Author/s:
Nesic, D;Nguyen, T;Tan, Y;Manzie, C

Title:
On a Shubert Algorithm-Based Global Extremum Seeking Scheme

Date:
2012

Citation:
Nesic, D., Nguyen, T., Tan, Y. & Manzie, C. (2012). On a Shubert Algorithm-Based Global Extremum Seeking Scheme. Proceedings of the 51st IEEE Annual Conference on Decision and Control (CDC), pp.1615-1620. IEEE. https://doi.org/10.1109/CDC.2012.6426507.

Persistent Link:
https://hdl.handle.net/11343/299766

51st IEEE Conference on Decision and Control
December 10-13, 2012. Maui, Hawaii, USA

On a Shubert Algorithm-Based Global Extremum Seeking Scheme

D. Nešić, T. Nguyen, Y. Tan and C. Manzie

Abstract— **This paper adapts the so-called Shubert algorithm for Extremum Seeking Control (ESC) to seek the global extremum (in presence of local extrema) of general dynamic plants. Different from *derivative based* methods that are widely used in ESC, the Shubert algorithm is a good representative of sampling optimization methods. With knowledge of the Lipschitz constant of an unknown static mapping, this deterministic algorithm seeks the global extremum. By introducing "waiting time" the proposed Shubert algorithm-based global extremum seeking guarantees the semi-global practical convergence (in the initial states) to the *global extremum* if compact sets of inputs are considered. Several numerical examples demonstrate how proposed method may be successfully deployed.**

I. INTRODUCTION

Many industrial optimization problems such as maximizing production in chemical and petrochemical facilities benefit immensely from a global optimum. Most extremum seeking controllers in the literature are based on optimization algorithms that require the derivatives of an unknown steady-state input-output map to be estimated online, see, for example, [8], [10], [20], [7], [1], [5], [12], [4], [14], [17], [21], [3], [11] and reference therein. Although under some conditions, the derivative-based global extremum-seeking is possible [19], these algorithms typically find local extrema only, and are not suitable in general for global optimization.

On the other hand, deterministic optimization methods based on *sampling techniques* may not require the derivatives of the map and can find a global extremum on a compact set even in the presence of local extrema, see [16], [15], [6]. These algorithms work well for static mappings, to the best of authors' knowledge, these algorithms have not been applied to dynamic systems.

In order to show how to apply these sampling based global optimization algorithms to dynamic systems, this paper focuses on one specific algorithm. The Shubert algorithm can be applied to nonlinear static mappings satisfying a Lipschitz condition when the input is within a compact set. The convergence of the Shubert algorithm was proven in [15]. Similar to [20], the tuning parameter "waiting time" is introduced in the proposed scheme for dynamic systems. That is, for each input computed from the Shubert algorithm, the controller enforces constant inputs for sufficiently long time (longer than this waiting time) until the output of the system settles down to a small neighborhood of the steady-state reference-to-output map. This output is then used to calculate the new value of the plant input. Intuitively, if the Shubert algorithm is robust to small measurement errors (due to waiting time), the proposed Shubert algorithm-based global extremum seeking can work on dynamic plants.

Our main result shows that the Shubert algorithm indeed is robust in this sense. Moreover, by selecting the waiting time properly, it is shown that the closed-loop system with the proposed algorithm yields convergence to an arbitrarily small neighborhood of the global extremum from a set of initial conditions within the given compact set.

This paper is organized as follows. Section 2 revisits the Shubert algorithm for static mappings. In Section 3, the Shubert algorithm-based global extremum seeking is proposed for dynamic plants. Section 4 provides some interesting numerical examples to show how the proposed global extremum seeking algorithm works. Conclusions are given in Section 5.

For convenience, the following notation is used. The set of real numbers is denoted by $\mathbb{R}$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is continuous, zero at zero and strictly increasing and for each $s \geq 0$ the function $\beta(s, \cdot)$ is strictly decreasing to zero. For any point $u \in \mathbb{R}$ and any set $\mathcal{A} \in \mathbb{R}$, the distance of this point to set $\mathcal{A}$ is defined as $|u|_{\mathcal{A}} = \inf_{z \in \mathcal{A}} |z - u|$.

II. REVISION OF THE SHUBERT ALGORITHM

First, we present the main assumption and an algorithm from [15] that can be used for global optimization of single-input-single-output (SISO) static systems. The algorithm uses *the Lipschitz constant* of the model and the output measurements to construct a sequence of inputs that converge to the global extremum. Thus this algorithm is ideally suited for online optimization such as extremum seeking. It is also worthwhile to highlight that the Shubert algorithm does not require derivatives of the map $Q(\cdot)$. Moreover, the algorithm finds *global extremum* on the compact set $[a, b]$ in presence of local extrema.

