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Author/s:

Yang, Y;Zhao, G

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The Cohomological Hall Algebras of a Preprojective Algebra with Symmetrizer

Yaping Yang¹ · Gufang Zhao¹

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Abstract

This paper aims at a geometric realization of the Yangian of non-simply laced type in terms of quiver with potentials. For every quiver with symmetrizer, there is an extended quiver with superpotential, whose Jacobian algebra is the generalized preprojective algebra of Geiß, Leclerc, and Schröer (*Inventiones Mathematicae* **209**(1), 61–158, 2017). We study the cohomological Hall algebra of Kontsevich and Soibelman associated to this quiver with potential. In particular, we prove a dimensional reduction result, and provide a shuffle formula of this cohomological Hall algebra. In the case when the quiver with symmetrizer comes from a symmetrizable Cartan matrix, we prove that this shuffle algebra satisfies the relations of the Yangian associated to this Cartan matrix.

Keywords Quiver with potential · Hall algebra · Yangian · Symmetrizable Cartan matrix

Mathematics Subject Classification (2010) Primary 16G20 · Secondary 17B37, 14F43

1 Introduction

For a Kac-Moody Lie algebra \mathfrak{g} , the Yangian of \mathfrak{g} is an affine type quantum algebra associated to \mathfrak{g} . In the case when \mathfrak{g} is simply-laced, a geometric construction of Yangian is given using Borel-Moore homology of Nakajima quiver variety [20] based on earlier works of Nakajima [13]. The present paper aims to construct geometrically the Yangian of non-simply laced type, which was previously unknown. The key observation is that these algebras are obtained in the framework of Kontsevich-Soibelman cohomological Hall algebras (COHA) for a quiver with potential, when the potential is so that the Jacobian algebra is

Presented by: Michela Varagnolo

✉ Yaping Yang
yaping.yang1@unimelb.edu.au

Gufang Zhao
gufangz@unimelb.edu.au

¹ School of Mathematics and Statistics, The University of Melbourne, 813 Swanston Street, Parkville VIC 3010, Australia

the generalized preprojective algebra of Geiß, Leclerc, and Schröer [5]. The latter is known as a quiver with symmetrizer, hence the title of the present paper. This can be viewed as the first step in studying cohomology of non-simply laced Nakajima quiver variety.

Let $Q = (I, H)$ be a quiver, with I the set of vertices, and H the set of arrows. A *symmetrizer* of the quiver Q is a collection of positive integers

$$L := \{l_{ij} \in \mathbb{N} \mid i, j \in I, \text{ and there is an arrow from } i \text{ to } j\}.$$

Associated to the quiver with symmetrizer (Q, L) , Geiß, Leclerc, and Schröer defined a generalized preprojective algebra [5].

Let \widehat{Q} be the extended quiver of Q . The set of vertices of \widehat{Q} is I , and the set of arrows is $H \sqcup H^{\text{op}} \sqcup B$ with H^{op} in bijection with H , and for each $h \in H$, the corresponding arrow in H^{op} , denoted by h^* , is h with orientation reversed. The set B is $\{B_i \mid i \in I\}$, with B_i an edge loop at the vertex $i \in I$. Consider the following potential of \widehat{Q}

$$W^L := \sum_{h \in H} (B_{\text{in}(h)}^{l_{\text{in}(h), \text{out}(h)}} hh^* - B_{\text{out}(h)}^{l_{\text{out}(h), \text{in}(h)}} h^* h).$$

The generalized preprojective algebra associated to (Q, L) is the Jacobian algebra of $(\widehat{Q}, \text{tr}(W^L))$ [5]. The representation variety of the Jacobian algebra of $(\widehat{Q}, \text{tr}(W^L))$ is the critical locus of $\text{tr } W^L$. This variety has been studied in detail in *loc. cit.*

The Jacobian algebra is naturally endowed with the structure of a DG-algebra [6]. In particular, the representation variety is naturally endowed with a complex of constructible sheaves, which is the vanishing cycle of the potential function $\text{tr}(W^L)$. The cohomology of the representation variety valued in this vanishing cycle complex has the structure of an algebra, constructed by Kontsevich and Soibelman [10], called the cohomological Hall algebra (COHA). In the present paper, we study this cohomological Hall algebra.

In the case when the quiver with symmetrizer comes from a symmetrizable Cartan matrix, we prove that a localized form of the COHA gives the positive part of the Yangian associated to the corresponding symmetrizable Kac-Moody algebra. The precise statement is given as Theorem 5.1. The proof uses similar calculations as in [21, § 7] and [22], where the latter significantly relies on [4, Appendix A]. Along the way, we give a more general version of dimensional reduction result describing the COHA using Borel-Moore homology instead of critical cohomology. This version is useful in other settings (see, e.g., [16]). We include a sketch of the proofs to make the present paper self-contained.

When the Cartan matrix is symmetric, and the symmetrizer $\{l_{ij} = l\}$ has the same order $l \geq 1$, the COHA is expected to be related to an l th zastava space from the work of Mirković [11, § 3.4]. This expectations is based on the comparison of the tautological line bundle on Grassmannian and the spherical COHA in the $l = 1$ case. The COHA, as well as the spherical subalgebra, are both coherent sheaves on the l -colored configuration space of points on \mathbb{C} , i.e., the moduli of finite subschemes of $\mathbb{C} \times I$. The l -colored configuration space has a map to the loop Grassmannian of the adjoint group via the Abel-Jacobi map. The restriction of the tautological line bundle $\mathcal{O}(1)$ via this map defines a line bundle with a *locality or factorization structure* [12, Proposition 3.5.4]. The sheaf of sections of this line bundle is identified with the spherical COHA [12, Proposition 3.5.2], without torus equivariant parameters. Under this identification, the locality structure of the line bundle corresponds to the algebra structure of spherical COHA. Turning on the equivariant parameters gives a quantization of this line bundle [12, Section 4]. For general $l \geq 1$, we expect the COHA for general symmetrizer l to give rise to a quantization of the line bundle $\mathcal{O}(l)$, and therefore lead to a quantization of the homogeneous coordinate ring of the zastava space. We pursue this elsewhere.

This quiver with potential and its relation to loop Grassmannian originates from the work of Nekrasov and Shatashvili [15, § 3.1.4]. There are already some mathematical work regarding the Jacobian algebra for a quiver with a symmetrizer [2, 3]. We further expect that the cohomology of the moduli space of stable framed representations of this quiver with potential, valued in the vanishing cycle, gives a global Weyl module of the Yangian. This is the analogue of Nakajima quiver variety for non-simply laced types. In the simply laced with equal symmetrizer case, notably in the work of Bykov and Zinn-Justin [1], the equivariant cohomology of moduli space of stable framed representations of this Jacobian algebra has been related to, in the \mathfrak{sl}_2 -case, the $(l + 1)$ -th spin representation.

2 The Cohomological Hall Algebra

In this section, we first review the cohomological Hall algebra (COHA) associated to a quiver with potential that was defined by Kontsevich and Soibelman [10, Section 7] (see also [4]). We then introduce the extended quiver with potential that is of particular interest in this paper.

2.1 The COHA

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, where Γ_0 is the set of vertices, and Γ_1 the set of arrows. For each $h \in \Gamma_1$, let $\text{in}(h)$ be the incoming vertex and $\text{out}(h)$ be the outgoing vertex. We denote the path algebra of Γ by $\mathbb{C}\Gamma$. Let W be a potential of Γ , that is, $W = \sum_u c_u u$ with $c_u \in \mathbb{C}$, and u 's are cycles in Γ .

Given a cycle $u = a_1 \dots a_n$ (here the cycles are considered up to cyclic order) and an arrow $a \in \Gamma_1$. The cyclic derivative is defined to be

$$\frac{\partial u}{\partial a} = \sum_{i:a_i=a} a_{i+1} \dots a_n a_1 \dots a_{i-1} \in \mathbb{C}\Gamma$$

as an element of $\mathbb{C}\Gamma$. We extend the cyclic derivative to the potential by linearity.

For any dimension vector $v = (v^i)_{i \in \Gamma_0} \in \mathbb{N}^{\Gamma_0}$, the representation space of Γ with dimension vector v is denoted by $\mathbf{M}_{\Gamma,v}$. That is, let $V = \{V^i\}_{i \in \Gamma_0}$ be a $|\Gamma_0|$ -tuple of vector spaces so that $\dim(V^i) = v^i$. Then,

$$\mathbf{M}_{\Gamma,v} := \bigoplus_{h \in \Gamma_1} \text{Hom}(V^{\text{out}(h)}, V^{\text{in}(h)}).$$

The group $G_v := \prod_{i \in \Gamma_0} \text{GL}(V^i)$ acts on the representation space $\mathbf{M}_{\Gamma,v}$ via conjugation.

For a quiver with potential (Γ, W) and dimension vector $v \in \mathbb{N}^{\Gamma_0}$, denote by $\text{tr}(W)_v$ the function on $\mathbf{M}_{\Gamma,v}$ given by the trace of the potential. Let $\text{Crit}(\text{tr } W_v)$ be the critical locus of $\text{tr } W_v$. Let $\varphi_{\text{tr } W_v}$ be the vanishing cycle complex on $\mathbf{M}_{\Gamma,v}$ associated to the function $\text{tr}(W)_v$, which is supported on $\text{Crit}(\text{tr } W_v)$. We refer the readers to [10] and [4] for the definition and details of the vanishing cycle complex. For a G_v -variety X , denote by $H_{c,G_v}^*(X)$ the equivariant cohomology with compact support. Let $H_{c,G_v}^*(X)^\vee$ be its dual. Similar for cohomology valued in a complex of sheaves, we have an isomorphism $H_{c,G_v}^*(\mathbf{M}_{\Gamma,v}, \varphi_{\text{tr } W_v}) \cong H_{c,G_v}^*(\text{Crit}(\text{tr } W_v), \varphi_{\text{tr } W_v})$.

