University Library

A gateway to Melbourne's research publications

Minerva Access is the Institutional Repository of The University of Melbourne

Author/s:
Demarzo, PM;Frankel, DM;Jin, Y

Title:
Portfolio Liquidity and Security Design with Private Information

Date:
2021-12-01

Citation:
Demarzo, P. M., Frankel, D. M. & Jin, Y. (2021). Portfolio Liquidity and Security Design with Private Information. Review of Financial Studies, 34 (12), pp.5841-5885. https://doi.org/10.1093/rfs/hhaa147.

Persistent Link:
https://hdl.handle.net/11343/261046

Portfolio Liquidity and Security Design with Private Information†

Peter M. DeMarzo (*Stanford University*)

David M. Frankel (*Melbourne Business School*)

Yu Jin (*Shanghai University of Finance and Economics*)

This Revision: November 5, 2020

ABSTRACT. A privately informed seller seeks to liquidate a portfolio to raise cash. Each asset's liquidity thus depends on the impact of its sale on the value of the entire portfolio. We demonstrate the importance of cross-signaling and derive sufficient conditions for a liquidity "pecking-order" that determines the order of sale. For assets backed by a common pool, liquidity naturally aligns with seniority. Finally, we extend the portfolio liquidation game to consider security design, and demonstrate the optimality of pooling securities and selling senior tranches or debt secured by the pool, with retention increasing in asset quality or informational asymmetry.

† DeMarzo: Stanford, CA 94305-5015; pdemarzo@stanford.edu. Frankel: 200 Leicester St., Carlton, VIC 3053, Australia, d.frankel@mbs.edu, (+61) 3-9349-8107. Sadly, our coauthor Yu Jin died in 2018; we dedicate this paper in his honor. We are grateful to Michael Fishman for useful comments, as well as seminar participants at Australian National University, the University of Cincinnati, Fudan University, the Haas School of Business, FRB-Minneapolis, University of Western Ontario, and the 2018 World Finance Conference, and to Ashwin Alankar for research assistance. This paper subsumes and generalizes the results of a similarly named working paper (DeMarzo 2003).

1. Introduction

This paper studies the problem of a privately informed seller who seeks to liquidate securities or assets from a portfolio in order to raise cash. The seller has a higher cost of capital than investors and has private information about a common underlying risk factor, such that bad news regarding one asset is bad news for all. For instance, a bank may hold a loan (or CDS) portfolio and have information regarding future volatility. An insider may seek to sell company stock or options and have information about the expected return of the firm. A seller of mortgage-backed securities may have information about default or prepayment risk.

The case in which an informed seller offers a single asset to investors has been studied extensively. Examples include Leland and Pyle (1979), Myers and Majluf (1984), and DeMarzo and Duffie (1999). Our analysis departs from the standard literature by considering a setting in which the seller can choose from a portfolio of assets. In this case, there is information in both the quantities *and* types of assets sold. In addition, the potential for cross-signaling arises, as the decision to sell more of one asset will affect the price received for the others.[1] A complete assessment of an asset's liquidity must thus consider the negative impact of its sale not only on its own price, but also on the prices of the other assets in the seller's portfolio.

Our first contribution is to define and analyze an Asset Sale game in which the seller has a fixed set of assets and chooses a quantity of each to put up for sale.[2] We characterize the unique equilibrium in this multi-dimensional signaling game.[3] We then show the key

---

[1] DeMarzo (2005) studies a setting with multiple assets in which the sale of one asset does not convey information about the values of the other assets, so cross-signaling does not arise. He (2009) considers a two-asset version of Leland and Pyle (1979). The issuer has distinct information about each asset, but because the issuer is risk-averse the cost of retaining one asset depends on retention of the other. The CARA-Normal framework precludes analysis of security design, however, which is a primary focus of our paper. Brown, Carlin, and Lobo (2010) study the problem of selling now vs. later when asset sales can have both temporary and permanent price impacts. As these impacts are exogenous rather than being derived from asymmetric information, their model is not directly comparable to ours.

[2] In the AS game, the seller cannot design new securities. Hence, this game is most relevant to the common situation in which the seller either urgently needs cash or holds standard securities, such as corporate bonds or stock that are rarely repackaged. In other situations, the seller may have the opportunity to design and issue a new security that relies on her assets as collateral. This case is studied in the General Security Design game in section 3.

[3] In order to obtain uniqueness, we assume the Intuitive Criterion of Cho and Kreps (1987): beliefs following an unexpected action must be concentrated only on types that could possibly hope to gain from the deviation.

contrasts with the single-asset case. First, increasing the quantity sold of one asset lowers not only its own price but also the prices of the other assets (and this cross-effect may be much larger than the own-price effect). Second, issuance need not be monotonic in the seller's information: a more pessimistic seller may sell *fewer* shares of some assets (and more of others) than an optimistic one, as the relative liquidity of securities may alternate. In contrast, liquidity cannot alternate if there is only one asset: an increase in pessimism always leads the seller to sell more of her single asset.

More formally, we show that as in the single asset case, the liquidity of a security is related to its informational sensitivity (the percentage change in value for a given change in information). Unlike standard single asset models, however, the ranking of informational sensitivity across securities can change depending on the seller's information. In addition, the relevant incentive constraints need not be local: the seller may choose her portfolio to separate from a much lower type (rather than an adjacent type). These features of the setting make the general characterization of the optimal portfolio choice quite complex.

We next show that this complexity is avoided if the ranking of the assets' informational sensitivity can be determined a priori. Specifically, if the value of asset X is always more sensitive than the value of asset Y to a change in the seller's information, then X is (globally) less liquid than Y: selling X is always a stronger signal that the seller's type is low and thus always has a larger effect on prices. In this case, the seller's portfolio choice becomes lexicographic: only the most pessimistic types will sell asset X, and will do so only after completely liquidating asset Y.

While the ability to rank securities by their informational sensitivity may seem special and rely on detailed knowledge of the seller's information structure, we show that it arises naturally in the following common setting. Suppose first that each asset's payout is a nondecreasing function of a common underlying cash flow.[4] Examples include prioritized debt and equity tranches that are secured by a common loan pool, or corporate debt, equity, and warrants that are secured by a firm's future operating profits. Assume also that an

[4] A common underlying cash flow is not needed for our preceding results, which assume only that the expected payout of each asset is nondecreasing in the issuer's information. For instance, each asset may be secured by a distinct cash flow, whose distribution is increasing in the issuer's information.

increase in the seller's information monotonically lowers the hazard rate of the underlying cash flow.[5] In this setting, we define a novel and generalized notion of seniority, and show that more senior assets will be more liquid: they will be liquidated first.[6] Moreover, as the assets are sold in strict priority order, in equilibrium the quantity sold of the most junior liquidated asset suffices to convey the seller's information to investors. These results provide the first formal theoretical verification of Myers's (1984) Pecking Order Hypothesis in a general multi-security context.[7]

The Asset Sale model assumes a fixed set of assets. The seller merely chooses how much of each asset to sell. In section 3, we show how the Asset Sale game results can be applied to study a General Security Design game in which an issuer can design new securities that use her initial assets as collateral. We permit her to design any number of securities both before and after she sees her information.[8] In this setting, the issuer maximizes her portfolio's liquidity by first pooling her initial assets and then following two equivalent, optimal strategies.[9]

1. She can see her information and then issue a single standard debt security, backed by the pool, whose face value is decreasing in her information.
2. She can tranche the pool into a maximal set of prioritized debt securities, see her information, and then sell those tranches whose seniority exceeds a threshold that is increasing in her information. This optimal strategy mirrors the usual structure of loan-pool securitization deals.[10]

---

[5] This hazard rate ordering property is stronger than first-order stochastic dominance, but weaker than the usual monotone likelihood ratio property.

[6] Our notion of seniority allows us to rank securities beyond just standard debt and equity (such as options, convertibles, etc.).

[7] While Myers's hypothesis has mixed empirical support in the corporate finance literature (e.g. Fama and French (2002), Flannery and Rangan (2006), Opler *et al* (1999), and Shyam-Sunder and Myers (1999)), our context is more suited to the sale of asset-backed securities, where retention rates decline and liquidity improves with seniority (see e.g. Begley and Purnanandam (2017) and Franke and Krahnen (2007)).

[8] This includes, as special cases, both DeMarzo and Duffie (1999) and the single-asset, ex-post security design model in the early working-paper version of this paper (DeMarzo 2003).

[9] They are equivalent in the sense that they yield the same type-contingent revenue for the issuer and promise the same aggregate final payout to investors for any cash flow realization.

[10] For empirical examples, see Begley and Purnanandam (2017) and Franke and Krahnen (2007). An alternative explanation appears in Dang, Gorton, and Holmström (2015): an uninformed issuer may retain an equity tranche in order to discourage information gathering by investors.

Our result generalizes the insight from the prior work of DeMarzo and Duffie (1999) (henceforth DD), who find that debt is the optimal *ex-ante* security design. However, their issuer signals greater optimism by selling fewer shares of a fixed debt security (whose face value is chosen ex ante), thus lowering the payout to investors by a constant proportion in all states. While our issuer could also choose this option, she instead issues debt with a lower face value. Intuitively, as this leaves investors' payouts unchanged in default states, it creates less temptation for imitation by more pessimistic issuers for whom default states are more likely. Hence, by adjusting the face value, the issuer can sell more of her cash flow while minimizing the impact on investors' beliefs. Moreover, for any type, the amount of cash flow sold to investors is greater when the face value is variable than when it is fixed: ex-post debt is more liquid than ex-ante debt.[11] Indeed, ex-post debt is better for the seller than *any* alternative set of securities.

The above security design results assume that the issuer's type and cash flow are discretely distributed. For wider applicability, we then consider the limit as the issuer's information and her realized cash flows become continuous.[12] In this limit, the equilibrium face value of the optimal ex-post debt security is characterized by a simple differential equation. We show that not only is this limit an equilibrium of the continuous model, but also that the issuer's expected profits in the discrete model converge uniformly to her expected profits in this equilibrium.[13] Hence, the continuous model is a good approximation to the discrete

[11] Nachman and Noe (1994) also show that debt is the optimal ex post security design when an issuer must raise a *fixed* amount of cash. The all-or-nothing nature of the financing problem leads to a pooling equilibrium in which each type of issuer sells standard debt with the same face value. A pooling equilibrium with debt also emerges when an informed issuer sells to a single large investor (Biais and Mariotti 2005); here, the issuer pools in order to shield her informational rents from the investor. Friewald, Hennessy, and Jankowitsch (2015) show that another advantage of debt is that it is more liquid in the resale market, which raises investors' initial willingness to pay. Finally, Axelson (2007) reverses our information assumptions: the issuer is uninformed and the investors are informed. He finds that either equity or debt may be optimal, depending on the degree of competition between the investors. Because of his different information structure, Axelson's model might be a good fit to an initial public offering of a firm whose value depends on the reactions of potential investors (as in the case of a new social networking platform). Our model, in contrast, would better suit a bank that has issued loans to borrowers whom the bank knows or has vetted.