In order to apply the Shubert algorithms to dynamic systems, the problem can be re-formulated as an online optimization (extremum seeking) problem for the discrete-time static SISO system (1). Results of [15] directly apply to static SISO plants (1) and they are recalled in this section.

Consider the model of a discrete-time static SISO system.

$$y_k = Q(u_k) , \qquad k = 1, 2, \ldots, \tag{1}$$

This research was supported by the Australia Research Council under the Discovery Project DP0985388

D. Nešić and Y. Tan are with Department of Electrical and Electronic Engineering, The University of Melbourne, VIC 3010, Australia. d.nesic@ee.unimelb.edu.au; yingt@unimelb.edu.au.

T. Nguyen is with Department of Engineering University of Exeter, EX4 4QF, UK. T.Nguyen-Tien@exeter.ac.uk.

C. Manzie is with Department of Mechanical Engineering, The University of Melbourne, VIC 3010, Australia. manziec@unimelb.edu.au.

978-1-4673-2066-5/12/$31.00 ©2012 IEEE

where $y \in \mathbb{R}$ and $u \in [a, b]$ are respectively the output and input of the system. It is noted that the system is considered for $k = 1, 2, \ldots$ instead of $k = 0, 1, 2, \ldots$ as done in [15]; this is done to have consistent notation with the next section where the former index set more naturally arises when considering dynamic plants.

Consider the system (1) on a compact interval $[a, b]$ with $a < b$; that is, $Q : [a, b] \to \mathbb{R}$. Without loss of generality we concentrate on finding the maximum of $Q(\cdot)$; indeed, finding a minimum of a function $\widetilde{Q}(\cdot)$ can be done by defining $Q(\cdot) := -\widetilde{Q}(\cdot)$ and then finding a maximum of $Q(\cdot)$.

The following assumption is needed for implementing Shubert algorithm

Assumption 1: The *unknown* mapping $Q(\cdot)$ satisfies the Lipschitz condition with a *known* positive constant L:

$$|Q(u_1) - Q(u_2)| \leq L|u_1 - u_2|, \quad \forall u_1, u_2 \in [a, b]. \tag{2}$$

Remark 1: Assumption 1 is a weak assumption as many functions encountered in engineering applications satisfy this condition. The knowledge of Lipschitz constant may not be available for all possible applications. In practice, the Lipschitz constant of $Q(\cdot)$ may need to be estimated. A more conservative estimate of L naturally leads to slower convergence of the algorithm as pointed out in [15]. ◦

Note that $Q(\cdot)$ attains a maximum on the compact interval $[a, b]$ since it is Lipschitz and, therefore, continuous. The maximum value of $Q(\cdot)$ is denoted as

$$y^* := \max_{u \in [a,b]} Q(u) \tag{3}$$

and the set of all u for which the global maximum is attained is defined as

$$\Phi := \{u \in [a, b] : Q(u) = y^*\} . \tag{4}$$

The Shubert algorithm generates a sequence of input points $u_1, u_2, \ldots$ within a closed interval $[a, b]$ by using measurements $y_1, y_2, \ldots$ as follows.

Shubert algorithm:

- Arbitrarily choose an initial input $u_1 \in [a, b]$. Usually, the initial point is selected as $u_1 = \frac{a+b}{2}$.
- Find the next u_{k+1} such that the following equation is satisfied:

$$F_k(u_{k+1}) = M_k, \quad k = 1, 2, \ldots, \tag{5}$$

where

$$F_k(u) := \min_{j=1,\ldots,k} \{y_j + L|u - u_j|\}, \tag{6}$$

$$M_k := \max_{u \in [a,b]} F_k(u). \tag{7}$$

As discussed in [15], the solution of (5) does exist and it is not unique. However, the different choices of solution from (5) will not affect the convergence properties of Shubert algorithm.