Let

$$\mathcal{H}(\Gamma, W) := \bigoplus_{v \in \mathbb{N}^{\Gamma_0}} \mathcal{H}(\Gamma, W)_v = \bigoplus_{v \in \mathbb{N}^{\Gamma_0}} H_{c,G_v}^*(\text{Crit}(\text{tr } W_v), \varphi_{\text{tr } W_v})^\vee.$$

There is an algebra structure on $\mathcal{H}(\Gamma, W)$ via Hall multiplications [10, Section 7.6]. The algebra $\mathcal{H}(\Gamma, W)$ is called the COHA associated to the quiver with potential (Q, W) . For a review of the COHA in the presence of a *cut*, see also [22, § 1.1, 1.2] and [4].

2.2 The Extended Quiver with Symmetrizer

For a quiver $Q = (I, H)$, with I the set of vertices, and H the set of arrows, we consider the extended quiver \widehat{Q} as defined in the introduction.

For any $i, j \in I$, denote the set of arrows of Q from i to j by $\{i \rightarrow j\}$. A collection of integers

$$L := \{l_{ij} \in \mathbb{N} \mid i, j \in I, \text{ and } \{i \rightarrow j\} \neq \emptyset\}$$

is called a symmetrizer of the quiver Q .

For a quiver Q with a symmetrizer L , we define a potential of \widehat{Q} to be

$$W^L := \sum_{h \in H} (B_{\text{in}(h)}^{l_{\text{in}(h), \text{out}(h)}} h h^* - B_{\text{out}(h)}^{l_{\text{out}(h), \text{in}(h)}} h^* h). \tag{1}$$

The cyclic derivatives of W^L are

$$\frac{\partial W^L}{\partial h} = h^* B_{\text{in}(h)}^{l_{\text{in}(h), \text{out}(h)}} - B_{\text{out}(h)}^{l_{\text{out}(h), \text{in}(h)}} h^* \tag{2}$$

$$\frac{\partial W^L}{\partial h^*} = B_{\text{in}(h)}^{l_{\text{in}(h), \text{out}(h)}} h - h B_{\text{out}(h)}^{l_{\text{out}(h), \text{in}(h)}} \tag{3}$$

$$\frac{\partial W^L}{\partial B_i} = \sum_{\{h \mid \text{in}(h)=i, \text{out}(h)=j\}} \sum_{e=0}^{l_{i,j}-1} B_i^{l_{ij}-1-e} h h^* B_i^e - \sum_{\{h \mid \text{out}(h)=i, \text{in}(h)=j\}} \sum_{e=0}^{l_{i,j}-1} B_i^{l_{ij}-1-e} h^* h B_i^e \tag{4}$$

We consider a *weight function* of Q :

$$\mathbf{m} : I \sqcup H \sqcup H^{\text{op}} \rightarrow \mathbb{Z}, i \mapsto \mathbf{m}_i, h \mapsto \mathbf{m}_h, h^* \mapsto \mathbf{m}_{h^*}, \text{ for } i \in I, h \in H \text{ and } h^* \in H^{\text{op}}.$$

For each $v \in \mathbb{N}^I$, on the space $\mathbf{M}_{\widehat{Q}, v}$, in addition to the action of GL_v , there is a 3-dimensional torus $(\mathbb{G}_m)^3$ action. Let (z_1, z_2, z_3) be the coordinates of $(\mathbb{G}_m)^3$,

$$(z_1, z_2, z_3)(h, h^*, B_i) = (z_1^{\mathbf{m}_h} h, z_2^{\mathbf{m}_{h^*}} h^*, z_3^{\mathbf{m}_i} B_i).$$

so that z_1 and z_2 scale h and h^* respectively with weights \mathbf{m}_h and \mathbf{m}_{h^*} for each $h \in H$, and z_3 scales B_i for each $i \in I$ with weight \mathbf{m}_i .

The action of $(z_1, z_2, z_3) \in (\mathbb{G}_m)^3$ preserves the potential function W^L , if and only if

$$z_1^{\mathbf{m}_h} z_2^{\mathbf{m}_{h^*}} z_3^{\mathbf{m}_{\text{in}(h)} l_{\text{in}(h), \text{out}(h)}} = z_1^{\mathbf{m}_h} z_2^{\mathbf{m}_{h^*}} z_3^{\mathbf{m}_{\text{out}(h)} l_{\text{out}(h), \text{in}(h)}} = 1. \tag{5}$$

In particular,

$$\mathbf{m}_i l_{ij} = \mathbf{m}_j l_{ji}, \text{ for any } i, j \in I \tag{6}$$

In the present paper, we make the assumption that there exists such a function \mathbf{m} , such that the condition (6) holds.

Let \mathcal{D} be a torus endowed with a group homomorphism $a : \mathcal{D} \rightarrow (\mathbb{G}_m)^3$ such that any element (z_1, z_2, z_3) in the image of a satisfies the condition (5). Thus, the torus \mathcal{D} acts on $\mathbf{M}_{\widehat{Q}, v}$ in a way which preserves the potential function W^L .

We now give an example of one choice of \mathcal{D} and $\{\mathbf{m}_h, \mathbf{m}_{h^*}, \mathbf{m}_i \mid i \in I, h \in H\}$ that satisfy the above assumptions.

Example 2.1 Let (Q, L) be a quiver with symmetrizer. Assume here Q has no oriented cycles. Let $\{\mathbf{m}_i \in \mathbb{Z}, i \in I\}$ be a set of integers, such that the condition (6) holds. For a pair of vertices $i, j \in I$, denote the number $\mathbf{m}_i l_{ij} = \mathbf{m}_j l_{ji}$ by d . Let n be the number of arrows between i and j (only in one direction because of the acyclicity assumption) in Q . We fix a numbering h_1, \dots, h_n of these arrows, and let

$$\mathbf{m}_{h_p} := (n + 2 - 2p)d, \quad \mathbf{m}_{h_p^*} := (-n + 2p)d, \quad \text{for } p = 1, \dots, n.$$

Consider the embedding $\mathcal{D} := \mathbb{G}_m \hookrightarrow (\mathbb{G}_m)^3, z \mapsto (z, z, z^{-2})$. The assumption (5) follows from the equality $\mathbf{m}_h + \mathbf{m}_{h^*} - 2\mathbf{m}_{\text{in}(h)}l_{\text{in}(h),\text{out}(h)} = 2d - 2d = 0$. This torus action will be used in the construction of Yangians associated to symmetrizable Cartan matrices in Section 5.

With \mathcal{D} satisfying condition (5), the Hall multiplication of Kontsevich and Soibelman is equivariant with respect to this \mathcal{D} -action. Therefore, we have the following equivariant COHA

$$\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L) = \bigoplus_{v \in \mathbb{N}^I} H_{c, G_v \times \mathcal{D}}^*(\text{Crit}(\text{tr } W_v^L), \varphi_{\text{tr } W_v})^{\vee}. \tag{7}$$

3 The Shuffle Algebra

In this section, we define the shuffle algebra associated to the COHA $\mathcal{H}(\widehat{Q}, W^L)$. In the shuffle algebra considered in this section, there are quantization parameters t_1, t_2, t_3 . Geometrically these parameters come from the torus $\mathcal{D} \hookrightarrow (\mathbb{G}_m)^3$ action on representation space of the quiver \widehat{Q} .

3.1 The Definition

To begin with, we fix some notations. Let (Q, L) be a quiver with symmetrizer. We fix a weight function $\mathbf{m} : I \sqcup H \sqcup H^{\text{op}} \rightarrow \mathbb{Z}$. Let $v = (v^i)_{i \in I} \in \mathbb{N}^I$ be a dimension vector of Q and $\mathfrak{S}_v := \prod_{i \in I} \mathfrak{S}_{v^i}$ be the product of symmetric groups. There is a natural action of \mathfrak{S}_v on the variables $\{\lambda_s^i \mid i \in I, s = 1, \dots, v^i\}$ by permutation.

For any pair (p, q) of positive integers, let $\text{Sh}(p, q)$ be the subset of \mathfrak{S}_{p+q} consisting of (p, q) -shuffles (permutations of $\{1, \dots, p+q\}$ that preserve the relative order of $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$). Given dimension vectors $v_1 = (v_1^i)_{i \in I}, v_2 = (v_2^i)_{i \in I}$ with $v = v_1 + v_2$, let $\text{Sh}(v_1, v_2) \subset \mathfrak{S}_v$ denote the product $\prod_{i \in I} \text{Sh}(v_1^i, v_2^i)$.

We now define the shuffle algebra \mathcal{SH} associated to the data (Q, L, \mathbf{m}) . The shuffle algebra \mathcal{SH} is an \mathbb{N}^I -graded $\mathbb{C}[t_1, t_2, t_3]$ -algebra. As a $\mathbb{C}[t_1, t_2, t_3]$ -module, we have $\mathcal{SH} = \bigoplus_{v \in \mathbb{N}^I} \mathcal{SH}_v$. The degree v piece is

$$\mathcal{SH}_v := \mathbb{C}[t_1, t_2, t_3][\lambda_s^i]_{i \in I, s=1, \dots, v^i}^{\mathfrak{S}_v}.$$

We will also consider specializations of the equivariant parameters, i.e., algebraic homomorphisms $\mathbb{C}[t_1, t_2, t_3] \rightarrow A$, for some algebra A . We assume the weight function \mathbf{m} is compatible with the specialization in the sense that

$$t_1 \mathbf{m}_h + t_2 \mathbf{m}_{h^*} + t_3 \mathbf{m}_{\text{in}(h)} l_{\text{in}(h),\text{out}(h)} = t_1 \mathbf{m}_h + t_2 \mathbf{m}_{h^*} + t_3 \mathbf{m}_{\text{out}(h)} l_{\text{out}(h),\text{in}(h)} = 0, \text{ for any } h \in H. \tag{8}$$

Identifying $\mathbb{C}[t_1, t_2, t_3]$ with $H_{\mathbb{G}_m^3}(\text{pt})$, then in the presence of $a : \mathcal{D} \rightarrow \mathbb{G}_m^3$, the specialization $\mathbb{C}[t_1, t_2, t_3] \rightarrow A$ can be taken to be $H_{\mathbb{G}_m^3}(\text{pt}) \rightarrow H_{\mathcal{D}}(\text{pt}) =: A$ induced by a . Condition (8) on the specialization is then equivalent to the condition (5) on a .

