Author/s:
Tan, Y;Moase, WH;Manzie, C;Nešić, D;Mareels, IMY

Title:
Extremum seeking from 1922 to 20101

Date:
2010-12-22

Citation:
Tan, Y., Moase, W. H., Manzie, C., Nešić, D. & Mareels, I. M. Y. (2010). Extremum seeking from 1922 to 20101. Chen, J (Ed.) Proceedings of the 29th Chinese Control Conference Ccc 10, pp.14-26. IEEE.

Persistent Link:
https://hdl.handle.net/11343/299847

Extremum Seeking From 1922 To 2010

Y. Tan[2], W.H. Moase[1], C. Manzie[1], D. Nešić[2], I.M.Y. Mareels[2]

1. Department of Mechanical Engineering, The University of Melbourne, VIC 3010, Australia
E-mail: moasew@unimelb.edu.au, manziec@unimelb.edu.au

2. Department of Electrical and Electronic Engineering, The University of Melbourne, Vic 3010, Australia
E-mail: y.tan@unimelb.edu.au, dnesic@unimelb.edu.au, i.mareels@unimelb.edu.au

Abstract: Extremum seeking is a form of adaptive control where the steady-state input-output characteristic is optimized, without requiring any explicit knowledge about this input-output characteristic other than that it exists and that it has an extremum. Because extremum seeking is model free, it has proven to be both robust and effective in many different application domains. Equally being model free, there are clear limitations to what can be achieved. Perhaps paradoxically, although being model free, extremum seeking is a gradient based optimization technique. Extremum seeking relies on an appropriate exploration of the process to be optimized to provide the user with an approximate gradient, and hence the means to locate an extremum. These observations are elucidated in the paper. Using averaging and time-scale separation ideas more generally, the main behavioral characteristics of the simplest (model free) extremum seeking algorithm are established.

Key Words: Adaptive Control, Extremum Seeking, Time Scale Separation, Averaging Analysis

1 INTRODUCTION AND EXTREMUM SEEKING HISTORY

Quoting from perhaps the first survey paper on the topic of extremum seeking, [58], extremum seeking is *a control system which is used to determine and to maintain the extremum value of a function*.

A more elaborate description, sufficient for the present paper, starts from a system with input u and output y_p that has a well defined steady state characteristic: that is for any constant input u, within the operational envelope, the output settles to a constant y_p. The situation is sketched in Figure 1. The steady state map may be expressed as $y_p = g(u, p)$. It may depend on many other influences as captured by the dependence on the parameter p. Assuming that the relationship g exhibits a desired extremal situation, say $y^*(p) = g(u^*(p), p)$, an *extremum seeking control* finds $u = u^*(p)$ and maintains this extremum condition despite (slow) variations in p. Importantly, an extremum seeking algorithm achieves these objectives without relying on any explicit knowledge about the system, its steady state input/output map g or the parameter p. In particular, the initial condition for u is not necessarily close to the desired u^*.

For simplicity, in this paper, only the case of scalar u and y_p is considered, and the presence of p is largely ignored.

In his 1922 paper, or invention disclosure, Leblanc [88] describes a mechanism to transfer power from an overhead electrical transmission line to a tram car using an ingenious non-contact solution. In order to maintain an efficient power transfer in what is essentially a linear, air-core, transformer/capacitor arrangement with variable inductance, due to the changing air-gap, he identifies the need to adjust a (tram based) inductance (the input) so as to maintain a *resonant* circuit, or maximum power (the output). Leblanc explains a control mechanism of how to maintain the desirable maximum power transfer using what is essentially an extremum seeking solution. The paper does not contain any analysis, nor does it provide a practical evaluation, it may well be that the ideas were never implemented.

During World War II, there was a significant research activity in Russia in the area of extremum seeking. Some of the early Russian work can be found in [68, 69].

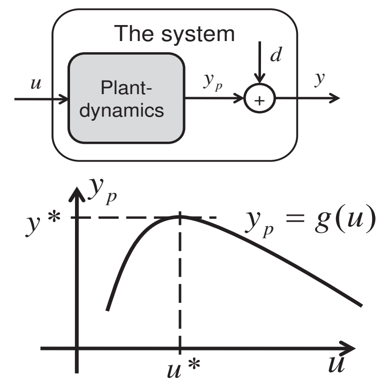

Fig. 1 Input-output system with steady state map, exhibiting a clear extremum

Probably the first, English literature paper detailing an extremum seeking control algorithm, and its performance, is the 1951 paper by Draper and Li [41]. This paper explores how to optimize an internal combustion engine, more particularly how to select ignition timing (the input) as to achieve maximum power output. Ever since this publication, internal combustion engines have remained a popular application domain for extremum seeking.

Extremum seeking, like all other forms of adaptive control, was a popular research topic in the 1950s and 1960s, see also Äström's 1995 review paper that describes this fertile decade for adaptive control research [9]. At its inception, extremum seeking went by many different names *extremum seeking regulator*, *optimalizing control system*, and *hill-climbing systems* to name but a few, e.g. [41, 100, 105, 111, 115] and references therein. Most results in the 1950's and 1960's focused on describing the algorithms, and exploring their performance as per the particular implementation or problem at hand, and there were indeed many variants. Design issues were prominent, but clear definitions, a precise

Already in 1960, this state of affairs was deplored by e.g. Eykhoff [44].

Over the next three decades, 1970-2000, extremum seeking related research continued but clearly the mainstream adaptive control research emphasis had shifted to studying other forms of adaptive control that address the more demanding and holistic problem of system stability with performance control. Nevertheless, steady progress continued to be made. The paper [95] presents a first Lyapunov based stability analysis (be it for a very special scheme). An interesting survey paper of this period is due to Sternby [144]. Until 1990, most of the extremum seeking algorithms use periodic excitation to explore the steady state map. Stochastic rather than deterministic excitation became somewhat popular in the 1990s, see for example, [139-141]. Whilst some progress was made on the theory of extremum seeking, the practice and industrial applications of extremum seeking grew much more rapidly, so that in their 1995 book, Äström and Wittenmark describe extremum seeking as one of the most promising adaptive control methods [10, Section 13.3].

The first rigorous assessment of the stability of a classic extremum seeking feedback scheme was published in 2000 by Wang and Krstić [165]. It appears that this paper sparked a renewed interest in the theory of extremum seeking. Using Google Scholar[1] it is estimated (although certainly not all conference papers have been located, due to terminology issues, and perhaps lack of digitization of existing libraries) that the number of publications (including patents and books) concerning extremum seeking in the last decade (2000-2009) is significantly larger than the total number of publications prior to this period. Figure 2 illustrates this. As mentioned, extremum seeking has been implemented successfully in many different engineering systems. Some of the popular application domains with some indicative references are: brake system control [40, 170, 175, 179]; autonomous vehicles and mobile robots [99, 143, 181] and [24, 34, 35, 112, 178]; yield optimization in bioprocesses [20, 52, 53, 98, 155, 166, 182]; bluff-body drag bility control [12, 15, 16, 84, 103, 104]; electromechanical valve [124]; internal combustion engines [27, 41, 60, 75, 84, 126, 135-137, 146, 162]; flow control problems [30, 61, 76, 77, 93, 94]; flocking and formation control [23, 24, 33, 63, 169, 186]; gyro control [5-7]; human exercise machines [183] optimizing neural network/fuzzy logic controllers [56, 62, 63]; maximum gain control in optical amplifiers [39]; particle accelerators and plasma control [134], [31, 116, 117] and also [29]; optimal power trackers in photovoltaic systems [28, 89]; process control [48, 64, 81, 127, 160]; tunable thermo-acoustic cooler [91]; weigh feeder control systems [8].

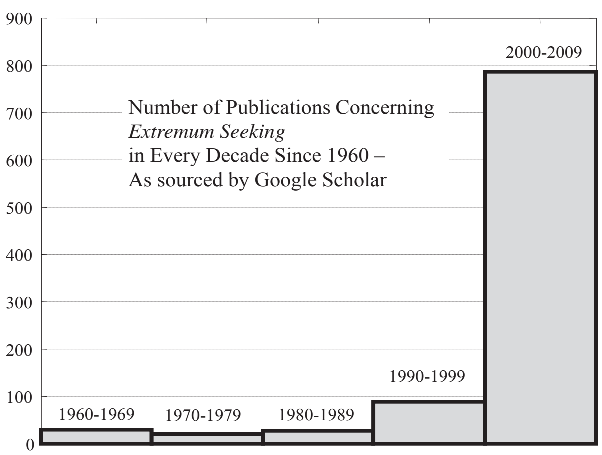

Fig. 2 Time line of extremum seeking publications

[1]The interface provided by the program *Harzing's Publish or Perish* was used. The results are based on the search phrase *extremum seeking*.

There are two main approaches to extremum seeking:

- using a continuous excitation signal to explore the steady state map, from which an approximate implicit gradient can be obtained, as described in [13].
- using a (repeated) sequence of constant probing inputs, that exploit the ideas and recipes from numerical optimization methods [154].

Either method relies heavily on an appropriate time-scale separation between learning and dynamics to be optimized. This paper deals exclusively with the more classic, continuous excitation signal case, which is inspired by the 1950s papers. The present paper does not attempt to contribute a precise, working definition of what is an adaptive or learning system, this remains an elusive goal, but a design framework for a family of extremum control algorithms will be sketched. It parallels the ideas expounded in [4].

The remainder of the paper is organized as follows. The next section sets the scene, and introduces the minimal extremum seeking algorithm in the simplest of circumstances in the context of scalar input/output maps, with underlying exponentially stable plant dynamics, and using periodic excitation signals. It is observed where and how the assumptions can be relaxed. In the following section this basic scheme is then (heuristically) analyzed using averaging and time scale separation ideas. The emphasis is to identify clearly the various time scales and their role in the design of extremum seeking. As an intermezzo, the gradient approximation at the heart of extremum seeking, and which is obtained from the preceding averaging analysis, is considered in some detail. An example is used to illustrate the main ideas, and to expose some of the design trade-offs. The final section summarizes and indicates avenues for further work.