The sequence of functions $F_k(u)$ and numbers M_k for $k = 1, 2, \ldots$ possess properties that play a key role in the convergence analysis of the Shubert algorithm, and lead to the following theorem taken from [15]:

Theorem 1: Suppose that Assumption 1 holds and that the input sequence for system (1) is generated by the Shubert algorithm. Then the following holds

$$\lim_{k \to \infty} y_k = y^* .$$

□

Remark 2: It was shown in [15] that a sequence $\hat{y}_k^*$ is generated to estimate of the maximum y^*:

$$\hat{y}_k^* := \max_{j=1,\ldots,k} \{y_j\}. \tag{8}$$

Hence, Theorem 1 implies that $\hat{y}_k^*$ also converges to y^*. Moreover, the rate of convergence of $y^* - \hat{y}_k^*$ is of order $O\left(\frac{1}{k}\right)$ for all Lipschitz functions satisfying (2). ◦

Remark 3: Note that the main result in [15] also shows that $\lim_{k \to \infty} M_k = y^*$ and $\lim_{k \to \infty} |u_k|_\Phi = 0$. For simplicity, we only state the convergence properties of y_k in Theorem 1 and in our main result (Theorem 2) which ensure an appropriate convergence to the global maximum y^*. ◦

III. Shubert algorithm-based global extremum seeking

The following single-input single-output (SISO) dynamic plant is considered

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, u), \quad \mathbf{x}(0) = \mathbf{x}_0, \tag{9}$$

$$y = h(\mathbf{x}), \tag{10}$$

where $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz in each argument. Assume that $u \in [a, b]$.

In order to prove the main results, we will assume that for each constant $u \in [a, b]$ there exists an equilibrium for (9) that is globally asymptotically stable. More precisely, we use the following:

Assumption 2: There exists a locally Lipschitz function $\ell : [a, b] \to \mathbb{R}^n$ such that

$$f(\ell(u), u) = 0, \qquad \forall u \in [a, b]. \tag{11}$$

Moreover, there exists $\beta \in \mathcal{KL}$ such that for any $u \in [a, b]$ and $\mathbf{x}_0 \in \mathbb{R}^n$, the following inequality holds:

$$|\mathbf{x}(t, \mathbf{x}_0, u) - \ell(u)| \leq \beta\left(|\mathbf{x}_0 - \ell(u)|, t\right), \tag{12}$$

for all $t \geq 0$. ◦

We denote

$$Q(\cdot) := h \circ \ell(\cdot) \tag{13}$$

as the steady-state input-to-output map of (9), (10). The same assumptions for $Q(\cdot)$ in (13) are used as in the previous section. Note that the domain of $Q(\cdot)$ is $[a, b]$, which is compact, and hence the function $Q(\cdot)$ achieves a maximum on the interval $[a, b]$. Moreover, note that $Q(\cdot)$ is locally Lipschitz since $h(\cdot)$ and $\ell(\cdot)$ are assumed to be locally Lipschitz. Hence, the first part of Assumption 1 holds for $Q(\cdot)$ given by (13). However, as we also need to know the Lipschitz constant L of $Q(\cdot)$, we will still explicitly state that Assumption 1 holds for $Q(\cdot)$ in (13).

The diagram of the Shubert algorithm-based global extremum seeking algorithm for the system (9–10) is shown in

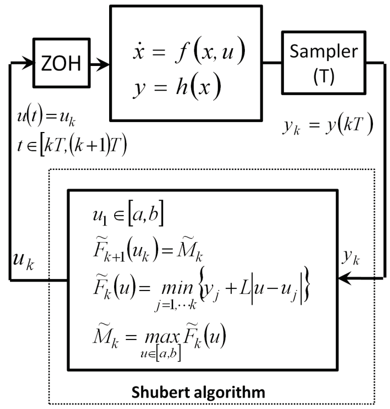

Fig. 1. The Shubert algorithm-based global extremum seeking algorithm

Figure 1. The Shubert algorithm is applied to this dynamical system but since the algorithm from the previous section is discrete-time and the plant (9) is continuous-time, we need to use analog-to-digital and digital-to-analog converters.

It is assumed that equidistant sampling time instants $t_k := kT$ are given, where $T > 0$ is a design parameter that will be determined later. T is referred to as the *waiting time* for reasons that will be explained in the sequel; in our implementation it plays the role of the classical sampling period. It is assumed that the input to the plant is piecewise constant, that is we have:

$$u(t) = u(t_k) =: u_{k+1}, \qquad \forall t \in [t_k, t_{k+1}) \ , \tag{14}$$

for $k = 0, 1, \ldots$. Note that we use a slight abuse of notation where u_1 is applied on the time interval $[t_0, t_1)$; in particular, the index of the input sequence u_k takes values $k = 1, 2, \ldots$. To formally state our system we introduce some notation. For any constant input $u \in [a, b]$, the solution of the state at time t and starting from an initial state $\mathbf{x}_0$ is denoted $\mathbf{x}(t, \mathbf{x}_0, u)$; obviously $\mathbf{x}(0, \mathbf{x}_0, u_1) = \mathbf{x}_0$. The notation $\mathbf{x}_k := \mathbf{x}(t_k, \mathbf{x}_{k-1}, u_k)$ for $k = 1, 2, \ldots$ is used. In particular, the sampler collects the output measurements:

$$y_k := h(\mathbf{x}_k) \qquad k = 1, 2, \ldots \ . \tag{15}$$

Note that with our notation we have that $\mathbf{x}_k$ and y_k can be defined for $k = 0, 1, \ldots$, whereas u_k only for $k = 1, 2, \ldots$. However, in our algorithm and analysis only y_k and u_k for $k = 1, 2, \ldots$ are needed.

