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*Classical discrete symplectic ensembles on the linear and exponential lattice: skew
orthogonal polynomials and correlation functions*

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CLASSICAL DISCRETE SYMPLECTIC ENSEMBLES ON THE LINEAR AND EXPONENTIAL LATTICE: SKEW ORTHOGONAL POLYNOMIALS AND CORRELATION FUNCTIONS

PETER J. FORRESTER AND SHI-HAO LI

ABSTRACT. The eigenvalue probability density function for symplectic invariant random matrix ensembles can be generalised to discrete settings involving either a linear or exponential lattice. The corresponding correlation functions can be expressed in terms of certain discrete, and q , skew orthogonal polynomials respectively. We give a theory of both of these classes of polynomials, and the correlation kernels determining the correlation functions, in the cases that the weights for the corresponding discrete unitary ensembles are classical. Crucial for this are certain difference operators which relate the relevant symmetric inner products to the skew symmetric ones, and have a tridiagonal action on the corresponding (discrete or q) orthogonal polynomials.

1. INTRODUCTION

1.1. Continuous invariant ensembles. In the theory of random matrices (see e.g. [15, Ch. 5]) an ensemble of Hermitian matrices is said to have a unitary symmetry if its eigenvalue probability density (PDF) is of the form

$$\frac{1}{Z_N} \prod_{l=1}^N w(x_l) \prod_{1 \leq j < k \leq N} (x_k - x_j)^2. \quad (1.1)$$

For example, choosing Hermitian matrices according to a Gaussian weight proportional to $e^{-\text{Tr } X^2}$ specifies a unitary invariant ensemble — known as the Gaussian unitary ensemble — with $w(x)$ in (1.1) equal to e^{-x^2} .

For general non-negative $w(x)$ — referred to as a weight — the k -point correlation $\rho_{N,k}(x_1, \dots, x_k)$ is specified by integrating (1.1) over x_{n+1}, \dots, x_N , and multiplying by $N(N-1) \cdots (N-n+1)$ as a normalisation. It is a standard result that

$$\rho_{N,k}(x_1, \dots, x_k) = \det \left[\hat{K}_N(x_{j_1}, x_{j_2}) \right]_{j_1, j_2=1}^k, \quad (1.2)$$

for a certain kernel function $\hat{K}_N(x, y)$ independent of k . The structure (1.2) specifies the eigenvalues of Hermitian matrices with a unitary symmetry as examples of determinantal point processes; see e.g. [7].

Significant too is the precise functional form of $\hat{K}_N(x, y)$. Denote by $\{p_l(x)\}_{l=0,1,\dots}$ the set of monic polynomials orthogonal with respect to the weight $w(x)$,

$$\int_I w(x) p_j(x) p_k(x) dx = h_j \delta_{j,k}. \quad (1.3)$$

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Here $h_j > 0$ denotes the normalisation, and I denotes the interval of support of $w(x)$. For $p_j(x)$ to be well defined up to degree N , we have to assume $\omega(x)$ decays fast at the boundary of the interval I so that the moments $\int_I x^i \omega(x) dx$ exist for all $i = 0, 1, \dots, N$. One then has (see e.g. [15, Props. 5.1.1 and 5.1.2])

$$\begin{aligned} \hat{K}_N(x, y) &= \left(w(x)w(y) \right)^{1/2} \sum_{l=0}^{N-1} \frac{p_l(x)p_l(y)}{h_l} \\ &= \frac{(w(x)w(y))^{1/2}}{h_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y}, \end{aligned} \quad (1.4)$$

where the identity implied by the final equality is known as the Christoffel-Darboux formula.

In classical random matrix theory the generalisation of (1.1) to the form

$$\frac{1}{Z_{N,\beta}} \prod_{l=1}^N w_\beta(x_l) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta \quad (1.5)$$

for $\beta = 1$ and 4 is also prominent. Thus (1.5) with $\beta = 1$ occurs as the eigenvalue PDF for real symmetric matrices with orthogonal symmetry. With $\beta = 4$ it occurs as the eigenvalue PDF for $2N \times 2N$ Hermitian matrices with entries of the block form

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \quad (1.6)$$

the latter being a 2×2 matrix representation of quaternions, assuming too an invariance of the distribution with respect to conjugation by unitary symplectic matrices. Both these cases are examples of Pfaffian point processes, with the general k -point correlation function having the form

$$\rho_{N,k}(x_1, \dots, x_k) = \text{Pf} \left[A_{N,\beta}(x_{j_1}, x_{j_2}) \right]_{j_1, j_2=1, \dots, k}$$

for a certain 2×2 matrix $A_{N,\beta}(x, y)$, anti-symmetric with respect to interchange of x and y , and independent of k . The latter has the particular structure

$$A_{N,\beta}(x, y) = \begin{bmatrix} I_{N,\beta}(x, y) & T_{N,\beta}(x, y) \\ -T_{N,\beta}(y, x) & D_{N,\beta}(x, y) \end{bmatrix}$$

with $I_{N,\beta}$, $D_{N,\beta}$ related to $T_{N,\beta}$ according to (see e.g. [15, Ch. 6; there $T_{N,\beta}(x, y)$ is denoted by $S_4(x, y)$ (for $\beta = 4$) and by $S_1(x, y)$ for $\beta = 1$])

$$I_{N,\beta}(x, y) = \int_x^y T_{N,\beta}(x, y') dy', \quad D_{N,\beta}(x, y) = \frac{\partial}{\partial x} T_{N,\beta}(x, y).$$

We emphasize that it is only for $\beta = 1, 2$ or 4 that the point process corresponding to (1.5) is known to possess a determinantal or Pfaffian structure.

While the PDFs (1.1) and (1.5) relate to continuous variables, there are prominent examples from the setting of combinatorial/ integrable probability (see e.g. [8], [17], [18], [19], [20], [25], [26], [29]) that give rise to PDFs of the same or an analogous form, but with the variables taking on discrete values, typically from $\{k\}_{k=-\infty}^{\infty}$ (linear lattice) or $\{q^k\}_{k=-\infty}^{\infty}$ (exponential lattice). For general $\beta > 0$, discretization of (1.5) of the type motivating our work shows up in the study of the so-called Jack and Macdonald measures; see e.g. [20]. Our interest in this paper is to identify special inter-relations that hold in such discrete settings between analogues of ensembles with unitary and symplectic symmetry. To be more explicit, further theory from the continuous case is required.

1.2. Inter-relations between continuous invariant ensembles with unitary and symplectic symmetry. In the case of unitary symmetry we know from (1.4) that the correlation kernel can be expressed in terms of orthogonal polynomials corresponding to the weight $w(x)$ in (1.1). It is similarly true that the quantity $T_{N,\beta}(x, y)$ determining the kernel matrix $A_{N,\beta}(x, y)$ for $\beta = 1$ and 4 can be expressed in terms of certain polynomials associated with a skew inner product. The details are different depending on whether $\beta = 1$ and $\beta = 4$. Our interest in the present paper is the case $\beta = 4$, when the relevant skew inner product reads

$$\langle f, g \rangle_4 := \int_I w_4(x) \left(f(x)g'(x) - f'(x)g(x) \right) dx.$$

The polynomials of interest, $\{Q_j(x)\}_{j=0,1,\dots}$ say, are required to have the skew orthogonality property

$$\langle Q_{2m}, Q_{2n+1} \rangle_4 = q_m \delta_{m,n}, \quad \langle Q_{2m}, Q_{2n} \rangle_4 = \langle Q_{2m+1}, Q_{2n+1} \rangle_4 = 0. \quad (1.7)$$

It is easy to see that skew orthogonality property holds if we make the replacement

$$Q_{2m+1}(x) \mapsto Q_{2m+1}(x) + \gamma_{2m} Q_{2m}(x) \quad (1.8)$$

for arbitrary γ_{2m} ; in practice γ_{2m} is chosen for convenience. In terms of these polynomials, one has (see e.g. [15, Prop. 6.1.6])

$$T_4(x, y) = \sum_{m=0}^{N-1} \frac{(w_4(x))^{1/2}}{q_m} \left(Q_{2m}(x) \frac{d}{dy} \left((w_4(y))^{1/2} Q_{2m+1}(y) \right) - Q_{2m+1}(x) \frac{d}{dy} \left((w_4(y))^{1/2} Q_{2m}(y) \right) \right).$$

Interplay between the above formulas holds if we first choose $w(x)$ as one of the so-called classical weights

$$w(x) = \begin{cases} e^{-x^2}, & \text{Hermite} \\ x^\alpha e^{-x} (x > 0), & \text{Laguerre} \\ (1-x)^\alpha (1+x)^\beta (-1 < x < 1), & \text{Jacobi} \\ (1+x^2)^{-\alpha}, & \text{Cauchy.} \end{cases}$$

These weights are distinguished by their logarithmic derivative being expressible as a rational function,

$$\frac{w'(x)}{w(x)} = -\frac{g(x)}{f(x)}, \quad (1.9)$$

with the degree of f less than or equal to 2, and the degree of g less than or equal to 1. Explicitly

$$(f, g) = \begin{cases} (1, 2x), & \text{Hermite} \\ (x, x-a), & \text{Laguerre} \\ (1-x^2, (a-b) + (a+b)x), & \text{Jacobi} \\ (1+x^2, 2\alpha x), & \text{Cauchy.} \end{cases}$$

It is a consequence of the low degrees of f and g that with

$$\mathcal{A} := f \frac{d}{dx} + \frac{f' - g}{2}, \quad (1.10)$$

and with $\{p_k(x)\}$ the corresponding set of monic orthogonal polynomials,

$$\mathcal{A}p_k(x) = -\frac{c_k}{h_{k+1}} p_{k+1}(x) + \frac{c_{k-1}}{h_{k-1}} p_{k-1}(x), \quad (1.11)$$

for normalisation constants $\{h_k\}$ in (1.3) and certain (easily determined) constants $\{c_k\}$; see [1].

With one of the classical forms of $w(x)$ assumed, choose

$$w_4(x) = f(x)w(x). \quad (1.12)$$

It is shown in [1] that then the corresponding skew orthogonal system satisfying (1.7) can be expressed in terms of the monic orthogonal polynomials corresponding to $w(x)$, $\{p_k(x)\}$, and the constants $\{c_k\}$ in (1.11) according to

$$\begin{aligned} Q_{2j+1}(x) &= p_{2j+1}(x), \\ Q_{2j}(x) &= \left(\prod_{p=0}^{j-1} \frac{c_{2p+1}}{c_{2p}} \right) \sum_{l=0}^j \prod_{p=0}^{l-1} \frac{c_{2p}}{c_{2p+1}} p_{2l}(x) \\ q_m &= c_{2m}. \end{aligned} \quad (1.13)$$

It is also shown in [1] that

$$T_4(x, y) = \frac{1}{2} \left(\frac{f(x)}{f(y)} \right)^{1/2} \left(K_{2N}(x, y) + (w(x)w(y))^{1/2} \frac{c_{2N-1}}{c_{2N}} \frac{p_{2N}(y)}{h_{2N}} Q_{2N-2}(x) \right). \quad (1.14)$$

Our primary aim in this paper is to introduce the notion of classical discrete weights in relation to discretisations of (1.1) and (1.6) — the latter restricted to $\beta = 4$ — on linear and exponential lattices, and to derive formulas analogous to (1.10), (1.11), (1.13) and (1.14) in this setting.

The notion of a classical weight relies on identifying a Pearson-type equation. This is given by (2.5) for the linear lattice, and by (2.8) for the exponential lattice. The analogue of the operator (1.10) is given by (3.10) and (4.25) respectively, with action on the corresponding family of orthogonal polynomials given by (3.13) and (4.28). We give the analogue of (1.13) for the linear and exponential lattice in Propositions 3.9 and 4.6, and that of (1.14) in Propositions 3.9 and 4.5.

2. PRELIMINARY: SOME FACTS ABOUT DISCRETE ORTHOGONAL POLYNOMIALS

In the first part of this section we collect together some basic facts about the classical discrete orthogonal polynomials, as required for later development. One can refer to [32] for more details. We conclude the section by giving the explicit form of the correlation functions for the discretisations of (1.1) on the linear and exponential lattices.

Definition 2.1. *Let $x(t) : \mathbb{R} \rightarrow \mathbb{R}$ be a monotonic function of t . The values $x_i := x(i)$, $i \in \mathbb{Z}$ are said to define lattice points. Consider a function $\rho(x)$ — referred to as a weight function — which has the property of being non-negative at all lattice points and permits finite moments. For general $h(x) = h(x(t))$ define*

$$\begin{aligned} \Delta h(x(t)) &= h(x(t+1)) - h(x(t)) \\ \nabla h(x(t)) &= h(x(t)) - h(x(t-1)). \end{aligned}$$

A set of monic polynomials $\{p_n(x)\}_{n=0}^\infty$, i.e. each $p_n(x)$ of degree n with coefficient of x^n unity, is said to be orthogonal with respect to the weight function $\rho(x)$ if for each n, m

$$\sum_{i \in \mathbb{Z}} p_m(x_i) p_n(x_i) \rho(x_i) \Delta x_{i-1/2} = h_n \delta_{n,m}. \quad (2.1)$$

Here h_n is the normalisation with the property that $h_n > 0$ for $\{x_i\}$ non-decreasing and $h_n < 0$ for $\{x_i\}$ non-increasing and $\Delta x_{i-1/2}$ is a notation defined on half-integer lattice as $\Delta x_{i-1/2} = x_{i+1/2} - x_{i-1/2}$, taking the place of dx in continuous case.

Proposition 2.2. *Suppose there are polynomials $f(x), g(x)$ such that*

$$\Delta[f(x_i)\rho(x_i)] = g(x_i)\rho(x_i)\Delta x_{i-\frac{1}{2}}, \quad (2.2)$$

and define

$$\rho_n(x_i) = \rho(x_{i+n}) \prod_{l=1}^n f(x_{i+l}).$$

Then, for suitable constants B_n , the monic orthogonal polynomials can be written as

$$p_n(x_i) = \frac{B_n}{\rho(x_i)} \left(\frac{\nabla}{\nabla x_{i+1/2}} \cdots \frac{\nabla}{\nabla x_{i+n/2}} \right) \rho_n(x_i). \quad (2.3)$$

Remark 2.3. *By analogy with the continuous case, (2.2) is referred to a Pearson-type equation for $\rho(x)$, while (2.3) is referred to as a Rodrigues-type equation for $p_n(x_i)$.*

There are two particular classes of lattices of interest in our study.

