Control Oriented Modeling of Turbocharged (TC) Spark Ignition (SI) Engine

***Abstract*— This paper proposes a new systematic procedure for model reduction of TC SI engines. The technique is based on the identification of time scale separation within the dynamics of various engine state variables. By classifying the complete engine dynamics into three time scales, fast, medium and slow, it is demonstrated by a two stage model reduction procedure that the fast and slow variables can be analytically expressed in terms of the medium time scale states. As a result a library of the control oriented engine models is obtained. The limitation of the reduced order models as a true representation of original model is first gauged qualitatively by the application of perturbation theory and then characterized quantitatively by means of simulations. Various assumptions under which these model reductions are applicable are presented and their validity in the context of the engine are discussed.**

I. INTRODUCTION

With the automotive industry growing and its market maturing, emission and fuel economy standards are becoming increasingly stringent. While minimizing emissions and improving fuel economy are legislative requirements, it is also of a paramount importance from the customers' point of view that there is a minimal (or no) sacrifice in terms of the drivability or performance. Developing such an engine control system which delivers these conflicting requirements in a most efficient way has been a major challenge during the last three to four decades.

The development of engine controllers requires the knowledge of the system in the form of its mathematical model referred to as *control oriented model*. Among the existing approaches for control oriented engine modeling, a major trend has been to use the measured engine data to establish functional relationships between inputs and outputs to obtain steady state engine maps. Such approaches, though widely used due to their simplicity and flexible implementation, yield models that are engine specific, leading to a lack of generalization capability in the model itself, and any engine components (such as engine controllers) that may be based upon it. Moreover, heavy dependence on the look-up tables is often undesirable as the accuracy is highly dependent on the amount of data available, extrapolation is unreliable, may induce unnecessary numerical noise (discontinuities) and leads to lengthy and costly calibration [8]. This is a major limitation of the empirical, quasi-static approach.

An alternative is to formulate a physics-based mathematical model of the system. This model should be suitable for controller design and at the same time ought to be appropriate for analyzing the system behavior via simulations, perturbation studies and sensitivity analysis. An important class of such engine models is the mean value engine models (MVEM) [3],[8]. MVEMs describe the average engine behavior over several engine event cycles. For *naturally aspirated* engines, such models have been well researched and are being successfully utilized in several aspects of engine operation including its control, torque management and supervision [2].

Recently there has been a push towards reducing fuel consumption and consequently CO_2 emissions by engine downsizing. This, however, comes at the expense of the maximum torque that the engine can generate and hence affects drivability. Torque reduction due to downsizing can be offset through the introduction of a turbocharger which can increase charge density and therefore power output while maintaining the lower pumping losses associated with reduced capacity engine [14].

Due to the proven efficacy of the MVEM of naturally aspirated engines in control related implementations, efforts have been directed in obtaining similar models for TC SI engines as well. Initial endeavours in this direction include [9] and [10] whereby the problem of control oriented modeling of the turbocharger is considered. In particular, [10] compares and discusses three key turbine and compressor MVEM models as described in [11], [12] and [13] . These works are followed by [1] where a fully validated comprehensive 13^{th} order TC SI engine model is developed. The modeling strategy is to first consider the physics of engine components (like air filter, compressor, intercooler, throttle, engine, turbine and exhaust system) which yields model structures as they behave in the engine setting. The resulting componentwise engine model is used by the same authors for further investigation with regard to cylinder air charge estimation and engine control and optimization [4], [5].

NOMENCLATURE	
$P_{cv}(Pa)$	Control volume Pressure
$T_{cv}(K)$	Control volume Temperature
$\dot{m}_{cv_{in}}(Kg/s)$	Mass-flow into the control volume
$\dot{m}_{cv_{out}}(Kg/s)$	Mass-flow out of the control volume
$T_{cv_{in}}(K)$	Control volume inlet temperature
γ	Ratio of the specific heats $\frac{C_P}{C_V}$
$V_{cv}(m^3)$	Volume of the control volume
$\omega_{tc}(rad/s)$	Turbocharger speed

SUBSCRIPTS	
cv	Control volume
af	Air filer
c	Compressor
ic	Intercooler
im	Intake manifold
em	Exhaust manifold
tb	Turbine

Nevertheless, none of the above referred papers utilize the dynamic and physical attributes of the engine to develop a systematic procedure for model reduction. Even though the reduced order engine models can be highly nonlinear, typically their lower dimension makes them more amenable to controller design techniques, provided we have the guarantee that any controller that stabilizes the reduced model will also stabilize a higher order model. Moreover, a rigorous procedure to build such reduced order models will improve portability of the engine models, leading to reduced engine calibration times as prior work in controller development on one turbocharged system can be utilized on multiple systems.

In this contribution we propose a new approach to develop a set of new reduced order but versatile mean value models for turbocharged SI engines. Approximation techniques, based on nonlinear systems theory, are employed which, on the one hand, can be theoretically justified and, on the other hand, the errors introduced by these approximation are characterized by performing simulations. Different sets of assumptions under which these model reductions are justified are presented and discussed. Interestingly, it becomes apparent that the procedure provides a great deal of modeling flexibility while the approximate solution of certain engine state variables are extracted in terms of rest of the variables. The eventual outcome is that a library of engine models is obtained which evince similar characteristics under a wide range of operating conditions provided certain sets of assumptions hold.

In section II below, the structure of 13^{th} order engine, as presented in [6], is briefly reviewed and summarized in the form of stage 1 model (Σ_1). Section III contains the main contribution of this paper and deals with the development of a procedure for model simplification by order reduction. This section is partitioned into subsections III-A and III-B which discuss two stages of model reduction and present the end results in the form of stage 2 (Σ_2) and stage 3 (Σ_3) models, respectively.

II. Stage 1 Engine Model

In this section, we briefly present the structure of high order componentwise engine model introduced in [1] and [6]. This model incorporates the physics due to individual constituent components and their interactions through pipes and/or manifolds (referred to as *control volumes*), builds on previous work in the engine modeling and has been extensively validated. Therefore, we believe this model represents state-of-the-art progress in control-oriented modeling of TC SI engines and we adopt it as our *starting point* for engine model reduction. It is worth mentioning that this higher order engine model is also strongly nonlinear which makes the task of controller design using this model much more cumbersome. In rest of this section, we summarize its structure in the form of stage 1 model denoted by Σ.