For any v_1 and $v_2 \in \mathbb{N}^I$, we consider $\mathcal{SH}_{v_1} \otimes \mathcal{SH}_{v_2}$ as a $\mathbb{C}[t_1, t_2, t_3]$ -submodule of

$$\mathbb{C}[t_1, t_2, t_3][\lambda_j^i]_{i \in I, j=1, \dots, (v_1+v_2)^i}$$

by sending λ_s^i to λ_s^i , and $\lambda_t^{i'}$ to $\lambda_{t+v_1}^i$. Here $\{\lambda_s^i \mid i \in I, s = 1, \dots, v_1^i\}$ and $\{\lambda_s^{i'} \mid i \in I, s = 1, \dots, v_2^i\}$ are the variables of \mathcal{SH}_{v_1} and \mathcal{SH}_{v_2} respectively. Define¹

$$\text{fac}_1 := \prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} \frac{\lambda_t^{i'} - \lambda_s^i + \mathbf{m}_i t_3}{\lambda_t^{i'} - \lambda_s^i}. \tag{9}$$

and

$$\text{fac}_2 := \prod_{h \in H} \left(\prod_{s=1}^{v_1^{\text{out}(h)}} \prod_{t=1}^{v_2^{\text{in}(h)}} (\lambda_t^{\text{in}(h)} - \lambda_s^{\text{out}(h)} + \mathbf{m}_h t_1) \prod_{s=1}^{v_1^{\text{in}(h)}} \prod_{t=1}^{v_2^{\text{out}(h)}} (\lambda_t^{\text{out}(h)} - \lambda_s^{\text{in}(h)} + \mathbf{m}_h t_2) \right). \tag{10}$$

The multiplication of $f_1(\lambda') \in \mathcal{SH}_{v_1}$ and $f_2(\lambda'') \in \mathcal{SH}_{v_2}$ is defined to be

$$f_1(\lambda') * f_2(\lambda'') := \sum_{\sigma \in \text{Sh}(v_1, v_2)} \sigma(f_1 \cdot f_2 \cdot \text{fac}_1 \cdot \text{fac}_2) \in \mathbb{C}[t_1, t_2, t_3][\lambda_j^i]_{i \in I, j=1, \dots, (v_1+v_2)^i}^{\mathbb{S}_{v_1+v_2}}. \tag{11}$$

It is a direct algebraic computation that \mathcal{SH} endowed with the above multiplication is an associative algebra. Note that although fac_1 has a denominator, the symmetrization over shuffle elements creates zeros on the numerator which cancels the poles introduced by the denominators. Therefore, the shuffle product is well-defined without introducing any localization.

3.2 COHA and the Shuffle Algebra

In this section, we take into account the torus \mathcal{D} -action on the representation space of \widehat{Q} . We compute $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)$ in terms of the shuffle algebra in Section 3.1. The main ingredient is the dimension reduction of Davison [4] and [22, Theorem 2.5] recalled in Appendix A. Here we follow the notations in Appendix A.

We take Γ to be \widehat{Q} , the potential to be W^L . We take the cut to be H , the set of arrows of the original quiver Q . Let $\widehat{Q} \setminus H$ be the quiver obtained from \widehat{Q} by removing H . Hence the set of vertices of $\widehat{Q} \setminus H$ is I , and its set of arrows is $H^{\text{op}} \sqcup B = \{h^*, B_i \mid h^* \in H^{\text{op}}, i \in I\}$. Consider the quotient of the path algebra $\mathbb{C}(\widehat{Q} \setminus H)$ by the relation

$$h^* B_{\text{in}(h)}^{\text{in}(h), \text{out}(h)} = B_{\text{out}(h)}^{\text{out}(h), \text{in}(h)} h^*, \text{ for any } h \in H.$$

¹The fac_1 in the present paper and fac_1 in [21, §3.1] differ by a sign $(-1)^{\sum_{i \in I} v_1^i v_2^i}$. The shuffle formula in the present paper is deduced from a 3-dimensional COHA, while the one in [21] is obtained from a 2-dimensional COHA. This sign naturally occurs when comparing the dimensional reduction of a 3d COHA to a 2d COHA [22, §5.1].

By Eq. 2, the representation variety of this quotient algebra is then

$$\begin{aligned} \mathbf{J}_{\widehat{Q}\setminus H,v} &:= \{x \in \mathbf{M}_{\widehat{Q}\setminus H,v} \mid \partial W^L/\partial h(x) = 0, \text{ for any } h \in H\} \\ &= \{(h^*, B_i) \in \mathbf{M}_{\widehat{Q}\setminus H,v} \mid h^* B_{\text{in}(h)}^{\text{in}(h),\text{out}(h)} = B_{\text{out}(h)}^{\text{out}(h),\text{in}(h)} h^*, \text{ for any } h \in H\}. \end{aligned}$$

Remark 3.1 Let $i : \mathbf{J}_{\widehat{Q}\setminus H,v} \times \mathbf{M}_{Q,v} \hookrightarrow \mathbf{M}_{\widehat{Q},v}$ be the natural embedding. Pushforward along i gives

$$i_* : H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\widehat{Q}\setminus H,v} \times \mathbf{M}_{Q,v}, \mathbb{Q}) \rightarrow H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{M}_{\widehat{Q},v}, \mathbb{Q}),$$

which is an isomorphism after localization. More precisely, take $T_v \subseteq G_v$ to be a maximal torus. Let $T = T_v \times \mathcal{D}$. Restricting to T gives a commutative diagram

$$\begin{array}{ccc} H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\widehat{Q}\setminus H,v} \times \mathbf{M}_{Q,v}, \mathbb{Q}) & \xrightarrow{i_*} & H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{M}_{\widehat{Q},v}, \mathbb{Q}) \\ \downarrow & & \downarrow \\ H_T^{\text{BM}}(\mathbf{J}_{\widehat{Q}\setminus H,v} \times \mathbf{M}_{Q,v}, \mathbb{Q}) & \xrightarrow{i_*} & H_T^{\text{BM}}(\mathbf{M}_{\widehat{Q},v}, \mathbb{Q}) \end{array}$$

The bottom row is a map of modules over $H_T^{\text{BM}}(\text{pt}) \cong \mathbb{C}[\text{Lie } T]$. Let $\mathbb{C}(\text{Lie } T)$ be the quotient field of $\mathbb{C}[\text{Lie } T]$. By the equivariant localization [8], the bottom map i_* induces an isomorphism when base changed from $\mathbb{C}[\text{Lie } T]$ to $\mathbb{C}(\text{Lie } T)$.

In Appendix A, Theorem A.2, we show $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)$ is canonically isomorphic to $\oplus_v H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\widehat{Q}\setminus H,v} \times \mathbf{M}_{Q,v}, \mathbb{Q})$.

Theorem 3.2 *The pushforward i_* induces an algebra homomorphism $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L) \rightarrow S\mathcal{H}$. It is an isomorphism after localization in the sense above.*

Proof With notations as in Appendix A, the first row of Diagram (25) becomes

$$\mathbf{M}_{\widehat{Q},v_1} \times \mathbf{M}_{\widehat{Q},v_2} \xleftarrow{p_1} (\mathbf{M}_{\widehat{Q}\setminus H,v_1} \times \mathbf{M}_{\widehat{Q}\setminus H,v_2}) \oplus \mathbf{M}_{Q,v_1,v_2} \xrightarrow{i_1} (\mathbf{M}_{\widehat{Q}\setminus H,v_1} \times \mathbf{M}_{\widehat{Q}\setminus H,v_2}) \oplus \mathbf{M}_{Q,v}$$

The first row of Eq. 27 becomes

$$(\mathbf{M}_{\widehat{Q}\setminus H,v_1}) \times (\mathbf{M}_{\widehat{Q}\setminus H,v_2}) \xrightarrow{\iota} \mathbf{Y} \xleftarrow{\omega} \mathbf{M}_{\widehat{Q}\setminus H,v_1,v_2} \xrightarrow{i_2} \mathbf{M}_{\widehat{Q}\setminus H,v}$$

with

$$\begin{aligned} \mathbf{Y} &= \{(y_h^*, x_h^*, B) \mid y_h^* \in \mathbf{M}_{Q^{\text{op}},v_1,v_2}, x_h^* \in (\mathbf{M}_{Q^{\text{op}},v_1} \times \mathbf{M}_{Q^{\text{op}},v_2}), B \in (\mathfrak{gl}_{v_1} \times \mathfrak{gl}_{v_2}) \\ &\quad \text{pr}(y_h^*) = B_{\text{in } h}^{\text{in}(h),\text{out}(h)} x_h^* - x_h^* B_{\text{out}(h)}^{\text{out}(h),\text{in}(h)}, \text{ for any } h \in H\}, \end{aligned}$$

where $\text{pr} : \mathbf{M}_{Q^{\text{op}},v_1,v_2} \rightarrow \mathbf{M}_{Q^{\text{op}},v_1} \times \mathbf{M}_{Q^{\text{op}},v_2}$ is the natural projection. Theorem A.2 gives an isomorphism of algebras

$$\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L) \cong \bigoplus_{v \in \mathbb{N}^I} H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\widehat{Q}\setminus H,v} \times \mathbf{M}_{Q,v}, \mathbb{Q}).$$

The Hall multiplication m^{crit} of the former is identified as

$$(\overline{i_2} \times \text{id}_{\mathbf{M}_{Q,v}})_* \circ \frac{1}{e(l)} (\omega \times \text{id}_{\mathbf{M}_{Q,v}})_{\omega \times \text{id}_{\mathbf{M}_{C,v}}}^{\sharp} \circ \overline{i_1}_* \circ \overline{p_1}^*. \tag{12}$$

of the latter. In particular, the latter is associative.