[12] Applications of these continuum results include section 4.2 of DeMarzo (2005) and section 7.2 of Frankel and Jin (2015). DeMarzo (2005) cites an earlier version of this paper, DeMarzo (2003), for these results.

[13] Manelli (1996, 1997) studies a general sequence of finite signaling games (which have finite type and message spaces) that converges to a continuous signaling game that, like ours, has compact type and message spaces. Manelli (1996) shows that any sequence of equilibria of the finite games has a subsequence that

model. While this continuous solution has been applied heuristically in DeMarzo (2005) and in Frankel and Jin (2015), we provide the first formal verification in a setting when the issuer can design any number of securities both before and after she becomes informed.

In the ex-post debt equilibrium, a higher face value signals that the issuer is pessimistic and thus leads investors to lower their valuation of her cash flow. Hence credit spreads rise (and revenues fall) more steeply as a function of face value, and this illiquidity induces high-value issuers to retain more equity. Begley and Purnanandam (2017) confirm this result empirically by showing that when the equity (retained) tranche of a pool of residential mortgage-backed securities (RMBS's) makes up a larger proportion of the pool's face value, the loans in the pool have lower subsequent delinquency rates conditional on observables and the securities that are sold fetch higher prices conditional on their credit ratings. They also find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation ("no-doc") loans, controlling for other loan observables. This prediction also follows from our analysis under the assumption that informational sensitivity is higher for no-doc loans. In this case, a no-doc issuer gains more from convincing investors that her information is higher. In order to deter such deviations, this greater benefit must be offset by a greater cost: the surplus sacrificed (which is proportional to the size of the equity tranche) must increase faster in the issuer's type.

We model the sale of assets as a simple signaling game. The seller chooses a portfolio, which specifies how much of each asset she will sell. The market then assigns a price to each asset, based on the informational content of the portfolio. DeMarzo and Frankel (2020, henceforth "DF") consider the sale of multiple assets using instead the more complex mechanism of Maskin and Tirole (1992; henceforth "MT"). In that mechanism, the seller first proposes a menu of contracts to a single uninformed investor. Each such contract specifies a portfolio and a price. If the investor accepts the menu, the seller

converges to an equilibrium of the continuous game. Similarly, Manelli (1997) shows that if each equilibrium of the finite game satisfies the Never a Weak Best Response criterion of Kohlberg and Mertens (1986), then the limiting equilibrium satisfies the same criterion. However, because our model has an infinite message space - the set of monotone securities - his results cannot be directly applied.

chooses any contract from the menu to implement. The seller then transfers the chosen portfolio to the investor at the selected price.

Focusing on the case of two assets and two seller types - optimistic and pessimistic - DF show that our results are robust to this change in mechanism. In particular, our unique outcome is also the unique outcome under MT's approach if the seller is not too likely to be optimistic. Otherwise, there exists a continuum of outcomes. For most parameters, these additional outcomes share the qualitative features of our outcome: the seller retains shares in order to signal optimism. However, in a small region of the parameter space there is also a pooling outcome in which the seller offers her entire portfolio for sale regardless of her beliefs. These results also carry over to security design. For most parameters, the seller sells standard debt and lowers the face value to signal optimism. For a small range of parameters, there is also a pooling outcome in which the optimistic and pessimistic sellers issue debt with the same face value.

The rest of the paper is organized as follows. The base model with a fixed set of assets is studied in section 2. The general security design game is analyzed in section 3. We relax a monotonicity assumption in section 4 and conclude in section 5.

2. The Asset Sale Game

Section 2.1 sets out a base model (the "Asset Sale game") in which a privately informed seller is endowed with a fixed portfolio of assets. Selling all her assets is efficient as the seller is relatively impatient; for instance, she may have attractive alternative investments or face liquidity or capital requirements. However, because there is asymmetric information, she may retain some of her assets in order to signal that she is optimistic and thus to obtain higher prices.

Signaling games often have multiple equilibria even with one-dimensional signals. The problem is potentially worse in our setting as the signal is multidimensional: the quantity of each asset to offer for sale. Nevertheless, we show in section 2.2 that the game has a unique equilibrium that satisfies the Intuitive Criterion.[14]

[14] The uniqueness argument will rely on an assumption that the issuer can set a price cap for each asset.

In the Asset Sale game, the most pessimistic type sells her entire portfolio. As her optimism grows, she signals her greater optimism by retaining a set of assets that she prizes the most *relative to lower types*. These are the seller's *least liquid* assets: as lower types place an especially low value on them, their sale constitutes the strongest signal that the seller's type is low and thus triggers the largest decline, per unit of revenue raised, in the market's valuation of the seller's portfolio. Since the identity of these least liquid assets may alternate as the seller's optimism grows, a given seller may retain an asset that a more optimistic seller liquidates. This nonmonotonic behavior, which cannot occur in the single-asset case, appears in two computed examples in section 2.3.

In section 2.4, we add a mild assumption that rules out such nonmonotonicity: that the seller's assets can be ranked globally in terms of informational sensitivity. That is, if one asset's expected value rises proportionally more than another's for a given increase in the seller's type, then it does so for *any* such increase. We show that in this case, a seller will not sell any portion of an asset unless she also sells all of her less informationally sensitive assets in their entirety. The intuition, as above, is that lower types have relatively low valuations of more informationally sensitive assets and thus a relatively strong incentive to sell them. Since selling them is thus a strong signal that the seller's type is low, it has a large impact on the market's valuation of her portfolio. The seller will thus sell them only as a last resort.

Section 2.5 then relates this prediction to commonly observed financial assets such as debt, equity, and the prioritized tranches of securitized loan pools. We show that under a mild distributional assumption – the hazard rate ordering property – senior assets are less informationally sensitive and thus more liquid than junior ones. Hence, by the preceding result, they will be sold first. Importantly, this result relies on a novel and weak notion of seniority: one asset is senior to another if its payout rises proportionally more slowly in the underlying cash flow – i.e., if its relative claim to the cash flow is stronger when the cash flow is low. This weak notion of seniority lets us extend Myers's (1984) pecking-order hypothesis to a much larger class of assets. For instance, a stock is senior to a call option written on the stock and thus will be sold before the option in order to raise cash.

2.1. The Base Model

The participants are a single seller and a continuum of investors. All participants are risk-neutral and fully rational. The seller is endowed with a portfolio of *n* assets represented by the vector $a \in \Re_+^n$, where $a_i > 0$ is the number of shares she owns of asset *i*. A holder of one share of asset $i \in \{1, \ldots, n\}$ is entitled to the random future payout F_i. The seller has private information about these payouts, which is summarized by her type $t \in \{0, \ldots, T\}$. Conditional on the seller's type, asset *i* has an expected payout per share[15] of $f_i(t) \overset{d}{=} E[F_i \mid t]$. Let $F = (F_1, \ldots, F_n) \in \Re_+^n$ denote the vector of random asset payouts and let $f(t) = E[F \mid t] \in \Re_+^n$ denote the vector of expected payouts conditional on *t*. We refer to (a, f) as the seller's *endowment*.

We assume throughout that (a) the seller's information can be ordered so that higher types *t* are more optimistic about the expected payout of each asset;[16] (b) an asset's expected conditional payout is never negative (e.g., because of limited liability); and (c) even the most pessimistic seller thinks that her portfolio has a positive expected value:

> **ASSUMPTION A (MONOTONE EXPECTED PAYOUTS).** For $t > s$, $f(t) \geq f(s)$. In addition, $f(0) \geq 0$ and $af(0) > 0$.

This information structure is natural when the seller is informed about a common factor that affects all of her assets. For instance, the seller might own a portfolio of securities backed by a common pool of assets, such as the debt and equity of a single firm or the prioritized debt tranches of a loan pool, and have private information about the future value of the underlying pool.[17]

---

[15] More precisely, the payout of asset *i* is a function $F_i(\omega)$ of an exogenous, unknown random state ω. Then $f_i(t)$ is simply the expectation of $F_i(\omega)$ with respect to the conditional distribution of ω given *t*.

[16] In the case of 2 types and 2 assets, this property is not needed; see section 4.

[17] The case of asset-backed securities is further developed in section 2.5. Other natural examples are a loan portfolio where the issuer has private information about market volatility or prepayment risk, and a portfolio of equities within an industry when the issuer is privately informed about future industry profits.

On seeing her information *t*, the seller chooses a quantity $q_i \in [0, a_i]$ of each asset *i* to offer for sale. Let $q = (q_1, \ldots, q_n) \in \Re_+^n$ denote the row vector of chosen quantities, where $0 \le q \le a$.[18] The seller may also set a price cap $\bar{p}_i \in [0, \infty]$ for each asset *i*.[19] If the market clearing price for asset *i* exceeds $\bar{p}_i$, she charges $\bar{p}_i$ and rations the asset.[20] Let $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n) \in \Re_+^n$ denote the vector of price caps.

The seller is less patient than the investors: she discounts future cash flows at some rate $\delta \in (0,1)$, while investors' discount factor is normalized to one.[21] For instance, the seller may face capital requirements or need cash to invest in worthwhile projects. This difference in discount factors is the source of gains from trade in the model.

We assume the seller has no other collateral which can be used to raise funds apart from her portfolio (a, f). We also assume, for now, that she is restricted to selling only her existing securities; section 3 will relax this assumption and allow her to design new securities (such as secured debt) using her existing securities as collateral.

The investors have a common, positive prior over the different possible realizations of the seller's type *t*. On seeing the seller's sale decision $(q, \bar{p})$, they form posterior beliefs

[18] For vectors *x* and *y*, the notation "$x \le y$" means that for all *i*, $x_i \le y_i$.

[19] There is a superficial similarity between the price caps in our model and the price-quantity menu chosen in Biais and Mariotti (2005). In the latter paper, a single investor chooses a price-quantity pair from a menu that the issuer specifies. Hence the issuer chooses one of the prices that the investor pays. In our model, in contrast, the price caps never bind. Rather, their role is merely to rule out implausible equilibria. Intuitively, a type *t* issuer who deviates could choose price caps low enough that any lower type must lose from the deviation. By the Intuitive Criterion, investors must then think her type is *t* or more so they cannot bid less than her valuation of the assets. This can make the deviation profitable, thus ruling out the equilibrium.

