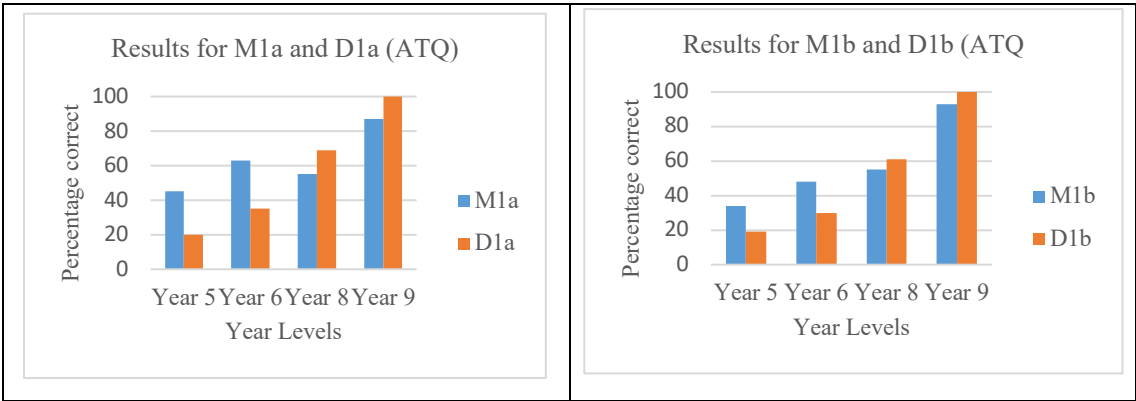


Figure 5-54: Comparison of Tasks M1a and D1a and Tasks M1b And D1b results (ATQ)

The graphs in Figure 5.55 shows that more primary students were successful with the multiplication of fraction tasks than the division of fraction tasks. The graph in the left-hand column shows that more secondary students were successful with the multiplication fraction task (Task M1c) than the division fraction task (Task D1c) but had similar results for the identity laws in Tasks M1d and D1d.

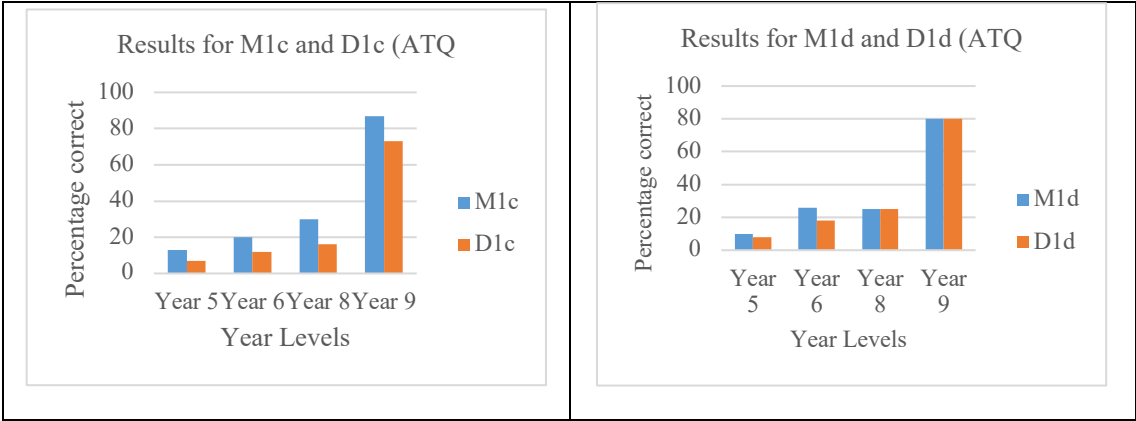


Figure 5-55: Comparison of Tasks M1c and D1c and Tasks M1d and D1d results (ATQ)

The graph in Figure 5.56 shows that more students from Years 5, 6 and 8 provided two responses for the multiplication task (Task M2a) than for the division task (Task D2a). These tasks tested whether students not only understood the sameness-relational understanding but the substitutive-relational understanding of the equals sign (Jones et al, 2013).

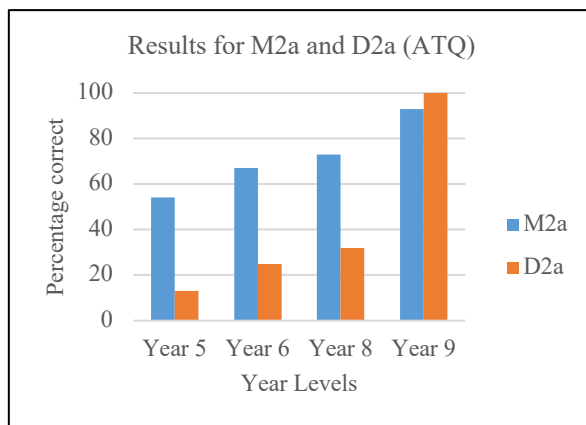


Figure 5-56: Comparison of Task M2a and Task D2a (ATQ) results

The graphs in Figure 5.57 compare the results for tasks that investigate the students' understanding about the relationship between the two expressions in Tasks M2b and D2b. The graph in the left-hand column of Figure 5.57 shows that more students in Years 5, 6 and 8 could explain the relationship between the two unknowns in the multiplication task (Task M2b) than the division task (Task D2b). A larger percentage of Year 6 students were successful with Task D2b than Year 8 students. More Year 9 students were successful with the division task rather than the multiplication task.

As shown in the right-hand column of Figure 5.57 more students in Years 5, 6 and 8 were to give an explanation for the multiplication task (Task M2c) than the division task (D2c) for the statement: "If you put any number in Box A can you still make a correct sentence". Students in Year 9 were more successful with the explanation for the division task than the multiplication task.

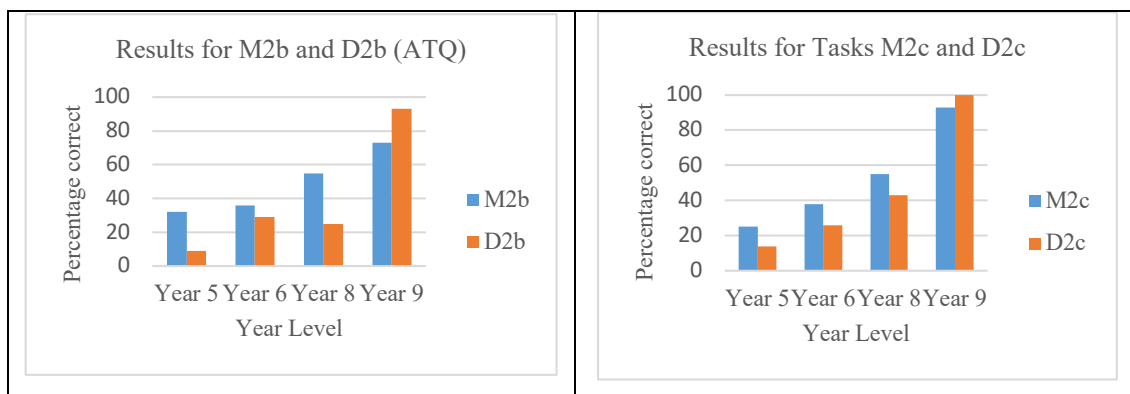


Figure 5-57: Comparison of Tasks M2b and D2b and Tasks M2c and D2c results (ATQ)

The graphs in Figure 5.58 compare the results for the only tasks where pronumerals, rather than boxes, are used to denote missing values. In Task M2d the pronumerals (c and d) are multiplied

by related whole numbers while in Task D2d the same pronumerals are divided by related whole numbers. In Task M2e the pronumerals (a and b) are multiplied by related fractions and in Task D2e the same pronumerals are divided by related by related fractions. Although the differences in the scores between M2d and D2d (left-hand side of Figure 5.58) more Year 6 students were successful with the multiplication task than the division task while for Year 5 and Year 8 students more were successful with the division task than the multiplication task.

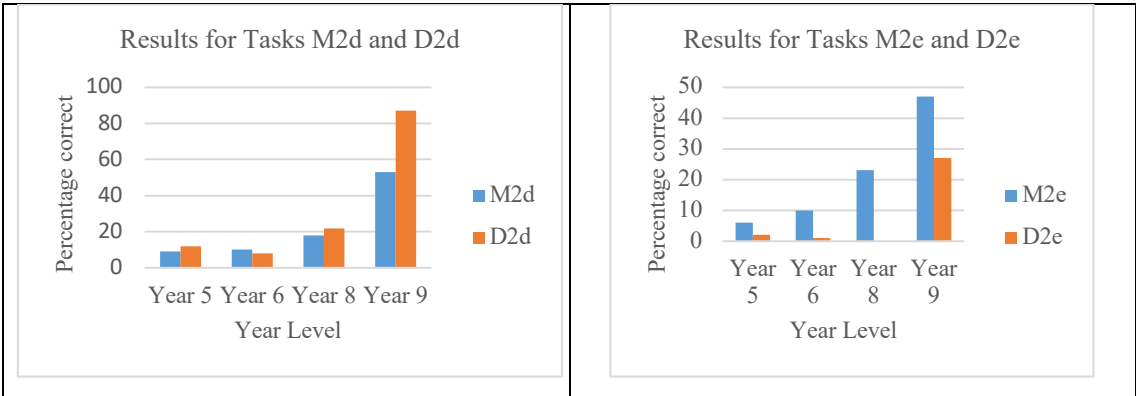


Figure 5-58: Comparison of Tasks M2d and D2d and Tasks M2e and D2e (ATQ)

For most of the ATQ tasks students found the division tasks more difficult than the multiplication tasks as shown in Tables 5.54 – 5.58. Overall there was a very poor response rate for the ATQ tasks making it difficult to make some clear and definitive comparisons between the two paper and pencil instruments.

5.4 Comparison of results for FST and ATQ

The detailed results for the paper and pencil assessment instruments are discussed in the previous sections of this chapter. The FST results are discussed in Section 5.1 and the ATQ results in Section 5.3. In this section the relationship between the total scores for both assessment instruments will be examined, followed by a discussion of the relationship between the scores for the individual parts of the FST and the ATQ. All correlations given in Section 5.4 are statistically significant at the .01 level.

5.4.1 Comparing the ATQ total scores and FST total scores

Scores for the ATQ ranged from a minimum of zero to a maximum of 26, with a mean of 6.86 and a standard deviation (SD) of 6.45. Scores for the FST ranged from a minimum of zero to a maximum of 35 with a mean of 21.91 (SD = 7.14).

Figure 5.59 shows the comparison of the total scores for both the ATQ and the FST. The relationship is modelled by a positive linear trend-line indicating that 45% of the variation in the ATQ total scores, may be explained by the FST total scores ($R^2=0.45$). Students who had low scores for the FST also achieved low scores on the ATQ and conversely students who achieved high scores on the FST achieved high scores on the ATQ.

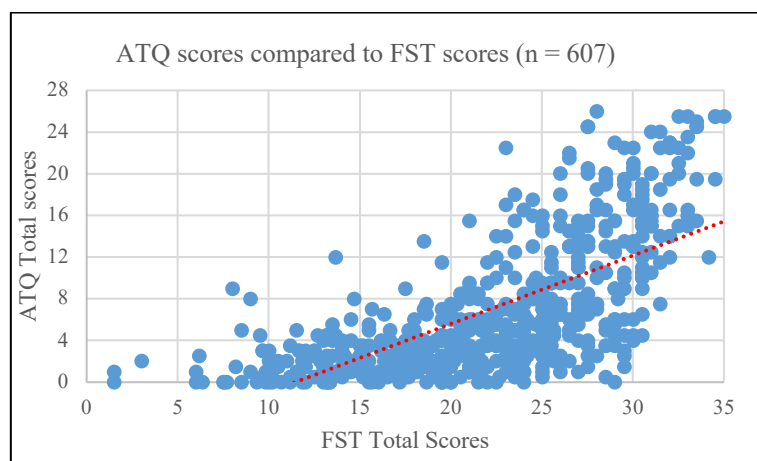


Figure 5-59: Comparison of total scores for ATQ and FST

To determine whether scores for individual parts of the FST had a greater impact on the total scores for the ATQ further scatterplots were drawn. In the next section scores for the two parts of the ATQ have been compared with the scores for the three individual parts of the FST.

5.4.2 Comparison of ATQ and Part A (FST) total scores

The routine fraction tasks in Part A (FST) include identifying shapes that represent the fraction three-quarters, finding fractional parts of groups and continuous areas, completing equivalent fractions and adding two related fractions. Scores for Part A ranged from a minimum of one to a maximum of 12, with a mean of 9.63 and SD of 2.22.

Figure 5.60 shows a scatterplot of the students' ATQ total scores plotted against their total scores for Part A (FST). The relationship is modelled by a positive linear trend-line indicating that nearly 25% of the variation in the ATQ scores may be explained by Part A (FST) scores ($R^2=0.25$). Students who had low scores for Part A (FST) also achieved low scores on the Algebraic Screening Test. Students who achieved high scores on the ATQ also achieved high scores on Part A (FST). However, many students who achieved high scores on Part A only achieved low scores on the ATQ. High scoring students for Part A (FST) tasks may have scored any of the full range of marks for the ATQ.

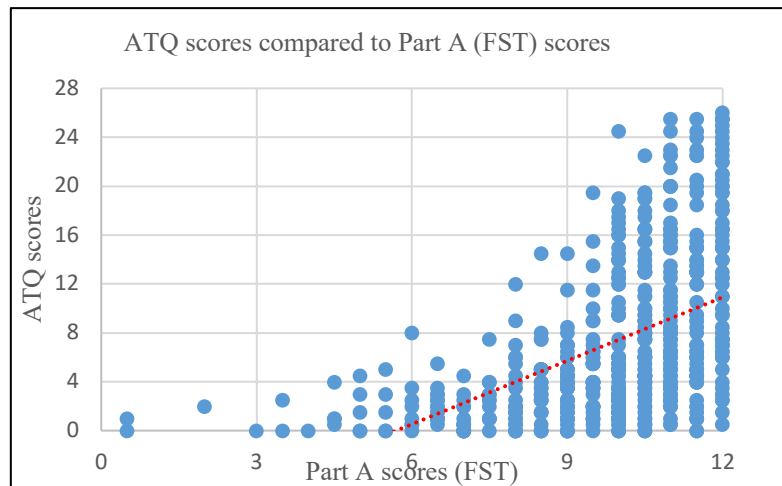


Figure 5-60: Comparison of ATQ and Part A (FST) total scores

Part M (ATQ) focused on multiplication with whole numbers, decimals and fractions. Scores for Part M ranged from a minimum of zero to a maximum of 14, with a mean of 4.20 and a SD of 3.34. Scores for Part A (FST) ranged from a minimum of one to a maximum of 12, with a mean of 9.63 and SD of 2.22. Figure 5.61 shows a scatterplot of the students' scores for Part M (ATQ) plotted against their Part A (FST) scores. The relationship is modelled by a positive linear trend-line indicating that approximately 21% of the variation of scores for Part M (ATQ) scores may be explained by Part A (FST) scores ($R^2=0.21$).

Students who scored low marks on Part A (FST) scored low marks on Part M (ATQ) however students who scored high marks on Part A could have scored any mark from the full range for Part M (ATQ).

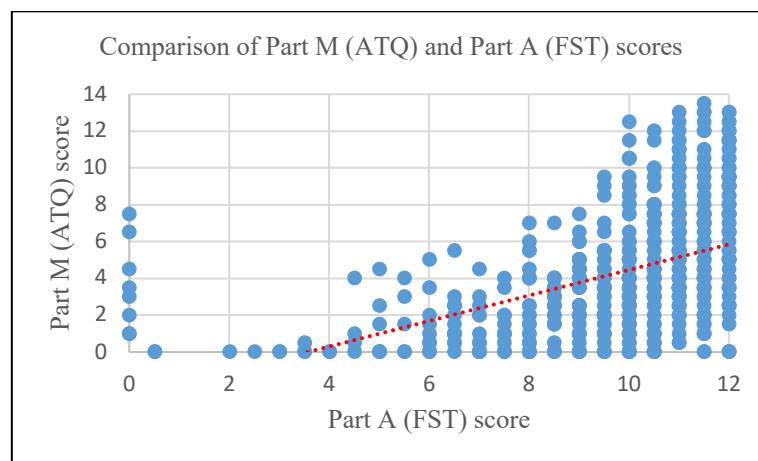


Figure 5-61: Comparison of Part M (ATQ) and Part A (FST) total scores

Part D (ATQ) focused on division with whole numbers and fractions. Scores for Part D ranged from a minimum of zero to a maximum of 13.5, with a mean of 2.66 and a SD of 3.54. Figure

5.62 shows a scatterplot of the students' scores for Part D (ATQ) plotted against their scores for Part A (FST). The relationship is modelled by a positive linear trend-line indicating that only about 14% of the variation of scores for Part D (ATQ) scores may be explained by Part A (FST) scores ($R^2=0.14$). Students with low scores for Part A (FST) scored low scores for Part D (ATQ). However, students who achieved a high score on Part A did not necessarily score highly on Part D.

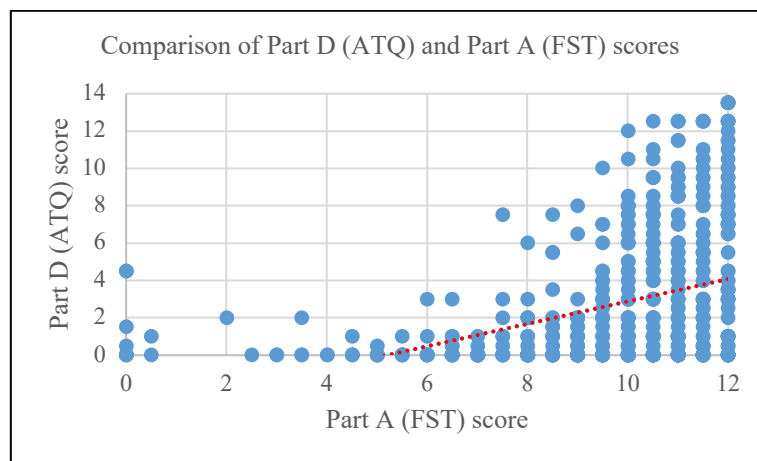


Figure 5-62: Comparison of total scores for Part D (ATQ) and Part A (FST)

5.4.3 Comparison of ATQ and Part B (FST) total scores

Part B (FST) includes number-line tasks where students are expected to find fractional parts of a number line as well as use the distance given to represent a fractional part to estimate where a specified whole number would be located on a given number line. Scores for Part B ranged from a minimum of zero to a maximum of ten, with a mean of 6.21 and SD of 2.67.

Figure 5.63 shows a scatterplot of the students' ATQ total scores plotted against their Part B (FST) scores. The relationship is modelled by a positive linear trend-line indicating that nearly 28% of the variation in the ATQ scores may be explained by Part B (FST) scores ($R^2=0.28$).

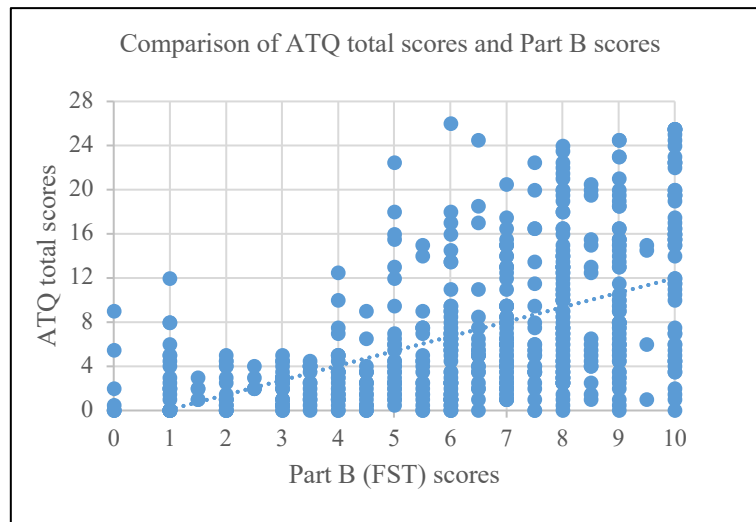


Figure 5-63: Comparison of scores for ATQ and Part B (FST).

Part M (ATQ) scores ranged from a minimum of zero to a maximum of 14, with a mean of 4.20 and a SD of 3.34. Scores for Part B (FST) ranged from a minimum of zero to a maximum of ten, with a mean of 6.21 and SD of 2.67. Figure 5.64 shows a scatterplot of the students' Part M (ATQ) scores plotted against their Part B (FST) scores. The relationship is modelled by a positive linear trend-line indicating that nearly 25% of the variation in Part M (ATQ) scores may be explained by Part B (FST) scores ($R^2=0.25$).

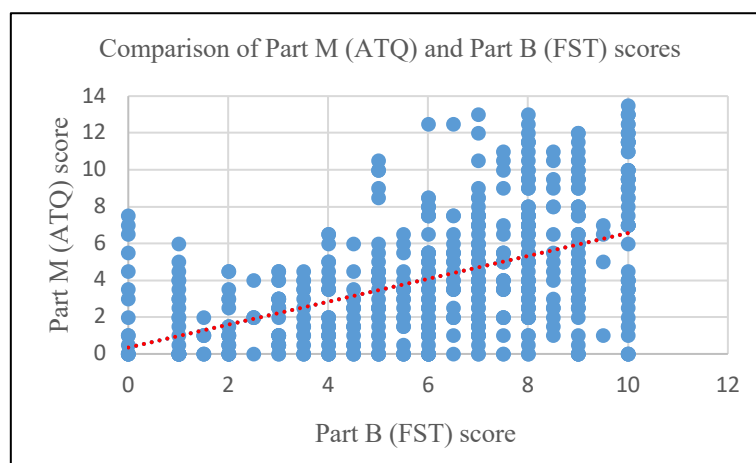


Figure 5-64: Comparison of Part M (ATQ) and Part B (FST) scores

Scores for Part D ranged from a minimum of zero to a maximum of 13.5, with a mean of 2.66 and a SD of 3.54. Figure 5.65 shows a scatterplot of the students' Part D (ATQ) scores plotted against their Part B (FST) scores. The relationship is modelled by a positive linear trend-line indicating that slightly more 21% of the variation in Part D (ATQ) scores may be explained by Part B (FST) scores ($R^2=0.21$).

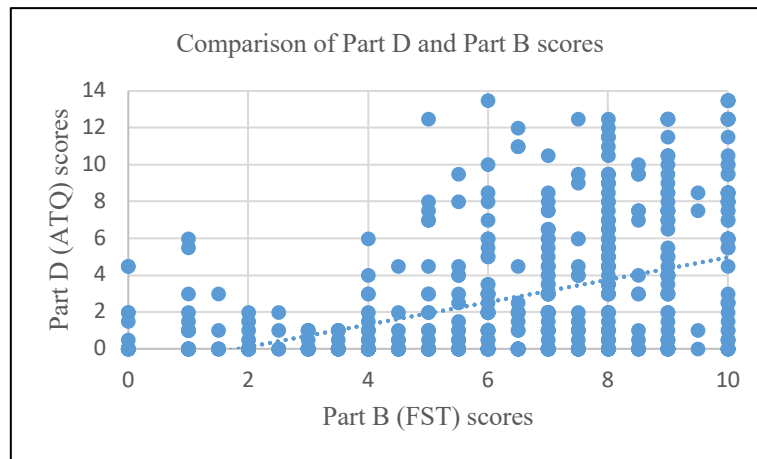


Figure 5-65: Comparison of total scores for Part D (ATQ) and Part B (FST)

5.4.4 Comparison of ATQ and Part C (FST) total scores

Part C (FST) includes tasks that test students' understanding of fractions and are presented in a less familiar format. Scores for Part C ranged from a minimum of zero to a maximum of 13, with a mean of 6.07 and SD of 3.29. Figure 5.66 shows a scatterplot of the students' ATQ total scores plotted against their scores for Part C (FST). The relationship is modelled by a positive linear trend-line indicating that nearly 46% of the variation in ATQ scores may be explained by Part C (FST) scores ($R^2 = 0.46$).

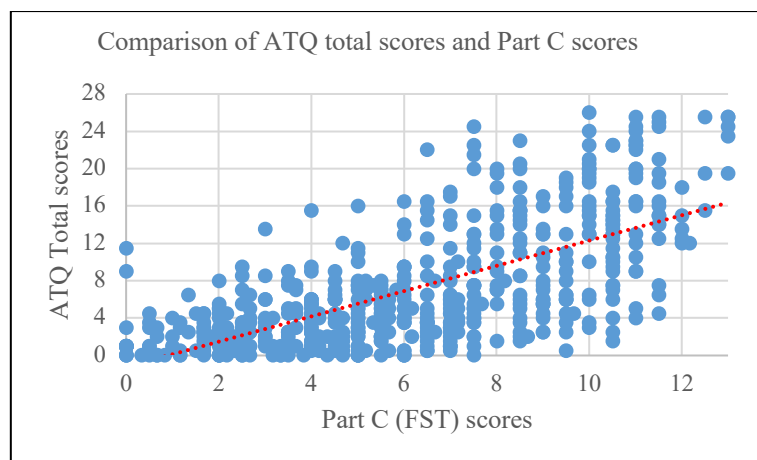


Figure 5-66: Comparison of ATQ and Part C (FST) scores

Figure 5.67 shows a scatterplot of the students' Part M (ATQ) scores plotted against their scores for Part C (FST). The relationship is modelled by a positive linear trend-line indicating that nearly 40% of the variation in Part M (ATQ) scores may be explained by Part C (FST) scores ($R^2 = 0.37$). Students who scored low marks on Part C (FST) scored low scores on Part M (ATQ) while students who scored high marks on Part C (FST) scored high marks on Part M (ATQ).

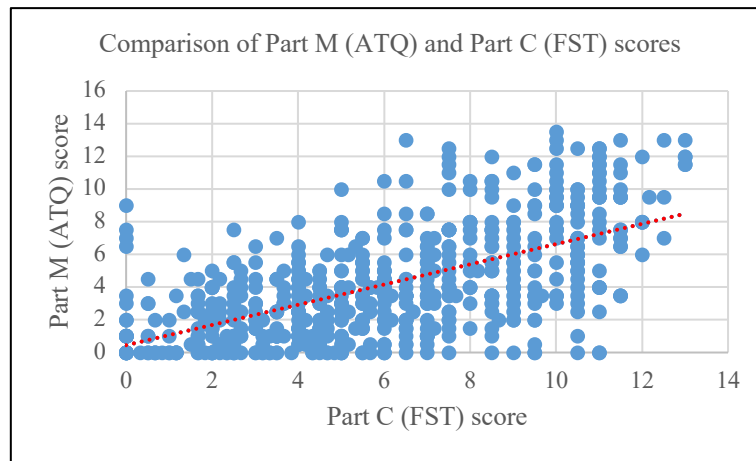


Figure 5-67: Comparison of Part M (ATQ) and Part C (FST) scores

Figure 5.68 shows a scatterplot of the students' Part D (ATQ) scores plotted against their scores for Part C (FST). Scores for Part D ranged from a minimum of zero and a maximum of 13.5, with a mean of 2.66 and a SD of 3.54. Scores for Part C ranged from a minimum of zero to a maximum of 13, with a mean of 6.07 and SD of 3.29. The relationship is modelled in Figure 5.68 by a positive linear trend-line indicating that nearly 38% of the variation in Part D (ATQ) scores may be explained by Part C (FST) scores ($R^2 = 0.38$).

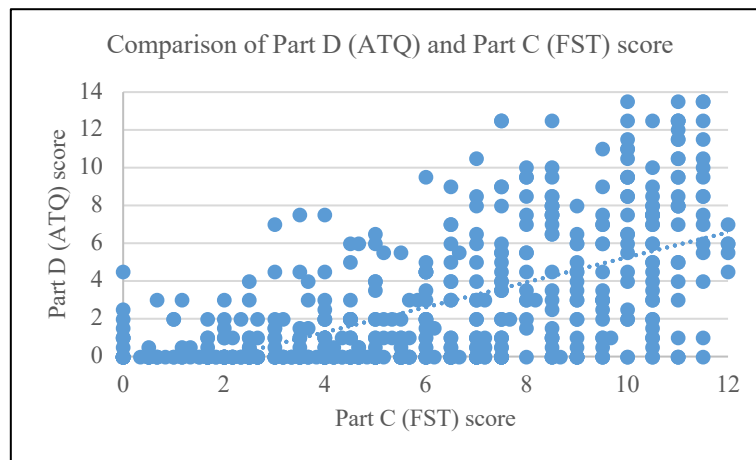


Figure 5-68: Comparison of Part D (ATQ) and Part C (FST) scores

5.5 Internal consistency for the two paper and pencil assessments

To determine the internal consistency of the FST tasks used in the main study the results were entered into SPSS (IBM SPSS Statistics, 2017). Cronbach's alpha was calculated using the scores of the 607 students for the 33 FST tasks. An analysis of the *Inter-item Correlation Matrix* showed that all Inter-item correlations were positive, so all items appear to be measuring the same underlying characteristic. The Cronbach's alpha (.888) given for the main study sample suggests

very good internal consistency for FST tasks. An analysis of the *Corrected Item-Total Correlation* values revealed that there was only one fraction task that had a value of less than 0.3 which indicates that the item is measuring something different to the scale as a whole. Task B2a asked students to mark on a number line the position for the number 75, where the length that represented 0 to 25, was given. Many students marked the position of 75 as though the whole number line represented 0 to 100 rather than iterating the length from 0 to 25 three times along the given number line. The corrected item-total correlation for B2a was 0.223 but when the result for the *Alpha If Item Deleted* values was checked, it was found that this task had the same Cronbach's alpha value (0.888) so it did not need to be removed.

As for the FST tasks Cronbach's alpha was calculated using the scores of the 607 students for the 28 ATQ tasks. An analysis of the *Inter-item Correlation Matrix* showed that all Inter-item correlations were positive, so all items appear to be measuring the same underlying characteristic. The Cronbach's alpha (.936) given for the main study sample suggests very good internal consistency for ATQ tasks. An analysis of the *Corrected Item-Total Correlation* values revealed that there was no task that had a value of less than 0.3 and thus no item needed to be removed.

5.6 Summarising the results for the paper and pencil assessments

In this chapter the results from the two paper and pencil assessments were analysed using descriptive statistics to assist the discussion about the research sub-questions in Chapter 8. Student work samples were included to illustrate both the range of strategies and misconceptions revealed in their responses to the written tasks from both assessments. Section 5.1 includes the analysis of results for the individual tasks of the FST while Section 5.2 describes the relationships between the three reverse fraction tasks and the results from the individual FST tasks. Both sections are needed to respond to research questions RQ1 and RQ2. Section 5.3 describes the relationships between the results for tasks from both the FST and the ATQ while Section 5.4 describes the relationship between the three reverse fraction tasks and the ATQ. Both Sections 5.3 and 5.4 will be used to respond to the research questions RQ3, RQ4, RQ5 and RQ6.

An important development in this chapter was the creation of the *Classification Framework for Reverse Fraction Tasks* (Table 5.15). In Section 5.1.4 students' responses for the three reverse fraction tasks were analysed revealing that all successful responses could be classified as one of five strategies: *Diagram Dependent*, *Additive/Subtractive*, *Partially Multiplicative*, *Fully Multiplicative* or *Advanced Multiplicative*.

Many students did not respond to questions that required them to explain their thinking about fraction and algebra tasks. According to Hadfield and McNeil (1994) these non-responses could be due to mathematics anxiety which includes environmental, intellectual or personality factors.

Environmental factors include classroom issues and the perception of mathematics as a rigid set of rules. Intellectual variables include a mismatch of learning styles and self-doubt. Personality factors include a reluctance to ask questions in class and low self-esteem. Any or all of these three factors could influence a learner and generate a “no attempt” or avoidance attitude.

Chinn (2012) concurs with this and suggests that:

... too many children and adults give up on mathematics learning by withdrawing from any task that is perceived as likely to result in failure, which in turn becomes a pervasive attitude. The withdrawal strategy avoids the learner being judged as wrong and thus adding to their sense of helplessness. Some of the demands of mathematics that contribute to a sense of failure, anxiety, and helplessness are based on beliefs, rather than academic necessity.

As many students were not confident with the tasks in the paper and pencil test on algebraic thinking, it was hoped that starting with their prior performances on the three reverse fraction tasks (ATQ) the semi-structured interview would provide opportunities for students to use more sophisticated algebraic reasoning. The tasks in the Structured Interview were carefully scaffolded in the expectation that students would move from a reliance on calculations and specific numbers and begin to generalise solution strategies.

The results for the 43 students who were interviewed using the Structured Interview will be discussed in Chapter 6. Nine case studies will be given in Chapter 7 where students' results for both the two paper and pencil assessments as well as their interview will be discussed. In Chapter 8 the results from the paper and pencil assessments as well as the Structured Interview will be synthesised in order to respond to the research question and sub-questions.

Chapter 6: Results from the Structured Interviews

Results from the two paper and pencil tests were discussed in Chapter 5 while Chapter 6 focuses on the students' responses to the semi-structured interviews. There were two specific aims for these interviews. One was to see whether students used the same methods in the interview as they had for their written responses. Did students use more sophisticated strategies when tasks were carefully scaffolded or with supportive questioning from an interviewer? The second aim was to see whether students could generalise and solve the tasks that included 'any number' of objects for the fractions they had used previously and in particular were they able to demonstrate emerging or established algebraic reasoning as they solved the final interview task with 'any fraction' and 'any number' of counters.

As previously stated in Chapter 4 there were 43 interviews were conducted in two schools including 17 primary students and 26 Year 8 students. Students were chosen because the methods they used to successfully solve at least two of the three reverse fraction tasks (Figure 3.7) represented a range of different strategies. The range of dominant methods was discussed in Chapter 4 and elaborated in Table 4.11.

Results from the Structured Interview will inform the discussion about the research questions: RQ3, RQ4, RQ5, RQ6 and RQ7 in Chapter 8. The results for the primary students' interviews are discussed in Section 6.1, the Year 8 students' results are in Section 6.2, Section 6.3 discusses the development of a framework for the analysis of the interviews and Section 6.4 compares the strategies students used to solve the three reverse fraction tasks in the written test with those they used for the interview questions. The results from the primary and secondary students are being analysed separately as formal algebraic notation is not introduced until Year 7 as shown in Table 1.4.

6.1 Primary Interviews

The 17 primary students used a variety of methods to successfully solve at least two of the three reverse fraction tasks (Tasks C5, C6, and C7) from the FST. As shown in Table 4.11 two primary students were dependent on diagrams, one student used additive methods only, seven used a mix of partially multiplicative methods, six consistently used fully multiplicative methods while one student used more abstract advanced multiplicative methods.

The first set of three questions of the Structured Interview uses the same fractions as those for the three reverse fraction tasks (Tasks C5 – C7) but the quantity representing the fraction has been

changed. For example, in Task C5 (FST) the fraction two-thirds is represented by a diagram of 10 counters (Figure 3.7) and in Question 1 (Table 6.1) the quantity representing two-thirds has been changed to 12 counters, but no diagram is given. In Table 6.1 the number of correct responses for Questions 1 – 3 are given along with those for Questions 4a, 5a and 6a which also include a specific number of objects representing the same three fractions as used in Questions 1 - 3. The numbers in Table 6.1 have not been converted to percentages as the number of students interviewed is relatively small.

Table 6-1: Number of correct primary responses by year level for SI questions (n = 17)

	Question 1	Question 2	Question 3	Question 4a	Question 5a	Question 6a
Year 5 (n = 9)	7	8	8	9	9	9
Year 6 (n = 8)	8	8	7	7	8	8
Primary (n = 17)	15	16	15	16	17	17

The strategies used for the individual questions of the Structured Interview have been classified according to the *Classification Framework for Reverse Fraction Tasks* shown in Table 5.14. Abbreviations are used in the tables detailing students' responses to the interview questions: *Diagram Dependent* (DD), *Additive/Subtractive* (AS), *Partially Multiplicative* (PM), *Fully Multiplicative* (FM), and *Advanced Multiplicative* (AM). Missing responses are labelled *Non-responses* (NR) and responses that are unclear or incomplete are deemed to be *Incomplete* (IN).

The dominant methods used for the three written reverse fraction tasks (Table 4.11) and the results for the six SI questions (Table 6.1) are very similar for Year 5 and Year 6 students so for the rest of this section the results will be combined and these Year 5 and Year 6 students will be referred to as the primary group.

6.1.1 Variation 1: Change of number 1

In the left-hand column of Table 6.2 is the first set of questions for the Structured Interview. These were initially presented in Table 4.5 with examples of the variations. The three fractions are the same as those used in the written test but the number of counters representing each fraction has been changed (*Change of number 1*). The other columns record the number of students who used the strategies according to the *Classification Framework for Reverse Fraction Tasks* (Table 5.14).

Primary students used multiplicative strategies (partially, fully and advanced) to answer Question 1 with the fraction two-thirds. It is interesting to note that Student LG and Student LC used

diagram dependent strategies for Question 2 involving the fraction four-sevenths with Student LC also using diagram dependent strategies for Question 3 with the fraction seven-sixths. After using a fully multiplicative strategy for Questions 1 and 2 Student KD started using *Advanced Multiplicative* strategies for all questions after Question 2.

Table 6-2: Primary students' strategies by year level for first set of SI questions (n = 17)

Structured Interview Questions	NR	IN	DD	AS	PM	FM	AM
1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	0	2	0	0	6	7	2
2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. How many CDs does Kay have? Explain your thinking.	0	1	2	0	4	8	2
3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	0	2	1	0	5	6	3

While 15 primary students gave correct responses for Question 1 two Year 5 students gave incorrect responses. This was unexpected as both students had written correct responses for Tasks C5 - C7 (FST). In her written responses Student CZ used additive strategies to solve Tasks C5 and C6 but multiplicative strategies to solve Task C7. For Question 1 (SI) she used her fingers to help her calculate the incorrect response (36) when she multiplied the number of counters (12) by the denominator (3). Student BM used additive/subtractive strategies and her faulty thinking for Question 1 is shown in Figure 6.1.

$\frac{2}{3}$ of 12 is 8 and half of that is 4
so if you add another 4 onto 12 you get 16

Figure 6-1: Student BM's responses to Question 1 (SI).