2 AN EXTREMUM SEEKING ALGORITHM

2.1 Notation

The set of real numbers is denoted $\mathbb{R}$. The continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is of class $\mathcal{KL}$ if $\beta(s,t)$ is for any t fixed, zero at zero, continuous and strictly increasing; and for any fixed s, it is function that decreases to zero as t grows without bound.

For a differentiable function g denote its derivative, or gradient by Dg; similarly D^2g denotes the Hessian, that is $D^2g = D(Dg)$ and so on.

The order notation $f(\epsilon) = O(\epsilon)$ is used to indicate that the function f, continuous on the interval $\epsilon \in (0, \epsilon^*)$ can

$\lim_{\epsilon\to 0} f(\epsilon) = 0$.

2.2 System and System Model

For simplicity, consider a single-input single-output system as in Figure 1. Here u is the input to the system and y_p is output of the plant, y is the measured output and d is a bounded disturbance. Input, output and disturbance are all functions of time. It is assumed that the system has a well defined *steady-state* characteristic (u, y_p). This relationship does not necessarily have to be a function, it could be a multi-valued steady state characteristic. Without loss of generality, assume that the steady-state characteristic has a (local) maximum (u^*, y_p^*). No other prior model knowledge is assumed of the system. Both the input u and the output y are available.

The control objective is to complement the system of Figure 1 so as to drive the input/output (u, y_p) pair to the extremum (u^*, y^*).

To fix the ideas, and assist the analysis, let the plant dynamics be modeled as

$$\dot{x} = f(x, u), \quad y_p = h(x), \quad y = y_p + d(t). \tag{1}$$

Here x is an (n-dimensional) state variable. The functions f, h are assumed to be differentiable[2] in their arguments.

To ensure that the notion of a steady state is well defined, the following assumptions may be imposed, see also [150].

Assumption 1 *There exists a (differentiable) function* $\ell : \mathbb{R} \to \mathbb{R}^n$ *such that*

$$f(x, u) = 0, \quad \text{iff} \quad x = \ell(u). \tag{2}$$

Assumption 2 *For each constant* u*, the corresponding equilibrium* $x = \ell(u)$ *of the system (1) is globally asymptotically stable, uniformly in* u*.*

Assumption 1 implies that the steady state characteristic is well defined, and a differentiable function

$$y_p = g(u) = h \circ \ell(u) = h\left(\ell(u)\right). \tag{3}$$

Assumption 2 ensures that the steady-state characteristic is stable and attractive in some equi-uniform manner (regardless of u) and unique.

At the cost of abandoning any attempt to achieve a global analysis, local (in x) results can be obtained by simply requiring a local uniqueness and a local stability property for the equilibria $x = \ell(u)$. Through this route multi-valued steady state characteristics can be analyzed as well.

Assumption 3 *Consider* ℓ *defined as in Assumption 2. Let* $g(u) = h \circ \ell(u)$*, be the steady state characteristic. There exists a unique* u^* *maximizing* g*:*

$$Dg(u^*) = 0 \qquad D^2 g(u^*) < 0 \tag{4}$$

$$Dg(u^* + \zeta)\zeta < 0 \qquad \forall \zeta \neq 0 \tag{5}$$

[2] Differentiability is not strictly necessary, Lipschitz continuity will do, at the expense of a few more technicalities.

teristic has a unique maximum (considering a maximum is without loss of generality). Again, if a local in-(x, u) result suffices, there is no need to insist on a global maximum, and indeed local extrema can be analyzed in this manner. Again differentiability conditions are somewhat stronger than necessary, but simplify greatly the analysis. In particular a condition like (5) is instrumental in establishing the stability of the objective u^* in the extremum seeking algorithm.

2.3 Extremum Seeking Control

The classic extremum seeking algorithm is represented in Figure 3. It is used in much of the literature but see in particular [13, 165]. In this classic diagram the design parameters are

- a, the gain determining the size of the dither or excitation signal.
- ω, the pulsation of the excitation signal.
- T_{LP} determines the cut-off frequency of the low pass filter.
- T_{HP} determines the cut-off frequency of the high pass filter.
- ϵ, which scales the gain of the integrator that determines the signal $\hat{u}$.

The high pass and low pass filters can be more complex (higher order filters) than suggested in Figure 3.

A meaningful algorithm requires that the dither signal $\sin(\omega t)$ belongs to the pass band of both the low pass and high pass filters, that is $T_{LP} < \frac{2\pi}{\omega} < T_{HP}$. Moreover, the period of the excitation signal must be small compared to the integration time, or $\omega \gg \epsilon$.

As the dither is essentially a nuisance signal as far as the plant and the control objective goes, it is normal to select its amplitude a to be much smaller than the expected $\hat{u}$.

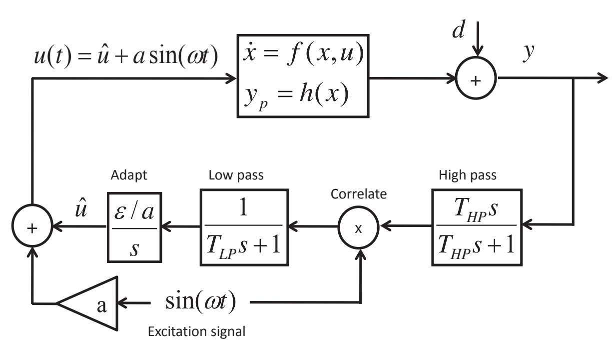

Fig. 3 Classic extremum seeking algorithm

The phase distortion introduced by the high pass and low pass filters is not without significance in this algorithm, moreover the correlation with a narrowband excitation signal as implied by the multiplication operation followed by tial to the operation of the extremum seeking scheme. They can indeed be removed. Doing so leads to the simpler extremum seeking control algorithm shown in Figure 4. This paper concentrates on elucidating the behavior of this minimal algorithm, see also [152]. The filters are not a complete nuisance however, and some of the benefits they may bring are briefly touched upon in Section 5.

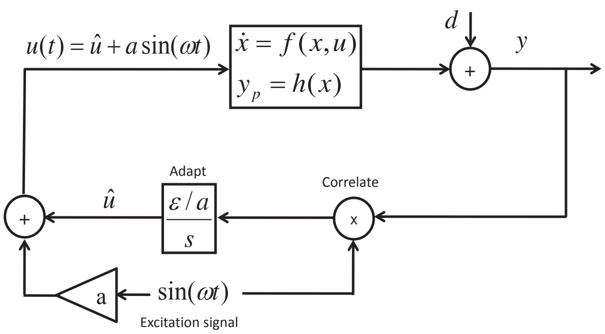

Fig. 4 Minimal extremum seeking algorithm

In the minimal algorithm, the only design parameters are the gain a and pulsation ω of the excitation signal and the gain ϵ. The design task in the minimal algorithm is therefore somewhat simpler and easier to explain.

Remark 1 *Although sinusoidal excitation signals are widely used in much of the extremum seeking literature, other signals can be explored as well. Both deterministic as well as stochastic signals have been proposed. Stochastic dither signals are discussed in [92, 97, 139-141] and references therein.*

The excitation signal does not necessarily have to be an external signal. In some applications where system noise, or other signals naturally occurring in the system have appropriate spectral content these can be used to advantage [29]. The main requirement, which will transpire from the sequel is that the set of values attained by the signal has an appropriate odd-symmetry with respect to zero *distribution.*

Obviously, the choice of dither signal is not without consequences. As indicated in [152] both the frequency content of the excitation signal as well as its amplitude distribution affect the overall behavior of the extremum seeking scheme, and in particular may affect how quickly the extremum may be located, and how well it can be tracked.

Finally, it is observed that the excitation signal cannot be correlated to the system noise, otherwise the direction in which to update u may be wrongly inferred.

Assumption 4 *The bounded disturbance d is uncorrelated with the dither signal $\sin(\omega t)$ i.e.*

$$\lim_{T\to 0} \frac{1}{T}\int_0^T \sin(\omega\tau) d(\tau) d\tau = 0. \qquad (6)$$

the above integral has to be sufficiently small. Also how fast the integral converges matters, as it will affect how slow the learning has to occur. The Assumption 4 may be relaxed, as

$$\lim_{T\to \infty} \frac{1}{T}\int_0^T \sin(\omega\tau) d(\tau) d\tau = O(a^2),$$

will suffice (for $1 \gg a$).

3 ANALYZING THE EXTREMUM SEEKING ALGORITHM

In an heuristic manner the key ideas, and steps required to understand the behavior of the minimal extremum seeking algorithm is presented. References to the key results, where rigorous proofs may be found, are provided. The extremum seeking system is summarized as follows:

$$\begin{aligned} \dot{x} &= f(x, \hat{u} + a\sin(\omega t)), \\ \dot{\hat{u}} &= \frac{\epsilon}{a}\left(h(x) + d(t)\right)\sin(\omega t). \end{aligned} \qquad (7)$$

3.1 The time scale of the x-transient dynamics

Assuming that both $\hat{u}$ and the excitation signal are slowly time varying compared to the transients in the x-dynamics (alternatively expressed, well inside the pass band of the x-system) it is reasonable, that the solution x may be approximated as follows:

$$x(t) = \ell\left(\hat{u} + a\sin(\omega t)\right) + \xi(t) + \eta(t), \qquad (8)$$

where the term $\xi(t)$ converges to zero (quickly) as t grows, and the η-term can be made small by selecting $a\omega$ and ϵ sufficiently small.

This can be made precise using a singular perturbation analysis, see e.g. [150, 151], ([80] describes the singular perturbation analysis technique).