On $\bigoplus_{v \in \mathbb{N}^I} H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{M}_{\widehat{Q}, v}, \mathbb{Q})$ we have the binary operation defined by

$$(i_2 \times \text{id}_{\mathbf{M}_{Q,v}})_* \circ (\omega \times \text{id}_{\mathbf{M}_{Q,v}})^* \circ \frac{1}{e(\iota)} (\iota \times \text{id}_{\mathbf{M}_{Q,v}})_* \circ i_{1*} \circ p_1^*. \tag{13}$$

Recall $i : \mathbf{J}_{\widehat{Q} \setminus H, v} \times \mathbf{M}_{Q,v} \rightarrow \mathbf{M}_{\widehat{Q}, v}$ is the natural embedding. Pushforward along i for each v gives

$$i_* : \bigoplus_{v \in \mathbb{N}^I} H_{G_v}^{\text{BM}}(\mathbf{J}_{\widehat{Q} \setminus H, v} \times \mathbf{M}_{Q,v}, \mathbb{Q}) \rightarrow \bigoplus_{v \in \mathbb{N}^I} H_{G_v}^{\text{BM}}(\mathbf{M}_{\widehat{Q}, v}, \mathbb{Q}),$$

which intertwines the above binary operations (12) and (13).

Now we identify the target with \mathcal{SH} , and Eq. 13 with Eq. 11. This follows from the same calculation of the Thom classes of ι , i_1 , and i_2 as in [21, § 1.4]. More precisely, the pushforward ι_* is the same as multiplication by $e(\iota)$. The normal bundle to i_1 and i_2 are identified with

$$\mathcal{H}om_Q(\mathcal{R}(v_1), \mathcal{R}(v_2)) \text{ and } \mathcal{H}om_{\widehat{Q} \setminus H}(\mathcal{R}(v_1), \mathcal{R}(v_2))$$

respectively, where $\mathcal{R}(r)$ is the tautological bundle of $\text{Grass}(r, \infty)$. Let $e(i)$ be the equivariant Euler class of the normal bundle of i and let $\{\lambda'_s\}_{s=1, \dots, v^i}$ be the Chern roots of the tautological bundle $\mathcal{R}(v^i)$. Thus, we have

$$e(i_1)e(i_2) = \prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} (\lambda''_t{}^i - \lambda'_s{}^i + \mathbf{m}_i t_3) \cdot \prod_{h \in H} \left(\prod_{s=1}^{v_1^{\text{out}(h)}} \prod_{t=1}^{v_2^{\text{in}(h)}} (\lambda''_t{}^{\text{in}(h)} - \lambda'_s{}^{\text{out}(h)} + \mathbf{m}_h t_1) \prod_{s=1}^{v_1^{\text{in}(h)}} \prod_{t=1}^{v_2^{\text{out}(h)}} (\lambda''_t{}^{\text{out}(h)} - \lambda'_s{}^{\text{in}(h)} + \mathbf{m}_h t_2) \right),$$

The denominator of fac_1 and the averaging over all the shuffle $\text{Sh}(v_1, v_2)$ come from pushing-forward from a Grassmannian bundle.

In the above we see one more time that Eq. 13 is well-defined without introducing denominators. The operation (12) is associative, and hence remains so after localization. On the other hand, the natural map from $H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{M}_{\widehat{Q}, v}, \mathbb{Q})$ to its localization is injective. In particular, Eq. 13 is associative after localization, and hence is so before localization. This shows that \mathcal{SH} is an algebra. The argument above then implies that i_* is an algebra homomorphism. This completes the proof of Theorem 3.2. \square

Remark 3.3 We expect a version of $\mathbf{J}_{\widehat{Q} \setminus H, v}$ with framing can be defined, together with suitable stability conditions, so that the cohomology groups of the stable framed representations carry representations of the double of the COHA.

4 Generalized Cartan Matrix

In this section, we consider a symmetrizable Cartan matrix. We associate to a Cartan matrix with symmetrizer a quiver with symmetrizer.

4.1 The Potential Associated to the Cartan Matrix

Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be a symmetrizable generalized Cartan matrix. Thus, $a_{ii} = 2$ for all $1 \leq i \leq n$; $a_{ij} \leq 0$ for any $1 \leq i \neq j \leq n$; $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$, and there

exists a diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ with positive integer entries such that DA is symmetric. In other words, we have $d_i a_{ij} = d_j a_{ji}$ for any $1 \leq i, j \leq n$. For simplicity, we drop the words “symmetrizable generalized” and call the pair (A, D) a *Cartan matrix with symmetrizer*.

Start with a Cartan matrix with symmetrizer (A, D) . We construct a quiver with symmetrizer as follows. The set of vertices is $I = \{1, 2, \dots, n\}$. For any $i < j \in I$, the number of arrows from i to j is

$$|\text{gcd}(a_{ij}, a_{ji})|.$$

We choose the symmetrizer of the quiver to be

$$l_{ij} := \left| \frac{a_{ij}}{\text{gcd}(a_{ij}, a_{ji})} \right|.$$

Thus, there exist weights $\mathbf{m}_i = d_i$ such that $\mathbf{m}_i l_{ij} = \mathbf{m}_j l_{ji}$, since DA is symmetric.

Example 4.1 When the Cartan matrix is $A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$. We have $d_1 = \mathbf{m}_1 = 2$ and $d_2 = \mathbf{m}_2 = 1$. In this case, the symmetrizer of the quiver is $l_{12} = 1$ and $l_{21} = 2$.

The following remark is kindly pointed out to us by Hiraku Nakajima.

Remark 4.2 When the Cartan matrix A in the present paper is taken to be the transpose of the Cartan matrix in [5], the quiver just defined with potential $W^L = \sum_{h \in H} (B_{\text{in}(h)}^{l_{\text{in}(h), \text{out}(h)}} h h^* - B_{\text{out}(h)}^{l_{\text{out}(h), \text{in}(h)}} h^* h)$ agrees with the quiver with potential in [5, §1.7.3], with l_{ij} in the present paper equal to f_{ij} of *loc. cit.* for any $i, j \in I$. The vanishing of the cyclic derivatives (2), (3), (4) gives the relations (P2) and (P3) of [5, Section 1.4]. In particular, the Yangian constructed in Section 5 is the Yangian of the Langlands dual to the Lie algebra constructed in *loc. cit.*

Starting from the same quiver with potential in the present paper, the Lie algebra associated to it in [9, 14] is Langlands dual to that of [5], thus it is the same Lie algebra as in the present paper.

4.2 A Sign Twist

We now work under the setting of Example 2.1. By specializing $t_1 = t_2 = \hbar/2$, and $t_3 = -\hbar$, we have a shuffle algebra \mathcal{SH} over $\mathbb{C}[\hbar]$. We define a sign-twisted shuffle algebra $\widetilde{\mathcal{SH}}$. As a $\mathbb{C}[\hbar]$ -module, $\widetilde{\mathcal{SH}}$ is the same as \mathcal{SH} . The weight function \mathbf{m} is given as in Example 2.1. In particular, we have $\mathbf{m}_i = d_i$, for $i = 1, 2, \dots, n$.

Define

$$\widetilde{\text{fac}}_1 := \text{fac}_1 = \prod_{i \in I} \prod_{s=1}^{v_1^i} \prod_{t=1}^{v_2^i} \frac{\lambda_s'^i - \lambda_t''^i - \mathbf{m}_i t_3}{\lambda_s^i - \lambda_t'^i} \tag{14}$$

$$\begin{aligned} \widetilde{\text{fac}}_2 &:= (-1)^{\sum_{h \in H} v_1^{\text{in}(h)} v_2^{\text{out}(h)}} \text{fac}_2 \\ &= \prod_{h \in H} \left(\prod_{s=1}^{v_1^{\text{out}(h)}} \prod_{t=1}^{v_2^{\text{in}(h)}} (\lambda_t''^{\text{in}(h)} - \lambda_s'^{\text{out}(h)} + \mathbf{m}_h t_1) \prod_{s=1}^{v_1^{\text{in}(h)}} \prod_{t=1}^{v_2^{\text{out}(h)}} (\lambda_s'^{\text{in}(h)} - \lambda_t''^{\text{out}(h)} - \mathbf{m}_{h^*} t_2) \right) \end{aligned} \tag{15}$$

For simplicity, we write the multiplication of $\widetilde{\mathcal{SH}}$ as \star . The multiplication of $f_1(\lambda') \in \widetilde{\mathcal{SH}}_{v_1} = \mathcal{SH}_{v_1}$ and $f_2(\lambda'') \in \widetilde{\mathcal{SH}}_{v_2} = \mathcal{SH}_{v_2}$ is defined to be

$$f_1(\lambda') \star f_2(\lambda'') = (-1)^{\sum_{h \in H} v_1^{\text{in}(h)} v_2^{\text{out}(h)}} f_1(\lambda') * f_2(\lambda'') = \sum_{\sigma \in \text{Sh}(v_1, v_2)} \sigma(f_1 \cdot f_2 \cdot \widetilde{\text{fac}}_1 \cdot \widetilde{\text{fac}}_2) \in \mathcal{SH}_{v_1+v_2}. \tag{16}$$

5 Shuffle Presentation of a Yangian for Symmetrizable Cartan Matrix

5.1 The Yangian

Let $\mathfrak{g}_{A,D}$ be the symmetrizable Kac-Moody Lie algebra associated to the Cartan matrix $A = (a_{ij})_{i,j \in I}$ with symmetrizer D . In this section we assume the collection of integers $(d_i)_{i \in I}$ to be relatively prime. This assumption is rather a conventional choice of normalization in the presentation of the Yangian we use [7]. Let (Q, L) be the quiver with symmetrizer corresponding to $\mathfrak{g}_{A,D}$. Without raising confusions, we will also write $\mathfrak{g}_{A,D}$ as \mathfrak{g}_Q .