[20] We could also allow the issuer to set reserve or minimum prices for the securities. However, since investors would refuse to buy overpriced securities, extending the strategy space in this way would play no role in equilibrium. (In other auction environments, a reserve price is useful to extract additional surplus from buyers. Our model differs in that investors are homogeneous and uninformed. Hence they earn no surplus even absent a reserve price.) Alternatively, rather than set individual price caps, we could set a cap on total issuance revenue. Such a cap would lead to the selection of the same intuitive equilibrium. Individual caps are more natural in practice so we use them to enhance the realism of our base model.

[21] More precisely, let the unnormalized discount factors of the investors and the issuer be $\hat{\delta} \in (0,1]$ and $\tilde{\delta} \in (0, \hat{\delta})$, respectively, and let $\hat{f}$ be the undiscounted expected payoffs of the assets. Then our model corresponds to $f = \hat{\delta}\hat{f}$ and $\delta = \tilde{\delta}/\hat{\delta}$; that is, we interpret *f* as the conditional expected present value of the asset using the investors' discount rate, and δ as the issuer's relative discount factor.

$\mu(t \mid q, \bar{p})$. Investors are risk-neutral, behave competitively, and have deep pockets, so the price of asset *i* is the minimum of its price cap $\bar{p}_i$ and its conditional expected value $\sum_t f_i(t)\,\mu(t \mid q, \bar{p})$. The vector of realized prices of the assets is thus the componentwise minimum of the vector $\bar{p}$ of price caps and the vector of conditional expected values:[22]

$$p(q, \bar{p}) = \bar{p} \wedge \left[\sum_t f(t)\,\mu(t \mid q, \bar{p})\right]. \tag{1}$$

Suppose a seller of type *t* sells the quantities *q* of her assets at prices $p = (p_1, \ldots, p_n) \in \Re^n_+$. She receives revenue *qp* from the asset sale, plus the discounted expected payout $\delta(a-q)\,f(t)$ of her retained assets.[23] Her total payoff – the sum of these terms – can be written as $U(t, q, p) = \delta a f(t) + q(p - \delta f(t))$.

Our equilibrium concept is as follows.

ASSET SALE EQUILIBRIUM. A *perfect Bayesian equilibrium* for the Asset Sale game is a strategy $(q(t), \bar{p}(t))$ for the seller, a price response function $p(q, \bar{p})$, and a posterior belief function $\mu(t \mid q, \bar{p})$ for the investors, such that the following conditions hold.[24]

1. Payoff Maximization: for any type *t*, the seller's chosen action $(q(t), \bar{p}(t))$ solves $\max_{q', \bar{p}'} U(t, q', p(q', \bar{p}'))$ subject to $0 \le q' \le a$.

2. Competitive Pricing: for any sale decision $(q, \bar{p})$, the price vector $p(q, \bar{p})$ satisfies equation (1).

3. Rational Updating: investors' posterior beliefs $\mu(t \mid q, \bar{p})$ are given by Bayes's rule if $(q, \bar{p})$ is chosen by some type of seller in equilibrium.

---

[22] The notation $x \wedge y$ denotes the componentwise minimum of vectors *x* and *y*.

[23] The concatenation of two vectors is always to be interpreted as a dot product; e.g., $pq = p \cdot q$.

[24] The arguments of these functions are included for clarity. For instance, $q(t)$ should be interpreted as a function $q : \{0, \ldots, T\} \to \Re^n_+$ that specifies the quantity vector chosen by each type of issuer.

Substituting the equilibrium price function $p(t) \overset{d}{=} p\left(q(t), \bar{p}(t)\right)$ for the price vector p in $U(t,q,p)$ and omitting the fixed term $\delta af(t)$, we obtain the seller's equilibrium payoff function $u(t) \equiv q(t)\left[p(t) - \delta f(t)\right]$, which we refer to as the *outcome* of the equilibrium.[25] It equals the additional surplus that a type-t seller receives from selling assets to investors. We also define an associated revenue function $\rho(t) = p(t)q(t)$: the seller's asset sale revenue when her type is t. Thus, the outcome of the equilibrium can also be written as $u(t) \equiv \rho(t) - \delta q(t) f(t)$: the seller's revenue less her opportunity cost of selling the assets.

2.2. Uniqueness and Computation

Like most signaling games, the above Asset Sale game has multiple equilibria. In order to obtain a unique prediction, we must use a signaling game refinement. The weakest such refinement is the Intuitive Criterion of Cho and Kreps (1987). In our context, this criterion states that if investors see an out-of-equilibrium sale decision $(q, \bar{p})$, their beliefs put weight only on those types who could possibly expect to gain from the deviation. More precisely, say a type-t seller deviates to $(q, \bar{p})$. A sufficient condition for her to lose from this deviation is that her equilibrium payoff $u(t)$ exceeds her maximum payoff $q\left[\bar{p} - \delta f(t)\right]$ from the deviation – or, equivalently, that[26]

$$q\bar{p} < u(t) + \delta q f(t). \tag{2}$$

[25] The outcome captures all payoff-relevant features of the equilibrium since competition drives investors' payoffs to zero. However, two equilibria may involve different actions but the same outcome; e.g., if the issuer were endowed with two *identical* assets 1 and 2, then the outcome would depend only on the sum $q_1 + q_2$ of quantities sold for each type t and not on the individual quantities q_1 and q_2.

[26] If type t issues q^t for the prices p^t in equilibrium, we can rewrite (2) as $q\bar{p} < q^t p^t + \delta(q - q^t) f(t)$, where the right hand side is the opportunity cost of deviating to $(q, \bar{p})$: the sum of type t's original revenue and her discounted cost of transferring q units of each asset to investors rather than q^t. Type t will not deviate to $(q, \bar{p})$ if this opportunity cost exceeds her maximum revenue $q\bar{p}$ from doing so, i.e. if (2) holds.

The Intuitive Criterion states that on seeing the deviation $(q, \bar{p})$, investors must put zero probability on type *t* if this type is not willing to choose $(q, \bar{p})$ but some other type might be: if condition (2) holds for *t* but fails for some other type *s*. That is:

> **THE INTUITIVE CRITERION.** A perfect Bayesian equilibrium of the asset sale game with posterior belief function $\mu(\cdot \mid \cdot, \cdot)$ and outcome $u(\cdot)$ is *intuitive* if, on seeing any quantity vector $0 \le q \le a$ and price cap vector $\bar{p}$, investors' posterior probability $\mu(t \mid q, \bar{p})$ is zero for any type *t* that satisfies (2) as long as there is some type *s* for which the inequality is reversed: for which $q\bar{p} \ge u(s) + \delta q f(s)$.

An equilibrium that satisfies the Intuitive Criterion will be called *intuitive*, as will investors' beliefs in that equilibrium. If an outcome $u(\cdot)$ is supported by an intuitive equilibrium, we will call the outcome intuitive as well.

Clearly, a simple way ensure intuitive beliefs is for investors to respond to a deviation $(q, \bar{p})$ by putting all of their weight on a type *t* for which the opportunity cost $u(t) + \delta q f(t)$ of the given deviation is at a minimum. Since no type has a lower opportunity cost of deviating to $(q, \bar{p})$ than this type *t*, there is no price vector $p \le \bar{p}$ that makes some type *s* willing to deviate to $(q, \bar{p})$ if it does not also make type *t* willing to choose this deviation: beliefs are intuitive. Since technically more than one type *t* may minimize the opportunity cost $u(t) + \delta q f(t)$ of deviating, we can break ties by putting all of investors' weight on the lowest such type. Summarizing, a sufficient condition for the belief function μ to be intuitive is that, following a deviation $(q, \bar{p})$, it satisfies

$$\mu\left(\tau(q) \mid q, \bar{p}\right) = 1 \tag{3}$$

where

$$\tau(q) = \min\left\{\arg\min_t \left[u(t) + \delta q f(t)\right]\right\} \tag{4}$$

is the lowest type *t* that minimizes $u(t) + \delta q f(t)$.[27]

---

[27] These beliefs survive not only the Intuitive Criterion, but also the stronger D1 refinement of Cho and Sobel (1990). This enhances their plausibility and allows comparisons with other research that assumes D1.

We will show that any intuitive equilibrium of the Asset Sale game satisfies the following property.[28]

Fair Pricing. An equilibrium is *fairly priced* if, for all i and t, $q_i(t) > 0$ implies $p_i(q(t), \bar{p}(t)) = f_i(t)$.

That is, the price assigned to any asset i that is sold in equilibrium equals the conditional expected payout of this asset. This property is related, but not identical, to the notion of a separating equilibrium. Unlike separation, fair pricing implies that the seller's price caps never bind and that investors' payoffs are identically zero, so social welfare coincides with the seller's payoff. And separation implies that investors can infer the seller's actual information t in equilibrium, while fair pricing implies only that they can infer the conditional expected payout of any asset that the seller chooses to sell – which is all they need in order to compute the asset's fair-market value.

We next recursively calculate upper bounds $u^*(t)$ and $\rho^*(t)$ on the payoff and issuance revenue of a type-t seller in any fairly priced equilibrium.[29] We later show that these bounds are tight by constructing a fairly priced equilibrium that attains them.[30]

[28] Fair pricing is a result, not an assumption: we assume the Intuitive Criterion, which implies fair pricing.
[29] That these are upper bounds is shown below in Proposition 1.
[30] Whether u^* corresponds to an equilibrium is not immediate, as RLP does not impose the incentive constraints that high types do not wish to mimic low types. This is shown in Proposition 1.

RECURSIVE LINEAR PROGRAM (RLP). For type $t=0$, let $q^*(0)$ be the seller's endowment vector a, and let $\rho^*(0)=af(0)$ and $u^*(0)=(1-\delta)af(0)$ be the value of this portfolio and the gains from trade from selling it. For any type $t>0$, let[31]

$$q^*(t)\in \underset{0\le q\le a}{\arg\max}\ (1-\delta)q\,f(t) \text{ such that } q\big(f(t)-\delta f(s)\big)\le u^*(s) \text{ for all } s<t$$

be the highest-value portfolio a type-t can sell that does not tempt lower types to imitate her; and let $\rho^*(t)=q^*(t)\,f(t)$ and $u^*(t)=(1-\delta)q^*(t)\,f(t)$ denote the value of this portfolio and the gains from trade from selling it.

We next construct a fairly priced equilibrium $e^*=\left(q^*,\bar{p}^*,p^*,\mu^*\right)$ that attains the bounds computed in RLP. For each t, let type t's quantity vector $q^*(t)$ be any solution to RLP and let type t's price cap vector be any vector that satisfies $\bar{p}^*(t)\ge f(t)$.[32] Let the posterior beliefs $\mu^*(\cdot\,|\,q,\bar{p})$ be given by Bayes's Rule if the profile e^* prescribes that some type choose $(q,\bar{p})$, and by (3) and (4) if not (with u replaced by u^* in (4)). Finally, let the market price function $p^*(q,\bar{p})$ be the result of substituting μ^* for μ in equation (1).