3.2 The learning time scale

The approximation (8) for x can be used to advantage, to eliminate the fast time response in the x-dynamics, from the slow learning dynamics:

$$\begin{aligned} \dot{\hat{u}} = & \frac{\epsilon}{a}\left[h\left(\ell\left(\hat{u} + a\sin(\omega t)\right) + \xi(t) + \eta(t)\right)\right. \\ & \left. + d(t)\right]\sin(\omega t). \end{aligned} \qquad (9)$$

In this equation, three different time scales may be distinguished. The ξ-term captures the fastest dynamics, the transients in the x-system. The medium fast time variations are represented by the excitation signal $\sin(\omega t)$ as well as the term $d(t)\sin(\omega t)$. The learning dynamics are the slowest, their time scale being governed by the small gain ϵ[3]. The equation (9) is in the standard form [132], ready for the application of averaging. Averaging out the time variations in (9), leads to a time-invariant averaged system that captures the main trend of the learning dynamics adequately.

[3]The fact that the $\dot{\hat{u}}$ is proportional to ϵ/a is only apparent, for it will transpire that the a can be eliminated, and that only ϵ determines the learning time scale

The averaged dynamics are described by

$$\frac{d}{dt}u_{av} = \epsilon g_{av}(u_{av}, a). \tag{10}$$

Here g_{av} is defined as:

$$g_{av}(u,a) = \lim_{S\to\infty} \frac{1}{aS}\int_0^S (h(\ell(u + a\sin(\omega t)) + \xi(t) + \eta(t)) + d(t))\sin(\omega t)dt.$$

Using the following observations

- ξ converges to zero,
- d is uncorrelated with $\sin(\omega t)$ (Assumption 4),
- η can be neglected by selecting $a\omega$ and ϵ sufficiently small,

the above expression can be simplified to:

$$g_{av}(u,a) = \frac{1}{aT}\int_0^T g(u + a\sin(\omega t))\sin(\omega t)dt \tag{11}$$

(Here $T = \frac{2\pi}{\omega}$.) Because g is differentiable (a consequence of the stated Assumptions), a Taylor series expansion of the integrand leads to $g(u + a\sin(\omega t)) \approx g(u) + aDg(u)\sin(\omega t) + O(a^2)$. It is clear that g_{av} is therefor approximately proportional to the required steepest descent direction:

$$g_{av}(u,a) = \frac{1}{2}Dg(u) + O(a). \tag{12}$$

This *approximate* gradient operator (11) is examined in some more detail in Section 4.

3.4 Main result

The above steps are now summarized. The main decomposition of the behavior of extremum seeking, follows from the time scale separation as expressed by

1. **Fast time variations:** the x-dynamics quickly settle down to the equilibrium manifold (8).
2. **Intermediate time variations:** the excitation signal $a\sin(\omega t)$, explores a neighborhood of the equilibrium manifold around the present estimate $\hat{u}$.
3. **The slow time variations:** the learning dynamics, with $\omega \gg \epsilon$, and a sufficiently small, and $\frac{1}{\epsilon}$ sufficiently large to be able to average out the influence of the disturbance d, $\hat{u}$ slowly evolves in the direction of the gradient $Dg(\hat{u})$ to seek the maximizer u^*.

For sufficiently small a, u_{av} (see equation (10)) converges (in a globally asymptotically stable manner) to an $O(a)$-neighbourhood of the (unique) maximum u^* (using Assumption 3 $Dg(u^*) = 0$ and $D^2g(u^*) < 0$). Moreover, standard averaging provides the estimate that $\hat{u}$ remains in an $o(\epsilon)$-small neighborhood of u_{av} and thus it converges to a $o(\epsilon) + O(a)$-sized neighborhood of u^*. It follows that u also converges to $o(\epsilon) + O(a)$-sized neighborhood of u^*. Finally, it may be concluded that the plant output y_p converges to an $o(\epsilon) + O(a)$-small neighborhood of $\ell(u^*)$ or $\ell(u^* + a\sin(\omega t))$.

These time-scale separation ideas, and general dynamics, are illustrated in Figure 5.

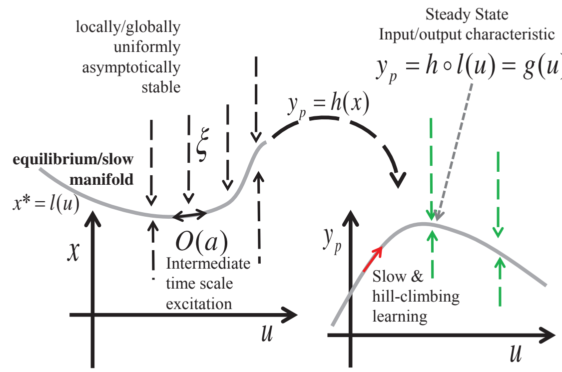

Fig. 5 Time-scale separation requirements underpinning extremum seeking

These heuristic arguments can be made precise. The corresponding, main result, see also [150], may be stated as follows:

Theorem 1 *Assume that Assumptions 1 to 4 hold. Select three positive scalars Δ, ν, δ. There exist class $\mathcal{KL}$ functions β_u and β_x and positive constants $a^*(\Delta,\nu,\delta)$ and $\epsilon^*(\Delta,\nu,\delta)$ such that for any $a \in (0, a^*)$ and any $\epsilon \in (0,\epsilon^*)$, there exists a positive constant $\omega^* = \omega^*(a,\epsilon)$ such that for any pulsation $\omega \in (0,\omega^*)$, the solutions of (7) satisfy:*

$$|\hat{u}(t) - u^*| \leqslant \beta_u(|\hat{u}(0) - u^*|, \epsilon t) + \nu \tag{13}$$
$$\|x(t) - \ell(u^*)\| \leqslant \beta_x(\|x(0) - \ell(u^*)\|, t) + \nu \tag{14}$$

for any $\|x(0) - \ell(u^*)\| \leqslant \Delta$, $|\hat{u}(0)| \leqslant \Delta$ and $\|d\|_\infty \leqslant \delta$.

A complete proof may be found in [107, 150].

This theorem may be paraphrased as follows. For any initial condition inside some ball of (possibly large) radius Δ, for any bounded disturbance $\|d(t)\| \leqslant \delta$ for all t and for any chosen (small) residual error $\nu > 0$, along the solutions of the extremum seeking system (7) the pair $(\hat{u}, y_p)$ will converge to a ν-sized ball centered on the desired extremum (u^*, y^*). Moreover, the x-solution will converge to a ν sized neighborhood of $\ell(u^*)$.

Clearly, the smaller ν is selected, the smaller a as well as ϵ have to be. The smaller ϵ the slower the learning progresses, as from (13) it follows that the learning dynamics converge in ϵt time scale, whereas the x-dynamics converge in t-time scale.

4 Extremum Seeking's Approximate Gradient

This section reconsiders the expression (11). As indicated for sufficiently small a and differentiable g, $g_{av}(u,a) = \frac{1}{2}Dg(u) + O(a)$.

sion:

$$G(u,a) = \frac{2}{Ta}\int_0^T g\left(u + a\sin(\omega t)\right)\sin(\omega t)dt, \tag{15}$$

appears to be a family (parametrized by a) of *approximate gradients* for the function g.

Clearly, G is well defined even when g is not differentiable. All that is required for G to exist is that g be integrable[4].

So, the obvious question arises, even when the underlying function g is not differentiable does this G still have properties that could be interpreted as an appropriate gradient in some sense, and would the introduced extremum seeking scheme still be useful?

Intuitively, because the integration in the definition of G smears any discontinuities at a point u out over a neighborhood of $O(a)$ radius around u, the answer is yes[5].

In Figure 6, several (g,G)-function pairs are illustrated in Figure 6, with various types of deficiency in differentiability or continuity of g. The figures consider g with an averaged minimum.

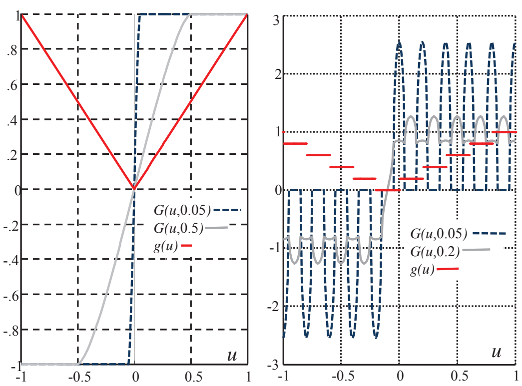

Fig. 6 Approximate gradient (g,G)-pairs.

5 TWO EXAMPLES

By way of example, consider a plant with an unknown output performance characteristic and first order actuator dynamics described by

$$h(x) = e^{-\frac{(x-3)^2}{0.5}} + 1.5e^{-\frac{(x-5)^2}{1.5}} \tag{16}$$

$$\dot{x} = 100(u - x) \tag{17}$$

This type of problem structure approximates many real world objectives including a 1-dimensional engine calibration whereby the optimal valve timing to maximise efficiency is sought. Note, that the performance map shown in Figure 7 has multiple maxima, and thus only regional convergence can be guaranteed.

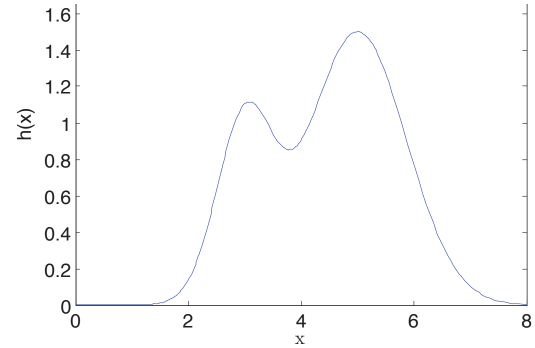

Fig. 7 Performance metric, $h(x)$

Following the procedure outlined in the earlier sections of the paper, the first step in designing the extremum seeking scheme is to recognise the three different time scales present. Since the plant dynamics have a bandwidth of 100 rad/s, the intermediate time scale (set by the perturbation frequency, ω) should be chosen slower, while the slow time variations are dictated by the choice of integrator gain, ϵ. Since in this case there is unity d.c. actuator gain, the dither signal amplitude can be based on the range of x likely to be encountered.