A. Control Volumes

The modeling approach utilized is to place *components* (like the air filter, compressor, intercooler, engine, turbine and turbo-shaft between the control volumes. The pressure (P_{cv}) and temperature T_{cv} within the control volumes are determined by mass flow into ($\dot{m}_{cv_{in}}$) and out ($\dot{m}_{cv_{out}}$) of the volume. On the other hand, mass flows and temperatures of the flows at the inlet of control volumes are determined by the components on the basis of the pressure and temperature in the control volumes before and after them. In other words, the behavior of the gas within the control volumes is dictated by filling and emptying dynamics of temperatures and pressures.

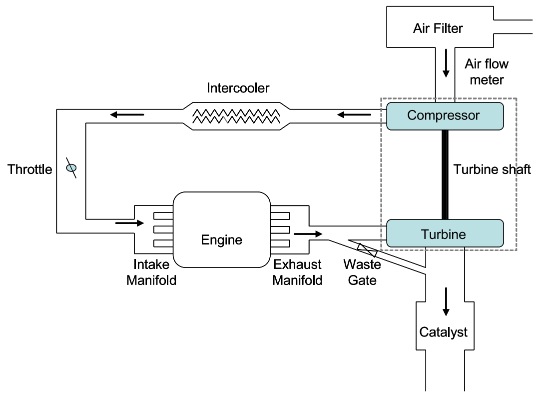

Fig. 1. Schematic of TC SI Engine

In a TC SI engine, there are six control volumes that are modeled the same way. Figure 1 shows the block diagram representation of the TC SI engine as a cascade connection of various components that are connected via pipes (control volumes). The equations describing the dynamics of pressure and temperature within each of the control volumes are derived, using the laws of conservation of mass and energy, as follows [3]:

$$\begin{aligned}\frac{dP_{cv}}{dt} &= \left(\frac{\gamma R}{V_{cv}}\right)\left[\dot{m}_{cv_{in}}\left(P,T,\omega_{tc}\right)T_{cv_{in}}\right.\\ &\qquad \left. - \dot{m}_{cv_{out}}\left(P,T,\omega_{tc}\right)T_{cv}\right] \qquad (1)\\ \frac{dT_{cv}}{dt} &= \left(\frac{T_{cv}}{P_{cv}}\right)\left(\frac{\gamma R}{V_{cv}}\right)\left[\dot{m}_{cv_{in}}\left(P,T,\omega_{tc}\right)T_{cv_{in}}\right.\\ &\quad -\dot{m}_{cv_{out}}\left(P,T,\omega_{tc}\right)T_{cv} - \frac{T_{cv}}{\gamma}\left(\dot{m}_{cv_{in}}\left(P,T,\omega_{tc}\right)\right.\\ &\qquad \left.\left. - \dot{m}_{cv_{out}}\left(P,T,\omega_{tc}\right)\right)\right] \qquad (2)\end{aligned}$$

where, vector $P = \left[P_{af}, P_c, P_{ic}, P_{im}, P_{em}, P_{tb}\right]^T$; vector $T = \left[T_{af}, T_c, T_{ic}, Tim, T_{em}, T_{tb}\right]^T$ and ω_{tc} denotes the speed of the turbocharger. γ denotes the ratio of the specific heats $\left(\frac{C_P}{C_V}\right)$.

B. Stage 1 model

This subsection presents the high fidelity 13^{th} order TC SI engine model as developed in [1], [6]. The following assumptions apply:

Assumption 1: No substantial heat or mass transfer through the control volume walls takes place.

Assumption 2: The temperature and pressure in each control volume are uniform.

Assumption 3: The fluid is a perfect gas.

In order to fully model every aspect of engine design and operation, an infinite dimensional mathematical model is needed which in practice is unrealizable. However, for the purpose of control oriented modeling, the need for an infinite dimensional model can be sufficiently alleviated by the Assumptions 1 to 3.

Accordingly, if Assumptions 1 to 3 hold, then the equations governing the dynamics of pressure and temperature within all the six control volumes can be depicted on the basis of equations (1)-(2). With turboshaft dynamics accounted for via the speed of the turbocharger, ω_{tc}, a 13^{th} order engine model, Σ, of the following form is obtained [1], [6]:

$$\Sigma : \begin{cases} \dot{P}_a f = f_{af}\left(P_{af}, T_{af}, P_c, \omega_{tc}\right) \\ \dot{T}_a f = \left(\frac{T_{af}}{P_{af}}\right) g_{af}\left(P_{af}, T_{af}, P_c, \omega_{tc}\right) \\ \dot{P}_c = f_c\left(P_{af}, T_{af}, P_c, T_c, P_{ic}, \omega_{tc}\right) \\ \dot{T}_c = \left(\frac{T_c}{P_c}\right) g_c\left(P_{af}, T_{af}, P_c, T_c, P_{ic}, \omega_{tc}\right) \\ \dot{P}_{ic} = f_{ic}\left(P_c, T_c, P_{ic}, T_{ic}, P_{im}\right) \\ \dot{T}_{ic} = \left(\frac{T_{ic}}{P_{ic}}\right) g_{ic}\left(P_c, T_c, P_{ic}, T_{ic}, P_{im}\right) \\ \dot{P}_{im} = f_{im}\left(P_{ic}, T_{ic}, P_{im}, T_{im}, P_{em}\right) \\ \dot{T}_{im} = \left(\frac{T_{im}}{P_{im}}\right) g_{im}\left(P_{ic}, T_{ic}, P_{im}, T_{im}, P_{em}\right) \\ \dot{P}_{em} = f_{em}\left(P_{im}, T_{im}, P_{em}, T_{em}, P_{tb}\right) \\ \dot{T}_{im} = \left(\frac{T_{em}}{P_{em}}\right) g_{em}\left(P_{im}, T_{im}, P_{em}, T_{em}, P_{tb}\right) \\ \dot{P}_{tb} = f_{tb}\left(P_{em}, T_{em}, P_{tb}, T_{tb}, \omega_{tc}\right) \\ \dot{T}_{tb} = \left(\frac{T_{tb}}{P_{tb}}\right) g_{tb}\left(P_{em}, T_{em}, P_{tb}, T_{tb}, \omega_{tc}\right) \\ \dot{\omega}_{tc} = f_{\omega_{tc}}\left(P_{af}, T_{af}, P_c, P_{em}, T_{em}, P_{tb}, \omega_{tc}\right) \end{cases} \tag{3}$$

Note the dynamics of ω_{tc} are governed by the driving torque due to the turbine $T_{q_{tb}}$ and loading torque from the compressor T_{q_c}. The full expressions of mass-flows and inlet temperatures in terms of engine states P_{cv}, T_{cv} and w_{tc} can be found in [1] and [6].