(1) The linear lattice $x(i) = i$. In this case the orthogonality condition (2.1) becomes

$$\sum_{x \in \mathbb{Z}} p_m(x)p_n(x)\rho(x) = h_n \delta_{n,m}. \quad (2.4)$$

At the same time, one can write the Pearson-type equation (2.2) as

$$\frac{\rho(x+1)}{\rho(x)} = \frac{f(x)+g(x)}{f(x+1)}. \quad (2.5)$$

Examples include the Hahn, Meixner, Krawtchouk and Charlier discrete orthogonal polynomial systems.

(2) The exponential lattice $x(i) = q^i$. Note that for $0 < q < 1$ the lattice points form a decreasing sequence. Now the orthogonality relation (2.1) reads

$$\sum_{s \in \mathbb{Z}} p_m(q^s)p_n(q^s)\rho(q^s)q^{s-\frac{1}{2}}(q-1) = h_n \delta_{n,m}.$$

According to the definition of Jackson's q -integral (see e.g. [23])

$$\int_0^\infty f(x)d_q x = (1-q) \sum_{s=-\infty}^\infty f(q^s)q^s, \quad (2.6)$$

showing the orthogonality relation is equivalent to

$$\int_0^\infty p_m(x)p_n(x)\rho(x)d_q x = \tilde{h}_n \delta_{n,m} \quad \text{with} \quad \tilde{h}_n = -q^{1/2}h_n. \quad (2.7)$$

Furthermore, the Pearson-type equation in this case reads

$$\frac{\rho(qx)}{\rho(x)} = \frac{f(x) - q^{-\frac{1}{2}}(1-q)xg(x)}{f(qx)}. \quad (2.8)$$

Examples include the q -analogue of the Hahn, Meixner, Krawtchouk and Charlier polynomial systems, and their degeneration cases, like the Al-Salam & Carlitz polynomials, little q -Jacobi polynomials and so on [28].

Associated with the orthogonalities (2.4) and (2.7) are the symmetric inner products $\langle \cdot, \cdot \rangle : \mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}$ specified by

$$\langle \phi(x), \psi(x) \rangle := \begin{cases} \sum_{i \in \mathbb{Z}} \phi(x_i)\psi(x_i)\rho(x_i), & \text{linear lattice} \\ \int_0^\infty \phi(x)\psi(x)\rho(x)d_q x, & \text{exponential lattice.} \end{cases} \quad (2.9)$$

It is of importance to point out that all of the orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ of interest span a Hilbert space \mathcal{H} . Because of this, the projection operator from $\mathcal{H} \rightarrow \mathcal{H}$ with respect to the inner product (2.9)

$$\delta(x, y) := \sum_{m=0}^{\infty} \frac{1}{\hat{h}_m} p_m(x) p_m(y), \quad (2.10)$$

where $\hat{h}_m = h_m$ as in (2.4) (linear case) and $\hat{h}_m = \tilde{h}_m$ (exponential case) has the reproducing property. In other words, we have

$$\langle \delta(x, y), \xi(x) \rangle = \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{h_m} a_i p_m(y) \langle p_m(x), p_i(x) \rangle = \sum_{i=0}^{\infty} a_i p_i(y) = \xi(y). \quad (2.11)$$

Note that as a function $\delta(x, y)$ is not well defined for $x = y$ as the sum diverges, but this does not effect the validity of (2.11).

Let \mathcal{H}_N be the subspace of \mathcal{H} spanned by $\{p_n(x)\}_{n=0}^{N-1}$, and let K_N denote the projection operator from $\mathcal{H} \rightarrow \mathcal{H}_N$ with respect to the inner product (2.9). Its kernel

$$K_N(x, y) = \sum_{m=0}^{N-1} \frac{1}{\hat{h}_m} p_m(x) p_m(y) \quad (2.12)$$

is formally the same as the polynomial part of the correlation kernel (1.4). Since it too can be summed according to the Christoffel-Darboux summation formula, it will be referred to as the Christoffel-Darboux kernel of the discrete unitary ensemble. As in the continuous case, this kernel specifies the general k -point correlation function of the ensemble.

Explicitly, in the case of the linear lattice the discrete unitary ensemble is defined by the PDF (1.1), with each $x_l = n_l$, $n_l \in \mathbb{Z}$. The same working as leading to (1.2) and (1.4) (see [15, Props. 5.1.1 and 5.1.2]) gives for the correlation function

$$\rho_{N,k}(n_1, \dots, n_k) = \prod_{l=1}^k w(n_l) \det \left[K_N(n_{j_1}, n_{j_2}) \right]_{j_1, j_2=1, \dots, k}. \quad (2.13)$$

The joint probability distribution for the discrete ensemble in the case of the exponential lattice is most conveniently written as a measure rather than a PDF,

$$\frac{1}{Z_N} \prod_{1 \leq j < k \leq N} (x_k - x_j)^2 \prod_{l=1}^N w(x_l) d_q x_l,$$

where each $x_l = q^{n_l}$ for some $n_l \in \mathbb{Z}$. The corresponding correlation functions then relate to the Christoffel-Darboux kernel by

$$\rho_{N,k}(n_1, \dots, n_k) = \prod_{l=1}^k w(q^{n_l}) \det \left[K_N(q^{n_{j_1}}, q^{n_{j_2}}) \right]_{j_1, j_2=1, \dots, k}.$$

3. THE DISCRETE SYMPLECTIC ENSEMBLE ON A LINEAR LATTICE

In this section, we consider the discrete symplectic ensemble on a linear lattice, with our aim being to develop a theory of the classical cases analogous to that done in [1] for the symplectic ensemble with continuous weights. We remark that a detailed study of the discrete symplectic ensemble on linear lattice has previously been given in [9], but from a different point of view. For completeness, we first consider general weights, before specialising to the classical cases.

Definition 3.1. Let $x_1 < \cdots < x_N$ be an ordered set of distinct integers. The joint probability density function of the discrete symplectic ensemble on a linear lattice with weight function $\omega(x)$ is specified by

$$\frac{1}{Z_N} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1) \prod_{i=1}^N \omega(x_i), \quad (3.1)$$

where Z_N is the partition function, assumed to be finite, and given by

$$Z_N = \sum_{\xi_N} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1) \prod_{i=1}^N \omega(x_i),$$

where ξ_N is a configuration space in \mathbb{Z}^N defined by

$$\xi_N = \{(x_1, \dots, x_N) | x_1 < \cdots < x_N, x_i \in \mathbb{Z}\}.$$

According to the [9, Lemma 5.1 and Lemma 5.2], one can see the partition function Z_N can be written as a Pfaffian

$$Z_N = \text{Pf}[A_{i,j}]_{i,j=0}^{2N-1}, \quad A_{i,j} = \sum_{x \in \mathbb{Z}} [\pi_i(x)\pi_j(x+1) - \pi_i(x+1)\pi_j(x)] \omega(x), \quad (3.2)$$

where the $\pi_i(x)$ are an arbitrary system of the monic polynomials of degree i , $i = 0, \dots, 2N - 1$. Now we proceed to use skew tridiagonalisation to show that a family of discrete skew-orthogonal polynomials are inherent in this model.

3.1. Discrete skew orthogonal polynomials. In this subsection, we are going to show how to relate the discrete symplectic ensemble on the integer lattice to the discrete skew orthogonal polynomials.

Definition 3.2. Consider a skew-symmetric inner product $\langle \cdot, \cdot \rangle_{s,\omega}$ with weight function $\omega(x)$, defined on $\mathbb{R}[x] \times \mathbb{R}[x] \rightarrow \mathbb{R}$, admitting the form

$$\begin{aligned} \langle \phi(x), \psi(x) \rangle_{s,\omega} &= \sum_{x \in \mathbb{Z}} [\phi(x)\psi(x+1) - \phi(x+1)\psi(x)] \omega(x) \\ &= \sum_{x \in \mathbb{Z}} [\phi(x)\Delta\psi(x) - \psi(x)\Delta\phi(x)] \omega(x) \end{aligned} \quad (3.3)$$

with $\Delta\phi(x) = \phi(x+1) - \phi(x)$ for arbitrary function $\phi(x) \in \mathbb{R}[x]$ (recall Definition 2.1).

From this definition, a moment matrix $(m_{i,j})_{i,j \geq 0}$ could be generated by the skew-symmetric inner product $\langle x^i, x^j \rangle_{s,\omega}$, and one sees $A_{i,j}$ in (3.2) can then be chosen as $m_{i,j}$. Next we introduce a family of discrete **monic** skew-orthogonal polynomials $\{Q_k(x)\}_{k=0}^{\infty}$, from which the moment matrix can be skew tridiagonalised.

Definition 3.3. Let $\langle \cdot, \cdot \rangle_{s,\omega}$ be specified as in Definition 3.2. Analogous to (1.7) in the continuous case, the discrete skew-orthogonal polynomials, unique up to the mapping (1.8), are specified by the relations

$$\begin{aligned} \langle Q_{2n}(x), Q_{2m+1}(x) \rangle_{s,\omega} &= -\langle Q_{2m+1}(x), Q_{2n}(x) \rangle_{s,\omega} = u_n \delta_{n,m}, \\ \langle Q_{2m}(x), Q_{2n}(x) \rangle_{s,\omega} &= \langle Q_{2m+1}(x), Q_{2n+1}(x) \rangle_{s,\omega} = 0. \end{aligned} \quad (3.4)$$

Note that the skew-orthogonal relation (3.4) is equivalent to the condition

$$\langle Q_{2n}(x), x^i \rangle_{s,\omega} = u_n \delta_{i,2n+1}, \quad \langle Q_{2n+1}(x), x^i \rangle_{s,\omega} = -u_n \delta_{i,2n+1}, \quad 0 \leq i \leq 2n+1.$$

Moreover, if we denote

$$Q_{2n}(x) = \sum_{k=0}^{2n} a_{2n,k} x^k, \quad Q_{2n+1}(x) = \sum_{k=0}^{2n+1} b_{2n+1,k} x^k$$

with $a_{2n,2n} = b_{2n+1,2n+1} = 1$, then the relations (3.4) are linear systems for solving $a_{2n,k}$ and $b_{2n+1,k}$. One can find a detailed computation in [13, Section 2.1], where it is shown that after solving the linear system and employing a determinant identity, the (discrete) skew-orthogonal polynomials can be written in terms of Pfaffians

$$Q_{2n}(x) = \frac{1}{\tau_{2n}} \text{Pf}(0, \dots, 2n, x), \quad Q_{2n+1}(x) = \frac{1}{\tau_{2n}} \text{Pf}(0, \dots, 2n-1, 2n+1, x)$$

with the elements

$$\text{Pf}(i, j) = m_{i,j} = \langle x^i, x^j \rangle_{s,\omega}, \quad \text{Pf}(i, x) = x^i, \quad \tau_{2n} = \text{Pf}(m_{i,j})_{i,j=0}^{2n-1}.$$

Here the arbitrary constant γ_{2n} in (1.8) has been chosen so that the coefficient of x^{2n} in $Q_{2n+1}(x)$ vanishes.

One important quantity here is the τ_{2n} , which is the normalisation factor of the skew orthogonal polynomials. It is also the partition function Z_N in (3.2) if one takes $N = n$. Moreover, one can compute that

$$\begin{aligned} \langle Q_{2n}(x), x^{2n+1} \rangle &= \frac{1}{\tau_{2n}} \langle \text{Pf}(0, \dots, 2n, x), x^{2n+1} \rangle_{s,\omega} = \frac{1}{\tau_{2n}} \sum_{i=0}^{2n} (-1)^i \text{Pf}(0, \dots, \hat{i}, \dots, 2n) \langle x^i, x^{2n+1} \rangle_{s,\omega} \\ &= \frac{1}{\tau_{2n}} \sum_{i=0}^{2n} (-1)^i \text{Pf}(0, \dots, \hat{i}, \dots, 2n) \text{Pf}(i, 2n+1) = \frac{\tau_{2n+2}}{\tau_{2n}} = u_n, \end{aligned}$$

where \hat{i} means the index i is omitted. And upon skew tridiagonalising the moment matrix, it follows

$$\text{Pf}[\langle x^i, x^j \rangle_{s,\omega}]_{i,j=0}^{2n-1} = \text{Pf}[\langle Q_i(x), Q_j(x) \rangle_{s,\omega}]_{i,j=0}^{2n-1} = \prod_{i=0}^{n-1} u_i = \tau_{2n}. \quad (3.5)$$

3.1.1. Skew orthogonal polynomials revisited—moment matrix realisation. In this section we focus attention to the skew-symmetric moment matrix $M_\infty = (m_{i,j})_{i,j=0}^\infty$. We always assume the even-order principal minor of this moment matrix is nonzero, and thus $\tau_{2n} \neq 0$ for each non-negative integer n .

Since M_∞ is a skew-symmetric matrix, one can apply a generalised LU decomposition [12], or so-called skew Borel decomposition [2], to the moment matrix M_∞ such that

$$M = S^{-1} J S^{-\top}, \quad (3.6)$$

where S is a lower triangular matrix with diagonals 1 and J is a block matrix with 2×2 block matrix, admitting the form

$$J = \begin{pmatrix} 0 & u_0 & & & \\ -u_0 & 0 & & & \\ & & 0 & u_1 & \\ & & -u_1 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Therefore, if we take $SM_\infty S^\top$, then this moment matrix is skew tridiagonalised. This has been used in (3.5).

On the other hand, by denoting $\chi(x) = (1, x, x^2, \dots)^\top$, we can define a family of polynomials $\{Q_n(x)\}_{n=0}^\infty$ by

$$Q_n(x) = (S\chi(x))_n,$$

where $(a)_n$ means the n -th component of the vector a . It is clear that $Q_n(x)$ is a monic polynomial of order n since S is a lower triangular matrix with diagonals 1. Moreover, the polynomials $\{Q_n(x)\}_{n=0}^\infty$ are skew orthogonal under the skew-symmetric inner product (3.3) so that

$$\langle S\chi(x), \chi(x)^\top S^\top \rangle_{s,\omega} = S\langle \chi(x), \chi(x)^\top \rangle_{s,\omega} S^\top = SM_\infty S^\top = J,$$

which corresponds to the definition (3.4).

3.2. Christoffel-Darboux kernel and Pfaffian point process. We know from [9] that the general k -point correlation function $\rho_{N,k}$ for the discrete symplectic ensemble (3.1) can be written as a Pfaffian. However no use was made of the skew orthogonal polynomials. It is our objective here to develop the associated theory with the skew orthogonal polynomials regarded as central.