REMARK. Since beliefs μ^* are intuitive by construction, the profile $e^*=\left(q^*,\bar{p}^*,p^*,\mu^*\right)$ satisfies the Intuitive Criterion if it is an equilibrium.

We now show that e^* is a fairly priced equilibrium with payoff and revenue functions u^* and ρ^*, and is optimal for the seller (and thus maximizes welfare) within the set of fairly priced equilibria.[33] Moreover, each intuitive equilibrium of the Asset Sale game gives a seller of each type t the same payoff $u^*(t)$ and revenue $\rho^*(t)$ as e^* does. Finally, in e^*, a higher quantity sold of any asset cannot raise the price of any asset:

[31] Any type s that chooses the same q as type t will earn revenue of $qf(t)$, and have cost $q\delta f(s)$, for total surplus $q(f(t)-\delta f(s))$, which must not exceed $u^*(s)$ in order to maintain incentive compatibility.

[32] Hence the price caps do not bind in any fairly priced equilibrium. However, they do restrict the set of equilibria that survive the Intuitive Criterion (see n. 19).

[33] Under fair pricing, the investors' payoff is zero so social welfare equals the issuer's payoff.

PROPOSITION 1.

1. The RECURSIVE LINEAR PROGRAM yields unique functions u^* and ρ^*, which are strictly positive and nonincreasing in the type t.

2. For each type t, $u^*(t)$ and $\rho^*(t)$ are upper bounds on the seller's payoff and revenue, respectively, in any fairly priced equilibrium.[34]

3. The profile e^* is a fairly priced, intuitive equilibrium with payoff and revenue functions u^* and ρ^* (whence the bounds in part 2 are tight).

4. Every intuitive equilibrium is fairly priced,[35] has payoff and revenue functions u^* and ρ^*, and has an asset sale function q^* that solves RLP.

5. The price function p^* in e^* is nondecreasing in q: for any price cap vector $\bar{p}$ and quantity vectors $q' \geq q$, $p^*(q', \bar{p}) \leq p^*(q, \bar{p})$.[36]

PROOF: Appendix. ◆

By parts 1 and 3, a seller's payoff in e^* is nonincreasing in her type. This property is far more general: it must hold in *any* equilibrium. Why? For a type-s seller not to deviate, her equilibrium payoff $u(s)$ cannot be less than the payoff – call it $u(t \mid s)$ - that she would get by imitating some type $t > s$. This payoff $u(t \mid s)$ equals a type t's equilibrium revenue $\rho(t)$ less the opportunity cost $\delta q(t) f(s)$ incurred by a type-s seller who sells type t's equilibrium quantities $q(t)$. But since type s places a lower value on the assets by ASSUMPTION A, her opportunity cost cannot exceed the opportunity cost $\delta q(t) f(t)$ incurred by type t from selling the quantities $q(t)$. This implies that the imitation payoff

[34] Intuitively, the IC constraints of a fairly priced equilibrium include the constraints used to define u^*, since fair pricing means that the payoff of any type t from imitating any type s must be $q(s)\left[f(s) - \delta f(t)\right]$ as assumed in RLP.

[35] The converse need not hold: there may be fairly priced equilibria that are not intuitive. In such an equilibrium, the threat of punishing beliefs can force a type-t issuer to sell a portfolio $q(t)$ that is less valuable than $q^*(t)$, thus holding her payoff below $u^*(t)$. Intuitively, fairly priced equilibria are like separating equilibria, many of which fail the Intuitive Criterion.

[36] This property holds regardless of whether $(q', \bar{p})$ and/or $(q, \bar{p})$ is chosen in equilibrium.

$u(t\,|\,s)$ of type s is not less than $\rho(t)-\delta q(t)\,f(t)$, which is just the equilibrium payoff $u(t)$ of type t. Combining inequalities we obtain $u(s)\geq u(t\,|\,s)\geq u(t)$: in any equilibrium, the seller's payoff is nonincreasing in her type.

2.3. Nonmonotonic Issuance

In the single-asset case, a more optimistic seller retains more of her asset in order to distinguish herself from more pessimistic types (e.g., Leland and Pyle 1979, DeMarzo and Duffie 1999). With multiple assets, this monotonicity property can fail to hold. We show this via two computed examples. The first is a simple example with three types of seller, whose solution can be easily calculated by hand.

> **EXAMPLE 1.** Assume the seller owns one share each of two assets: $a_1 = a_2 = 1$. The seller has three possible types $t = 0,1,2$ and her discount factor is $\delta = 1/2$. The conditional expected payouts of the assets are $f(0)=(1,1)$, $f(1)=(4,2)$, and $f(2)=(5,5)$. That is, asset 1 (resp., 2) responds relatively more when the seller's type rises from 0 to 1 (resp., from 1 to 2). One can then verify that the solution to RLP is $q^*(0)=(1,1)$, $q^*(1)=(0,2/3)$, and $q^*(2)=(2/9,0)$: the seller retains no shares when her type is 0, all shares of asset 1 and 1/3 share of asset 2 when her type is 1, and 2/9 share of asset 1 and no shares of asset 2 when her type is 2.[37]

Intuitively, each type retains a set of assets that she prizes the most relative to lower types, so as not to be mistaken for those types. In the present example, asset 1 is most prized by type 1, while asset 2 is most prized by type 1 (relative to the next lower type in each case). Accordingly, asset 2 is issued nonmonotonically: when her type falls from 2 to 1, the seller sells *fewer* shares of asset 1. This contrasts with the single-asset case, in which more pessimistic sellers always retain fewer shares of their one asset.

The next example illustrates this principle in a more realistic setting with many seller types.

> **EXAMPLE 2.** Suppose the seller holds 1 share of each of 2 assets, with expected payouts of each (conditional on t = 0, 0.1, …, 200) given in Figure 1. While the

[37] A proof is available from the authors on request.

overall sensitivity to information for each asset is similar (a 10% increase in value over the range of t), the right panel shows that asset 2's return is more sensitive to increases in t for $t < 50$ and asset 1 is more sensitive when $t > 50$.

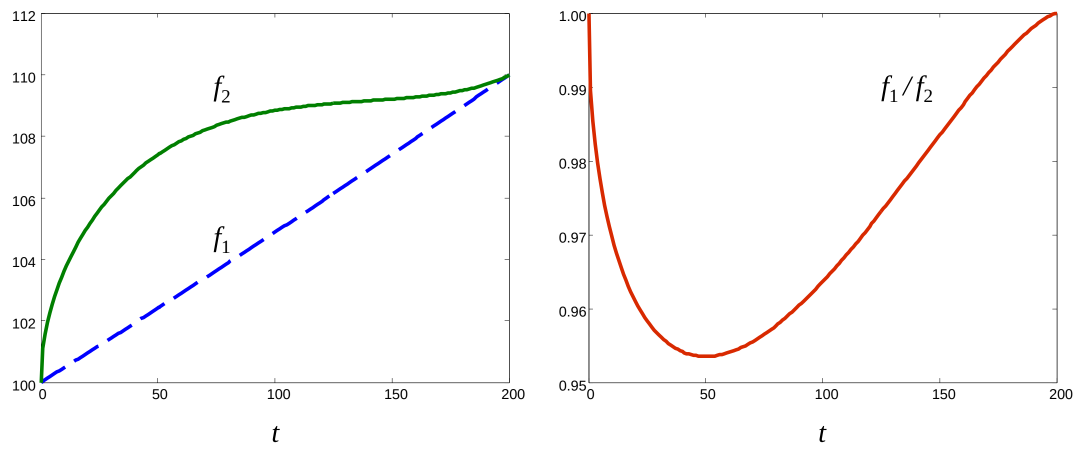

Figure 1: Expected Payoffs of Assets for EXAMPLE 2.
Asset 1's returns are less sensitive to information than asset 2's returns for low t, but are more sensitive for high t.

Figure 2 depicts the equilibrium strategies for $\delta = 0.9$ (given by the solid line with circles for $t = 0, 10, \ldots 200$). The lowest type sells all shares of both assets. Initially, she signals type increases by selling fewer shares of asset 2, which is more sensitive than asset 1 is to her information when the seller's type is low. Her approach changes when her type rises above 50: asset 1 now becomes more information-sensitive, so she signals better information by selling fewer shares of asset 1 and more shares of 2. Finally, when her type is so high that asset 1 is retained in its entirety, the seller has no choice but to sell less of asset 2 as a signal (even though asset 2 is less informationally sensitive than asset 1 in this range).

Also depicted in Figure 2 is the equilibrium price function, as a contour plot showing the type investors infer given any quantity q; darker shades indicate lower types t, while white lines indicate the iso-price contours for $t = 0, 10, \ldots 200$. On each such contour, investors' posterior beliefs and thus the prices of both assets are constant. As the contours are downward sloping, investors become more pessimistic as the seller sells more shares of either asset. Moreover, for types t between 50 and 120, beliefs drop discontinuously for a small increase in either quantity. For example, starting from $q^*(60)$, a small increase in the quantity sold of either security will cause beliefs to drop discontinuously from 60 to below 20; similarly, a small increase from $q^*(90)$ would cause beliefs to drop to 0. Discontinuities occur when the binding incentive constraint in RLP is non-local; for example, the type with the greatest incentive to mimic type 60 is type 18. The possibility that non-local constraints may bind distinguishes our setting from standard signaling models in which a single crossing property holds.