***Remark** 1:* This 13^{th} order engine model, Σ, does not include some important phenomena like engine oil temperature, engine warm up dependence and wall wetting, that are quite significant for describing the complete engine operation. However, our main aim in this paper is to demonstrate a systematic and rigorous procedure for model reduction. Once a reduced order model has been obtained any additional features can be easily incorporated to account for such phenomena.

III. Model Reduction

In this section we investigate the simplification of the engine model Σ via order reduction. The high order model Σ has been shown to be very effective in examining the behavior of various engine components and carrying out simulation studies. Nevertheless, it is too complex for usage in tasks involving optimization and control using model based techniques. Thus, in order to design improved controllers with the objective of achieving best possible engine performance, model simplification via order reduction may be quite useful in simplifying the problem.

The order reduction is accomplished in two main stages and is based on the use of *perturbation theory* [7]. Figure 2 summarizes the sequence of model reduction steps. The first reduced order control oriented model (as elaborated in subsection III-A) is developed on the basis of dynamic characteristics of pressure and temperature of mass in the control volumes. In this direction, first regular perturbation theory is applied on Σ to obtain a preliminary 13^{th} order system denoted by Σ_1. Under certain assumptions, Σ_1 is identified to demonstrate two time scale separation. Specifically, the pressure dynamics are much fasters than those of the temperatures and hence by the application of singular perturbation theory the pressure dynamics can be parameterized by fixed values of temperatures to obtain approximate solutions. The corresponding reduced order system is denoted by Σ_2. The stage 2 model reduction, as demonstrated in subsection III-B, is based on relative sizes of the various engine control volumes. It is shown that, under some conditions on the engine geometry, the order of Σ_2 can be further reduced by eliminating the fast pressures and expressing them in terms of rest of the variables. The resulting reduced order model is represented by Σ_3.

A. Stage 2 Engine Model: Model reduction I

This model simplification is based on the observation that in all engine control volumes the magnitude of the derivative of pressure is significantly larger in comparison to the magnitude of derivative of temperature. This difference in magnitudes can be attributed to the $\left(\frac{T_{cv}}{P_{cv}}\right)$ term in (2). Typically, the ratio $\left(\frac{T_{cv}}{P_{cv}}\right)$ in the context of engine control volumes is a very small positive quantity. Since variations in the temperatures are negligibly small in comparison to the pressures, temperatures can be thought to belong to a compact set which contains the equilibrium point. In order to investigate this time scale separation let us rewrite equations (1)-(2), which govern the dynamics of temperature and pressure in all six control volumes, in the following form:

$$\frac{dP}{dt} = f\left(P, T, \omega_{tc}\right) \tag{4}$$

$$\frac{d}{dt}\begin{bmatrix} T_{af} \\ T_c \\ T_{ic} \\ T_{im} \\ T_{em} \\ T_{tb} \end{bmatrix} = \begin{bmatrix} \varepsilon_{af}(P_{af}, T_{af}) g_{af}\left(P, T, \omega_{tc}\right) \\ \varepsilon_c(P_c, T_c) g_c\left(P, T, \omega_{tc}\right) \\ \varepsilon_{ic}(P_{ic}, T_{ic}) g_{ic}\left(P, T, \omega_{tc}\right) \\ \varepsilon_{im}(P_{im}, T_{im}) g_{im}\left(P, T, \omega_{tc}\right) \\ \varepsilon_{em}(P_{em}, T_{em}) g_{em}\left(P, T, \omega_{tc}\right) \\ \varepsilon_{tb}(P_{tb}, T_{tb}) g_{tb}\left(P, T, \omega_{tc}\right) \end{bmatrix} \tag{5}$$

where,
$P := \left[P_{af}, P_c, P_{ic}, P_{im}, P_{em}, P_{tb}\right]^T$;
$T := \left[T_{af}, T_c, T_{ic}, T_{im}, T_{em}, T_{tb}\right]^T$;
$f := [f_{af}, f_c, f_{ic}, , f_{em}, f_{tb}]^T$;
$\forall\ cv \in \{af, c, ic, im, em, tb\}$; $\varepsilon_{cv}(P_{cv}, T_{cv}) = \left(\frac{T_{cv}}{P_{cv}}\right)$;

$$f_{cv}\left(P, T, \omega_{tc}\right) = \left(\frac{\gamma R}{V_{cv}}\right)\left[\dot{m}_{cv_{in}}\left(P, T, \omega_{tc}\right) T_{cv_{in}} - \dot{m}_{cv_{out}}\left(P, T, \omega_{tc}\right) T_{cv}\right]$$

and

$$g_{cv}\left(P, T, \omega_{tc}\right) = \left(\frac{\gamma R}{V_{cv}}\right)\Big[\dot{m}_{cv_{in}}\left(P, T, \omega_{tc}\right) T_{cv_{in}} - \dot{m}_{cv_{out}}\left(P, T, \omega_{tc}\right) T_{cv} - \frac{T_{cv}}{\gamma}\left(\dot{m}_{cv_{in}}\left(P, T, \omega_{tc}\right) - \dot{m}_{cv_{out}}\left(P, T, \omega_{tc}\right)\right)\Big].$$

Further, by defining

$$\varepsilon_T(P, T) = \underset{cv \in \{af, c, ic, im, em, tb\}}{min} \varepsilon_{cv}(P_{cv}, T_{cv}) \tag{6}$$

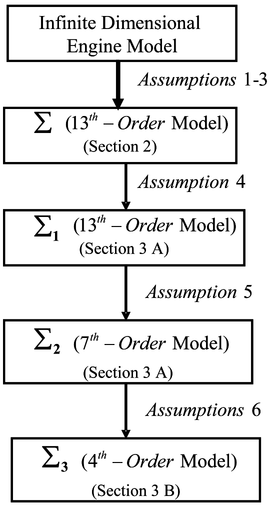

Fig. 2. Flowchart depicting the sequence of model reduction

and

$$G(P,T,\omega_{tc}) = \begin{bmatrix} \left(\frac{\varepsilon_{af}(P_{af},T_{af})}{\varepsilon_T(P,T)}\right) g_{af}(P,T,\omega_{tc}) \\ \left(\frac{\varepsilon_{c}(P_{c},T_{c})}{\varepsilon_T(P,T)}\right) g_{c}(P,T,\omega_{tc}) \\ \left(\frac{\varepsilon_{ic}(P_{ic},T_{ic})}{\varepsilon_T(P,T)}\right) g_{ic}(P,T,\omega_{tc}) \\ \left(\frac{\varepsilon_{im}(P_{im},T_{im})}{\varepsilon_T(P,T)}\right) g_{im}(P,T,\omega_{tc}) \\ \left(\frac{\varepsilon_{em}(P_{em},T_{em})}{\varepsilon_T(P,T)}\right) g_{em}(P,T,\omega_{tc}) \\ \left(\frac{\varepsilon_{tb}(P_{tb},T_{tb})}{\varepsilon_T(P,T)}\right) g_{tb}(P,T,\omega_{tc}) \end{bmatrix}, \quad (7)$$