The k -point correlation function corresponding to (3.1) is specified by the multi-dimensional summation

$$\rho_{N,k}(x_1, \dots, x_k) = \frac{1}{\tau_{2N}} \sum_{(x_{k+1} < \dots < x_N) \subset \mathbb{Z}^{N-k}} \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1) \prod_{i=1}^N \omega(x_i).$$

As for the correlations of the continuous symplectic symmetry ensemble (1.6) with $\beta = 4$, $\rho_{N,k}$ can be expressed as a Pfaffian, and is fully determined by a particular 2×2 anti-symmetric matrix.

Proposition 3.4. *Let M denote the moment matrix $M = (m_{i,j})_{i,j=0}^{2N-1}$, and define $\Delta x^i = (x+1)^i - x^i$. The correlation function $\rho_{N,k}$ could be written as a Pfaffian, admitting the form $\rho_{N,k} = \prod_{i=1}^n \omega(x_i) Pf \left[\tilde{S}_N(x_{j_1}, x_{j_2}) \right]_{j_1, j_2=1}^k$, where $\tilde{S}_N(x, y)$ is a 2×2 anti-symmetric matrix*

$$\tilde{S}_N(x, y) = \begin{pmatrix} \sum_{0 \leq i, j \leq 2N-1} x^i M_{i,j}^{-\top} y^j & \sum_{0 \leq i, j \leq 2N-1} x^i M_{i,j}^{-\top} \Delta y^j \\ \sum_{0 \leq i, j \leq 2N-1} \Delta x^i M_{i,j}^{-\top} y^j & \sum_{0 \leq i, j \leq 2N-1} \Delta x^i M_{i,j}^{-\top} \Delta y^j \end{pmatrix}. \quad (3.7)$$

This proposition is essentially a special case of [33, Corollary 1.3], if one takes the measure λ therein as supported on the integer lattice, and $\phi_i(x) = x^{i-1}$, $\psi_i(x) = \Delta \phi_i(x) = (x+1)^{i-1} - x^{i-1}$. The slight difference is that we have considered the ordered set $x_{k+1} < \dots < x_N$ and therefore a factor of $(N-k)!$ is missed. Since the matrix M is skew tridiagonalisable (which is equivalent to $\tau_{2n} \neq 0$), the inverse of M exists.

Remark 3.5. *With respect to the formula for the correlations [33, Corollary 1.3], the authors of [9] took $\phi_i(x) = p_j(x)$ — the discrete orthogonal polynomials $\{p_j(x)\}$ from (2.4) — and $\psi(x) = \Delta\phi(x)$. This choice provides one strategy to investigate the relation to Christoffel-Darboux kernel using the theory of [34]. However in this article we are viewing the discrete skew orthogonal polynomials as an inherent structure of the discrete symplectic ensemble and therefore seek to analyse the kernel within this framework.*

Therefore, according to the skew Borel decomposition (3.6), one can obtain

$$\begin{aligned} \sum_{0 \leq i, j \leq 2N-1} x^i M_{i,j}^{-\top} y^j &= -\chi^\top(x) S^\top J^{-1} S \chi(y) = \sum_{i=0}^{N-1} \frac{1}{u_i} (Q_{2i}(x) Q_{2i+1}(y) - Q_{2i}(y) Q_{2i+1}(x)) \\ &:= S_N(x, y). \end{aligned} \quad (3.8)$$

By analogy with (2.12) this will be referred to as the symplectic Christoffel-Darboux kernel. One can simplify the correlation kernel (3.7) with the help of the symplectic Christoffel-Darboux kernel to obtain

$$\tilde{S}_N(x, y) = \begin{pmatrix} S_N(x, y) & \Delta_x S_N(x, y) \\ \Delta_y S_N(x, y) & \Delta_x \Delta_y S_N(x, y) \end{pmatrix}. \quad (3.9)$$

We remark that the symplectic Christoffel-Darboux kernel admits the reproducing property

$$\begin{aligned} &\langle S_N(x, y), S_N(y, z) \rangle_{s, \omega} \\ &= \sum_{i, j=0}^{N-1} \frac{1}{u_i u_j} [-Q_{2i}(x) Q_{2j+1}(z) \langle Q_{2i+1}(y), Q_{2j}(y) \rangle_{s, \omega} - Q_{2i+1}(x) Q_{2j}(z) \langle Q_{2i}(y), Q_{2j+1}(y) \rangle_{s, \omega}] \\ &= \sum_{i, j=0}^{N-1} \frac{1}{u_i u_j} [u_i \delta_{i, j} Q_{2i}(x) Q_{2j+1}(z) - u_i \delta_{i, j} Q_{2i+1}(x) Q_{2j}(z)] = S_N(x, z). \end{aligned}$$

In the continuous case, theory developed in [1] gives a relationship between the symplectic Christoffel-Darboux kernel and the unitary Christoffel-Darboux kernel, in the case of classical weights related by (1.12). Our objective in the subsequent subsections is to give an analogous theory relating (3.8) to (2.12) for certain classical weights.

3.3. Relationship between discrete skew orthogonal polynomials and discrete orthogonal polynomials. To discover the relationship between skew orthogonal polynomials and discrete orthogonal polynomials is an effective way to depict the linkage between the kernels and one can see [1] and [15, Chapter 6] for more details in continuous case.

We know that in the continuous case a crucial role is played by the operator (1.10). For the discrete case on a linear lattice, as an analogue define the operator

$$\mathcal{A}_1 = g(x)T + f(x)(\Delta + \nabla), \quad (3.10)$$

where $g(x)$ and $f(x)$ are as in (2.5), T is the shift operator defined by $T\phi(x) = \phi(x+1)$, and $\Delta\phi(x) = \phi(x+1) - \phi(x)$ and $\nabla\phi(x) = \phi(x) - \phi(x-1)$ are the same as in Definition 2.1. This operator has the following key properties.

Proposition 3.6. *Define the symmetric inner product*

$$\langle \phi(x), \psi(x) \rangle = \sum_{x \in \mathbb{Z}} \phi(x) \psi(x) \rho(x)$$

as is consistent with the linear lattice case of (2.9). Define the skew symmetric inner product $\langle f(x), g(x) \rangle_{s, \omega}$ according to (3.3). Under the assumption that $\rho(x)f(x)$ vanishing at the end points of support, the operator \mathcal{A}_1 defined in (3.10) satisfies

$$(1) \quad \langle \mathcal{A}_1 \phi(x), \psi(x) \rangle = -\langle \phi(x), \mathcal{A}_1 \psi(x) \rangle; \quad (3.11)$$

$$(2) \quad \langle \phi(x), \mathcal{A}_1 \psi(x) \rangle = \langle \phi(x), \psi(x) \rangle_{s, f(x+1)\rho(x+1)}. \quad (3.12)$$

Proof. Firstly, we prove the equality (3.11). Decompose

$$\begin{aligned} & \langle \mathcal{A}_1 \phi(x), \psi(x) \rangle + \langle \phi(x), \mathcal{A}_1 \psi(x) \rangle \\ &= \sum_{x \in \mathbb{Z}} (\phi(x+1)\psi(x) + \phi(x)\psi(x+1))g(x)\rho(x) + \sum_{x \in \mathbb{Z}} (\Delta\phi(x)\psi(x) + \phi(x)\Delta\psi(x))f(x)\rho(x) \\ &+ \sum_{x \in \mathbb{Z}} (\nabla\phi(x)\psi(x) + \phi(x)\nabla\psi(x))f(x)\rho(x) := A_1 + A_2 + A_3. \end{aligned}$$

Now from the discrete Pearson equation (2.5), we get

$$A_2 = -2 \sum_{x \in \mathbb{Z}} \phi(x)\psi(x)f(x)\rho(x) + \sum_{x \in \mathbb{Z}} [\phi(x+1)\psi(x) + \phi(x)\psi(x+1)][\rho(x+1)f(x+1) - g(x)\rho(x)],$$

$$A_3 = 2 \sum_{x \in \mathbb{Z}} \phi(x)\psi(x)f(x)\rho(x) - \sum_{x \in \mathbb{Z}} [\phi(x-1)\psi(x) + \phi(x)\psi(x-1)]\rho(x)f(x).$$

Moreover, since by assumption $\rho(x)f(x)$ vanishes at the end points of the supports, we see

$$A_2 + A_3 = - \sum_{x \in \mathbb{Z}} [\phi(x+1)\psi(x) + \phi(x+1)\psi(x)]g(x)\rho(x) = -A_1,$$

showing $\langle \mathcal{A}_1 \phi(x), \psi(x) \rangle + \langle \phi(x), \mathcal{A}_1 \psi(x) \rangle = 0$.

Now we turn to (3.12). This equation can be proved by noting

$$\begin{aligned} & \langle \phi(x), \mathcal{A}_1 \psi(x) \rangle \\ &= \sum_{x \in \mathbb{Z}} \phi(x)\psi(x+1)\rho(x)g(x) + \sum_{x \in \mathbb{Z}} \phi(x)\Delta\psi(x)f(x)\rho(x) + \sum_{x \in \mathbb{Z}} \phi(x)\nabla\psi(x)f(x)\rho(x) \\ &= \sum_{x \in \mathbb{Z}} \phi(x)\psi(x+1)f(x+1)\rho(x+1) - \sum_{x \in \mathbb{Z}} \phi(x)\psi(x-1)f(x)\rho(x) \\ &= \sum_{x \in \mathbb{Z}} [\phi(x)\psi(x+1) - \phi(x+1)\psi(x)]f(x+1)\rho(x+1) = \langle \phi(x), \psi(x) \rangle_{s, f(x+1)\rho(x+1)}. \end{aligned}$$

□

Remark 3.7. *In this proof, we have assumed $\rho(x)f(x)$ vanishes at the end points of support, which is always valid for the classical weights. The specific examples, such as Hahn polynomials supported on $[0, N]$, Charlier polynomials and Meixner polynomials supported on $[0, \infty)$ are demonstrated in Section 3.3.1.*

Now we specialise to the classical cases, for which $f(x)$ and $g(x)$ in the discrete Pearson equation (2.5) are of order no bigger than 2 and 1 respectively. With this assumption \mathcal{A}_1 maps polynomials of degree n to polynomials of degree $n+1$ and thus

$$\mathcal{A}_1 p_n(x) = \sum_{j=0}^{n+1} a_{n,j} p_j(x),$$

for some coefficients $a_{n,j}$. Noting from the anti-commutativity property (3.11) and discrete orthogonality (2.4) that

$$\langle \mathcal{A}_1 p_n, p_i \rangle = -\langle p_n, \mathcal{A}_1 p_i \rangle = 0, \quad \text{if } i < n - 1,$$

it follows $a_{n,i} = 0$ for $i < n - 1$. For $i = n - 1$ and $i = n$, we have

$$\begin{aligned} a_{n,n-1} h_{n-1} &= \langle \mathcal{A}_1 p_n, p_{n-1} \rangle = -\langle p_n, \mathcal{A}_1 p_{n-1} \rangle = -a_{n-1,n} h_n, \\ a_{n,n} h_n &= \langle \mathcal{A}_1 p_n, p_n \rangle = -\langle p_n, \mathcal{A}_1 p_n \rangle = -a_{n,n} h_n, \quad \text{and thus } a_{n,n} = 0, \end{aligned}$$

where h_n is the normalisation constant in the discrete orthogonality (2.4). From these equalities, one can conclude the following result.

Proposition 3.8. *For the discrete orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ and operator \mathcal{A}_1 defined in (3.10), we have*

$$\mathcal{A}_1 p_n(x) = -\frac{c_n}{h_{n+1}} p_{n+1}(x) + \frac{c_{n-1}}{h_{n-1}} p_{n-1}(x), \quad (3.13)$$

where c_n is a constant, depending on the degree of polynomials p_n and the functions $f(x)$, $g(x)$ appearing in the discrete Pearson equation (2.5) (cf. (1.11)). Equivalently

$$[\langle p_j, \mathcal{A}_1 p_k \rangle]_{j,k=0}^{2N-1} = \mathbf{c}^\top \Lambda - \Lambda^\top \mathbf{c} := \mathbf{C}, \quad (3.14)$$

where $\mathbf{c} = (c_0, \dots, c_{2N-1})^\top$ and Λ is the shift operator, whose entries on the leading upper diagonal are all 1 and the other entries are all 0.

Since each $Q_j(x)$ and $p_j(x)$ is a monic polynomial of degree j we can introduce a lower triangular matrix \mathbf{T} with 1's on the diagonal such that

$$[Q_j(x)]_{j=0}^{2N-1} = \mathbf{T} [p_j(x)]_{j=0}^{2N-1}. \quad (3.15)$$

Moreover, the equations (3.12) and (3.15) tell us

$$\mathbf{U} := [\langle Q_j, Q_k \rangle_{s,f(x+1)\rho(x+1)}]_{j,k=0}^{2N-1} = [\langle Q_j, \mathcal{A} Q_k \rangle]_{j,k=0}^{2N-1} = \mathbf{T} [\langle p_j, \mathcal{A} p_k \rangle]_{j,k=0}^{2N-1} \mathbf{T}^\top = \mathbf{T} \mathbf{C} \mathbf{T}^\top,$$

and from the skew orthogonality (3.4) we know \mathbf{U} is a skew symmetric 2×2 block diagonal matrix, admitting the form

$$\mathbf{U} = \text{diag} \left(\begin{bmatrix} 0 & u_0 \\ -u_0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & u_{N-1} \\ -u_{N-1} & 0 \end{bmatrix} \right).$$

Since \mathbf{U} and \mathbf{C} are the tridiagonal matrices, and \mathbf{T} is a lower triangular matrix, from the equation $\mathbf{T}^{-1} \mathbf{U} = \mathbf{C} \mathbf{T}^\top$, we know the left hand side is an upper Hessenberg matrix and the right one is a lower Hessenberg matrix, which means both sides of the equation are in fact tridiagonal matrices. If we set $\mathbf{T}^{-1} := [t_{j,k}]_{j,k=0}^{2N-1}$ and consider the completely lower triangular part (not including the diagonal), to be denoted $(\cdot)_-$, one has

$$(\mathbf{T}^{-1} \mathbf{U})_- = \begin{bmatrix} \star & & & & & \\ -u_0 & \star & & & & \\ -u_0 t_{21} & u_0 t_{20} & \star & & & \\ -u_0 t_{31} & u_0 t_{30} & -u_1 & \star & & \\ -u_0 t_{41} & u_0 t_{40} & -u_1 t_{43} & u_1 t_{42} & \star & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

On the other hand, from equation (3.14), one has

$$(\mathbf{CT}^\top)_- = ((\mathbf{c}^\top \Lambda - \Lambda^\top \mathbf{c})\mathbf{T}^\top)_- = (-\Lambda^\top \mathbf{c}\mathbf{T}^\top)_- = -\Lambda^\top \mathbf{c}.$$

Equating these two formulas gives

$$\begin{aligned} c_{2n} &= u_n, & n &= 0, \dots, N-1, \\ t_{2n+1, j} &= 0, & j &= 0, \dots, 2n-1, \\ t_{2n, j} &= 0, & j &= 0, \dots, 2n-3, 2n-1, \\ t_{2n, 2n-2} &= -\frac{c_{2n-1}}{c_{2n-2}}, \end{aligned}$$

and therefore, one can get

$$\begin{aligned} p_{2n+1}(x) &= Q_{2n+1}(x) + t_{2n+1, 2n} Q_{2n}(x), \\ p_{2n}(x) &= Q_{2n}(x) - \frac{c_{2n-1}}{c_{2n-2}} Q_{2n-2}(x). \end{aligned} \tag{3.16}$$

Solving this equation backwards and setting $t_{2n+1, 2n} = 0$ as permitted by the non-uniqueness of the skew-orthogonal polynomials $Q_{2n+1}(x)$, we finally get an expression for the discrete skew orthogonal polynomials in terms of the corresponding discrete orthogonal polynomials.