Student BM was the only student to give an incorrect response to Question 2. Her response: 'I think it is around 15' is quite different to her written response for Task C6 (FST) where she used a partially multiplicative strategy shown on the left-hand side of Figure 6.2. She calculated that one-seventh was three which she multiplied by three to get three-sevenths (9) and added that to four-sevenths (12) to get the correct response of 21. The right-hand side of Figure 6.2 shows her

elaborated incorrect response for Question 2. As the interviewer stated: “Started optimistically, looked puzzled, almost imperceptible counting on fingers and stated that ‘four-sevenths was ‘unfriendly’”. She was able to give the correct response for Question 3 but said: ‘I’m not quite sure but it worked out like that’.

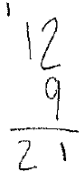
	Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. a. How many CDs does Kay have? <u>15</u> b. Explain your thinking. I think it is around 15 because it seems like it's a likely number because it has to be more than 8. $\frac{4}{7}$ is about half. <i>started optimistically looked puzzled, almost imperceptible counting on fingers friendly 4/7</i>
---	---

Figure 6-2: Comparison of BM's responses to written task C6 (FST) and Question 2 (SI)

Two students gave incorrect responses for Question 3. As shown in the right-hand column of Figure 6.3 Student CZ's response for Question 3 shows that she recognised that seven-sixths could be rewritten as mixed number $1\frac{1}{6}$. However, she wrote the division expression in reverse. Instead of writing $21 \div 7$ she wrote $7 \div 21$. She wrote the counting sequence of seven threes, then added another sequence of five threes to get the incorrect response of 36. This is similar to the response she gave for Task C7 where she drew extra circles to make a total of 24 counters (left-hand side of Figure 6.3). While she has written the correct response of 12 her explanation does not really make sense.

7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  a. How many counters did I start with? b. Explain how you decided that your answer is correct. because it is $\frac{7}{6}$ and she/he might of lost 10 from her/his 2 sets of counters	3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with. a. How many counters did I start with? b. Explain your thinking. 3 6 9 12 15 18 21 24 27 30 33 36 $\therefore 36$ $7 \div 21 = 3^{eq}$ $1\frac{1}{6}$ $\textcircled{2}$
--	--

Figure 6-3: Student CZ uses same strategies for Task C7 (FST) and Question 3 (SI)

Although Student LG gave the correct response for Task C7 (FST) she was unsuccessful with Question 3. Student LG recognised that $\frac{7}{6}$ could be written as $1\frac{1}{6}$ and asked whether she should take away one-sixth of the group which is what she did for Task C7. She drew three rows of seven

circles and tried to form them into six groups. She tried to make groups of four then said it was ‘three and a bit counters’ and wrote $3\frac{1}{6}$.

3.

Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with.

a. How many counters did I start with?

b. Explain your thinking.

$1\frac{1}{6}$ 3 $\frac{1}{6}$ counters in each group

Figure 6-4: Student LG's response to Question 3 (SI)

6.1.2 Variation 2: Change of number 2 and ‘any number’

Table 6.3 shows the results for the second set of questions where initially there is a change of number of counters representing the fraction (*change of number 2*) and then students are asked to describe what they would do if there was ‘any number of counters’ representing the given fraction (*‘any number’*).

Student KJ was the only student to give an incorrect response for Question 4a when she divided 18 by two to get six (rather than 9) which she added to 18 to get 24 rather than the correct response of 27. This partially multiplicative strategy is the same as she used for all three of the reverse fractions and attempted to use for all interview questions. Note that for each question

Table 6-3: Primary students’ strategies by year level for second set of SI questions (n = 17)

Structured Interview Questions	NR	IN	DD	AS	PM	FM	AM
4a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with?	0	1	0	0	5	8	3
4b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?	0	0	0	1	6	6	4
5a. If Susie had 20 CDs, which was $\frac{4}{7}$ of her friend Kay’s CDs, how would you find the number of CDs Kay has?	0	0	2	1	4	7	3

5b. If it was any number of CDs that Susie had, and this was still $\frac{4}{7}$ of the number CDs Kay had, what would you need to do to find the number of CDs Kay had?	1	0	1	0	5	6	4
6a. If I gave you 70 counters, which was $\frac{7}{6}$ of the number of counters I started with, how would you find the number of counters I started with?	1	0	0	0	5	8	3
6b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?	3	0	0	0	4	6	4

6.1.3 Variation 3: ‘Any fraction’ and ‘any number’

Many students were unable to respond to Question 7 where they were asked to think about the scenario with ‘any number’ of counters that represented ‘any fraction’. The responses appear to fall into three categories. Some students tried to generalise their additive methods, which had worked previously for questions where the quantity representing the fractional part was given. The other two responses were linked to algebraic thinking or reasoning where students either verbally wrote the generalised method in words or alternatively wrote in symbols. This research aims to determine whether students use algebraic thinking or reasoning to solve fraction tasks. Kieran (1989) stated that algebraic thinking or reasoning requires algebraic symbolism:

For algebraic thinking to be different from generalisation, I propose that a necessary component in the use of algebraic symbolism is to reason about and to express that generalization (p. 165)

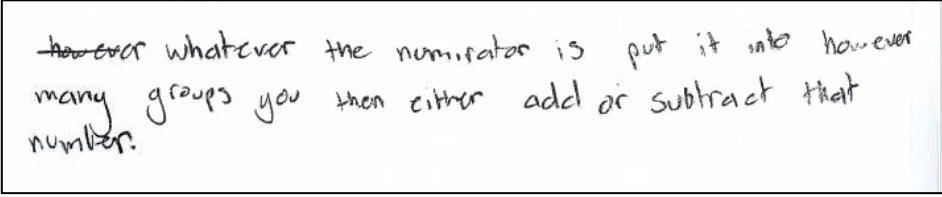
However, Radford (2018) suggests that algebraic symbolism can include both the verbal explanations as well as the alphanumeric symbolism:

Genuine algebraic symbolism includes the alphanumeric symbolism but also non-conventional semiotic systems – like natural language ... through which, as recent research shows, students signify generality (p.7)

6.1.3.1 Additive methods

Some students attempted to use additive methods to solve Question 7 where neither the fraction, nor the quantity representing that fraction, was given. For example, Student LG (Figure 6.5) has

determined that the unit fraction can be found by dividing by the numerator but needs to know how many times the quantity equivalent to the unit fraction needs to be added or subtracted before calculating the number of objects representing the whole group.



however whatever the numerator is put it into however many groups you then either add or subtract that number.

Figure 6-5: Student LG's additive response to Question 7 (SI)

It appears that Student LG has determined that the number of times the quantity corresponding to the unit fraction needs to be added to the given quantity can be obtained by subtracting the numerator from the denominator (with a subtractive compensation when the numerator is larger than the denominator). Student LG's response to Question 7 has the appearance of generalisability and demonstrates a deep perception of the problem when the fraction and the quantity are both unknown, but this method cannot be fully generalised either in words or written as an algebraic solution. This encapsulates the fundamental limitation of the additive method employed by students like Student LG who are reliant on additive methods.

In Figure 6.6 are Student CE's written responses for Questions 6b (left-hand side) and Question 7 (right-hand side). For the response given in the left-hand side of Figure 6.6 she stated that the letter A stands for the number of counters representing the fraction seven-sixths, and B the number of counters in one-sixth of this quantity. To find the number of counters in the whole group she subtracts the number of counters representing one-sixth from the number representing seven-sixths.

In the right-hand side of Figure 6.6 Student CE has again used letters as a shorthand way of expressing the same strategy she used for Question 6b. She stated that for this example, the letter A represents 'any number of counters', B represents the numerator of the 'any fraction' while C represents the number of counters in the unit fraction. She has stated that this amount can be subtracted or added to give the number in the whole quantity. However, this will only work for specific cases where the given fraction is one part less or more than the whole. While this student confidently solved all preceding interview tasks, she was unable to give an appropriate multiplicative response to Question 7.

b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with? $A \div 7 = B$ $A - B = C$	What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with? a. Can you tell me what you would do? $A \div B = C$ b. Please write your explanation in your own words. $A - C = D$ or $A + C = D$
--	--

Figure 6-6: Student CE's responses to Question 6b and 7 (SI)

6.1.3.2 Verbal multiplicative method

Student PK used a similar method to Student CE (left-hand side of Figure 6.6) to answer Question 6b (left-hand side of Figure 6.7). However, when presented with Question 7, Student PK realised that the partially multiplicative method she used for the previous interview questions would no longer work. She gave a fully generalisable answer to Question 7 (right-hand side of Figure 6.7): “Divide the (number of) counters by the numerator, then use the answer by multiplying it by the denominator”. When Student PK wrote: “This is the old method” she is referring to the fully multiplicative method that she had used in her written responses to Tasks C5 – C7 (FST).

b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with? See what goes into that number 7 times, then subtract the answer from the original number to get the answer	divide the counters by the numerator, then use the answer by multiplying it by the denominator this is the old method
--	--

Figure 6-7: Student PK's responses to Question 6b and Question 7 (SI)

6.1.3.3 Symbolic multiplicative method

Student MJ consistently solved all interview questions by dividing the quantity representing the fraction by the given fraction. In other words, to solve the task he multiplied the quantity by the reciprocal of the given fraction. For Question 7 Student MJ represented all quantities and fractions algebraically in a fully generalised solution as shown in Figure 6.8.

$\frac{a}{1} \div \frac{b}{c} = \frac{a}{1} \times \frac{c}{b} = d$	a is the number I have b is the numerator of the starting fraction c is the denominator of the starting fraction d is the number I had
---	---

Figure 6-8: Student MJ's symbolic multiplicative response to Question 7 (SI)

Table 6.4 summarises the results for Question 7 of the Structured Interview using 0 for non-responses, 1 for responses that were incorrect or unclear, 2 for responses where students attempted

to use *Additive Methods* (see Figure 6.6), 3 for *Verbal Multiplicative Methods* (see Figure 6.7) and 4 for *Symbolic Multiplicative Methods* (see Figure 6.8).

Two primary students did not attempt Question 7, two attempted an answer but were incorrect while three attempted to use additive methods. However, ten of the students stated generalisable multiplicative responses with seven students using verbal responses only and three writing appropriate algebraic symbols.

Table 6-4: Primary students' strategies for final SI question (n = 17)

Structured Interview Question 7	0	1	2	3	4
What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write in your own words.	2	2	3	7	3

Table 6.5 summarises the primary students' responses to the Structured Interview questions. Questions 1 – 6b have been coded using the same codes as in Tables 6.2 and 6.3. For example, Student CZ gave incorrect responses for Questions 1, and 3 but was unable to give any response for Questions 5b, 6a and 6b. She used a range of strategies for the other questions including an additive strategy for Question 5a, partially multiplicative strategies for Questions 4a and 4b and a fully multiplicative strategy for Question 2. Her incorrect response for Question 7 was based on her responses to Questions 6a and 6b and was scored as 1.

So you have to divide the top number of the fraction by the amount of counters you have then the answer is representing each one-sixth or one-half or just first increment. Then you count in the amount of one-seventh until you get to the bottom number of the fraction. For example, if 10 counters equals one-seventh, $10 \div 1 = 10$ so 10 20 30 40 50 60 70 which is seven counts (Student CZ).

Eight primary students used the same strategies for the first six interview questions. Students CE and PK consistently used *Partially Multiplicative* strategies. Student CE attempted to use the same strategy for Question 7 (right-hand column of Figure 6.6) while Student PK used a generalisable method (right-hand column of Figure 6.7). The four students who consistently used *Fully Multiplicative* strategies for the first six questions used generalisable *Verbal Multiplicative* strategies for Question 7. Students MJ and GR consistently used generalisable *Advanced*

Multiplicative strategies for the first six questions. While Student MJ was able to use *Symbolic Multiplicative* strategies Student GR used *Verbal Multiplicative* strategies for Question 7.

Table 6-5: Strategies used by the primary students for the SI Questions (n = 17)

	Year	Structured Interview Questions									
		1	2	3	4a	4b	5a	5b	6a	6b	7
CZ	5	IN	FM	IN	PM	PM	AS	NR	NR	NR	1
LC	5	PM	DD	DD	PM	PM	DD	PM	PM	NR	1
BM	5	IN	IN	PM	FM	AS	FM	FM	FM	NR	0
LG	6	PM	DD	IN	PM	PM	DD	DD	PM	PM	2
KJ	6	PM	PM	PM	IN	PM	PM	PM	PM	PM	2
CE	6	PM	PM	PM	PM	PM	PM	PM	PM	PM	2
PK	5	PM	PM	PM	PM	PM	PM	PM	PM	PM	3
DJ	6	FM	PM	PM	FM	FM	PM	PM	FM	FM	3
RA	6	PM	FM	FM	FM	FM	FM	FM	FM	FM	0
GS	5	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
PA	5	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
EO	5	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
CL	6	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
HF	6	FM	FM	FM	FM	AM	FM	AM	FM	AM	4
KD	5	FM	FM	AM	AM	AM	AM	AM	AM	AM	4
MJ	6	AM	AM	AM	AM	AM	AM	AM	AM	AM	4
GR	5	AM	AM	AM	AM	AM	AM	AM	AM	AM	3

Diagram Dependent
 Partially Multiplicative
 Fully Multiplicative
 Advanced Multiplicative

Two students were unable to respond to Question 7. Student RA used a fully multiplicative strategy for the first six questions but was unable to respond to Question 7. Student BM gave incorrect responses for the first two questions and used different strategies for the questions answered correctly but was unable to answer either Question 6b or Question 7.

6.2 Year 8 Interviews

Twenty-six Year 8 students were selected to participate because the methods they used to successfully solve at least two of the three reverse fraction tasks (Tasks C5, C6, and C7) represented a range of different methods. The range of dominant methods was elaborated in Table 5.15. Table 6.1 shows the number of Year 8 students, using the dominant methods for the set of three reverse fraction tasks (Figure 3.9). One student used pictorial methods across the set of three reverse fractions, two used only additive strategies, 12 used a mixture of additive and

multiplicative methods while 11 used multiplicative methods consistently for their written solutions to the three reverse fraction tasks (C5, C6 and C7).

In Table 6.6 the number of correct Year 8 responses for Questions 1 – 3 are given along with Questions 4a, 5a and 6a all of which state the specific number of objects that represent the given fraction. As the number of students is relatively small the numbers have not been converted to percentages. Note that one student gave an incorrect response to Question 2 and one did not respond to Question 5a both of which focused on the fraction four-sevenths. Four students did not respond to Question 6a the task which focuses on the improper fraction seven-sixths.

Table 6-6: Number of correct Year 8 responses for SI questions (n = 26)

Structured Interview	Question 1	Question 2	Question 3	Question 4a	Question 5a	Question 6a
Number of correct responses	26	25	26	26	25	22

Year 8 students also used a range of strategies in their responses to the Structured Interview questions. As for the primary analysis the Year 8 students' responses for each individual question of the Structured Interview have been classified according to the *Classification Framework for Reverse Fraction Tasks* shown in Table 5.14. Abbreviations are used in the tables detailing the students' responses to the interview questions: *Diagram Dependent* (DD), *Additive/Subtractive* (AS), *Partially Multiplicative* (PM), *Fully Multiplicative* (FM), and *Advanced Multiplicative* (AM). Student responses that were unclear or incomplete are deemed to be *Incomplete* (IN) while Non-responses were labelled NR.

6.2.1 Variation 1: Change of number 1

The Structured Interview Questions with the variations are presented initially in Table 4.5. In the left-hand column of Table 6.7 is the initial set of questions for the Structured Interview (*Change of number 1*). The three fractions are the same as those used in the written test but the number of counters representing each fraction has been changed. In the other columns are the strategies the Year 8 students used for this initial set of three tasks according to the *Classification Framework for Reverse Fraction Tasks* (Table 5.14).

Note that no Year 8 student used *Advanced Multiplicative* strategies and slightly more than one-half of the students used *Fully Multiplicative* strategies for the questions where the number of objects representing the fraction was stated.

Table 6-7: Analysis of Year 8 students' strategies for first set of SI questions (n = 26)

Questions 1 – 3 (Structured Interview)	NR	IN	DD	AS	PM	FM	AM
1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	0	0	2	1	9	14	0
2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. How many CDs does Kay have? Explain your thinking.	0	1	4	0	5	16	0
3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with. How many counters did I start with? Explain your thinking.	0	0	4	0	5	17	0

6.2.2 Variation 2: Change of number 2 and 'any number'

In Table 6.8 are the Year 8 students' results for the second set of questions where initially there is a change of number of objects (counters or CDs) representing the fraction (*change of number 2*) and then students are asked to describe what they would do if there was 'any number' of objects representing the given fraction (*any number*). Several Year 8 students used *Advanced Multiplicative* strategies to solve the tasks with 'any number' of objects but no Year 8 student used *Advanced Multiplicative* strategies for tasks where the number of objects was specified.

Table 6-8: Analysis of Year 8 students' strategies for second set of SI questions (n = 26)

Questions 4a – 6b (Structured Interview)	NR	IN	DD	AS	PM	FM	AM
4a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with?	0	0	3	1	2	20	0
4b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?	1	0	0	0	4	15	6
5a. If Susie had 20 CDs, which was $\frac{4}{7}$ of her friend Kay's CDs, how would you find the number of CDs Kay has?	1	0	0	0	5	20	0
5b. If it was any number of CDs that Susie had, and this was still $\frac{4}{7}$ of the number CDs Kay had, what would you need to do to find the number of CDs Kay had?	2	0	0	0	3	14	7

6a. If I gave you 70 counters, which was $\frac{7}{6}$ of the number of counters I started with, how would you find the number of counters I started with?

4	0	0	0	5	17	0
---	---	---	---	---	----	---

6b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?

4	0	0	0	4	12	6
---	---	---	---	---	----	---

6.2.3 Variation 3: ‘Any fraction’ and ‘any number’

Several Year 8 students were unable to respond to Question 7 of the Structured Interview where they were asked to think about the scenario where there was: ‘any number’ of counters that represented ‘any fraction’. Just like the primary students Year 8 students who attempted a response appear to fall into three categories. Some students tried to generalise their additive methods, which had worked previously for the questions where the quantity representing the fractional part was given. The other two responses were linked to generalised multiplicative methods where students either verbally wrote the generalised method in words (*Verbal multiplicative method*) or alternatively wrote in symbols (*Symbolic multiplicative method*).

The codes used for Question 7 are the same as those used for the primary students and include 0 for non-responses, 1 for responses that were incorrect or unclear, 2 for responses where students attempted to use *Additive Methods*, 3 for *Verbal Multiplicative Methods* and 4 for *Symbolic Multiplicative Methods*.

As shown in Table 6.9 seven Year 8 students were unable to give an appropriate generalizable multiplicative response for the ‘any fraction’ and ‘any quantity’ question. Four students did not attempt Question 7 while three attempted to use additive methods. However, 19 students stated generalisable multiplicative responses; 15 used verbal responses only and four wrote appropriate algebraic symbols.

Table 6-9: Year 8 student responses to Question 7 (Structured Interview)

Question 7 (Structured Interview)	0	1	2	3	4
What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with? Can you tell me what you would do? Please write in your own words.	4	0	3	15	4

Table 6.10 summarises the Year 8 students' responses to the Structured Interview questions. Questions 1 – 6b have been coded using the same codes as for the primary students where NR indicates no response, IN for incorrect attempts, DD for *Diagram Dependent*, PM for *Partially Multiplicative*, FM for *Fully Multiplicative*, and AM for *Advanced Multiplicative*. The Question 7 codes include 0 for non-responses, 1 for responses that were incorrect or unclear, 2 for *Additive Methods*, 3 for *Verbal Multiplicative Methods* and 4 for *Symbolic Multiplicative Methods*.

Table 6-10: Strategies used by Year 8 students for the SI Questions (n = 26)

Student	Structured Interview Questions									
	1	2	3	4a	4b	5a	5b	6a	6b	7
TA	PM	PM	DD	DD	NR	NR	NR	NR	NR	0
RV	DD	DD	DD	DD	PM	PM	NR	NR	NR	0
GA	PM	DD	DD	DD	PM	PM	PM	FM	PM	2
HD	PM	PM	PM	FM	FM	PM	AM	NR	NR	0
CM	AS	FM	PM	AS	PM	PM	PM	PM	PM	2
MM	PM	PM	PM	PM	PM	PM	PM	PM	PM	2
BL	FM	DD	DD	PM	FM	FM	AM	PM	AM	3
AH	FM	FM	FM	FM	AM	FM	AM	NR	NR	0
HJ	FM	FM	PM	FM	FM	FM	FM	PM	PM	3
HV	DD	DD	FM	FM	AM	FM	AM	FM	AM	3
CE	FM	FM	PM	FM	FM	FM	FM	PM	FM	3
GB	PM	PM	FM	FM	FM	FM	FM	FM	FM	3
TZ	FM	IN	FM	FM	FM	FM	AM	FM	AM	3
BS	PM	FM	FM	FM	FM	FM	FM	FM	FM	3
MG	PM	FM	FM	FM	FM	FM	FM	FM	FM	4
MO	PM	PM	FM	FM	FM	FM	FM	FM	e	4
CC	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
TL	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
DH	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
IT	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
RS	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
AJ	FM	FM	FM	FM	FM	FM	FM	FM	FM	4
GS	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
WL	FM	FM	FM	FM	AM	FM	FM	FM	FM	3
TS	PM	FM	FM	FM	AM	FM	AM	FM	AM	3
BL	FM	FM	FM	FM	AM	FM	AM	FM	AM	4

Diagram Dependent Partially Multiplicative Fully Multiplicative Advanced Multiplicative

6.3 Scoring the Structured Interviews

Each interview questions can be classified using the *Classification Framework for Reverse Fraction Tasks*. While the initial Structured Interview Scoring Framework (Table 4.12) allowed students to be classified according to their success with the actual tasks the overall interview results need to be classified in terms of algebraic reasoning in order to answer the main research question about the connections between fractional competence and algebraic reasoning.

Some students were only able to attempt the questions where both the fraction, and the quantity representing the fraction were given. These responses have been defined in terms of computational fluency (Level 1 and Level 2). Students are deemed to be beginning to generalise when consistently using additive or multiplicative methods to solve the questions with both a given fraction and equivalent quantity and the ‘any number’ questions. These students sometimes attempted an additive response to the more general question of ‘any fraction’ and ‘any number’ (Level 3 and Level 4). Students deemed to be using algebraic reasoning used either a verbal response (Level 5) or articulated their symbolic representations (Level 6). Table 6.11 elaborates the six levels that form the *Emerging Algebraic Reasoning Framework*.

Table 6-11: The Emerging Algebraic Reasoning Framework

Level	Description
1 Computational fluency – Partial	Solved only some questions with method restricted to given fractions and quantities.
2 Computational fluency – Complete	Solved all questions with given fractions and quantities but unable to answer more than one question with ‘any number’ of objects.
3 Generalising – Additive	Solved all questions with given fractions and quantities. Used additive or mixed methods to solve questions with ‘any number’ of objects. No appropriate generalised multiplicative response for ‘any fraction’ and ‘any number’ of objects.
4 Generalising- Multiplicative	Solved all questions with given fraction and ‘any number’ of objects using multiplicative methods. No appropriate generalised response to ‘any fraction’ and ‘any number’.
5 Algebraic generalisation – Verbal	Solved all questions with known fractions and ‘any number’ using consistent multiplicative methods. Students verbalised but did not symbolise full generalisation to ‘any fraction’ and ‘any number’.
6 Algebraic generalisation – Symbolic	Solved all questions with known fractions and ‘any number’ and generalised using consistent multiplicative methods. Appropriate algebraic notation used to solve ‘any fraction’ and ‘any number’ task.

Table 6.12 shows the number of students at each level of the *Emerging Algebraic Reasoning Framework*. Seven students are still reliant on using either diagrams or computational strategies

and unable to answer questions with ‘any quantity’. Six students are starting to generalise using additive strategies and three consistently use multiplicative strategies for the tasks with ‘any quantity’. Twenty students were able to give an appropriate verbal generalised solution for the question with ‘any fraction’ and ‘any quantity’ while seven students used appropriate written algebraic expressions. The primary students have not been introduced to formal algebra, but the Year 8 students would have been introduced to formal algebra in Year 7 and would be solving algebraic expressions in Year 8.

Table 6-12: Classification of students' interview responses

	Computational		Generalising		Algebraic	
	Partial	Complete	Additive	Multiplicative	Verbal	Symbolic
Primary (n = 17)	2	1	3	2	6	3
Year 8 (n = 26)	2	2	3	1	14	4

6.3.1 Comparison of primary students’ written and interview responses

The primary students’ methods for the set of three reverse fraction tasks (C5, C6 and C7) were classified according to the dominant method they used in their written responses and are shown in the left-hand column of Table 6.13. These results are then compared to the classifications for the *Emerging Algebraic Reasoning Framework* (right-hand columns of Table 6.13).

Table 6-13: Comparison of primary students’ written (FST) and interview responses (SI)

Tasks C5 – C7 (FST)		Emerging Algebraic Reasoning Framework					
Dominant Methods		Computational		Generalising		Algebraic	
		Partial	Complete	Additive	Multiplicative	Verbal	Symbolic
Diagram Dependent	2		1	1			
Additive/Subtractive	1	1					
Partial Multiplicative	7	1		2	1	3	
Fully Multiplicative	6				1	3	2
Advanced Multiplicative	1						1
Total	17	2	1	3	2	6	3

The Structured Interview was designed to allow the primary students to treat variations in the given fractions as ‘quasi-variables’ (Fujii & Stephens, 2001); that is, recognizing that the same multiplicative operations applied regardless of the fraction. In responding to Question 7 with 'any

fraction' and 'any number' students typically referred to dividing by the numerator and multiplying by the denominator. Three primary students, who used partially multiplicative methods to solve specific written fraction problems, recognised that these methods were not appropriate when dealing with unknown quantities and were able to shift to fully generalizable thinking using either additive or multiplicative strategies.

The six primary students whose overall responses to the interview questions were deemed to demonstrate verbal algebraic generalisations appear well positioned for formal algebra as expected in Year 7 as given in the Australian Curriculum: Mathematics (ACARA 2016) where they will be introduced to: “the concept of variables as a way of representing numbers using letters (ACMNA175)”. The three primary students who were already using algebraic reasoning and writing algebraic equations showed evidence of being able to create and “simplify algebraic expressions involving the four operations (ACMNA192)” as recommended for Year 8 (Table 1).

It needs to be noted that 14 of the 46 students from this school who completed the paper and pencil tests were unable to complete more than one of the reverse fraction tasks in the written test. Six of these were Year 6 students about to transition into secondary school. These students appear to be at risk in subsequent years when meeting linear equations involving rational coefficients and in relation to proportional reasoning.

6.3.2 Comparison of Year 8 students' written and interview responses

As shown in Table 6.14 seven of the Year 8 students who used partially multiplicative strategies for the written tasks (C5 - C7) are deemed to have used verbal algebraic strategies when solving the interview questions. Students who used fully multiplicative strategies in their written responses to Tasks C5 - C7 used either verbal or symbolic strategies in their interviews.

Table 6-14: Comparison of Year 8 students' written (FST) and interview responses (SI)

Tasks C5 – C7 (FST) Dominant Methods		Emerging Algebraic Reasoning Framework					
		Computational		Generalising		Algebraic	
		Partial	Complete	Additive	Multiplicative	Verbal	Symbolic
Diagram Dependent	1	1					
Additive/Subtractive	2		1	1			
Partial Multiplicative	12	1	1	2	1	7	
Fully Multiplicative	11					7	4
Advanced Multiplicative	0						
Total	26	2	2	2	2	14	4

Many Year 8 students were able to treat variations in the given fractions as ‘quasi-variables’ recognising that the same multiplicative operations applied regardless of the fraction. For the question with ‘any fraction’ and ‘any quantity’ seven Year 8 students verbally referred to dividing by the numerator and multiplying by the denominator while four used the appropriate algebraic notation while verbally interpreting their symbolic response. Seven Year 8 students, who had used both additive and multiplicative methods to solve specific fraction problems, recognised that additive methods were not appropriate when dealing with unknown quantities and were able to shift to fully generalisable multiplicative thinking.

These Year 8 students should have been introduced to: “the concept of variables as a way of representing numbers using letters” (ACMNA175, ACARA, 2018) in Year 7. Only four of the 26 students used algebraic expressions for Question 7 demonstrating that they could create and “simplify algebraic expressions involving the four operations” (ACMNA192) as recommended for Year 8 (Table 1.4).

6.4 Implications from the analysis of the Structured Interview

Students who are dependent on pictorial or additive methods are likely to experience difficulty in adopting a multiplicative approach when given tasks with ‘any number’ of objects and describing a rule as stated in the Content Descriptor ACMNA133 (ACARA, 2016) and shown in Table 1.1. For these students, each question appears to be a new problem that had to be considered on its own terms. These students are most likely to experience difficulty in transitioning to algebra.

A necessary precursor to being able to generalise a solution was to recognise, implicitly or explicitly, an equivalence relationship between the given fraction and its related quantity. This allows students to find the quantity related to the unit fraction that can then be scaled up to a whole additively or multiplicatively. Even when an equivalence relationship had been identified additive methods were less easily generalised as students needed to know how many parts to add or subtract. Several students failed to give a correct response despite using this method due to their faulty computation.

Multiplicative methods were clear precursors to generalisation. Students typically divided by the numerator to find the quantity equivalent to the unit fraction and then multiplied by the denominator to find the quantity in the whole group. Some students divided by the given fraction or multiplied by the reciprocal to obtain a whole number equivalent. Generalisable methods

provided evidence of algebraic thinking when students could describe what needed to be done if a given fraction was related to 'any quantity'.

Fully generalisable methods demonstrated algebraic reasoning when students could describe verbally in non-symbolic terms how to find the whole given 'any fraction' and 'any quantity'. Some students demonstrated clear algebraic thinking by using symbols such as a/b to represent any given fraction and c to represent any given quantity in order to generalise their solutions.

Chapter 7: Case Studies

This chapter focuses on nine case study students chosen from the 45 students who were interviewed. These case studies include four primary students with two students from Year 5 and two from Year 6 and five students from Year 8. Each student has been given a pseudonym to retain anonymity and these are shown in Column 1 of Table 7.1 with their corresponding year level in Column 2. Students were only selected to be interviewed if they had correctly answered at least two of the three reverse fraction tasks, that is, Tasks C5, C6 and C7 from the FST. These students' solution strategies were able to be classified in contrast to other students who may have given correct answers but whose thinking was not so transparent due, for example, to incomplete written explanations. These nine case study students provide examples of common misconceptions and the types of strategies used in both the written assessments and the interviews. These case studies are not meant to be considered as representing the achievement levels of the overall sample of students but rather each has used a different range of strategies.

The strategies that students used for each task from Question 1 to Question 6b in the Structured Interview are classified according to the *Framework for Reverse Fraction Task Strategies* (Table 5.15) and have been elaborated in Table 7.1. Students' responses to the interview questions have been coded as given in Tables 6.6 and 6.11: *Diagram Dependent* (DD), *Additive/Subtractive* (AS), *Partially Multiplicative* (PM), *Fully Multiplicative* (FM), and *Advanced Multiplicative* (AM). Missing responses are labelled *Non-responses* (NR) and responses that are unclear or incomplete are deemed to be *Incomplete* (IN).

The codes used for Question 7 (last column in Table 7.1) include 0 for non-responses, 1 for responses that were incorrect or unclear, 2 for responses where students have attempted to use *Additive Methods*, 3 for *Verbal Multiplicative Methods* and 4 for *Symbolic Multiplicative Methods*.

As shown in Table 7.1 four of the case study students used the same strategies for Questions 1 - 6b of the Structured Interview. Kate used partially multiplicative strategies, while Alex and Tanya used fully multiplicative strategies before responding verbally to Question 7 and Jack used advanced multiplicative strategies for each question before using a symbolic response for Question 7. Andrew consistently used fully multiplicative strategies for all questions but was unable to answer Question 7.

Table 7-1: Structured Interview results for case study students (n = 9)

Student	Year Level	Structured Interview Questions									
		1	2	3	4a	4b	5a	5b	6a	6b	7
Violet	8	DD	DD	DD	DD	PM	PM	NR	NR	NR	0
Gloria	6	PM	DD	IN	PM	PM	DD	DD	PM	PM	2
Kate	5	PM	PM	PM	PM	PM	PM	PM	PM	PM	3
Daniel	8	PM	PM	PM	FM	FM	PM	AM	NR	NR	0
Andrew	8	PM	FM	FM	FM	FM	FM	FM	FM	FM	0
Alex	5	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
Tanya	8	FM	FM	FM	FM	FM	FM	FM	FM	FM	3
Lewis	8	FM	FM	FM	FM	AM	FM	AM	FM	AM	4
Jack	6	AM	AM	AM	AM	AM	AM	AM	AM	Am	4

Diagram Dependent
 Partially Multiplicative
 Fully Multiplicative
 Advanced Multiplicative

Two students moved between *Diagram Dependent* and *Partially Multiplicative* strategies. For example, Violet (Year 8) moved from using diagram dependent strategies to partially multiplicative strategies before her interview was discontinued. Gloria (Year 6) switched between using diagram dependent strategies and partially multiplicative strategies before giving a partially multiplicative response for Question 7. Daniel (Year 8) used three different strategies for the interview questions. He initially used a partially multiplicative strategy for Question 1 but then used fully multiplicative strategies for Question 4 with the fraction two-thirds, partially multiplicative strategies for the fractions four-sevenths and seven-sixths where the number of counters were given but advanced multiplicative strategies for the more generalised tasks that refer to ‘any number’ of counters given the fraction four-sevenths. However, his interview was terminated after Question 5b as the interviewer felt he was becoming too stressed to complete any further tasks.

Table 7.2 compares the overall results of the nine case study students for the three assessment protocols: FST, ATQ and the Structured Interview. Total scores are given for the FST and the ATQ while the dominant strategies are included for the set of three reverse fraction tasks. The last column in Table 7.2 draws on the six levels that form the *Emerging Algebraic Reasoning Framework* described in detail in Table 6.12.

Table 7-2: Results from the three assessment protocols for the case study students

Student	Year	Fraction Screening Test		ATQ Total score	Emerging Algebraic Reasoning Framework
		Dominant Reverse Fraction Strategies	Total score		
Violet	8	Diagram dependent	24	7.5	Computational fluency - Partial
Gloria	6	Partially Multiplicative	17.7	0.5	Generalising - Additive
Kate	5	Partially multiplicative	26.5	5	Algebraic Generalisation - Verbal
Daniel	8	Partially multiplicative	18	6	Computational fluency - Complete
Andrew	8	Fully Multiplicative	27	8	Generalising - Multiplicative
Alex	5	Fully Multiplicative	34.2	12	Algebraic Generalisation - Verbal
Tanya	8	Partially Multiplicative	27.5	13	Algebraic Generalisation - Verbal
Lewis	8	Fully Multiplicative	33.5	15.5	Algebraic Generalisation - Symbolic
Jack	6	Advanced Multiplicative	33.5	19.5	Algebraic Generalisation - Symbolic

The case study students whose results are presented in Table 7.2 provide examples of the six classifications from the *Emerging Algebraic Reasoning Framework*. Violet is an example of the *Computational Fluency-Partial* group while Daniel represents the *Computational Fluency-Complete* group. Gloria represents the *Generalising-Additive* group and Andrew the *Generalising-Multiplicative* group. Alex and Tanya represent the *Algebraic Generalisation-Verbal* group and used multiplicative strategies while Jack and Lewis representing the *Algebraic Generalisation-Symbolic* group used advanced multiplicative strategies, consistently and successfully, for all tasks in the Structured Interview. Kate has been included as she was classified as being in *Algebraic Generalisation-Verbal* group despite using partially multiplicative strategies for all questions prior to Question 7.

The aim of these case studies is to explore, at the level of each student, how algebraic thinking or reasoning may be inferred in its different stages as each student completed the reverse fraction tasks, initially in the written assessment, and then during the Structured Interview. The Algebraic Thinking Questionnaire was generally quite difficult for most students, with a large percentage of missing responses, and therefore could not provide a clear description of algebraic thinking for

many students. For students who performed poorly on the reverse fraction tasks in their written responses for the FST, and who also performed poorly on the Algebraic Thinking Questionnaire, little evidence is available to confirm the presence of algebraic thinking. The case studies will demonstrate that evidence of algebraic thinking is emergent at different levels but is still far from consolidated. Evidence of algebraic thinking appears clearest for students classified as either generalising or algebraic (last column of Table 7.2) but it will be evident from the analysis of their responses that even these students have some gaps or misunderstandings in their algebraic thinking.

In Table 7.3 the case study students' scores for all parts of the FST and the Algebraic Thinking Questionnaire are compared to those of the whole cohort. The range of the possible scores are given along with the mean and standard deviation (SD) for the individual parts and the total scores for both assessments. These scores are given here to allow the elaboration of the case studies and not as a point of comparison or as levels expected by a different cohort of students.

Table 7-3: Comparison of scores for case study students for FST and ATQ

	Fraction Screening Test				Algebraic Thinking Questionnaire		
	Part A	Part B	Part C	Total	Part M	Part D	Total
<i>Mean</i>	9.63	6.21	6.07	21.91	4.20	2.66	6.86
<i>SD</i>	2.22	2.67	3.29	7.11	3.34	3.54	6.45
Violet	8.5	8.0	7.5	24.0	4.0	3.5	7.5
Gloria	10.0	2.0	5.7	17.7	0.5	0	0.5
Kate	10.5	9.0	7.0	26.5	0	5.0	5.0
Daniel	9.0	5.0	4.0	18.0	6.0	0	6.0
Andrew	9.0	9.0	9.0	27.0	6.0	2.0	8.0
Alex	12.0	10.0	12.2	34.2	9.5	2.5	12.0
Tanya	10.5	7.0	10.0	27.5	7.0	6.0	13.0
Lewis	11.0	10.0	12.5	33.5	7.0	8.5	15.5
Jack	10.5	10.0	13.0	33.5	11.5	8.0	19.5

 scores less than the mean
  scores between mean + 1SD
  scores greater than mean + 1SD

In Figure 7.1 the case study students' results for ATQ and FST total scores are compared. Violet and Daniel who are deemed to be at the Computational Fluency level are indicated by the red shape, Gloria and Andrew who are deemed to be at the Generalising level are indicated by the blue shape while Kate, Alex and Tanya deemed to be at the Algebraic Generalisation – Verbal level are indicated by the green shape. The two students, Lewis and Jack, deemed to be at the

Algebraic Generalisation level are indicated by the orange shape and achieved the best results for both the FST and the ATQ.