As an initial demonstration, the extremum seeking parameters are chosen to be $\omega = 10$, $a = 0.1$, and $\epsilon = 0.001$. Figure 8 illustrates the convergence of the scheme for different initial conditions of $\hat{u}$. It is clear that the convergence is directed towards local maxima, but also the trajectories highlight the different convergence rates are directly impacted by the local gradients (e.g. slow initial convergence is observed for $\hat{u}(0) = 7$ as the performance metric is relatively flat around this point).

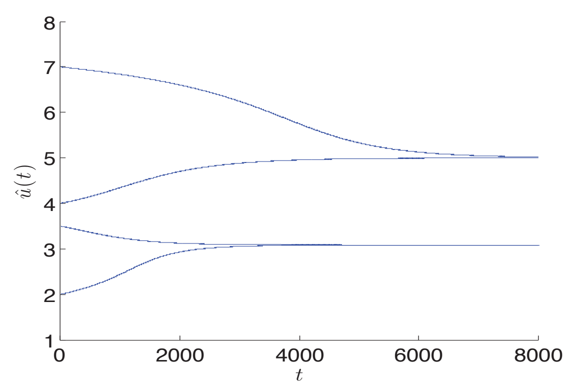

Fig. 8 Convergence of $\hat{u}$ to local maxima for different $\hat{u}(0)$

The speed of convergence to the extremum for different ϵ is illustrated in Figure 9. It is clear that increasing ϵ increases the rate of convergence, however at $\epsilon = 0.01$, the time scale separation between the perturbation frequency and the learning rate starts to break down and small oscillations are observed in the response.

This can be addressed by the inclusion of appropriate filters, as discussed in Section 2.3. To illustrate their effect, in Figure 10 a low pass filter with $T_{LP} = 0.4$ and high pass filter with $T_{HP} = 10$ have been included and the results compared to the case without filters for ϵ further increased to 0.1. It is apparent that the filters do not directly impact

[4]Actually, for the above averaging analysis to hold it is equally not necessary that g be differentiable or even Lipschitz continuous, all that is required is that g_{av} be Lipschitz continuous, [132].

[5]In order to apply extremum seeking where the steady state characteristic is not Lipshitz continuous, it is important that the extremal conditions in Assumption 3 are restated in terms of the averaged quantities g_{av}, not the original steady state characteristic g.

duce the oscillations in $\hat{u}$ and $h(x)$ – the latter is not shown explicitly here.

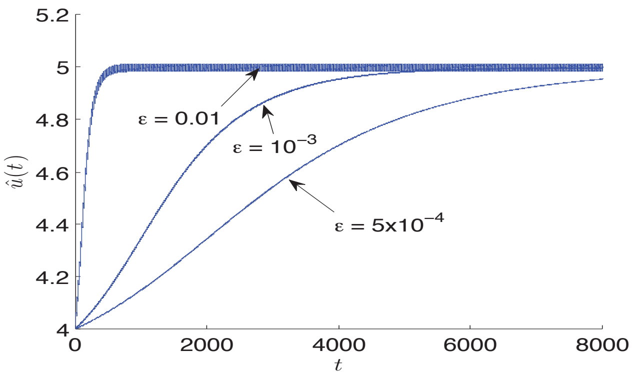

Fig. 9 Convergence to extremum using different ϵ

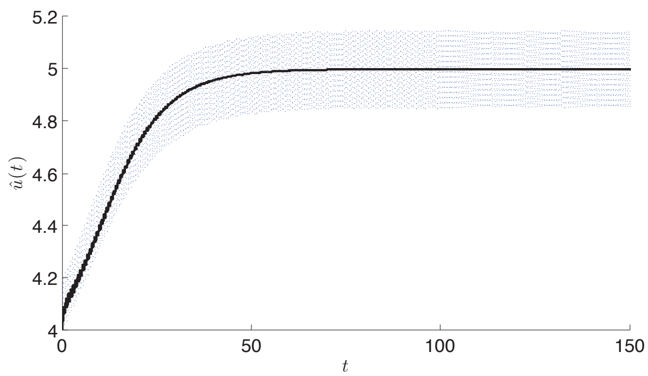

Fig. 10 Convergence to extremum (Black line) with LPF and HPF; and (Dotted blue) without filters.

Just to illustrate the capacity to deal with non-differentiable steady state characteristics, consider the following toy example:

$$\begin{aligned} \dot{x} &= -3x(1+u^2)+3f(u); \\ \dot{\hat{u}} &= -\tfrac{\epsilon}{a}\left(x+\sin(3\pi t)\right)\sin(\omega t); \\ u &= \hat{u}+a\sin(\omega t); \\ f(u) &= \begin{cases} 3u & \forall u<1 \\ 3e^{(1-u)} & \forall u>1 \end{cases} \end{aligned}$$

Select $a=0.1$, $\omega=1$ and $\epsilon=0.01$.

Here $\omega=1$ is selected well inside the bandwidth of the x dynamics (about 3), a is of the order of 10% of the expected size of x or u (which is rather large, but chosen so as to illustrate the approximation point in the above result). Also ϵ is relatively large compared to a. Clearly the disturbance $d(t)=\sin(3\pi t)$ is independent of the dither signal. Notice that it is relatively large compared to x, but observe that

$$|\epsilon\int_0^{1/\epsilon} d(t)\sin(\omega t)dt| << \epsilon,$$

from which it can be concluded that the disturbance's influence is minimal.

The theory sketched above provides $O(a)+o(\epsilon)\approx O(a)$ approximations to the overall behavior, and the Figures 11 and 12 below illustrate how the steady state (dashed line) tremum is approximated (again $O(a)$). The convergence time to steady state is $O(1/\epsilon)$ as illustrated in Figure 13. The three different time scales are clearly visible in the response.

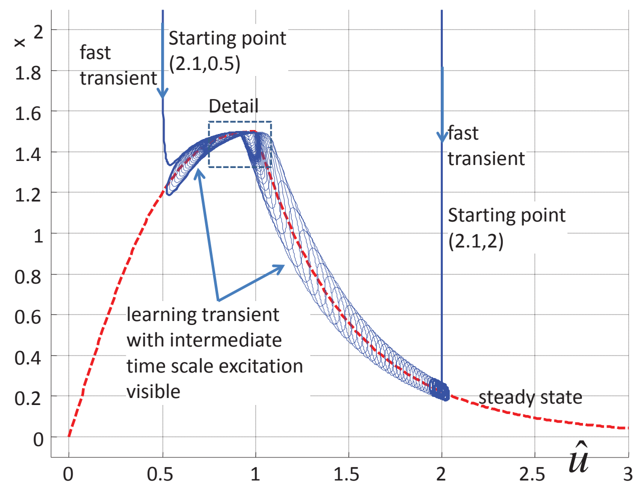

Fig. 11 Intermediate transients hug the steady state.

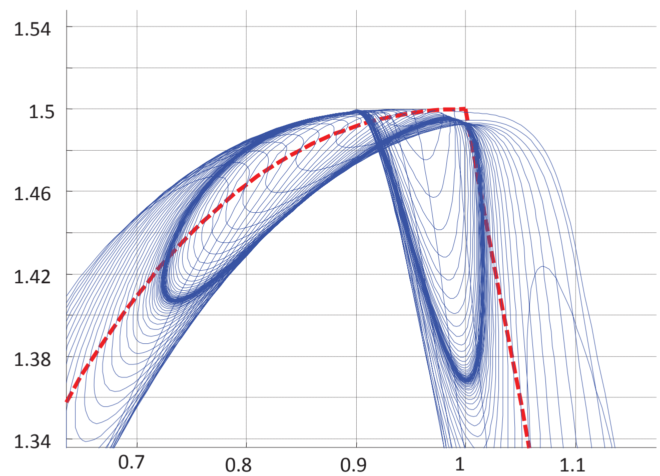

Fig. 12 Enlarged detail from Figure 11.

6 CONCLUSION

Extremum seeking is a popular adaptive control technique. It is truly model free. The underlying assumptions enabling the approach are easily satisfied and verified in a wide variety of applications: a steady state input-output map must exist, and this map must have a desired extremum that persists, remains stable, under minor (dynamic) perturbations. Moreover, due to the approximation involved, the steady state characteristic does not have to be differentiable in the traditional sense at the extremum.

Designing a successful extremum seeking approach within a specific context is relatively straightforward as long as the principles of time scale separation, as revealed in the analysis, can be, and are indeed adhered to: learning or adaptation happens in the slowest time scale, the (periodic) excitation's period is significantly shorter than the horizon over which adaptation is effective, and the excitation frequency is in the pass band of the plant, and uncorrelated with other inputs driving the plant's response. In case the extremum is

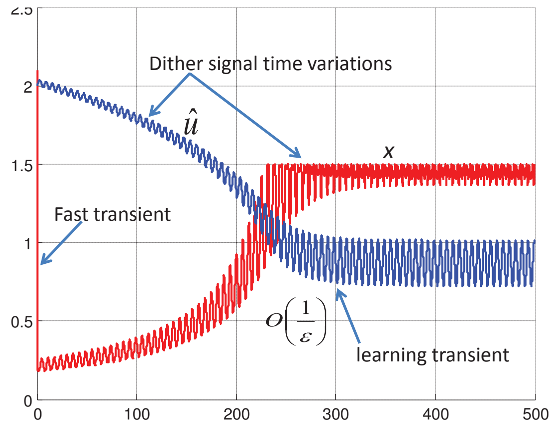

Fig. 13 Convergence to (averaged) extremum.

time varying, it is essential that its time variation exists in an even slower time scale than the learning dynamics themselves. The time scale separation ideas apply mutatis mutandis. These same time scale separation ideas work in the broader context of adaptive systems as explained in [4].

As extremum seeking is at its core a gradient based optimization method, it inherits all the shortcomings of such methods. In the presence of local extrema, a global extremum will not be found without exploring many different initializations. Modifications dealing with local extrema and passage through local extrema using ideas from simulated annealing have been explored, [151] but much work remains to be done. In particular design in the context of multi-valued steady state relationships (for example [20]), and when the steady state relationship is not differentiable requires further work.