The Kac-Moody presentation of \mathfrak{g}_Q is given by $\mathfrak{g}_Q = \langle e_i, f_i \mid i \in I \rangle$ such that

$$\begin{aligned} (\text{ad } e_i)^{1-a_{ij}} e_j = 0 = (\text{ad } f_i)^{1-a_{ij}} f_j, \quad [e_i, f_j] = 0, \quad i \neq j, \\ [[e_i, f_i], e_j] = a_{ij} e_j, \quad [[e_i, f_i], f_j] = -a_{ij} f_j, \quad i, j \in I. \end{aligned}$$

Recall that the Yangian of \mathfrak{g}_Q , denoted by $Y_{\hbar}(\mathfrak{g}_Q)$, is an associative algebra over $\mathbb{Q}[\hbar]$, generated by the variables

$$x_{k,r}^{\pm}, h_{k,r}, \quad (k \in I, r \in \mathbb{N}),$$

subject to certain relations. A complete list of relations can be found in [7], which we refrain from recalling in this paper. Relevant to us, define $Y_{\hbar}^+(\mathfrak{g}_Q)$ to be the quotient of the free algebra on the generators $x_{k,r}^+$, for $k \in I, r \in \mathbb{N}$ by the following relations. Define the generating series $x_k^+(u) \in Y_{\hbar}^+(\mathfrak{g}_Q)[[u^{-1}]]$ by $x_k^+(u) = \hbar \sum_{r \geq 0} x_{k,r}^+ u^{-r-1}$. The following is a complete set of relations defining $Y_{\hbar}^+(\mathfrak{g}_Q)$:

$$\begin{aligned} (u - v - d_k a_{kl} \hbar/2) x_k^+(u) x_l^+(v) = (u - v + d_k a_{kl} \hbar/2) x_l^+(v) x_k^+(u) \\ + \hbar \left([x_{k,0}^+, x_l^+(v)] - [x_k^+(u), x_{l,0}^+] \right), \quad \text{for any } k, l \in I. \end{aligned} \tag{Y1}$$

$$\sum_{\sigma \in \mathfrak{S}_{1-a_{kl}}} [x_k^+(u_{\sigma(1)}), [x_k^+(u_{\sigma(2)}), [\dots, [x_k^+(u_{\sigma(1-a_{kl})}), x_l^+(v)] \dots]] = 0, \quad \text{for } k \neq l \in I. \tag{Y2}$$

Denote by $Y_{\hbar}^{sub}(\mathfrak{g}_Q)$ the subalgebra of $Y_{\hbar}(\mathfrak{g}_Q)$ generated by $x_{k,r}^+$, for $k \in I, r \in \mathbb{N}$. It is clear that there is a surjective homomorphism $Y_{\hbar}^+(\mathfrak{g}_Q) \twoheadrightarrow Y_{\hbar}^{sub}(\mathfrak{g}_Q)$, and they may differ in general.

5.2 Main Result

In this section, we compare the Yangian of non-simply laced type with the COHA $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)$ associated to quiver with symmetrizer.

Let $\widetilde{\mathcal{SH}}^{\text{sph}}$ be the spherical subalgebra of $\widetilde{\mathcal{SH}}$ generated by \mathcal{SH}_{e_k} as k varies in I .

Theorem 5.1 *The assignment*

$$Y_{\hbar}^+(\mathfrak{g}_Q) \ni x_{k,r}^+ \mapsto (\lambda^{(k)})^r \in \mathcal{SH}_{e_k} = \mathbb{C}[\hbar][\lambda^{(k)}]$$

extends to a well-defined algebra epimorphism $Y_{\hbar}^+(\mathfrak{g}_Q) \rightarrow \widetilde{\mathcal{SH}}^{\text{sph}}|_{t_1=t_2=\hbar/2, t_3=-\hbar}$.

Remark 5.2 The proof of Theorem 5.1 is similar as that of [21, Theorem 7.1], taking into account the shuffle formulas in the present setting. The main difference here, compared to the proof in [21] are the values of n, d, a from Example 2.1, the definitions of S and S' . Note also that the Serre relation (Y2) is asymmetric with respect to k and l . See Remark 5.3 for a detailed discussion. We only include the reduction of the Serre relation to a form to which [21, Corollary A2] can be directly applied.

In order to prove Theorem 5.1, we need to verify the relations Eq. Y1 and Eq. Y2 in the algebra $\widetilde{\mathcal{SH}}$. We include here the detailed computation parallel to the proof of [21, Theorem 7.1] to keep track of $D = \text{diag}(d_1, \dots, d_n)$ from the Yangian side and the symmetrizer L from the COHA side.

5.2.1 The Quadratic Relation (Y1)

We now check the relation (Y1) in the shuffle algebra. We have

$$x_k^+(u) \mapsto \hbar \sum_{r \geq 0} (\lambda^{(k)})^r u^{-r-1} = \frac{\hbar}{u - \lambda^{(k)}},$$

where the equality means the expansion of the rational function $\frac{\hbar}{u - \lambda^{(k)}}$ at $u = \infty$.

To check the quadratic relation (Y1), we need to show

$$\begin{aligned} & \left(u - v - \frac{\hbar d_k a_{kl}}{2}\right) \frac{\hbar}{u - \lambda^{(k)}} \star \frac{\hbar}{v - \lambda^{(l)}} - \left(u - v + \frac{\hbar d_k a_{kl}}{2}\right) \frac{\hbar}{v - \lambda^{(l)}} \star \frac{\hbar}{u - \lambda^{(k)}} \\ &= \hbar \left(1^{(k)} \star \frac{\hbar}{v - \lambda^{(l)}} - \frac{\hbar}{v - \lambda^{(l)}} \star 1^{(k)} - \frac{\hbar}{u - \lambda^{(k)}} \star 1^{(l)} + 1^{(l)} \star \frac{\hbar}{u - \lambda^{(k)}} \right). \end{aligned} \tag{17}$$

We first consider the case when $k \neq l$. We spell out the formula of the multiplication $\widetilde{\mathcal{SH}}_{e_k} \otimes \widetilde{\mathcal{SH}}_{e_l} \rightarrow \widetilde{\mathcal{SH}}_{e_k + e_l}$ as a map $\mathbb{C}[\hbar][\lambda^{(k)}] \otimes \mathbb{C}[\hbar][\lambda^{(l)}] \rightarrow \mathbb{C}[\hbar][\lambda^{(k)}, \lambda^{(l)}]$. Plugging-in $v_1 = e_k$, and $v_2 = e_l$ to Eq. 14, we have $\text{fac}_1 = 1$. If there is no arrow between k and l , then both sides of Eq. 17 are zero. Without loss of generality, we assume there are n arrows from k to l . As in Example 2.1, we have

$$n = |\text{gcd}(a_{kl}, a_{lk})|, \quad d = \mathbf{m}_k l_{kl} = \left| \frac{d_k a_{kl}}{\text{gcd}(a_{kl}, a_{lk})} \right|, \text{ thus } a := nd = -d_k a_{kl} = -d_l a_{lk},$$

$$\text{and the weights } \mathbf{m} \text{ are : } \mathbf{m}_{h_p} := a + 2d - 2pd \quad \mathbf{m}_{h_p^*} := -a + 2pd, \quad \text{for } p = 1, \dots, n.$$

Let S be the set of integers $\{a, a - 2d, a - 4d, \dots, -a + 4, -a + 2d\}$. Then, $S = \{\mathbf{m}_{h_p} \mid 1 \leq p \leq n\} = \{\mathbf{m}_{h_p^*} \mid 1 \leq p \leq n\}$. Set $S' := \{a - 2d, a - 4d, \dots, -a + 2d\}$, we then have

$$S' \sqcup \{a\} = S, \text{ and } S' = -S'. \tag{18}$$

Plugging the weight function \mathbf{m} into Eq. 15, we have $\widetilde{\text{fac}}_2 = \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2})$. Therefore, by the shuffle formula (16), the Hall multiplication is given by

$$(\lambda^{(k)})^p \star (\lambda^{(l)})^q = (\lambda^{(k)})^p (\lambda^{(l)})^q \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2} \right).$$

Similarly, the multiplication $\widetilde{\mathcal{S}}\mathcal{H}_{e_l} \otimes \widetilde{\mathcal{S}}\mathcal{H}_{e_k} \rightarrow \widetilde{\mathcal{S}}\mathcal{H}_{e_k+e_l}$ is given by

$$(\lambda^{(l)})^p \star (\lambda^{(k)})^q = (\lambda^{(l)})^p (\lambda^{(k)})^q \prod_{m \in S} (\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2}).$$