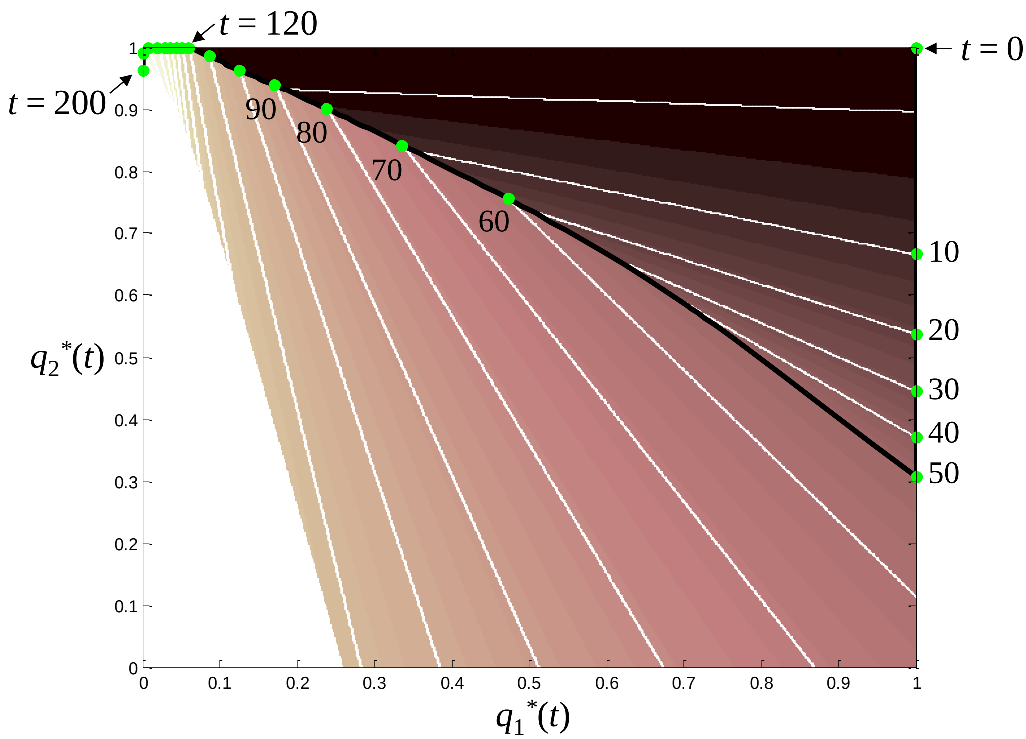

Figure 2: Equilibrium for EXAMPLE 2.
Circles show equilibrium quantities along the solid curve; shading indicates iso-price contours given the beliefs in (3). Note the discontinuity in prices along the upper right boundary of the equilibrium curve.

Figure 3 illustrates the demand curves for asset 1 as a function of the level of asset 2 sales. Note the cross-signaling effect: higher sales of asset 2 make investors pessimistic and thereby lower the demand curve for asset 1.[38] The demand curves also demonstrate the discontinuous price declines for asset 1 that occur, as noted above, when the binding incentive constraint in RLP is nonlocal.

---

[38] This illustrates a property stated in PROPOSITION 1: the price of each asset is nonincreasing in the quantity sold of any asset.

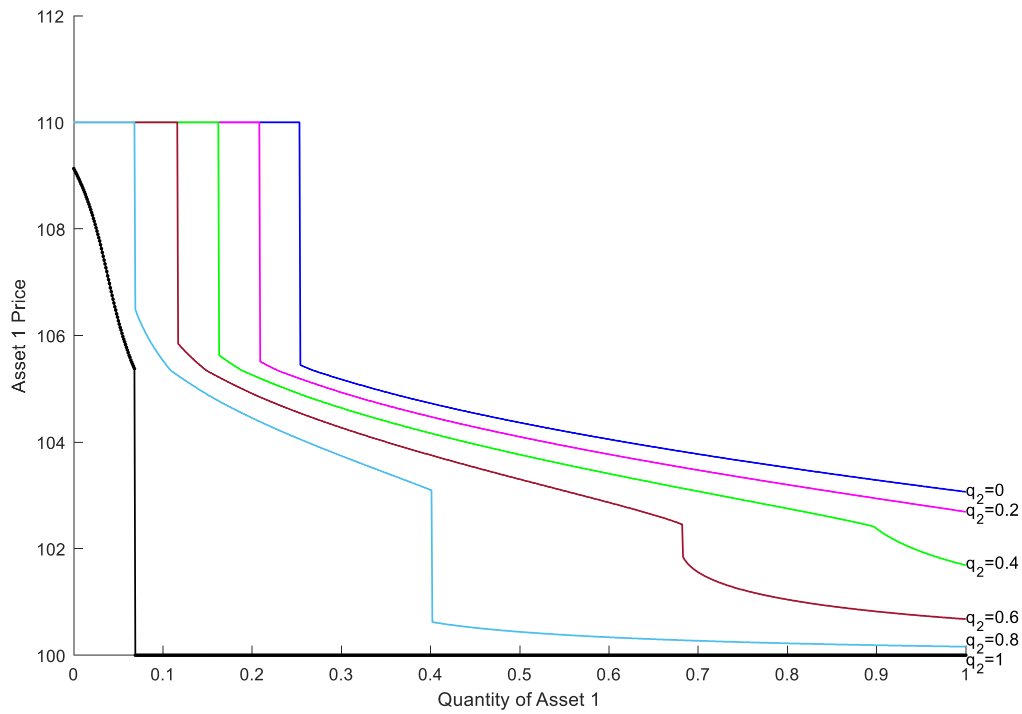

Figure 3: Asset 1 Demand Curves for EXAMPLE 2.
Demand curves for asset 1 given different quantities of asset 2 sold. Note the cross-signaling effect of asset 2 sales on asset 1 demand. Vertical portions of demand curves correspond to situations in which the binding incentive constraint in RLP is non-local.

2.4. Informationally Ordered Assets

We saw in section 2.3 that greater pessimism may sometimes lead a seller to sell fewer shares of a given asset. This occurs because assets can alternate in their informational sensitivity. In the remainder of the paper, we focus on settings in which the assets' sensitivity can be unconditionally ranked: if one asset is more sensitive than another to a given change in information, then it is more sensitive to any such change.

In this setting, the Asset Sale game has a simple outcome. The lowest type seller sells all her assets as before. As her type rises, she retains her more informationally sensitive assets first. Intuitively, these assets are less liquid as lower types have a stronger incentive to sell them. As a result, the model predicts that a firm will never sell an asset unless it also sells *all* of its less sensitive assets in their entirety, as predicted by Myers's (1984) Pecking Order hypothesis.

2.4.1. Information Sensitivity

We will say that one asset is more informationally sensitive than another if its expected value changes proportionally more for a given change in the seller's type. To ensure that proportional changes are well-defined, we assume the conditional expected payoff of each asset is always positive:

> **ASSUMPTION B (POSITIVE EXPECTED PAYOFFS).** For all *i*, $f_i(0) > 0$.[39]

The formal definition is as follows.

> **INFORMATIONAL SENSITIVITY.** Asset *i* is *more informationally sensitive than asset j at type t* if, for any lower type *s*, $f_i(t)/f_i(s) > f_j(t)/f_j(s)$. If this holds for each type *t*, then asset *i* is *more informationally sensitive* than asset *j*.[40] The assets $1,\ldots,n$ display *increasing information sensitivity* (*IIS*) if each asset *i* is more informationally sensitive than any lower-indexed asset $j < i$.[41]

In this setting, the seller prefers to retain her more informationally sensitive assets first as her optimism grows. Formally:

> **PROPOSITION 2.** Suppose asset *i* is more informationally sensitive at *t* than asset *j*. Then in any solution q^* to the RECURSIVE LINEAR PROGRAM, if $q_j^*(t) < a_j$ then $q_i^*(t) = 0$: the seller will not retain any shares of asset *j* unless she also retains all shares of asset *i*.
>
> **PROOF:** Appendix. ◆

2.4.2. Hurdle Class Strategies

If we further assume IIS then, by PROPOSITION 2, the seller will choose to sell all of her less informationally sensitive assets and retain all of her more informationally sensitive ones, with the exact cutoff, or *hurdle class*, determined by her type:

---

[39] Under this requirement, a security can have zero payouts in some states as long as, for any type *t*, there are states in which the security's payout is strictly positive.

[40] Equivalently, *i* is more informationally sensitive than *j* if $f_i(t)/f_j(t)$ is increasing in *t*.

[41] IIS is equivalent to the functions $f_i(t)$ being log-supermodular in (i,t).

PROPOSITION 3. Let the seller's assets display Increasing Information Sensitivity. Then a seller of each type will choose a hurdle class quantity vector: a vector of the form $q^*(t)=(a_1,\dots,a_{c-1},q_c(t),0,\dots,0)$ for some hurdle class $c\in\{1,\dots,n+1\}$ and some hurdle class quantity $q_c(t)\in[0,a_c)$.[42] Moreover, for any types $t > s$, $q^*(t)\le q^*(s)$, whence t chooses a weakly lower hurdle class than s does.

PROOF: Appendix. ◆

Intuitively, a low type has a stronger incentive to sell additional assets by choosing a higher hurdle class than a high type because, while the impact on revenue is the same, the opportunity cost of parting with the additional shares is smaller for the low type. Hence, a low type cannot choose a lower hurdle class than a high type does in equilibrium.

Any hurdle class vector $(a_1,\dots,a_{c-1},q_c,0,\dots,0)$ for $q_c(t)\in[0,a_c)$ is uniquely identified by the total number $q_c+\sum_{i=0}^{c-1}a_c$ of shares that the seller sells (of all assets combined). This number can be thought of as a one-dimensional signal. Thus, under IIS the seller's signaling problem reduces to the familiar and tractable one-dimensional case.

One remaining complication is that for each type $t>0$, RLP includes $t-1$ incentive compatibility (IC) constraints: one for each lower type s. The next result shows that, in fact, the IC constraint for the next lower type $t-1$ is binding and, moreover, identifies a unique hurdle class vector for type t:

PROPOSITION 4. If the assets display Increasing Information Sensitivity, then $q^*(0)=a$ and, for each $t>0$, $q^*(t)$ is the unique hurdle class vector that satisfies the incentive compatibility constraint for type $t-1$, which is

$$q^*(t)\left[f(t)-\delta f(t-1)\right]=(1-\delta)q^*(t-1)f(t-1). \tag{5}$$

PROOF: Appendix. ◆

[42] When c equals n + 1, q equals a: the issuer sells her entire portfolio.

2.5. Application: Prioritized Assets

To apply the preceding result, one must know whether one asset is more sensitive than another to a seller's information. In practice, this condition may be hard to check. In this section we give a simple rule that can be used to make this determination: under a weak distributional condition, *more senior assets* are less informationally sensitive. Hence, they are more liquid and will be sold first, as predicted by Myers's (1984) pecking-order hypothesis.

Importantly, we develop this result using a novel, generalized notion of seniority that permits the ranking of assets that are not ordered by strict priority. A debt asset has strict priority over another asset if its owners must be paid its face value before the owners of the other asset are paid anything. While our notion agrees with strict priority in this case, it also applies when neither asset has strict priority.

Generally speaking, seniority refers to the claim of one asset versus another to a future cash flow that secures both assets. Hence, we assume that the payout of each asset $i \in \{1,\ldots,n\}$ of the seller is given by some nondecreasing, nonnegative function $F_i(Y)$ of a common, stochastic cash flow $Y \in \Re_+$. The cash flow Y might represent a firm's future operating profits or the aggregate repayments on a bank's loan portfolio.

The distributional assumption we need is as follows.