There is a significant literature that deals with extremum seeking and the enclosed bibliography, although not insubstantial, is but a minor subset of the literature (the bibliography identifies less than 20% of the existing literature) (a Google Scholar search locates 990 distinct publications[6] over the period 1960-2010). In this paper only the simplest form of extremum seeking has been pursued as to reveal the essence of the extremum seeking methodology without the clutter of much technicalities. Multivariable versions, considering pareto optimality, optimality in the face of operational constraints, dynamic as well as static, as well as higher order methods rather than simple gradient techniques have been explored, but also form the topic of much ongoing research.

ACKNOWLEDGEMENTS

The authors acknowledge the support of the Australian Research Council through the Discovery Project Grant scheme as well as the support received through the Future Fellowship Scheme for Dr Ying Tan and Prof Dragan Nešić.

REFERENCES

[1] V. Adetola, D. Dehann, M. Guay, "Adaptive extremum-seeking receding horizon control of nonlinear systems", 2937–2942, 2004.

[2] V. Adetola and M. Guay, "Adaptive output feedback extremum seeking receding horizon control of linear systems", *Journal of Process Control*, vol. 16, pp. 521–533, 2006.

[3] V. Adetola and M. Guay, "Parameter convergence in adaptive extremum-seeking control", *Automatica*, vol. 43, pp. 105–110, 2007.

[4] Brian D. O. Anderson, Robert R. Bitmead, C. Richard Johnson, Jr., Petar V. Kokotovic, Robert L. Kosut, Iven M.Y. Mareels, Laurent Praly, Bradley D. Riedle, *Stability of adaptive systems: passivity and averaging analysis*, MIT Press, 1986

[5] R. Antonello, R. Oboe, "Mode-matching in vibrating microgyros using extremum seeking control", *the IEEE 33rd Annual Conference of Industrial Electronics Society*, pp. 2301–2306, 2007.

[6] R. Antonello and R. Oboe, "Mode-matching in vibrating microgyros using an extremum seeking controller with switching logic" *34th Annual Conference of IEEE on Industrial Electronics*, pp. 3425–3430, 2008.

[7] R. Antonello, R. Oboe, L. Prandi and F. Biganzoli, "Automatic mode matching in MEMS vibrating gyroscopes using extremum-seeking control", *IEEE Transactions on Industrial Electronics*, vol.56, No. 10, pp. 3880–3891, 2009.

[8] N. Araki, T. Sato, Y. Kumamoto, Y. Iwai, Y Konishi, "Design of weigh feeder control system using extremum-seeking method" *Proceedings of 2009 Asian Control Conference* pp. 250–255, 2009.

[9] K. J. Aström, Adaptive Control Around 1960, *Proceedings of the 34th IEEE CDC*, New Orleans, Dec 1995. Also in *IEEE Control Systems*, June 1996, pp44-49.

[10] K. J. Aström and B. Wittenmark, *Adaptive control (2nd ed.)* Reading, MA: Addison-Wesley, 1995.

[11] K. B. Ariyur and M. Krstić, "Slope seeking and application to compressor instability control", *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 3690–3697, 2002.

[12] K. B. Ariyur and M. Krstić, "Analysis and design of multivariable extremum seeking", *Proceedings of the 2002 American Control Conference*, Anchorage, AK May 8-1, pp. 2903–2908, 2002.

[13] K. B. Ariyur and M. Krstić, *Real-Time Optimization by Extremum- Seeking Control*. Hoboken, NJ: Wiley-Interscience, 2003.

[14] W. Bamberger and R. Isermann, "Adaptive on-line steadystate optimization of slow dynamic processes", *Automatica*, vol. 14, pp. 223–232, 1978.

[15] A. Banaszuk, Y. Zhang, C. A. Jacobson, "Adaptive control of combustion instability using extremum-seeking", *Proceedings of the 2000 American Control Conference*, pp. 416–422, 2000.

[16] A. Banaszuk, K. B. Ariyur, M. Krstić and C. A. Jacobson, "An adaptive algorithm for control of combustion instability and application to compressor instability control", *Automatica*, vol. 40, No. 11, pp. 1965–1972, 2003.

[17] R. N. Banavar, D. F. Chichka, J. L. Speyer, "Functional feedback in an extremum seeking loop", *Proceedings of the 40th IEEE Conference on Decision and Control*, pp. 1316–1321, 2001.

[18] R. N. Banavar, "Extremum seeking loops with assumed functions: estimation and control", *Proceedings of the American Control Conference*, Anchorage, AK May 8-10, pp. 3159–3164, 2002.

[6] In this context it may be of some interest to observe that the h-index for the phrase *extremum seeking* is 32.

functions: estimation and control", *International Journal of Control*, vol. 76, No. 14, pp. 1475-1482, 2009.

[20] G. Bastin, D. Nešić, Y. Tan and I. Mareels, "On extremum seeking in bioprocesses with multi-valued cost functions", *Biotechnology Progress*, May/June, Vol. 25, No 3, pp.683–690, 2009.

[21] R. Becker, R. King, R. Petz, and W. Nitsche, "Adaptive closed-loop separation control on a high-lift configuration using extremum seeking", *AIAA J.*, vol. 45, No. 6, pp. 1382-1392, 2007.

[22] J. F. Beaudoin, O. Cadot, J. L. Aider, J. E. Wesfreid, "Bluff-body drag reduction by extremum-seeking control", *Journal of Fluids and Structures*, vol. 22, pp. 973–978, 2006.

[23] P. Binetti, K. B. Ariyur, M. Krstić and F. Bernelli, "Control of formation flight via extremum seeking", *Proceedings of the 2002 American Control Conference* Anchorage, AK May 8-1 pp. 2848–2853, 2002.

[24] E. Biyik and M. Arcak, "Gradient climbing in formation via extremum seeking and passivity-based coordination rules", *Proc. 46th IEEE Conf. Decis. Contr.*, New Orleans, USA, 2007.

[25] P. F. Blackman, "Extremum-seeking regulators", In J. H. Westcott, *An exposition of adaptive control* New York: The Macmillan Company, 1962.

[26] A. I. Bratcu, I. Munteanu, E. Ceanga, S. Epure "Energetic optimization of variable speed wind energy conversion systems by extremum seeking control" *Proceedings of EUROCON, 2007, The International Conference on "Computer as a Tool"*, pp. 2536–2541, 2007.

[27] J. Bredenbeck, "Statistical design of experiments for on-line optimization of internal-combustion engines", (in German), *MTZ Motortech. Z.*, vol. 60, No. 11, pp. 740-744, 1999.

[28] S. L. Brunton, C. W. Rowley, S. R. Kulkarni, C. Clarkson, "Maximum power point tracking for photovoltaic optimization using extremum seeking", *2009 34th IEEE Conferene on Photovoltaic Specialists Conference (PVSC)*, pp. 000013–000016, 2009.

[29] D. Carnevale, A. Astolfi, C. Centioli, S. Podda, V. Vitale, L. Zaccarian, "A new extremum seeking technique and its application to maximize RF heating on FTU", *Fusion Engineering and Design*, vol. 84, Jpp. 554–558, 2009.

[30] Y. Y. Chang and S. J. Moura, "Air flow control in fuel cell systems: an extremum seeking approach", *Proceedings of 2009 American Control Conference*, Hyatt Regency Riverfront, St. Louis, MO, USA June 10-12, pp. 1052–1059, 2009.

[31] C. Centioli, F. Iannone, G. Mazza, M. Panella, L. Pangione, S. Podda, A. Tuccillo, V. Vitale, and L. Zaccarian, "Extremum seeking applied to the plasma control system of the Frascati Tokamak Upgrade", *Proc. 44th IEEE Conf. Decision Control Eur. Control Conf.*, Dec. pp. 8327-8232, 2005.

[32] J. Y. Choi; M. Krstić, K. B. Ariyur and J. S. Lee, "Extremum seeking control for discrete-time systems", *IEEE Transactions on Automatic Control*, vol. 47, No. 2, pp. 318–323, 2002.

[33] D. F. Chichka, J. L. Speyer, and C. G. Park, "Peak-seeking control with application to formation flight", *Proceedings of the 38th IEEE conference on decision and control*, December, 1999.

[34] M. R. Cistelecan, "Power control in mobile wireless networks using sliding mode extremum seeking control implementing bifurcations", *IEEE International Conference on Control Applications 2008*, pp. 67–72, 2008.

with tuning of angular velocity", *IEEE Transactions on Automatic Control*, vol. 54, No. 4, pp. 717–731, 2009.

[36] C. Danielson, S. Lacy, B. Hindman, P. Collier, J. Hunt, R. Moser, "Extremum seeking control for simultaneous beam steering and wavefront correction", *Proceedings of 2006 American Control Conference*, 2006.

[37] D. DeHaan and M. Guay, "Extremum seeking control of nonlinear systems with parametric uncertainties and state constraints", *Proceedings of the 2004 American Control Conference*, pp. 596–601, 2004.

[38] C. Dixon and E. W. Frew, "Controlling the mobility of network nodes using decentralized extremum seeking" *Proceedings of the 45th IEEE Conference on Decision & Control*, Manchester Grand Hyatt Hotel San Diego, CA, USA, December 13-15, pp. 1291-1296, 2006.

[39] P. M. Dower, P. Farrell and D. Nešić, "Extremum seeking control of cascaded Raman optical amplifiers", *IEEE Transactions on Control Systems Technology*, 2007.

[40] S. Drakunov, U. Özgüner, P. Dix and B. Ashra, "ABS control using optimum search via sliding modes", *IEEE Transactions on Control Systems Technology*, vol. 3 pp. 79–85, 1995.

[41] C. S. Draper and Y. T. Li, "Principles of Optimalizing Control Systems and an Application to the Internal Combustion Engine", *In R. Oldenburger, Ed., Optimal and selfoptimizing control*. Boston, MA: The M.I.T. Press, 1951.