Plugging into Eq. 17, Eq. Y1 becomes the following identity

$$\begin{aligned} & \left(u - v - \frac{\hbar d_k a_{kl}}{2} \right) \frac{\hbar}{u - \lambda^{(k)}} \frac{\hbar}{v - \lambda^{(l)}} \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2} \right) \\ & - \left(u - v + \frac{\hbar d_k a_{kl}}{2} \right) \frac{\hbar}{v - \lambda^{(l)}} \frac{\hbar}{u - \lambda^{(k)}} \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2} \right) \\ & = \hbar \left(\frac{\hbar}{v - \lambda^{(l)}} \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2} \right) - \frac{\hbar}{v - \lambda^{(l)}} \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2} \right) \right. \\ & \quad \left. - \frac{\hbar}{u - \lambda^{(k)}} \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2} \right) + \frac{\hbar}{u - \lambda^{(k)}} \prod_{m \in S} \left(\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2} \right) \right). \end{aligned} \tag{19}$$

Using Eq. 18, we have the common factor

$$\hbar^2 \prod_{m \in S'} (\lambda^{(l)} - \lambda^{(k)} + m \frac{\hbar}{2}) = \hbar^2 \prod_{m \in S'} (\lambda^{(l)} - \lambda^{(k)} - m \frac{\hbar}{2}).$$

Canceling the above common factor, the equality (19) becomes

$$\begin{aligned} & \left(u - v + \frac{\hbar a}{2} \right) \frac{\lambda^{(l)} - \lambda^{(k)} + a \frac{\hbar}{2}}{(u - \lambda^{(k)})(v - \lambda^{(l)})} - \left(u - v - \frac{\hbar a}{2} \right) \frac{\lambda^{(l)} - \lambda^{(k)} - a \frac{\hbar}{2}}{(v - \lambda^{(l)})(u - \lambda^{(k)})} \\ & = a \hbar \left(\frac{1}{v - \lambda^{(l)}} - \frac{1}{u - \lambda^{(k)}} \right). \end{aligned}$$

Both sides of the above identity are equal to $a \hbar \frac{u-v+\lambda^{(l)}-\lambda^{(k)}}{(u-\lambda^{(k)})(v-\lambda^{(l)})}$. This shows the relation (Y1) for the case when $k \neq l$.

We now check the relation (Y1) when $k = l$. A similar calculation using the shuffle formula (16) shows that Eq. 17 becomes the following identity in $\widetilde{\mathcal{S}}\mathcal{H}_{2e_k} = \mathbb{C}[\hbar][\lambda_1, \lambda_2]$

$$\begin{aligned} & (u - v - d_k \hbar) \sum_{\sigma \in \mathfrak{S}_2} \sigma \left(\frac{\hbar}{u - \lambda_1} \frac{\hbar}{v - \lambda_2} \frac{\lambda_1 - \lambda_2 + d_k \hbar}{\lambda_1 - \lambda_2} \right) \\ & - (u - v + d_k \hbar) \sum_{\sigma \in \mathfrak{S}_2} \sigma \left(\frac{\hbar}{v - \lambda_1} \frac{\hbar}{u - \lambda_2} \frac{\lambda_1 - \lambda_2 + d_k \hbar}{\lambda_1 - \lambda_2} \right) \\ & = \hbar \sum_{\sigma \in \mathfrak{S}_2} \sigma \left(\left(\frac{\hbar}{v - \lambda_2} - \frac{\hbar}{v - \lambda_1} - \frac{\hbar}{u - \lambda_1} + \frac{\hbar}{u - \lambda_2} \right) \frac{\lambda_1 - \lambda_2 + d_k \hbar}{\lambda_1 - \lambda_2} \right). \end{aligned}$$

It is straightforward to show that both sides of the above identity can be simplified to

$$\frac{2d_k \hbar^3}{\lambda_1 - \lambda_2} \left(\frac{1}{v - \lambda_2} - \frac{1}{v - \lambda_1} - \frac{1}{u - \lambda_1} + \frac{1}{u - \lambda_2} \right).$$

This completes the proof of relation (Y1) for $k = l$.

5.2.2 The Serre Relation (Y2)

By an argument similar to [13, § 10.4], to show Eq. Y2, without loss of generality it suffices to check

$$\sum_{p=0}^{1-akl} (-1)^p \binom{1-akl}{p} x_{k,0}^{*p} \star x_{l,0} \star x_{k,0}^{*(1-akl-p)} = 0, \tag{Y2'}$$

where $x^{*n} = x \star x \star \dots \star x$, the shuffle product of n -copies of x . We use the shuffle formula (16) to check the Serre relation (Y2').

For any i, j , let $\lambda_{i,j} = \lambda_i - \lambda_j$. By the shuffle formula (16), we have the recurrence relation

$$(x_{k,0})^{*n+1} = \sum_{\sigma \in \text{Sh}(n,1)} \sigma \left((x_{k,0})^{*n} \prod_{i=1}^n \frac{\lambda_{i,n+1}^{(k)} + d_k \hbar}{\lambda_{i,n+1}^{(k)}} \right).$$

Therefore, inductively, we get a formula of $(x_{k,0})^{*n}$:

$$(x_{k,0})^{*n} = \sum_{\sigma \in \mathfrak{S}_n} \sigma \left(\frac{\lambda_{12}^{(k)} + d_k \hbar}{\lambda_{12}^{(k)}} \cdot \frac{\lambda_{13}^{(k)} + d_k \hbar}{\lambda_{13}^{(k)}} \dots \frac{\lambda_{n-1,n}^{(k)} + d_k \hbar}{\lambda_{n-1,n}^{(k)}} \right). \tag{20}$$

Note that $k \neq l$. By the shuffle formula (16), the multiplication $\widetilde{\mathcal{SH}}_{ne_k} \otimes \widetilde{\mathcal{SH}}_{el} \rightarrow \widetilde{\mathcal{SH}}_{ne_k+el}$ is given by

$$(x_{k,0})^{*n} \star x_{l,0} = (x_{k,0})^{*n} \prod_{i=1}^n \prod_{m \in S} \left(\lambda^{(l)} - \lambda_i^{(k)} + m \frac{\hbar}{2} \right), \tag{21}$$

where $S = \{a, a - 2d, a - 4d, \dots, -a \pm 4d, -a \pm 2d\}$. Here again we write $a = -d_k a_{kl}$.

For the multiplication $\widetilde{\mathcal{SH}}_{pe_k+el} \otimes \widetilde{\mathcal{SH}}_{qe_k} \rightarrow \widetilde{\mathcal{SH}}_{e_l+(p+q)e_k}$ considered as a map

$$\mathbb{C}[\hbar][\lambda_1^{(k)}, \dots, \lambda_p^{(k)}, \lambda_1^{(l)}] \otimes \mathbb{C}[\hbar][\lambda_{p+1}^{(k)}, \dots, \lambda_{p+q}^{(k)}] \rightarrow \mathbb{C}[\hbar][\lambda_1^{(k)}, \dots, \lambda_{p+q}^{(k)}, \lambda_1^{(l)}],$$

we have

$$\begin{aligned} ((x_{k,0})^{*p} \star x_{l,0}) \star (x_{k,0})^{*q} &= \sum_{\sigma \in \text{Sh}(p,q)} \sigma \left((x_{k,0})^{*p} \star x_{l,0} \cdot (x_{k,0})^{*q} \right. \\ &\quad \cdot \prod_{s=1}^p \prod_{t=p+1}^{p+q} \frac{\lambda_s^{(k)} - \lambda_t^{(k)} + d_k \hbar}{\lambda_s^{(k)} - \lambda_t^{(k)}} \cdot \left. \prod_{t=p+1}^{p+q} \prod_{m \in S} \left(\lambda^{(l)} - \lambda_t^{(k)} - m \frac{\hbar}{2} \right) \right). \end{aligned} \tag{22}$$

Plugging the formulas of Eqs. 20, 21 into Eq. 22 with $q = 1 - a_{kl} - p$, we get

$$\begin{aligned} x_{k,0}^{*p} \star x_{l,0} \star x_{k,0}^{*(1-akl-p)} &= \sum_{\pi \in \text{Sh}(p,1-akl-p)} \pi \left(\left(\sum_{\sigma \in \mathfrak{S}_p} \sigma \cdot \prod_{1 \leq i < j \leq p} \frac{\lambda_{i,j}^{(k)} + d_k \hbar}{\lambda_{i,j}^{(k)}} \right) \right. \\ &\quad \cdot \left(\sum_{\sigma \in \mathfrak{S}_{1-akl-p}} \sigma \cdot \prod_{\{p+1 \leq i < j \leq 1-akl\}} \frac{\lambda_{i,j}^{(k)} + d_k \hbar}{\lambda_{i,j}^{(k)}} \right) \left(\prod_{s=1}^p \prod_{t=p+1}^{1-akl} \frac{\lambda_{s,t}^{(k)} + d_k \hbar}{\lambda_{s,t}^{(k)}} \right) \\ &\quad \cdot \left. \prod_{m \in S} \left(\prod_{i=1}^p \left(\lambda^{(l)} - \lambda_i^{(k)} + m \frac{\hbar}{2} \right) \prod_{t=p+1}^{1-akl} \left(\lambda^{(l)} - \lambda_t^{(k)} - m \frac{\hbar}{2} \right) \right) \right). \end{aligned}$$

Re-arranging the above summation, we have:

$$\begin{aligned} & \sum_{p=0}^{1-a_{kl}} (-1)^p \binom{1-a_{kl}}{p} x_{k,0}^{\star p} \star x_{l,0} \star x_{k,0}^{\star(1-a_{kl}-p)} = \sum_{p=0}^{1-a_{kl}} (-1)^p \binom{1-a_{kl}}{p} \\ & \times \sum_{\sigma \in \mathfrak{S}_{1-a_{kl}}} \left(\prod_{1 \leq i < j \leq 1-a_{kl}} \frac{\lambda_{\sigma(i),\sigma(j)}^{(k)} + d_k \hbar}{\lambda_{\sigma(i),\sigma(j)}^{(k)}} \right. \\ & \cdot \left. \prod_{m \in S} \left(\prod_{i=1}^p \left(\lambda^{(l)} - \lambda_{\sigma(i)}^{(k)} + m \frac{\hbar}{2} \right) \prod_{t=p+1}^{1-a_{kl}} \left(\lambda^{(l)} - \lambda_{\sigma(t)}^{(k)} - m \frac{\hbar}{2} \right) \right) \right). \end{aligned}$$