> **HAZARD RATE ORDERING (HRO).** For any type t, the posterior support of Y coincides with the prior support. Moreover, for any types $t > s$, the ratio $\Pr(Y \geq y \mid s)/\Pr(Y \geq y \mid t)$ is decreasing in y in the support of Y whenever the ratio is well-defined.[43]

While stronger than first-order stochastic dominance, HRO is weaker than the monotone likelihood ratio property,[44] which is commonly assumed in signaling environments. If the

[43] The only point y in the support of Y at which the ratio may be undefined is the upper boundary of Y's support, where $\Pr(Y \geq y \mid t)$ may equal zero.

[44] See Shaked and Shanthikumar (2007), p. 43, Thm. 1.C.1.

conditional hazard rate of Y given t is well defined, HRO states that this hazard rate is decreasing in the type t; however, HRO also applies when the hazard rate is undefined.[45]

As for seniority, let us say that asset i is senior to asset j if asset i has a relatively stronger claim on the cash flow when the cash flow is low: if the payout ratio $r(Y) = F_j(Y)/F_i(Y)$ is increasing in the cash flow Y. The precise definition, which permits the denominator to be zero and the ratio to be locally constant, is as follows.

> **PRIORITIZED ASSETS.** Asset i is *senior* to asset j (and j is *junior* to i) if the payout $F_j(Y)$ to asset j can be written as the product of the payout $F_i(Y)$ of asset i and a nonnegative, nondecreasing function $r(Y)$ of the cash flow that takes more than one value with positive probability on that portion of the support of the cash flow where asset i's payout is positive.[46] The assets $\{F_i\}$ are *prioritized* if each asset i is senior to any higher-indexed asset $j > i$.

The standard notion of strict priority is implied by the above definition: if asset i has strict priority over asset j, the function $r(Y)$ is zero while asset i is being repaid, grows while asset j is being repaid, and becomes constant once asset j has been repaid in full. However, the definition is more general: asset i may be senior to asset j even if the payouts to both rise with the cash flow, as long as the payout to asset j rises relatively faster.

Some examples appear in Figure 4. The securities are ranked in order, with F_1 the most senior and F_5 the most junior, with the only exception that F_2 is noncomparable with F_3 or F_4. Thus (F_1, F_2, F_5) and (F_1, F_3, F_4, F_5) are prioritized sets.

---

[45] The conditional hazard rate is $h(y\,|\,t)/\left[1-H(y\,|\,t)\right]$ where H denotes the conditional distribution of the cash flow Y given the type t, and h is the corresponding density. This hazard rate is undefined if the conditional distribution is discrete or has atoms, while the condition in HRO is still well defined in this case.

[46] I.e., there is a constant $c > 0$ such that $\Pr\left(F_i(Y) > 0 \text{ and } r(Y) < c\right)$ and $\Pr\left(F_i(Y) > 0 \text{ and } r(Y) > c\right)$ are both positive. This requirement rules out trivial cases where securities differ only on measure zero outcomes.

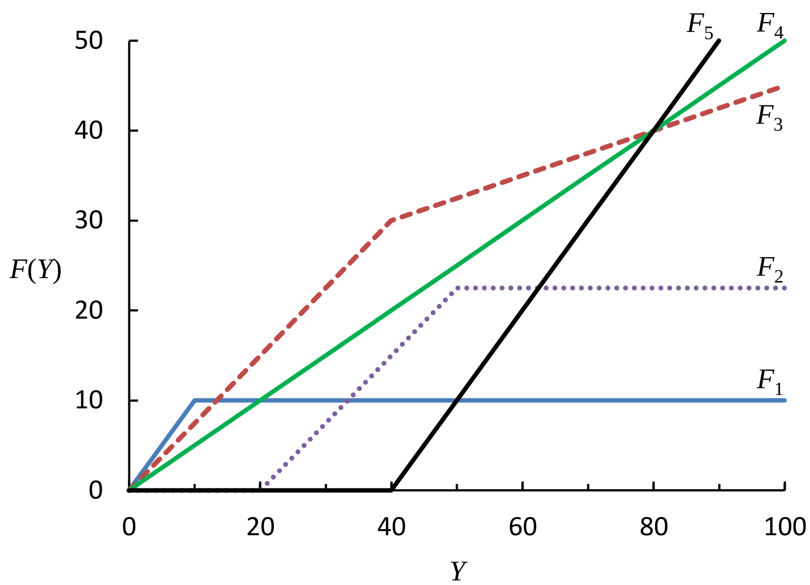

Figure 4: Examples of Prioritized Securities.
Securities are ranked with F_1 most senior and F_5 most junior, with the exception that F_2 is non-comparable with F_3 and F_4.

We can now relate a security's seniority to the order in which it will be sold: under HRO, junior securities will be retained first as a seller becomes more optimistic. Or, equivalently, senior securities will be sold first:

> **PROPOSITION 5.** Suppose HRO holds and the securities are prioritized. Then the assets satisfy Increasing Informational Sensitivity, so their liquidation is as set out in PROPOSITION 3 and PROPOSITION 4. In particular, the seller will not sell any shares of a given asset unless she also sells all more senior assets in their entirety.
>
> **PROOF:** Appendix. ◆

We now apply this result to our early example by allowing the seller to issue debt and equity backed by her existing portfolio of securities:

> **EXAMPLE 3.** Suppose the seller holds assets as in **EXAMPLE 2**. Rather than sell the assets directly, she pools them and designs two new securities: standard debt with a face value of 150, and equity which is the residual claimant on the pool. The seller then sees her information t and decides how much of each security to sell.

The final value of the underlying aggregate asset pool is normal with a standard deviation of 50 (with same aggregate mean as in **EXAMPLE 2**).

Figure 5 shows the demand curve for debt (asset 1) given the quantity of equity (asset 2) sold. As before, additional sales of equity lower the market price of the debt. Unlike the prior example, only local incentive constraints bind, so large discontinuities in the demand curves are not observed: as the seller sells more debt, investors' beliefs fall one type at a time. In equilibrium, the seller sells all of the debt if she sells any equity at all: if $q_2 > 0$, then $q_1 = 1$.

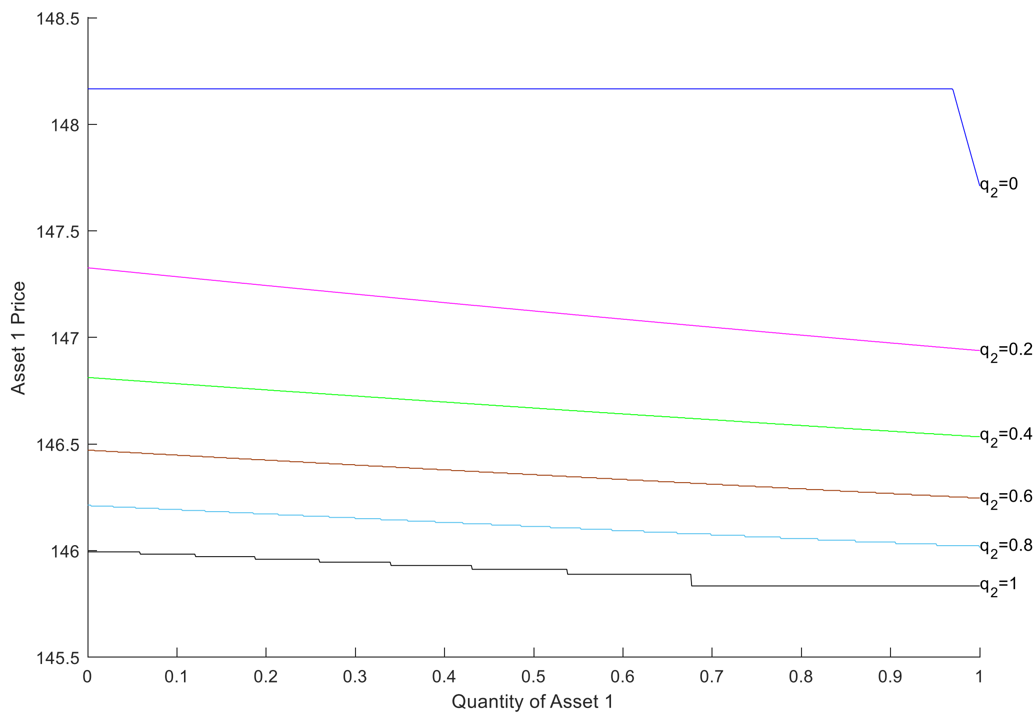

Figure 5: Debt Demand Curves for EXAMPLE 3.
Demand curves for debt given different quantities of equity sold. While cross-signaling is still important, demand curves become continuous (relative to the type space) as only local incentive constraints bind.

Figure 6 compares the liquidity of the portfolio in **EXAMPLE 3** with that in **EXAMPLE 2** by showing the valuation of the seller's portfolio as a function of the revenue raised in the sale. In this case, pooling and tranching leads to significantly greater liquidity: at each level of issuance revenue, the market's valuation of the whole portfolio is higher. We will study the optimal way to pool and tranche a portfolio in the next section.

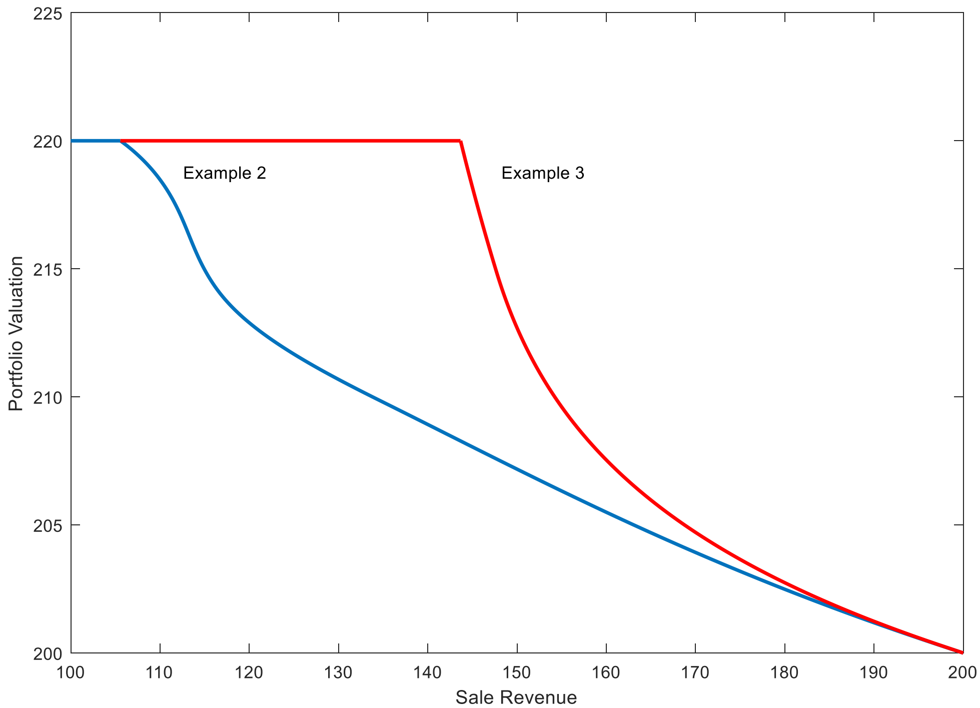

Figure 6: Portfolio Liquidity in EXAMPLE 2 versus EXAMPLE 3.
Pooling and tranching allows a given amount of sales revenue to be raised with a smaller impact on the market's valuation of the seller's portfolio.