Proposition 3.9. *Let $\{Q_j(x)\}$ be the skew orthogonal polynomials associated with (3.1), and choose for the weight $\omega(x) = f(x+1)\rho(x+1)$, where $\rho(x)$ is the original weight function of orthogonal polynomials $\{p_j(x)\}$ defined in (2.4) and $(f(x), g(x))$ satisfies the Pearson-type equation (2.2) with $f(x)$ of degree no bigger than 2, and $g(x)$ of degree no bigger than 1. We have*

$$Q_{2n+1}(x) = p_{2n+1}(x), \quad Q_{2n}(x) = \left(\prod_{j=0}^{n-1} \frac{c_{2j+1}}{c_{2j}} \right) \sum_{l=0}^n \prod_{j=0}^{l-1} \frac{c_{2j}}{c_{2j+1}} p_{2l}(x), \tag{3.17}$$

with normalisation $u_n = c_{2n}$.

Remark 3.10. *The relations (3.17) are formally the same as those for skew orthogonal polynomials the continuous symplectic invariant ensemble (1.5) with $\beta = 4$ and $w_4(x)$ given by (1.12) [1].*

3.3.1. *Examples.* In this part, Pearson pairs (f, g) of Meixner, Charlier and Hahn polynomials are constructed. With the help of the Pearson pairs, we can obtain the coefficients c_n in (3.13) such that the discrete skew orthogonal polynomials with classical weight can be explicitly written down.

(1) Meixner case. For the Meixner weight

$$\rho(x) = \frac{(\beta)_x}{x!} a^x, \quad (\beta)_x = \beta \cdots (\beta + x - 1)$$

supported on $x \in \mathbb{N}$ with positive real number β and $0 < a < 1$, we can get the Pearson pair

$$(f, g) = (x, (a-1)x + a\beta).$$

Moreover, the coefficients in (3.13) can be written as

$$c_n = (1-a)h_{n+1},$$

where h_n is the normalisation constant of monic Meixner polynomials.

(2) Charlier case. For the Charlier weight

$$\rho(x) = \frac{a^x}{x!}, \quad a > 0$$

supported on $x \in \mathbb{N}$, one can compute the Pearson pair

$$(f, g) = (x, a - x).$$

One can show the coefficients c_n in (3.13) for the skew orthogonal Charlier polynomials are given by

$$c_n = h_{n+1},$$

where h_n is the normalisation constant of monic Charlier polynomials.

(3) Hahn case. For the Hahn weight

$$\rho(x) = \binom{\alpha + x}{x} \binom{N + \beta - x}{N - x}, \quad \alpha > 0, \beta > 0$$

supported on $\{0, 1, \dots, N\}$, one can compute the Pearson pair

$$(f, g) = (-x^2 + (N + \beta + 1)x, -(\alpha + \beta + 2)x + N(\alpha + 1)).$$

It is easy to determine the coefficients c_n in (3.13) as

$$c_n = (n + \alpha + \beta + 2)h_{n+1},$$

with h_n the normalisation constant of monic Hahn polynomials.

3.4. Kernels between discrete unitary ensembles and discrete symplectic ensembles. To express the discrete symplectic Christoffel-Darboux kernel (3.8) in terms of its unitary counterpart (2.12), we need to use the relationship between the skew-symmetric inner product and symmetric inner product obtained in (3.12).

Firstly, consider the skew symmetric inner product

$$\langle \phi(x), \psi(x) \rangle_{s, f(x+1)\rho(x+1)} = \sum_{x \in \mathbb{Z}} [\phi(x)\psi(x+1) - \phi(x+1)\psi(x)]f(x+1)\rho(x+1)$$

and take $\phi(x) = \delta(x, y)$, where $\delta(x, y)$ is defined in (2.10). On the one hand, according to the equations (3.12) and (2.11), one knows

$$\langle \delta(x, y), \psi(x) \rangle_{s, f(x+1)\rho(x+1)} = \langle \delta(x, y), \mathcal{A}_1\psi(x) \rangle = \mathcal{A}_1\psi(y).$$

On the other hand, from the definition of $\delta(x, y)$

$$\langle \delta(x, y), \psi(x) \rangle_{s, f(x+1)\rho(x+1)} = \sum_{n=0}^{\infty} \frac{p_n(y)}{h_n} \langle p_n(x), \psi(x) \rangle_{s, f(x+1)\rho(x+1)}.$$

If we take $\psi(x) = Q_{2m}(x)$, combining these gives

$$\mathcal{A}_1 Q_{2m}(y) = \sum_{n=0}^{\infty} \frac{p_n(y)}{h_n} \langle p_n(x), Q_{2m}(x) \rangle_{s, \rho(x+1)f(x+1)} = -\frac{u_m}{h_{2m+1}} p_{2m+1}(y)$$

where we used the equation (3.16) for the final equality, and here u_m is the normalisation factor defined in (3.4). Also, if we take $\psi(x) = Q_{2m+1}(x)$, we can get

$$\mathcal{A}_1 Q_{2m+1}(y) = u_m \left(\frac{p_{2m}(y)}{h_{2m}} + \frac{t_{2m+2, 2m}}{h_{2m+2}} p_{2m+2}(y) \right).$$

To summarise, we have the following proposition.

Proposition 3.11. *In the setting of Proposition 3.9, one has the following relations between the discrete orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ and discrete skew orthogonal polynomials $\{Q_n(x)\}_{n=0}^{\infty}$*

$$(1) \quad p_{2m+1}(x) = Q_{2m+1}(x); \quad (3.18a)$$

$$(2) \quad \frac{1}{u_m} \mathcal{A}_1 Q_{2m+1}(y) = \frac{p_{2m}(y)}{h_{2m}} + t_{2m+2,2m} \frac{p_{2m+2}(y)}{h_{2m+2}}; \quad (3.18b)$$

$$(3) \quad p_{2m}(x) = Q_{2m}(x) + t_{2m,2m-2} Q_{2m-2}(x); \quad (3.18c)$$

$$(4) \quad \frac{1}{u_m} \mathcal{A}_1 Q_{2m}(y) = -\frac{p_{2m+1}(y)}{h_{2m+1}}. \quad (3.18d)$$

To make use of these in relation to the correlation kernels, consider first the discrete unitary Christoffel-Darboux kernel (2.12) with $N \mapsto 2N - 1$, and rewrite to read

$$K_{2N-1}(x, y) = \sum_{m=0}^{N-1} \frac{1}{h_{2m}} p_{2m}(x) p_{2m}(y) + \sum_{m=0}^{N-1} \frac{1}{h_{2m+1}} p_{2m+1}(x) p_{2m+1}(y).$$

Use of (3.18a) and (3.18d) shows

$$\sum_{m=0}^{N-1} \frac{1}{h_{2m+1}} p_{2m+1}(x) p_{2m+1}(y) = -\sum_{m=0}^{N-1} \frac{1}{u_m} Q_{2m+1}(x) \mathcal{A}_1 Q_{2m}(y).$$

Similarly, by using both (3.18b) and (3.18c), one can obtain

$$\sum_{m=0}^{N-1} \frac{1}{h_{2m}} p_{2m}(x) p_{2m}(y) = \sum_{m=0}^{N-1} \frac{1}{u_m} Q_{2m}(x) \mathcal{A}_1 Q_{2m+1}(y) - \frac{t_{2N,2N-2}}{h_{2N}} p_{2N}(y) Q_{2N-2}(x).$$

Combining these gives

$$K_{2N-1}(x, y) = \mathcal{A}_1^{(y)} S_N(x, y) - \frac{t_{2N,2N-2}}{h_{2N}} p_{2N}(y) Q_{2N-2}(x), \quad (3.19)$$

where $S_N(x, y)$ is the Christoffel-Darboux kernel of the discrete symplectic ensemble defined in (3.8), and $\mathcal{A}_1^{(y)}$ means the operator acts on the variable y .

Use of the discrete Pearson equation (2.5) allows the operator \mathcal{A}_1 to be rewritten

$$\begin{aligned} \mathcal{A}_1 &:= g(x)T + f(x)(\Delta + \nabla) = \rho^{-1}(x)[\rho(x+1)f(x+1)T - \rho(x)f(x)T^{-1}] \\ &:= \rho^{-1}(x)(\omega(x+1)T - \omega(x)T^{-1}) := \rho^{-1}(x)\mathcal{R}, \end{aligned}$$

where T^{-1} is defined as $T^{-1}\phi(x) = \phi(x-1)$. Furthermore, as suggested by [9, Equation (2.3)], by inspection there exists an inverse for the operator $\mathcal{R} := (\omega(x+1)T - \omega(x)T^{-1})$, called ϵ , whose action is specified by

$$\begin{aligned} (\epsilon \cdot \phi)(2m) &= -\sum_{k=m}^{\infty} \frac{\omega(2m+2) \cdots \omega(2k)}{\omega(2m+1) \cdots \omega(2k+1)} \phi(2k+1), \\ (\epsilon \cdot \phi)(2m+1) &= \sum_{k=-\infty}^m \frac{\omega(2k+2) \cdots \omega(2m)}{\omega(2k+1) \cdots \omega(2m+1)} \phi(2k). \end{aligned}$$

Therefore, the inverse of \mathcal{A}_1 can be written as $\epsilon\rho$, and we denote it as \mathcal{D} .

With this notation, firstly, we can rewrite the equation (3.18d) as

$$Q_{2m}(y) = -\frac{u_m}{h_{2m+1}} (\mathcal{D} \cdot p_{2m+1})(y),$$

and moreover, we have the following proposition, which coincides with the result in [9, Corollary 2.8].

Proposition 3.12. *In the setting of Proposition 3.9 we have the relation between the Christoffel-Darboux kernels of the discrete unitary ensemble and discrete symplectic ensemble*

$$S_N(x, y) = \mathcal{D}^{(y)} K_{2N-1}(x, y) + \frac{c_{2N-1} u_{N-1}}{c_{2N-2} h_{2N} h_{2N-1}} (\mathcal{D} \cdot p_{2N})(y) (\mathcal{D} \cdot p_{2N-1})(x).$$

A standard concern in random matrix theory and its applications is the study of scaled limits of correlation functions. According to Proposition 3.4 and (3.11), for the class of point processes specified in Proposition 3.9, this study is reduced to analysing the symplectic Christoffel-Darboux kernel $S_N(x, y)$. Although outside the scope of the present work, the fact that methods to analyse the unitary Christoffel-Darboux kernel are well established (see e.g. [25]) shows that this identity between kernels is well suited for such a purpose. In fact, such a program has been carried out in [18] for a model with discrete orthogonal (not symplectic) symmetry.

We remark that by taking the explicit c_N in Section 3.3.1, the kernels of Meixner, Charlier and Hahn polynomials can be obtained as

$$\begin{aligned} S_N^{(Meixner)}(x, y) &= \mathcal{D}^{(y)} K_{2N}^{(Meixner)} + \frac{1-a}{h_{2N-1}^{(Meixner)}} (\mathcal{D} \cdot p_{2N})(y) (\mathcal{D} \cdot p_{2N-1})(x), \\ S_N^{(Charlier)}(x, y) &= \mathcal{D}^{(y)} K_{2N}^{(Charlier)} + \frac{1}{h_{2N-1}^{(Charlier)}} (\mathcal{D} \cdot p_{2N})(y) (\mathcal{D} \cdot p_{2N-1})(x), \\ S_N^{(Hahn)}(x, y) &= \mathcal{D}^{(y)} K_{2N}^{(Hahn)} + \frac{2N + \alpha + \beta + 1}{h_{2N-1}^{(Hahn)}} (\mathcal{D} \cdot p_{2N})(y) (\mathcal{D} \cdot p_{2N-1})(x). \end{aligned}$$

4. THE q -ANALOGY TO SYMPLECTIC ENSEMBLE

The choice $w(x) = x^a(1-x)^b$, supported on $0 < x < 1$ in (1.5) defines the Selberg weight in the theory of the Selberg integral; see e.g. [15, Section 4] and [21]. As noted in the latter references, there are natural generalisation of the continuous Selberg weight and integral to a discretisation on the exponential lattice known as the q -Selberg weight and integral. We will use this in the case corresponding to $\beta = 4$ to motivate a study of discrete symplectic ensembles on the exponential lattice. In particular, since a direct computation of the q -skew orthogonal polynomials for the q -Selberg weight is possible, we are presented with the challenge of interpreting their structure in the context of a theory applicable to all q -weights satisfying a Pearson-type equation, as will be taken up below.