Now we can apply the Shubert algorithm from the previous section to the dynamic plant (9), (10) with sampler (15) and zero order hold (14).

Shubert algorithm-based global extremum seeking:

- Arbitrarily choose an initial input $u_1 \in [a, b]$
- Find the next u_{k+1} such that the following equation is satisfied:

$$\widetilde{F}_k(u_{k+1}) = \widetilde{M}_k, \quad k = 1, 2, ..., \tag{16}$$

where

$$\widetilde{F}_k(u) := \min_{j=1,\ldots,k} \{y_j + L|u - u_j|\}, \tag{17}$$

$$\widetilde{M}_k := \max_{u\in[a,b]} \widetilde{F}_k(u). \tag{18}$$

Remark 4: Note that in (17), (18), we use a different notation $\widetilde{F}_k(u)$ and $\widetilde{M}_k$ as opposed to $F_k(u)$ and M_k used in (6) and (7). This is because in this section y_j in (17) is the sampled output measurement of a dynamic system. ◦

The closed loop system consists of the plant (9), (10), the above algorithm and the sampler and zero order hold that are described above. The goal is to show that this closed loop system would achieve global extremum seeking under certain conditions.

Remark 5: It is easy to extend our results to infinite dimensional systems in a manner similar to [20]. In this case, we would need to appropriately generalize Assumption 2. This level of generality is not pursued in order to keep the presentation simpler. ◦

Our goal is to show that the algorithm described above can find approximately the global maximum of $Q(\cdot)$ from an arbitrary set of initial conditions and to within an arbitrary prescribed margin if the waiting time T is sufficiently large.

In order to show our main result (Theorem 2), the following proposition is needed. This proposition indicates that after waiting for sufficiently long time, the sampled output approximates the output $Q(u)$ with input u generated from the Shubert algorithm. Due to space limitation, the proof is omitted.

Proposition 1: Consider the closed loop system consisting of the plant (9), (10), sampler (15), zero order hold (14) and the extremum seeking algorithm. Suppose that Assumption 2 holds. Then, for any strictly positive pair (Δ, ν) there exists $T > 0$ such that for any $|\mathbf{x}_0| \leq \Delta$ and any $u_k \in [a, b]$, $k = 1, \ldots$ we have that

- $|\mathbf{x}_k| \leq \ell_{max} + 1$ for $k = 1, 2, \ldots$
- $|y_k - Q(u_k)| \leq \nu$ for all $k = 1, 2 \ldots$,

where $\ell_{max} := \max_{u\in[a,b]} |\ell(u)|$. □

With the help of Proposition 1, the main result is stated next. The proof is an appropriate generalization of the proof in [15] as the sampled behavior of the Shubert algorithm-based global ESC is a perturbed version of the algorithm for static plants given by (1). More precisely, we show that the Shubert algorithm is robust to small additive perturbations. Hence, we can almost "recover" the result of Theorem 1 if we consider static plants. Due to space limitation, the proof is omitted.

Theorem 2: Consider the closed loop system consisting of the plant (9), (10), sampler (15), zero order hold (14) and the extremum seeking algorithm. Suppose Assumptions 1 and 2 hold. Then, for any strictly positive (Δ, ν), there exists $T > 0$ such that the discrete-time approximation of the closed loop system satisfy

$$y^* - \nu \leq \liminf_{k\to\infty} y_k \leq \limsup_{k\to\infty} y_k \leq y^* + \nu \ . \tag{19}$$

for any $|\mathbf{x}_0| \leq \Delta$. □

Remark 6: Theorem 2 establishes semi-global practical convergence of the closed loop in the parameter T. In particular, the domain of attraction $|\mathbf{x}_0| \leq \Delta$ can be arbitrarily large and the accuracy of the algorithm that is measured by ν can be arbitrarily small. However, the waiting time T is necessarily larger for larger Δ and/or smaller ν and hence the convergence to a neighborhood of the maximum y^* is slower. This tradeoff was observed in other, e.g. gradient based, extremum seeking schemes, see [17]. ∘

Remark 7: Note that the domain $[a, b]$ of the map $Q(\cdot)$ that we are optimizing is compact. Hence, global optimization is with respect to this compact interval. ∘

Theorem 2 shows that the convergence discrete-time approximation of the sampled-data system shown in Figure 1. With Assumption 1 and Assumption 2, the following corollary is obtained by applying [9, Theorem 2].