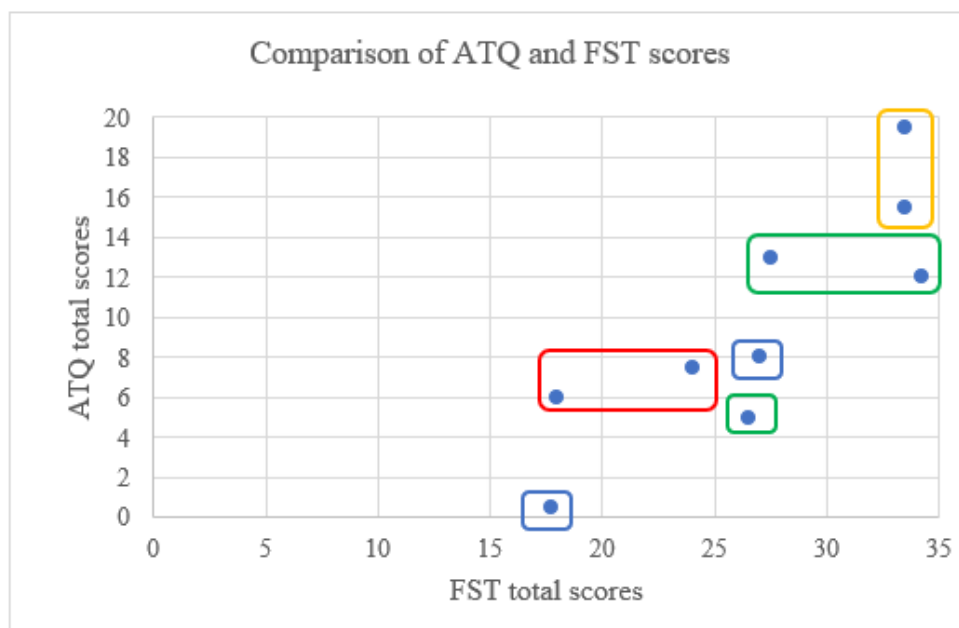


Figure 7-1: Comparison of ATQ and FST total scores for case study students

The teachers of the primary students who were interviewed indicated that their students had not yet been introduced to formal algebraic notation. However, there were some primary students classified as advanced multiplicative thinkers, who used either verbal or symbolic notation to express their algebraic reasoning, sometimes in idiosyncratic ways, as will be evident from the case studies. Conversely, for the Year 8 students, algebraic notation should have been consistently used in mathematics classes and met in class texts for nearly two years.

In the elaboration of each case study, the aim is to show which multiplicative and generalisable strategies each student used as they attempted to answer questions with increasing levels of abstraction embodied in the interview. The first level of abstraction was a change of number representing each of the three given fractions. The next level was to have ‘any number’ representing each of the three given fractions, and finally students were asked to consider how they would calculate the whole “if they had any number of objects representing any fraction”. In each case study, the intention is to investigate more closely their demonstrated overall performance on the both the FST and ATQ as well as the strategies they used to solve the reverse fraction tasks in both the FST and the Structured Interview. In particular, did the strategies these students use change when the levels of abstraction changed in the Structured Interview?

7.1 Case Study 1: Violet

Violet (Year 8) was chosen for a case study as she is an example of the *Computational Fluency-Partial* group at Level 1 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. She depended on diagrams for the first four tasks of the Structured Interview, used partially multiplicative strategies for the next two tasks and was unable to respond to any further questions. She was chosen to be interviewed as she had successfully used diagrams with a range of strategies for the three reverse fraction tasks (C5 - C7) from the FST. Violet's scores for the FST and ATQ are slightly above the mean score for the whole cohort.

Pearn, Stephens, and Pierce (2018a; 2018b) included a brief description of Violet's responses to the three reverse fraction tasks and the Structured Interview questions. Section 7.1.1 will discuss her responses to the FST, Section 7.1.2 the strategies she used for the reverse fraction tasks (Tasks C5 - C7), Section 7.1.3 the strategies she used for the Structured Interview and Section 7.1.4 her responses to the ATQ.

7.1.1 Fraction Screening Test

Violet's solutions to the tasks from the three parts of the FST will be discussed in this section. Overall, Violet scored 24 out of a possible 35 marks for the FST where the mean was 21.91 and the Standard Deviation was 7.14 for the cohort of students. This is approximately 69% of the possible total marks for the FST.

Violet's score for Part A tasks (FST) was 8.5 out of a possible 12 marks where the mean was 9.63 and the standard deviation was 2.22. This means that for Part A her score was nearly 71% of the total possible score and less than the mean for all students. She chose the two correct shapes representing three-quarters as well as one incorrectly marked shape, circled one-third of a group of beetles, and indicated that there would be eight lollies in the whole group when four lollies represented one-half of the group. Violet accurately shaded one-third of a rectangle and correctly marked, and shaded, three-quarters of a length. She correctly completed the equivalent fractions and calculated that the whole was \$28 if \$4 represented one-seventh. However, she was unable to calculate the fraction of pizza that each person would receive if three pizzas were shared between four and did not attempt to complete the counting sequence. When attempting to solve the task of addition of fractions with related denominators she converted $\frac{1}{5}$ to $\frac{2}{10}$ but did not use this information to complete the calculation, instead writing: $\frac{3}{10} + \frac{1}{5} = \frac{4}{10}$

Violet scored 8 marks out of a possible 10 marks (80%) for the Part B tasks (FST) where the mean for all students was 6.21 and the standard deviation was 2.67. So, her score for the Part B tasks

was above the mean for all students. She successfully wrote the four missing numbers on the evenly spaced marks on the number line marked zero to 100. Violet accurately marked the position of the number three-fifths on the number line marked zero to two. She used the distance from zero to one-third to mark in the number two-thirds and then the number one on a number line. Violet chose the correct option for the placement of the number one on a number line marked zero and $\frac{3}{4}$ and stated: “I decided it wasn’t a or b because they are a lot closer to $\frac{3}{4}$ than c is”. Violet incorrectly marked where the number 75 would be on a number line marked zero and 25 as she assumed the whole line was from zero to 100 and justified her choice saying: “Because 25 is one quarter of the way and 75 is three quarters of the way”. Violet did not respond to any of the questions which asked her to explain how she decided that her answer was correct.

Violet scored 7.5 marks out of a possible 13 marks (58%) for the Part C tasks where the mean was 6.07 and the standard deviation was 3.29 for the whole cohort. Thus, for Part C Violet’s score was above the mean for all students. Violet successfully ordered four fractions and matched all four fractions with their equivalent decimals. In the two tasks that asked her to circle the symbols that did not belong to the set of fractions, decimals and percentages she circled both decimal representations. One of her choices was correct as she recognised that 0.4 was not equivalent to the various representations of one-quarter. However, she incorrectly chose 1.2 as the one that did not belong to the group including $\frac{6}{5}$ and $1\frac{6}{30}$ which are equivalent to 1.2.

7.1.2 Reverse Fraction Tasks

Violet used a fully multiplicative method to solve Task C5 as shown in Figure 7.2. She circled five of the dots given in the diagram and wrote the symbol for one-half above the circled dots. While her written explanation was brief it demonstrated that she knew that one-third was represented by five dots and she multiplied five by three to find three-thirds or one-whole.

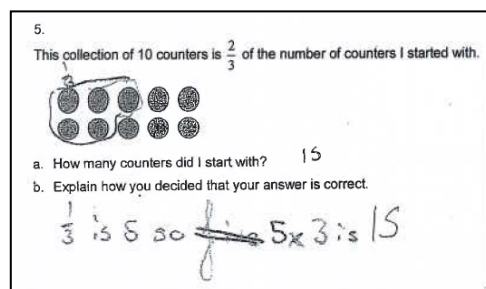


Figure 7-2: Violet's response to Task C5 (FST)

No diagram was given for Task C6, but Violet drew her own diagrams to solve the task as shown in Figure 7-3. She initially drew three rows of four circles, drew around four groups of three circles, wrote 12 on the left-hand side of her drawing and the fraction four-sevenths on the right-
182

hand side of the drawing. Although she did not specifically state that three CDs represented one-seventh of the group it is implied through this diagrammatic solution. Violet then replicated the three rows of four circles, drew around the groups of three and added another three groups of three circles. The total from the three rows of seven circles (21) is written in the top right-hand corner of Figure 7.3. Violet relied on the diagram where she drew the initial four groups of three circles to represent four-sevenths then drew another three groups of three circles to represent the extra three-sevenths needed to represent seven-sevenths or the whole.

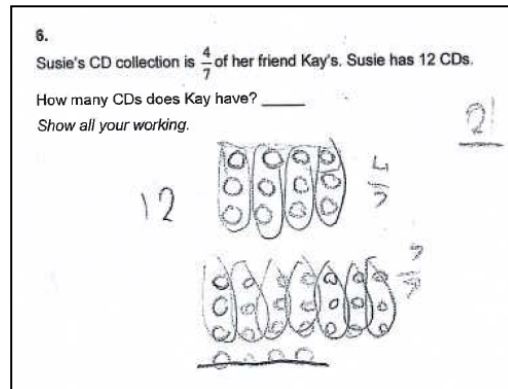


Figure 7-3: Violet's response to Task C6 (FST)

As shown in Figure 7.4, Violet had several attempts at circling sets of dots in her attempt to solve Task C7. Her written explanation is an attempt to explain how she decided that her answer was correct, but she did not state why she divided 12 by two. It is also unclear how she determined that one-sixth was represented by two dots, but she multiplied six by two to get 12 ($\frac{6}{6}$) and seven by two to get 14 which was originally stated as representing seven-sixths. There is an element of uncertainty in her response as she wrote: "started with 12?".

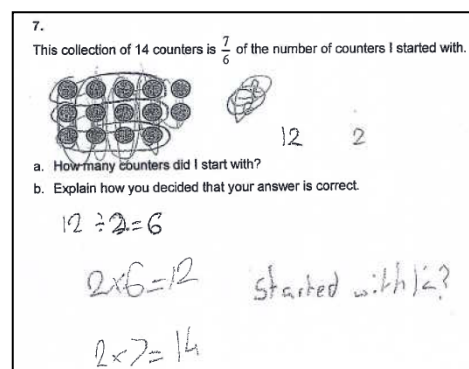


Figure 7-4: Violet's response to Task C7 (FST)

Although Violet was dependent on diagrams to solve the part C three reverse fraction from the FST, she appears to use multiplicative methods for Tasks C5 and C7 but a partially multiplicative method for Task C6.

7.1.3 Structured Interview

Violet appears to rely on the use of diagrams or drawings to find the number of objects that represent the whole if given the number of objects representing the fractional part and has been classified as *Diagram Dependent* (Table 7.1). In Question 1 (Figure 7.5) where 12 counters represent two-thirds of the whole Violet drew three rows of six circles then drew around two of those rows to indicate two-thirds. She correctly wrote that the initial number was 18 counters. Unfortunately, her explanation seems a little strange as she wrote that: “Both 12 and 18 are divisible by 3 and 6” referring to the number of groups (three) and the number in each group (six).

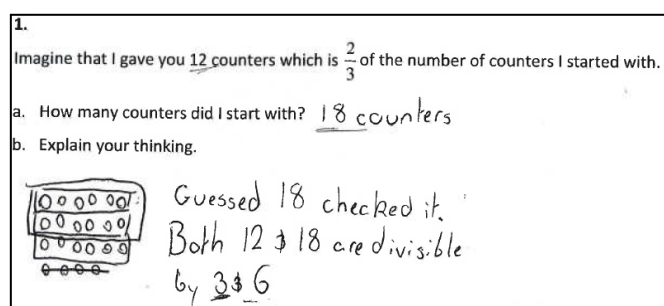


Figure 7-5: Violet's responses to Question 1 (SI)

Violet's initial attempt at using a diagram in Question 2 is incorrect as shown on the left-hand side of Figure 7-6. She initially drew an array of five rows of four circles then crossed out one circle from each row. She then added two more rows of three circles to make seven rows of three circles. Violet then drew an additional array of seven rows of three circles before attempting to draw around groups of seven circles. She drew around three groups of seven circles, four groups of eight circles and one group of five circles. At this stage, she was encouraged by the interviewer to re-read the question.

After re-reading Question 2 Violet then correctly drew a row of eight circles to represent the four-sevenths, and divided these eight circles into four equal groups, writing the fraction four-sevenths beside the drawing. She then added a further four groups of two circles underneath the first diagram, then crossed out one group of two circles, leaving three groups with two circles to represent three-sevenths and correctly stated that the total is 14 CDs (See the right-hand side of Figure 7.6).

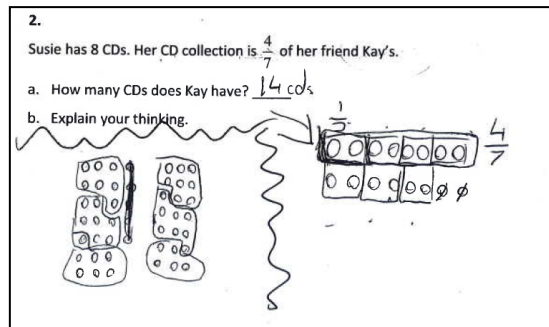


Figure 7-6: Violet's responses to Question 2 (SI)

In Question 3 Violet initially drew four rows of seven dots, correctly partitioned the 21 dots into seven groups of three and recognised that seven-sixths was equivalent to one and one-sixth and correctly stated that there were 18 counters in the whole group (Figure 7.7).

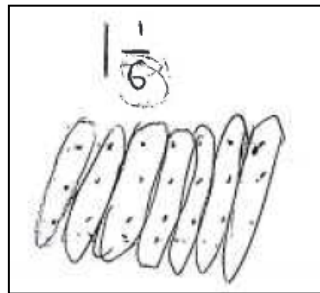


Figure 7-7: Violet's responses to Question 3 (SI)

In Question 4, where 18 counters represent the fraction two-thirds, Violet constructed four diagrams (Figure 7.8) where the first, second and fourth are incorrect. In the first diagram, she drew three rows of six circles and drew around each row stating that each row represented one-third. In the second diagram she again drew three rows of six circles and attempted to divide these in two parts but unfortunately, she ended up with one group of seven dots and one of 11 dots. The fourth diagram shows three rows of seven dots, which she divided into three groups of seven, then wrote the answer as 14, which is two of the groups of seven or two-thirds of the 21 dots she drew. The third diagram shows three groups of nine, but only after many corrections have been made. She then correctly decided that three-thirds was 27.

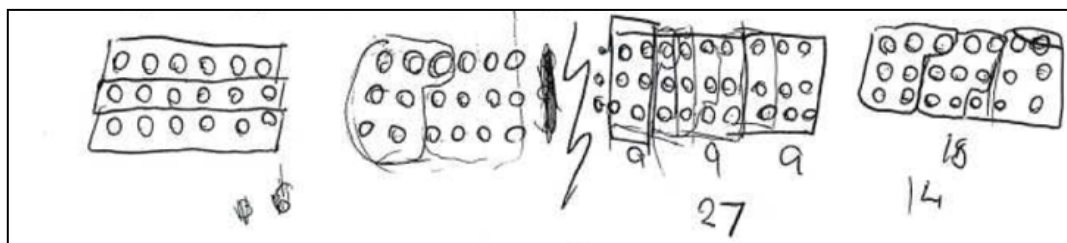


Figure 7-8: Violet's responses to Question 4 (SI)

After finally succeeding with Question 4a using the third diagram from the left in Figure 7.8, Violet correctly used an additive solution for Question 4b which asks about ‘any number of counters’ representing two-thirds saying: “You would halve the number and then add the result to the number you started with”.

In Question 5 (Figure 7.9) Violet correctly found one quarter of 20 by halving, and halving again, but represented this as three equivalent expressions. She started to draw a diagram for Question 5 which she scribbled out before using a partially multiplicative method to correctly determine three-sevenths as 15 (5×3) adding this to the original four-sevenths (20) to calculate the total number (35).

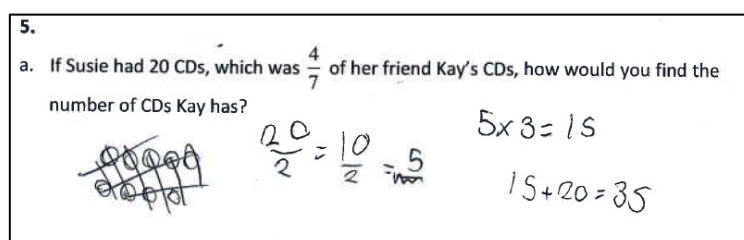


Figure 7-9: Violet's responses to Question 5 (SI)

Violet was unable to complete any further questions from the Structured Interview. However, it is evident from the tasks she has successfully solved that her partially multiplicative thinking is frequently tied to a diagrammatic representation. Moreover, Violet has difficulty in creating appropriate diagrams to represent the situation, often requiring several attempts to construct the diagram as shown in her response to Question 4 (Figure 7.8). She provided a solution to ‘any number of counters’ representing two-thirds but could not provide a generalised solution for either four-sevenths or seven-sixths. Consequently, Violet is deemed to be at the *Computational Fluency-Partial* level of the *Emerging Algebraic Reasoning Framework*.

7.1.4 Algebraic Thinking Questionnaire

In the Algebraic Thinking Questionnaire, Violet was limited in her ability to deal with multiplicative relationships. She scored 4 out of a possible 14 marks (29%) for Part M, which was less than the mean of 4.20 for the multiplication tasks. She scored 3.5 marks out of a possible 14 marks (25%) for Part D, which was more than the mean for the division tasks in Part D. Overall Violet scored 7.5 out of a possible 28 marks (27%) for the Algebraic Thinking Questionnaire which was more than the mean of 6.86 for the total score for the whole cohort.

Despite attempting calculations for all Question 1 tasks from Part M, Violet was unable to successfully complete any task. She drew a diagram for the task $\frac{2}{3} \times \square = 18$ (Figure 7.10) in

which she drew nine shapes with each shape divided into three parts. She then counted the shaded parts of each shape (9) but if she had counted the total number of parts i.e. 27, rather than one part of each shape, she would have given the correct response.

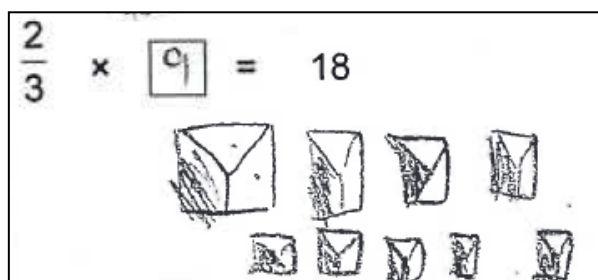


Figure 7-10: Violet's response to Task M1c (ATQ)

Violet gave two correct responses for Question 2a as shown in the left-hand column of Figure 7.11 which shows that Violet understood the sameness-relational meaning of the equals sign (Jones et al., 2013). However, Violet wrote the incorrect reverse relationship between Box A and Box B for Question 2b (right-hand column of Figure 7.11) but as this is a written test not an interview it is unclear as to her reason for her response.

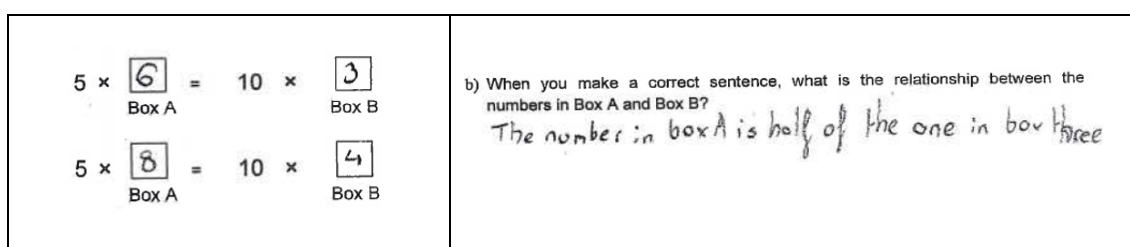


Figure 7-11: Violet's response to Tasks M2a and M2b (ATQ)

Violet's correct response for Question 2c (Figure 7.12) stating that the number in Box B is half of the number in Box A contradicts her incorrect response for Question 2b (Figure 7.11) where she stated that the number in Box A is half the number in Box B.

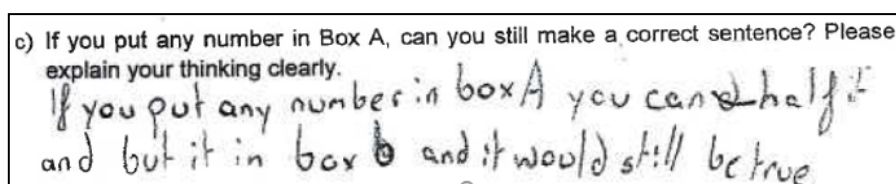


Figure 7-12: Violet's response to Task M2c (ATQ)

In Part D Violet correctly responded to the first two tasks: $3 \div 4 = 15 \div 20$ and $8 \div 15 = 16 \div 30$ but as there was no working out on the page it is difficult to determine how she decided on her responses. Violet was unable to answer any further questions in Part D.

Violet's answers for the ATQ confirms the judgements made from the analysis based on her responses for both the FST and the Structured Interview tasks. While there is some evidence of multiplicative thinking, and some capacity to generalise, Violet's apparent dependence on diagrams to solve the calculations means she would need to think about each problem as a new situation to be represented in a diagram. Although she tried to consistently apply the same approach there is not enough evidence to determine the extent of her algebraic thinking.

7.2 Case Study 2: Gloria

Gloria (Year 6) was chosen as a case study as she is an example of the *Generalising-Additive* group at Level 3 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. She depended on diagrams for three tasks, used partially multiplicative strategies for five tasks and attempted to use additive strategies for Question 7 of the Structured Interview (see Table 7.1). Gloria was chosen to be interviewed as the language she used to solve the three reverse fraction tasks from the FST was unclear and, in some cases, did not make sense. It was hoped that her dominant solution strategy for solving the reverse fraction tasks could be more clearly revealed during the Structured Interview. Gloria's score for the FST is below the mean score and her ATQ score substantially below the mean score for the whole cohort (see Table 7.2).

Section 7.2.1 will discuss her responses to the FST, Section 7.2.2 the strategies she used for the reverse fraction tasks (Tasks C5 - C7), Section 7.2.3 the strategies she used for the Structured Interview and Section 7.2.4 her responses to the ATQ.

7.2.1 Fraction Screening Test

Gloria scored 17.7 out of a possible 35 marks (51%) for the FST which is less than the mean for the whole cohort. Gloria's score for the Part A tasks was 10 out of a possible 12 marks (83%) which is slightly more than the mean of 9.63 for the whole cohort. She chose shapes that correctly represented three-quarters, circled one-third of a group of beetles, and wrote that there would be eight lollies in the whole group when four lollies represented one-half of the group. Gloria accurately shaded one-third of a rectangle, marked and shaded three-quarters of a length, and calculated that each person would receive three-quarters of a pizza if three pizzas were shared between four. She correctly stated the appropriate equivalent fractions and calculated that the whole was \$28 if \$4 represented one-seventh. However, she was unable to complete a backwards counting sequence for two-fifths and added numerators and denominators to solve the addition task of fractions with related denominators writing: $\frac{3}{10} + \frac{1}{5} = \frac{4}{15}$. Thus, Gloria was able to

successfully complete routine fraction tasks but demonstrated a common misconception (adding numerators and adding denominators) when adding two fractions.

Gloria had limited success with the number line tasks in Part B. She scored 2 marks out of a possible 10 marks (20%) where the mean was 6.21 and the standard deviation was 2.67 for the whole cohort. This means that her score for the number-line tasks was less than the mean score for the whole cohort. Gloria correctly marked where the number 75 would be on a number line marked zero to 25 and chose the correct position for the number one on a number line marked zero and $\frac{3}{4}$. However, she was unable to write the missing numbers for the four equally spaced marks on the number line marked zero to 100. She demonstrated a common misconception when she marked three-fifths of the whole number line marked zero to two, instead of marking the position of the number three-fifths. Gloria incorrectly placed the number one halfway between zero and one-third on a number line and was unable to respond to any question that asked her to explain how she decided her answer was correct.

Gloria had limited success with the conceptual fraction tasks from Part C that tested her understanding of fractions and their equivalent representations with decimals and percentages and included the three reverse fraction tasks. She scored a total of 5.7 for the Part C tasks out of a possible 13 marks (44%) which is slightly less than the mean of 6.07 for the whole cohort. Gloria successfully ordered the four fractions but was only able to match two fractions with their equivalent decimals. She successfully matched 0.1 and $\frac{1}{10}$, and 0.5 and $\frac{1}{2}$, but incorrectly matched 0.4 with $\frac{1}{4}$, and $\frac{2}{5}$ with 0.25 which are commonly held misconceptions. She circled the percentages for both tasks asking her to circle the symbol that did not belong to the group of four and consequently was correct with one choice and incorrect with the other. Gloria's responses suggest that she does not understand the structure or numeric representation of fractions.

7.2.2 Reverse Fraction Tasks

While Gloria gave the correct numerical solutions to Tasks C5, C6 and C7 her solution strategies were not explicitly stated nor completely clear. The written response shown in Figure 7.13 gives one example of this lack of clarity. When Gloria wrote: "I added 5 more circles" the assumption is that she had decided that five dots or circles represented one-third of the original quantity and used an additive strategy to mentally add on another five dots or circles to correctly determine the whole collection of 15. When the second part of the question asked how she decided that her answer was correct she wrote: "... then see if it split evenly and it did". Again, her thinking is not clear, but the assumption is that when she wrote 'split evenly' she meant that the total of 15 dots or circles could be split into three equal groups.

5.
This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.



15

a. How many counters did I start with?
b. Explain how you decided that your answer is correct.

I added 5 more circles and
then see if it split evenly/
and it did

Figure 7-13: Gloria's solution to Task C5 (FST)

In Gloria's solution to Task C6, shown in Figure 7.14, it can be assumed that she has found one-seventh of the whole quantity by finding one-quarter of 12 when she confirms her solution: "I knew that 3 goes into 12 four times". She then added on the three-sevenths, given by nine CDs, to the original four-sevenths (12 CDs) to get 21 CDs. Her strategy would be classified as partially multiplicative for this task.

6.
Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs.
How many CDs does Kay have? 21
Show all your working.

I knew that 3 goes into 12 4 times
and then I added another 9

Figure 7-14: Gloria's solution to Task C6 (FST)

Gloria's written response for Task C7 (FST), shown in Figure 7.15 demonstrates her lack of clarity about the strategy she used when she states: "... 7 can't go into six". She successfully calculated that one-seventh was represented by two counters which she subtracted from the 14 to get 12 as shown in her statement: "I just took away one-seventh of the dots".

7.
This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.



12

a. How many counters did I start with?
b. Explain how you decided that your answer is correct.

well 7 can't go into six so I
just took away 1 7th of
the dots

Figure 7-15: Gloria's solution to C7 (FST)

Gloria consistently used a partially multiplicative strategy for each of the three reverse fraction tasks so her dominant strategy for reverse fraction tasks would be classified as being partially multiplicative.

7.2.3 Structured Interview

Gloria used partially multiplicative strategies for the interview questions using the fraction two-thirds but relied on diagrams to solve the questions using the fraction four-sevenths. She gave an incorrect response for Question 3 which used the improper fraction of seven-sixths but used partially multiplicative strategies to solve subsequent questions using seven-sixths. During the Structured Interview Gloria transitioned from being dependent on diagrams to being able to answer the questions using a partially multiplicative strategy without drawing any diagrams.

In Question 1 where 12 counters represent the fraction two-thirds there is no accompanying diagram unlike Task C5 on the FST (Figure 7.13). Gloria used a partially multiplicative response as she stated: “Half of 12 = 6. Plus 6 to 12 = 18”. That is, she halved the number representing two-thirds (12) to find the number of counters representing one-third (6) and then added both numbers together to get three-thirds or the whole (18).

In Question 2 where 8 CDs represent four-sevenths of the group Gloria initially drew eight circles (left hand side of Figure 7-16) and tried to place them in seven groups. She then reread the question and drew the eight circles in four groups of two and circled each pair (left-hand side of Figure 7.16). She then added three more pairs of circles as shown on the right-hand side of Figure 7.16). While she did not write that the total was 14, she told the interviewer that the answer was 14 CDs.

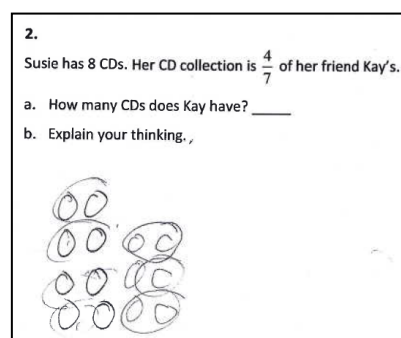


Figure 7-16: Gloria's response to Question 2 (SI)

In Question 3 where 21 counters represent seven-sixths of the group, Gloria was unable to give a correct response. As shown in Figure 7.17, she changed the improper fraction of seven-sixths to the mixed number $1\frac{1}{6}$, and asked: “Would I take away one-sixth?” She then drew 21 circles in

three rows of seven. She initially attempted to put them into six groups and remarked that she had seven groups of 3. She kept trying to divide the 21 into six groups and eventually tried groups of four. She then decided that there were six groups with $3\frac{1}{6}$ counters in each as shown in Figure 7.17.

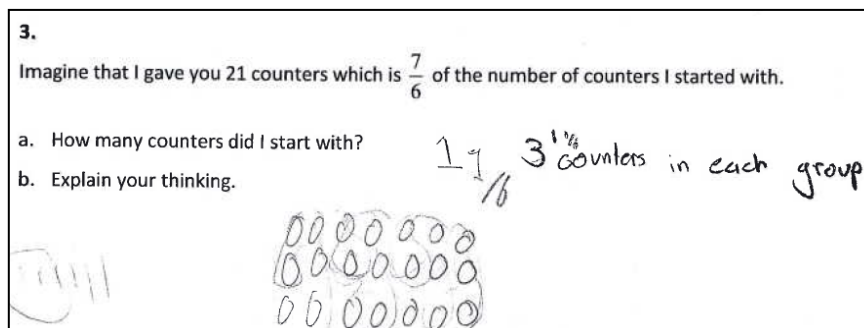


Figure 7-17: Gloria's response to Question 3 (SI)

Gloria was more confident working with the fraction two-thirds as shown in her response to Question 4: "Half of 18 is 9 so if I add the nine to 18 I get 27 counters". When asked what she would do if she was given 'any number of counters which was $\frac{2}{3}$ of the number', Gloria confidently responded: "You would halve the number and then add it to that number".

Gloria used a similar partially multiplicative strategy for Question 5 to the one she used for Question 2 (Figure 7.16). As shown in Figure 7.18, Gloria drew five rows of four circles, then drew vertical lines highlighting the columns of five to show one-seventh of the whole group. She then verbally stated that she would need to add on three more groups of five (15) to the original 20 to get 35 CDs. She explained to the interviewer, then wrote, that to calculate the number of CDs needed in the general case of any number of CDs: "whatever number that you have you have to put it into 4 groups and then add another 3 of the groups".

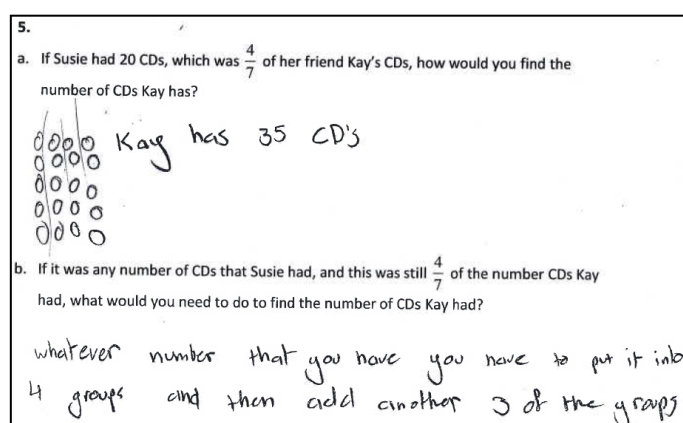


Figure 7-18: Gloria's response to Question 5 (SI)

Unlike her incorrect attempt to solve Question 3, Gloria's response to Question 6a, which also focuses on seven-sixths, was successful. As shown in Figure 7.19 she drew 7 circles to represent the 70 counters and stated that there would be seven groups with 10 counters in each group. To find the number of counters in the whole group she said that she would need to remove one group of ten to get the answer 60. For the general case of 'any number of counters' representing the fraction seven-sixths in Question 6b she stated: "Put it into 7 groups. However, many in that group take it away from the original number". While this demonstrates her use of the partially multiplicative strategy as she calculates one-sixth and subtracts that number of counters away from seven-sixths to find six-sixths she still appeared to rely on a diagram to assist in using this method.

6.

a. If I gave you 70 counters, which was $\frac{7}{6}$ of the number of counters I started with, how would you find the number of counters I started with?

put it into 7 groups and, ten in each group, take 10
 ○○○○○○○○
 away from 70, it gave me 60

b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?

put it into 7 groups however many in that group
 take it away from the original number

Figure 7-19: Gloria's response to Question 6 (SI)

Gloria's response for Question 7, shown in Figure 7.20, involving 'any fraction' with 'any number of counters representing that fraction' showed that she used the same partially multiplicative strategy she had used for the previous tasks. "Whatever the numerator is put it into however many groups you then either add or subtract that number".

7.

Think about the tasks you have just done.

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?

a. Can you tell me what you would do?

b. Please write your explanation in your own words.

~~however~~ whatever the numerator is put it into however many groups you then either add or subtract that number.

Figure 7-20: Gloria's response to Question 7 (SI)

Gloria initially used diagram dependent strategies indicating a clearly established pattern of representing a whole as a composite of its fractional parts. The underlying conception is that of part-part-whole. Diagrams, often with circling, are used to identify, usually successfully, the component relationships; recognising that it is necessary to deduce the value of the unit fraction, in order to scale up (or down) the number of fractional parts to make a whole. This is most clearly illustrated in her response to Question 5 (Figure 7.18) where she created four groups (each representing one-seventh) and then added three more groups to give a correct answer of 35 CDs. These extra groups are not specifically shown but indicated by the vertical lines on the diagram. Apart from the first multiplicative step to create one-seventh, all other operations are performed additively. When presented with a known fraction representing ‘any number’ this student explained how the separate parts or components can be combined to make a whole. But in Question 7 when presented with ‘any fraction’, Gloria’s clearly expressed part-part-whole conceptions cannot be generalised: “Whatever the numerator is, put it into however many groups. You then either add or subtract that number”.

Gloria's strategies changed depending on the fraction used in the task. She used partially multiplicative strategies for the fraction two-thirds, diagrams for the fraction four-sevenths and successfully used a partially multiplicative strategy for the final seven-sixths task before suggesting the same strategy for the final task with 'any fraction' and 'any number'. The overall analysis of her responses to the Structured Interview questions suggest she is at the *Generalising-Additive* level of the *Emerging Algebraic Reasoning Framework* as described in Table 6.12.

Gloria’s reliance on a part-part-whole strategy gave her a clear advantage over Violet who needed diagrams to complete the calculations in the Structured Interview questions and was deemed to be at the *Computational Fluency-Partial* level of the *Emerging Algebraic Reasoning Framework*.

7.2.4 Algebraic Thinking Questionnaire

In the Algebraic Thinking Questionnaire, Gloria struggled with multiplicative relationships. She scored 0.5 marks (4%) for Part M, which is well below the mean score (4.20) for the whole cohort. and zero marks for Part D. Overall Gloria scored 0.5 marks out of a possible 28 marks (2%) for the Algebraic Thinking Questionnaire which is below the mean score for the whole cohort.

Gloria attempted the first two tasks in Question 1 of Part M. Her response to Question 1a is shown in Figure 7.21. Her initial computational error occurred when she multiplied 25 by six and incorrectly wrote 180 instead of the correct response (150) as shown in the first line of the

multiplication in the right-hand column. Her second error is shown in the left-hand column where she equated 9×95 to 930. There is no working out to show how she decided on her answer.

$36 \times 25 = 9 \times \boxed{95}$ 930	$\begin{array}{r} 1 \\ 3 \overline{) 25} \\ \underline{\times 36} \\ 180 \\ 750 \\ \hline 930 \end{array}$
--	--

Figure 7-21: Gloria's response to Task M1a (ATQ)

Despite correctly completing the multiplication of 48×2.5 in Question 1b Gloria was unable to divide 120 by 10 writing instead the incorrect response, 20, as shown in the left-hand side of Figure 7.22. As there is no other written detail it is difficult to know how she decided on this response.

$48 \times 2.5 = \boxed{20} \times 10$ 120.0	$\begin{array}{r} 4 \\ 48 \\ \times 2.5 \\ \hline 240 \\ 760 \\ \hline 1200 \end{array}$
--	--

Figure 7-22: Gloria's incorrect response for Question 1b, Part M (ATQ)

Gloria did not attempt Question 1c but her solution to Question 1d demonstrates her additive thinking when she incorrectly wrote $\frac{3}{5}$ in the empty box in Figure 7.23. This response would have been correct if the two fractions were being added but is incorrect as the two fractions are supposed to be multiplied. This was a common incorrect response for this question.

$$\frac{2}{5} \times \frac{\boxed{3}}{\boxed{5}} = 1$$

Figure 7-23: Gloria's incorrect additive response for Task M1d (ATQ)

Gloria successfully completed Question 2a (Part M). She gave two correct responses for the missing boxes as shown in Figure 7.24. This contrasts with her inability to correctly complete any

equation with missing numbers in Question 1, Part M (Figures 7.21, 7.22 and 7.23). However, the numbers were much larger in Question 1 and the calculations involved more steps.

$5 \times \boxed{10} = 10 \times \boxed{5}$ Box A Box B	$5 \times \boxed{20} = 10 \times \boxed{10}$ Box A Box B
--	---

Figure 7-24: Gloria's response to Task M2a (ATQ)

When asked to express the relationship between the numbers in Box A and Box B in Question 2b (Figure 7.25), Gloria stated that the number must be halved. However, she did not indicate whether it is the number in Box A, or the number in Box B, that needed to be halved.

b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

The number must be halved

c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

Not sure

Figure 7-25: Gloria's response to Tasks M2b and M2c (ATQ)

Gloria gave incorrect calculations for Questions 1a and 1b (Part D). In the left-hand column of Figure 7.26 she has had more than one attempt at the solution before dividing four by three. She used the same incorrect method for Question 1b shown in the right-hand column.