equation (4)-(5) can be rewritten as

$$\frac{dP}{dt} = f(P,T,\omega_{tc}) \quad (8)$$

$$\frac{dT}{dt} = \varepsilon_T(P,T) G(P,T,\omega_{tc}) \quad (9)$$

If $\varepsilon_{T_{max}}$ and $\varepsilon_{T_{min}}$ denote the maximum and minimum values of $\varepsilon_T(P,T)$, then its average value, ε_{av}, in its *domain of operation* becomes

$$\varepsilon_{av} = \frac{\varepsilon_{T_{max}} + \varepsilon_{T_{min}}}{2} \quad (10)$$

From (10), we deduce that there exists $\Delta\varepsilon_T(P,T)$, which signifies the variation of ε_{av} from its average value, such that

$$\varepsilon_T(P,T) = \varepsilon_{av} + \Delta\varepsilon_T(P,T) \quad (11)$$

By substituting (11) in (9) we obtain

$$\frac{dT}{dt} = \varepsilon_{av} G(P,T,\omega_{tc}) + \Delta\varepsilon_T(P,T) G(P,T,\omega_{tc}) \quad (12)$$

Assumption 4: The magnitude of $\Delta\varepsilon_T(P,T)$ is sufficiently small in comparison to ε_{av}.

Assumption 5: The average value $\varepsilon_{av} << 1$.

***Remark** 2:* Satisfaction of Assumptions 4 and 5 ensures (as demonstrated in sequel) that the high dimensional 13^{th} order system Σ can be approximated by a reduced (7^{th}) order system. The response of the reduced order model approaches that of 13^{th} order model Σ as $\Delta\varepsilon_T(P,T)$ and ε_{av} tend to zero.

***Remark** 3:* It may be noted that the conditions on $\Delta\varepsilon_T(P,T)$ and ε_{av}, as per Assumptions 4 and 5 above, is a qualitative requirement. However, we investigate their smallness quantitatively by carrying out comprehensive simulation studies. The simulations first demonstrate that, in the engine setting, both $\Delta\varepsilon_T(P,T)$ and ε_{av} are very small relative to the magnitudes of pressures and temperatures. Then, the errors introduced by approximations based on the smallness of $\Delta\varepsilon_T(P,T)$ and ε_{av} are examined.

***Remark** 4:* Satisfaction of Assumption 4 with a sufficiently small $\Delta\varepsilon_T(P,T)$ allows for the effective use of *regular perturbation theory* to obtain a simplified engine model (which yields an approximate response) [7]. One way to analyze the smallness of $\Delta\varepsilon_T(P,T)$ is to examine the response of its nominal value under changing operating conditions.

Figure 3 shows the responses of $\frac{\Delta\varepsilon_T(P,T)}{\varepsilon_{av}}$ for three different sets of initial conditions and under widely changing throttle position. The simulation results depict the worst case scenario as the throttle position is varied from almost closed to wide open. It is clear from the simulation results that the magnitude of $\frac{\Delta\varepsilon_T(P,T)}{\varepsilon_{av}}$ is of the order of 10^{-1}. So, it can be deduced that magnitude of $\Delta\varepsilon_T(P,T)$ is much smaller than ε_{av} under widely varying operating conditions. A sample plot of ε_{av} and $\Delta\varepsilon_T(P,T)$ is given in Figure 4.

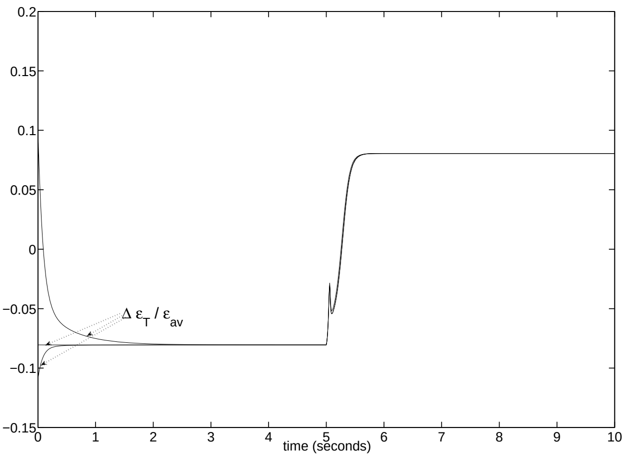

Fig. 3. $\frac{\Delta\varepsilon_T(P,T)}{\varepsilon_{av}}$ (for 3 different initial conditions)

***Remark** 5:* Figure 4 also reveals that the magnitude of ε_{av} is of the order of 10^{-3} and can be seen as "sufficiently" small. This permits the implementation of *singular perturbation theory* for model order reduction (demonstrated later in this section) [7].

***Remark** 6:* It may be noted that there is a sufficient flexibility in terms of the choice of $\varepsilon_T(P,T)$. An alternative to existing choice of $\varepsilon_T(P,T)$, as in (6), could be to use the average value of $\varepsilon_{cv}(P_{cv},T_{cv})$ over all the six control volumes and express each of $\varepsilon_{cv}(P_{cv},T_{cv})$ in terms of the average value term and a term depicting the deviation from the

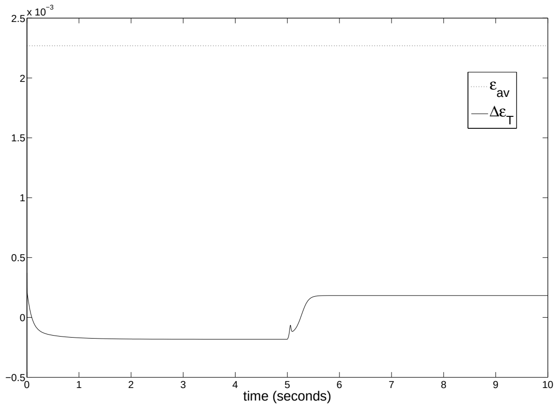

Fig. 4. $\Delta\varepsilon_T(P,T)$ and ε_{av} responses

average value. The deviation term (due to its smallness) can be eliminated by regular perturbation techniques. This choice of $\varepsilon_T(P,T)$ will eliminate the dependence on $\left(\frac{\varepsilon_{cv}(P_{cv},T_{cv})}{\varepsilon_T(P,T)}\right)$ terms in the equations governing the temperature dynamics.