Corollary 1: Consider the closed loop system consisting of the plant (9), (10), sampler (15), zero order hold (14) and the extremum seeking algorithm. Suppose Assumptions 1 and 2 hold. Then, for any strictly positive (Δ, ν), there exists $T > 0$ such that solutions of the closed loop system satisfy

$$y^* - \nu \leq \liminf_{t\to\infty} y(t) \leq \limsup_{t\to\infty} y(t) \leq y^* + \nu \ . \tag{20}$$

for any $|\mathbf{x}_0| \leq \Delta$. □

Remark 8: Our results can be easily extended to the case when the measured output is contaminated by measurement noise, which is a more realistic scenario. We note that all our results can be appropriately adjusted in the following manner. Suppose that the measured output is $y(t) = h(x(t)) + n(t)$, where $n(t)$ is the noise satisfying

$$\text{ess sup}_{t\in[0,\infty)} |n(t)| \leq \nu_m \tag{21}$$

for some $\nu_m > 0$. Theorem 2 can be rephrased as follows.

Suppose Assumptions 1, 2 and (21) hold. Then, for any strictly positive (Δ, ν), there exist $T > 0$ and $\nu_m > 0$ such that for any $|x_0| \leq \Delta$ we have that the following holds for the closed loop system:

$$y^* - \nu \leq \liminf_{k\to\infty} y_k \leq \limsup_{k\to\infty} y_k \leq y^* + \nu \ . \tag{22}$$

The proof of the above fact follows almost the same steps as the proof of Theorem 2 and it is omitted for simplicity.

IV. Illustrative Examples

In order to illustrate the effectiveness of the proposed Shubert algorithm-based global extremum seeking algorithm, two illustrative examples are presented. The dynamics of the system take the following linear-time-invariant form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \mathbf{x}_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \tag{23}$$

A simple calculation yields $\ell(u) = -(A)^{-1}Bu$, where $A := \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}$ and $B := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. By choosing different $h(\cdot)$, the reference-to-output mapping $Q(\cdot)$ will be different.

A. Example 1: Shubert function

Our first example is a one-dimensional Shubert function, which is widely used as a test function for unconstrained global optimization problems. The output function is selected as

$$\begin{aligned} h_1(\mathbf{x}) \;\; = \;\; & \sin\left(\frac{1}{3}x_1 + 1\right) + 2\sin\left(\frac{1}{2}x_1 + 2\right) \\ & +3\sin\left(\frac{2}{3}x_1 + 3\right) \\ & +4\sin(\frac{5}{6}x_1 + 4) + 5\sin(x_1 + 5), \end{aligned}$$

which leads to

$$\begin{aligned} Q_1(u) \;\; = \;\; & \sin(2u + 1) + 2\sin(3u + 2) + 3\sin(4u + 3) \\ & +4\sin(5u + 4) + 5\sin(6u + 5). \end{aligned}$$

As shown in Figure 2, in the compact set $[-10, 10]$, there are 19 local maxima and one global maximum when $u^* = 5.792$ with $Q_1(u^*) = 12.03$, though two local maxima: $Q_1(-6.775) \approx 12.03, Q_1(-0.491) \approx 12.03$ are very close this global maximum. The estimation of Lipschitz constant is $L = 70$.

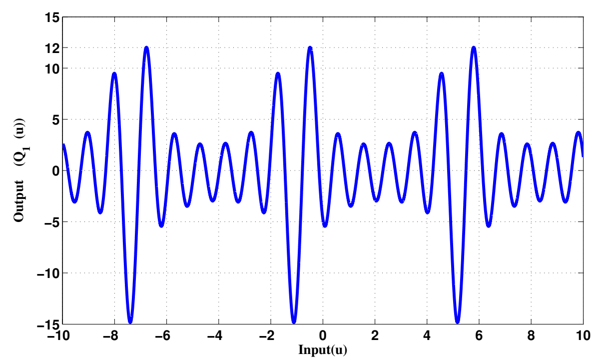

Fig. 2. A Shubert function

When the waiting time is selected as $T = 2$, the performance of Shubert algorithm-based global extremum seeking is shown in Figure 3. The output $y(t)$ converges to a small neighborhood ($\nu = 0.057$) of the optimal value (see Figure 3). In order to see clearly the sampler and zero order hold controller in the proposed algorithm, Figure 4 shows the output and input for the first 50 seconds (or $k = 1, \cdots, 50$ in Shubert algorithm-based global extremum seeking).