3. Security Design

Having solved the general portfolio liquidation problem, we now apply the above tools to the problem of security design. More precisely, we generalize the single *ex-ante* security case studied by DeMarzo and Duffie (1999) (henceforth DD) by permitting an issuer to design multiple securities before and/or after she receives her information. We refer to this as the General Security Design (GSD) game. A key finding of this section is that the option to design the security ex post eliminates any benefit of ex-ante design.

In particular, DD assume the issuer designs a security in a symmetric-information ex ante stage, learns her information, and then chooses how many shares to sell of this fixed security. We show that when ex-post design is available, the ex-ante strategy of DD is suboptimal. More precisely, the issuer prefers to wait until she learns her information and then design a single standard debt security whose face value is decreasing in her type. She prefers to use the face value of her security - rather than, as in DD, the quantity of shares - to signal her information. The intuition is that lowering the face value is a more efficient signal of optimism since it lowers the payout only in high states, which are more likely when the issuer is more optimistic.

Our analysis of the GSD game proceeds in two steps. First, we first show that any intuitive equilibrium of the GSD game is equivalent to one in which the issuer pools her initial assets, sees her information, and then designs and sells a single security whose payout equals the aggregate payout of the securities she sells in the original equilibrium.[47]

To further sharpen these predictions, we then impose the above Hazard Rate Ordering property. We also assume *monotonicity*: the payout received by investors, as well as that retained by the issuer, is a nonnegative and nondecreasing function of the final cash flow.[48]

Under these two assumptions, we show that the issuer's optimal ex-post security is standard debt with a face value that is decreasing in her information. We establish this by proving a formal equivalence between ex-post security design and a strategy in which the issuer pools her initial assets, designs a maximal set of prioritized debt securities or "tranches" that are secured by the pool, sees her information, and then chooses how much of each tranche to sell as in the asset sale game.[49] By PROPOSITION 5, the issuer will sell those

[47] One may wonder, if the issuer can design a security during an ex-ante stage in which it is common knowledge that she has no private information, then why can't she sell her assets during this stage as well – thus realizing all the gains from trade? We have three responses. First, as noted by DD, some issuers "shelf-register" a security before they acquire the assets that will serve as collateral. In this case, selling the security before acquiring the collateral would presumably be ruled out for moral hazard reasons. Second, the presence of the ex ante stage does not alter our model's predictions since the issuer has an optimal strategy that does not use it. Finally, a model without an ex ante stage is indeed studied in the working-paper version of our paper, DeMarzo (2003). However, as that model does not generalize DD, it does not show (as we do) that ex-post design is better for the issuer than DD's ex-ante strategy.

[48] Monotonicity is a common assumption in the security design literature; see, e.g., DeMarzo (2005), DeMarzo and Duffie (1999), Frankel and Jin (2015), Hart and Moore (1995), Matthews (2001), and Nachman and Noe (1994). It can be justified by supposing the issuer has free disposal over her cash flow Y and can also contribute cash to inflate it. Hence, if her payout were decreasing in Y, she would freely dispose of some cash in order to raise this payout. And if, alternatively, the payout to investors were falling in Y, the issuer would contribute cash to inflate Y, thus paying investors less and raising her payout by more than the amount contributed. Finally, nonnegativity is motivated by the common limited-liability feature of financial assets.

[49] The two strategies are equivalent because any monotone security can be mimicked by a suitable portfolio of tranches and vice-versa. Equivalence means that the portfolios sold by the issuer in the two strategies

tranches whose seniority lies above a threshold that is increasing in her information. But this strategy is equivalent to selling a single standard debt security whose face value is decreasing in the issuer's information. Thus, the two strategies are equivalent and optimal.

In each such strategy, the issuer begins by pooling her initial assets. In DeMarzo (2005), in contrast, it is sometimes better not to pool. Why? In our setting, the initial assets partition a common underlying cash flow. Hence, each pre-existing asset can be exactly replicated by a new asset that is secured by the pool of initial assets. As pooling does not restrict the issuer, it cannot harm her. In DeMarzo (2005), in contrast, the initial assets are backed by distinct cash flows. This gives rise to two new effects. The first is an information destruction effect: pooling prevents the issuer from signaling that one cash flow is high while another is low, which is harmful to her. The second is a diversification effect: by diversifying away the residual risk in the cash flows, pooling allows the issuer to issue debt with a lower default risk and thus to reap more of the gains from trade with investors.[50] Whether pooling is optimal depends on the tradeoff between these two forces.

More precisely, DeMarzo (2005, section 4.1) assumes an issuer who has independent *n*-dimensional information about the payoffs of a given set of *n* assets. The issuer can either design a single debt contract that is secured by the pool of *n* assets, or *n* separate debt contracts, each secured by a different asset. Each security is ex-ante debt: its face value is chosen before the issuer sees her information. Conditions are given under which the diversification effect outweighs the information destruction effect: the issuer fares better with a single security than with *n* separate ones.

DeMarzo (2005, section 4.2) then extends this result to ex-post debt: debt whose face value is chosen by the issuer *after* she learns her information. Importantly, DeMarzo (2005) does not show that debt is the optimal ex-post security. Rather, he cites an earlier version of our paper, "DeMarzo (2003b)", for this result. The present version of this paper proves a stronger result: that a single ex-post debt contract is the optimal security even when the issuer can design arbitrary numbers of ex-ante and ex-post securities.

---

raise the same revenue from investors and, for any realization of the underlying cash flow, entail the same final aggregate payout to investors.

[50] The second effect can also be seen in the present model, as a benefit to the issuer from a reduction in the conditional volatility of her underlying cash flow; see EXAMPLE 7 below.

As they are somewhat technical, the proofs of this section's results appear in our online appendix (DeMarzo, Frankel, and Jin 2017).

3.1. The General Security Design Game

In the Asset Sale game introduced in section 2, the issuer chooses how many shares to sell of each of a given set of assets. This limits her; for instance, she cannot borrow money using some of her assets as collateral. Doing so is equivalent to selling a debt security that is secured by her assets. In this section we discard this restriction: the issuer can design general securities (not only debt) that are secured by her assets. She can do so both before and after she sees her information. Thus, the model is a generalization of both DeMarzo and Duffie (1999), in which the issuer designs a single security before seeing her information, and an earlier version of this paper (DeMarzo 2003) in which she designs a single security ex post.

As in section 2.5, we assume the issuer is endowed with a set of initial assets that are secured by a common, stochastic cash flow Y.[51] Before learning her type, the issuer designs a set of interim securities whose payouts are functions of the payouts of the initial assets. After she sees her type, she then designs a set of ex-post securities whose payouts are functions of the payouts of the interim securities.[52]

In this section, we show that any intuitive equilibrium of this game is equivalent to one in which the issuer pools her initial assets and sells a single ex-post security whose payout is a function of the pool's final value. In subsequent sections we show that under additional mild assumptions, this security is standard debt.

Formally, the general security design game is as follows.

[51] This means that the sum of initial asset payouts equals the realization Y of the cash flow. This assumption is equivalent to assuming that, in addition to her initial assets, the issuer can also use any residual cash flow as collateral for her securities. Hence, for all intents and purposes, this residual cash flow is an initial asset.

[52] Requiring the issuer to issue interim securities is not restrictive: any of them can be a pass-through security whose payout is identically equal to the payout of some initial asset. However, this flexibility is needed in order to reduce the general security design problem to the asset liquidation problem of our base model.

THE GENERAL SECURITY DESIGN (GSD) GAME. The issuer is endowed with a finite set $F(Y)=\left(F_i(Y)\right)_{i=1}^{n}$ of initial assets whose payout functions partition the common cash flow: $\sum_{i=1}^{n} F_i(Y)=Y$. Timing is as follows.

1. The issuer designs a finite set $I=\left(I^k\right)_{k=1}^{K}$ of *interim* securities; the payout of each such security *k* is some function $I^k\left(F(Y)\right)$ of her initial asset payouts which we write, for brevity, as $I^k(Y)$.[53] The interim assets satisfy limited liability: $\sum_{k=1}^{K} I^k(Y)\in[0,Y]$.

2. The issuer sees her type *t* and offers investors a finite vector $P=\left(P^j\right)_{j=1}^{J}$ of some number of ex-post securities, where the payout of ex-post security *j* is a function $P^j\left(I(Y)\right)$ of her interim assets' payouts. These payouts also satisfy limited liability: $\sum_{j=1}^{J} P^j\left(I(Y)\right)\in\left[0,\sum_{k=1}^{K} I^k(Y)\right]$. At the same time, the issuer commits to a global revenue cap $\bar{\rho}\in\Re_+$: if her issuance revenue exceeds the cap, she keeps $\bar{\rho}$ and discards the rest.[54]

3. Investors assign a price p^j to each ex-post security *j*.

Let

$$W_P^I(Y)=\sum_{j=1}^{J} P^j\left(I(Y)\right) \tag{6}$$

denote the aggregate payout promised to investors, given the interim security vector *I* and the ex-post security vector *P*. By limited liability, this payout lies in $[0,Y]$. If a type-*t* issuer chooses the interim security vector *I*, the ex-post security vector *P*, and the revenue cap $\bar{\rho}$, her expected surplus from issuance equals $\left(\bar{\rho}\wedge\sum_{j=1}^{J} p^j\right)-\delta E\left[W_P^I(Y)\mid t\right]$: her issuance revenue less the discounted expected value of the securities issued.

[53] Formally, we define the new function $\hat{I}^k$ satisfying $\hat{I}^k(Y)=I^k\left(F(Y)\right)$ and then rename it to I^k.

[54] Like the price caps in the Asset Sale model, we use the revenue cap in order to rule out implausible equilibria (see n. 19). The cap never binds in equilibrium.

The following two examples illustrate the generality and flexibility of the GSD game. We begin with a two-security generalization of DeMarzo and Duffie (1999).

EXAMPLE 4. The issuer may design interim assets consisting of two prioritized securities: a senior tranche $I^1(Y)=\min\{D,Y\}$ with face value D and a junior tranche $I^2(Y)=\min\{D',Y-I^1(Y)\}$ with face value D'. After discovering her type *t*, she may then sell a quantity z_t^i of each tranche $i=1,2$. Formally, her two ex-post securities are then given by $P_t^i(I(Y))=z_t^i I^i(Y)$ for $i=1,2$.