Throughout this section, we fix q as $0 < q < 1$ and use the symbols

$$(a; q)_N = \prod_{i=0}^{N-1} (1 - aq^i), \quad (a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$$

and

$$(a; q)_\alpha = \frac{(a; q)_\infty}{(aq^\alpha; q)_\infty}.$$

We will require the q -analogous of gamma function defined as

$$\Gamma_q(x) = (1-q)^{1-x} \frac{(q; q)_\infty}{(q^x; q)_\infty},$$

the q -difference operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad (4.1)$$

and for finite a , the particular q -Jackson integral ¹

$$\int_0^a f(z) d_q z := (1-q) \sum_{n=0}^{\infty} a q^n f(a q^n). \quad (4.2)$$

Also required is the double integral formula

$$\int_0^1 \int_0^{q^\gamma z_1} f(z_1, z_2) \frac{dz_1 dz_2}{z_1 z_2} = (1-q) \int_0^1 \sum_{\mu_2=0}^{\infty} f(z_1, q^{\mu_2} q^\gamma z_1) \frac{dz_1}{z_1} = (1-q)^2 \sum_{\mu_1, \mu_2=0}^{\infty} f(q^{\mu_1}, q^{\mu_1} q^{\mu_2} q^\gamma)$$

if the function $f(z, q^\gamma q^{\mu_2} z)$, $\mu_2 \in \mathbb{N}$ is summable for $z \in q^{\mathbb{N}}$, and the multiple integral generalisation of the latter is

$$\int_0^1 \cdots \int_{z_n=0}^{q^\gamma z_{n-1}} f(z) \frac{d_q z_1}{z_1} \cdots \frac{d_q z_n}{z_n} = (1-q)^n \sum_{\langle \xi_F \rangle} f(t_1, \dots, t_n)$$

with the summation region $\langle \xi_F \rangle$

$$t_1 = q^{\mu_1}, t_2/t_1 = q^{\mu_2} q^\gamma, \dots, t_n/t_{n-1} = q^{\mu_n} q^\gamma, \quad \mu_j \in \mathbb{Z}_{\geq 0}.$$

4.1. The model of q -symplectic ensemble. We begin with the q -generalisation of the Selberg integral, given by Aomoto [3],

$$\begin{aligned} & \int_{z_1=0}^1 \cdots \int_{z_n=0}^{q^\gamma z_{n-1}} \prod_{i=1}^n z_i^\alpha \frac{(qz_i; q)_\infty}{(q^\beta z_i; q)_\infty} \prod_{1 \leq j < k \leq n} z_j^{2\gamma-1} \frac{(q^{1-\gamma} z_k/z_j; q)_\infty}{(q^\gamma z_k/z_j; q)_\infty} (z_j - z_k) \frac{d_q z_n}{z_n} \cdots \frac{d_q z_1}{z_1} \\ & = q^{\alpha\gamma \binom{n}{2} + 2\gamma^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)\gamma) \Gamma_q(\beta + (j-1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\alpha + \beta + (n+j-2)\gamma) \Gamma_q(\gamma)} \end{aligned}$$

for $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $|q^{\alpha+(i-1)\gamma}| < 1$ for $i = 1, \dots, n$.

This has the property that when γ is a positive integer it reduces to the formula [4, 24, 27]

$$\begin{aligned} & \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n z_i^{\alpha-1} (qz_i; q)_{\beta-1} \prod_{\substack{1 \leq j < k \leq n \\ 1-\gamma \leq l \leq \gamma-1}} (z_j - q^l z_k) \prod_{1 \leq j < k \leq n} (z_j - z_k) d_q z_1 \cdots d_q z_n \\ & = q^{\alpha\gamma \binom{n}{2} + 2\gamma^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)\gamma) \Gamma_q(\beta + (j-1)\gamma) \Gamma_q(j\gamma)}{\Gamma_q(\alpha + \beta + (n+j-2)\gamma) \Gamma_q(\gamma)}. \end{aligned} \quad (4.3)$$

This latter form suggests a joint probability measure generalising (1.5) in the case $\beta = 4$,

$$\frac{1}{\tau_{2n}} \prod_{1 \leq j < k \leq n} (x_j - q^{-1} x_k) (x_j - x_k)^2 (x_j - q x_k) \prod_{i=1}^n \omega(x_i; q) d_q x_i, \quad (4.4)$$

on the configuration space $I^n \subset \mathbb{R}^n$. Here I is the supported set of the q -weight function $\omega(x; q)$, which is assumed to be summable for the moments $\int_I x^i \omega(x; q) d_q x$, $i \in \mathbb{N}$. Moreover, the partition

¹Please note the definitions of Jackson's q -integral for finite a and infinite a are different. One can refer [23] for details.

function τ_{2n} is given by

$$\tau_{2n} = \frac{1}{n!} \int_{I^n} \prod_{1 \leq j < k \leq n} (x_j - q^{-1}x_k)(x_j - x_k)^2(x_j - qx_k) \prod_{i=1}^n \omega(x_i; q) d_q x_i, \quad (4.5)$$

Use of the equality

$$\prod_{1 \leq j < k \leq n} (x_j - x_k)^2(x_j - q^{-1}x_k)(x_j - qx_k) = (-q)^{-\binom{n}{2}} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2(x_k - qx_j)(x_j - qx_k) \quad (4.6)$$

in (4.4) gives us a more symmetric form to work with.

We begin by expressing τ_{2n} as a Pfaffian.

Lemma 4.1. *Let τ_{2n} be given by (4.5). We have*

$$\tau_{2n} = Pf \left[\int_I (x^i D_q x^j - x^j D_q x^i) \omega(x; q) d_q x \right]_{i,j=0}^{2n-1}. \quad (4.7)$$

Proof. We first rewrite the product of differences in (4.6). Using the Vandermonde determinant

$$\prod_{1 \leq j < k \leq 2n} (y_k - y_j) = \det[y_j^{k-1}]_{j,k=1}^{2n}$$

and setting $y_j = x_j$ for $j = 1, \dots, n$ and $y_{n+j} = qx_j$ for $j = 1, \dots, n$, one can see that the left hand side of the Vandermonde determinant is equal to

$$\begin{aligned} & \prod_{1 \leq j < k \leq n} (x_k - x_j) q(x_k - x_j) \prod_{j,k=1}^n (qx_k - x_j) \\ &= q^{\binom{n}{2}} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 (qx_k - x_j) (qx_j - x_k) \prod_{j=1}^n (qx_j - x_j). \end{aligned}$$

By using this formula, we see

$$\prod_{1 \leq j < k \leq n} (x_j - x_k)^2 (x_j - qx_k) (x_k - qx_j) = (-q)^{-\binom{n}{2}} \det \left[x_i^j, D_q x_i^j \right]_{i=1, \dots, n}^{j=0, \dots, 2n-1},$$

where D_q is the q -difference operator defined by (4.1). The stated result now follows from the de Bruijn formula [11, 16]. \square

The most systematic way to evaluate the Pfaffian in (4.7) is to tridiagonalise the skew symmetric moment matrix, which leads us to consider q -skew orthogonal polynomials. In next subsection, we will give the definition of the q -skew orthogonal polynomials and formulate the correlation kernel in terms of the q -skew orthogonal polynomials, using a method similar to that given in the Section 3.1 and Section 3.2.

4.2. q -skew orthogonal polynomials and correlation function. We denote $\mathbb{R}_q[x]$ as the ring of polynomials in x over the field $\mathbb{R}(q)$. Consider the skew symmetric inner product $\langle \cdot, \cdot \rangle_{s,\omega} : \mathbb{R}_q[x] \times \mathbb{R}_q[x] \mapsto \mathbb{R}(q)$, with the form

$$\langle f(x; q), g(y; q) \rangle_{s,\omega} = \int_I (f(x; q) D_q g(x; q) - g(x; q) D_q f(x; q)) \omega(x; q) d_q x. \quad (4.8)$$

In terms of this inner product one has for the sequence of moments

$$m_{i,j}(q) = \langle x^i, y^j \rangle_{s,\omega} = ([j]_q - [i]_q) \int_I x^{i+j-1} \omega(x; q) d_q x, \quad [i]_q = \frac{1 - q^i}{1 - q}. \quad (4.9)$$

Assuming the even-order principal minors of moment matrix $M_\infty(q) = (m_{i,j}(q))_{i,j=0}^\infty$ are nonsingular (i.e. $\tau_{2n} \neq 0$ for $n \in \mathbb{Z}_{\geq 0}$), we can apply the skew-Borel decomposition to this moment matrix to obtain

$$M_\infty(q) = S^{-1}(q)J(q)S^{-\top}(q). \quad (4.10)$$

Here $S(q)$ is a lower triangular matrix with diagonals 1 and $J(q)$ a skew symmetric 2×2 block matrix, denoted as

$$J(q) = \begin{pmatrix} 0 & u_0(q) & & & & \\ -u_0(q) & 0 & & & & \\ & & 0 & u_1(q) & & \\ & & -u_1(q) & 0 & & \\ & & & & \ddots & \\ & & & & & \ddots \end{pmatrix}.$$

By introducing $\chi(x) = (1, x, x^2, \dots)^\top$ we can therefore define the q -skew orthogonal polynomials $Q(x; q) := \{Q_n(x; q)\}_{n=0}^\infty$ by

$$Q(x; q) = S(q)\chi(x),$$

satisfying the defining properties

$$\begin{aligned} \langle Q_{2n}(x; q), Q_{2m+1}(x; q) \rangle_{s,\omega} &= -\langle Q_{2m+1}(x; q), Q_{2n}(x; q) \rangle_{s,\omega} = u_n(q)\delta_{n,m}, \\ \langle Q_{2m}(x; q), Q_{2n}(x; q) \rangle_{s,\omega} &= \langle Q_{2m+1}(x; q), Q_{2n+1}(x; q) \rangle_{s,\omega} = 0. \end{aligned} \quad (4.11)$$

In this setting one can compute $u_n(q)$ in (4.11) as $u_n(q) = \frac{\tau_{2n+2}}{\tau_{2n}}$. Moreover, τ_{2n} defined in (4.7) can then be written as

$$\text{Pf} \left[\int_I (x^i D_q x^j - x^j D_q x^i) \omega(x; q) d_q x \right]_{i,j=0}^{2n-1} = \text{Pf} [\langle Q_i^s(x; q), Q_j^s(y; q) \rangle_{s,\omega}]_{i,j=0}^{2n-1} = \prod_{i=0}^{n-1} u_i = \tau_{2n}.$$

It is also the case that this family of polynomials have the following q -integral representation (cf. [14], [22]).

Proposition 4.2. *In terms of multidimensional q -integrals,*

$$\begin{aligned} Q_{2n}(x; q) &= \frac{1}{n! \tau_{2n}} \int_{I^n} \Delta_q^4(x) \prod_{i=1}^n (x - x_i)(x - qx_i) \omega(x_i; q) d_q x_i, \\ Q_{2n+1}(x; q) &= \frac{1}{n! \tau_{2n}} \int_{I^n} \Delta_q^4(x) \left(x + (1+q) \sum_{i=1}^n x_i + c \right) \prod_{i=1}^n (x - x_i)(x - qx_i) \omega(x_i; q) d_q x_i \end{aligned}$$

with $\Delta_q^4(x) := \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 (x_j - qx_k)(q^{-1}x_k - x_j)$ and τ_{2n} being defined in (4.7). Moreover, c in the second equality is an arbitrary constant as is γ_{2m} in (1.8).

Now, we turn to the correlation function of the q -symplectic case. Define the correlation function on the phase space I^k , where I is the support of $\omega(x; q)$, as

$$\rho_{n,k}(x_1, \dots, x_k) = \frac{1}{(n-k)! \tau_{2n}} \int_{I^{n-k}} \Delta_q^4(x) \prod_{i=k+1}^n \omega(x_i; q) d_q x_i.$$

According to [33, Corollary 1.4], if we take the measure as the discrete measure on the exponential lattice, then this correlation function can be written as a Pfaffian specified by a particular 2×2 skew symmetric kernel.

Proposition 4.3. *The statistical state corresponding to the probability measure (4.4) is a Pfaffian point process. Explicitly, the k -point correlation function $\rho_{n,k}$ admits the form*

$$\rho_{n,k} = \prod_{l=1}^k \omega(x_l; q) \text{Pf}(\tilde{R}_n(x_i, x_j))_{i,j=1}^k,$$

where

$$\tilde{R}_n(x, y) = \begin{pmatrix} \sum_{0 \leq i, j \leq 2n-1} x^i M(q)_{i,j}^{-\top} y^j & \sum_{0 \leq i, j \leq 2n-1} x^i M(q)_{i,j}^{-\top} D_q y^j \\ \sum_{0 \leq i, j \leq 2n-1} D_q x^i M(q)_{i,j}^{-\top} y^j & \sum_{0 \leq i, j \leq 2n-1} D_q x^i M(q)_{i,j}^{-\top} D_q y^j \end{pmatrix},$$

with $M(q)$ the moment matrix with entries $m_{i,j}(q)$ in (4.9).

Analogous to (3.11), if we denote

$$R_n(x, y; q) := \sum_{0 \leq i, j \leq 2n-1} x^i M(q)_{i,j}^{-\top} y^j = \sum_{i=0}^{n-1} \frac{1}{u_i(q)} (Q_{2i}(x; q) Q_{2i+1}(y; q) - Q_{2i}(y; q) Q_{2i+1}(x; q)),$$

we can write

$$\tilde{R}_n(x, y) = \begin{pmatrix} R_n(x, y; q) & D_{q,y} R_n(x, y; q) \\ D_{q,x} R_n(x, y; q) & D_{q,x} D_{q,y} R_n(x, y; q) \end{pmatrix}.$$

4.3. Two examples of classical q -skew orthogonal polynomials. To guide our study in the more general case, we begin by analysing two specific examples of q -skew orthogonal polynomials as examples. A general discussion will then be given in the next subsection.