1) The performance comparison with different choice of waiting time T: The waiting time "T" plays an important role in our proposed Shubert algorithm-based global extremum seeking. As shown in Theorem 2, the longer waiting time leads to slower convergence with a better accuracy (more close to the optimal value). On the other hand, the smaller T will lead to fast convergence with less accuracy. In order to illustrate the importance of this design parameter, a larger waiting time is selected, i.e. $T = 15$.

The performance of the proposed Shubert algorithm-based global extremum seeking is shown in Figure 5. Compared with Figure 3, the accuracy has been improved ($\nu = 8.5 \times 10^{-6}$), though convergence speed slows down. Figure 5 (b)

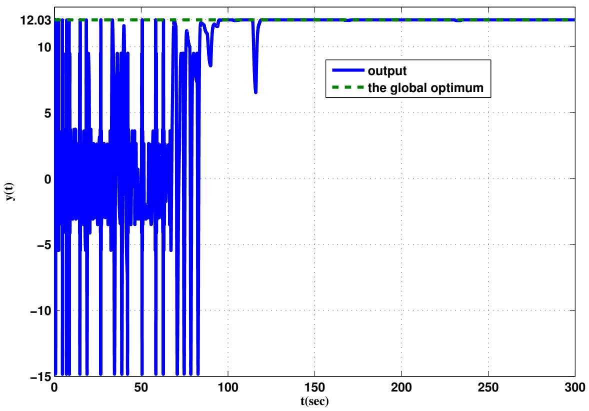

Fig. 3. The convergence of $y(t)$ to the global optimum

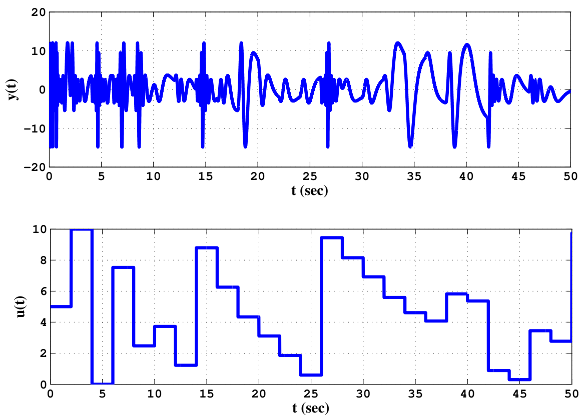

Fig. 4. The output and input over the first 50 seconds

shows the output $y(t)$ for the first 30 seconds. It can be seen clearly that with sampled-data input, the output reaches a small neighborhood of the steady-state when the waiting time T is large enough.

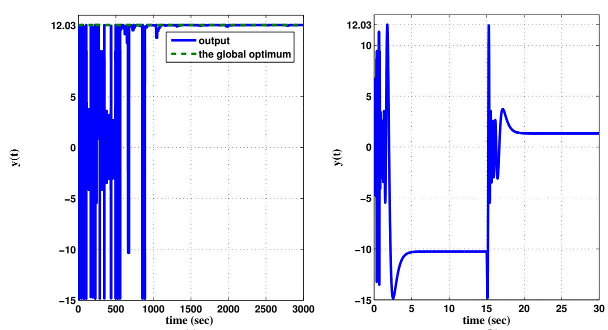

Fig. 5. The convergence of output to global maximum using Shubert algorithm-based global ES

2) Different estimate of Lipschitz constant: Another important parameter in the proposed Shubert algorithm-based global extremum seeking is the Lipschitz constant L. Instead of using $L = 70$, we use a more conservative estimation $L = 100$. In the simulation, the waiting time is selected as $T = 2$.

From Figure 6, it is clearly seen that by using a conservative estimate of L, the Shubert algorithm-based global extremum seeking converges slowly. It oscillated around the global maximum value for longer time (more than 200 seconds, compared with 130 seconds when $L = 70$) before it settles to a small neighborhood of the optimal value. This is consistent with what had been reported in [15]. Therefore, it is very useful to get less conservative estimate of the Lipschitz constant. As indicated in [16], some adaptive techniques might be used to estimate a proper Lipschitz constant. We will focus on using some adaptive techniques in our future work.