[42] E. Elong, M. Krstić and K. B. Ariyur, "A case study of performance improvement in extremum seeking control", *Proceedings of the American Control Conference*, Chicago, Illinois June,pp. 428–432, 2000

[43] F. Esmaeilzadeh Azar, M. Perrier, B. Srinivasan, "A global optimization method based on multi-unit extremum-seeking for scalar nonlinear systems", *Computers & Chemical Engineering*, In Press, Corrected Proof, Available online 10 April 2010.

[44] P. Eykhoff, Adaptive and Optimilazing Systems, *IRE Transactions on Automatic Control*, June 1960, pp148-151

[45] A. Favache, D. Dochain, M. Perrier and M. Guay, "Extremum-seeking control of retention for a microparticulate system", *The Candian Jounal of Chemical Engineering*, vol. 86, pp. 815–827, 2008.

[46] J. J. Floretin "An approximately optimal extremal regulator", *J. Electron. and Control*, vol 17, No. 2, pp. 211–310, 1964.

[47] J. S. Frait and P. Eckman, "Optimizing control of single input extremum systems", *J. Basic Engrg.*, vol. 84, pp. 85–90, 1962.

[48] A. L. Frey, W. B. Deem and R. J. Altpeter, "Stability and optimal gain in extremum-seeking adaptive control of a gas furnace". *Proceedings of the Third IFAC World Congress*, London, (p. 48A), 1966.

[49] L. Fu and U. Ozguner, "Variable structure extremum seeking control based on sliding mode gradient estimation for a class of nonlinear systems" *Proceedings of 2009 American Control Conference*, pp. 8–13, 2009.

[50] S. Fujii and N. Kanda, "An optinmalizing control of boiler pebble mills", *Proc. 2nd. IFAC Conf.*, Butterworth, London, 1963.

[51] N. Ghods and M. Krstić, "Extremum seeking with very slow or drifting sensors", *2009 American Control Conference*, Hyatt Regency Riverfront, St. Louis, MO, USA June 10-12, pp. 1946–1951, 2009.

[52] M. Guay, D. Dochain and M. Perrier, "Adaptive extremum seeking control of continuous stirred tank bioreactors with

2004.

[53] M. Guay, D. Dochain and M. Perrier,, “Adaptive extremum seeking control of nonisothermal continuous stirred tank reactors”, *Chemical Engineering Science*, vol. 60, No. 13, pp. 3671-3680, 2005.

[54] M. Guay and T. Zhang, “Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties”, *Automatica*, vol. 39, 1283–1293, 2003.

[55] M. Guay, D. Dochain, M. Perrier, and N. Hudon, “Flatness-based extremum-seeking control over periodic orbits”, *IEEE Transactions on Automatic Control*, vol. 52, No. 10, pp. 2005–2012, 2007.

[56] L. Gurvich, “Fuzzy logic base extremum seeking control system”, *Proceedings of 2004 23rd IEEE Convention of Electrical and Electronics Engineers*, pp. 18-21, 2004.

[57] L. Gurvich, I. Gurvich, “Analysis and methods for control of objects with quasi-static time-dependent characteristic - objects with elastic and plastic static characteristic”, *Proceedings of IEEE 25th Convention of Electrical and Electronics Engineers in Israel*, pp. 125–129, 2008.

[58] M. Hamza, “Extremum control of continuous systems”, *IEEE Transactions on Automatic Control*, vol. 11,No. 2, pp. 182–189, 1966.

[59] M. H. Hamza, Synthesis of extremum-seeking regulators, in Azdotnalion and Remote Control, vol. XXI., pt. 8 (in Russian). USSR: The Academy of Sciences, 1964.

[60] I. Haskara, G. G. Zhu and J. Winkelman, “Multivariable EGR/spark timing control for IC engines via extremum seeking”, *Proceedings of 2006 American Control Conference*, 2006.

[61] L. Henning, R. Becker, G. Feuerbach, R. Muminovic, R. King, A. Brunn and W. Nitsche, “Extensions of adaptive slope-seeking for active flow control”, *Proc. IMechE Part I: J. Systems and Control Engineering*, vol. 22, pp. 309–922, 2008.

[62] Y. A. Hu and B. Zuo, “An annealing recurrent neural network for extremum seeking control”, *International Journal of Information Technology*, vol. 11, No. 6, pp. 45–52, 2005.

[63] Y. A. Hu, B. Zuo and X. D. Li, “The application of an annealing recurrent neural network for extremum seeking algorithm to optimize UAV tight formation flight”, *IMACS Multiconference on Computational Engineering in Systems Applications*, pp. 613–620, 2006.

[64] N. Hudon, M. Guay, M. Perrier and D. Dochain, “Adaptive extremum seeking control of a tubular reactor with limited actuation”, *Proceedings of the 2005 American Control Conference*, pp. 4563–4568, 2005.

[65] O. L. R. Jacobs and G. C. Shering, “Design of a single-input sinusoidal-perturbation extremum-control system”. *Proceedings IEE*, vol. 115, pp. 212–217, 1968.

[66] O.L.R. Jacobs, S.M. Langdon, “An optimal extremal control system”, *Automatica*,vol. 6, No. 2, pp. 297–301, 1970.

[67] Y. Iwai, Y. Konishi, N. Araki, H. Ishigaki, “Optimum integrated design of mechanical structure/controller using bargaining game theory”, *Proceedings of International Conference on Control, Automation and Systems*, pp. 1–6, 2008.

[68] V.V.Kazakevich, “Technique of automatic control of different processes to maximum or to minimum”, Avtorskoe svidetelstvo, (USSR Patent), No 66335, 25 Nov 1943.

[69] V.V.Kazakevich, *On extremum seeking*, PhD Thesis, Moscow High Technical University, 1944.

[70] V. V. Kazakevich, “Extremum control of objects with inertia and of unstable objects”, *Soviet Physics*, vol. l, No. 5, pp. 658–661, 1960.

some algorithms for the control of inertial objects of optimization in the presence of drift”, *Automat. Remote Contr.*, No. 11, pp. 49-56, 1974.

[72] V. V. Kazakevich, “Joint identification and accelerated optimization of plants with lag”, *Automat. Remote Contr.*, No. 9, pp. 62-73, 1984.

[73] V. V. Kazakevich and Y. V. Shcherbina, “Design of continuous-discrete extremal control systems that are stable under low-frequency disturbances”, *Automat. Remote Contr.*, No. 2, pp. 59-64, 1979.

[74] N. J. Killingsworth and M. Krstić, “PID tuning using extremum seeking: online, model-free performance optimization”, *IEEE Transactions on Control Systems Magazine*, vol. 26, No. 1, pp. 70–79, 2006.

[75] N. J. Killingsworth, S. M. Aceves, D. L. Flowers, F. Espinosa-Loza, and M. Krstić, “HCCI engine combustion-timing control: optimizing gains and fuel consumption via extremum seeking”, *IEEE Transactions on Control Systems Technology*, vol. 17, No. 6, pp. 1350–1361, 2009.

[76] K. Kim, C. Kasnakoglu, A. Serrani and M. Samimy, “Extremum-seeking control of subsonic cavity flow”, *Proceedings of the 46th AIAA aerospace sciences meeting and exhibit*, 2008.

[77] R. King, R. Becker, G. Feuerbach, L. Henning, R. Petz, W. Nitsche, O. Lemke, W. Neise, “Adaptive flow control using slope seeking”, *Proceedings of 14th Mediterranean Conference on Control and Automation*, pp. 1–6, 2006.

[78] S. K. Korovin and V. I. Utkin, “The use of the slip mode in problems of static optimization”, *Automatic and Remote Control*, pp. 50-60, 1972.

[79] S. K. Korovin and V. I. Utkin, “Using sliding modes in static optimization and nonlinear programming”, *Automatica*, vol. 10, pp. 525-532, 1974.

[80] H. K. Khalil, *Nonlinear Systems*, Third edition Prentice Hall, Upper Saddle River, New Jersey, 2002.

[81] A . A. Krasovsky, “Problems in the theory of continuous systems of extremum process control”, *Proc. Internatl Federation of Automatic Control Congress*, Basle, Switzerland, 1963.

[82] M. Krstić, “Towards faster adaptation in extremum seeking control”, *Proceedings of the 38th IEEE Conference on Decision & Control* Phoenix, Arizona USA December, pp. 4766–4771, 1999.

[83] M. Krstić, “Performance improvement and limitations in extremum seeking control”, *Systems and Control Letters*, vol. 39, 313–326, 2000.

[84] M. Krstić and A. Banaszuk, “Multivariable adaptive control of instabilities arising in jet engines”, *Control Engineering Practice*, vol. 14, pp. 833–842, 2006.

[85] M. Krstić, and J. Cochran, “Extremum seeking for motion optimization: From bacteria to nonholonomic vehicles”, *Chinese Control and Decision Conference*, pp. 18–27, 2008.

[86] M. Krstić and H.-H. Wang, “Stability of extremum seeking feedback for general nonlinear dynamic systems”, *Automatica*, vol. 36, 595–601, 2000.

[87] E. Lavretsky, E. Hovakimyan, A. Calise, “Adaptive extremum seeking control design”, *Proceedings of 2003 American Control Conference*, pp. 567–572, 2003.

[88] M. Leblanc, “Sur l’electri”cation des chemins de fer au moyen de courants alternatifs de frequence elevee”, *Revue Generale de l’Electricite*, 1922.

[89] R. Leyva, C. Alonso, I. Queinnec, A. Cid-Pastor, D. Lagrange, L. Martinez-Salamero, “MPPT of photovoltaic systems using extremum seeking control”, *IEEE Transations*

249–258, 2006.

[90] P. F. Li, Y. Y. Lu and J. E. Seem, “Extremum seeking control for efficient and reliable operation of air-side economizers”, *Proceedings of 2009 American Control Conference*, pp. 20–25, 2009.