$3 \div 4 = 15 \div \boxed{16}$ $\begin{array}{r} 3 \overline{) 12} \\ \underline{9} \\ 3 \end{array}$ $\begin{array}{r} 15 \overline{) 48} \\ \underline{30} \\ 18 \end{array}$	$\boxed{1} \div 15 = 16 \div \boxed{30}$ $\begin{array}{r} 1 \overline{) 15} \\ \underline{15} \\ 0 \end{array}$ $\begin{array}{r} 16 \overline{) 480} \\ \underline{320} \\ 160 \end{array}$
---	--

Figure 7-26: Gloria's response to Tasks D1a and D1b (ATQ)

Gloria wrote an incorrect solution to Question 1d (Part D) as shown in Figure 7.27 and there are no written calculations to indicate her thinking or how she decided on her response.

$\frac{7}{6} \div \frac{\boxed{3}}{\boxed{6}} = 1$

Figure 7-27: Gloria's response to Task D1d (ATQ)

For Question 2a (Part D) Gloria only attempted one task and her response was incorrect. Like many other students in this study, Gloria appears to be applying the rule of commutativity inappropriately for division. As shown in the left-hand column of Figure 7.28 she stated that $3 \div 15 = 15 \div 3$ and suggested that the “relationship is they have a factor of five” (right-hand column). As for the related multiplication task, she did not specifically state the relationship.

$3 \div \boxed{15} = 15 \div \boxed{3}$ Box A Box B	<i>the relationship is they have a factor of 5</i>
---	---

Figure 7-28: Gloria’s response to Tasks D2a and D2b (ATQ)

As for the similar multiplication task, Gloria was unsure whether any number could be placed in Box A and still make a correct number sentence.

Gloria’s partially multiplicative approaches to the reverse fraction tasks from the FST, and her *Generalising-Additive* classification for the Structured Interview, need to be viewed against her very limited multiplicative thinking for the Algebraic Thinking Questionnaire tasks. Gloria has many misconceptions both with fractions and division. Some algebraic thinking may be evident, but it appears highly unlikely that Gloria would be able to use algebraic thinking or reasoning in a range of different contexts until she overcomes her misconceptions.

7.3 Case Study: Kate

Kate (Year 5) was chosen for a case study as she is an example of the *Algebraic Generalisation-Verbal* group at Level 5 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. Kate was the only student deemed to be at Level 5 despite using partially multiplicative strategies for all tasks in the Structured Interview except the final task. For the final task she realised that her partially multiplicative strategy would not work so she used a verbal multiplicative method similar to the method she had used previously for the reverse fraction tasks from the FST (see Figure 7.31). She was chosen to be interviewed as she had used a variety of strategies, including additive, multiplicative and ‘guess and check’ and successfully solved all three reverse fraction tasks (Figure 7.29). Kate’s score for the FST is above the mean score and her ATQ score below the mean score for the whole cohort (see Table 7.2).

Section 7.3.1 will discuss her responses to the FST, Section 7.3.2 the strategies she used for the reverse fraction tasks (Tasks C5 - C7), Section 7.3.3 the strategies she used for the Structured Interview and Section 7.3.4 her responses to the ATQ.

7.3.1 Fraction Screening Test

In this section Kate's solutions to the tasks from the three parts of the FST will be discussed. Overall, she scored 26.5 out of a possible 35 marks (76%) for the FST where the mean was 21.91 for the whole cohort.

Kate's score for Part A tasks was 10.5 out of a possible 12 marks (88%) which is above the mean for the whole cohort. She chose the two correct shapes representing three-quarters, circled one-third of the group of beetles, and correctly stated there would be eight lollies in the whole group if four lollies represented one-half of the group. Kate accurately marked and shaded three-quarters of a length and shaded one-third of a rectangle. She correctly stated the appropriate equivalent fractions and calculated the whole if \$4 was one-seventh but did not include the dollar sign in her response. Kate calculated that each person would receive three-quarters of a pizza if three pizzas were shared between four people and successfully completed the backwards counting sequence for the fraction two-fifths. Like Gloria she had the common misconception when adding fractions and added the numerators and denominators separately: $\frac{3}{10} + \frac{1}{5} = \frac{4}{15}$

Kate was very successful with the number-line tasks in Part B. Kate scored 9 marks out of a possible 10 marks (90%) which is more than the mean for all students. She successfully wrote the four missing numbers on the equally spaced marks on the number line marked zero to 100 stating: "Because there are 5 marks from zero and the last is 100, so all I had to do was find 4 numbers that fit evenly: 20, 40, 60 and 80 all fit". Kate correctly marked where the number 75 would be on a number line marked zero and 25 using the length from 0 to 25 to mark the position for 50 and then the correct position for 75, hence demonstrating that she could iterate the length from 0 to 25 two more times. However, her explanation did not match her written estimation for the number 75. Her explanation suggested that she assumed the whole line was 100: "Because it is (about) $\frac{3}{4}$ of the way along the line. 75 is $\frac{3}{4}$ of the number 100". Kate accurately marked the number three-fifths on the number line marked zero to two. She iterated two more times using the distance from zero to one-third to mark the position of the number two-thirds and then again to position the number one. Kate chose the correct option for the placement of one on a number line marked zero and $\frac{3}{4}$ and stated that the two other options: "were too close to the $\frac{3}{4}$ mark to be correct".

For Part C Kate successfully matched all four fractions with their equivalent decimals. She correctly identified that 0.4 was not equivalent to the various representations of one-quarter but incorrectly chose $1\frac{6}{30}$ not recognising that $\frac{6}{5}$ and $1\frac{6}{30}$ are equivalent. Kate appears to have ordered the four fractions using the 'gap' method where she has calculated the difference between

each fraction's numerator and denominator: $\frac{1}{10}$ $\frac{1}{4}$ $\frac{3}{5}$ $\frac{1}{2}$. If she has used the gap method, she would consider one-tenth as the smallest fraction as it has the 'largest gap' and one-half as the largest fraction as it has the 'smallest gap'. From the FST tasks discussed previously there is clear evidence that Kate is able to use multiplicative thinking, but she does not do so consistently.

7.3.2 Reverse Fraction Tasks

Kate used three different methods to correctly solve the three reverse fraction tasks. For Task C5 Kate used a partially multiplicative method to justify her answer of 15: "Because half of the number there is 5, so I added five to that number and got my answer" (left-hand side of Figure 7.29). For Task C6 Kate used a fully multiplicative method where she divided the number of CDs (12) by the numerator (4) to find one-seventh of the whole (3) and then multiplied by seven to get the whole i.e. seven-sevenths (21) as shown in the right-hand side of Figure 7.29.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? 15 b. Explain how you decided that your answer is correct. because half of the number there is 5, so I added five to that number and got my answer	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs How many CDs does Kay have? <u>21</u> Show all your working. $12 \div 4 = 3$ $3 \times 7 = 21$ <u>21</u>
--	---

Figure 7-29: Kate's responses to the Tasks C5 and C6 (FST)

Despite getting the correct answer for Task C7 Kate's explanation demonstrates her 'trial and error' method: "Because 14 divided by 6 doesn't work, neither does $13 \div 6$. But $12 \div 6$ does work so I got the answer of 12 as the original number of counters" (Figure 7.30).

7.
This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.



a. How many counters did I start with? 12
b. Explain how you decided that your answer is correct.

Because 14 divided by 6 doesn't work, neither does $13 \div 6$. But $12 \div 6$ does work so I got the answer of 12 as the original number of counters

Figure 7-30: Kate's responses to Task C7 (FST)

7.3.3 Structured Interview

Kate used a partially multiplicative strategy for Questions 1 – 6b of the Structured Interview questions. For the tasks with a focus on the fraction two-thirds, Kate halved the number of counters representing two-thirds and then added that number of counters onto the original number of counters to find the whole collection. For example, in Question 1 she stated: “Because half of 12 is six, so I added 6 to twelve”. Similarly, in Question 4b, where she was asked about ‘any number of counters’ she responded: “Divide the counters by 2, then add the answer on the original number of counters”.

For the tasks with a focus on the fraction four-sevenths, Kate calculated the number of objects needed to represent one-seventh, multiplied by three to calculate the three-sevenths, then added that number to the number representing four-sevenths, to calculate the number of objects needed to represent seven-sevenths or the whole. This partially multiplicative approach contrasts with the fully multiplicative approach shown in the right-hand column of Figure 7.29.

Kate also used the partially multiplicative method to solve the Structured Interview tasks with a focus on seven-sixths. She initially found the number of objects that represented one-sixth then subtracted that number of objects from the number representing seven-sixths to find the whole or six-sixths. In Question 6a shown in the top row of Figure 7.31, Kate calculated that ten counters represented one-sixth by dividing 70 by seven and then subtracted that number from 70 to get 60 which is six-sixths or one-whole. Kate’s response to Question 6b, in the bottom row of Figure 7.31 demonstrates her partially multiplicative strategy where she describes her method of finding one-sixth as “see what goes into that number 7 times” and then subtract one-sixth from the seven-sixths to get six-sixths or one whole.

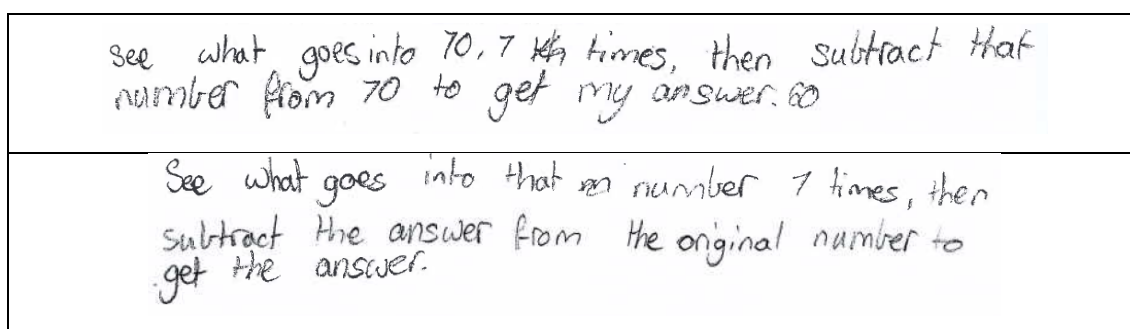
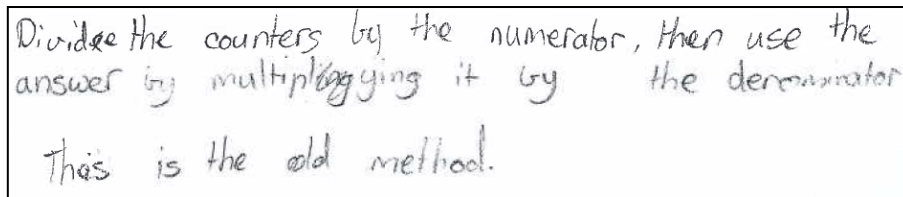


Figure 7-31: Kate’s responses to Questions 6a and 6b (SI)

However, Kate adjusted the partially multiplicative strategy that she used for all six of the previous questions to a fully multiplicative strategy for Question 7 (Figure 7.32). She states that:

“This is the old method” which is the fully multiplicative method she had used previously for Task C6 (FST).



Divide the counters by the numerator, then use the answer by multiplying it by the denominator.
This is the old method.

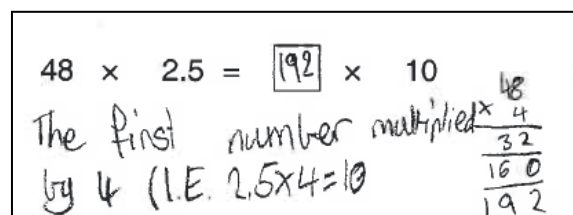
Figure 7-32: Kate's response to Question 7 (SI)

This apparent change of thinking demonstrates that Kate can transition to a fully multiplicative strategy when it is clear to her that a partially multiplicative approach is not applicable and will not work. As she was able to respond to the question with ‘any fraction’ and ‘any quantity’ Kate is classified as being *Algebraic Generalisation – Verbal*. This demonstrates that students can be encouraged to use algebraic reasoning if the task demands it.

7.3.4 Algebraic Thinking Questionnaire

In the Algebraic Thinking Questionnaire, Kate struggled to solve multiplicative tasks as she only attempted two of the Part M tasks and three of the Part D tasks. She was unsuccessful with the multiplication tasks but more successful with the division tasks on the ATQ. She scored 0 marks for Part M, 5 marks for Part D with a total score of 5 for the ATQ. This score placed her below the mean for the whole cohort.

Kate unsuccessfully attempted two multiplication tasks in Part M (ATQ). In Question 1a she calculated the relationship between 36 and nine as being four but did not complete the task. In Question 1b (Figure 7.33) Kate correctly identified that ten was four times 2.5 but instead of dividing 48 by four she multiplied by four.



$48 \times 2.5 = 192 \times 10$
The first number multiplied by 4 (i.e. $2.5 \times 4 = 10$)

$$\begin{array}{r} 48 \\ \times 4 \\ \hline 32 \\ 160 \\ \hline 192 \end{array}$$

Figure 7-33: Kate's response to Question M1b (ATQ)

Kate successfully solved the first three division tasks in Question 1 Part D. All three responses demonstrate her ability to reason multiplicatively and relationally for division tasks. For example, her correct answer for Question 1a shows that she has recognised the relationship between 3 and 15: “It’s 5 times the number that comes first (i.e. $3 \times 5 = 15$)”. In this case she has used the word

‘first’ to indicate the number three in the same relative position on the left-hand side of the expression to the 15 on the right-hand side of the expression. She used this relationship successfully to determine that the missing number was five times four (20), the four being on the left-hand side of the expression (see Figure 7.34)

$3 \div 4 = 15 \div \boxed{20}$
 It's 5 times the number that comes first (I.E. $3 \times 5 = 15$)

Figure 7-34: Kate's response to Task D1a (ATQ)

Kate's response to Question 1b, shown in Figure 7.35, also demonstrates her understanding of the relationship between 15 and 30 where the second number on the left-hand side of the equation (15) is half the second number on the right-hand side of the equation (30). As evidence for the relationship of one-half between the two numbers she used the subtraction fact: $30 - 15 = 15$.

$\boxed{8} \div 15 = 16 \div 30$
 Half the second number is the first.
 (I.E. ~~16/18~~ $30 - 15 = 15$)

Figure 7-35: Kate's response to Task D1b (ATQ)

Kate was one of the few students to correctly respond to Question 1c. She determined the relationship between 18 and 6 as shown in her statement: “Multiply the second number by 3 to get the first number (i.e. $6 \times 3 = 18$)” and successfully used this relationship to multiply two-thirds by three to get the correct response of two (Figure 7.36).

$18 \div \boxed{12} = 6 \div \frac{2}{3}$
 Multiply the second number by 3 to get the first number
 (I.E. $6 \times 3 = 18$)

Figure 7-36: Kate's response to Task D1c (ATQ)

Kate did not attempt any of the Box A and Box B questions for either Part M or Part D. While it is possible to be more confident about Kate's multiplicative thinking, there is limited evidence of her capacity for algebraic thinking in the Algebraic Thinking Questionnaire.

7.4 Case Study 4: Daniel

Daniel (Year 8) was chosen for a case study as he is an example of the *Computational Fluency - Complete* group at Level 2 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. Daniel used partially multiplicative strategies for the first three tasks, the fully multiplicative strategies for the fraction two-thirds with both a change of number and 'any number' of objects but was unable to complete similar tasks for seven-sixths nor the final task with 'any fraction' and 'any number', as he had successfully solved all three reverse fraction tasks using a variety of solution strategies, including additive, partially multiplicative and multiplicative methods respectively as shown in Figures 7.38 and 7.39. Daniel's scores for the FST and the ATQ are less than the mean scores for the whole cohort (see Table 7.2).

Section 7.4.1 will discuss his responses to the FST, Section 7.4.2 the strategies he used for the reverse fraction tasks (Tasks C5 - C7), Section 7.4.3 the strategies he used for the Structured Interview and Section 7.4.4 his responses to the ATQ.

7.4.1 Fraction Screening Test

In this section Daniel's solutions to the tasks from the three parts of the FST will be discussed. Overall, he scored 18 out of a possible 35 marks (51%) for the FST which is slightly less than the mean for the whole cohort.

Daniel's score for Part A tasks was 9 out of a possible 12 marks (75%) where the mean was 9.63 and the standard deviation was 2.22. He chose the two correct shapes representing three-quarters and indicated that there would be eight lollies in the whole group when four lollies represented one-half of the group. He correctly marked, and shaded, three-quarters of a length and calculated that each person would receive three-quarters of a pizza if three pizzas were shared between four people and successfully completed the counting sequence. However, while Daniel accurately shaded one-third of a rectangle he circled three beetles rather than one-third of the group of six beetles. He correctly stated that one equivalent fraction for $\frac{8}{10}$ is $\frac{4}{5}$ but incorrectly stated that an equivalent fraction for $\frac{7}{9}$ is $\frac{25}{27}$. Like Gloria and Kate, Daniel also incorrectly added the numerators and denominators separately when adding three-tenths and one-fifth (left-hand column of Figure 7.37). Daniel correctly calculated that the whole was \$28 if \$4 was one-seventh but counted by fours to determine the answer was 28 as shown in the right-hand column of Figure 7.37. Daniel was reasonably successful with the routine type fraction tasks but displayed the identical misconception as Gloria and Kate when adding two fractions.

11. Complete $\frac{3}{10} + \frac{1}{5} = \frac{\boxed{4}}{\boxed{5}}$	12. To buy a new workbook I spent \$4. This is $\frac{1}{7}$ of what I saved up. How much did I save up? <u>\$28</u>
--	---

Figure 7-37: Daniel's responses to Tasks A11 and A12 (FST)

Daniel had limited success with the number-line tasks. He scored 5 marks out of a possible 10 marks (50%) for the Part B tasks where the mean was 6.21 and the standard deviation was 2.67. He successfully wrote the missing numbers on the equally spaced marks on the number line marked zero to 100 stating: "Because they fit in those spaces". Daniel incorrectly marked where the number 75 would be on a number line marked zero and 25. His explanation suggested that he assumed the whole line was 100: "because it is the same distance from 100 as 25 is from 0". Daniel did not attempt any of the other number line tasks.

Daniel did not attempt the Part C tasks that included ordering a set of four fractions from largest to smallest, drawing a line to connect fractions and their equivalent decimals and circling the 'one that does not belong' from the set of fractions, decimals and percentages but attempted the three reverse fraction tasks: Tasks C5, C6, and C7.

7.4.2 Reverse Fraction Tasks

Daniel's solution methods for the three reverse fraction tasks appeared to be mainly additive (Figures 7.38 & 7.39). The language he has used for Task C5 is unusual when he states there are "10 more, 2 of the 3" (left-hand side of Figure 7.38). It is possible he was adding 10 more counters, which represents two-thirds, onto the five counters, that represent one-third of the original number of counters to calculate his correct total of 15 counters.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? <u>15</u> b. Explain how you decided that your answer is correct. <i>because there are 10 more 2 of the 3</i>	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. <i>12 ÷ 4 = 3</i> <i>12 15 18 21 24</i>
---	---

Figure 7-38: Daniel's responses to Task C5 and Task C6 (FST)

In response to Task C6 (right-hand side of Figure 7.38) Daniel divided the given number of counters (12) by the numerator (4) to determine that three counters represented one-seventh of the original group. He then counted by threes starting at 12 and correctly wrote the answer of 21. While there is evidence here of some initial multiplicative thinking to determine one-seventh there is a concern that he needs to use the additive strategy of counting by threes to determine 7×3 in Task C6 (right-hand side of Figure 7.38).

For Task C7 Daniel determined that one-seventh of 14 was two and gave the correct response of 12 but there is no explanation of how he worked out the correct solution (Figure 7.39).

7.
This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.

(14 counters arranged in two rows of seven)

a. How many counters did I start with? 12
b. Explain how you decided that your answer is correct.
because 14 gives us 2 for each 1 of 7

Figure 7-39: Daniel's responses to Task C7 (FST)

There is some evidence of multiplicative thinking, but Daniel also relies on additive methods to establish answers to the fraction tasks as shown in the middle column of Figure 7.38.

7.4.3 Structured Interview

Daniel used a variety of strategies for the first five questions of the Structured Interview moving from partially multiplicative for the first three questions to fully multiplicative strategies, but the interview was discontinued by the interviewer after Question 5 because he was becoming stressed.

Daniel correctly answered Question 1 using a partially multiplicative strategy similar to the method he used for Task C5. That is, he halved the number of counters representing two-thirds to calculate the number of counters that would represent one-third. He then added the number of counters representing one-third (6) to the number of counters representing two-thirds (12) to get three-thirds or one whole (18). In his written response shown in the top row of Figure 7.40 he describes his method of finding one-third of the group but did not clearly explain his method of calculating three thirds instead using the term "hence 18".

The bottom row of Figure 7.40 shows the steps Daniel took to solve Question 2. He divided the number of CDs that represented four-sevenths (8) by four to calculate one-seventh of the number

of CDs. He then subtracted four (sevenths) from seven (sevenths) to calculate that he needed three-sevenths to make the required seven-sevenths or the whole. He then multiplied two by three, where the two represents the number of CDs that represent one-seventh of the original number and the three is three-sevenths, to calculate that six counters represent the fraction three-sevenths. Finally, he added the number of CDs representing three-sevenths and four-sevenths together to decide that Kay had 14 CDs.

1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? 18 b. Explain your thinking. because half of 12 is 6 this means that $\frac{1}{3}$ is 6 - hence 18
2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. a. How many CDs does Kay have? 14 b. Explain your thinking. $8 \div 4 = 2$ $7 - 4 = 3$ $3 \times 2 = 6$ $6 + 8 = 14$

Figure 7-40: Daniel's response to Questions 1 and 2 (SI)

As shown in Figure 7.41 Daniel used a partially multiplicative method to calculate the number of counters in the whole group. He divided the 21 counters by seven to calculate one-sixth of the collection (3) and then subtracted the number representing one-sixth (3) from the original number to determine six-sixths.

3. Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with. a. How many counters did I start with? 18 b. Explain your thinking. $21 \div 7 = 3$ $21 - 3 = 18$
--

Figure 7-41: Daniel's response to Question 3 (SI)

Daniel used a fully multiplicative strategy to solve Question 4a as shown in the left-hand column of Figure 7.42. He divided the number of counters representing two-thirds (18) by two to find one-third (9) and then multiplied the number representing one-third (9) by three to correctly calculate three-thirds (27). Daniel used an advanced multiplicative strategy shown in the recording of the response to Question 4b (right-hand column of Figure 7.42). He defined x as the

‘mystery number’ and a as the answer. His interviewer wrote that when solving this task Daniel was “very reluctant to put anything into words, withdrew slightly in body language”.

4. a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with? $18 \div 2 = 9$ $9 \times 3 = 27$	b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with? $(x \div 2) \times 3 = a$ $x = \text{mystery number}$ $a = \text{answer}$
--	---

Figure 7-42: Daniel’s response to Question 4 (SI)

Daniel’s response to Structured Interview Question 5a was identical to his response to Question 2 where he found the number of counters that represented the one-seventh, calculated the number of counters needed for three-sevenths, and added this to the number given for four-sevenths to calculate seven-sevenths (left-hand side of Figure 7.43).

Daniel thought for a long time about Question 5b and said he could not say the answer in words but wrote the response algebraically in terms of a and y which he defined as shown in the right-hand column of Figure 7.43. When asked about the numbers four and seven he was happy to explain and define them as the numerator and denominator respectively.

5. a. If Susie had 20 CDs, which was $\frac{4}{7}$ of her friend Kay’s CDs, how would you find the number of CDs Kay has? $20 \div 4 = 5 \quad 7 - 4 = 3 \quad 3 \times 5 = 15$ $20 + 15 = 35$	b. If it was any number of CDs that Susie had, and this was still $\frac{4}{7}$ of the number CDs Kay had, what would you need to do to find the number of CDs Kay had? $(y \div 4) \times 7 = a$ $y = \text{unknown number}$ $a = \text{answer}$ 4 is the numerator 7 is the denominator
---	---

Figure 7-43: Daniel's response to Questions 5a and 5b (SI)

The interviewer stopped the interview at this point: “In anticipation of his reluctance to use words increasing with the progression of difficulty of questions, I decided to avoid any confrontation and stop the interview”.

7.4.4 Algebraic Thinking Questionnaire

In the ATQ, Daniel attempted most questions with varying degrees of success. He scored 6 marks out of a possible 14 marks (43%) for Part M but zero marks for Part D. This means that Daniel scored a total of 6 out of a possible 28 marks (21%) for the Algebraic Thinking Questionnaire.

Daniel successfully solved Task M1a by doubling 25 and halving 36 as shown in the left-hand column of Figure 7.44. To complete the calculation, he multiplied 18 by 50 by splitting 18 into

$10 + 8$ and multiplying each part separately before adding the results i.e. $10 \times 5 = 50$ and $8 \times 50 = 400$ and $50 + 400 = 450$. He then divided 450 by nine, which enabled him to correctly write 100 in the empty box. Daniel's answer to Task M1b is correct but it is difficult to interpret how he got this correct response from his written calculations (right-hand column of Figure 7.44).

$36 \times 25 = 9 \times \boxed{100}$ 18×50 $\underline{500} \quad 400 \quad 400$	$48 \times 2.5 = \boxed{12} \times 10$ $24 \times 2.5 = 60$ $12 \times 2 = 24 + 6 = 30$
--	---

Figure 7-44: Daniel's response to Tasks M1a and M1b (ATQ)

As shown in the left-hand column of Figure 7.45, Daniel calculated two-thirds of 18 for Task M1c, rather than finding the number of objects in the whole group when 18 represents two-thirds. To calculate two-thirds of 18 he wrote down the counting sequence for three starting at three and stopping at 18. Underneath the counting sequence by three he wrote down the number two six times (see bottom of left-hand column of Figure 7.45) and then counted by twos to get the answer 12. It is unclear why he has written the decimal 0.75.

Daniel was one of the few students to successfully complete Task M1d, but his response is brief as shown in the right-hand column of Figure 7.45 and thus it is not clear how he correctly decided the answer was $\frac{5}{2}$. Again, it is interesting to note that he has added two zeros to the five.

$\frac{2}{3} \times \boxed{12} = 18$ 0.75 $\frac{3}{2} \quad \frac{6}{2} \quad \frac{9}{2} \quad \frac{12}{2} \quad \frac{15}{2} \quad \frac{18}{2}$	$\frac{2}{5.00} \times \frac{\boxed{5}}{\boxed{2}} = 1$
--	---

Figure 7-45: Daniel's response to Tasks M1c and M1d (ATQ)

In Question 2a Daniel was only able to write one set of numbers as shown in Figure 7.46 (left-hand side) and stated that one number was one-half of the other without specifying whether it was Box A that was one-half of Box B or vice versa (right-hand side of Figure 7.46).

$5 \times \boxed{10} = 10 \times \boxed{5}$ Box A Box B	b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? <i>one is half of the other</i>
---	--

Figure 7-46: Daniel's responses to Tasks M2a and M2b (ATQ)

Daniel obviously thought that only whole numbers could be placed in the empty boxes as shown in the example he used in his response to Task M2c in Figure 7.47. In the example in Figure 7.47 he appears to have swapped the numbers he used in Figure 7.46.

c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.	<i>5 x 5 = 25 then you can't go 10 x 10 = 25</i> <i>So no</i>
---	---

Figure 7-47: Daniel's response to Task M2c (ATQ)

In Task M2d Daniel gave only one specific example for the missing numbers, c and d , where he suggested that c could be seven and d could be one (top row of Figure 7.48). However, he was unable to give an appropriate response to Question 2e as he did not believe that "the letters a and b are related" (bottom row of Figure 7.48).

d) What can you say about c and d in this mathematical sentence? $c \times 2 = d \times 14$ <i>it could be 7x2 = 1x14</i>	e) What can you say about a and b in this mathematical sentence? $a \times \frac{3}{4} = b \times 1\frac{1}{2}$ <i>I don't believe they're related</i>
---	--

Figure 7-48: Daniel's responses to Tasks M2d and M2e (ATQ)

Despite attempting all four tasks in Question 1 of Part D, Daniel was not successful with any of them. No working was provided so it was difficult to know what his thinking entailed and how his answers were calculated. Figure 7.49 shows his incorrect responses for Task D1a and Task D1b.

$3 \div 4 = 15 \div \boxed{16}$	$\boxed{16} \div 15 = 16 \div 30$
---------------------------------	-----------------------------------

Figure 7-49: Daniel's response to Tasks D1a and D1b (ATQ)

Figure 7.50 shows Daniel's incorrect responses to Tasks D1c and D1d. As there is no working out it is difficult to determine how he decided on his responses.

$18 \div \boxed{6} = 6 \div \frac{2}{3}$	$\frac{7}{6} \div \frac{\boxed{6}}{\boxed{5}} = 1$
--	--

Figure 7-50: Daniel's responses to Tasks D1c and D1d (ATQ)

In Figure 7.49, Daniel's response shows his misunderstanding about commutativity and division. As for his response for Task M2b he recognised that there is a relationship between the numbers needed for Box A and Box B but has not been specific: "Multiply one by five and you get the other". He did not consider that the general case where numbers could be placed in the two boxes but focused on the need to maintain the relationship.

$3 \div \boxed{15} = 15 \div \boxed{3}$ Box A Box B	(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? x one by 5 you get the other
---	--

Figure 7-51: Daniel's response to Tasks D2a and D2b (ATQ)

Daniel was unable to state the relationship, or give any specific values, for the letters in Tasks D2d and D2e as he did not think that either pair of numbers were related.

Although Daniel did not complete the interview and despite his increasing reluctance to articulate his reasoning he was obviously in transition from the *Computational Fluency-Complete* level to the generalising level the *Emerging Algebraic Reasoning Framework* as indicated by his responses to Questions 4 and 5.

7.5 Case Study 5: Andrew

Andrew (Year 8) was chosen for a case study as he is an example of the *Generalising-Multiplicative* group at Level 4 of the *Emerging Algebraic Reasoning Framework* shown in Table 210

6.12. After using a partially multiplicative strategy for the first task of the Structured Interview he successfully used a fully multiplicative strategy for the remaining interview tasks. He was unable to answer the final question with 'any fraction' and 'any number' of objects. He was chosen to be interviewed as he had successfully used a partially multiplicative strategy for Task C5 and fully multiplicative strategies for Tasks C6 and C7. Andrew's scores for the FST and ATQ are more than the mean score for the cohort (see Table 7.2).

Section 7.5.1 will discuss his responses to the FST, Section 7.5.2 the strategies he used for the reverse fraction tasks (Tasks C5 - C7), Section 7.5.3 the strategies he used for the Structured Interview and Section 7.5.4 his responses to the ATQ.

7.5.1 Fraction Screening Test

In this section Andrew's solutions to the tasks from the three parts of the FST will be discussed. Overall, he scored 27 out of a possible 35 marks for the FST where the mean was 21.91 and the standard deviation was 7.11 for the whole cohort.

Andrew's score for Part A tasks was 9 out of a possible 12 marks (75%) where the mean was 9.63 and the standard deviation was 2.22. He chose the two correct shapes representing three-quarters, circled one-third of a group of beetles, and indicated the number in the whole group when four lollies represented one-half of the group. Andrew accurately shaded one-third of a rectangle and three-quarters of a length. He correctly stated the equivalent fractions, calculated the whole if \$4 was one-seventh and gave the next number to complete the counting backwards sequence. He incorrectly stated that each person would receive two-thirds of a pizza if three pizzas were shared between four people. Andrew, like Gloria, Kate and Daniel, incorrectly added the numerators and denominators separately when adding three-tenths and one-fifth.

Andrew scored 9 marks out of a possible 10 marks (90%) for the Part B tasks where the mean was 6.21 and the Standard Deviation was 2.67. He successfully wrote the missing numbers on the number line marked zero to 100 and placed the numbers 75 and 5 on a number line marked zero and 25 and justified his response: "I measured with my fingers to make every space the same amount." As shown in Figure 7.52 Andrew struggled to accurately mark three-fifths of the number line marked zero to two having many attempts before marking three-fifths of the entire line rather than the number three-fifths.

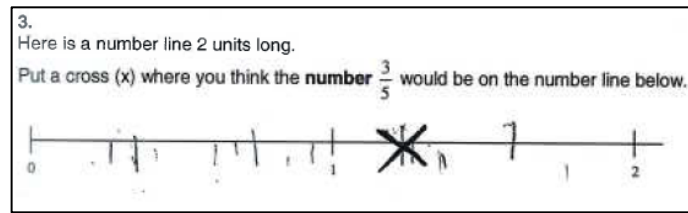


Figure 7-52: Andrew's response to Task B3 (FST)

Andrew accurately marked where the number one would be given the distance from zero to one-third but there are no interim marks. He also chose the correct option for the placement of one on a number line marked zero and $\frac{3}{4}$ as he marked in the $\frac{1}{4}$ and $\frac{2}{4}$. In his explanation about how he decided the other two choices were incorrect he said: "because they are too close to the $\frac{3}{4}$ mark".

Andrew scored 9 marks out of a possible 13 for the Part C tasks which is more than the mean for the whole cohort. Andrew incorrectly ordered the four fractions by the gap between the numerator and the denominator as shown in Figure 7.53. As 'gap thinkers' usually believe that the smaller the gap the larger the fraction he also appears to have ordered from smallest to largest.

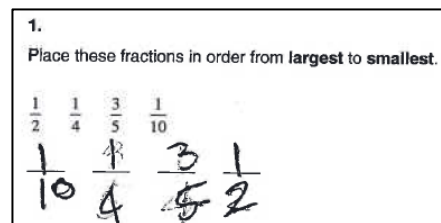


Figure 7-53: Andrew's 'gap thinking' response to Task C1 (FST)

Andrew correctly matched all four fractions with their equivalent decimals and correctly circled the representations that did not belong to the set of alternatives offered for one-quarter but chose the incorrect response for the equivalent representations for 1.2 (Figure 7.54).

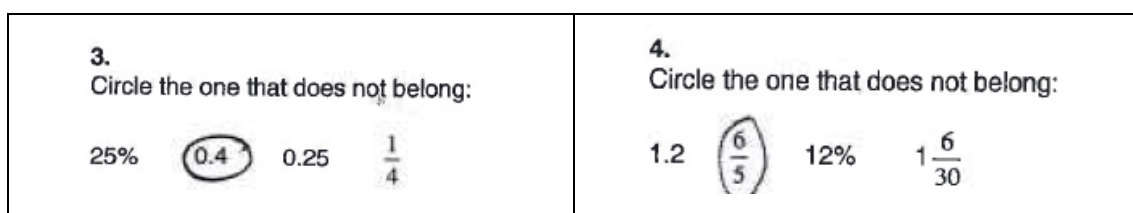


Figure 7-54: Andrew's responses for Tasks C3 and C4 (FST)

7.5.2 Reverse Fraction Tasks

Andrew's strategies for the three reverse fraction tasks moved from partially multiplicative for Task C5 to fully multiplicative for Tasks C6 and C7. His partially multiplicative response, both in words and symbols, for Task C5 is shown in Figure 7.54.

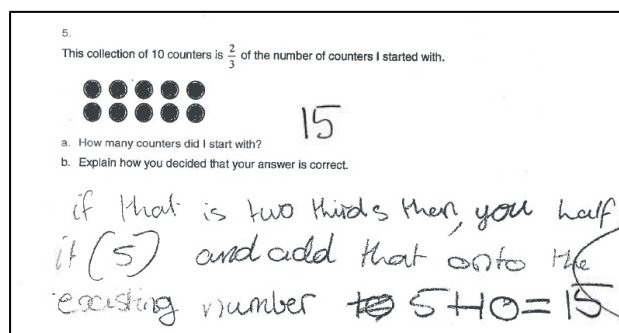


Figure 7-55: Andrew's partially multiplicative response to Task C5 (FST)

In his initial response to Task C6 Andrew drew 12 circles and wrote seven lots of 12 which he crossed out before using a fully multiplicative strategy and divided 12 by four and multiplied the result by seven (Figure 7.56).

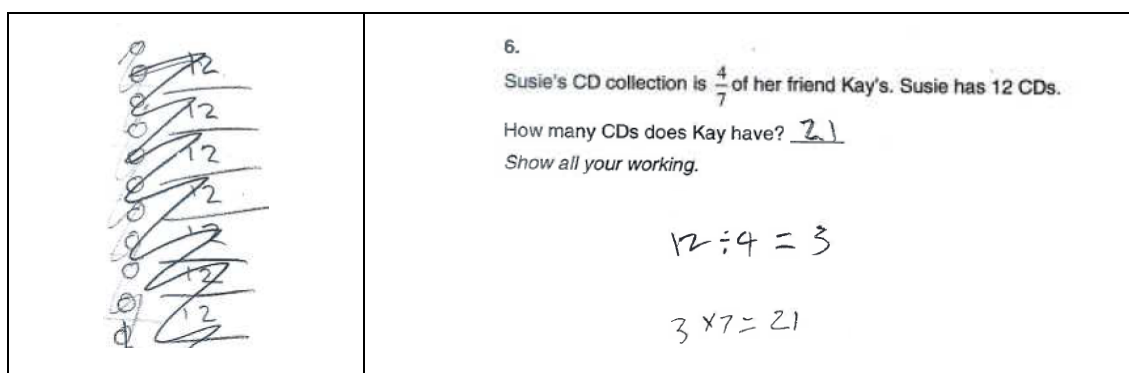


Figure 7-56: Andrew's solution for Task C6

For Task C7 shown in Figure 7.57 Andrew used a fully multiplicative strategy and calculated that one-seventh of 14 was two and then multiplied by six to give the correct solution of 12.

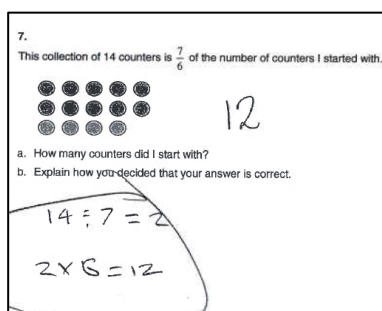


Figure 7-57: Andrew's fully multiplicative response to Task C7 (FST)

7.5.3 Structured Interview

Andrew used a partially multiplicative strategy for Question 1 of the Structured Interview (left-hand column of Figure 7.58) and a fully multiplicative strategy for Question 2 (right-hand column of Figure 7.58)

1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? <u>18</u> b. Explain your thinking. I halved the ammount of counters (12) and got 6, with 1 then added on to 12.	2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. a. How many CDs does Kay have? <u>14</u> b. Explain your thinking. $8 \div 4 = 2$ $2 \times 7 = 14$
---	--

Figure 7-58: Andrew's correct responses for Questions 1 and 2 (SI)

For the rest of the interview questions, where the number of objects was given, he used the same fully multiplicative strategy as that shown in the right-hand side of Figure 7.58. For the questions with 'any number' of objects he used and defined the symbols to represent the number of objects given in the task and 'the answer from doing the first part of the equation' similar to that shown for Question 4b in Figure 7.59.

b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?