In light of Remarks 4 and 5 above, it is reasonable to assume that the Assumptions 4 and 5 are justified. Our simulations of the error responses, introduced by the approximate engine model, will further confirm this fact.

Due to the smallness of $\Delta\varepsilon_T(P,T)$, solving equation (12) can be seen as a *regular perturbation* problem. The smallness of $\Delta\varepsilon_T(P,T)$ will now be exploited to establish an approximate solution to (12).

By setting $\Delta\varepsilon_T(P,T) = 0$, the following nominal or unperturbed system is obtained:

$$\frac{d\bar{T}}{dt} = \varepsilon_{av} G\left(P,\bar{T},\omega_{tc}\right) \tag{13}$$

where, $\bar{T}$ represents the temperature.

Closeness of solutions of the dynamic equation (12) and (13) can be ensured by the application of Theorem 3.4 of [7]. As a result, the equation (12) which governs dynamics of the temperatures can be approximated by equations (13). Accordingly, the original 13^{th} order system Σ can be approximated by the following 13^{th} order system

$$\Sigma_1 : \begin{cases} \frac{dP}{dt} = f\left(P,\bar{T},\omega_{tc}\right), & P(t_0) = P_0 \\ \frac{d\bar{T}}{dt} = \varepsilon_{av} G\left(P,\bar{T},\omega_{tc}\right), & \bar{T}(t_0) = \bar{T}_0 \\ \frac{d\omega_{tc}}{dt} = f_{\omega_{tc}}\left(P,\bar{T},\omega_{tc}\right), & \omega_{tc}(t_0) = \omega_{tc_0} \end{cases} \tag{14}$$

By introducing a change of time variable as $\varepsilon_{av} t = \tau$, Σ_1 can be rewritten as

$$\varepsilon_{av}\frac{d\tilde{P}}{d\tau} = F(\tilde{P},\bar{T}) \tag{15}$$

$$\frac{d\bar{T}}{d\tau} = G(\tilde{P},\bar{T}) \tag{16}$$

where, $\tilde{P} = \left[P^T \;\; \omega_{tc}^T\right]^T$ and $F = \left[f^T \;\; f_{\omega_{tc}}^T\right]^T$.

The system description (15)-(16) is in standard *singularly perturbed form* [7] and demonstrates the time scale separation between the dynamics of pressure and temperature. Specifically, due to the smallness of ε_{av} the dynamics of pressures and turbocharger speed are much faster than those of temperatures. Thus, the smallness of ε_{av} permits the application of singular perturbation theory to interpret the system Σ_1 in two separate time scale subsystems, namely, slow time scale subsystem and fast time scale subsystem. For that, we set $\varepsilon_{av} = 0$ to obtain the following reduced order *slow time scale subsystem*

$$0 = F(\tilde{P},\bar{T}) \tag{17}$$

$$\frac{d\bar{T}}{d\tau} = G(h(\bar{T}),\bar{T}) \tag{18}$$

where, $h = [h_{af}, h_c, h_{ic}, h_{im}, h_{em}, h_{tb}, h_{\omega_{tc}}]^T$ is the solution of (17) for $\tilde{P}$ in terms of $\bar{T}$ ($\tilde{P} = h(\bar{T})$). One way to analytically evaluate function $h(\bar{T})$ is to expand the right hand side of (17) in its Taylor series expansion and solve for $\tilde{P}$ by equating it to zero.

The closeness of solutions of systems (15)-(16) and (18) is ascertained by the application of Theorem 11.1 of [7] (commonly known as *Tikhnov's theorem*), whose natural consequence for the case at hand can be expressed as follows.

Corollary 1: Noting that Assumption 5 holds with small ε_{av} and $\tilde{P} = h(\bar{T})$ is an isolated stable root of reduced order system (17)-(18), therefore, by Tikhnov's Theorem solution of the degenerate system (18) tends to the solution of (15)-(16).

On the other hand, the *fast time scale subsystem* is obtained by parameterizing the pressure dynamics by fixed values of temperatures ($\bar{T}_0$) (in the original time variable (t)):

$$\frac{d\tilde{P}}{dt} = F\left(\tilde{P},\bar{T}_0\right) \tag{19}$$

The usual practice in singular perturbation theory is to approximate the fast dynamic with its quasi steady state value and reduce the order of the system by considering only slow dynamic. However, for the purpose of control oriented modeling, we pursue the alternative direction by focusing on the fast dynamic (pressure) and approximating the slow dynamic (temperature) by its quasi steady state value. This course of action is motivated by the following:

- The transient fluctuations in temperatures are much smaller than those in pressures. This makes approximation of temperatures a more viable choice. By excluding temperatures from the state vector and replacing them by fixed values ($\bar{T}_0$) allows the complete engine to be described by fewer state variables.
- Also most engine control algorithms are related to torque, applications which in turn require scheduling of pressures.

As a result, the following control oriented 7^{th} order model is yielded (Σ_2):