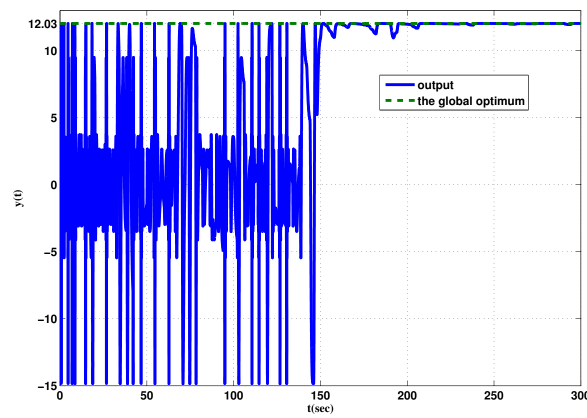

Fig. 6. The convergence of output when $L = 100$

B. Example 2

The second output mapping is selected as $h_2(\mathbf{x}) = -e^{\frac{1}{1+\frac{1}{300}x_1^2}} + e^{\frac{1}{1+5\left(\frac{x_1}{6}-15\right)^2}}$, which leads to

$$Q_2(u) = -e^{\frac{1}{1+.02u^2}} + e^{\frac{1}{1+5(u-15)^2}}. \qquad (24)$$

This example was proposed [18, Example 5] to show that the traditional gradient based methods with dither scheduling for global ES discussed in [18] are unable to locate the global maximum.

As shown in Figure 7, $Q(\cdot)$ has 2 maxima over a compact set $[-5, 20]$: the global maximum at $x = 15$ and the local maximum at $x = 0$ (see Figure 7).

It was shown in [18, Example 5] via bifurcation diagram analysis that the scheme in [18] would **not** converge to the global extremum no matter how to choose parameters. When $a_0 = 10$, $\epsilon = 0.01$, $\omega = 0.1$ and $\delta = 0.1$, by applying global ESC proposed in [18], the output converges to local maximum as shown in Figure 8.

When Shubert algorithm-based global ESC proposed is applied to system with $Q_2(\cdot)$ with waiting time $T = 2$ and Lipschitz constant $L = 2$, the performance is shown in Figure 9. The output $y(t)$ converges to a small neighborhood

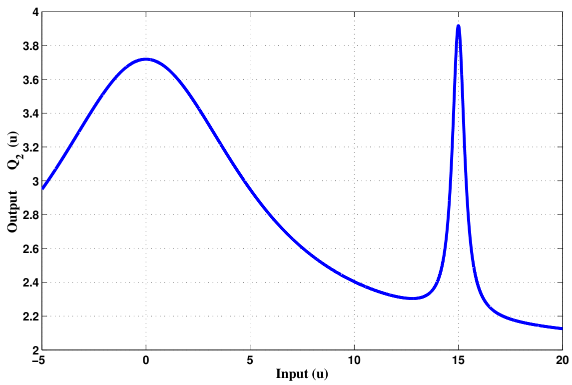

Fig. 7. $Q_2(\cdot)$ does not satisfy conditions in [18, Assumption 2]

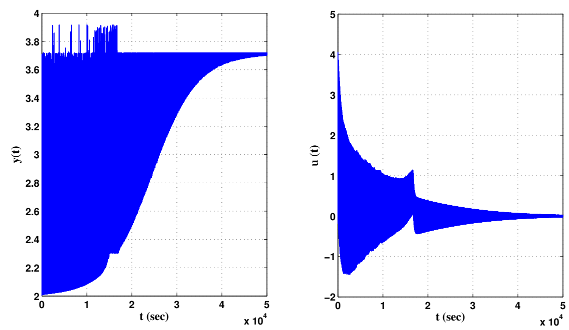

Fig. 8. The output of system converges to the local maximum by global ESC proposed in [18]

of the optimal value. This example shows the Shubert algorithm-based global extremum seeking can be applied to a more general nonlinear system, compared with the global extremum seeking proposed in [18].

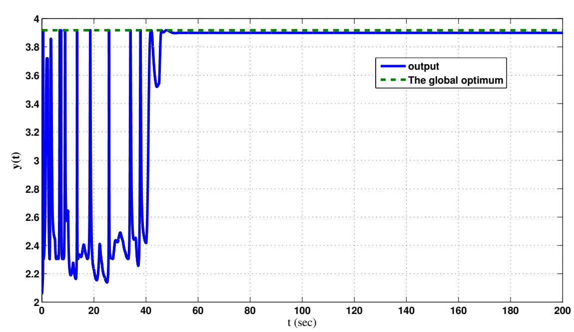

Fig. 9. The output converges to global maximum using Shubert algorithm-based global ES

V. Conclusions and Future Work

In this note, a Shubert algorithm-based global extremum seeking control scheme is proposed for general nonlinear dynamic systems. This work opens new research opportunities for the adaptation of sampling optimization algorithms in [16] in the context of extremum seeking control. Our future work will focus on weakening Assumption 1 by using some adaptive techniques to estimate the Lipschitz constant as well as generating a design framework which combines arbitrary sampling optimization algorithm with extremum seeking control.