Note that the factor

$$\prod_{m \in S'} \left(\prod_{i=1}^p \left(\lambda^{(l)} - \lambda_{\sigma(i)}^{(k)} + m \frac{\hbar}{2} \right) \prod_{t=p+1}^{1-a_{kl}} \left(\lambda^{(l)} - \lambda_{\sigma(t)}^{(k)} - m \frac{\hbar}{2} \right) \right) = \prod_{m \in S'} \prod_{i=1}^{1-a_{kl}} \left(\lambda^{(l)} - \lambda_i^{(k)} - m \frac{\hbar}{2} \right),$$

is independent of $\sigma \in \mathfrak{S}_{1-a_{kl}}$, hence a common factor. Here again $S' = \{a - 2d, a - 4d, \dots, -a + 2d\}$. Let $\lambda_i^{(k)} = \lambda_i^{(k)} - \lambda^{(l)}$. After canceling the above common factor, to show the Serre relation (Y2'), it suffices to show

$$\sum_{p=0}^{1-a_{kl}} (-1)^p \binom{1-a_{kl}}{p} \sum_{\sigma \in \mathfrak{S}_{1-a_{kl}}} \sigma \left(\prod_{s=1}^p \left(\lambda_s^{(k)} - \frac{a\hbar}{2} \right) \prod_{t=p+1}^{1-a_{kl}} \left(\lambda_t^{(k)} + \frac{a\hbar}{2} \right) \prod_{1 \leq i < j \leq 1-a_{kl}} \frac{\lambda_{i,j}^{(k)} + d_k \hbar}{\lambda_{i,j}^{(k)}} \right) = 0. \tag{23}$$

The identity (23) is [21, Corollary A2]. This proves Eq. Y2.

Remark 5.3 Note that for non-simply laced Cartan matrix A , when $a_{kl} \neq a_{lk}$, the Serre relation (Y2) is asymmetric switching k and l . This is reflected in the fact that in Eq. 23, the last factor $\prod_{1 \leq i < j \leq 1-a_{kl}} \frac{\lambda_{i,j}^{(k)} + d_k \hbar}{\lambda_{i,j}^{(k)}}$ is different switching k and l . In addition, although the number a is the same, the decompositions $a = -a_{kl}d_k$ and $a = -a_{lk}d_l$ used in the proof above are different switching k and l .

As a consequence of Theorems 3.2 and 5.1, we obtain an algebra epimorphism

$$\Phi : Y_{\hbar}^+(\mathfrak{g}_{A,D}) \rightarrow (\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)/\text{tors})^{\text{sph}} \cong \widetilde{\mathcal{SH}}^{\text{sph}},$$

where tors means the torsion elements of $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)$ which are defined to be elements in the kernel of the morphism $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L) \rightarrow \mathcal{SH}$. Assume the quiver Q is of finite type and simply laced, it follows from the work of Schiffmann and Vasserot [18, Theorem A (c)] that $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)_v$ is torsion free over $H_{\mathcal{D} \times \text{GL}_v}^{BM}(\text{pt})$. The Yangian action on the cohomology of quiver varieties is faithful [21, Lemma 8.1]. The argument in the proof of [21, Theorem 8.3] shows that the morphism Φ is an isomorphism. For general quiver Q , after tensoring the fractional field K of $H_{\mathcal{D}}^{BM}(\text{pt})$, we have $\mathcal{H}_{\mathcal{D}}(\widehat{Q}, W^L)_v \otimes_{H_{\mathcal{D}}^{BM}(\text{pt})} K$ is torsion free over $H_{\mathcal{D} \times \text{GL}_v}^{BM}(\text{pt}) \otimes_{H_{\mathcal{D}}^{BM}(\text{pt})} K$ [18, Theorem B (d)]. The comparison of the Yangian of Maulik and Okounkov and COHA can be found in [19].

We do not know if either statement holds in general. To investigate these questions, we need to consider a faithful representation of the Yangian, which is expected to be obtained from the cohomology of Nakajima quiver varieties in the present setting, generalizing the construction of [20]. In the present paper, we do not consider frames or stability conditions of representations of the quiver with symmetrizer, hence do not address these questions.

Appendix A: Review of Dimensional Reduction

We review a dimensional reduction procedure that describes the cohomological Hall algebra in the presence of a cut. Such a dimensional reduction on the level of cohomology groups is obtained by Davison [4, Appendix A], which is the main ingredient here.

In this section we study the behaviour of the algebra structure under the dimensional reduction, which is needed in the present paper. The argument here is similar to that in the proof of [22, Theorem 2.5]. However, the statement here is more general and more useful (see e.g., [16, Section 7.1]). We include a brief sketch of the argument in present generality, highlighting the difference to that in [22, Theorem 2.5].

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, and W be the potential. In general, a cut C of (Γ, W) is a subset $C \subset \Gamma_1$ such that W is homogeneous of degree 1 with respect to the grading defined on arrows by

$$\text{deg } a = \begin{cases} 1 & : a \in C, \\ 0 & : a \notin C. \end{cases}$$

Consider the quotient of the path algebra $\mathbb{C}(\Gamma \setminus C)$ by the relations $\{\partial W / \partial a \mid a \in C\}$. The representation variety of this quotient algebra is denoted by

$$\mathbf{J}_{\Gamma \setminus C, v} := \{x \in \mathbf{M}_{\Gamma \setminus C, v} \mid \partial W / \partial a(x) = 0, \forall a \in C\}.$$

We view C as forming the edges of a new quiver, which we still denote by C for simplicity. Let $\mathbf{M}_{C, v}$ be the representation variety of C with dimension vector v . Consider the trivial vector bundle $\pi : \mathbf{M}_{\Gamma, v} = \mathbf{M}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v} \rightarrow \mathbf{M}_{\Gamma \setminus C, v}$ carrying a scaling \mathbb{G}_m action of weight one on the fiber $\mathbf{M}_{C, v}$. Let $\text{tr}W_v : \mathbf{M}_{\Gamma, v} = \mathbf{M}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v} \rightarrow \mathbb{A}^1$ be the function which is \mathbb{G}_m -equivariant. Define $Z \subset \mathbf{M}_{\Gamma \setminus C, v}$ to be the reduced scheme consisting of points $z \in \mathbf{M}_{\Gamma \setminus C, v}$, such that $\pi^{-1}(z) \subset (\text{tr}W_v)^{-1}(0)$. Then, we have $Z = \{x \in \mathbf{M}_{\Gamma \setminus C, v} \mid \text{tr}(W_v)(x, l) = 0, \forall l \in \mathbf{M}_{C, v}\}$. To summarize the notations, we have the diagram:

$$\begin{array}{ccccc} Z \times \mathbf{M}_{C, v} & \xrightarrow{i \times \text{id}} & \mathbf{M}_{\Gamma \setminus C, v} \times \mathbf{M}_{\Gamma \setminus C, v} & & \\ \downarrow \pi_Z & & \downarrow \pi & \searrow & \\ Z & \xrightarrow{i} & \mathbf{M}_{\Gamma \setminus C, v} & \xrightarrow{p} & \text{pt.} \end{array}$$

Lemma A.1 *The subvariety Z of $\mathbf{M}_{\Gamma \setminus C, v}$ is naturally identified with $\mathbf{J}_{\Gamma \setminus C, v}$.*

Proof The lemma follows from the same proof of [22, Lemma 3.1]. The difference in the current setting is the non-degenerate pairing

$$\text{tr}(- \cdot -) : \text{Hom}(\mathbb{C}^{v_{\text{in}}(a)}, \mathbb{C}^{v_{\text{out}}(a)}) \times \text{Hom}(\mathbb{C}^{v_{\text{out}}(a)}, \mathbb{C}^{v_{\text{in}}(a)}) \rightarrow \mathbb{C}.$$

given by the trace. □

Let \mathcal{D} be a torus \mathbb{G}_m^r for some $r \in \mathbb{N}$. To each arrow in Γ_1 , we associate a \mathcal{D} -weight such that $\text{tr}W_v$ is \mathcal{D} -invariant for any v . Then, $\partial W / \partial a$ is homogeneous for any $a \in C$. In particular, $\mathbf{J}_{\Gamma \setminus C, v}$ is a \mathcal{D} -equivariant subvariety of $\mathbf{M}_{\Gamma \setminus C, v}$.

There is a canonical isomorphism of vector spaces [4, Theorem A.1]

$$H_{C, G_v \times \mathcal{D}}^*(\mathbf{M}_{\Gamma, v}, \varphi_{\text{tr}W_v})^\vee \cong H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}, \mathbb{Q}), \text{ for } v \in \mathbb{N}^{\Gamma_0},$$

Following [22], we now describe a multiplication $m^{\mathbf{J}}$ on the graded vector space

$$\bigoplus_{v \in \mathbb{N}^{\Gamma_0}} H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}, \mathbb{Q}).$$

Let $v_1, v_2 \in \mathbb{N}_0^\Gamma$ be dimension vectors such that $v = v_1 + v_2$, let $V_1 \subset V$ be a $|\Gamma_0|$ -tuple of subspaces of V with dimension vector v_1 . Define $\mathbf{M}_{\Gamma, v_1, v_2} := \{x \in \mathbf{M}_{\Gamma, v} \mid x(V_1) \subset V_1\}$. We write $G := G_v \times \mathcal{D}$ for short. Let $P \subset G_v \times \mathcal{D}$ be the parabolic subgroup preserving the subspace V_1 and $L := G_{v_1} \times G_{v_2} \times \mathcal{D}$ be the Levi subgroup of P . We have the following correspondence of L -varieties.

$$\mathbf{M}_{\Gamma, v_1} \times \mathbf{M}_{\Gamma, v_2} \xleftarrow{p_\Gamma} \mathbf{M}_{\Gamma, v_1, v_2} \xrightarrow{\eta_\Gamma} \mathbf{M}_{\Gamma, v_1 + v_2}, \tag{24}$$

where p_Γ is the natural projection and η_Γ is the embedding.