$$\Sigma_2 : \begin{cases} \frac{dP_{af}}{dt} = \left(\frac{\gamma R}{V_{af}}\right)\left[\dot{m}_{af_{in}}\left(P,\bar{T}_0,\omega_{tc}\right)T_{af_{in}}\right. \\ \qquad\qquad \left.-\dot{m}_{af_{out}}\left(P,\bar{T}_0,\omega_{tc}\right)\bar{T}_{af_0}\right] \\ \frac{dP_c}{dt} = \left(\frac{\gamma R}{V_c}\right)\left[\dot{m}_{c_{in}}\left(P,\bar{T}_0,\omega_{tc}\right)T_{c_{in}}\right. \\ \qquad\qquad \left.-\dot{m}_{c_{out}}\left(P,\bar{T}_0,\omega_{tc}\right)\bar{T}_{c_0}\right] \\ \frac{P_{ic}}{dt} = \left(\frac{\gamma R}{V_{ic}}\right)\left[\dot{m}_{ic_{in}}\left(P,\bar{T}_0,\omega_{tc}\right)T_{ic_{in}}\right. \\ \qquad\qquad \left.-\dot{m}_{ic_{out}}\left(P,\bar{T}_0,\omega_{tc}\right)\bar{T}_{ic_0}\right] \\ \frac{dP_{im}}{dt} = \left(\frac{\gamma R}{V_{im}}\right)\left[\dot{m}_{im_{in}}\left(P,\bar{T}_0,\omega_{tc}\right)T_{im_{in}}\right. \\ \qquad\qquad \left.-\dot{m}_{out}\left(P,\bar{T}_0,\omega_{tc}\right)\bar{T}_{im_0}\right] \\ \frac{dP_{em}}{dt} = \left(\frac{\gamma R}{V_{em}}\right)\left[\dot{m}_{em_{in}}\left(P,\bar{T}_0,\omega_{tc}\right)T_{em_{in}}\right. \\ \qquad\qquad \left.-\dot{m}_{em_{out}}\left(P,\bar{T}_0,\omega_{tc}\right)\bar{T}_{em_0}\right] \\ \frac{dP_{tb}}{dt} = \left(\frac{\gamma R}{V_{tb}}\right)\left[\dot{m}_{tb_{in}}\left(P,\bar{T}_0,\omega_{tc}\right)T_{tb_{in}}\right. \\ \qquad\qquad \left.-\dot{m}_{tb_{out}}\left(P,\bar{T}_0,\omega_{tc}\right)\bar{T}_{tb_0}\right] \\ \frac{d\omega_{tc}}{dt} = \left(\frac{1}{I_{tc}}\right)\left[T_{q_t}\left(P,\bar{T}_0,\omega_{tc}\right) - T_{q_c}\left(P,\bar{T}_0,\omega_{tc}\right)\right. \\ \qquad\qquad \left.-\omega_{tc}c_{fr}\right] \end{cases}$$

1) Comparison of Σ and Σ_2: In this subsection, we examine the errors in the pressures introduced by the stage 1 model reduction with respect to the original engine model Σ. Figures 5 and 6 show the deviations in the responses of pressures and temperatures when the temperature dynamics are excluded and replaced by their quasi-steady state values. In the simulation, we assume that at time $t = 5s$ a step change in the throttle takes place whereby the throttle status changes from almost closed to wide open. This is followed by instantaneous opening of wastegate at time $t = 10s$. It is clear from the error responses that the behavior of pressures generated by Σ and Σ_2 and are identical in steady state. On the other hand, as expected the values are different during the transient phases but the errors are small.

Due to the non-zero errors during the transient phases, Σ_2 cannot be seen as a *perfect* representation of Σ. However, since the errors introduced by Σ_2 relative to the original 13^{th} order validated model Σ are negligibly small, it can be treated as a good representation for many practical purposes including controller design and simulations.

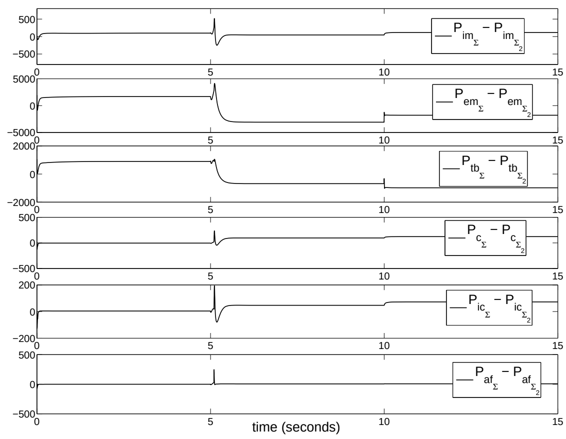

Fig. 5. Discrepancies in pressure responses due to Σ and Σ_2

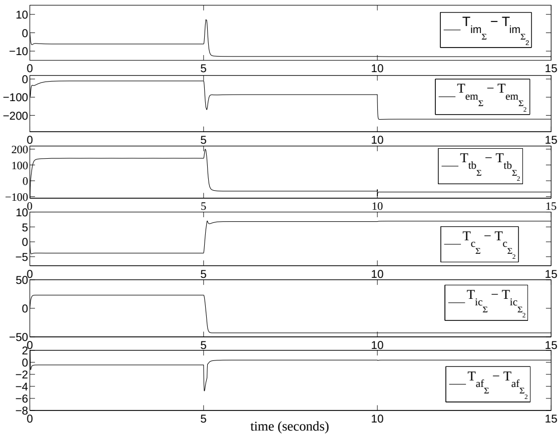

Fig. 6. Discrepancies in temperature responses due to Σ and Σ_2

B. Stage 3 Engine Model: Model Reduction II

In this section, we explore the possibility of further model reduction of engine model Σ_2 derived in the previous section. While reductions leading to Σ_2 are based on the dynamic characteristics of engine control volumes, now we consider their physical properties, specifically, the relative control volumes. The motivation for this follows from the fact that the magnitude of the pressure dynamics is inversely proportional to volume V_{cv}. Thus, smaller V_{cv} will lead to faster transients and hence control volumes can be classified to possess slow and fast pressure dynamics depending upon their volumes.

Assumption 6: The volumes of the intercooler control volume, intake manifold and exhaust manifold are sufficiently larger than those of the air filter, compressor and turbine control volumes and the turbine control volume is the smallest.

***Remark** 7:* Satisfaction of Assumption 6 permits the engine model Σ_2 to be approximated by a reduced (4^{th}) order model. The errors introduced due to this approximation will be small if the difference between the small and large volumes is high.

***Remark** 8:* Assumption 6 is justified as in the context of turbocharged engine, sizes of intake manifold, exhaust manifold and intercooler control volumes are typically much larger than those of turbine, compressor and air filter with turbine control volume being the smallest. In the situations where this is not strictly true, but an *alternative assumption* is justified, a similar procedure for model reduction can be adopted.