$C =$ ~~any~~ number of counters
 $A =$ Answer ~~loger~~ from doing the first part of the equation?

$C \div 2$
 $A \times 3$

Figure 7-59: Andrew's response for Question 4b

However, Andrew was unable to answer Question 7 with 'any fraction' and 'any number' of counters. According to the interviewer Andrew was willing to have a go, looked puzzled and after a brief hesitation said: "Don't think I can do this". Andrew is classified as being at the *Generalising Multiplication* level of the *Emerging Algebraic Reasoning Framework*.

7.5.4 Algebraic Thinking Questionnaire

In the ATQ, Andrew was limited in his ability to deal with multiplicative relationships. Overall Andrew scored 8 out of a possible 28 marks (29%) which is more than the mean (6.86) for the whole cohort for the Algebraic Thinking Questionnaire. He scored 6 marks for Part M which focused on multiplication, and 2 marks for Part D which focused on division tasks.

Andrew used lattice multiplication to correctly calculate 36×25 as shown in the left-hand column of Figure 7.60. While he has not written the total for this lattice multiplication, he was able to write 100 in the box to give the correct total of 900 thus it can be assumed his total was correct.

Thus, he demonstrated that he understands the sameness-relationship of the equals sign (Jones et al., 2013). He used the same procedure of lattice multiplication for 48×2.5 as shown in the right-hand column of Figure 7.60. However, he has recorded 5×4 as 02 instead of 20 which would have given him an incorrect solution of 102. It is unclear how he then managed to write the correct answer in the box.

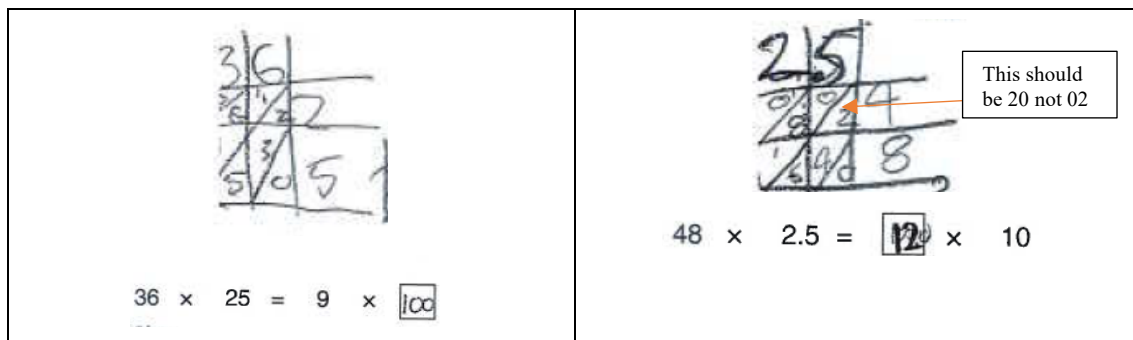


Figure 7-60: Andrew's responses for tasks M1a and M1b (ATQ)

Andrew did not respond to either Tasks M1c or Task M1d. However, Andrew gave two correct responses for Task M2a. He doubled both the numbers used in the left-hand side of Figure 7.61 and was able to state the relationship between the pairs of numbers as: "Box A is double Box B". He agreed that any number could be put in Box A as the answer could include decimals.

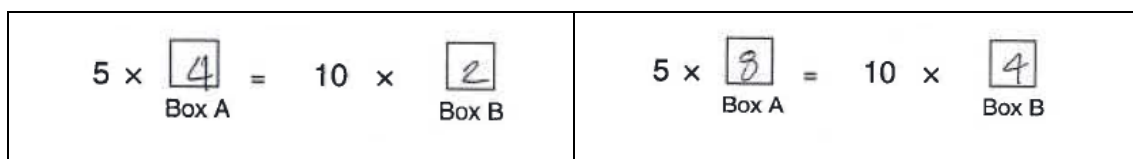


Figure 7-61: Andrew's two correct responses for Task M2a (ATQ)

Andrew did not respond to Task M2d and gave the incorrect reverse relationship for the fraction task in Task M2e as shown in Figure 7.62.

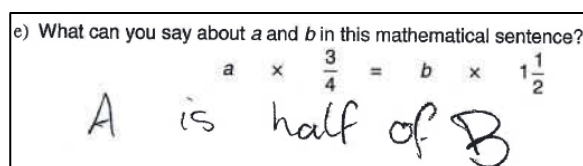


Figure 7-62: Andrew's incorrect response for Task M2e (ATQ)

Andrew was unable to answer any of Question 1 in Part D as shown by his question marks beside each task. However, he gave two correct responses for Task D2a as shown in Figure 7.63 and gave an appropriate answer describing the relationship: "You have to times them both by the same number".

$3 \div \boxed{9} = 15 \div \boxed{45}$ Box A Box B	$3 \div \boxed{12} = 15 \div \boxed{60}$ Box A Box B
--	---

Figure 7-63: Andrew's correct responses to Task M2a (ATQ)

Andrew struggled with many of the tasks from the Algebraic Thinking Questionnaire. After using a partially multiplicative strategy for the first task of the Structured Interview he successfully used a fully multiplicative strategy for the remaining interview tasks, but he was unable to answer the final question with 'any fraction' and 'any number' of objects. Andrew is deemed to be an example of the *Generalising-Multiplicative* group on the *Emerging Algebraic Reasoning Framework*

7.6 Case Study 6: Alex

Alex (Year 5) was chosen for a case study as he is an example of the *Algebraic Generalisation-Verbal* group at Level 5 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. Despite only being in Year 5 he consistently used fully multiplicative strategies for all Structured Interview tasks and used a verbal multiplicative strategy for the final question with 'any fraction' and 'any number' of objects. He was chosen to be interviewed as he had successfully solved all three reverse fraction tasks using fully multiplicative methods as shown in Figure 7.61. Alex's score for the FST is the highest for the case study students and his ATQ score was above the mean score for the whole cohort (see Table 7.2).

Section 7.6.1 will discuss his responses to the FST, Section 7.6.2 the strategies he used for the reverse fraction tasks (Tasks C5 - C7), Section 7.6.3 the strategies he used for the Structured Interview and Section 7.6.4 his responses to the ATQ.

7.6.1 Fraction Screening Test

Examples of Alex's solutions to tasks from the three parts of the FST will be discussed. Overall, he scored 34.2 out of a possible 35 marks (98%) for the FST which was the highest score achieved by the case study students.

Alex's score for Part A tasks was 12 out of a possible 12 marks where the mean for all students was 9.63 and the standard deviation was 2.22. Alex successfully completed all the routine tasks given in Part A. He chose the two correct shapes representing three-quarters, circled one-third of a group of beetles, and indicated that there would be eight lollies in the whole group when four lollies represented one-half of the group. Alex accurately shaded one-third of a rectangle and marked, and shaded, three-quarters of a length. He correctly stated the equivalent fractions and

calculated the total cost if \$4 was one-seventh. He calculated the fraction of pizza that each person would receive if three pizzas were shared between four people and gave the next number to complete the counting backwards sequence for two-fifths. Alex successfully added the fractions with related denominators and simplified the answer.

Alex scored 10 marks out of a possible 10 marks for the Part B tasks where the mean for all students was 6.21 and the standard deviation was 2.67. Alex successfully wrote the missing numbers on the four equally spaced marks on the number line marked zero to 100, accurately placed the numbers 75 and 5 on a number line marked zero and 25 using interim marks and accurately marked three-fifths of the number line marked zero to two including the interim marks for one-fifth and two-fifths. Alex used the distance from zero to one-third to mark in the number two-thirds and then the number one.

He chose the correct option for the placement of one on a number line marked zero and $\frac{3}{4}$ as he trialled the positions of one-quarter and one-half. As shown in Figure 7.64 Alex made several attempts at marking the number line in quarters before choosing position C as representing one on the number line and stated that he used “trial and error as you can see”. Alex successfully answered all number-line tasks.

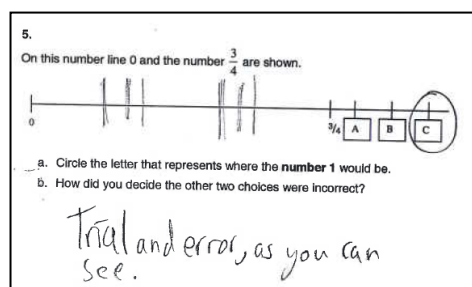


Figure 7-64: Alex's response to Task B5 (FST)

Alex scored 12.2 out of a possible 13 marks for the Part C tasks which included ordering a set of four fractions from largest to smallest, drawing a line to connect fractions and their equivalent decimals and circling the ‘one that does not belong’ from the set of fractions, decimals and percentages. Alex successfully ordered the four fractions using equivalent fractions where he converted all given fractions to twentieths. He successfully matched two of the four fractions with their equivalent decimals but joined both $\frac{1}{4}$ and $\frac{1}{10}$ with 0.25 as shown in Figure 7.65. As this was a written test it is unclear whether this was a mistake rather than a misconception but as he has done so well on the rest of the assessment it is possibly a mistake.

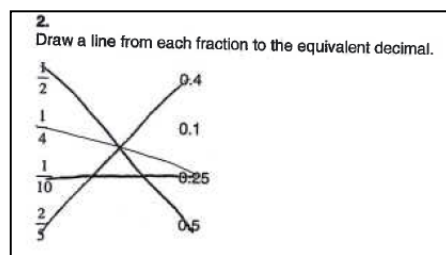


Figure 7-65: Alex's response to Task C2 (FST)

For Tasks C3 and C4 he correctly circled the representations that did not belong to the set of alternatives offered for the fraction one-quarter and the decimal 1.2.

7.6.2 Reverse Fraction Tasks

Alex consistently used a fully multiplicative strategy for all three reverse fraction tasks. He found the number of objects representing the unit fraction by dividing by the numerator and then calculated the whole by multiplying the number representing the unit fraction by the denominator as shown in Figure 7.66. Despite using different fractions, and different number of objects representing these fractions, he used the same strategy for all three tasks. This raises the question about whether Alex's strategies are simply a rule he has learnt or whether it demonstrates a stronger grasp of multiplicative and algebraic thinking and reasoning.

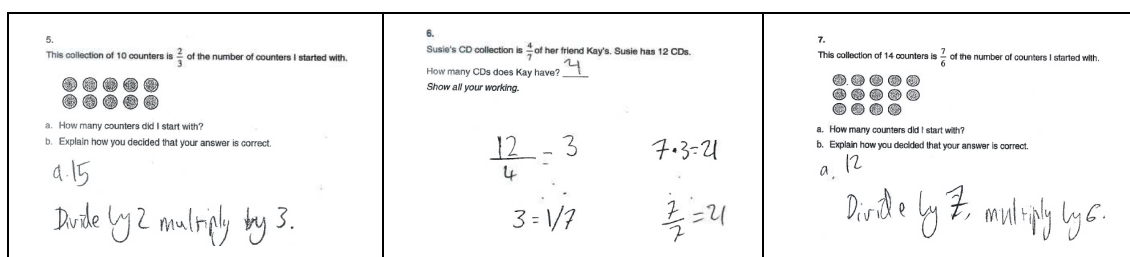


Figure 7-66: Alex's strategies for the Tasks C5, C6 and C7 (FST)

7.6.3 Structured Interview

Alex used the same fully multiplicative strategy for the Structured Interview questions that he had used for the written tasks (Tasks C5 - C7) on the FST. The interviewer encouraged him to write his explanation which clearly demonstrated that he could determine the number of counters that represented one-third and then multiplied that by three to get the whole (Figure 7.67).

In Question 1 Alex recorded his response (left-hand side of Figure 7.67) and then justified the division of the 12 counters by two to find one-third of the original group. Alex's justification shows that he is not merely applying a learned rule. He clearly understands the underlying structure of fractions and the relationship between numerators and denominators.

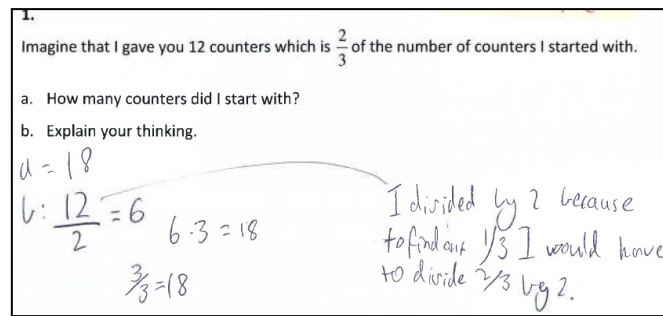


Figure 7-67: Alex's response to Question 1 (SI)

In Questions 2 – 6 Alex used the same fully multiplicative strategy as shown for Question 1 in Figure 7.67. The interviewer wrote that Alex had read and written his response to Question 7 “in 10 secs”. As shown in Figure 7.68 Alex used the relationship between the fraction, and the number of objects representing that fraction, to calculate the number of objects required to represent the whole for Question 7. This strategy will work for any fraction representing any number of objects.

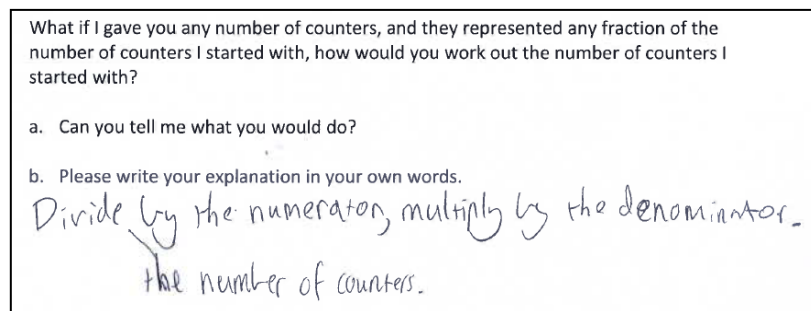


Figure 7-68: Alex's response to Question 7 (SI)

Alex consistently used fully multiplicative strategies to solve reverse fraction tasks on the written FST and for each of the interview questions with both given quantities and ‘any quantity’. As he verbally responded to Question 7 of the Structured Interview after consistently using fully multiplicative strategies he is deemed to be at the *Algebraic Generalisation – Verbal* level on the *Emerging Algebraic Reasoning Framework* (Table 6.12).

7.6.4 Algebraic Thinking Questionnaire

Alex attempted every ATQ task. He scored 9.5 marks for Part M and 2.5 marks for Part D so overall scored 12 out of a possible 28 marks (43%). His score for the ATQ was higher than the mean for the whole cohort.

Alex successfully calculated the total of the left-hand side of the equation in Task M1a as shown in the right-hand column of Figure 7.69. He then successfully divided by nine and placed the correct number (100) in the box in the left-hand column. He has demonstrated that he understands

the sameness-relational understanding of the equals sign (Jones et al., 2013) and has ensured that both sides of the expression are equal.

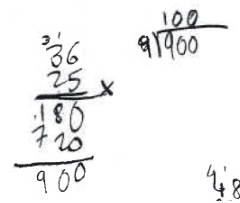
$36 \times 25 = 9 \times \boxed{100}$	
---------------------------------------	--

Figure 7-69: Alex's response to Task M1a (ATQ)

In Question 1b Alex accurately multiplied the two numbers on the left-hand side of the equation and then divided by ten mentally to write the correct response in the empty box on the right-hand side of the equation (Figure 7.70).

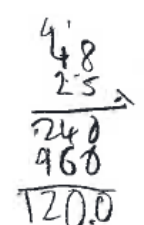
$48 \times 2.5 = \boxed{12} \times 10$	
--	--

Figure 7-70: Alex's response to Task M1b (ATQ)

As shown in the left-hand column of Figure 7.71 Alex used the same strategy that he used for both the three reverse fraction tasks in Part C (FST), and all the Structured Interview questions to solve Task M1c. He divided the number representing two-thirds (18) by the numerator (2) and then multiplied by the denominator (3). It would be highly unlikely that a Year 5 student would have been given a question in this format and he certainly would not have been taught any rules for solving tasks such as this one. In the right-hand column of Figure 7.71 he has written the correct response for Question M1d, but no working out has been provided.

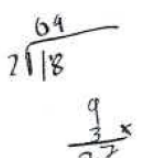
$\frac{2}{3} \times \boxed{27} = 18$ 	$\frac{2}{5} \times \frac{\boxed{5}}{2} = 1$
--	--

Figure 7-71: Alex's response to Task M1c (ATQ)

Alex gave two correct responses for Question M2a as shown in Figure 7.72. In this case he has not only demonstrated that he understands the sameness-relational understanding of the equals sign but also the substitutive-relational understanding (Jones et al, 2013).

$5 \times \boxed{4} = 10 \times \boxed{2}$ Box A Box B	$5 \times \boxed{7} = 10 \times \boxed{4}$ Box A Box B
--	--

Figure 7-72: Alex's response to Task M2a (ATQ)

Alex correctly noted that the number in Box B was half the number in Box A (left-hand column of Figure 7.73) and while he stated that not any number could be put in Box A, he correctly wrote that the relationship between the number in Box A and the number in Box B needed to be maintained so that the "number in Box A must be two times the number in Box B" (right-hand column of Figure 7.73). Alex's response of 'No' really stands for: 'No - unless the relationship is maintained' which could be interpreted as: 'yes - provided the relationship has been maintained'.

$\text{Box B} = \frac{1}{2} \text{ of Box A}$	No No, because the number in Box A must be 2 times the number in Box B.
---	--

Figure 7-73: Alex's responses to Tasks M2b and M2c (ATQ)

As shown in Figure 7.74 Alex identified, and correctly stated, the relationship between c and d in Task M2d and a and b in Task M2e. As a Year 5 student Alex had not been exposed to the use of pronumerals at school. When solving M2d he has recognised that both sides of the expression need to be the same and has also substituted 7×2 or 2×7 for 14. As he has not formally recorded any working out it could be assumed that this was done mentally. Thus, if $c \times 2 = d \times 7 \times 2$ then $c = d \times 7$.

d) What can you say about c and d in this mathematical sentence?

$c \times 2 = d \times 14$

c would be 7 times larger than d .

Figure 7-74: Alex's response to Task M2d (ATQ)

In Task M2e (Figure 7.75) Alex has shown that he has recognised that $1\frac{1}{2}$ is two times $\frac{3}{4}$. This means he has successfully demonstrated his understanding of both the sameness-relational and substitutive relational understanding of the equals sign (Jones et al., 2013).

e) What can you say about a and b in this mathematical sentence?

$$a \times \frac{3}{4} = b \times 1\frac{1}{2}$$

a would be 2 times the size of b.

Figure 7-75: Alex's correct response to M2e (ATQ)

Alex was not as successful with the Part D division tasks as he was with the Part M multiplication tasks. As shown in Figure 7.76 he correctly answered Task D1a (left-hand side) but gave an incorrect response for Task D1b (right-hand side). While there are no calculations to show his thinking as there had been for Tasks M1a (Figure 7.69) and M1b (Figure 7.70) it appears that he doubled rather than halved the 16 on the left-hand side of the expression shown in the right-hand column of Figure 7.76.

$3 \div 4 = 15 \div \boxed{20}$	$\overset{32}{\cancel{30}} \div 15 = 16 \div 30$
---------------------------------	--

Figure 7-76: Alex's responses to Tasks D1a and D1b (ATQ)

Alex was not successful with Tasks D1c and D1d as shown Figure 7.77 and as there is no working out it is difficult to determine his thinking. His number formation is poor for the numbers in the right-hand column of Figure 7.77, but it would appear he has written seven-sixths divided by six-sevenths equals one.

$18 \div \boxed{4} = 6 \div \frac{2}{3}$	$\frac{7}{6} \div \frac{\boxed{6}}{\boxed{7}} = 1$
--	--

Figure 7-77: Alex's incorrect responses to Tasks D1c and D1d (ATQ)

Alex attempted two solutions for Task D2a. The solution in the left-hand side of Figure 7.78 is correct but it appears that he has written 36 rather than 30 in the box on the right-side of the equation and shown in the right-hand column of Figure 7.78. However, it could have been a poorly written zero rather than six as his explanation for Task D2b is correct.

$3 \div \boxed{3} = 15 \div \boxed{15}$ Box A Box B	$3 \div \boxed{6} = 15 \div \boxed{30}$ Box A Box B
--	--

Figure 7-78: Alex's responses to Task D2a (ATQ)

Alex correctly described the relationship between the number in Box A and the number in Box B in Task D2b as the number in Box A is the number in Box B divided by five as shown in Figure 7.79.

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

Box A = Box B ÷ 5

Figure 7-79: Alex's correct response to Task D2b (ATQ)

As he had done for the equivalent multiplication task Alex suggested that not any number could be put in Box A. In Figure 7.80 he appears to suggest that the number in Box A 'has to be one-third of Box B'. This seems a strange response when he has clearly defined the relationship in the previous task as the number in Box A being one-fifth of the number in Box B. Again, it may be careless or incorrect recording of the number 'five'.

(c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

No. It has to be 1/3 of Box B.

Figure 7-80: Alex's response to Task D2c (ATQ)

Alex incorrectly reversed the relationship between c and d stating that c would be three times d rather than d being three times c (left-hand column of Figure 7.81) and was unable to identify the relationship between a and b in Task D2e saying "I don't know" (right-hand column).

(d) What can you say about c and d in this mathematical sentence? $c \div 8 = d \div 24$ <i>c would be 3 times d.</i>	(e) What can you say about a and b in this mathematical sentence? $a \div \frac{2}{7} = b \div \frac{3}{7}$ <i>I don't know.</i>
--	---

Figure 7-81: Alex's responses to Tasks D2d and D2e (ATQ)

While Alex shows clear evidence of algebraic thinking in both the FST and the Structured Interview, there are also evident strengths and limitations in his responses to the Algebraic Thinking Questionnaire as he struggled with the division tasks. However, for a Year 5 student this is not surprising. Alex is already thinking multiplicatively and algebraically and is deemed to be at *Algebraic Generalisation-Verbal* on the *Emerging Algebraic Reasoning Framework*.

7.7 Case Study 7: Tanya

Tanya (Year 8) was chosen for a case study as she is an example of the *Algebraic Generalisation-Verbal* group at Level 5 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. Like Alex she used fully multiplicative strategies for all Structured Interview tasks and used a verbal multiplicative method for the final task with 'any fraction' and 'any number' of objects. While Alex used fully multiplicative methods for the three reverse fraction tasks she successfully solved them using a partially multiplicative method as shown in Figure 7.79. Tanya's score for the FST is similar to Andrew's score and above the mean score for the whole cohort. Her ATQ score is similar to Alex's and also above the mean score for the whole cohort (see Table 7.2).

Section 7.7.1 will discuss her responses to the FST, Section 7.7.2 the strategies she used for the reverse fraction tasks (Tasks C5 - C7), Section 7.7.3 the strategies she used for the Structured Interview and Section 7.7.4 her responses to the ATQ.

7.7.1 Fraction Screening Test

Tanya scored 27.5 out of a possible 35 marks (79%) for the FST which was above the mean score (21.91) for the whole cohort.

Tanya's score for Part A tasks was 10.5 out of a possible 12 marks (88%) where the mean for all students was 9.63 and the standard deviation was 2.22. She chose the two correct shapes representing three-quarters, circled one-third of a group of beetles, and indicated that there would be eight lollies in the whole group when four lollies represented one-half of the group. Tanya accurately shaded one-third of a rectangle and three-quarters of a length. She correctly stated the equivalent fractions and calculated the total amount as \$28 if \$4 represented one-seventh. Tanya gave the next number to complete the counting backwards sequence for two-fifths. Like many other students she calculated the fraction of one pizza that each person would receive rather than the fractional part of three pizzas shared between four. Tanya also added the fractions with related denominators by incorrectly adding numerators and denominators separately thus stating $\frac{3}{10} + \frac{1}{5} = \frac{4}{15}$.

Tanya scored 7 marks out of a possible 10 marks for the Part B tasks where the mean for all students was 6.21 and the standard deviation was 2.67. She successfully wrote the four missing numbers on the equally spaced marks on the number line marked zero to 100. However, she placed the number 75 on a number line marked zero and 25 as though the ends of the number-line should be marked zero and 100 and stated that she knew she was correct "because you divide 100 by 4". She accurately placed the number 5 on the same number-line marked zero to 25. Tanya

accurately marked three-fifths of the number line marked zero to two without drawing any interim marks. She also chose the correct option for the placement of one on a number line marked zero and $\frac{3}{4}$ and stated that the other two options “were too close to $\frac{3}{4}$ ”.

Tanya scored 10 out of a possible 13 marks for the Part C tasks where the mean was 6.07 and the standard deviation was 3.29 for all students. Part C tasks included ordering a set of four fractions from largest to smallest, drawing a line to connect fractions and their equivalent decimals and circling the ‘one that does not belong’ from the set of fractions, decimals and percentages. Tanya successfully ordered the four fractions by converting them to percentages, matched all four fractions with their equivalent decimals and correctly circled the representations that did not belong to the set of alternatives offered.

7.7.2 Reverse Fraction Tasks

Tanya used a partially multiplicative strategy to solve all three reverse fraction tasks on the FST. For each task, she divided the given number of objects by the numerator to find the number representing the unit fraction, calculated the number for the missing fractional part and either added or subtracted to determine the number in the whole collection. In the left-hand column of Figure 7.82, showing her response to Task C5, Tanya halved the number given for two-thirds and added this number to the number given for two-thirds thus giving the number in the whole group.

For Task C6, shown in the right-hand column of Figure 7.82, Tanya found one-seventh by dividing the number of counters given for four-sevenths by four, multiplied the number by three to get three-sevenths. She then added the number of counters for three-sevenths to the number for four-sevenths to get seven-sevenths. This showed a clear understanding of the multiplicative relationships between four-sevenths and one-seventh, and between seven and 21. She also used a part-part-whole strategy to combine four-sevenths and three-sevenths to find the whole.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? <u>15</u> b. Explain how you decided that your answer is correct. $\frac{1}{3}$ must equal 5, so you add 5 to 10.	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. $12 \div 4 = 3$ $3 \times 3 = 9$ $12 + 9$
--	--

Figure 7-82: Tanya's responses to Task C5 and Task C6 (FST)

For Task C7, Tanya again used a partially multiplicative strategy and subtracted the number of counters representing one-seventh from the number given for seven-sixths as shown in the right-hand column of Figure 7.83.

7.
This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.



a. How many counters did I start with? 12
b. Explain how you decided that your answer is correct.

Because $\frac{1}{6}$ equals 2, so you subtract 2 from the total amount.

Figure 7-83: Tanya's responses to the three reverse fraction tasks (Tasks C5 - C7)

7.7.3 Structured Interview

Tanya used a fully multiplicative strategy for all questions of the Structured Interview, which was a more sophisticated strategy than she had used for the original three reverse fractions on the FST. In other words, she could move beyond her earlier part-part-whole solution strategy as evident in Figure 7.78. She divided by the numerator to find the number of objects needed to represent the unit fraction and then multiplied this number by the denominator to find the number needed for the whole group. For Question 1 (left-hand column of Figure 7.84) Tanya divided the number of counters representing two-thirds by two to find one-third and then multiplied by three to find three-thirds. For Question 2 (right-hand column of Figure 7.84) she divided by four to find one-seventh and then multiplied by seven to calculate the number of CDs needed for the whole collection.

1. Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with. a. How many counters did I start with? b. Explain your thinking. $12 \div 2 = 6$ $6 \times 3 = 18$	2. Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's. a. How many CDs does Kay have? ____ b. Explain your thinking. $8 \div 4 = 2$ $2 \times 7 = 14$
--	--

Figure 7-84: Tanya's response to Questions 1 and 2 (SI)

Tanya also used the fully multiplicative strategy for the questions that asked her to think about 'any number' of counters. Her responses to Question 4 shown in Figure 7.85 demonstrate her ability to move from specific numbers to the more generalised question that includes 'any number'.

4. a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with? $18 \div 2 = 9$ $9 \times 3 = 27$	b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with? divide by 2 and then multiply by 3.
--	--

Figure 7-85: Tanya's responses to Questions 4a and 4b (SI)

In Question 7 Tanya stated the generalisable case for 'any number of counters representing any fraction' but used the terms 'top number' and 'bottom' respectively to describe the correct terms of numerator and denominator (Figure 7.86).

7.

Think about the tasks you have just done.

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?

a. Can you tell me what you would do?

b. Please write your explanation in your own words.

depending on the number of counters, you would ~~multiply~~ ^{divide} it by the top number and multiply by the bottom.

Figure 7-86: Tanya's response to Question 7 (SI)

Tanya moved from using partially multiplicative strategies in her written solutions to the three reverse fraction tasks to using fully multiplicative strategies for the interview questions. She verbally responded to Question 7 of the Structured Interview after consistently using fully multiplicative strategies and hence is deemed to be at the *Algebraic Generalisation – Verbal* level on the *Emerging Algebraic Reasoning Framework* (Table 6.12).

7.7.4 Algebraic Thinking Questionnaire

Tanya attempted all tasks given in the ATQ. She scored 7 marks for Part M, and 6 marks for Part D. Overall Tanya scored 13 out of a possible 28 marks for the ATQ where the mean was 6.86 and the standard deviation was 6.45 for the whole cohort.

In Question M1a Tanya used lattice multiplication (right-hand side of Figure 7.87) to complete the calculation for 36×25 and then divided the result by nine to give a correct response for the empty box on the right-hand side of the equation.

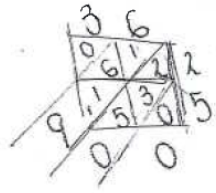
$\begin{array}{r} 36 \times 25 = 9 \times \boxed{100} \\ \hline = 900 \div 9 \end{array}$	
---	--

Figure 7-87: Tanya's response to Task M1a (ATQ)

Tanya's correct solution for 48×2.5 is shown in the left-hand column of Figure 7.88 but there is no written calculation, so her method of computation is not obvious. However, in the right-hand column her response for Task M1c shows her fully multiplicative strategy, similar to the one she used in the Structured Interview, but more sophisticated than the partially multiplicative strategy she used in the FST completed at the same time as the ATQ.

$\begin{array}{r} 48 \times 2.5 = \boxed{12} \times 10 \\ \hline = 120 \div 10 \end{array}$	$\begin{array}{r} \frac{2}{3} \times \boxed{27} = 18 \\ 18 \div 2 \times 3 \end{array}$
---	---

Figure 7-88: Tanya's responses to Tasks M1b and M1c (ATQ)

Tanya did not respond to Question M1d and left it blank. In Question M2a Tanya gave three correct responses, two of which are shown in Figure 7.89 demonstrating that she understands both the sameness-relational and the substitutive understanding of the equals sign (Jones et al., 2013).

$5 \times \boxed{2} = 10 \times \boxed{1}$ Box A Box B	$5 \times \boxed{4} = 10 \times \boxed{2}$ Box A Box B
---	---

Figure 7-89: Tanya's response to Task M2a (ATQ)

Tanya identified and wrote the correct relationship between the numbers in Box A and Box B for Task M2b as shown in Figure 7.90. For Question 2c she has assumed that each empty box must contain a whole number as evidenced by her response which stated that "the number in Box A would have to be even".

b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?
Box B is half of box A

c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.
In terms of whole numbers, the number in box a would have to be even.

Figure 7-90: Tanya's response to Tasks 2b and 2c (ATQ)

Tanya's incorrect responses for Tasks M2d and M2e are shown in Figure 7.91. Both her responses state an incorrect inverse relationship as c should be seven times d and not one-seventh of d and a should be twice b not one-half of a .

d) What can you say about c and d in this mathematical sentence? $c \times 2 = d \times 14$ c is $\frac{1}{7}$ of d.	e) What can you say about a and b in this mathematical sentence? $a \times \frac{3}{4} = b \times 1\frac{1}{2}$ a is half of b.
---	---

Figure 7-91: Tanya's responses to Tasks 2d and 2e (ATQ)

Tanya used arrows and labels to indicate the multiplicative relationships between the terms on both sides of the equation in both Tasks D1a and D1b as shown in Figure 7.92. This assisted her to correctly determine the missing numbers for each of the boxes. This shows clearer understanding of the equivalence relationship for division.

$3 \div 4 = 15 \div \boxed{20}$ $\times 5$ $\times 5$	$\boxed{8} \div 15 = 16 \div 30$ $\times 2$ $\times 2$
---	--

Figure 7-92: Tanya's response to Tasks D1a and D1b (ATQ)

Unfortunately, Tanya did not include arrows or labels for Task D1c and thus expressed the incorrect relationship. Instead of dividing the unknown number by three she divided two-thirds by three as shown in the left-hand column of Figure 7.93. She wrote the incorrect answer in the boxes in Task D1d and changed the sign to end up with a correct equation as shown in the right-hand column of Figure 7.93.

$18 \div \boxed{\frac{2}{3}} = 6 \div \frac{2}{3}$ $\frac{2}{3} \div 3$	$\frac{7}{6} \times \boxed{\frac{6}{7}} = \frac{1}{1}$
---	--

Figure 7-93: Tanya's response to Tasks D1c and D1d, Part A (ATQ)

In Task D2a Tanya completed the equations as though they were multiplication tasks rather than the division tasks as shown in Figure 7.94.

$3 \div \boxed{10} = 15 \div \boxed{2}$ Box A Box B	$3 \div \boxed{15} = 15 \div \boxed{3}$ Box A Box B
---	---

Figure 7-94: Tanya's response to Task M2a (ATQ)

Tanya's responses to Tasks D2b and D2c, shown in Figure 7.90 are correct but do not relate to her two incorrect responses to Task M2a (Figure 7.91).

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? Box A is $\frac{1}{5}$ of box b
(c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly. No, it has to be a multiple of 5

Figure 7-95: Tanya's responses to Tasks D2b and D2c (ATQ)

Tanya gave the reverse relationship for c and d in Task D2d shown in the left-hand column of Figure 7.96 and used additive thinking in Task D2e (right-hand side of Figure 7.96).

(d) What can you say about c and d in this mathematical sentence? $c + 8 = d + 24$ c is 3 times d	(e) What can you say about a and b in this mathematical sentence? $a + \frac{2}{7} = b + \frac{3}{7}$ a is $\frac{1}{7}$ more than b.
---	--

Figure 7-96: Tanya's responses to Tasks D2d and D2e (ATQ)

It is clear from Tanya's response to Task D2d that she is not able or confident to apply the same clear relational thinking to these expressions containing two unknowns that she demonstrated for Tasks M1a (Figure 7.87) and M1b (left-hand side of Figure 7.88) where there was only one unknown in the expression. Tanya's algebraic thinking, while evident in the Structured Interview, is not as robust across the range of questions in the ATQ. She used fully multiplicative strategies for all Structured Interview tasks and used a verbal multiplicative method for the final task with 'any fraction' and 'any number' of objects and is an example of the *Algebraic Generalisation-Verbal* group of the *Emerging Algebraic Reasoning Framework*.

7.8 Case Study 8: Lewis

Lewis (Year 8) was chosen for a case study as he is an example of the *Algebraic Generalisation Symbolic* group at Level 6 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. He used fully multiplicative strategies for all interview tasks where the number of objects was given for the fraction and advanced multiplicative strategies for the tasks with a given fraction but 'any number' represented that given fraction. He used a symbolic multiplicative method for the task with 'any fraction' and 'any number' of objects. He was chosen to be interviewed as he used either fully multiplicative or advanced multiplicative strategies to solve the three reverse fraction tasks (C5 - C7). Lewis's scores for the FST and ATQ are both more than one Standard Deviation above the mean score for the whole cohort.

Section 7.8.1 will discuss his responses to the FST, Section 7.8.2 the strategies he used for the reverse fraction tasks (Tasks C5 - C7), Section 7.8.3 the strategies he used for the Structured Interview and Section 7.8.4 his responses to the ATQ.

7.8.1 Fraction Screening Test

Overall, Lewis scored 33.5 out of a possible 35 marks (96%) for the FST where the mean was 21.91 and the standard deviation was 7.11 for the whole cohort. This is the same score as Jack from Year 6 and 0.7 less than Alex in Year 5 who was from the same primary school as Jack.

Lewis's score for Part A tasks was 11 out of a possible 12 marks (92%) where the mean for all students was 9.63. He chose the two correct shapes representing three-quarters, circled one-third of a group of beetles, and indicated there would be eight lollies in the whole group when four lollies represented one-half of the group. Lewis accurately shaded one-third of a rectangle and three-quarters of a length. He correctly stated the equivalent fractions and calculated the total as \$28 if \$4 was one-seventh. He calculated the fraction of pizza that each person would receive if three pizzas were shared between four and gave the next number to complete the backwards counting sequence for two-fifths. Lewis correctly added the fractions with related denominators but did not simplify the answer. Lewis was successful with the routine tasks of Part A.

Lewis was successful with all the number-line tasks and scored 10 marks out of a possible 10 marks for the Part B tasks where the mean for all students was 6.21 and the standard deviation was 2.67. He successfully wrote the missing numbers on the equally spaced marks on the number-line marked zero to 100, placed the numbers 75 and 5 on a number line marked zero and 25 and accurately marked three-fifths of the number line marked zero to two drawing in the interim marks. Lewis used the distance from zero to one-third to mark in the number two-thirds and then the number one. Thus, he iterated the original length from zero to one-third two more times to mark where the number one should be placed. He also chose the correct option for the placement of one on a number line marked zero and $\frac{3}{4}$. He partitioned the distance from zero to three-quarters into three parts as indicated by the marks shown for $\frac{1}{4}$ and $\frac{2}{4}$. His explanation stated: "If you take the distance from 0 to $\frac{1}{4}$ it would be the same as $\frac{3}{4}$ to C". These questions allowed Lewis to demonstrate clear and confident multiplicative thinking.