We have the following commutative diagram

$$\begin{CD} \mathbf{M}_{\Gamma, v_1} \times \mathbf{M}_{\Gamma, v_2} @<p_1<< (\mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2}) \times \mathbf{M}_{C, v_1, v_2} @>i_1>> (\mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2}) \times \mathbf{M}_{C, v} \\ @VVV @VVV @VVV \\ (\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2}) \times (\mathbf{M}_{C, v_1} \times \mathbf{M}_{C, v_2}) @<\bar{p}_1<< (\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2}) \times \mathbf{M}_{C, v_1, v_2} @>\bar{i}_1>> (\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2}) \times \mathbf{M}_{C, v} \end{CD} \tag{25}$$

where $p_1 = \text{id}_{\mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2}} \times p_C$, and $i_1 = \text{id}_{\mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2}} \times \eta_C$. Here p_C, η_C are maps in the correspondence $\mathbf{M}_{C, v_1} \times \mathbf{M}_{C, v_2} \xleftarrow{p_C} \mathbf{M}_{C, v_1, v_2} \xrightarrow{\eta_C} \mathbf{M}_{C, v}$. The vertical maps are natural inclusions, and \bar{p}_1, \bar{i}_1 are the restrictions of p_1, i_1 .

Identify $\mathbf{M}_{C^{\text{op}}, v}$ with $\mathbf{M}_{C, v}^*$ via $\mathbf{M}_{C^{\text{op}}, v} \cong \mathbf{M}_{C, v}^*, x \mapsto (y \mapsto \text{tr}(xy))$. For $x \in \mathbf{M}_{\Gamma \setminus C, v}$, the cyclic derivative $\partial W / \partial a(x)$ is an element in $\mathbf{M}_{C^{\text{op}}, v}$, for any $a \in C$. Thus, for any $l \in \mathbf{M}_{C, v}$, we have the pairing $(\partial W / \partial a(x), l) = \text{tr}(\partial W / \partial a(x) \cdot l)$.

Recall that $p_{C^{\text{op}}} : \mathbf{M}_{C^{\text{op}}, v_1, v_2} \rightarrow \mathbf{M}_{C^{\text{op}}, v_1} \times \mathbf{M}_{C^{\text{op}}, v_2}$ is the natural projection. Introduce the following subvariety $\mathbf{Y} \subset \mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2} \times \mathbf{M}_{C^{\text{op}}, v_1, v_2}$.

$$\mathbf{Y} := \{(x, l) \mid x \in \mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2}, l \in \mathbf{M}_{C^{\text{op}}, v_1, v_2}, \text{ such that } (\partial W / \partial a)_{a \in C}(x) = p_{C^{\text{op}}}(l)\}. \tag{26}$$

There are two maps

$$\begin{aligned} \iota : \mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2} &\hookrightarrow \mathbf{Y}, \text{ given by } x \mapsto (x, (\partial W / \partial a)_{a \in C}(x)) \\ \omega : \mathbf{M}_{\Gamma \setminus C, v_1, v_2} &\rightarrow \mathbf{Y}, \text{ given by } x \mapsto (p_{\Gamma \setminus C}(x), (\partial W / \partial a)_{a \in C}(x)). \end{aligned}$$

Let $\mathbf{J}_{\Gamma \setminus C, v_1, v_2} \subset \mathbf{M}_{\Gamma \setminus C, v_1, v_2}$ be the subvariety defined by the equation $(\partial W / \partial a)_{a \in C}(x)$, for all $a \in C$. We then have an embedding $\bar{i}_2 : \mathbf{J}_{\Gamma \setminus C, v_1, v_2} \subset \mathbf{J}_{\Gamma \setminus C, v}$. They fit into the following commutative diagram.

$$\begin{CD} \mathbf{M}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{\Gamma \setminus C, v_2} @>\iota>> \mathbf{Y} @<\omega<< \mathbf{M}_{\Gamma \setminus C, v_1, v_2} @>i_2>> \mathbf{M}_{\Gamma \setminus C, v} \\ @VVV @VVV @VVV @VVV \\ \mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2} @<\bar{\omega}<< \mathbf{J}_{\Gamma \setminus C, v_1, v_2} @>\bar{i}_2>> \mathbf{J}_{\Gamma \setminus C, v} \end{CD} \tag{27}$$

where the map $\bar{\omega}$ is the restriction of ω . Note that by introducing the variety \mathbf{Y} , the pullback of the two maps $\omega : \mathbf{M}_{\Gamma \setminus C, v_1, v_2} \rightarrow \mathbf{Y}$ and $\bar{\iota} : \mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2} \rightarrow \mathbf{Y}$ is $\mathbf{J}_{\Gamma \setminus C, v_1, v_2}$. In other words, the square in the diagram (27) formed by $\omega, \bar{\iota}, \bar{\omega}$ is Cartesian.

The multiplication $m^{\mathbf{J}}$ is defined to be

$$m^{\mathbf{J}} := (\bar{i}_2 \times \text{id}_{\mathbf{M}_{C,v}})_* \circ \frac{1}{e(\iota)} (\omega \times \text{id}_{\mathbf{M}_{C,v}})_{\bar{\omega} \times \text{id}_{\mathbf{M}_{C,v}}}^{\sharp} \circ \bar{i}_{1*} \circ \bar{p}_1^*. \tag{28}$$

The maps in the composition are the following.

- (1) The Künneth morphism $H_{G_{v_1} \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{M}_{C, v_1}) \otimes H_{G_{v_2} \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_2} \times \mathbf{M}_{C, v_2}) \rightarrow H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2} \times \mathbf{M}_{C, v_1} \times \mathbf{M}_{C, v_2})$. Here the tensor is over $H_{\mathcal{D}}^{\text{BM}}(\text{pt})$.
- (2) $\bar{i}_{1*} \circ \bar{p}_1^* : H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2} \times \mathbf{M}_{C, v_1} \times \mathbf{M}_{C, v_2}) \rightarrow H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2} \times \mathbf{M}_{C, v})$.
- (3) Denote by $(\omega \times \text{id}_{\mathbf{M}_{C,v}})_{\bar{\omega} \times \text{id}_{\mathbf{M}_{C,v}}}^{\sharp}$ the refined Gysin pullback of $\omega \times \text{id}_{\mathbf{M}_{C,v}}$ along $\bar{\omega} \times \text{id}_{\mathbf{M}_{C,v}}$. Let $e(\iota)$ be the L -equivariant Euler class of the normal bundle of ι . We have the following map

$$\frac{1}{e(\iota)} \omega_{\bar{\omega}}^{\sharp} : H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1} \times \mathbf{J}_{\Gamma \setminus C, v_2} \times \mathbf{M}_{C, v}) \rightarrow H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1, v_2} \times \mathbf{M}_{C, v}) \left[\frac{1}{e(\iota)} \right].$$

- (4) The pushforward $(\bar{i}_2 \times \text{id}_{\mathbf{M}_{C,v}})_* : H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v_1, v_2} \times \mathbf{M}_{C, v}) \rightarrow H_L^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v})$.
- (5) Pushforward along $G \times P (\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}) \rightarrow \mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}$, $(g, m) \mapsto gmg^{-1}$, we get $H_P^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}) \cong H_G^{\text{BM}}(G \times P (\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v})) \rightarrow H_G^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v})$.

This map $m^{\mathbf{J}}$ a priori is only defined after inverting $e(\iota)$. However, it follows from Theorem A.2 that it is well-defined before localization.

Theorem A.2 *There is an isomorphism of algebras*

$$\mathcal{H}_{\mathcal{D}}(\Gamma, W) \cong \bigoplus_{v \in \mathbb{N}^{\Gamma_0}} H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}, \mathbb{Q}),$$

where $\mathcal{H}_{\mathcal{D}}(\Gamma, W)$ endowed with the Hall multiplication of Kontsevich-Soibelman and $\bigoplus_{v \in \mathbb{N}^{\Gamma_0}} H_{G_v \times \mathcal{D}}^{\text{BM}}(\mathbf{J}_{\Gamma \setminus C, v} \times \mathbf{M}_{C, v}, \mathbb{Q})$ has multiplication given by $m^{\mathbf{J}}$.

As has been mentioned, a special case of this is [22, Theorem 2.5] and [17, Appendix, Corollary 4.5]. The proof of [22, § 2] goes through in the setting verbatim, with the following substitutions.

- (1) We need to distinguish between C and C^{op} in the present paper, where in [22], C consists of edge loops and hence $C = C^{\text{op}}$.
- (2) Consequently we need to identify $\mathbf{M}_{C^{\text{op}}, v}$ with $\mathbf{M}_{C, v}^*$ using the trace map.
- (3) The variety \mathbf{Y} is changed to Eq. 26.
- (4) The [22, Lemma 3.1] is replaced by Lemma A.1.
- (5) The torus \mathcal{D} -action in the present generality is given in Section 2.2.

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Declarations

Conflict of Interests The authors have no conflicts of interest to declare that are relevant to the content of this article.

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