Therefore, the control volume pressures can be separated into two time scales. That is, P_{af}, P_c and P_{tb} are much faster in comparison with P_{ic}, P_{im} and P_{em}. In order to conceptualize this time scale separation, let us express the reduced order model Σ_2 in the following form:

$$\frac{d\tilde{P}_1}{dt} = f_1\Big(\tilde{P},\bar{T}_0\Big) \tag{20}$$

$$\varepsilon_{P_{tb}}\frac{dP_2}{dt} = f_2\Big(\tilde{P},\bar{T}_0\Big) \tag{21}$$

where, $\tilde{P} = \left[\tilde{P}_1^T,\ P_2^T\right]^T$; $\tilde{P}_1 = [P_{ic},\ P_{im},\ P_{em},\ \omega_{tc}]^T$; $P_2 = [P_{af},\ P_c,\ P_{tb}]^T$; $\varepsilon_{P_i} = \left(\frac{V_i}{V_{max}}\right)$ for each $i = [af, c, tb]$ and $V_{max} = max(V_{ic}, V_{im}, V_{em})$; $\alpha_{af} = \left(\frac{\varepsilon_{P_{af}}}{\varepsilon_{P_{tb}}}\right)$; $\alpha_c = \left(\frac{\varepsilon_{P_c}}{\varepsilon_{P_{tb}}}\right)$. The functions f_1 and f_2 are defined as follows:

$$f_1\left(\tilde{P}_1, P_2\right) = \begin{bmatrix} \frac{\gamma R}{V_{max}}\left(\dot{m}_{ic_{in}}\left(\tilde{P},\bar{T}_0\right)T_{ic_{in}} - \dot{m}_{ic_{out}}\left(\tilde{P},\bar{T}_0\right)\bar{T}_{ic_0}\right) \\ \frac{\gamma R}{V_{max}}\left(\dot{m}_{im_{in}}\left(\tilde{P},\bar{T}_0\right)T_{im_{in}} - \dot{m}_{im_{out}}\left(\tilde{P},\bar{T}_0\right)\bar{T}_{im_0}\right) \\ \frac{\gamma R}{V_{max}}\left(\dot{m}_{em_{in}}\left(\tilde{P},\bar{T}_0\right)T_{em_{in}} - \dot{m}_{em_{out}}\left(\tilde{P},\bar{T}_0\right)\bar{T}_{em_0}\right) \\ \left(\frac{1}{I_{tc}}\right)\left(T_{q_t}\left(\tilde{P},\bar{T}_0\right) - T_{q_c}\left(\tilde{P},\bar{T}_0\right) - \omega_{tc}c_{fr}\right) \end{bmatrix}$$

and

$$f_2\left(\tilde{P},\bar{T}_0\right) = \begin{bmatrix} \left(\frac{1}{\alpha_{af}}\right)\left(\frac{\gamma R}{V_{max}}\right)\left(\dot{m}_{af_{in}}\left(\tilde{P},\bar{T}_0\right)T_{af_{in}} - \dot{m}_{af_{out}}\left(\tilde{P},\bar{T}_0\right)\bar{T}_{af_0}\right) \\ \left(\frac{1}{\alpha_c}\right)\left(\frac{\gamma R}{V_{max}}\right)\left(\dot{m}_{c_{in}}\left(\tilde{P},\bar{T}_0\right)T_{c_{in}} - \dot{m}_{c_{out}}\left(\tilde{P},\bar{T}_0\right)\bar{T}_{c_0}\right) \\ \left(\frac{\gamma R}{V_{max}}\right)\left(\dot{m}_{tb_{in}}\left(\tilde{P},\bar{T}_0\right)T_{tb_{in}} - \dot{m}_{tb_{out}}\left(\tilde{P},\bar{T}_0\right)\bar{T}_{tb_0}\right) \end{bmatrix}.$$

As a consequence of Assumptions 6, we have $\varepsilon_{P_{tb}}$ to be sufficiently small. The smallness of $\varepsilon_{P_{tb}}$ allows the application of singular perturbation theory and, hence, dynamics of P_2 can be approximated by their quasi-steady value. By setting $\varepsilon_{P_{tb}} = 0$, from (20)-(21) following reduced order system is obtained

$$\frac{d\tilde{P}_1}{dt} = f_1\left(\tilde{P}_1, \psi_2(\tilde{P}_1), \bar{T}_0\right) \tag{22}$$

$$0 = f_2\left(\tilde{P}_1, P_2, \bar{T}_0\right) \tag{23}$$

where, $\psi_2(\tilde{P}_1) = [\psi_{af}, \psi_c, \psi_{tb}]^T$ is the solution of (23) for P_2 in terms of $\tilde{P}_1$, expressed in the following form:

$$P_{af} = \psi_{af}\left(\tilde{P}_1\right) \tag{24}$$

$$P_c = \psi_c\left(\tilde{P}_1\right) \tag{25}$$

$$P_{tb} = \psi_{tb}\left(\tilde{P}_1\right) \tag{26}$$

The functions $\psi_{af}\left(\tilde{P}_1\right)$, $\psi_c\left(\tilde{P}_1\right)$ and $\psi_{tb}\left(\tilde{P}_1\right)$ can be approximated by expanding $f_2(.)$ in its Taylor series and solving it for P_2 by equating it to zero. Then, as in case of the development of stage 2 reduced order model Σ_2, here again the closeness of the solutions of (20)-(21) and (22)-(23) is ascertained by the application of *Tikhnov's Theorem* [7], whose validity in this case is elaborated in the form of the following corollary.

Corollary 2: Provided Assumption 6 holds with small $\varepsilon_{P_{tb}}$ and $P_2 = \psi_2\left(\tilde{P}_1\right)$ is an isolated stable root of reduced order system (22)-(23), therefore, by Tikhnov's Theorem solution of the degenerate system (22) approaches that of (20)-(21).

***Remark** 9:* At this point, it can be highlighted that the procedure allows for a sufficient flexibility in the manner in which functions $\psi_{af}\left(\tilde{P}_1\right)$, $\psi_c\left(\tilde{P}_1\right)$ and $\psi_{tb}\left(\tilde{P}_1\right)$ can be obtained. Expanding in the Taylor series and solving, as demonstrated in this paper, is but one of the possible ways.

***Remark** 10:* While higher order terms in the Taylor series expansion are considered, the functions $\psi_{af}\left(\tilde{P}_1\right)$, $\psi_c\left(\tilde{P}_1\right)$ and $\psi_{tb}\left(\tilde{P}_1\right)$ may attain very long and complicated forms making them less favorable for the control oriented purposes. One way to overcome this problem is to approximate these expressions by simpler polynomials. MATLAB tools like MBC (*Model Based Calibration*) Toolbox or *polyfitn* are found to be quite useful in generating such simplified polynomials.