4.3.1. Al-Salam & Carlitz type skew-orthogonal polynomials. Consider the Al-Salam & Carlitz weight [6, 28]

$$\omega(x; q) = \frac{(qx; q)_{\infty} (qx/\alpha; q)_{\infty}}{(q; q)_{\infty} (\alpha; q)_{\infty} (q/\alpha; q)_{\infty}}, \quad \alpha < 0 \quad (4.12)$$

and its corresponding integral region $I = [\alpha, 1]$, interpreted as

$$\int_{\alpha}^1 f(z) d_q z := (1-q) \left(\sum_{n=0}^{\infty} q^n f(q^n) - \alpha \sum_{n=0}^{\infty} \alpha q^n f(\alpha q^n) \right). \quad (4.13)$$

We say $\{U_n^{(\alpha)}(x; q)\}_{n=0}^{\infty}$ are the monic Al Salam & Carlitz orthogonal polynomials with weight function $\omega(x; q)$ in the integral I if they satisfy

$$\int_{\alpha}^1 U_m^{\alpha}(x; q) U_n^{\alpha}(x; q) \omega(x; q) d_q x = (1-q) (-\alpha)^n q^{\binom{n}{2}} (q; q)_n \delta_{n,m} := h_n \delta_{n,m}. \quad (4.14)$$

It is known that these polynomials satisfy the following lowering equation (see e.g. [5])

$$D_q U_n^{\alpha}(x; q) = [n]_q U_{n-1}^{\alpha}(x; q), \quad [n]_q = \frac{1-q^n}{1-q}. \quad (4.15)$$

Being monic polynomials of successive degree, they form basis for the polynomials space $\mathbb{R}_q[x]$, and thus one can expand the corresponding skew orthogonal polynomials $\{Q_m^\alpha(x; q)\}_{m=0}^\infty$ in the form

$$Q_{2m}^\alpha(x; q) = \sum_{p=0}^{2m} a_{m,p} U_{2m-p}^\alpha(x; q), \quad Q_{2m+1}^\alpha(x; q) = \sum_{p=0}^{2m+1} b_{m,p} U_{2m+1-p}^\alpha(x; q)$$

with $a_{m,0} = b_{m,0} = 1$. We seek the explicit form of the coefficients.

Firstly, consider polynomials of even degree. From the skew orthogonal relation (4.11), we know

$$\langle Q_{2m}^\alpha(x; q), U_i^\alpha(x; q) \rangle_{s,\omega} = 0, \quad 0 \leq i \leq 2m,$$

or equivalently,

$$\int_I (Q_{2m}^\alpha(x; q) D_q U_i^\alpha(x; q) - D_q Q_{2m}^\alpha(x; q) U_i^\alpha(x; q)) \omega(x; q) d_q x = 0.$$

Then from the lowering relation (4.15) and orthogonality (4.14) of the Al-Salam & Carlitz orthogonal polynomials, one can obtain

$$a_{m,2m-i+1} [i]_q h_{i-1} - a_{m,2m-i-1} [i+1]_q h_i = 0, \quad 0 \leq i \leq 2m.$$

This is an explicit iterative relation for the coefficients $\{a_{m,p}, p = 0, \dots, 2m\}$. Inserting the explicit value of h_i , it reads

$$a_{m,2m-i+1} = -\alpha(1-q)q^{i-1} [i+1]_q a_{m,2m-i-1}, \quad 0 \leq i \leq 2m.$$

Notice that $a_{m,-1} = 0$ implies $a_{m,\text{odd}} = 0$, while $a_{m,0} = 1$ implies

$$a_{m,2p} = \prod_{l=1}^p \{-\alpha(1-q)q^{2m-2l} [2m-2l+2]_q\} = \prod_{l=1}^p \{-\alpha(1-q^2)q^{2(m-l)} [m-l+1]_{q^2}\}.$$

Changing the variable $2p \mapsto 2(m-p)$, one can show

$$a_{m,2(m-p)} = \prod_{l=1}^{m-p} \{-\alpha(1-q^2) [m-l+1]_{q^2} q^{2(m-l)}\} = (-\alpha(1-q^2))^{m-p} q^{(m-p)(m+p-1)} \frac{[m]_{q^2}!}{[p]_{q^2}!},$$

and thus

$$Q_{2m}^\alpha(x; q) = (-\alpha(1-q^2))^m [m]_{q^2}! \sum_{p=0}^m \frac{q^{(m-p)(m+p-1)}}{(-\alpha(1-q^2))^p [p]_{q^2}!} U_{2p}^\alpha(x; q).$$

For polynomials of odd degree, we make particular use of the skew orthogonal relations

$$\langle Q_{2m+1}^\alpha(x; q), Q_{2m+1}^\alpha(x; q) \rangle_{s,\omega} = 0, \tag{4.16a}$$

$$\langle Q_{2m+1}^\alpha(x; q), U_i^\alpha(x; q) \rangle_{s,\omega} = 0, \quad 0 \leq i \leq 2m-1. \tag{4.16b}$$

The equation (4.16b) gives rise to

$$b_{m,2m-i+2} [i]_q h_{i-1} - b_{m,2m-i} [i+1]_q h_i = 0, \quad 0 \leq i \leq 2m-1 \tag{4.17}$$

by using the orthogonal relation (4.14). Also, noting that $b_{m,2m+2} = 0$, it follows

$$b_{m,2m+2} = \dots = b_{m,2} = 0. \tag{4.18}$$

The equation (4.16a) is equivalent to

$$\langle Q_{2m+1}^\alpha(x; q), U_{2m+1}^\alpha(x; q) + b_{m,1} U_{2m}^\alpha(x; q) \rangle_{s,\omega} = 0.$$

By using (4.16b) again, it follows that

$$b_{m,1}[2m+1]_q h_{2m} + b_{m,2} b_{m,1} [2m]_q h_{2m-1} - b_{m,1} [2m+1]_q h_{2m} = 0.$$

This shows $b_{m,1} = 0$ or $b_{m,2} = 0$ or both of them are equal to zero. Without generality, we can assume $b_{m,1}$ is not equal to zero ($b_{m,2} = 0$ has been obtained in (4.18)). Further, one can see the iterative process (4.17) is the same as for $a_{m,2p}$. Therefore we get

$$Q_{2m+1}^\alpha(x; q) = U_{2m+1}^\alpha(x; q) + b_{m,1} Q_{2m}^\alpha(x; q).$$

Using the freedom implied by (1.8) we can take $b_{m,1} = 0$ for brevity.

In summary, we have found that

$$\begin{aligned} Q_{2m+1}^\alpha(x; q) &= U_{2m+1}^\alpha(x; q), \\ Q_{2m}^\alpha(x; q) &= (-\alpha(1-q^2))^m [m]_{q^2}! \sum_{p=0}^m \frac{q^{(m-p)(m+p-1)}}{(-\alpha(1-q^2))^p [p]_{q^2}!} U_{2p}^\alpha(x; q), \end{aligned} \quad (4.19)$$

are q -skew orthogonal with respect to the skew symmetric inner product (4.8) under the Al-Salam & Carlitz weight (4.12). Moreover, the normalisation constant can be evaluated by

$$\begin{aligned} & \frac{1}{n!} \int_I \prod_{1 \leq j < k \leq n} (x_j - q^{-1}x_k)(x_j - x_k)^2 (x_j - qx_k) \prod_{i=1}^n \omega(x_i; q) d_q x_i \\ &= \prod_{i=0}^{n-1} \langle Q_{2i}^\alpha(x; q), Q_{2i+1}^\alpha(x; q) \rangle = \prod_{i=0}^{n-1} \{c_{2i}[2i+1]_q\} = \frac{1}{2^n} \alpha^{n(n-1)} q^{\frac{n(2n-1)(n-1)}{6}} \prod_{i=0}^{n-1} (q; q)_{2i+1}, \end{aligned}$$

which corresponds to the results in [6, Equation 4.27].

4.3.2. *Little q -Jacobi skew orthogonal polynomials.* The monic little q -Jacobi polynomials $\{p_n^{(\alpha, \beta)}(x; q)\}_{n=0}^\infty$ are defined by the orthogonality relation [28, 30]

$$\int_0^1 p_n^{(\alpha, \beta)}(x; q) x^k \omega^{(\alpha, \beta)}(x; q) d_q x = 0, \quad 0 \leq k \leq n-1$$

with the weight function

$$\omega^{(\alpha, \beta)}(x; q) := (qx; q)_\beta x^\alpha, \quad \alpha > -1, \beta > -1. \quad (4.20)$$

This is the same weight as in (4.3) upon incrementing α and β by 1 in the latter, and one can easily find

$$\omega^{(\alpha, \beta)}(x; q) = x(1 - q^\beta x) \omega^{(\alpha-1, \beta-1)}(x; q).$$

For the normalisation, one has

$$\begin{aligned} & \int_0^1 p_n^{(\alpha, \beta)}(x; q) p_m^{(\alpha, \beta)}(x; q) \omega^{(\alpha, \beta)}(x; q) d_q x \\ &= q^{n(n+\alpha+2)} [\alpha + \beta + 2n + 1]_q^{-1} \frac{(q; q)_{n+\alpha+\beta} (q; q)_{n+\alpha} (q; q)_{n+\beta} (q; q)_n}{(q; q)_{2n+\alpha+\beta}^2} \delta_{n,m} =: h_n^{(\alpha, \beta)} \delta_{n,m}. \end{aligned} \quad (4.21)$$

It is well known that the little q -Jacobi polynomials permit the lowering operation [30]

$$D_q p_n^{(\alpha, \beta)}(x; q) = [n]_q p_{n-1}^{(\alpha+1, \beta+1)}(x; q) \quad (4.22)$$

and the raising operation

$$D_p \left[\omega^{(\alpha, \beta)}(x; q) p_n^{(\alpha, \beta)}(x; q) \right] = -\frac{1 - q^{n+\alpha+\beta}}{(1-q)q^{n+\alpha-1}} \omega^{(\alpha-1, \beta-1)}(x; q) p_{n+1}^{(\alpha-1, \beta-1)}(x; q), \quad p = \frac{1}{q}.$$

From this the Rodrigues formula enables us to give an explicit expression for this polynomial,

$$p_n^{(\alpha,\beta)}(x; q) = \frac{q^{(n+\alpha)n}(q^{-n-\alpha}; q)_n}{(q^{n+\alpha+\beta+1}; q)_n} \sum_{k=0}^n \frac{(q^{-n}; q)_k (q^{n+\alpha+\beta+1}; q)_k}{(q^{\alpha+1}; q)_k} \frac{q^k x^k}{(q; q)_k} =: x^n + \sum_{i=1}^n \gamma_{n,i}^{(\alpha,\beta)} x^{n-i}.$$

Note in particular

$$\gamma_{n,1}^{(\alpha,\beta)} = -\frac{(1-q^n)(1-q^{n+\alpha})}{(1-q)(1-q^{2n+\alpha+\beta})},$$

which will be used later.

To construct the relations between little q -Jacobi orthogonal polynomials and q -skew orthogonal polynomials, the following proposition is needed.

Lemma 4.4. *One has the relation between $\{p_n^{(\alpha-1,\beta-1)}(x; q)\}_{n=0}^\infty$ and $\{p_n^{(\alpha,\beta)}(x; q)\}_{n=0}^\infty$,*

$$p_n^{(\alpha-1,\beta-1)}(x; q) = p_n^{(\alpha,\beta)}(x; q) + a_{n,n-1}^{(\alpha,\beta)} p_{n-1}^{(\alpha,\beta)}(x; q) + a_{n,n-2}^{(\alpha,\beta)} p_{n-2}^{(\alpha,\beta)}(x; q),$$

where $a_{n,n-1}^{(\alpha,\beta)}$ and $a_{n,n-2}^{(\alpha,\beta)}$ are given in (4.24).

Proof. Since $\{p_n^{(\alpha,\beta)}(x; q)\}_{n=0}^\infty$ form a basis of $\mathbb{R}_q[x]$, one can assume

$$p_n^{(\alpha-1,\beta-1)}(x; q) = \sum_{k=0}^n a_{n,k}^{(\alpha,\beta)} p_k^{(\alpha,\beta)}(x; q).$$

From the orthogonal relation (4.21), one knows

$$\begin{aligned} & \int_0^1 p_n^{(\alpha-1,\beta-1)}(x; q) x^i \omega^{(\alpha-1,\beta-1)}(x; q) d_q x \\ &= \sum_{k=0}^n a_{n,k}^{(\alpha,\beta)} \int_0^1 p_k^{(\alpha,\beta)}(x; q) x^i \omega^{(\alpha-1,\beta-1)}(x; q) d_q x = 0, \quad 0 \leq i \leq n-1. \end{aligned}$$

Moreover, the relation (4.20) leads to $a_{n,0} = \dots = a_{n,n-3} = 0$, and hence

$$p_n^{(\alpha-1,\beta-1)}(x; q) = p_n^{(\alpha,\beta)}(x; q) + a_{n,n-1}^{(\alpha,\beta)} p_{n-1}^{(\alpha,\beta)}(x; q) + a_{n,n-2}^{(\alpha,\beta)} p_{n-2}^{(\alpha,\beta)}(x; q).$$

Now, from the equations

$$\begin{aligned} & \int_0^1 p_n^{(\alpha-1,\beta-1)}(x; q) x^n \omega^{(\alpha-1,\beta-1)}(x; q) d_q x = h_n^{(\alpha-1,\beta-1)}, \\ & \int_0^1 p_n^{(\alpha-1,\beta-1)}(x; q) x^{n+1} \omega^{(\alpha-1,\beta-1)}(x; q) d_q x = -\gamma_{n+1,1}^{(\alpha-1,\beta-1)} h_n^{(\alpha-1,\beta-1)}, \end{aligned}$$

it follows

$$\begin{aligned} a_{n,n-2}^{(\alpha,\beta)} h_{n-2}^{(\alpha,\beta)} &= -q^\beta h_n^{(\alpha-1,\beta-1)}, \\ a_{n,n-1}^{(\alpha,\beta)} h_{n-1}^{(\alpha,\beta)} &= c_n^{(\alpha-1,\beta-1)} + q^\beta h_n^{(\alpha-1,\beta-1)} (\gamma_{n+1,1}^{(\alpha-1,\beta-1)} - \gamma_{n-1,1}^{(\alpha,\beta)}), \end{aligned} \tag{4.23}$$

allowing us to compute

$$\begin{aligned} a_{n,n-2}^{(\alpha,\beta)} &= -q^{2n+2\alpha+\beta-1} \frac{(1-q^{n+\alpha-1})(1-q^{n+\beta-1})(1-q^{n-1})(1-q^n)}{(1-q^{\alpha+\beta+2n-1})(1-q^{\alpha+\beta+2n-2})^2(1-q^{\alpha+\beta+2n-3})}, \\ a_{n,n-1}^{(\alpha,\beta)} &= \frac{q^{n+\alpha+1}(1-q^n)}{1-q^{n+\alpha+\beta-1}} \left[1 + \frac{q^\beta}{1-q} \left(\frac{(1-q^{n-1})(1-q^{n+\alpha-1})}{1-q^{2n+\alpha+\beta-2}} - \frac{(1-q^{n+1})(1-q^{n+\alpha})}{1-q^{2n+\alpha+\beta}} \right) \right], \end{aligned} \tag{4.24}$$

and thus complete the proof. \square

Now consider the monic little q -Jacobi skew orthogonal polynomials $\{Q_m^{(\alpha,\beta)}(x; q)\}_{m=0}^\infty$, which have the expansion with the basis $\{p_m^{(\alpha,\beta)}(x; q)\}_{m=0}^\infty$

$$Q_{2m}^{(\alpha,\beta)}(x; q) = \sum_{j=0}^{2m} \xi_{m,j}^{(\alpha,\beta)} p_j^{(\alpha,\beta)}(x; q), \quad Q_{2m+1}^{(\alpha,\beta)}(x; q) = \sum_{j=0}^{2m+1} \eta_{m,j}^{(\alpha,\beta)} p_j^{(\alpha,\beta)}(x; q).$$