Lewis scored 12.5 out of a possible 13 marks for the Part C tasks included ordering a set of four fractions from largest to smallest, drawing a line to connect fractions and their equivalent decimals and circling the 'one that does not belong' from the set of fractions, decimals and percentages. Part C also included the three reverse fraction tasks. Lewis successfully ordered the four fractions,

matched all four fractions with their equivalent decimals and correctly circled the representations that did not belong to the set of alternatives offered.

7.8.2 Reverse Fraction Tasks

Lewis used the same fully multiplicative method to solve Tasks C5, C6 and C7. In Tasks C5 and C6 (Figure 7.97) Lewis found the number of objects representing the whole collection, starting with the known quantities (10 and 12 respectively), then divided by the fraction's numerator and multiplied by the denominator in a single operation.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  a. How many counters did I start with? <u>15</u> b. Explain how you decided that your answer is correct. $= 10 \div 2 \times 3$ $= 5 \times 3$ $= 15$	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? <u>21</u> Show all your working. $= 12 \div 4 \times 7$ $= 3 \times 7$ $= 21$
---	---

Figure 7-97: Lewis's responses to Task C5 and Task C6 (FST)

In Task C7 these steps are performed separately (Figure 7.98). Lewis expressed the relationship between all parts of the tasks multiplicatively. Lewis's method of dividing by the numerator and multiplying by the denominator is very similar to that used by Jack where he divided by the fraction (Figures 7.110 and 7.111).

7.

This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.



a. How many counters did I start with? 12

b. Explain how you decided that your answer is correct.

Because $14 \div 7 = 2$
and $2 \times 6 = 12$

Figure 7-98: Lewis's responses to Tasks C5 – C7 (FST)

7.8.3 Structured Interview

Lewis's responses to each question of the Structured Interview were expressed in the same way as his responses to the three reverse fraction tasks and the multiplication and division tasks on the ATQ. On the left-hand side of Figure 7.99 is his written response to Question 1 and on the right-hand side is the additional explanation he gave when persuaded by the interviewer to give more detail about his response.

1.
Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with.

a. How many counters did I start with?
b. Explain your thinking.

$$\begin{aligned}
 &= 12 \div 2 \times 3 \\
 &= 6 \times 3 \\
 &= 18
 \end{aligned}$$

$$\begin{aligned}
 &12 \div 2 = \frac{1}{3} \text{ of the total} \\
 &\quad \text{counters} \\
 &\therefore 6 \times 3 \text{ or } \frac{3}{3} = 18
 \end{aligned}$$

Figure 7-99: Lewis's response to Question 1 (SI)

In Question 4b when he was asked what he needed to do if there were ‘any number’ of counters that represented two-thirds of the number, he wrote $x \div 2 \times 3 = y$, x = number of counters and y = number of counters started with. This was without any prompting from the interviewer. For Question 7 he wrote: $x \div n \times d = y$ and again defined the pro-numerals without prompting from the interviewer: x = number of counters, y = number of counters started with, n = numerator and d = denominator. This fully generalised algebraic strategy is supported by some of Lewis’s responses to the ATQ tasks. Since Lewis symbolically responded to Question 7 of the Structured Interview after consistently using fully multiplicative strategies for all other interview questions he is deemed to be at the *Algebraic Generalisation – Symbolic* level on the *Emerging Algebraic Reasoning Framework* (Table 6.12).

7.8.4 Algebraic Thinking Questionnaire

In the ATQ, Lewis scored 7 marks for Part M, 8.5 marks for Part D, and 15.5 overall out of a possible 28 marks for the ATQ, where the mean was 6.86 and the standard deviation was 6.45 for the whole cohort.

Task M1a requires students to “write a number in the box to make a true statement” and explain their working. When solving Tasks M1a and M1b Lewis set up an equation and wrote the first line of his solution in one step as shown in Figure 7.100. Lewis was able to transpose numerical terms to find the missing number, after setting up an equation for x .

$36 \times 25 = 9 \times \boxed{100}$ $x = 36 \times 25 \div 9$ $= 900 \div 9$ $= 100$	$48 \times 2.5 = \boxed{12} \times 10$ $x = 48 \times 2.5 \div 10$ $= 120 \div 10$ $= 12$
---	--

Figure 7-100: Lewis's responses to Tasks M1a and M1b (ATQ)

Lewis set up his one-step solution equation for Task M1c as shown in Figure 7.101 demonstrating that in order to find the unknown number he needs to divide by the fraction ($\frac{2}{3}$), that is, multiply

by the reciprocal of the fraction ($\frac{3}{2}$). He successfully completed the calculation first completing the division ($18 \div 2$) and then multiplying by 3 to get 27. This is a close parallel to the solution strategies he used to solve the three reverse fraction tasks.

$$\frac{2}{3} \times [27] = 18 \quad x = 18 \div 2 \times 3$$

Figure 7-101: Lewis's response to Task M1c (ATQ)

Lewis, however, is unsuccessful with Task M1d as he incorrectly divided one by 0.2 to get the incorrect answer of 0.2 as shown in Figure 7.102. On two occasions he equates 0.2 with two-fifths. Lewis's equation model seems to ignore a simpler strategy based on multiplicative inverses. A similar problem was encountered by Jack.

$$\frac{2}{5} \times \frac{[2]}{[5]} = 1 \quad x = 1 \div 0.2 = 0.2 = \frac{2}{5}$$

Figure 7-102: Lewis's response to Task M1c ATQ

In Task M2a students are required to put numbers in Box A and Box B so that the expressions on both sides of the equal sign are equivalent. Lewis gave two incorrect examples for the missing numbers. He treated the expressions on each side of the equals sign as though they were division rather than multiplication as given. He stated the appropriate relationship between the numbers he chose to place in Box A and Box B rather than the correct one (left-hand side of Figure 7.103). He stated that any number could be placed in Box A “as long as $B = 2 \times A$ ” (right-hand side of Figure 7.103) which relates to the numbers he chose rather than the correct ones.

a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct? $5 \times \frac{[1]}{\text{Box A}} = 10 \times \frac{[2]}{\text{Box B}}$ $5 \times \frac{[3]}{\text{Box A}} = 10 \times \frac{[6]}{\text{Box B}}$	b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B? $A \times 2 = B$ c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly. I + doesn't matter what number is A as long as $B = 2 \times A$.
---	---

Figure 7-103: Lewis's solution to Tasks M2a, M2b and M2c (ATQ)

When the empty box representation of the previous tasks was changed to a pronumeral Lewis consistently gave the inverse relationship as shown in Figure 7.104 instead of one of the correct responses: $c = 7 \times d$ or $d = \frac{1}{7} c$.

d) What can you say about c and d in this mathematical sentence?

$$c \times 2 = d \times 14$$

$$d = 6 \times 7$$

Figure 7-104: Lewis's solution to Task M2d (ATQ)

Lewis's unusual calculation to Task M2e is shown in Figure 7.105. It is unclear why he chose to subtract 1.5 from 6 when his initial line showing $1.5 \div 0.75$ appeared to be leading to a correct calculation.

e) What can you say about a and b in this mathematical sentence?

$$a \times \frac{3}{4} = b \times 1\frac{1}{2}$$

$$b = a + 4.5$$

$$1.5 \div 0.75 = 2$$

$$6 - 1.5 = 4.5$$

Figure 7-105: Lewis's response to Task M2e (ATQ)

Lewis's solution methods for Part D tasks were similar to those he used for Part M tasks. For example, for Task M1a ($3 \div 4 = 15 \div \square$) he wrote $x = 15 \div 3 \times 4$ and solved the division first ($15 \div 3$) before multiplying by 4 to get the correct response of 20.

Lewis then attempted to use the same strategy to solve Tasks D1b and D1c shown in Figure 7.106. He wrote an incorrect equation in his attempt to solve Task D1b. The equation in the left-hand side should be $x = 16 \div 30 \times 15$ rather than the one he wrote: $x = 30 \div 15 \times 16$. The equation in the right-hand side is correct, $18 \div 6 \times \frac{2}{3}$, but unfortunately Lewis made a computational error when he multiplied $\frac{9}{3}$ and $\frac{2}{3}$ and wrote $\frac{18}{3}$ instead of $\frac{16}{3}$. However, a check of his completed responses would have highlighted that his answers were incorrect: $\frac{32}{15}$ does not equal $\frac{16}{30}$ and $\frac{18}{6}$ does not equal $6 \div \frac{2}{3}$.

$\boxed{32} \div 15 = 16 \div 30$ $x = 30 \div 15 \times 16$ $= 2 \times 16$ $= 32$	$18 \div \boxed{6} = 6 \div \frac{2}{3}$ $x = 18 \div 6 \times \frac{2}{3}$ $= 3 \times \frac{2}{3}$ $= \frac{9}{3} \times \frac{2}{3}$ $= \frac{18}{3}$ $= 6$
---	--

Figure 7-106: Lewis's response to Tasks D1b and D1c (ATQ)

In Tasks D1b and D1c Lewis realised that the unknown indicated by the empty box in the left-hand expression can be denoted by x . He then attempted to use the equivalence relationship between the left and right side of this expression to determine the value of x . However, he clearly misunderstands the nature of inverse operations. The right-hand side of the expression would need to stay the same and be multiplied by 15 giving $16 \div 30 \times 15$ giving an answer of 8. His emerging algebraic understanding was demonstrated by his successful completion of other questions on the ATQ and the Structured Interview. Instead, his more complicated method could have been easily completed using relational thinking or by converting the left side and right side into equivalent fractions.

In Task D2a Lewis gave two correct examples for the empty boxes as shown in Figure 7.107. He has shown his understanding of the sameness-relational understanding of the equals sign (Jones et al., 2013).

$$3 + \boxed{11} = 15 + \boxed{5}$$

Box A Box B

$$3 + \boxed{3} = 15 + \boxed{16}$$

Box A Box B

Figure 7-107: Lewis's response to Task D2a (ATQ)

Lewis identified the relationship between the numbers in Box A and Box B as $B = A \times 5$ and stated that any number could be placed in Box A: “as long as $B = 5 \times A$ ”. This showed confident use of symbols to denote Box A and Box B and a clear understanding of commutativity for multiplication between $A \times 5$ and $5 \times A$ (Figure 7.108)

(b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

$B = A \times 5$

(c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

As long as B is $5 \times A$ it will be correct.

Figure 7-108: Lewis responses to Tasks D2b and D2c (ATQ)

In Task D2d Lewis correctly identified that d would need to be three times larger than c to ensure that $c \div 8 = d \div 24$ was correct but in Task D2e wrote that a was one-seventh of b (Figure 7.109) rather than the correct response of a is two-thirds of b or b is one and a half times a . As there was no working out shown on his paper it is difficult to ascertain his thinking about this task, but it

suggests an additive relationship rather than a multiplicative one. This question was one of the most challenging questions in the ATQ.

(e) What can you say about a and b in this mathematical sentence?

$$a + \frac{2}{7} = b + \frac{3}{7}$$

$b = a + \frac{1}{7}$

Figure 7-109: Lewis's response to Task D2e (ATQ)

Lewis's confident multiplicative thinking clearly underpins his algebraic thinking. He is confident in using symbolic expressions and solving for unknowns using an equation method but unfortunately, he sometimes sets up an incorrect equation. He expressed generalisations symbolically as shown in the Structured Interview, but his confident algebraic thinking exhibits some incorrect thinking. Lewis is better positioned to overcome these areas of weakness than most other students described in these case studies. Lewis is deemed to be at *Algebraic Generalisation- Symbolic* level of the *Emerging Algebraic Reasoning Framework*.

7.9 Case Study 9: Jack

Jack (Year 6) was chosen for a case study as he is an example of the *Algebraic Generalisation - Symbolic* group at Level 6 of the *Emerging Algebraic Reasoning Framework* shown in Table 6.12. He used advanced multiplicative strategies for all tasks of the Structured Interview and used Symbolic multiplicative methods for the final task with 'any fraction' and 'any number' of objects. He was chosen to be interviewed as he used the same method for all three reverse fraction tasks. He divided each number of objects by the fraction that those objects represented. Jack's score for the FST was the same as Lewis's and more than one standard deviation above the mean score for the whole cohort. Jack had the highest ATQ score of the case study students (see Table 7.2).

Section 7.9.1 will discuss his responses to the FST, Section 7.9.2 the strategies he used for the reverse fraction tasks (Tasks C5 - C7), Section 7.9.3 the strategies he used for the Structured Interview and Section 7.9.4 his responses to the ATQ.

7.9.1 Fraction Screening Test

Jack scored 33.5 out of a possible 35 marks (96%) for the FST which was similar to that achieved by Lewis.

Jack's score for Part A tasks was 10.5 out of a possible 12 marks where the mean for all students was 9.63 and the standard deviation was 2.22. He chose the two correct shapes representing three-

quarters, circled one-third of a group of beetles, and indicated that there would be eight lollies in the whole group when four lollies represented one-half of the group. Jack accurately shaded one-third of a rectangle and three-quarters of a length. He correctly stated the equivalent fractions and calculated the total amount was \$28 if \$4 was one-seventh. He calculated the fraction of pizza that each person would receive if three pizzas were shared between four and gave the next number to complete the backwards counting sequence for two-fifths. However, Jack was unable add the fractions with related denominators but appears instead to have found the difference between the two fractions: $\frac{3}{10} + \frac{1}{5} = \frac{1}{10}$

Jack scored 10 marks out of a possible 10 marks for the Part B tasks where the mean for all students was 6.21 and the standard deviation was 2.67. He successfully wrote the missing numbers on the four equally spaced marks on the number line marked zero to 100 and stated that “I counted all the lines excluding the first one and there was five, so I then divided five by 100 and got 20. I just started from 0 and added 20 to each line”. However, he divided 100 by five, and not “five by 100”, as he stated in his explanation. Jack placed the number 75 on a number line marked zero and 25 using the length from 0 to 25 as a measure: “I marked from 0 to 15 on a rubber and used that information to find 50 then 75”. On the same number-line he successfully marked in the number five “I roughly made 5 equally apart lines to find the number 5”. On the number-line he marked the four lines he made between 0 and 25 correctly: 5, 10, 15 and 20.

Jack accurately marked three-fifths of the number line marked zero to two drawing in the interim marks. Jack used the distance from zero to one-third to mark in the number two-thirds and then the number one. He also chose the correct option for the placement of one on a number line marked zero and $\frac{3}{4}$ as he marked in the $\frac{1}{4}$ and $\frac{2}{4}$. His explanation stated: “I first roughly estimated $\frac{1}{4}$ and $\frac{2}{4}$ and with that answer I got the answer of C”. Jack successfully completed all number-line tasks and his ability to use multiplicative relationships is strongly evident in all these tasks.

Part C tasks included ordering a set of four fractions from largest to smallest, drawing a line to connect fractions and their equivalent decimals and circling the ‘one that does not belong’ from the set of fractions, decimals and percentages. Jack successfully ordered the four fractions, matched all four fractions with their equivalent decimals and correctly circled the representations that did not belong to the set of alternatives offered.

7.9.2 Reverse Fraction Tasks

Jack used the same calculation method for all three of the reverse fraction tasks. (Figure 7.110 and Figure 7.111). He divided the given number of counters by the given fraction, applying this method for all quantities and all fractions.

5. This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.  $\frac{10}{1} \div \frac{2}{3} = \frac{10}{1} \times \frac{3}{2} = 15$ a. How many counters did I start with? 15 b. Explain how you decided that your answer is correct. I simply divided $\frac{10}{1}$ by $\frac{2}{3}$ which led me to the answer of 15.	6. Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? 21 Show all your working. $\frac{12}{1} \div \frac{4}{7} = \frac{12}{1} \times \frac{7}{4} = 21$
--	--

Figure 7-110: Jack's responses to Task C5 and Task C6 (FST)

This 'short cut' method raises the question as to whether Jack is applying a rule that he may have been taught without fully understanding why the rule works. On the other hand, it may be that Jack is employing an inverse relationship which underpins a truly algebraic solution strategy. This question can only be resolved by probing Jack's responses on the Structured Interview and the ATQ.

7. This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.  $\frac{14}{1} \div \frac{7}{6} = \frac{14}{1} \times \frac{6}{7} = 12$ a. How many counters did I start with? 12 b. Explain how you decided that your answer is correct. Just like I did with number 5 a simply divided $\frac{14}{1}$ by $\frac{7}{6}$ which gave me the answer of 12.

Figure 7-111: Jack's response to Task C7 (FST)

7.9.3 Structured Interview

Jack's responses to each of the questions on the Structured Interview where the number of objects were given for each of three fractions: two-thirds, four-sevenths, seven-sixths were completed in the same way as his responses to the three reverse fraction tasks. That is, he divided the number representing the fraction, by the given fraction to obtain a correct solution. His responses to Question 1 are shown in both columns of Figure 7.112. In the left-hand column is his response to

the specific question asked and in the right-hand column is his response for ‘any number’ of counters that represent two-thirds (a).

$12 \div \frac{2}{3} = \frac{12}{1} \times \frac{3}{2} = 18$	$a \div \frac{2}{3} = \frac{a}{1} \times \frac{3}{2} = x$ a is the number I have now. x is the number I had.
--	---

Figure 7-112: Jack's response to Question 1 (SI)

In Question 4b, 5b and 6b, where the fractions were the same, but students were asked to consider what they would do for ‘any number’ of counters, Jack used similar responses to those in the right-hand column of Figure 7.112. When asked about what he would do if he had ‘any number’ of counters that represented ‘any fraction’ in Question 7 his response is given in Figure 7.113.

$\frac{a}{1} \div \frac{b}{c} = \frac{a}{1} \times \frac{c}{b} = d$	a is the number I have b is the numerator of the starting fraction. c is the denominator of the starting fraction. d is the number I had.
---	--

Figure 7-113: Jack's response to Question 7 (SI)

This fully generalised and well-articulated algebraic response refutes any suggestion that Jack is simply repeating a rule he has been taught. He is deemed to be at the *Algebraic Generalisation-Symbolic* level of the *Emerging Algebraic Reasoning* Framework. It is important, however, to consider his written responses for the ATQ.

7.9.4 Algebraic Thinking Questionnaire

In the ATQ, Jack scored 11.5 marks for Part M and 8 marks for Part D. Overall Jack scored 19.5 out of a possible 28 marks (70%) for the ATQ, where the mean is 6.86 and the standard deviation is 6.45 for the whole cohort.

For Task M1a Jack demonstrated a clear understanding of equivalence using the relationship between 36 and 9 as shown in Figure 7.114 when he said: “They divided 36 by 4 so to get the same answer you must multiply 25 by 4”. He has demonstrated his understanding of the sameness-relational and substitutive-relational understanding of the equals sign.

$$36 \times 25 = 9 \times \boxed{100}$$

They divided 36 by 4 so to get the same answer you must $\times 25$ by 4.

Figure 7-114: Jack's response to task M1a (ATQ)

However, for Task M1b he needed to complete the multiplication of 48 by 2.5 then divided the result (120) by 10 to calculate the correct answer of 12 as shown in Figure 7.115.

$$48 \times 2.5 = \boxed{12} \times 10$$

$$\begin{array}{r} 48 \\ \times 2.5 \\ \hline 240 \\ 960 \\ \hline 120.0 \end{array}$$

$$\frac{120}{10} = 12$$

Figure 7-115: Jack's responses to Task M1b (ATQ)

To calculate the answer for the two tasks involving multiplication by fractions Jack used the same method as he had used for the three reverse fraction tasks (Figures 7.110 and 7.111). To find the missing number for the empty box Jack divided the number on the right-hand side of the equation by the fraction (Figure 7.116). It seems that his reliance on this method, which is totally appropriate in the first question, may prevent him from seeing a simpler solution in the second question where $\frac{5}{2}$ is simply the multiplicative inverse of $\frac{2}{5}$.

$$\frac{2}{3} \times \boxed{27} = 18 \quad \frac{18}{\frac{2}{3}} = \frac{18}{1} \times \frac{3}{2} = 27$$

$$\frac{2}{5} \times \boxed{\frac{5}{2}} = 1 \quad \frac{1}{\frac{2}{5}} = \frac{1}{1} \times \frac{5}{2} = \frac{5}{2}$$

Figure 7-116: Jack's responses to Tasks M1c and M1d (ATQ)

In Question 2a, Part A, students were required to put numbers in Box A and Box B so that the expressions on both sides of the equal sign were equivalent. Jack gave three correct examples for the missing numbers when only two were required (see Figure 7.117). He also gave 2-digit examples when many other students chose much easier examples. The examples in Figure 7.117 are evidence of Jack's sameness-relational and substitutive-relational understanding of the equals sign (Jones et al., 2013).

2. Can you think about the following mathematical sentence:

$$5 \times \boxed{82} = 10 \times \boxed{41}$$

Box A Box B

a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$$5 \times \boxed{72} = 10 \times \boxed{36}$$

Box A Box B

$$5 \times \boxed{76} = 10 \times \boxed{38}$$

Box A Box B

Figure 7-117: Jack's responses to Task M2a (ATQ)

Jack correctly identified that Box A “is just twice the size of Box B” and suggested that any number could be placed in Box A “I guess it can because then Box B will just simply be a decimal answer”. Jack gave a correct relationship between c and d in Figure 7.118 but wrote the symbolic expression incorrectly.

d) What can you say about c and d in this mathematical sentence?

$$c \times 2 = d \times 14$$

$d = 7c$

That c will be 7 times the value of d .

Figure 7-118: Jack's response to Task M2d (ATQ)

However, when both sides of the equation included fractions Jack's response was an additive one where he subtracted $1\frac{1}{2}$, written as both $1\frac{2}{4}$ and $\frac{6}{4}$, from $\frac{3}{4}$ (Figure 7.119).

e) What can you say about a and b in this mathematical sentence?

$$a \times \frac{3}{4} = b \times 1\frac{1}{2}$$

$$\frac{3}{4} - 1\frac{2}{4} = \frac{3}{4} - \frac{6}{4} = \frac{3}{4}$$

~~base~~ a will be $\rightarrow$ ~~3~~ times bigger than b

Figure 7-119: Jack's response to M2e (ATQ)

Despite very competent algebraic thinking demonstrated in the Structured Interview, and earlier questions of the ATQ there appears to be a ceiling on what Jack is able to do. This is evident in some subsequent questions.

In Part D of the ATQ the focus was on division tasks. In Question 1 the focus was on the relationship between the two expressions. In the first task, Jack rewrote three divided by four as the fraction three-quarters. He then divided 15 by three-quarters by multiplying by $\frac{4}{3}$ which gave 242

him the correct answer of 20. In the second task, Jack divided 16 by 30 and simplified it to get $\frac{8}{15}$. To complete the calculation, he multiplied 15 by $\frac{8}{15}$ to get the correct answer (8) as shown in Figure 7.120. This competent arithmetical approach shows that Jack is less confident to transfer his understanding of equivalence using multiplication to include division relationships.

Figure 7-120 shows handwritten work for two tasks. The first task is $3 \div 4 = 15 \div \boxed{20}$. The student has written $\frac{3}{4} = \frac{6}{8} = \frac{3}{4}$ and $\frac{15}{1} \times \frac{8}{4} = \frac{15}{1} \times \frac{2}{1} = 20$. The second task is $\boxed{8} \div 15 = 16 \div 30$. The student has written $\frac{16}{30} = \frac{8}{15}$ and $\frac{15}{1} \times \frac{8}{15} = \frac{15}{1} \times \frac{8}{15} = 8$.

Figure 7-120: Jack's responses to Tasks D1a and D1b (ATQ)

In Task D1c Jack completes each side of the equation separately. Initially he solves the right-hand side of the equation (6 divided by $\frac{2}{3}$) and then divides 18 by the answer from the right-hand side to correctly write the number two in the empty box. Taking two steps to solve this equation guaranteed his success as shown in Figure 7.121.

Figure 7-121 shows handwritten work for Task D1c. The equation is $18 \div \boxed{2} = 6 \div \frac{2}{3}$. The student has written $\frac{6}{1} \div \frac{2}{3} = \frac{6}{1} \times \frac{3}{2} = 9$ and $\frac{18}{1} \div \frac{9}{1} = \frac{18}{1} \times \frac{1}{9} = 2$.

Figure 7-121: Jack's response to Task D1c (ATQ)

Jack was unsuccessful with Task D1d as he set up his equation incorrectly. On the left-hand side of the equation he wrote $\frac{6}{7}$ (Figure 7.122) but it looks as though he may initially have thought that he should multiply by $\frac{7}{6}$ but changed his mind and divided by $\frac{7}{6}$.

Figure 7-122 shows handwritten work for Task D1d. The equation is $\frac{7}{6} \div \boxed{\frac{6}{7}} = 1$. The student has written $\frac{1}{1} \times \frac{7}{6} = \frac{1}{1} \times \frac{6}{7} = \frac{6}{7}$.

Figure 7-122: Jack's response to D1d (ATQ)

Jack gave two incorrect responses to Task D2a as shown in Figure 7.123. When asked about the relationship between Box A and Box B his response that the numbers in Box A were five times

larger than the numbers in Box B was true for the incorrect numbers he had written. As he had for the multiplication task, he recognised that any number could be used in Box A provided: “you’re ready to go to the decimal places”.

$$\begin{array}{ccc} 3 \div \boxed{25} = 15 \div \boxed{5} \\ \text{Box A} & & \text{Box B} \\ 3 \div \boxed{75} = 15 \div \boxed{15} \\ \text{Box A} & & \text{Box B} \end{array}$$

Figure 7-123: Jack's response to Task D2a (ATQ)

For Task D2d Jack correctly identified that d would need to be three times larger than c to ensure that the following was correct: $c \div 8 = d \div 24$. In Figure 7.124 Jack found the difference between three-sevenths and two-sevenths and wrote that a would one-seventh bigger than b .

(e) What can you say about a and b in this mathematical sentence?
 $\frac{6}{7}a = b$ $a \div \frac{2}{7} = b \div \frac{3}{7}$
 $\frac{3}{7} - \frac{2}{7} = \frac{1}{7}$
 a will be $\frac{1}{7}$ times bigger than b .

Figure 7-124: Jack's response to D2e (ATQ)

Jack’s algebraic thinking is stronger than most students in the Year 6 sample, and as strong as many competent Year 8 students interviewed. However, some limitations are clearly evident as shown in Figure 7.123 and Figure 7.124. Jack is deemed to be at *Algebraic Generalisation-Symbolic* level of the *Emerging Algebraic Reasoning Framework*.

7.10 Summarising the case studies

The FST provided students with opportunities to display multiplicative reasoning for many of the tasks. In particular, the three reverse fraction tasks offered the best window into some students’ capacity to apply strategies that were partly multiplicative but still depending on part-part-whole approaches to construct an unknown whole, and other students to apply strategies that were fully multiplicative. It was evident that a few students relied on pictures and diagrams to search for multiplicative relationships. While some students used pictures and diagrams effectively to solve individual tasks, these students appeared to consider each task as a new problem requiring a new diagram. These solutions are not a sound foundation for generalisations and algebraic thinking.

These two areas were probed more deeply in the Structured Interview. For students who were interviewed, the Structured Interview successfully ‘moved’ some students away from part-part-whole strategies towards fully multiplicative approaches. Those students who were able to make this transition were well placed to offer fully generalised solutions which exhibited clear elements of algebraic thinking which was sometimes expressed symbolically but sometimes was not. Reliance on part-part-whole strategies frustrated some students’ attempts to deal with the most generalised form of the reverse fraction task, from ‘some fraction’ and ‘some number’ of equivalent objects.

Reference to the ATQ allowed the researcher to draw on a wider range of tasks incorporating using the key idea of equivalence. These tasks used number sentences having one unknown, sentences with two unknowns and sentences using letter-symbolic terms representing two unknowns. Where students had demonstrated generalisable algebraic thinking on the Structured Interview, their performance on this wider range of tasks embodying algebraic thinking allowed the researcher to identify how widely students could apply algebraic thinking and algebraic operations. Even cases where it had been demonstrated that a student’s algebraic thinking was well developed across a variety of tasks it was still clear that some of the ATQ tasks, typically involving division, presented challenges even to the most accomplished students at Year 6 and Year 8. However, there were students who correctly answered some of these more difficult questions even though they were not as successful overall on the ATQ. It was also evident that some students performed well on the ATQ tended to rely on solutions that they had used confidently to solve other tasks but were not the most appropriate methods for these more challenging tasks.

Chapter 8: Discussing the Research Questions

Chapter 8 focuses on the integration of the quantitative and qualitative data results from all three assessment instruments: The Fraction Screening Test (FST), the Algebraic Thinking Questionnaire (ATQ) and the Structured Interview (SI). The results from these assessment instruments are used to inform the responses to the seven research sub-questions designed to address the overall research question: “How does middle-years students' fractional competence and reasoning show evidence of algebraic reasoning?” Table 8.1 indicates the instruments used to construct responses to the research sub-questions.

Table 8.1: Assessment instruments analysed for responses to research sub-questions

Research Questions	FST			ATQ		SI
	A	B	C	M	D	
RQ1. Is overall fraction ability as measured by a broad fraction screening test a predictor of success on reverse fraction tasks?	X	X	X			
RQ2. How are performances on specific fraction topics linked to success on the reverse fraction tasks?	X	X	X			
RQ3. To what degree do proficient multiplicative thinkers for fraction tasks (i.e. students who demonstrate relational thinking, represent fractions in various ways and use reverse or reciprocal thinking when solving rational number tasks) use solution strategies that demonstrate emerging or established generalised algebraic reasoning			X	X	X	X
RQ4. How does middle-years students' fractional competence and reasoning show evidence of non-symbolic algebraic thinking and its progressive transition towards more explicit forms of symbolic thinking typically introduced in lower secondary classes?			X	X	X	X
RQ5. How is competence with reverse fraction tasks associated with equivalence, transformation using equivalence, and generalisation anticipated in algebraic thinking?			X	X	X	X
RQ6. What is the relationship between verbal fluency in describing mathematical reasoning in the context of fraction tasks and demonstrated generalisations that are anticipated in algebraic thinking?			X	X	X	X
RQ7. How can students' level of emerging algebraic reasoning be described and identified so that teachers may monitor and prompt students' progress?			X			X

In this chapter descriptive statistics are calculated using the Statistical Package for the Social Sciences (SPSS) and reported following the guidelines given by Pallant (2011). That is, small correlations (weak relationships) are values between 0.1 and 0.29, medium correlations (moderate relationships) are values between 0.3 and 0.49 while large correlations (strong relationships) are values between 0.5 and 1.0 (Pallant, 2011, p.134).

8.1 Research Question 1

RQ1: Is overall fraction ability as measured by a broad fraction screening test a predictor of success for the reverse fraction tasks?

As stated earlier, for this research, fraction ability or competence is deemed to be the demonstrated understanding of fractional topics appropriate for students in the middle-years as stated in the Australian Curriculum: Mathematics (ACARA, 2018) and shown in Table 1.4. The FST is a broad screening test as it includes tasks relating to fraction size and relationships and successful arithmetic computation, for example: geometric representations of fractions, counting by fractions, addition of fractions with related denominators, partitioning and iterating number-lines, equivalence of fractions, decimals and percentages and ordering fractions. Fraction ability for the FST is reflected by students' total scores. That is, it is assumed that the higher the total score achieved by students in response to all three parts of the FST the greater the overall ability with fractions. The tasks in the FST are described in detail in Chapter 3 and the results for each individual task are given in Chapter 5.

To determine whether overall fraction ability is a predictor of success for the three reverse fraction tasks (Tasks C5, C6 and C7) two scatterplots were created to compare scores for the three reverse fraction tasks with scores for the FST. The first scatterplot shown in Figure 5.23 compares the scores for the three reverse fraction scores (Tasks C5, C6 and C7) with the total scores for the FST. This graph shows that there is a strong positive correlation (0.84) between these two scores and nearly 70% of the variation of the scores for the three reverse fraction tasks may be explained by the total scores for the FST ($R^2=0.67$). However, the three reverse fraction tasks are a subset of the FST, so a second scatterplot was drawn (Figure 5.24) to enable a comparison to be made for the scores of three reverse fraction tasks with the total scores of the FST tasks (without the three reverse fraction tasks). The total possible score for the FST without the three reverse fraction tasks is 26 marks. The relationship, shown by a positive linear trend-line, indicates that there is a strong positive relationship (0.63) and that nearly 40% of the variation for the three reverse fraction scores may be explained by the total scores for the remaining FST tasks ($R^2=0.39$).

The graph in Figure 5.24 shows that students who scored low marks on the FST also scored low marks for the reverse fraction tasks. For example, students who scored in the bottom one-third of scores for the FST scored in the bottom one-third for the reverse fraction tasks. Students who scored more than six out of nine on the reverse fraction tasks scored more than 14 out of a possible 26 on the FST. Students who used more sophisticated strategies to solve the three reverse fraction tasks demonstrated a greater overall fraction ability or competence on the FST. The level of success on the reverse fraction tasks appears to be related to the level of success demonstrated by the total scores for the FST. So, in response to RQ1 *overall fraction ability* as measured by a broad fraction screening test (The FST) appears to be *a predictor of success on reverse fraction tasks*.

8.2 Research Question 2

RQ2. *How are performances on specific fraction topics linked to success on the reverse fraction tasks?*

To determine whether performances on specific fraction topics were linked to success with the reverse fraction tasks scores for the three reverse fraction tasks were compared with the scores for each individual task of the FST. Table 5.17 shows the relationships as correlations between total scores for the three reverse fraction tasks (Tasks C5, C6 and C7) and the individual scores for the routine fraction tasks in Part A. Table 5.18 shows the correlations between the three reverse fraction tasks total scores and scores for the number-line tasks in Part B while Table 5.19 shows the correlations between the three reverse fraction tasks total score and the four remaining Part C task (i.e. Tasks C1 – C4) which test understanding of fractions.

8.2.1 Relationship between reverse fraction tasks and Part A tasks

As shown in Table 5.17 the relationship between the scores for the three reverse fraction tasks and the individual Part A tasks ranged from little correlation (0.09), a small correlation or weak relationship (0.14 - 0.26) and medium correlation or moderate relationship (0.33 – 0.39) with no Part A task demonstrating a strong relationship (> 0.5) with the scores for the three reverse fraction tasks. There is a weak positive relationship for eight of the Part A tasks (A1, A2, A4, A5, A7, A8a, A8b & A10) and the total scores for the three reverse fraction tasks. These correlations ranged from 0.14 for the finding the number of objects in the whole group if given the number in one-half of the group (A4), to 0.26 for shading a fractional part of a grid (Task A5).

There is a moderate positive relationship with the three reverse fraction tasks and four Part A tasks (A6, A9, A11 and A12). Task A6, the sharing pizza task, has a correlation of 0.33 while completing the equivalent fraction task (Task A9) has a correlation of 0.36 with the reverse fraction tasks. Task A12 requiring students to find the total amount of money, if given the amount representing one-seventh, has a correlation of 0.38 with the three reverse tasks and Task A11, where students are required to add fractions with related denominators, has a correlation of 0.39.

Part A tasks have been included in the FST as they represent the types of fraction tasks middle-years students would be used to completing in class. The four Part A tasks that have a moderate positive relationship with the three reverse fraction tasks were the tasks that students found to be the most difficult in Part A. It is concerning to note that there were students who achieved above average scores on the FST and successfully answered the reverse fraction tasks but who held the misconception that addition of fractions (Task A11) involves adding numerators and adding denominators. This suggests that some students who appear to be competent with some fraction tasks still hold fundamental misconceptions as they appear to rely on rules and procedures rather than understanding. For example, in Task A11 students needed to add three-tenths and one-fifth. Those who added numerators and denominators would have had a solution of four-fifteenths demonstrating that they are using their whole number knowledge for this fraction task.

8.2.2 Relationship between reverse fraction tasks and Part B tasks

As shown in Table 5.18 the correlations between the reverse fraction tasks and the individual Part B number-line tasks had either a small correlation or weak relationship (0.15 - 0.27) or a medium correlation or moderate relationship (0.30 – 0.39). As for the Part A tasks there were no results for the Part B tasks showing a strong relationship (> 0.5) with the three reverse fraction tasks.

There is a weak positive relationship between the three reverse fraction tasks and four of the Part B tasks (B1a, B2a, B2b, B2c) ranging from a small correlation or weak relationship of 0.15 for the three reverse fractions and Task B2a (placing 75 on the number line marked zero to 25), 0.19 for Task B1a (placing numbers on the 4 marks between zero and 100), 0.25 for an explanation of the students' responses to Task B2a (i.e. B2b) and 0.27 for marking where the number 5 was on the number line marked 0 – 25 (B2c).

Three Part B tasks (B1b, B2d, B5b), which required students to explain their responses to number-line tasks using whole numbers, had moderate positive relationships with the three reverse fraction tasks. Tasks B1b and B2d both had the same correlation (0.41) with the reverse fraction tasks while Task B5b had a correlation of 0.44 with the three reverse fraction tasks. All three

tasks asked students to explain their responses to the previous task. Tasks B1b and B2d required students to: “Explain how you decided that your answer is correct” while Task B5b asked students to explain: “How did you decide the other two choices were incorrect?”

There were similar moderate positive relationships between the three reverse fraction task scores and the three number-line questions using fractions. The correlations were 0.30 for Task B3 which required students to partition a number line marked zero to two to mark the number place three-fifths and 0.37 for B4 and B5a respectively. Task B4 requires students to iterate the length marked zero and one-third to place the number 1 accurately on a number-line. Task B5a requires students to choose the position of number 1 from three alternatives when given the distance from zero to three-quarters.

A stronger relationship was expected when comparing the results for the three reverse fraction tasks and the number-line tasks, especially for the number-line tasks where the length for a fractional piece was given and students were asked to mark the position of the whole. For example, when responding to Task B4 (FST) many students just used a 'best guess' for the placement of the number one while other students iterated the length given for one-third two more times. (See Figure 5.11). Since students started with a length representing a fractional piece and were required to mark the position of the whole these tasks could also be deemed to be 'reverse fraction' tasks.

8.2.3 Relationship between reverse fraction and other Part C tasks

Table 5.19 shows that there is a moderate positive correlation between the scores for each of the other Part C tasks (C1 – C4) and the total scores for the three reverse fraction tasks (C5 – C7). The relationships ranged from a correlation of 0.32 for Task C4 (choosing the symbol which is not equivalent to 1.2), 0.33 for Task C3 (choosing the symbol that is not equivalent to one-quarter), 0.36 for Task C2 (matching equivalent fractions and decimals) and 0.39 for ordering fractions (Task C1) with the total score for the three reverse fractions.