[91] Y. Y. Li, M. A. Rotea, G. T. C. Chiu, l. G. Mongeau and I. S. Paek, “Extremum seeking control of tunable thermoacoustic cooler”, *IEEE Transactions on Control System Technology* vol. 13, No. 4, pp. 527–536, 2005.

[92] S. J. Liu and M. Krstić, “Stochastic averaging in continuous time and its applications to extremum seeking”, *IEEE Transactions on Automatic Control*, to appear, 2010.

[93] L. X. Luo ad E. Schuster, “Mixing enhancement in 2D magnetohydrodynamic channel flow by extremum seeking boundary control” *Proceedings of 2009 American Control Conference*, Hyatt Regency Riverfront, St. Louis, MO, USA June 10-12, pp. 1530–1535, 2009.

[94] L. X. Luo and E. Schuster, “Boundary feedback control for heat exchange enhancement in 2D magnetohydrodynamic channel flow by extremum seeking”, *Proceedings of the 48th IEEE Conference on Decision and Control, jointly with the 2009 28th Chinese Control Conference*, pp. 8272–8277, 2009.

[95] J. Luxat, L. Lees, “Stability of peak-holding control systems”, *IEEE Transactions on Industrial Electronics and Control Instrumentation*, pp. 11–15, 1971.

[96] J. Luxat, “Design and analysis of an electronic peak-holding controller”, *IEEE Transactions on Industrial Electronics and Control Instrumentation*, pp. 173–179, 1974.

[97] C. Manzie and M. Krstić, “Extremum seeking with stochastic perturbations”, *IEEE Transactions on Automatic Control*, vol. 54, NO. 3, pp. 580–585, 2009.

[98] N. Marcos, M. Guay, T. Zhang and D. Dochain, “Adaptive extremum-seekingcontrol of a continuous stirred tank bioreactor with Haldane kinetics”, *Journal of Process Control*, vol. 14, No. 3, pp. 317-328, 2004.

[99] C. G. Mayhew, R. G. Sanfelice and A. R. Teel, “Robust source-seeking hybrid controllers for autonomous vehicles”, *Proceedings of the 2007 American control conference*, pp. 1185–1190, 2007.

[100] S. M. Meerkov, “Asymptotic methods for investigating a class of forced states in extremal systems”, *Automation and Remote Control*, vol, 12, pp. 1916–1920, 1967.

[101] S. M. Meerkov, “Asymptotic methods for investigating quasistationary states in continuous systems of automatic optimization”, *Automation and Remote Control*, vol. 11, pp. 1726–1743, 1967.

[102] S. M. Meerkov, “Asymptotic methods for investigating stability of continuous systems of automatic optimization subjected to disturbance action”, (in Russian), *Avtomatika i Telemekhanika*, vol. 12, pp. 14–24, 1968.

[103] W. H. Moase, C. Manzie and M. J. Brear, “Newton-Like extremum-seeking for the control of thermoacoustic instability”, *IEEE Transactions on Automatic Control*, vol. 55, 2010, Digital Object Identifier: 10.1109/TAC.2010.2042981, 2010.

[104] J. P. Moeck, M. R. Bothien, C. O. Paschereit, G. Gelbert and R. King, “Two-parameter extremum seeking for control of thermoacoustic instabilities and characterization of linear growth”, *AIAA paper* pp. 1416, 2007.

[105] I. S. Morosanov, “Method of extremum control”, *Automatic and Remote Control*, vol. 18, pp. 1077–1092, 1957.

[106] H. Nakadoi, D. Sobey, M. Yamakita, T. Mukai, “Liquid environment-adaptive IPMC fish-like robot using extremum seeking feedback”, *Proceedings of 2008. IEEE/RSJ International* 3089–3094, 2008

[107] D. Nešić, “Extremum seeking control: convergence analysis”,*Europ. J. Contr.*, vol. 15, No. 3-4, pp. 331–347, 2009.

[108] D. Nešić, A. Mohamadi and C. Manzie, “A unifying approach to extremum seeking based on parameter estimation”, submitted to Conf. Decis. Contr., Atlanta, Georgia, USA, 2010.

[109] Nusawardhana, S. H. Zak, “Extremum seeking using analog nonderivative optimizers”, *Proceedings of the 2003 American Control Conference*, pp. 3242–3247, 2003.

[110] Nusawardhana, S. H. Zak, “Simultaneous perturbation extremum seeking method for dynamic optimization problems”, *Proceedings of the 2004 American control conference*, pp. 2085–2810, 2004.

[111] V. K. Obabkov, “Theory of multichannel extremal control systems with sinusoidal probe signals”, *Automation and Remote Control*, vol. 28, pp. 48-54, 1967.

[112] P. Ögren, E. Fiorelli and N.E. Leonard, “Cooperative control of mobile sensor networks: adaptive gradient climbing in a distributed environment”, *IEEE Trans. Automat. Contr.*, vol. 49, 1292–1302, 2004.

[113] R. Oldenburger, Ed., “Optimal and Self-Optimizing Control”, MIT Press, 1966.

[114] C. Olalla, M. I. Arteaga, R. Leyva, A. E. Aroudi, “Analysis and comparison of extremum seeking control techniques”, *Proceedings of IEEE International Symposium on Industrial Electronics*, pp. 72–76, 2007.

[115] I. I. Ostrovskii, “Extremum regulation”, *Automatic and Remote Control*, vol. 18, 900–907, 1957.

[116] Y. Ou, C. Xu, E. Schuster, T. C. Luce, J. R. Ferron, M. L. Walker, “Extremum-seeking finite-time optimal control of plasma current profile at the DIII-D tokamak” *Proceedings of 2007 American Control Conference*, pp. 4015-4020, 2007.

[117] Y. Ou, C. Xu, E. Schuster, T. C. Luce, J. R. Ferron, M. L. Walker and D. A. Humphreys, “Design and simulation of extremum-seeking open-loop optimal control of current profile in the DIII-D tokamak”, *Plasma Physics and Controlled Fusion*, vol. 50, pp. 1–24, 2008.

[118] T. L. Pan, Z. C. Ji and Z. H. Jiang, “Conversion systems based on sliding mode extremum seeking control”, *Proceedings of IEEE Energy 2030*, Atlanta, GA USA 17-18 November, pp. 1–5, 2008.

[119] Y. D. Pan, T. Acarman and Ümit Özgüner, “Nash solution by extremum seeking control approach”, *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 329–334, 2002.

[120] Y. D. Pan and K. Furuta, “Variable structure control by switching among feedback control laws”, *Proceedings of 45th IEEE Conference on Decision and Control*, pp. 789–794, 2006.

[121] Y. D. Pan and Ümit Özgüner, “Discrete-time extremum seeking algorithms”, *Proceedings of 2002 American Control Conference*, Anchorage, AK, May 8-10, pp. 3147–3152, 2002.

[122] Y. D. Pan and Ümit Özgüner, “Sliding mode extremum seeking control for linear quadratic dynamic game”, *Proceedings of the 2004 American Control Conference* Boston, Massachusetts, pp. 614–619, 2004.

[123] A. A. Pervozvanskii, “Continuous extremum control system in the presence of random noise”, *Automatic and Remote Control*, vol. 21, 673–677, 1960.

[124] K. S. Peterson and A. G. Stefanopoulou, “Extremum seeking control for sof landing of an electromechanical valve actuator”, *Automatica*, vol. 40, 1063–1069, 2004.

and training algorithms", *Automation and Remote Control*, vol. 34, No. 3, pp. 377-397, 1973.

[126] D. Popović, M. Janković, S. Manger and A. R. Teel, "Extremum seeking methods for optimization of variable cam timing engine operation", *IEEE Transactions on Control Systems Technology*, vol. 14, 2006.

[127] V. P. Putsillo, "The principles of construction of one class of extermal control systems for automatized circuits in production process", *Proceedings of Internatl Federation of Automatic Control Congress*, Moscow, USSR, 1960.

[128] L, Pun, "Comments on adaptation concepts and on analysis of control systems", *IEEE Transations on Automatic Control*, vol 8,No. 3,pp. 266–269, 1963.

[129] J. D. Roberts, "Extremum or hill-climbing regulation: a statistical theory involving lags, disturbances and noise", *Proc. IEE*, vol. 112, pp. 137–150, 1965.

[130] M. A. Rotea, "Analysis of multivariable extremum seeking algorithms", *Proceedings of the 2000 American Control Conference*, pp. 433–437, 2000.

[131] P. Sadegh, "Constrained optimization via stochastic approximation with a simultaneous perturbation gradient approximation", *Automatica*, vol. 33, No. 5, pp. 889-892, 1997.

[132] Sanders, J. A. and Verhulst, F. *Averaging methods in nonlinear dynamical systems*, Springer-Verlag, New York, 1985.

[133] G. Schneider, K. B. Ariyur, M. Krstić, "Tuning of a combustion controller by extremum seeking: a simulation study", *Proceedings of the 39th IEEE Conference on Decision and Control*, pp. 5219–5223, 2000.

[134] E. Schuster, E. Morinaga, C. K. Allen, M. Krstić, "Optimal beam matching in particle accelerators via extremum seeking", *Proceedings of 2006 American Control Conference*, Minneapolis, Minnesota, USA, June 14-16, pp. 1962–1967, 2006.

[135] P. G. Scotson, "Self-tuning optimisation of automotive engines", PhD thesis, Control Systems Centre, UMIST, 1986.

[136] P. Scotson and P. E. Wellstead, "Self-tuning optimisation of spark ignition automobile engines", *Proc. 1989 ACC*, Pittsburgh, USA, 1989.

[137] P. Scotson and P. E. Wellstead, "Self-tuning optimization of spark ignition automotive engines", *IEEE Control Syst. Mag.*, vol. 3, No. 10, pp. 94-101, 1990.

[138] S. Skogestad, "Plantwide control: The search for the self-optimizing control structure", *Journal of Proceedings Control*, vol. 10, pp. 487-507, 2000.