Using the q -skew orthogonality relation

$$\int_0^1 \left(Q_{2m}^{(\alpha,\beta)}(x; q) D_q p_i^{(\alpha,\beta)}(x; q) - D_q Q_{2m}^{(\alpha,\beta)}(x; q) p_i^{(\alpha,\beta)}(x; q) \right) \omega^{(\alpha+1,\beta+1)}(x; q) d_q x = 0, \quad 0 \leq i \leq 2m$$

and taking the lowering operation for the little q -Jacobi polynomials (4.22), one can find

$$\begin{aligned} & \int_0^1 Q_{2m}^{(\alpha,\beta)}(x; q) D_q p_i^{(\alpha,\beta)}(x; q) \omega^{(\alpha+1,\beta+1)}(x; q) d_q x \\ &= \sum_{j=0}^{2m} [i]_q \xi_{m,j}^{(\alpha,\beta)} \int_0^1 p_j^{(\alpha,\beta)}(x; q) p_{i-1}^{(\alpha+1,\beta+1)}(x; q) \omega^{(\alpha+1,\beta+1)}(x; q) d_q x \\ &= [i]_q h_{i-1}^{(\alpha+1,\beta+1)} (\xi_{m,i-1}^{(\alpha,\beta)} + a_{i,i-1}^{(\alpha+1,\beta+1)} \xi_{m,i}^{(\alpha,\beta)} + a_{i+1,i-1}^{(\alpha+1,\beta+1)} \xi_{m,i+1}^{(\alpha,\beta)}). \end{aligned}$$

A similar strategy shows

$$\begin{aligned} & \int_0^1 D_q Q_{2m}^{(\alpha,\beta)}(x; q) p_i^{(\alpha,\beta)}(x; q) \omega^{(\alpha+1,\beta+1)}(x; q) d_q x \\ &= \sum_{j=0}^{2m} \xi_{m,j}^{(\alpha,\beta)} [j]_q \int_0^1 p_{j-1}^{(\alpha+1,\beta+1)}(x; q) p_i^{(\alpha,\beta)}(x; q) \omega^{(\alpha+1,\beta+1)}(x; q) d_q x \\ &= [i+1]_q h_i^{(\alpha+1,\beta+1)} \xi_{m,i+1}^{(\alpha,\beta)} + [i]_q h_{i-1}^{(\alpha+1,\beta+1)} a_{i,i-1}^{(\alpha+1,\beta+1)} \xi_{m,i}^{(\alpha,\beta)} + [i-1]_q h_{i-2}^{(\alpha+1,\beta+1)} a_{i,i-2}^{(\alpha+1,\beta+1)} \xi_{m,i-1}^{(\alpha,\beta)}. \end{aligned}$$

Moreover, the skew orthogonal relation implies

$$\begin{aligned} & ([i+1]_q h_i^{(\alpha+1,\beta+1)} - [i]_q h_{i-1}^{(\alpha+1,\beta+1)} a_{i+1,i-1}^{(\alpha+1,\beta+1)}) \xi_{m,i+1}^{(\alpha,\beta)} \\ &= ([i]_q h_{i-1}^{(\alpha+1,\beta+1)} - [i-1]_q h_{i-2}^{(\alpha+1,\beta+1)} a_{i,i-2}^{(\alpha+1,\beta+1)}) \xi_{m,i-1}^{(\alpha,\beta)}, \quad 0 \leq i \leq 2m. \end{aligned}$$

Taking (4.23) into the above equation and noting $\xi_{m,2m}^{(\alpha,\beta)} = 1$ gives

$$\xi_{m,2m-2j}^{(\alpha,\beta)} = \prod_{k=1}^j \frac{[2m-2k+2]_q h_{2m-2k+1}^{(\alpha+1,\beta+1)} + q^{\beta+1} [2m-2k+1]_q h_{2m-2k+2}^{(\alpha,\beta)}}{[2m-2k+1]_q h_{2m-2k}^{(\alpha+1,\beta+1)} + q^{\beta+1} [2m-2k]_q h_{2m-2k+1}^{(\alpha,\beta)}}.$$

The odd indexed coefficients vanish as follows by noting $\xi_{m,2m+1}^{(\alpha,\beta)} = 0$. Therefore,

$$Q_{2m}^{(\alpha,\beta)}(x; q) = \sum_{j=0}^m \prod_{k=1}^{m-j} \frac{[2m-2k+2]_q h_{2m-2k+1}^{(\alpha+1,\beta+1)} + q^{\beta+1} [2m-2k+1]_q h_{2m-2k+2}^{(\alpha,\beta)}}{[2m-2k+1]_q h_{2m-2k}^{(\alpha+1,\beta+1)} + q^{\beta+1} [2m-2k]_q h_{2m-2k+1}^{(\alpha,\beta)}} p_{2j}^{(\alpha,\beta)}(x; q).$$

For the odd ones, firstly consider the skew orthogonal relation

$$\langle Q_{2m+1}^\alpha(x; q), p_i^{(\alpha,\beta)}(x; q) \rangle_{s,\omega^{(\alpha+1,\beta+1)}} = 0, \quad 0 \leq i \leq 2m-1.$$

Following the above working, one can easily obtain the equation

$$\begin{aligned} & ([i+1]_q h_i^{(\alpha+1,\beta+1)} - [i]_q h_{i-1}^{(\alpha+1,\beta+1)} a_{i+1,i-1}^{(\alpha+1,\beta+1)}) \eta_{m,i+1}^{(\alpha,\beta)} \\ &= ([i]_q h_{i-1}^{(\alpha+1,\beta+1)} - [i-1]_q h_{i-2}^{(\alpha+1,\beta+1)} a_{i,i-2}^{(\alpha+1,\beta+1)}) \eta_{m,i-1}^{(\alpha,\beta)}, \quad 0 \leq i \leq 2m-1. \end{aligned}$$

Since $\eta_{m,-1}^{(\alpha,\beta)} = 0$, this equation demonstrates that $\eta_{m,\text{odd}}^{(\alpha,\beta)} = 0$ except $\eta_{m,2m+1}^{(\alpha,\beta)} = 1$. Moreover, the coefficients $\eta_{m,\text{even}}^{(\alpha,\beta)}$ enjoy the same recurrence relation with the coefficients of polynomials of even degree $\xi_{m,\text{even}}^{(\alpha,\beta)}$, which suggests the odd family can be chosen as

$$Q_{2m+1}^{(\alpha,\beta)}(x; q) = p_{2m+1}^{(\alpha,\beta)}(x; q) + \gamma Q_{2m}^{(\alpha,\beta)}(x; q)$$

with arbitrary constant γ . For simplicity, we take $\gamma = 0$.

Therefore, the skew orthogonal little q -Jacobi polynomials with weight function $\omega^{(\alpha+1,\beta+1)}(x; q)$ in (4.8) have the form

$$Q_{2m+1}^{(\alpha,\beta)}(x; q) = p_{2m+1}^{(\alpha,\beta)}(x; q),$$

$$Q_{2m}^{(\alpha,\beta)}(x; q) = \sum_{j=0}^m \prod_{k=1}^{m-j} \frac{[2m-2k+2]_q h_{2m-2k+1}^{(\alpha+1,\beta+1)} + q^{\beta+1}[2m-2k+1]_q h_{2m-2k+2}^{(\alpha,\beta)}}{[2m-2k+1]_q h_{2m-2k}^{(\alpha+1,\beta+1)} + q^{\beta+1}[2m-2k]_q h_{2m-2k+1}^{(\alpha,\beta)}} p_{2j}^{(\alpha,\beta)}(x; q).$$

These two classical q -skew orthogonal polynomials reflect a structure similar to (3.17), and so suggest a general theory relating classical q -skew orthogonal polynomials to classical q -orthogonal polynomials.

4.4. Relationship between q -skew orthogonal polynomials and q -orthogonal polynomials: General theorem. The relationship of q -skew orthogonal polynomials and q -orthogonal polynomials is heavily dependent on the Pearson-type equation (2.8). Recall that in the classical case, the functions $f(x)$ and $g(x)$ in the Pearson-type equation (2.8) are polynomials in x of order 2 and 1 at most, respectively.

Define the operator

$$\mathcal{A}_q = q^{-\frac{1}{2}}g(x)T_q + q^{-1}f(x)D_{q^{-1}} + f(x)D_q, \quad (4.25)$$

where T_q is the q -shift operator defined by $T_q\phi(x) = \phi(qx)$, $D_q\phi(x) = \frac{\phi(x)-\phi(qx)}{(1-q)x}$ and $D_{q^{-1}}\phi(x) = \frac{\phi(x)-\phi(q^{-1}x)}{(1-q^{-1})x}$. The following analogue of Proposition 3.6 holds.

Proposition 4.5. *Denote the inner product $\langle \cdot, \cdot \rangle$ as a symmetric bilinear form from $\mathbb{R}_q[x] \times \mathbb{R}_q[x] \mapsto \mathbb{R}(q)$, which is defined by*

$$\langle \phi(x; q), \psi(x; q) \rangle = \int_I \phi(x; q)\psi(x; q)\rho(x; q)d_qx.$$

One has

$$(1) \langle \mathcal{A}_q\phi(x; q), \psi(x; q) \rangle = -\langle \phi(x; q), \mathcal{A}_q\psi(x; q) \rangle; \quad (4.26)$$

$$(2) \langle \phi(x; q), \mathcal{A}_q\psi(x; q) \rangle = \langle \phi(x; q), \psi(x; q) \rangle_{s,f(qx;q)\rho(qx;q)}, \quad (4.27)$$

provided $f(x; q)\rho(x; q)$ vanishes at the end points of supports.

Proof. Here we just prove the case $I = [0, \infty)$; the other cases can be verified similarly. To do so, take the summation of these two inner products and find

$$\begin{aligned} & \langle \mathcal{A}_q\phi(x), \psi(x) \rangle + \langle \phi(x), \mathcal{A}_q\psi(x) \rangle \\ &= q^{-1} \int_0^\infty f(x)\rho(x)(D_{q^{-1}}\phi(x)\psi(x) + \phi(x)D_{q^{-1}}\psi(x))d_qx + \int_0^\infty f(x)\rho(x)(D_q\phi(x)\psi(x) \\ &+ \phi(x)D_q\psi(x))d_qx + q^{-\frac{1}{2}} \int_0^\infty g(x)\rho(x)(\phi(qx)\psi(x) + \phi(x)\psi(qx))d_qx := A_1 + A_2 + A_3. \end{aligned}$$

Then from the definition of q -Jackson integral (4.13), we know

$$A_3 = (1-q)q^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} g(q^n)\rho(q^n)q^n (\phi(q^{n+1})\psi(q^n) + \phi(q^n)\psi(q^{n+1}))$$

and by noting the equation (2.8) at the points $x = q^n$, this term can then be written as

$$A_3 = \sum_{n=-\infty}^{\infty} (\phi(q^{n+1})\psi(q^n) + \phi(q^n)\psi(q^{n+1})) (\rho(q^n)f(q^n) - \rho(q^{n+1})f(q^{n+1})).$$

On the other hand, one can compute

$$\begin{aligned} A_1 &= -2 \sum_{n=-\infty}^{\infty} \phi(q^n)\psi(q^n)f(q^n)\rho(q^n) + \sum_{n=-\infty}^{\infty} f(q^n)\rho(q^n) (\phi(q^{n-1})\psi(q^n) + \psi(q^{n-1})\phi(q^n)), \\ A_2 &= 2 \sum_{n=-\infty}^{\infty} \phi(q^n)\psi(q^n)f(q^n)\rho(q^n) - \sum_{n=-\infty}^{\infty} f(q^n)\rho(q^n)(\phi(q^{n+1})\psi(q^n) + \phi(q^n)\psi(q^{n+1})). \end{aligned}$$

If we combine these three equalities and assume $f(x; q)\rho(x; q)$ vanishes at the boundary points, then equation (4.26) is proved.

For equation (4.27), it should be noted that

$$\begin{aligned} \langle \phi(x), \mathcal{A}_q \psi(x) \rangle &= q^{-\frac{1}{2}} \int_I g(x)\rho(x)\psi(qx)\phi(x)d_qx \\ &\quad + q^{-1} \int_I f(x)\rho(x)D_{q^{-1}}\psi(x)\phi(x)d_qx + \int_I f(x)\rho(x)\phi(x)D_q\psi(x)d_qx \\ &= - \int_I \frac{\psi(qx)\phi(x)f(qx)\rho(qx)}{(1-q)x}d_qx + \int_I \frac{\psi(q^{-1}x)\phi(x)f(x)\rho(x)}{(1-q)x}d_qx. \end{aligned}$$

Moreover, if $f(x; q)\rho(x; q)$ vanishes at the end points of supports, then

$$\int_I \frac{\psi(q^{-1}x)\phi(x)f(x)\rho(x)}{(1-q)x}d_qx = \int_I \frac{\psi(x)\phi(qx)f(qx)\rho(qx)}{(1-q)x}d_qx,$$

and therefore

$$\langle \phi(x), \mathcal{A}_q \psi(x) \rangle = \int_I (\phi(x)D_q\psi(x) - D_q\phi(x)\psi(x)) f(qx)\rho(qx)d_qx,$$

as required. \square

With the help of (4.26) and orthogonal relation (2.7), one can see (cf. (1.11) and (3.13))

$$\mathcal{A}_q p_n(x; q) = -\frac{c_n(q)}{\tilde{h}_{n+1}} p_{n+1}(x; q) + \frac{c_{n-1}(q)}{\tilde{h}_{n-1}} p_{n-1}(x; q), \quad (4.28)$$

where $c_n(q)$ is a constant dependent on Pearson pair $(f(x; q), g(x; q))$ and the order of polynomials n only, and \tilde{h}_n is the normalisation constant defined in (2.7) before. Similar to the derivation in (3.15)-(3.17), we finally get the sought relationship between q -orthogonal polynomials and q -skew orthogonal polynomials (cf. Proposition 3.9).