References

[1] K.B. Ariyur and M. Krstić, *Real-Time Optimization by Extremum Seeking Control*, Wiley-Interscience, Hoboken, NJ, 2003.

[2] F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer Verlag, New York, Inc. 1998.

[3] P. M. Dower, P.M. Farrell and D. Nešić, "Extremum seeking control of cascaded Raman optical amplifiers", *IEEE Transactions on Control Systems Technology*, vol. 16(3), pp. 396–407, 2008.

[4] M. Guay, D. Dochain and M. Perier, "Adaptive extremum seeking control of continuous stirred tank bioreators with unknown growth kinetics", *Automatica*, vol. 40, pp. 881–888, 2004.

[5] M. Guay and T. Zhang, "Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties", *Automatica*, vol. 40, pp. 1283–1293, 2004.

[6] P. Hansen and B. Jaumard and S.-H. Lu, "Global optimization of univariate Lipschitz functions: I. Survey and properties", *Mathematical Programming*, vol. 55(1), pp. 252–272, 1992.

[7] M. Krstic and H. H. Wang, "Stability of extremum seeking feedback for general nonlinear dynamic systems", *Automatica*, vol.36 (4), pp. 595–601, 2000.

[8] I.S. Morosanov, "Method of extremum control", *Automatic & Remote Control*, vol. 18, pp. 1077–1092, 1957.

[9] D. Nesic, A. R. Teel and E. D. Sontag, "Formulas relating KL-stability estimates of discrete-time and sampled-data nonlinear systems", *Sys. Contr. Lett.*, vol. 38 (1), pp. 49–60, 1999.

[10] I.I. Ostrovskii, "Extremum regulation", *Automatic & Remote Control*, vol. 18, pp. 900–907, 1957.

[11] W.H. Moase, C. Manzie and M. J. Brear, "Newton-Like Extremum-Seeking for the Control of Thermoacoustic Instability", *IEEE Transactions on Automatic Control*, vol. 55(9), pp. 2094–2105, 2010.

[12] K.S. Peterson and A.G. Stefanopoulou, "Extremum seeking control for soft landing of an electromechanical valve actuator", *Automatica*, vol. 40, pp. 1063–1069, 2004.

[13] S.A. Piyavskii, "An algorithm for finding the absolute extremum of a function", *USSR Computational Mathematics and Mathematical Physics*, vol.12, pp 57–67, 1972.

[14] D. Popović, M Janković, S. Magner and A. R. Teel, "Extremum seeking methods for optimization of variable cam timing engine operation", *IEEE Transactions on Control Systems Technology*,, vol. 14(0, pp. 398–407, 2006.

[15] B.O. Shubert, "A sequential method seeking the global maximum of a function", *SIAM Journal on Numerical Analysis*, vol. 9 (3), pp. 379–388, 1972.

[16] R.G. Strongin and Y.D. Sergeyev, *Global Optimization with Non-Convex Constraints - Sequential and Parallel Algorithms*, Springer-Verlag, New York, 2000.

[17] Y. Tan, D. Nešić and I.M.Y. Mareels, "On non-local stability properties of extremum seeking control", *Automatica*, vol. 42 (6), pp. 889–903, 2006.

[18] Y. Tan, D. Nešić, I.M.Y. Mareels and A. Astolfi, "On global extremum seeking in the presence of local extrema", *Proceedings of the* 45^{th} *IEEE Conference on Decision and Control*, pp. 5663–5668, 2006.

[19] Y. Tan, D. Nešić, I.M.Y. Mareels and A. Astolfi, "On global extremum seeking in the presence of local extrema", *Automatica*, vol. 45(1). pp. 245–252, 2009.

[20] A.R. Teel and D. Popović, "Solving smooth and nonsmooth multivariable extremum seeking problems by the methods of nonlinear programming", *Proceedings of the 2001 American Control Conference*, vol. 3, pp. 2394–2399, 2001.

[21] X.T. Zhang, D.M. Dawson, W.E. Dixon and B. Xian, "Extremum-seeking nonlinear controllers for a human exercise machine", *IEEE/ASME Transactions on Mechatronics*, vol. 11 (2), pp. 233–240, 2006.