The highest correlations with the scores for the reverse fraction tasks included the tasks which tested students' understanding of fractions. For example, the tasks which asked students to explain their responses to the Part B number-line tasks and the four, Part C tasks where students needed to choose equivalent expressions and order fractions. In response to RQ2 the performances on some individual fraction tasks were only weakly related, with some moderately related, but no task was strongly related to success with the reverse fraction tasks.

However further analysis of the relationships for the three parts of the FST with the three reverse fraction tasks have revealed that there is a stronger relationship with the scores for the Parts than the individual tasks.

8.2.4 Relationship between reverse fraction tasks and Parts A, B, C (FST)

After comparing the scores for individual tasks with the reverse fraction tasks, the overall results for the three Parts of the FST were compared with the three reverse fraction task scores. Table 5.16 shows the correlations between the total score for the three reverse fraction tasks and the total scores for Part A tasks, total Part B task scores, and total scores for the Part C tasks: C1 – C4.

Figure 5.25 shows the graph of the relationship between the students' total scores for Part A (FST) and the three reverse fraction tasks (0.54). The relationship is modelled by a positive linear trend-line, indicating that nearly 30% of the variation for the three reverse fraction scores, may be explained by the scores for Part A tasks ($R^2=0.29$).

Figure 5.26 shows a scatterplot of the students' scores for the Part B (FST) scores and the three reverse fraction tasks (0.57). The relationship is modelled by a positive linear trend-line, indicating that nearly one-third of the variation of the three reverse fraction scores FST scores, may be explained by the scores for Part B tasks ($R^2=0.32$).

Part C tasks were mainly non-traditional tasks that tested students' fractional understanding. Figure 5.27 compares the scores from the first four tasks of Part C (Tasks C1 –C4) with the scores from the reverse fraction tasks (0.53). The relationship is modelled in Figure 5.27 by a positive linear trend-line, indicating that nearly 30% of the variation of the scores for the three reverse fraction tasks, may be explained by the scores for the first four Part C tasks ($R^2=0.28$).

While no individual task from the FST was found to have a strong relationship with the three reverse fraction tasks the scores for many tasks were found to be moderately positively correlated with the scores for the Part C reverse fraction tasks.

In response to RQ2 performances on some individual fraction tasks were only weakly related, with some moderately related, but no task was strongly related to success with the reverse fraction tasks. However, there is a positive strong relationship between the scores for each Part of the FST and the reverse fraction tasks. Many students were successful with some tasks but demonstrated well documented misconceptions for others. This, together with their lack of explanations,

suggests that many students do not expect mathematics to make sense but just apply a rule or procedure which may or may not be the correct one.

8.3 Research Question 3

RQ3. To what degree do proficient multiplicative thinkers for fraction tasks (i.e. students who demonstrate relational thinking, represent fractions in various ways and use reverse or reciprocal thinking when solving rational number tasks) use solution strategies that demonstrate emerging or established generalised algebraic reasoning?

The written responses for the three reverse fraction tasks from the FST (Tasks C5, C6 and C7) and the responses given both orally and written for the Screening Interview will be examined for evidence of emerging and established algebraic reasoning. In particular, the emphasis will be on the responses deemed to demonstrate fully or advanced multiplicative strategies.

8.3.1 The written responses to the three reverse fraction tasks (FST)

The students' written solutions to the three reverse fractions (FST) were analysed according to the *Framework for reverse fraction task strategies* (Table 5.15). Successful strategies for the three reverse fraction tasks ranged from the concrete (diagram dependent), strictly computational (additive or partly multiplicative) to the generalisable (fully multiplicative) and algebraic (advanced multiplicative).

Many students used non-multiplicative approaches successfully to solve all three reverse fraction tasks. For example, Student BM (Figure 4.1) for Task C5 depended on the diagram to determine the number of counters in the whole group. Student JB (Figure 4.3) used a partially multiplicative strategy for Task C6 as he divided 12 by four to find one-seventh then multiplied by three to find three-sevenths. He then added the number of CDs representing three-sevenths (9) to the original number of CDs representing four-sevenths (12) to get the correct response of 21. Students using these strategies are using computational methods and not demonstrating either emerging or generalised algebraic reasoning. These strategies are appropriate for the three reverse fraction tasks given in the FST but cannot be applied when either the fraction or the number representing the fraction is not known as given in the Structured Interview.

Proficient multiplicative thinkers for fraction tasks did not just apply a rule or procedure but instead demonstrated relational thinking, represented fractions in various ways and used reverse or reciprocal thinking. Students using fully multiplicative strategies used division to find the quantity representing the unit fraction and then multiplied this quantity to calculate the whole.

Students using fully multiplicative methods tended to use the same method regardless of the fraction presented in the task. For example, Student DC uses the same multiplicative solution method consistently for the three reverse fraction tasks (Figure 4.4).

Students whose solution strategies were classified as advanced multiplicative used either a one-step method to find the whole by dividing the given quantity by the fraction (reverse or reciprocal thinking) or used the appropriate algebraic notation to find the whole. Student JP divided the number of objects representing the given fraction by the fraction to successfully calculate the number of objects in the whole group (Figure 4.8). This student used consistent advanced multiplicative methods for all three reverse fraction tasks. Since this was not a method this student had been taught in school and these were unfamiliar tasks this student is demonstrating evidence of emerging algebraic reasoning.

8.3.2 The Structured Interview

In the Structured Interview fraction tasks were sequenced, moving beyond a known fraction with the related known quantity of objects to a known fraction with an unknown quantity objects to finally an unknown fraction and its related unknown quantity. The format and development of the Structured Interview is described in detail in Chapter 4 and the results of the interviews are documented in detail in Chapter 6 with detailed case studies of students' responses in Chapter 7.

The responses to each individual task for the Structured Interview were scored using the *Framework for reverse fraction task strategies* (Table 5.15). As for the written responses for the three reverse fraction tasks, students' responses to the Structured Interview tasks ranged from the concrete (diagram dependent), strictly computational (additive or partly multiplicative) to the generalisable (multiplicative) and algebraic (advanced multiplicative). The interviewers were able to ask more probing questions to clarify the students' responses, unlike the written responses to the FST where no clarification could be sought. Not all students were able to move to the more generalised fraction tasks with "any number of objects" or "any fraction", "any number" of objects.

The next step was to analyse the overall results of the Structured Interview to see whether the students used responses that demonstrated generalised algebraic reasoning. The *Emerging Algebraic Reasoning Framework* (Table 6.12) allowed students' responses to the overall Structured Interview to be classified as strictly arithmetic (computational fluency), beginning to generalise (additive, multiplicative) to the fully generalised demonstration of algebraic reasoning (verbal, symbolic).

Students who used arithmetic responses were only able to respond correctly to tasks where all the information was given. That is, if they were given the fraction, and the number of objects representing the fraction, students could then use the appropriate rule or procedure to calculate the number of objects representing the whole group. Some students showed that they were starting to generalise when they consistently used additive, multiplicative or a mix of methods in response to tasks with a given fraction and ‘any quantity’. Some students were able to move beyond their arithmetical strategies to more generalizable strategies when interviewed. However, some students were unable to give an appropriate generalised response to the fraction task with ‘any fraction’ and ‘any quantity’. These students were classified as demonstrating emerging generalised multiplicative thinking. While students who successfully solved ‘any fraction’ and ‘any quantity’ either orally or symbolically were deemed to be demonstrating algebraic reasoning.

Both the written responses to the three reverse fraction tasks from the FST and the verbal and written responses from the Structured Interview allowed students to use solution strategies that demonstrate emerging or established generalised algebraic reasoning as demonstrated in Table 6.12. These students recognised that the procedures and strategies they used to solve one reverse fraction task could be repeated for all fractions and quantities and then used with the task with ‘any fraction’ represented by ‘any quantity’. In these cases, the given fractions were seen by the students as what Fujii & Stephens (2001) refer to as ‘quasi-variables’.

In response to RQ3 students in this study deemed to be proficient multiplicative thinkers represented fractions in various ways and used relational, reverse or reciprocal thinking to solve rational number tasks. Students using fully multiplicative strategies or advanced multiplicative thinking to solve the reverse fraction tasks, either orally or in writing, used solution strategies that demonstrated emerging or established generalised algebraic reasoning.

8.4 Research Question 4

RQ4. How does middle-years students’ fractional competence and reasoning show evidence of non-symbolic algebraic thinking and its progressive transition towards more explicit forms of symbolic thinking typically introduced in lower secondary classes?

The three reverse fraction tasks provided an initial lens into the types of strategies students use when presented with these types of tasks. In both the FST and the Structured Interview students’ responses to the reverse fraction tasks, both written and verbal, were able to be classified according to the *Framework for Reverse Fraction Task Strategies* given in Table 5.15. Students deemed to be using fully multiplicative strategies found the quantity representing the unit fraction

using division and then scaled up (or down) to find the number of objects representing the whole. These types of multiplicative strategies show evidence of algebraic thinking and its progressive transition towards more explicit forms of symbolic thinking (Radford, 2012) as expected in lower secondary classes. Students using advanced multiplicative strategies correctly used the appropriate algebraic notation to find the whole, or a one-step method to find the whole by dividing the given quantity by the known fraction.

In the Structured Interview students who solved Question 7 with ‘any fraction’ and ‘any quantity’ either verbally or algebraically, demonstrated that they had established generalised algebraic reasoning. These students would be classified as being at either *Algebraic Generalisation-Verbal* or *Algebraic Generalisation-Symbolic* on the *Emerging Algebraic Reasoning Framework* given in detail in Table 6.12.

8.5 Research Question 5

RQ5. Is competence with reverse fraction tasks associated with equivalence, transformation using equivalence, and generalisation anticipated in algebraic reasoning?

Researchers such as Kieran (1981), Jacobs et al. (2007), Mason et al. (2009) and Stephens and Ribeiro (2012) suggested that students' understanding of equivalence, transformation using equivalence and the use of generalisable methods are important aspects of algebraic thinking. These three aspects of algebraic thinking underpinned this research. Thus, competence with the reverse fraction tasks included having a correct response for each task as well as a focus on the strategies that students used to solve these tasks. Students' responses to the three reverse fraction tasks have shown that successful strategies varied from the concrete (diagram dependent), strictly arithmetic (additive, partially multiplicative) to the generalisable (fully multiplicative) and algebraic (advanced multiplicative). Students successfully responded to unfamiliar situations by employing a range of mathematical strategies.

8.5.1 Three reverse fraction tasks

When applying the *Framework for Reverse Fraction Task Strategies* (Table 5.15) to students' responses for the three reverse fraction tasks some patterns emerged. There were three strategies that would be deemed to show evidence that students were using equivalence, transformation using equivalence and moving towards generalisation: partially multiplicative, fully multiplicative and advanced multiplicative strategies.

Students who used partially multiplicative strategies demonstrated their understanding of equivalence as they divided the number of objects representing the given fraction by the numerator of the given fraction to determine the number of objects in the unit fraction. Initially students found the number of objects representing the missing fractional part by scaling up the number of objects representing the unit fraction. The number of objects in the whole group was then achieved additively by adding the two quantities representing the two fractions. For example, in Task C6 students are given the number of objects that represent four-sevenths. Students using partially multiplicative strategies find the number of counters that represent one-seventh and multiply by three to find the quantity for three-sevenths. They then add the original quantity representing four-sevenths with the quantity for seven-sevenths to find the whole. See for example Student JB's response for Task C6 in Figure 4.3. These students demonstrate their understanding of the sameness-relational meaning of the equals sign (Jones et al., 2013).

Students who used fully multiplicative strategies used equivalence to transform the number of objects representing a given fraction to obtain the number of objects representing the unit fraction. These students continued to use transformations using equivalence to successfully scale up the unit fraction and its related quantities multiplicatively to find the whole. For example, in Task C6 students found the number of objects representing one-seventh by dividing by four (the numerator) and then multiplying the answer by seven to find the number representing seven-sevenths or the whole. These fully multiplicative methods mirror the thinking needed to solve the corresponding algebraic equations and can be seen as evidence of emerging algebraic thinking.

Students using the advanced multiplicative strategies appeared to be able to generalise. The size of the fraction and the size of the given number of objects representing that fraction appear to be irrelevant as their solution methods or strategies remain the same regardless of the fraction or the size of the group the fraction represents.

Some students used different solution strategies when either the fraction, or the corresponding number of objects changed. Some students using concrete or additive strategies were able to move to using multiplicative methods. Conversely, when a fraction and/or a quantity changed other students were unable to successfully complete the task. In the FST many students were successful with Task C5 because the diagram allowed them to proceed using diagrams but did not attempt, or gave an incorrect response, for the following reverse fraction tasks. Students using multiplicative strategies for Task C5 tended to use multiplicative strategies for all three reverse fraction tasks. Some students were struggling to make the successful transition to algebraic

thinking while others were able to generalise and use the same multiplicative strategies regardless of the size of the fraction or the number of objects the given fraction represented.

8.5.2 The Algebraic Thinking Questionnaire

The ATQ was designed to assess whether students were able to demonstrate algebraic reasoning. The results from the ATQ are discussed in more detail in Chapter 5. Competence with tasks from the ATQ was deemed to be the total score for the written test. Total scores ranged from 0 to 25.5 with a mean of 8.36 and standard deviation of 7.48. Part M of the ATQ had a mean of 5.14 (SD = 3.94) and Part D had a mean of 3.22 (SD = 3.88). Figure 5.70 shows that there is a strong relationship between the ATQ and Part C (FST) scores for the whole group of students. One consideration would need to be the number of non-responses for the ATQ and whether this may have had an impact on the results.

8.5.3 The Structured Interview

The Structured Interview specifically required students to use equivalence and transformation to deal with more generalised situations including those where the fraction and/or the number of objects were unknown. *The Emerging Algebraic Reasoning Framework* documents the levels and descriptions based on the interview data and is shown in Table 6.12.

Table 6.14 compares the interviewed primary students' dominant method for the three reverse fraction tasks compared to their level on the *Emerging Algebraic Reasoning Framework*. Students using *Diagram Dependent* or *Additive* methods for the three reverse fraction tasks did not show evidence of algebraic reasoning while three students who had used *Partially Multiplicative* methods for the written reverse fraction tasks were able to use verbal algebraic reasoning in the interview responding to the task with 'any fraction' and 'any number'. Five of the six primary students used algebraic reasoning for the last task while the student who had used advanced multiplicative methods for the written tasks used symbolic algebraic reasoning for the final interview question.

The Year 8 students' dominant method for the three reverse fraction tasks are compared to their level on the *Emerging Algebraic Reasoning Framework* and shown in Table 6.15. Altogether 18 of the 26 Year 8 students who were interviewed were deemed to be using either verbal or symbolic algebraic reasoning. All Year 8 students who used fully multiplicative strategies for the three reverse fraction tasks used algebraic reasoning for the interview tasks while three of the seven students who had used partially multiplicative strategies for the reverse fractions used verbal algebraic reasoning with 'any fraction' and/or 'any number'.

8.6 Research Question 6

RQ6: *What is the relationship between verbal fluency in describing mathematical reasoning in the context of fraction tasks and demonstrated generalisations that are anticipated in algebraic reasoning?*

In Australia the emphasis on the four proficiencies: *Fluency, Understanding, Problem Solving* and *Reasoning* for students solving mathematical tasks means there is also a need for well-developed communication skills so that students can demonstrate these proficiencies in their solutions to mathematical tasks. In the PISA report Thomson, De Bortoli and Underwood (2016) describe mathematical literacy as:

... an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective citizens.

According to Mason (2018) students need to be able to express their understandings about mathematical topics in order that they can make sense of the underpinning mathematics.

In order to make sense of mathematics, in order to appreciate and comprehend the topics that learners meet, it is necessary to use and develop one's natural powers to imagine and to express, to specialize and to generalize, to conjecture and to convince (p. 334).

Students had many opportunities to express their reasoning for the responses they gave to the written tasks in the FST and the ATQ. In the Structured Interview students were encouraged to articulate their responses before recording on paper.

However, it was disappointing to note that many students either chose not, or were unable, to justify their response to the written tasks particularly as there is now a greater focus on the proficiencies including reasoning (Table 1.3). Many tasks requiring a written verbal response remained unanswered in both the FST and the ATQ. For example, Task M2a (ATQ) shown in Table 5.28 relies on students understanding of equivalence of related whole numbers. Fifty-four percent of Year 5 students wrote two correct number sentences but 19% did not attempt a response for Task M2a while 73% of Year 8 students gave two correct responses only 6% of Year 8 did

not respond. These scores are unexpected as the Content Description from the Australian Curriculum: Mathematics (ACARA, 2018) states that Year 5 students:

Find unknown quantities in number sentences involving multiplication and division and identify equivalent number sentences involving multiplication and division (ACMNA121)

As the numbers involved are numbers students at these levels should be familiar with i.e. 5 and 10 this highlights the importance of a greater focus on equality as highlighted by Jones et al., (2013).

Task M2b requires students to give a verbal explanation of the whole number relationship in Task M2a. Fewer students attempted this part with 9% of non-responses at Year 8 increasing to 32% at Year 5. All students in Year 9* attempted the task although 27% responded with the incorrect relationship despite giving the correct answers for Task M2a. Again, the Achievement Standard from the Australian Curriculum (ACARA, 2018) states that Year 5 students:

... identify and explain strategies for finding unknown quantities in number sentences involving the four operations.

The non-response rate was much higher for the equivalent division tasks from Part D (ATQ). For example, in Question D2a students were asked to give two responses to the equivalence relationship shown in the left-hand column of Table 5.42. Forty-six percent of Year 5 students made no attempt to respond to this task with a similar percentage of non-responses from Year 8 students (43%), while 33% of Year 6 students did not respond. Nearly 60% of Year 5 students did not respond to D2b when they had to explain the relationship between the numbers in the boxes and this percentage decreased to 36% of Year 8 students not explaining the relationship.

This research has demonstrated that there are many students who either did not attempt any, or all, of the reverse fraction tasks or gave incorrect responses to those they attempted, or could not explain their solution methods, or needed a diagram in order to use a 'guess and check' method to solve one or more of the reverse fraction tasks. This included both primary and Year 8 students. Chinn (2012) stated that: "too many children and adults give up on mathematics learning by withdrawing from any task that is perceived as likely to result in failure".

Most of the tasks in the FST did not expect students to describe or elaborate their strategies. For example, in the number-line tasks students' partitioning skills could be deduced from their written responses. However, it was possible to distinguish students who used limited and sometimes

inaccurate understandings of fraction size for example, placing three-quarters very close to the mark that indicated the number one as “three-quarters is nearly one”.

On the other hand, responses to the reverse fraction tasks from the FST provided a window into a more explicit repertoire of additive and multiplicative reasoning. Many students were able to obtain correct results using their own informal methods. Sometimes this written response involved slightly idiosyncratic representations e.g. $3 = \frac{1}{7}$ as shown in Figure 4.4.

Students’ written solutions for the three reverse fractions (Task C5, C6 and C7) were analysed using the *Framework for reverse fraction task strategies* (Table 5.15). Students who used fully and advanced multiplicative strategies demonstrated generalisations that are anticipated in algebraic reasoning. See Figure 4.4 for Student DC’s response for Task C6 and Figure 4.5 for Student DC’s response to Task C7. Both students used symbols rather than words to explain their reasoning.

However, the Structured Interview was deliberately designed to enable students to demonstrate and expand their repertoire of strategies that could be used to demonstrate algebraic reasoning. Students who demonstrated their verbal fluency in describing mathematical reasoning in the context of fraction tasks demonstrated generalisations that are anticipated and expected in algebraic reasoning.

Students need opportunities to check and self-correct their solutions to the fraction tasks and to connect their verbal descriptions to both diagrams and symbolic representations. Long, De Temple and Millman (2012) have highlighted the importance of students being able to explain their thinking:

Students need to explain and justify their thinking and learn how to detect fallacies and critique others’ thinking. They need to have ample opportunity to apply their reasoning skills and justify their thinking in mathematical discussions. They will need time, many varied and rich experiences, and guidance to develop the ability to construct valid arguments and to evaluate the arguments of others (p. 49).

If students continue to rely on arithmetical rules and procedures and are unable to demonstrate verbal fluency in describing mathematical reasoning in the context of fraction tasks, they are unable to demonstrate generalisations that are necessary for algebraic reasoning. The format in which the questions were asked in the FST needed to be further extended and developed in order to enable students to verbalise their mathematical thinking.

8.7 Research Question 7

RQ7. *How can students' level of emerging algebraic reasoning be described and identified so that teachers may monitor and prompt students' progress?*

The two frameworks developed as part of this research allow students' level of algebraic reasoning to be described and identified. The first, the *Framework for Reverse Fraction Task Strategies* (Table 5.15) allows teachers to analyse students' responses to reverse fraction tasks such as those in Part C (FST). Students' strategies can be classified as *Diagram Dependent*, *Additive/subtractive*, *Partially Multiplicative*, *Fully Multiplicative* and *Advanced Multiplicative*. The students' strategies can then be monitored. The second, the *Emerging Algebraic Reasoning Framework* (Table 6.12) was developed to classify students' responses to the Structured Interview and consequently the levels of algebraic reasoning. The Structured Interview included tasks where initially the numbers representing the given fractions were changed and then included tasks that included 'any number' and 'any fraction' and 'any number'.

The responses to the research sub-questions discussed in this chapter have highlighted the importance of tasks such as the reverse fraction tasks which allow students to demonstrate evidence of algebraic reasoning. The Structured Interview used similar reverse fraction tasks but moved beyond the change of numbers to 'any number' then to 'any fraction and any number'. For many students the final interview task highlighted the need to change their strategy as an additive strategy would no longer work.

Chapter 9 includes the conclusion, key findings and implications for teaching and for further research.

Chapter 9: Conclusion and Implications

One of the challenges of teaching mathematics in the middle years is moving students' thinking from the specific, concrete and additive strategies to generalised, abstract and multiplicative strategies. Early algebra, with an emphasis on generalising, is important because "our students deserve the chance to develop to the best of their potential" (Lins & Kaput, 2004, p.64). While some topics in the mathematics curriculum offer specific opportunities to address this challenge of moving from additive to multiplicative strategies many do not. The tasks used in this research demonstrate that tasks can be presented that allow students to move beyond the additive strategies to the more generalizable multiplicative strategies.

Previous related research investigated the types of strategies primary students use to solve fraction tasks and the connections between fraction knowledge and whole number knowledge (Hunting, Davis & Pearn, 1996). Other research (Pearn & Stephens, 2007) highlighted the misconceptions revealed by middle-years students and their reliance on rules and procedures even when these were deemed to be inefficient and incorrect. Building on these earlier findings this current research investigated the links between fractional competence and algebraic reasoning to answer the main research question addressed in this study: "How does middle years students' fractional competence and reasoning show evidence of emerging algebraic reasoning?" As discussed in Section 2.3 fractional competence for this research is deemed to be understanding fraction size and relationships, and basic arithmetic competence with simple fractions (Pearn & Stephens, 2015a; 2016) while algebraic reasoning is based on the definition from Kaput and Blanton (2005), that states:

... a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways. (p. 99)

In this mixed method research students attempted two paper and pencil assessments: The Fraction Screening Test (Pearn & Stephens, 2015a) and the Algebraic Thinking questionnaire (Pearn & Stephens, 2015b; 2016). The results from the two paper and pencil tests have been described in Chapter 3, analysed in Chapter 5 and discussed in Chapter 8. After analysing the results from the three reverse fraction tasks from the FST (Figure 3.7) 43 students were interviewed. The Interview is described in Chapter 4, analysed in Chapter 6 and discussed in Chapter 8. Case studies of nine students who were interviewed are given in Chapter 7.

The key findings of this research will be highlighted in the next section where the importance of the three reverse fraction tasks and the two frameworks developed through the process of analysing the data will be discussed.

9.1 Key findings from this research

The results for the FST and the ATQ discussed in Chapter 5 confirm that middle-years students in this study were still displaying the same misconceptions regarding fractions as were reported in previous research (see for example, Pearn & Stephens, 2007; Mitchell & Horne, 2010; Wong & Evans, 2007, 2011).

However, the inclusion of the three reverse fraction tasks (Figure 3.7) in the broad FST allowed new insights into the different solution strategies, both successful and unsuccessful, which middle-years' students used to solve such unfamiliar fraction tasks, where they were asked to find the number in the whole collection if given the number representing a specific fractional part. Grounded in the data, the *Framework for Reverse Fraction Task Strategies* (Table 5.15) classifies the strategies students used to solve the reverse fraction tasks. Their strategies (Table 5.15) ranged from a concrete, additive strategy (*Diagram Dependent*) to more generalised and algebraic strategies classified as *Advanced Multiplicative* where students used either algebraic methods or divided by the given fraction.

The Structured Interview, developed for data collection, allows teachers as well as research interviewers to observe more closely the strategies students used to solve fraction tasks that were similar to those shown in Figure 3.7 then progresses to tasks that are more generalised (see Table 4.5 and Figure 4.9). While the strategies students used for individual tasks from the Structured Interview could be classified using the *Framework for Reverse Fraction Task Strategies* (Figure 5.15) a second framework, grounded in the interview data was developed to map students' progress towards algebraic reasoning: *The Emerging Algebraic Reasoning Framework* (Table 6.12).

9.1.1 The three reverse fraction tasks

Three reverse fraction tasks (see Figure 3.7), not used in previous fraction tests reported in the literature, were trialled and proved successful in providing an initial lens into the different types of students' solution strategies. Students successfully responded to unfamiliar situations by employing a range of mathematical strategies. Their strategies could be hierarchically classified,

and this observation led to the development of the *Framework for Reverse Fraction Task Strategies* as detailed Table 5.15.

A limitation of the presentation of these items as a written test was that while students may have correctly interpreted the task and used an appropriate rule or procedure this may or may not indicate algebraic reasoning. It may represent simply a learned rule, or it may represent a deeper understanding of the structure of fractions. This cannot be decided on the basis of their written responses to FST alone. One on one “interview” dialogue is needed to further probe students’ thinking.

9.1.2 The Structured Interview

The Structured Interview protocol was developed to enable the researcher to observe more closely the types of strategies students used to solve reverse fraction tasks. It also allowed researchers to probe the student’s thinking if unsure of the strategy and reasoning being used by the student. The Structured Interview included reverse fraction tasks similar to those of the written test but with progressive levels of abstraction to capture students’ ability to generalise (see Pearn, Stephens & Pierce, 2018a; 2018b). This Structured Interview is described in detail in Chapter 4, the results analysed in Chapter 6 and elaborated in the nine case studies in Chapter 7.

The Structured Interview was designed to encourage students to demonstrate and expand their repertoire of algebraic reasoning using fraction tasks. While responses to each individual task from the Structured Interview could be classified according to the *Framework for Reverse Fraction Task Strategies* (Figure 5.15) the overall results for each interview also needed to be classified in terms of the development of algebraic reasoning. Students’ overall responses to the Structured Interview were analysed using a thematic analysis approach (Braun & Clarke, 2006). These responses varied from computational reasoning to fully generalised algebraic reasoning. *The Emerging Algebraic Reasoning Framework* with levels and descriptions based on the interview data is shown in Table 6.12.

9.1.3 The two Frameworks

This study was designed to determine how middle-years students’ fractional competence and reasoning shows evidence and guides monitoring of emerging algebraic reasoning. Based on the data from the FST, the ATQ and the Structured Interview, two frameworks have been developed: The *Framework for Reverse Fraction Task Strategies* (Table 5.15) and the *Emerging Algebraic Reasoning Framework* (Table 6.12).

Analysis of students' solution strategies using these frameworks has shown that many students did not attempt any or all of these reverse fraction tasks, or gave incorrect responses to those they attempted, or could not explain their solution methods, or needed a diagram in order to use a 'guess and check' method to solve one or more of the tasks. This might be expected at Year 5, but according to the Australian Curriculum: Mathematics (ACARA, 2018) Year 8 students should be solving rational number tasks and simplifying equations written in algebraic form.

Students relying on additive or partially multiplicative strategies were unable to solve the last task in the Structured Interview asking them to think about 'any fraction' and 'any number'. Some students' solution methods changed in the course of the interview when a fraction, and the corresponding number of objects, changed. Some students who appeared to rely on concrete or additive strategies were able to move confidently to using multiplicative methods. Some students were unable to successfully complete the tasks when presented with 'any number' instead of a given quantity during the interview.

Although students used a variety of methods to solve reverse fraction tasks on the written test, *The Emerging Algebraic Reasoning Framework* (Table 6.12) identified students who were computationally proficient but unable to generalise, as distinct from those who were beginning to generalise, and those who could fully generalise their solutions. This allowed the researcher to identify students whose proficient generalisations showed clear evidence of algebraic reasoning.

The two frameworks, used in this study, highlight the connection between middle-years students' fractional competence and reasoning and emerging evidence of their algebraic reasoning. For teachers, these frameworks serve a double purpose. First by providing indicators that enable teachers to identify the stage where students are at, and second to monitor students' progress by giving clear suggestions for how students can and need to be prompted to make the next steps.

9.2 Implications for Teaching

The main research question of this research was: "How does middle years students' fractional competence and reasoning show evidence of emerging algebraic reasoning?" However, in order to assist teachers a subsequent question could be: "How can this evidence of emerging algebraic reasoning help teachers monitor and prompt students' progress towards reasoning algebraically?" While some topics in the curriculum offer specific opportunities to address the challenge of moving middle-years students' thinking from the specific, concrete and additive strategies to generalised, abstract and multiplicative strategies the tasks in this study have already shown that

this can be achieved using fraction tasks that have been designed to find the number of objects in the whole collection if given the number of objects in a given fractional part.

The practical implications of this research for teaching will be discussed in the rest of this section. These include the importance of using appropriate assessment and monitoring, a broader interpretation of the curriculum, appropriate sequencing of tasks, the use of transferable models, valuing students' strategies and being aware of common misconceptions.

9.2.1 Assessments and frameworks monitor progress towards algebraic reasoning

The *Framework for Reverse Fraction Task Strategies* (Table 5.15) and the *Emerging Algebraic Reasoning Framework* (Table 6.12) provide a guide for the analysis of individual students' responses to the reverse fraction tasks from the FST and the Structured Interview. This information will make a valuable addition to tools for monitoring students' progress in demonstrating their understanding of fraction concepts. For students who have demonstrated *Complete Computational-Fluency* (*The Emerging Algebraic Reasoning Framework*) the task sequence used in the Structured Interview using 'any number of counters' and 'any number of counters and any fraction' provides a model for prompting students' thinking towards generalisation and hence emergent algebraic reasoning. Although the Structured Interview focussed on reverse fraction tasks this model could be used with other fraction topics and in fact other mathematical topics.

The Structured Interview was designed so that students would be given all the tasks so that the tasks themselves encouraged students to move from strategies that demonstrated their computational fluency to the articulation of their strategies that demonstrate their algebraic reasoning. However, some students' interviews were discontinued when they were unable to complete tasks that required them to use multiplicative or algebraic reasoning.

The two frameworks: *The Framework of Reverse Fraction Task Strategies* (Table 5.15) and the *Emerging Algebraic Reasoning Framework* (Table 6.12) were recently trialled with a group at a well-regarded large, annual, mathematics teachers' conference. The participants completed both the set of three reverse fraction tasks and the Structured Interview questions and were asked to classify their responses against the two frameworks. Many of these teachers volunteered positive feedback and stated that they were going to use both the assessments and the frameworks with their students next year.

9.2.2 Broader interpretation of the curriculum

The Australian Curriculum: Mathematics (ACARA, 2018) takes a carefully structured approach to fractions from Years 5 to 7: starting by comparing fractions then working through arithmetic processes with the same and different denominators. However, nowhere in the curriculum does it suggest that students find the whole if they have been given a fraction and the number of objects that represents that fraction. This research has highlighted the importance of including reverse fraction tasks that allow students to demonstrate their understanding of fractions and their algebraic reasoning. If we want teachers to include tasks similar to the reverse fraction tasks, we need a broader interpretation of the curriculum. A further content descriptor or elaboration at Year 6 could be: “Find the number of objects in the whole group, if given the number of objects representing a specific fractional part of that whole group”. Items such as the following give a context for using reverse fraction reasoning.

In a forest there are many nocturnal creatures. Three volunteers, each checking one-tenth of forest, found a total of 12 nocturnal creatures. How many nocturnal creatures would you expect to find in the whole forest? What assumptions did you make?

9.2.3 Appropriate sequencing – going beyond the numbers

The content descriptions for fractions in the Australian Curriculum: Mathematics (ACARA, 2018) for comparing, ordering and adding and subtracting fractions are presented in a trajectory from comparing and computing fractions with the same denominators at Year 5 to including unrelated denominators at Year 7.

In contrast the *Emerging Algebraic Reasoning Framework* (Table 6.12) draws attention to increasing levels of variation and abstraction for students to generalise their thinking. The first level changes the number of counters given for a fraction, building on from “How would you find the number of objects in the whole if 12 counters represented two-thirds?” to “What if 16 counters represented two-thirds of the whole collection?”

The second level of abstraction introduces the idea that a fraction can be represented by ‘any number’ of counters. For example, after trying several different numbers of objects representing the fraction two-thirds then asking: “How would you find the number of objects in the whole if there were ‘any number of counters’ that represented two-thirds?” Students demonstrate understanding if they find the number of counters for one-third and then scale up either additively or multiplicatively. Those who use multiplicative methods are beginning to demonstrate algebraic

reasoning which can support the next level of abstraction. For those who use additive or subtractive strategies the third level is not possible.

The third level of abstraction presents ‘any fraction’ and ‘any number of counters’. For example, “How would you find the number of counters in the whole group if there are any number of counters representing any fraction of that group?” Students who are successful with this task stating that they would need to divide by the numerator to find the unit fraction then multiply by the respective denominator have understood the structure of fractions and are demonstrating fully generalised algebraic reasoning as are those who understand that these two steps can be performed in one step by dividing the fraction or multiplying by its reciprocal.

Students who were able to move from the second level of abstraction to the third level of abstraction demonstrated that the Structured Interview tasks allowed them to use the tasks as “assessment for learning” (Wiliam, 2011).

9.2.4 Use transferable models

Peter Gould (2013) suggested that: “fractions should be introduced through equal sharing contexts that use countable or continuous quantities that can be cut or divided” (p.33). This research has highlighted the need to extend the concrete models to include starting with a part of the collection or continuous quantity to calculate the whole. The outcomes of this research suggest that consideration also needs to be given to transferability of fraction models to apply the same reverse relationships with decimals and percentages. For example, in the question above, students need to understand that three volunteers have covered three-tenths of the forest identifying a total of 12 nocturnal creatures. The total population can be estimated by finding one-tenth (four creatures) and then deducing the total (40); or divide 12 by three-tenths to get the same result.

The development of both the *Framework for Reverse Fraction Task Strategies* (Table 5.15) and the *Emerging Algebraic Reasoning Framework* (Table 6.12) have been informed by the analysis of students’ responses to reverse fraction tasks, but these frameworks could also be adapted to model students’ emergent algebraic thinking in other areas of the curriculum. For example, assessment tasks can be designed so that student responses can be classified according to the strategies being used and each task can be scaffolded upwards along the same lines as the Structured Interview so that students can start to see commonality and pattern, whether they are using fractions, decimals or percentages.

9.2.5 Value students' strategies

The research has identified possible strategies students may use to solve reverse fraction tasks with written tasks and during a semi-structured interview. Teachers need to recognise and value students' thinking strategies which are actually correct but may not be the most efficient or have limited applicability. Part of the teacher's role then becomes to provide a task for which their strategy becomes far too cumbersome or one that just will not work. This is exemplified in the Structured Interview where the tasks went from changing numbers to 'any number' then 'any fraction' and 'any number'. In this case the partially multiplicative strategy no longer worked.

9.2.6 Awareness of common misconceptions and difficulties

Teachers need to be vigilant regarding possible misconceptions regarding both whole numbers and fractions. This research has exposed the misconceptions and difficulties that have been documented in the research for decades. Common misconceptions include adding numerators and denominators to add two fractions as shown by the results from the FST and reported in Chapter 5. Division of whole numbers and fractions appears to be more difficult for students than multiplication of whole numbers as shown by the results from the ATQ in Chapter 5.

These common misconceptions and difficulties were even obvious in students who demonstrated algebraic reasoning. Many students appeared to be responding quickly using incorrect rules and procedures with little evidence they had any understanding of the concept being tested. Since the mathematics curriculum now places emphasis on the four proficiencies, teachers need to encourage students to demonstrate their understanding as well as their fluency. The choice of appropriate examples for mathematics tasks is very important so that these misconceptions are not accidentally taught.

9.3 Limitations of this study

This research was conducted with a convenient sample of Victorian teachers who volunteered to take part in this study after they had attended a professional development session presented by the researcher. These teachers had already shown that they were interested in the topic of fractions. The schools in this study were either at or above the state average ICSEA value thus most students were from above socioeconomic average households. As parent permission was required for this research some schools had high response rates while others had very low response rates. There were no Year 7 students involved in this study as there were no Year 7 teachers who volunteered. This study needs to be extended to a wider range of schools across Victoria, Australia and internationally.

Not all the testing was done with the researcher present. Some of the paper and pencil tests were conducted by the classroom teachers or a member of the school leadership team while in other schools the researcher oversaw or conducted the assessments. As discussed in Chapter 4 there were multiple interviewers, including the researcher, conducting the Structured Interview with the primary and Year 8 students. The interviewers were briefed prior to the interviews and debriefed after the interviews. Multiple interviewers were used to minimise the disruption for the two schools involved.

As discussed in Chapter 5 the ATQ appears to be too difficult for some students in the middle years as evidenced by the number of non-responses, question marks and comments such as: "I don't know". While the ATQ tasks were in line with curriculum expectations the ATQ may need some easier questions or better scaffolding so students will at least be more willing to attempt the tasks.

This research assessed a volunteer group of students whose teachers and parents agreed to their participation. Consequently, only descriptive statistics were used for the analysis of the data collected. There was no opportunity to perform inferential statistics with a multilevel analytical approach that acknowledged the possible hierarchical structure of the data. A more structured research study could be initiated where data was collected from different classrooms, different schools and different classrooms that are more representative of the wider community.

9.4 Implications for further research

The Algebraic Thinking Questionnaire needs to be modified and presented in such a way that students will at least attempt the tasks rather than leaving the spaces blank or writing question marks or statements such as "I don't know". Both paper and pencil instruments, the FST and the ATQ, may need to be updated to ensure that the contexts are locally relevant. Possible research questions could focus on identifying emerging algebraic reasoning, and opportunities to promote this, in other areas of the mathematics curriculum. Are these two Frameworks useful for identifying emerging algebraic reasoning in other areas of mathematics and in other levels of schooling?