[139] J. C. Spall, "Multivariate stochastic approximation using simultaneous perturbation gradient approximation", *IEEE Trans. Automat. Contr.*, vol. 37, pp. 332-341, 1992.

[140] J. Spall, "A one-measurement form of simultaneous perturbation stochastic approximation", *Automatica*, vol. 33, pp. 109-112, 1997.

[141] J. Spall, "Implementation of the simultaneous perturbation algorithm for stochastic optimization", *IEEE Trans Aerosp. Electron. Syst.*, pp. 817-823, 1998.

[142] J.L. Speyer, R. N. Banavar, D. F. Chichka, I. Rhee, "Extremum seeking loops with assumed functions", *Proceedings of the 39th IEEE Conference on Decision and Control*, Sydney, Australia December, pp. 142–147, 2000.

[143] M. S. Stankovic and D. M. Stipanovic, "Discrete time extremum seeking by autonomous vehicles in a stochastic environment", *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference*, pp. 4541–4546, 2009.

[144] J. Sternby, "Extremum control systems: An area for adaptive control?", *Proc. Joint Amer. Contr. Conf.*, San Francisco, CA, 1980.

extremum seeking method for non-isometric neuromuscular electrical stimulation", *Proceedings of the 41st IEEE Conference on Systems, Man and Cybernetics 2007*, pp. 2528–2532, 2007.

[146] S. Sugihira, K. Ichikawa, H. Ohmori, "Starting speed control of si engine based on online extremum control", *Proceedings of SICE 2007 Annual Conference*, pp. 2569–2573 , 2007.

[147] H. Takata, D. Matsumoto and T. Hachino, "An extremum seeking control via Chebyshev polynomial identification and LQ control", *Journal of Signal Processing*, vol. 7, No. 6, pp. 509–515, 2003.

[148] H. Takata, T. Hachino, R. Tamura and K. Komatsu, "Design of extremum seeking control with accelerator", *The Institute of Electronics, Information and Communication Engineers. Trans. Fundamentals*, vol. E88-A, No. 10, pp. 2535–2540, 2005.

[149] Y. Tan, D. Nešić, I. M.Y. Mareels, "On non-local stability properties of extremum seeking control", *Proc. 16th IFAC World Congress*, Prague, Czech Republic, pp. 2483–2489, 2005.

[150] Y. Tan, D. Nešić and I. M. Mareels, "On non-local stability properties of extremum seeking controllers", *Automatica*, vol. 42, 889–903, 2006.

[151] Y. Tan, D. Nešić, I. M. Mareels and A. Astolfi, "On the global extremum seeking control", Automatica, Vol. 45, No. 1, pp. 245–251, 2009.

[152] Y. Tan, D. Nešić, I. M. Mareels, "On the dither choice in extremum seeking control", Automatica, 2007.

[153] M. Tanelli, A. Astolfi and S. M. Savaresi, "Non-local extremum seeking control for active braking control systems", *Proceedings of the 2006 IEEE International Conference on Control Applications*, Munich, Germany, October 4-6, pp. 891–896, 2006.

[154] A. R. Teel and D. Popović, "Solving smooth and nonsmoooth multivariable extremum seeking problems by the methods of nonlinear programming" *Proc. 2000 Amer. Contr. Conf.*, Arlington, VA, June 25-27, (2001), 2394-2399, 2000.

[155] M. Titica, D. Dochain and M. Guay, "Adaptive extremum-seeking control of fed-batch bioreactors", *European Journal of Control*, vol. 9, No. 6, pp. 614-627, 2003.

[156] D. M. Titterington, "A method of extremum adaptation", *J. Inst. Math. Applics* vol. 11, pp. 297–315, 1973.

[157] D. M. Titterington, "An example of a perturbation-type extremum controller", *Journal of Applied Probability*, vol. 11, No. 3 pp. 513–520, 1974.

[158] H. S. Tsien and S. Serdengecti, "Annalysis of peak-holding optimalizing control", *J. Aero. Sci.*, vol. 22, pp. 561–570, 1955.

[159] I. Tunay, "Antiskid control for aircraft via extremum-seeking", *Proceedings of the 2001 American Control Conference*, pp. 665–670, 2001.

[160] V. Tyagi, H. Sane and S. Darbha, "An extremum seeking algorithm for determining the set point temperature for condensed water in a cooling tower", *Proceedings of 2006 American Control Conference*, 2006.

[161] S. van der Meulen, B. de Jager, E. van der Noll, F. Veldpaus, F. van der Sluis, M. Steinbuch, "Improving pushbelt continuously variable transmission efficiency via extremum seeking control," *Proceedings of 2009 IEEE Conference on Control Applications, (CCA) & Intelligent Control, (ISIC)*, pp. 357–362, 2009.

[162] G. Vasu, "Optimalizing control applied to control of engine pressure", *Transactions ASME*, vol. 79, pp. 481-488, 1957.

seeking control", *Proceedings of the 2000 American Control Conference*, pp. 411–415, 2000.

[164] H.-H. Wang, S. Yeung, and M. Krstić, "Experimental application of extremum seeking on an axial-flow compressor", *IEEE Transactions on Control Systems Technology*, **8**, pp. 300–309, 2000.

[165] H.-H. Wang and M. Krstić, "Extremum seeking for limit cycle minimization", *IEEE Transactions on Automatic Control*, vol. 45, 2432–2437, 2000.

[166] H.-H. Wang, M. Krstić, and G. Bastin, "Optimizing bioreactors by extremum seeking", *International Journal on Adaptive Control and Signal Processing*, vol. 13, pp. 651-669, 1999.

[167] P.E. Wellstead, P.G. Scotson, "Self-tuning extremum control", *IEE Proceedings*, vol. 137, No. 3, 1990.

[168] D. J. Wilde, *Optimum Seeking Methods*. Englewood NJ: Prentice-Hall Inc., 1964.

[169] F. Xie, X. M. Zhang, R. Fierro, M. Motter, "Autopilot-based Nonlinear UAV Formation Controller with Extremum-Seeking", *Proceedings of 44th IEEE Conference on Decision and Control, 2005 and 2005 European Control Conference*, pp. 4933–4938, 2005.

[170] H. Yu and U. Ozguner, "Extremum-seeking control strategy for ABS system with time delay", *Proceedings of the 2002 American Control Conference*, Anchorage, AK May 8-1, pp. 3753–3758, 2002.

[171] H. Yu and U. Ozguner, "Extremum-seeking control via sliding mode with periodic search signals", *Proceedings of the 41st IEEE Conference on Decision and Control*, pp. 323–328, 2002.

[172] H. Yu and U. Ozguner,"Smooth extremum-seeking control via second order sliding mode", *Proceedings of the 2003 American Control Conference*, pp. 3248–3253, 2003.

[173] S. Yamanaka and H. Ohmori, "Nonlinear adaptive extremum seeking control for time delayed index in the presence of deterministic disturbance", *Proceedings of PhysCon 2005*, St. Petersburg, Russia, 2005.

[174] C. Zhang and R. Ordóñez, "Numerical optimization-based extremum seeking control of LTI systems", *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference 2005*, Seville, Spain, December 12-15, pp. 4428–4433, 2005.

[175] C. Zhang and R. Ordóñez, "Non-gradient extremum seeking control of feedback linearizable systems with application to ABS design", *Proceedings of the 45th IEEE Conference on Decision & Control*, Manchester Grand Hyatt Hotel, San Diego, CA, USA, December 13-15, pp. 6666–6671, 2006.

based on numerical optimization and state regulation - Part I: theory and framework ", *Proceedings of the 45th IEEE Conference on Decision & Control*, Manchester Grand Hyatt Hotel, San Diego, CA, USA, December 13-15, pp. 4466-4471, 2006.

[177] C. Zhang and R. Ordóñez, "Extremum seeking control based on numerical optimization and state regulation - Part II: robust and adaptive control design", *Proceedings of the 45th IEEE Conference on Decision & Control*, Manchester Grand Hyatt Hotel San Diego, CA, USA, December 13-15, pp. 4460–4465, 2006.

[178] C. Zhang, D. Arnold, N. Ghods, A. Siranosian and M. Krstić, "Source seeking with non-holonomic unicycle without position measurement and with tuning of forward velocity", *Systems and Control Letters*, vol. 56, pp. 245–252, 2007.

[179] C. Zhang and R. Ordóñez, "Numerical optimization-based extremum seeking control with application to ABS design", *IEEE Transactions on Automatic Control*, vol. 52, No. 3, pp. 454–467, 2007.

[180] C. Zhang and R. Ordóñez, "Robust and adaptive design of numerical optimization-based extremum seeking control", *Automatica*, vol. 45, pp. 634–646, 2009.

[181] C. Zhang, A. Siranosian and M. Krstić, "Extremum seeking for moderately unstable systems and for autonomous vehicle target tracking without position measurements", *Automatica*, vol. 43, pp. 1832–1839, 2007.

[182] T. Zhang, M. Guay, and D. Dochain, "Adaptive extremum seeking control of continuous stirred-tank bioreactors", *AIChe Journal*, vol. 17, No. 10, pp. 113–123, 2003.

[183] X. T. Zhang, D. M. Dawson, W. E. Dixon, and B. Xian, "Extremum-seeking nonlinear controllers for a human exercise machine", *IEEE/ASME Transactions on mechatronics*, vol. 11. 2006.

[184] Y. Zhang, "Stability and performance tradeoff with discrete time triangular search minimum seeking", *Proceedings of the 2000 American control conference*, vol. 1, pp. 423–427, 2000.

[185] Z. Zhong, H. Huo, X. Zhu, G. Cao and Y. Ren, "Adaptive maximum power point tracking control of fuel cell power plants", *Journal of Power Sources*, vol. 176, pp. 259–269, 2008.

[186] B. Zuo and Y. A. Hu, "Optimizing UAV close formation flight via extremum seeking", *Proceedings of Fifth World Congress on Intelligent Control and Automation*, pp. 3302–3305, 2004.