The reduced order stage 3 model thus obtained after the exclusion of fast pressures and replacing them with their quasi-steady state values $\psi_2(\tilde{P}_1)$

$$\Sigma_3 : \begin{cases} \dot{P}_{ic} = \left(\frac{\gamma R}{V_{ic}}\right)\left[\dot{m}_{ic_{in}}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right)T_{ic_{in}} - \dot{m}_{ic_{out}}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right)T_{ic_0}\right] \\ \dot{P}_{im} = \left(\frac{\gamma R}{V_{im}}\right)\left[\dot{m}_{im_{in}}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right)T_{im_{in}} - \dot{m}_{out}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right)T_{im_0}\right] \\ \dot{P}_{em} = \left(\frac{\gamma R}{V_{em}}\right)\left[\dot{m}_{em_{in}}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right)T_{em_{in}} - \dot{m}_{em_{out}}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right)T_{em_0}\right] \\ \dot{\omega}_{tc} = \left(\frac{1}{I_{tc}}\right)\left(T_{q_t}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right) - T_{q_c}\left(\tilde{P}_1, \psi_2\left(\tilde{P}_1\right), \bar{T}_0\right) - \omega_{tc}c_{fr}\right) \end{cases}$$

***Remark** 11:* It may be noted that the model reduction is based on the existence of *three time scale* separation within the dynamics of control volume temperatures and pressures. Whereas Σ_2 is obtained by considering the fast dynamics (control volume pressures) and eliminating the slow dynamics (control volume temperatures), Σ_3 is derived by eliminating the fast pressures and considering the slow ones. Thus, to obtain reduced order control oriented model Σ_3 from Σ we are focusing on the *middle* time scale. The middle time scale is physically the most relevant time scale with respect to both slow as well as fast time scales. The control inputs, the throttle angle and waste gate control, which play a vital role in the control design are directly associated with the dynamical equations governing the variables in the middle time scale, namely P_{ic}, P_{im} and P_{em}. Furthermore, conceptually control oriented mean value model should capture the average behavior and, hence, considering the middle time scale is more feasible from the point of view of controller design.

C. Comparison of Σ and Σ_3 models

The key to the accuracy of stage 3 engine model lies in the accuracy with which the fast pressure, P_{af}, P_c, P_{tb} can be approximated. In this section we demonstrate the solution of (23) using Taylor series expansion (shown in Figures 7, 8 and 9). It is easy to see that the approximated fast pressures (as obtained in the form of ψ_2) approaches the original responses (as per Σ) as the order of approximation is increased. For the generation of results in Figures 7 to 9, MATLAB Symbolic toolbox is extensively used and the functions $\psi_{(.)}$ are obtained by using the command *solve*. It is worth mentioning that the authors had to stop at 4^{th} order approximation due to the computational and display limitations of MATLAB Symbolic Toolbox. With an alternative computational tool, it may be possible to achieve even higher level of accuracy.

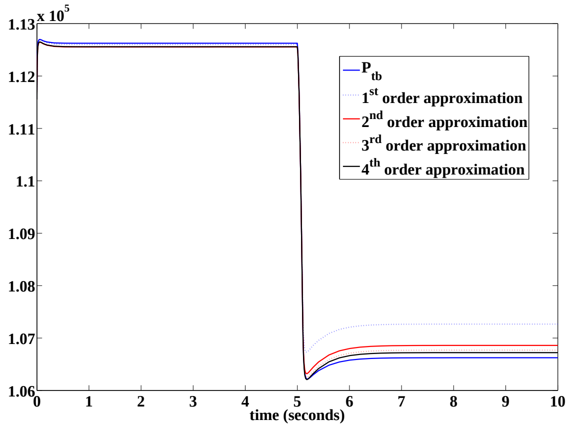

Fig. 7. Approximation of turbine pressure

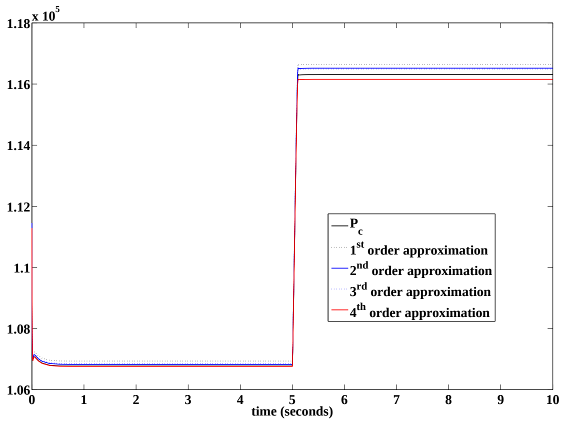

Fig. 8. Approximation of compressor pressure

IV. Conclusions

In this paper, a novel technique to obtain reduced order engine models is demonstrated. It is constructively shown that if certain set of assumptions are satisfied then it is possible to sequentially eliminate some of the state variable and represent the engine behavior by fewer states to obtain reduced order models. The assumptions, imposed on the

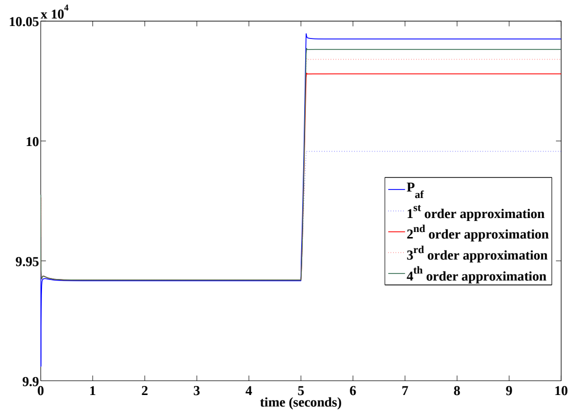

Fig. 9. Approximation of air filter pressure

physical and dynamic characteristics of the engine, are quantified by means of simulations. Model reduction is achieved by the application of perturbation theory, first to eliminate slow temperatures and then to reject fast pressures. The main advantage of this modeling approach is that it is based on the solid theoretical bases and as a result the reduced order models closely approximate the original model under a wide range of operating conditions. Moreover, the obtained model, because they heavily rely on analytical approximations of the original engine model than empirical findings, can be carried to represent a number of different engines. However, the portability of the reduced order models to other engines is governed by the operating conditions and engine geometry.

While investigating the model reduction a number of interesting modeling scenarios are revealed. In the situations where the ratio of temperature and pressure stays relatively constant with the changing operating conditions, reduction from Σ to Σ_2 is an close description of true behavior. Further, if the difference in the relative sizes of various control volumes is large, then the solutions of reduced order model Σ_3 tend to the solution of Σ_2. It is also interesting to note that the procedure leading up to stage 2 and stage 3 models can be implemented independent of each other depending upon which sets of assumptions hold.

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