While there were some interesting and important findings from this research there was no opportunity to perform a multilevel analytical approach that acknowledged the possible hierarchical structure of data that could be collected if a more comprehensive study was done where data was collected from different classrooms, different schools and different classrooms nationally and internationally.

9.5 The importance of this study

Previous studies have shown the importance of carefully constructed fraction tasks that allow students to demonstrate that they understand the concepts of fractions and not just perform carefully rehearsed rules and procedures. This study was designed to determine how middle-years students' fractional competence and reasoning shows evidence and guides monitoring of emerging algebraic reasoning. Sequencing tasks that initially allowed students to use diagrams to support specific additive strategies, to tasks without diagrams where students can choose to use either additive or partially multiplicative strategies, then to tasks that can only be solved using multiplicative strategies and finally tasks that require generalisations and can only be solved using fully multiplicative strategies.

Carefully scaffolding tasks beyond giving students the numbers to use in calculations but instead asking them to think about 'any number of objects' to 'any number of objects representing any fraction' allowed some students to move beyond the additive strategies they were relying on to the more generalised algebraic strategies as they applied their understanding of fractions and provided the reasoning for their choices. The second and third levels of abstraction of the Structured Interview tasks prompted students to say out loud, in their own words, their generalised response. Articulating their thinking to an interviewer allowed students to reason for themselves their understanding of the generalization and demonstrate their algebraic reasoning.

Teachers can use *the Emerging Algebraic Reasoning Framework* to analyse the strategies used by their students in response to assessment tasks and then provide them with the types of tasks that will allow their students to move to the next level of the *Emerging Algebraic Reasoning Framework*.

References

- Australian Curriculum Assessment and Reporting Authority [ACARA] (2018). *Australian Curriculum: Mathematics*. Retrieved from <https://www.australiancurriculum.edu.au/f-10-curriculum/mathematics/>
- Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *The Elementary School Journal*, 84(2), 199-212.
- Baturo, A. (1997). The implication of multiplicative structure for students' understanding of decimal-number numeration. In F. Biddolf & K. Carr (Eds.), *People in mathematics* (Proceedings of the 20th annual conference of the Mathematics Education Research Group of Australasia, pp.88-95). Rotorua, NZ: MERGA.
- Behr, M. J., & Post, T. R. (1988). Teaching rational number and decimal concepts. In T. R. Post (Ed.), *Teaching Mathematics in Grades K - 8*. Pp.201-248. Boston, MA: Allyn and Bacon.
- Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 296-333). New York, NY: Macmillan.
- Behr, M., Wachsmuth, I., Post, T., & Lesh, R. (1984). Order and equivalence of rational numbers. *Journal for Research in Mathematics Education*, 15(1), 323-341.
- Bezuk, N. S. (1988). Fractions in early childhood mathematics curriculum. *Arithmetic Teacher*, 35(6), 56-59.
- Blanton, M., & Kaput, J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds), *Early Algebraization: Advances in Mathematics Education*. pp. 5-23 Berlin, Germany: Springer-Verlag.
- Blanton, M., Brizuela, B., Stephens, A., Knuth, E., Isler, I., Gardiner, A. M., Stroud, R., Fonger, N., & Stylianou, D. (2018). Implementing a framework for early algebra. In C. Kieran (Ed.), *Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice*, pp. 27-49 Cham, Switzerland: Springer.
- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology, *Qualitative Research in Psychology*, 3(2), 77-101.
- Bryman A. (2006). Integrating quantitative and qualitative research: How is it done? *Qualitative Inquiry*, 6(1), 97-113.
- Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.
- Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first-grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education* 12 (1), 27-39.

- Carraher, D., & Schliemann, A. (2018). Cultivating early algebraic thinking. In C. Kieran (Ed.), *Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice*. (pp. 107-138). Cham, Switzerland: Springer.
- Cathcart, G., Pothier, Y., Vance, N., & Bezuk, N. (2011). *Learning mathematics in elementary and middle school: A learner-centered approach*. Upper Saddle River, NJ: Merrill.
- Charalambous, C., & Pitta-Pantazi, D. (2005). Revisiting a theoretical model on fractions: Implications for teaching and research. In Chick, H. L. & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 233-240). Melbourne, Australia: PME.
- Chinn, S. (2012). Beliefs, anxiety and avoiding failure in mathematics. *Child Development Research*. 1-8.
- Clarke, D. M., Roche, A., Mitchell, A., & Sukenik, M. (2006). In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds), *Proceedings 30th Conference of the International Group of Psychology of Mathematics Education* (Vol. 4, pp. 425-432). Prague, Czech Republic: PME.
- Clarke, D. M., Sullivan, P., Cheeseman, J., & Clarke, B. A. (2000). The early numeracy research project: Developing a framework for describing early numeracy learning. In J. Bana & A. Chapman (Eds.), *Mathematics education beyond 2000*. (Proceedings of the 23rd annual conference of the Mathematics Education Research Group of Australasia, pp. 180-187). Fremantle, Australia: MERGA.
- Creswell, J. W. (2003). *Research design: Qualitative, quantitative, and mixed methods approaches* (2nd ed.). Thousand Oaks, CA: Sage.
- Creswell, J., & Plano Clark, V. (2011). *Designing and conducting mixed methods research* (2nd ed.). Thousand Oaks, CA: Sage.
- Creswell, J., Plano Clark, V., Guttman, M., & Hanson, W. (2003). Advanced mixed methods research designs. In A. Tashakkori & C. Teddlie (Eds.), *Handbook of mixed methods in social & behavioral research* (pp. 209-240). Thousand Oaks, CA: Sage.
- Davis, G. E., Hunting, R. P., & Pearn, C. (1993a). What might a fraction mean to a child and how would a teacher know? *The Journal of Mathematical Behavior*, 12(1), 63-76.
- Davis, G. E., Hunting, R. P., & Pearn, C. (1993b). Iterates and relations: Elliot and Shannon's fraction schemes. In I. Hirabayashi, N. Nohda, K. Shigematsu, & F. Lin (Eds.), *Proceedings of the 17th conference of the International Group for the Psychology of Mathematics Education* 3, 154-161). Tsukuba, Japan: PME.
- Empson, S. B., Levi, L., & Carpenter, T. P. (2010). The algebraic nature of fractions: developing relational thinking in elementary school. In J. Cai and E. Knuth (Eds), *Early Algebraization: Cognitive, Curricular and Instructional Perspectives*. (pp. 409- 428) New York, NY: Springer.

- Falkner, K., Levi, L., & Carpenter, T. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, 6(4), 232-236.
- Fetters, Curry, & Creswell, J. W. (2013). Achieving integration in mixed methods designs: Principles and practices. *Health Services Research*, 48(6,2), 2134-2156.
- Fuchs, L., Schumacher, R., Long, J., Namkung, J., Hamlett, C., Cirino, P., ... & Chngas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*, 105(3), 683-700.
- Fujii, T., & Stephens, M. (2001). Fostering an understanding of algebraic generalisation through numerical expressions: the role of quasi-variables. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th conference of the International Commission on Mathematics Instruction* (Vol. 1, pp. 258-264). Melbourne, Australia: ICMI.
- Gagnon, J., & Maccini, P. (2001). Preparing students with disabilities for algebra. *Teaching Exceptional Children*, 34(1), 8-15.
- Goldin, G. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In R. Lesh & A. Kelly (Eds.), *Handbook of Research Design in Mathematics and Science Education*, (pp. 517-545). Mahwah: Erlbaum.
- Gunderson, A., & Gunderson, E. (1957). Fraction concepts held by young children. *Arithmetic Teacher* 4(4), 168-173.
- Hackenberg, A. (2005). *Construction of algebraic reasoning and mathematical caring relations*. Retrieved from: https://getd.libs.uga.edu/pdfs/hackenberg_amy_j_200508_phd.pdf
- Hackenberg, A. J., & Lee, M. (2015). Relationships between students' fractional knowledge and equation writing. *Journal for Research in Mathematics Education*, 46(2), 196-243.
- Hadfield, O., & McNeil, K. (1994). The relationship between Myers-Briggs personality type and mathematics anxiety among pre-service elementary teachers. *Journal of Instructional Psychology*, 21(4), 375-384.
- Harel, G., & Confrey, J. (1994). *The development of multiplicative reasoning in the learning of mathematics*. New York: SUNY Press.
- Hart, K. (1981). *Children's Understanding of Mathematics: 11-16*. London, UK: John Murray.
- Herscovics, N. & Kieran, C. (1980). Constructing meaning for the concept of equation. *The Mathematics Teacher*, 73(8), 572-580.
- Hunting, R. P. (1986). Rachel's schemes for constructing fraction knowledge. *Educational Studies in Mathematics*, 17(1), 49-66.
- Hunting, R. P., & Davis, G. (1991). (Eds.) *Early fraction learning*. New York, NY: Springer Verlag.

- Hunting, R. P., Davis, G., & Pearn, C. (1996). Engaging whole number knowledge for rational number learning using a computer-based tool. *Journal for Research in Mathematics Education*, 27(3), 354-379.
- IBM Corporation (2017). *IBM SPSS Statistics for Windows, Version 25.0*. Armonk, NY: IBM Corporation.
- Jacobs, V., Franke, M., Carpenter, T., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education*, 38(3), 258-288.
- Jones, I., Inglis, M., Gilmore, C., & Evans, R. (2013). Teaching the substitutive conception of the equals sign. *Research in Mathematics Education*, 15(1), 34-49.
- Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. Carraher & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235-272). New York, NY: Lawrence Erlbaum Associates.
- Kaput, J., & Blanton, M. (2005). Algebrafying the elementary mathematics experience in a teacher-centered, systemic way. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding Mathematics and Science Matters*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Kaput, J., Blanton, M., & Moreno, L. (Eds.) (2008). *Algebra in the early grades*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Kervin, L., Vialle, W., Herrington, J., & Okely, A. (2006). *Research for Educators*. South Melbourne, Australia: Thomson/Social Science Press.
- Kieran, C. (2004): Algebraic thinking in the early grades: What is it? *The Mathematics Educator* 8(1), 139-151.
- Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, 12(3), 317-326.
- Kieran, C. (1989). A perspective on algebraic thinking. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proceedings of the 13th Conference of the International Group for the Psychology of Mathematics Education* (Vol 2, pp. 163-171). Paris, France: PME.
- Kieren, T. E. (1976). On the mathematical, cognitive and instructional foundations of rational numbers. In R. Lesh (Ed.), *Number and measurement: Papers from a research workshop* (pp. 101-144). Columbus, OH: ERIC/SMEAC.
- Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125-150). Columbus, OH: ERIC/SMEAC.
- Kieren, T. E. (1983). Partitioning, equivalence, and the construction of rational number ideas. In M. Zweng (Ed.), *Proceedings of the 4th International Congress on Mathematical Education* (pp. 506-508). Boston, MA: Birkhauser.

- Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 162-181). Hillsdale, NJ: Erlbaum.
- Knuth, E., Alibali, M., Hattikudur, S., McNeil, N., & Stephens, A. (2008). The importance of equal sign understanding in the middle grades. *Mathematics Teaching in the Middle School*, 13(9), 514-519.
- Lamon, S. J. (2001). Presenting and representing: From fractions to rational numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 146-165). Reston, VA: National Council of Teachers of Mathematics.
- Lamon, S. J. (1999). *Teaching Fractions and Ratios for Understanding: Essential Knowledge and Instructional Strategies for Teachers*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Lamon, S. J. (1993). Children's cognitive and metacognitive processes. In Carpenter, T., Fennema, E., Romberg, T. (Eds.) *Rational numbers: An integration of research* (pp. 131-156). Hillsdale, NJ: Erlbaum.
- Lee, M.Y. (2012). *Fractional Knowledge and equation writing: The cases of Peter and Willa*. Paper presented at the 12th International Congress on Mathematical Education 8 – 15th July, 2012, Seoul, Korea.
Last accessed from <http://www.icme12.org/upload/UpFile2/TSG/0766.pdf>
- Lee, M.Y., & Hackenberg, A. (2014). Relationships between fractional knowledge and algebraic reasoning: The case of Willa. *International Journal of Science and Mathematics Education*, 12, 975-1000.
- Leech, N., & Onwuegbuzie, A. (2008) A typology of mixed methods research designs, *Quality and Quantity*, 43(2). pp. 265-275.
- Lesh, R. A., Landau, M., & Hamilton, E. (1983). Conceptual Models and Applied Mathematical Problem-Solving Research. In R. A. Lesh & M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 263-341). Orlando: Academic Press, Inc.
- Lins, R. & Kaput, J. (2004). The early development of algebraic reasoning: the current state of the field. In K. Stacey, H. Chick & M. Kendal, (Eds.), *The Future of the Teaching and learning of Algebra. The 12th ICMI Study*. pp. 45 – 70. Norwood, MA: Kluwer Academic Publishers
- Long, C., De Temple, D., & Millman, R. (2012). *Mathematical reasoning for elementary teachers*. (6th Ed). Boston, Mass.: Pearson Addison Wesley.
- Mack, N. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, 21, 16-33.
- Marton, F., Runesson, U., & Tsui, A. (2004). The space of learning. In F. Marton, A. Tsui, P. Chik, P. Ko, & M. Lo (Eds.), *Classroom discourse and the space of learning* (pp. 43-62). Mahwah, NJ: Lawrence Erlbaum.

- Mason, J. (2018). How early is too early for thinking algebraically? In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5 – to 12- year olds: The global evolution of an emerging field of research and practice*. pp. 329-350 Springer, Cham, Switzerland.
- Mason, J., Graham. A., Pimm, D., & Gowar, N. (1985). *Routes to/Roots of algebra*. Milton Keynes, UK: Open University Press.
- Mason, J., Stephens, M., & Watson, A. (2009). Appreciating mathematical structure for all. *Mathematics Education Research Journal*, 21(2), 10-32.
- McCloskey, A., & Norton, A. (2009). Using Steffe's advanced fraction schemes. *Mathematics Teaching in the Middle School*, 15(1), 44-50.
- Mitchell, A., & Clarke, D. M. (2004). When is three quarters not three quarters? Listening for conceptual understanding in children's explanations in a fractions interview. In I. Putt, R. Farragher & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010*. (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 367-373). Townsville, Australia: MERGA.
- Mitchell, A., & Horne, M. (2008). Fraction number line tasks and the additivity concept of length measurement. In M. Goos, R. Brown & K. Makar (Eds), *Navigating currents and charting directions*. (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, pp. 353-360). Brisbane, Australia: MERGA.
- Mitchell, A., & Horne, M. L. (2010). Gap thinking in fraction pair comparisons is not whole number thinking: Is this what early equivalence looks like? In Sparrow, B. Kissane, & C. Hurst (Eds.), *Shaping the future of mathematics education*. (Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia. pp. 414-421) Fremantle, Australia: MERGA.
- National Mathematics Advisory Panel [NMAP] (2008). *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*. U.S. Department of Education. Retrieved from <http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf>
- Norton, A., & McCloskey, A. (2008). Modeling students' mathematics using Steffe's fraction schemes. *Teaching Children Mathematics*. 15, 48-54.
- Olive, J. (1999). *From Fractions to Rational Numbers of Arithmetic: A reorganization hypothesis*, *Mathematical Thinking and Learning*, 1(4), 279-314.
- Onwuegbuzie, A., & Combs. J. (2010). Emergent data analysis techniques in mixed methods research: A synthesis. In A. Tashakkori and C. Teddlie (Eds), *Sage handbook of mixed methods in Social & Behavioral research* (2nd ed., pp. 397-430). Thousand Oaks, CA: SAGE Publications.
- Pallant, J. (2011). *SPSS survival manual: A step by step guide to data analysis using the SPSS program* (4th ed.), Berkshire, UK: Allen & Unwin.
- Pearn, C., Pierce, R., & Stephens, M. (2017a). Questions to reveal and guide fractional thinking. In V. Barker (Ed.), *Capital Maths*. (Proceedings of the 26th Biennial Conference of the

Australian Association of Mathematics Teachers Inc., pp. 125-322). Adelaide, Australia: AAMT.

- Pearn, C., Pierce, R., & Stephens, M. (2017b). Reverse fraction tasks reveal algebraic thinking. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*. (Vol 4 pp. 1-8) Singapore: PME.
- Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what Year 8 students really think about fractions. In I. Putt, R. Faragher, & M. McLean (Eds.), *Mathematics Education for the Third Millennium: Towards 2010* (Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia, pp. 430-437) Sydney, Australia: MERGA.
- Pearn, C., & Stephens, M. (2007). Whole number knowledge and number lines help develop fraction concepts. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, 2, pp. 601-610). Adelaide, Australia: MERGA.
- Pearn, C., & Stephens, M. (2014). Fractional knowledge as a signpost to algebraic readiness. *Maths Rocks*. (Proceedings of the annual conference of the Mathematical Association of Victoria) Melbourne, Australia.
- Pearn, C., & Stephens, M. (2015a). Strategies for solving fraction tasks and their link to algebraic thinking. In M. Marshman, V. Geiger, & A. Bennison (Eds.), *Mathematics Education in the Margins*. (Proceedings of the 38th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 493-500). Sunshine Coast, Australia: MERGA
- Pearn, C., & Stephens, M. (2015b). The importance of fractional thinking as a bridge to algebraic reasoning. In N. Davis, K. Manuel & T. Spencer (Eds.), *Mathematics: learn, lead, link*. (Proceedings of the 25th Biennial Conference of the Australian Association of Mathematics Teachers Inc., pp. 103-111). Adelaide, Australia: AAMT
- Pearn, C., & Stephens, M. (2016). Competence with fractions in fifth or sixth grade as a unique predictor of algebraic thinking? In B. White, M. Chinnappan & S. Trenholm (Eds.), *Opening up mathematics education research*. (Proceedings of the 39th Annual Conference of the Mathematics Education Research Group of Australasia, pp. 519-526). Adelaide, Australia: MERGA.
- Pearn, C., & Stephens, M. (2017). Generalising fraction structures as a means for engaging in algebraic thinking. In A. Downton, S. Livy & J. Hall (Eds.), *40 years on: We are still learning*. (Proceedings of the 40th Annual Conference of Mathematics Education Research Group of Australasia, pp. 405-412). Adelaide, Australia: MERGA.
- Pearn, C., & Stephens, M. (2018). Generalizing fractional structures: A critical precursor to Algebraic Thinking. In C. Kieran (Ed.), *Teaching and Learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice*. (pp. 237-260). Cham, Switzerland: Springer.

- Pearn, C., Stephens, M., & Lewis, G. (2002). Assessing rational number knowledge in the middle years of schooling. In M. Goos & T. Spencer (Eds.), *Mathematics making waves*. (Proceedings of the 19th Biennial Conference of the Australian Association of Mathematics Teachers. pp. 170-178). Adelaide: AAMT.
- Pearn, C., Stephens, M., & Pierce, R. (2018a). Diagram effective or diagram dependent? In J. Hunter, L. Darragh, & P. Perfer, (Eds.), *Making Waves, Opening Spaces*. (Proceedings of the 41st Annual Conference of the Mathematics Education Research Group of Australasia. pp. 631-638). Auckland, NZ: MERGA.
- Pearn, C., Stephens, M., & Pierce, R. (2018b). Interviews reveal Year 8 students' struggle to generalise solutions to reverse fraction tasks. In F. J. Hsieh (Ed). *ICMI-EARCOME 8 Proceedings of the 8th ICMI-East Asia Regional Conference on Mathematics Education*. (pp. 215-226). Taipei, Taiwan: ICME.
- Pearn, C., Stephens, M., & Pierce, R. (2019). Monitoring and prompting emergent algebraic reasoning in the middle years: Using reverse fraction tasks. In G. Hine, S. Blackley & A. Cooke (Eds.), *Mathematics Education Research Impacting Practice*. (Proceedings of the 42nd Annual Conference of The Mathematical Research Group of Australasia. pp. 564-571) Perth, Australia: MERGA.
- Pearn, C., Stephens, M., Zhang, Q., & Pierce, R. (2019). Identifying emerging algebraic reasoning. In M. Graven, H. Venkat, A. Essien, & P. Vale, (Eds.), *Proceedings of the 43rd Conference of the International Group for the Psychology of Mathematics Education*. (Vol 2, pp. 201-208). Pretoria, South Africa: PME.
- Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, 16, 18-36.
- Powell, S. (2012). Equations and the equal sign in elementary mathematics textbooks. *Elementary School Journal*, 112(4) 627-648.
- Powell, S., & Fuchs, L. (2010). Contribution of equal-sign instruction beyond word-problem tutoring for third-grade students with mathematics difficulty. *Journal of Educational Psychology*, 102(2), 381-394.
- Radford, L. & Pierce, C. S. (2006). Algebraic thinking and the generalization of patterns: A semiotic perspective. In S. Alatorre, J. Cortina, M. Sáiz, & A. Méndez, (Eds). *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. 1*, 2-21) Mérida, México: Universidad Pedagógica Nacional.
- Radford, L. (2018). The emergence of symbolic algebraic thinking in primary school. In C. Kieran (Ed.), *Teaching and learning algebraic thinking with 5 – to 12-year olds: The global evolution of an emerging field of research and practice*. pp. 3-25 Springer, Cham, Switzerland.
- Radford, L. (2012). Early algebraic thinking: Epistemological, semiotic and developmental issues. Seoul, (*Proceedings of the 12th International Congress on Mathematical Education* (pp. 209–227). Seoul, Korea.

Retrieved from www.icme12.org/upload/submission/1942_f.pdf

- Rittle-Johnson, B., Matthews, P. G., Taylor, R. S., & McEldoon, K. L. (2011). Assessing knowledge of mathematical equivalence: A construct-modeling approach. *Journal of Educational Psychology*, 103(1), 85-104.
- Roche, A. (2005). Longer is larger: Or is it? *Australian Primary Mathematics Classroom*, 10(3), 1-16.
- Saenz-Ludlow, A. (1994). Michael's fraction schemes. *Journal for Research in Mathematics Education*, 25, 50-85.
- Siegler, R., Duncan, G., Davis-Kean, P., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M., & Chen, M. (2012). Early Predictors of High School Mathematics Achievement. *Psychological Science*, 23(7), 691-697.
Retrieved from <http://pss.sagepub.com/content/early/2012/06/13/0956797612440101>
- Siegler, R., Fazio, L., Bailey, D., and Zhou, X. (2013). Fractions: the new frontier for theories of numerical development. *Trends in Cognitive Sciences*. 17(1), 13 - 19
- Siegler, R. S., Thompson, C., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*. 62, 273 - 296
- Siemon, D., Virgona, J., & Corneille, K. (2001). *The Final Report of the Middle Years Numeracy Research Project: 5-9*. Retrieved from:
<http://www.eduweb.vic.gov.au/edulibrary/public/curricman/middleyear/MYNumeracyResearchFullReport.pdf>
- Siemon, D. (2003). Partitioning: The missing link in building fraction knowledge and confidence. *Australian Mathematics Teacher*, 59(2), 22-24.
- Siemon, D. Breed, M., & Virgona, J. (2005). From additive to multiplicative thinking – The big challenge of the middle years. In J. Mousley, L Bragg, & C. Campbell, (Eds.) *Mathematics – Celebrating Achievement*, Proceedings of the 42nd Conference of the Mathematical Association of Victoria, pp. Melbourne, Australia: MAV
- Smith, E. (2008). Representational thinking as a framework for introducing functions in the elementary curriculum. In J. Kaput, D. Carraher, M. Blanton, *Algebra in the early grades*. 133-160. New York, NY, Lawrence Erlbaum
- Steffe, L. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior*. 20, 267-307
- Steffe, L.P., & Olive, J. (2010). *Children's Fractional Knowledge*. New York: Springer
- Steffe, L. P., & Olive, J. (1990). Constructing fractions in computer microworlds. In G. Booker, P. Cobb, and T. N. de Mendicutti (Eds.), *Proceedings of the Fourteenth International Conference for the Psychology of Mathematics Education*, (pp. 59-66). Mexico City, Mexico: CONACYT.

- Steffe, L. P., & Olive, J. (1993, January). *Children's construction of the rational numbers of arithmetic*. Paper presented at the International Conference on Rational Number Learning, University of Georgia, Athens, GA.
- Stephens, M. (2015). Algebraic Thinking Questionnaire. *Reframing Mathematical Futures 11*: RMIT University.
- Stephens, M. (2006). Describing and exploring the power of relational thinking. In P. Grootenboer, R. Zevenbergen, & M. Chinnappan (Eds.), *Identities, Cultures and Learning Spaces*. (Proceedings of the 29th annual conference of the Mathematics Education Research Group of Australasia, pp. 479-486). Canberra, Australia: MERGA
- Stephens, M., & Pearn, C. (2003). Probing whole number dominance with fractions. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics Education Research: Innovation, Networking, Opportunity*. (Proceedings of the Twenty-sixth Annual Conference of the Mathematics Education Research Group of Australasia, pp. 650-657). Sydney, Australia: MERGA.
- Stephens, M., & Ribeiro, A. (2012). Working towards Algebra: The importance of relational thinking. *Revista Latinoamericana de Investigacion en Matematica Educativa*, 15(3), 373 – 402
- Streefland, L. (1991). *Fractions in realistic mathematics education. A paradigm of developmental research*. Dordrecht, Netherlands: Kluwer Academic Publishers.
- Streefland, L. (1984). Unmasking N- distractors as a source of failures in learning fractions. In B. Southwell, R. Eyland, M. Cooper, J. Conroy, & K Collis (Eds.), *Proceedings of the Eighth International Conference for the Psychology of Mathematics Education*. (pp. 142-152). Sydney, Australia: Mathematical Association of New South Wales.
- Tall, D. (1991). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced Mathematical Thinking*. (pp. 3 – 21) Dordrecht, Netherlands: Springer
- Teddlie, C., & Tashakkori, A. (2009). *Foundations of mixed methods research: Integrating quantitative and qualitative approaches in the Social and Behavioral Sciences*. Los Angeles, CA: Sage.
- Thomson, S., De Bortoli, L., Underwood, C. (2016). *PISA 2015: A first look at Australia's results*, Camberwell, Australia: Australian Council for Education Research [ACER].
- Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education* (30)4, 390-416
- Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education*, 15, 154-164.
- Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.). *Acquisition of mathematics concepts and processes*. (pp.127-174). New York, NY: Academic Press.

- Victorian Curriculum and Assessment Authority [VCAA]. *The Victorian Curriculum: Mathematics*. Retrieved from <https://victoriancurriculum.vcaa.vic.edu.au/mathematics/introduction/rationale-and-aims>
- Wiliam, D. (2011). What is assessment for learning? *Studies in Educational Evaluation*. 37, 3 - 14.
- Wong, M. (2013a). Identifying fractions on a number line. *Australian Primary Mathematics Classroom*, 18(3), 13–18.
- Wong, M. (2013b). Locating fractions on a number line. *Australian Primary Mathematics Classroom*, 18(4), 22–26.
- Wong, M. (2009). *Assessing students' knowledge and conceptual understanding of fraction equivalence*. Unpublished doctoral thesis, University of Sydney, Australia
- Wong, M., & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In J. Watson & K. Beswick (Eds), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Group of Australasia, pp. 824–833). Adelaide, Australia: MERGA.
- Wong, M. M., & Evans, D. (2011). Assessing students' understanding of fraction equivalence. *Fractions: Teaching for Understanding* (pp. 81-90). Australia: Australian Association of Mathematics Teachers.
- Wu, H. (2001). How to prepare students for algebra. *American Educator*, 25(2), 10 - 17.
- Yin, R. K. (2006). Case Study Methods. In J. L. Green, G. Camilli, & P. B. Elmore (Eds.), *Handbook of complementary methods in education research* (pp. 111–122). Mahwah, NJ: Lawrence Erlbaum

Appendix A: Fraction Screening Test

Student:

School:

Year:

Date:

Fractions

Screening Test

Catherine Pearn

Max Stephens

The University of Melbourne

Part A

1.

Circle the picture(s) that represent $\frac{3}{4}$



2.

Draw a circle around $\frac{1}{3}$ of these beetles.



3.

I want to share 20 fish equally between 4 friends.

How many fish would each friend get?

4.

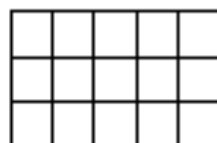
This is one-half of the lollies I started with.



How many lollies did I start with?

5.

Shade one-fifth ($\frac{1}{5}$) of this rectangle.



|

6.



Four friends shared three pizzas equally.

What fraction of a pizza would each friend get? _____

7.

Here is a counting pattern:

$$1\frac{3}{5} \quad 1\frac{1}{5} \quad \frac{4}{5} \quad \dots$$

What is the next term in this counting pattern? _____

8.

Here is a picture of a ribbon.

Shade $\frac{3}{4}$ of the ribbon.



9.

What number do you need to write in the box so that the fractions are equivalent (have the same value)?

$$\frac{7}{9} = \frac{\square}{27}$$

10.

What number do you need to write in the box so that the fractions are equivalent?

$$\frac{8}{10} = \frac{\square}{5}$$

11.

Complete

$$\frac{3}{10} + \frac{1}{5} = \frac{\square}{\square}$$

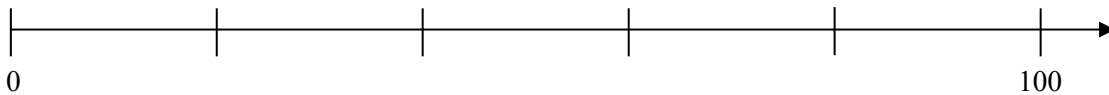
12.

To buy a new workbook I spent \$4. This is $\frac{1}{7}$ of what I saved up. How much did I save up? _____

Part B

1.

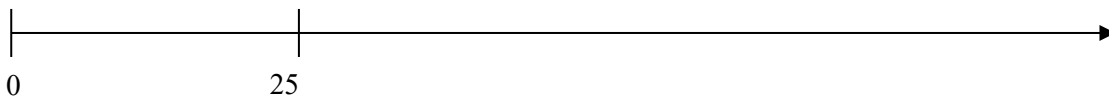
This number line is marked 0 to 100 and has been divided up.



- Write the numbers that should go on the marks.
- Explain how you decided that your answer is correct.

2.

This number line shows 0 to 25.

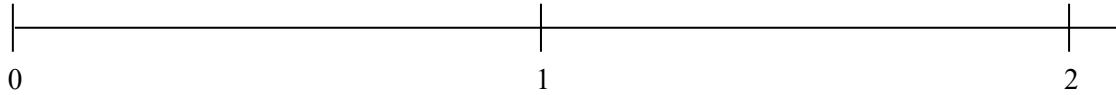


- Place the **number 75** on the number line as accurately as you can.
- Explain how you decided that your answer is correct.
- Mark the number 5 on the number line as accurately as you can.
- Explain how you decided that your answer is correct.

3.

Here is a number line 2 units long.

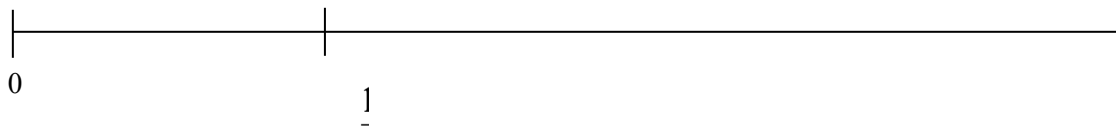
Put a cross (x) where you think the **number** $\frac{3}{5}$ would be on the number line below.



4.

This number line shows where the number $\frac{1}{3}$ is.

Put a cross (x) where you think the number 1 would be on the number line



5.

On this number line 0 and the number $\frac{3}{4}$ are shown.



- Circle the letter that represents where the **number 1** would be.
- How did you decide the other two choices were incorrect?

Part C

1.

Place these fractions in order from largest to smallest.

$$\frac{1}{2} \quad \frac{1}{4} \quad \frac{3}{5} \quad \frac{1}{10}$$

2.

Draw a line from each fraction to the equivalent decimal.

$$\frac{1}{2} \quad 0.4$$

$$\frac{1}{4} \quad 0.1$$

$$\frac{1}{10} \quad 0.25$$

$$\frac{2}{5} \quad 0.5$$

3.

Circle the one that does not belong:

$$25\% \quad 0.4 \quad 0.25 \quad \frac{1}{4}$$

4.

Circle the one that does not belong:

$$1.2 \quad \frac{6}{5} \quad 12\% \quad 1\frac{6}{30}$$

5.

This collection of 10 counters is $\frac{2}{3}$ of the number of counters I started with.



- How many counters did I start with?
- Explain how you decided that your answer is correct.

6.

Susie's CD collection is $\frac{4}{7}$ of her friend

Kay's. Susie has 12 CDs.

How many CDs does Kay have? _____

Show all your working.

7.

This collection of 14 counters is $\frac{7}{6}$ of the number of counters I started with.



- How many counters did I start with?
- Explain how you decided that your answer is correct.



Appendix B: Algebraic Thinking Questionnaire

Algebraic Thinking Questionnaire

NAME _____ YEAR LEVEL ____ SCHOOL _____

Part M

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$$36 \times 25 = 9 \times \boxed{}$$

$$48 \times 2.5 = \boxed{} \times 10$$

$$\frac{2}{3} \times \boxed{} = 18$$

$$\frac{2}{5} \times \frac{\boxed{}}{\boxed{}} = 1$$

2. Can you think about the following mathematical sentence:

$$5 \times \boxed{} = 10 \times \boxed{}$$

Box A Box B

a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$$5 \times \boxed{} = 10 \times \boxed{}$$

Box A Box B

$$5 \times \boxed{} = 10 \times \boxed{}$$

Box A Box B

b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?

c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

d) What can you say about c and d in this mathematical sentence?

$$c \times 2 = d \times 14$$

e) What can you say about a and b in this mathematical sentence?

$$a \times \frac{3}{4} = b \times 1\frac{1}{2}$$

Part D

1. For each of the following number sentences, write a number in the box to make a true statement. Explain your working briefly.

$$3 \div 4 = 15 \div \square$$

$$\square \div 15 = 16 \div 30$$

$$18 \div \square = 6 \div \frac{2}{3}$$

$$\frac{7}{6} \div \frac{\square}{\square} = 1$$

2. Can you think about the following mathematical sentence:

$$3 \div \boxed{} = 15 \div \boxed{}$$

Box A Box B

- (a) In each of the sentences below, can you put numbers in Box A and Box B to make each sentence correct?

$$3 \div \boxed{} = 15 \div \boxed{}$$

Box A Box B

$$3 \div \boxed{} = 15 \div \boxed{}$$

Box A Box B

- (b) When you make a correct sentence, what is the relationship between the numbers in Box A and Box B?
- (c) If you put any number in Box A, can you still make a correct sentence? Please explain your thinking clearly.

- (d) What can you say about c and d in this mathematical sentence?

$$c \div 8 = d \div 24$$

- (e) What can you say about a and b in this mathematical sentence?

$$a \div \frac{2}{7} = b \div \frac{3}{7}$$

Appendix C: Structured Interview

Student Interviews: Notes for Interviewers

You will need:

1. Interview sheet for student to complete.
2. Notes for interviewers.

Please note that the instructions for interviewers are in bold.

Say:

- I'm really interested in the strategies you used so I'm going to ask you some questions and I'd like you to explain your thinking as you solve these fraction tasks.
- Your responses are going to help me understand how students think about fractions.

Hand the students the interview sheet.

Say: Please write your name, class and school on the top of Page 1.

Read Question 1 out loud.

Say:

Write your answers, and working, to each question in the space provided.

Include enough detail to explain your thinking.

[Note: *Ensure students write detailed answers. Students are allowed to draw diagrams.*]

Wait until student have completed Question 1 before reading Question 2.

Similarly, for Question 3.

Note: If student is struggling discontinue the interview after Question 3 and make a note of where they stopped and why.

After students have successfully completed Page 1 turn to Page 2 of the interview script.

Say:

Here are some more tasks that are similar to the tasks you have just done.

Read Question 4 out loud for student.

If student is unable to complete Question 4 stop the interview and make a note of where they stopped and why.

If student successful with Question 4 read Question 5 out loud for student.

If student is unable to complete Question 5 stop the interview and make a note of where they stopped and why.

If student successful with Question 5b turn to Page 3.

Say:

Here are some more tasks that are similar to the tasks you have just done.

Read Question 6 (Page 3) out loud for student.

If student is unable to complete Question 6 stop the interview and make a note of where they stopped and why.

If student successfully answers 6b then ask Question 7.

[Note: If necessary prompt students to write detailed answers. If students refer to “top number” and “bottom number” ask if they know the correct terms for these numbers and include these terms in their explanations.]

Student Name _____ Class _____ School _____

1.

Imagine that I gave you 12 counters which is $\frac{2}{3}$ of the number of counters I started with.

- a. How many counters did I start with?
- b. Explain your thinking.

2.

Susie has 8 CDs. Her CD collection is $\frac{4}{7}$ of her friend Kay's.

- a. How many CDs does Kay have? _____
- b. Explain your thinking.

3.

Imagine that I gave you 21 counters which is $\frac{7}{6}$ of the number of counters I started with.

- a. How many counters did I start with?
- b. Explain your thinking.

Here are some more tasks that are similar to the ones we have just done.

4.

a. If I gave you 18 counters, which is $\frac{2}{3}$ of the number of counters I started with, how would you find the number of counters I started with?

b. If I gave you any number of counters, which is also $\frac{2}{3}$ of the number I started with, what would you need to do to find the number of counters I started with?

5.

a. If Susie had 20 CDs, which was $\frac{4}{7}$ of her friend Kay's CDs, how would you find the number of CDs Kay has?

b. If it was any number of CDs that Susie had, and this was still $\frac{4}{7}$ of the number CDs Kay had, what would you need to do to find the number of CDs Kay had?

6.

- a. If I gave you 70 counters, which was $\frac{7}{6}$ of the number of counters I started with, how would you find the number of counters I started with?
- b. If it was any number of counters, which was $\frac{7}{6}$ of the number of counters I started with, what would you need to do to find the number of counters I started with?

7.

Think about the tasks you have just done.

What if I gave you any number of counters, and they represented any fraction of the number of counters I started with, how would you work out the number of counters I started with?

- a. Can you tell me what you would do?
- b. Please write in your own words.