Proposition 4.6. *Let $\rho(x; q)$ be a classical weight function in the sense of the Pearson-type equation (2.8) with appropriate restriction on the degree of f and g . Let the weight in (4.4) be related*

to $\rho(x)$ by $\omega(x; q) = f(qx; q)\rho(qx; q)$. We have

$$p_{2n+1}(x; q) = Q_{2n+1}(x; q), \quad p_{2n}(x; q) = Q_{2n}(x; q) - \frac{c_{2n-1}(q)}{c_{2n-2}(q)}Q_{2n-2}(x; q), \quad (4.29a)$$

$$Q_{2n+1}(x; q) = p_{2n+1}(x; q), \quad Q_{2n}(x; q) = \left(\prod_{j=0}^{n-1} \frac{c_{2j+1}(q)}{c_{2j}(q)} \right) \sum_{l=0}^n \prod_{j=0}^{l-1} \frac{c_{2j}(q)}{c_{2j+1}(q)} p_{2l}(x; q) \quad (4.29b)$$

with normalisation $u_n(q) = c_{2n}(q)$.

4.4.1. *Several Examples.* In this part, we show the with explicit Pearson pairs (f, g) and coefficients c_n of Al-Salam & Carlitz polynomials and Little q -Jacobi polynomials. The idea is to obtain the Pearson pairs from the weight functions with the Pearson-type equation (2.8), and from the equation (4.28), the latter quantities c_n can then be given.

- (1) Al-Salam & Carlitz polynomials. For the weight function (4.12), one can obtain the Pearson pair

$$(f, g) = \left(x^2 - (1 + \alpha)x + \alpha, q^{\frac{1}{2}} \frac{x - (1 + \alpha)}{1 - q} \right)$$

from (2.8). The coefficients c_n can then be obtained as

$$c_n = -\frac{q^{-n}}{1 - q} h_{n+1}.$$

By taking the explicit normalisation constant h_n in (4.14), one can obtain

$$c_n = -(-\alpha)^{n+1} q^{\binom{n}{2}} (q; q)_{n+1}.$$

The weight $\tilde{\omega}(x; q)$ of skew orthogonal Al-Salam & Carlitz polynomials is determined by

$$\tilde{\omega}(x; q) = f(qx; q)\omega(qx; q) = \alpha\omega(x; q),$$

where $\omega(x; q)$ is the weight of Al-Salam & Carlitz polynomials defined by (4.12).

- (2) Little q -Jacobi polynomials. For the weight function (4.20), the Pearson pair is

$$(f, g) = \left(-x^2 + x, -q^{\frac{1}{2}}([\alpha + \beta + 2]_q x - [\alpha + 1]_q) \right),$$

which degenerates to the Pearson pair of Jacobi polynomials as $q \rightarrow 1^-$. With the help of the leading term in (4.28), one can obtain

$$c_n = (q^n [\alpha + \beta + 2]_q + [n]_q - [-n]_q) h_{n+1}^{(\alpha, \beta)} = q^{-n} [2n + \alpha + \beta + 2]_q h_{n+1}^{(\alpha, \beta)},$$

where $h_n^{(\alpha, \beta)}$ is the normalisation constant of orthogonal relation (4.21). We remark that in this case, the weight of skew orthogonal little q -Jacobi polynomials is taken as

$$\tilde{\omega}(x; q) := f(qx; q)\omega(qx; q) = -(qx)^{\alpha+1} (qx; q)_{\beta+1} = -q^{\alpha+1} \omega^{(\alpha+1, \beta+1)}(x; q)$$

with $\omega^{(\alpha, \beta)}(x; q)$ being the weight of little q -Jacobi polynomials defined in (4.20). As we consider the weight $\omega^{(\alpha+1, \beta+1)}(x; q)$ in Section 4.3.2, there is only a scalar transformation between it and $\tilde{\omega}(x; q)$.

4.5. Kernels between q -unitary ensemble and q -symplectic ensemble. This part is devoted to the relationship between the kernels of q -unitary ensemble and q -symplectic ensemble. To this end, we firstly need to modify the inner product in (2.9) and delta function given in (2.10).

Consider the symmetric inner product (as above we assume the interval $I = [0, \infty)$; the other cases can be done similarly)

$$\langle \phi(x; q), \psi(x; q) \rangle = (1 - q) \sum_{i \in \mathbb{Z}} \phi(q^i) \psi(q^i) \rho(q^i) q^i = \int_0^\infty \phi(x; q) \psi(x; q) \rho(x; q) d_q x.$$

Under this symmetric inner product, we can define a family of q -orthogonal polynomials $\{p_n(x; q)\}_{n=0}^\infty$ such that $\langle p_n(x; q), p_m(x; q) \rangle = \tilde{h}_n(q) \delta_{n,m}$ with some proper normalisation constant \tilde{h}_n . In this case, we can define the delta function on $\mathcal{H} \rightarrow \mathcal{H}$ as

$$\delta(x, y; q) = \sum_{n=0}^\infty \frac{p_n(x; q) p_n(y; q)}{\tilde{h}_n(q)},$$

such that $\langle \delta(x, y; q), \xi(x; q) \rangle = \xi(y; q)$, and a finite dimensional projection operator

$$K_N(x, y; q) = \sum_{n=0}^{N-1} \frac{p_n(x; q) p_n(y; q)}{\tilde{h}_n(q)},$$

which is the Christoffel-Darboux kernel of the q -unitary ensemble.

Consider now the equation $\langle \delta(x, y; q), \psi(x; q) \rangle_{s, \rho(qx) f(qx)}$. According to the equation (4.27) and the property of the delta function, one can get

$$\langle \delta(x, y; q), \psi(x; q) \rangle_{s, \rho(qx) f(qx)} = \langle \delta(x, y; q), \mathcal{A}_q \psi(x; q) \rangle = \mathcal{A}_q \psi(y; q).$$

On the other hand, if we take $\psi(x; q)$ as the q -skew orthogonal polynomials $Q_{2m}(x; q)$, which is skew orthogonal with the weight $\rho(qx) f(qx)$, then

$$\langle \delta(x, y; q), Q_{2m}(x; q) \rangle_{s, \rho(qx) f(qx)} = \sum_{n=0}^\infty \frac{p_n(y; q)}{\tilde{h}_n(q)} \langle p_n(x; q), Q_{2m}(x; q) \rangle_{s, \rho(qx) f(qx)}.$$

By using the equality (4.29a) and the skew orthogonality (4.11), it follows from this that

$$\mathcal{A}_q Q_{2m}(y; q) = -\frac{u_m(q)}{\tilde{h}_{2m+1}(q)} p_{2m+1}(y; q) \quad (4.30)$$

(cf. (3.18d)). And if one takes $\psi(x; q) = Q_{2n+1}(x; q)$, by again using the equations (4.11) and (4.29a), we obtain

$$\mathcal{A}_q Q_{2m+1}(y; q) = u_m(q) \left(\frac{p_{2m}(y; q)}{\tilde{h}_{2m}(q)} - \frac{c_{2m+1}(q)}{c_{2m}(q)} \frac{p_{2m+2}(y; q)}{\tilde{h}_{2m+2}(q)} \right) \quad (4.31)$$

(cf. (3.18b)).

To make use of the above formulae, first rewrite the Christoffel-Darboux kernel for the q -unitary ensemble in the case $N \mapsto 2N - 1$ as

$$K_{2N-1}(x, y; q) = \sum_{n=0}^{N-1} \frac{1}{\tilde{h}_{2n}(q)} p_{2n}(x; q) p_{2n}(y; q) + \sum_{n=0}^{N-1} \frac{1}{\tilde{h}_{2n+1}(q)} p_{2n+1}(x; q) p_{2n+1}(y; q).$$

Making use of (4.29a), (4.30) and (4.31) shows

$$\begin{aligned} \sum_{n=0}^{N-1} \frac{1}{\tilde{h}_{2n+1}(q)} p_{2n+1}(x; q) p_{2n+1}(y; q) &= - \sum_{n=0}^{N-1} \frac{1}{u_n(q)} Q_{2m+1}(x; q) \mathcal{A}_q Q_{2m}(y; q), \\ \sum_{n=0}^{N-1} \frac{1}{\tilde{h}_{2n}(q)} p_{2n}(x; q) p_{2n}(y; q) &= \sum_{n=0}^{N-1} \frac{1}{u_n(q)} Q_{2m}(x; q) \mathcal{A}_q Q_{2m+1}(y; q) \\ &\quad + \frac{c_{2N-1}(q)}{\tilde{h}_{2N}(q) c_{2N-2}(q)} p_{2N}(y; q) Q_{2N-2}(x; q), \end{aligned}$$

and hence

$$K_{2N-1}(x, y; q) = \mathcal{A}_{q,y} R_N(x, y; q) + \frac{c_{2N-1}(q)}{\tilde{h}_{2N}(q) c_{2N-2}(q)} p_{2N}(y; q) Q_{2N-2}(x; q). \quad (4.32)$$

Inspired by the continuous case [1] and discrete case [9], we expect to find an inverse operator of \mathcal{A}_q , allowing the kernel of q -symplectic ensemble be written as the kernel of the q -unitary ensemble plus a rank 1 decomposition, dependent on the q -orthogonal polynomials only.

For this purpose we note the operator \mathcal{A}_q , as defined in (4.25), can be also written as

$$\mathcal{A}_q = \frac{\rho^{-1}(x; q)}{(q-1)x} [\rho(qx; q) f(qx; q) T_q - f(x; q) \rho(x; q) T_{q^{-1}}] := \frac{\rho^{-1}(x; q)}{(q-1)x} \mathcal{R}_q$$

with the shift operator $T_q \phi(x) = \phi(qx)$ and $T_{q^{-1}} \phi(x) = \phi(q^{-1}x)$. We see by inspection that the operator \mathcal{R}_q has an inverse operator ϵ_q specified by

$$\begin{aligned} (\epsilon_q \cdot \phi)(aq^{2m}) &= - \sum_{k=m}^{+\infty} \frac{\omega(aq^{2m+2}; q) \cdots \omega(aq^{2k}; q)}{\omega(aq^{2m+1}; q) \cdots \omega(aq^{2k+1}; q)} \phi(aq^{2k+1}), \\ (\epsilon_q \cdot \phi)(aq^{2m+1}) &= \sum_{k=-\infty}^m \frac{\omega(aq^{2k+2}; q) \cdots \omega(aq^{2m}; q)}{\omega(aq^{2k+1}; q) \cdots \omega(aq^{2m+1}; q)} \phi(aq^{2k}), \end{aligned} \quad (4.33)$$

where $\omega(x; q) = f(x; q) \rho(x; q)$ and $a \in \mathbb{C}^\times$. Hence we can write the inverse of operator \mathcal{A}_q as $\mathcal{D}_q := \epsilon_q(q-1)x\rho(x)$, allowing us to solve (4.32) for $R_N(x, y; q)$ (cf. Proposition 3.12).

Proposition 4.7. *The Christoffel-Darboux kernel of the q -symplectic ensemble can be written in terms of the Christoffel-Darboux kernel of the q -unitary ensemble according to*

$$R_N(x, y; q) = \mathcal{D}_{q,y} K_{2N-1}(x, y; q) + \frac{c_{2N-1}(q) u_{N-1}(q)}{c_{2N-2}(q) \tilde{h}_{2N}(q) \tilde{h}_{2N-1}(q)} (\mathcal{D}_q \cdot p_{2N})(y; q) (\mathcal{D}_q \cdot p_{2N-1})(x; q).$$

This formula suggests one can analyse the asymptotic results for the q -symplectic ensemble via the q -unitary case. In particular, the kernel of skew orthogonal Al-Salam & Carlitz polynomials can then be formulated as

$$R_N^{(AC)}(x, y) = \mathcal{D}_{q,y} K_{2N-1}^{(AC)}(x, y) - \frac{q^{1-2N}}{(1-q) \tilde{h}_{2N-1}^{(AC)}} (\mathcal{D}_q \cdot p_{2N})(y; q) (\mathcal{D}_q \cdot p_{2N-1})(x; q),$$

and the one of skew orthogonal little q -Jacobi polynomials can be written as

$$R_N(x, y) = \mathcal{D}_{q,y} K_{2N-1}(x, y) + \frac{q^{1-2N} [2N + \alpha + \beta + 1]_q}{\tilde{h}_{2N-1}^{(\alpha, \beta)}} (\mathcal{D}_q \cdot p_{2N})(y; q) (\mathcal{D}_q \cdot p_{2N-1})(x; q).$$

5. CONCLUDING REMARKS

Consideration of discretisations of the eigenvalue PDF for Hermitian ensembles with symplectic symmetry lead to the skew symmetric inner products (3.3) (linear lattice) and (4.8) (exponential lattice). We have presented here a theory of the corresponding discrete, and q , skew orthogonal polynomials in the cases that the weight function in the corresponding discretised unitary ensemble PDFs are classical. The theory is based on identifying the appropriate analogues of the differential operator (1.10), and developing their properties, and direct analogues of the formulas (1.13) known in the continuous case are found.

In the classical, continuous cases it is also known [1] that we have

$$\left(f(x)w(x)\right)^{1/2} Q_{2j}(x) = -\frac{q_m}{h_{2m+1}} \int_x^\infty \left(\frac{w(t)}{f(t)}\right)^{1/2} p_{2j+1}(t) dt. \quad (5.1)$$

An analogue of this formula does not appear possible in the discrete setting. It is (5.1) which leads to the kernel summation formula (1.14) known for the continuous case [1]. In the discrete cases, there is no known analogue of (1.14). Rather we have the less explicit kernel summations of Proposition 3.12 (first derived in [9]) and Proposition 4.5.

The question of deducing the analogue of the kernel summation in Proposition 3.12 on a linear lattice discretisation of (1.5) with $\beta = 1$ (orthogonal symmetry case) has been solved in [9]. The question of developing a theory of the corresponding skew orthogonal polynomials remains to be addressed (one application of knowledge of the corresponding normalisations would be to give an explanation of the (some of) the multiple summations found in [10]; another would be to explain the structure of the particular discrete orthogonal symmetry skew orthogonal polynomials found in [18] and that of the summation formula for the corresponding Christoffel-Darboux kernel), as does the task of developing a theory for the q analogue of the orthogonal symmetry case.

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