The following is a two-security generalization of the one-security case considered by DeMarzo (2005) and Frankel and Jin (2015).

EXAMPLE 5. The issuer may sell senior and junior debt ex post with type-contingent face values D_t and D_t' respectively. Formally, she designs a single ex-ante pass-through security $I^1(Y)=Y$, while her two ex-post securities are then given by $P_t^1(I(Y))=\min\{D_t,Y\}$ and $P_t^2(I(Y))=\min\{D_t',Y-P_t^1(I(Y))\}$.

We will refer to the action $(P,\bar{\rho})$ that the issuer chooses after learning her type as her *ex-post action*. A *strategy* for the issuer is a triplet $(I,P_{\cdot}(\bullet),\bar{\rho}_{\cdot}(\bullet))$, which specifies her interim asset vector *I* and, for any type *t* and interim asset vector I' (which equals *I* unless the issuer deviated), her ex-post action $(P_t(I'),\bar{\rho}_t(I'))$. A *pricing function* for investors is a vector $p=(p^j)_{j=1}^J$ where $p^j=p^j(I,P,\bar{\rho})$ is the price of ex-post security *j* when the issuer chooses the action $(I,P,\bar{\rho})$. The issuer's *conditional expected payoff* from this action, given her type *t* and the investors' pricing function *p*, is thus

$$U(I,P,\rho\,|\,t,p)=\left[\bar{\rho}\wedge\sum\nolimits_{j=1}^{J}p^j(I,P,\rho)\right]-\delta\,E\left[W_P^I(Y)\,|\,t\right]. \tag{7}$$

Similarly, her *unconditional expected payoff* from the strategy $(I,P_{\cdot}(\bullet),\bar{\rho}_{\cdot}(\bullet))$, given the investors' pricing function *p*, is $U(I,P_{\cdot}(\bullet),\bar{\rho}_{\cdot}(\bullet)\,|\,p)=\sum\nolimits_t g(t)U(I,P_t(I),\bar{\rho}_t(I)\,|\,t,p)$, where $g(t)$ denotes the prior probability that her type is *t*.

EQUILIBRIUM: GSD GAME. An equilibrium of the General Security Design game consists of a strategy $\left(\bar{I}, P_{\cdot}(\bullet), \bar{\rho}_{\cdot}(\bullet)\right)$ for the issuer,[55] together with a price function p and a belief function μ, such that the following conditions hold.

1. Payoff Maximization: (a) the prescribed interim asset vector $\bar{I}$ is contained in $\arg\max_I U\left(I, P_{\cdot}(\bullet), \bar{\rho}_{\cdot}(\bullet) \mid p\right)$ and (b) for each interim asset vector *I*, the ex-post action $\left(P_t(I), \bar{\rho}_t(I)\right)$ chosen by each type t lies in $\arg\max_{(P,\rho)} U\left(I, P, \bar{\rho} \mid t, p\right)$.

2. Rational Updating: let $T\left(P, \bar{\rho} \mid I\right)$ denote the set of types who choose the ex-post action $\left(P, \bar{\rho}\right)$ conditional on having chosen any given interim asset vector *I*. Upon seeing the action $\left(I, P, \bar{\rho}\right)$, the investors' posterior probability $\mu(t \mid I, P, \bar{\rho})$ equals the probability that the issuer's type is *t* conditional on her type being in $T\left(P, \bar{\rho} \mid I\right)$: it equals $g(t)\left[\sum_{s \in T(P,\rho|I)} g(s)\right]^{-1}$ if $t \in T\left(P, \bar{\rho} \mid I\right)$ and zero otherwise.[56]

3. Competitive Pricing: for any action $\left(I, P, \bar{\rho}\right)$ of the issuer, the price of each ex-post security equals its expected payout given investors' posterior beliefs:

$$p\left(I, P, \bar{\rho}\right) = \sum_t E[P\left(I\left(Y\right)\right) \mid t]\ \mu(t \mid I, P, \bar{\rho})\,. \tag{8}$$

The *outcome* $u(t)$ of the above equilibrium – a function that gives the expected payoff of each type t from playing her prescribed strategy - is just her payoff $U\left(\bar{I}, P_t\left(\bar{I}\right), \bar{\rho}_t\left(\bar{I}\right) \mid t, p\right)$ from issuing the prescribed interim asset vector $\bar{I}$, learning her type *t*, and then choosing her prescribed ex-post action $\left(P_t\left(\bar{I}\right), \bar{\rho}_t\left(\bar{I}\right)\right)$.

---

[55] This strategy instructs the issuer to choose the interim asset vector $\bar{I}$ and then, following any interim asset vector *I*, to issue the ex-post action $\left(P_t(I), \rho_t(I)\right)$ when her type is *t*.

[56] In particular, if an unexpected interim asset vector *I* is chosen, investors believe that it was simply a mistake that conveys no information about the issuer's type. This is in the spirit of the "no signaling what you don't know" condition of perfect Bayesian equilibrium (Fudenberg and Tirole 1991, p. 332): the issuer's interim asset vector *I* cannot signal her type *t* as she chooses *I* before learning *t*.

3.2. Reduction to the Ex-Post Security Design Game

The GSD game must be simplified so as to be tractable. We do this in two steps. In this section, we show that one can focus on simpler strategies in which the issuer pools her initial assets and sells a single ex-post security that is secured by her initial asset pool. Formally, for any equilibrium E of the GSD game, let $S_t(Y)$ be the aggregate payout of the ex-post securities that a type t issuer sells in E when her underlying cash flow is Y. Then it is also an equilibrium of the GSD game for each type t issuer to pool her initial assets and sell a single ex-post security with payout function $S_t(Y)$.[57] Moreover, the new equilibrium is intuitive if the original one is:

> **PROPOSITION 6**. Let $E = \left(\hat{I}, P_{\cdot}(\bullet), \bar{\rho}_{\cdot}(\bullet), p, \mu\right)$ be any equilibrium of the GSD game, in which the aggregate payout function of the ex-post securities of a type-t issuer is $S_t(Y)$. Then there is an equilibrium $\tilde{E}$ of the GSD game, with the same outcome, in which the issuer pools her initial assets and, on seeing her type t, issues a single ex-post security with payout function $S_t(Y)$. This security yields the same payoff and issuance revenue as a type-t issuer gets in E. And if the original equilibrium E is intuitive, then so is $\tilde{E}$.[58]
>
> **PROOF:** Online appendix. ◆

3.3. Solving the Ex-Post Security Design Game

By PROPOSITION 6, we may restrict to a setting in which the issuer pools her initial assets and then, on seeing her type, designs a single ex-post security whose payout is some function of the payout of the pool. We will refer to this restriction of the GSD Game as the *Ex-Post Security Design* (EPSD) game.

---

[57] Technically, the GSD framework requires us to write the ex-post security's payout as a function of the payouts of some interim securities. To do so, we can assume the issuer designs a single pass-through interim security whose payout equals the aggregate value Y of her initial assets.

[58] The definition of Intuitive Equilibrium in the context of the GSD game appears in the proof.

While the EPSD game is simpler than the GSD game, the issuer's security is still a multidimensional signal of her type. The multidimensional signaling problem has not been solved in general.[59] However, we can solve it in an important special case, using results from prior sections. As in section 2.5, we will assume the issuer's information satisfies the Hazard Rate Ordering property. Moreover, we will restrict the issuer to the following set of *monotone securities*:

$$M = \left\{ S\text{: } S(y) \text{ and } y - S(y) \text{ are nonnegative and nondecreasing in } y \in \text{supp}(Y). \right\}.$$

We will also assume, for now, that the cash flow is discrete.[60] For convenience, we also normalize the lowest realization to zero:[61]

ASSUMPTION C (DISCRETE CASH FLOW). $Y \in \{y_0, \ldots, y_n\}$ where $y_0 = 0$.

Given her information *t*, if the issuer sells security $S \in M$ for the price *p*, her total payoff is $p + \delta E[Y - S(Y) \mid t]$ which can be written as the sum of the exogenous component $\delta E[Y \mid t]$, which we ignore, and her issuance profit $p - \delta E[S(Y) \mid t]$.

The key insight of this section is that under the above assumptions, the EPSD game can be solved as an instance of the Asset Sale game. Suppose that, ex ante, the issuer pools and tranches her initial assets into *n* interim securities or "tranches". The payout of tranche $i = 1, \ldots, n$ is defined by

$$F_i^*(Y) = (y_i - y_{i-1}) \times 1[Y \geq y_i] \tag{9}$$

[59] There is a small literature on multidimensional signaling. The closest papers are Cho and Sobel (1990) and Ramey (1996). Like us, they show that equilibria exist that satisfy an equilibrium refinement, and that all such equilibria have the same payoffs. The main difference is that they use the D1 refinement, while we assume only the weaker Intuitive Criterion.

[60] The continuous case is studied below in section 3.5.

[61] We can recover the unnormalized case as follows. Suppose that there is an equilibrium of the normalized model, in which the issuer sells a security with payout $S_t(Y)$ for price p_t and the lowest realization of *Y* is zero. Now suppose her cash flow is instead the unnormalized $Y' = Y + y_0$ for some positive y_0. Since each type of issuer can commit to transfer the lowest cash flow y_0 to investors, the issuer will sell a risk-free bond with face value y_0 in order to reap the full gains from trade $(1-\delta) y_0$ from this transfer. As for the residual cash flow $Y = Y' - y_0$, the original equilibrium analysis applies: a type *t* issuer will sell a security with payout $S_t(Y)$ for the price p_t. Her issuance revenue is simply the sum $y_0 + p_t$ of the prices of these two securities.

where the indicator function $1[Y \geq y_i]$ equals one if $Y \geq y_i$ and zero otherwise. That is, tranche *i* pays the incremental cash flow $y_i - y_{i-1}$ if the cash flow Y is at least y_i, and zero otherwise. The payouts of the *n* tranches clearly sum to the cash flow *Y*. Let

$$f^*(t) = E\left[F^*(Y) \mid t\right] \tag{10}$$

denote the corresponding conditional expected payout vector.

After tranching her asset pool, the issuer sees her type *t* and chooses a quantity q_i of each tranche *i* to sell. This subgame can be thought of as an Asset Sale game with endowment (a^*, f^*) where a^* denotes a vector consisting of *n* ones. Crucially, there is a one-to-one correspondence between portfolios *q* in this Asset Sale game, on the one hand, and monotone securities *S* in the EPSD game, on the other. The correspondence is as follows.

1. For any monotone security *S*, let q^S denote the row *n*-vector whose *i*th component q_i^S equals the proportion $\left[S(y_i) - S(y_{i-1})\right]/(y_i - y_{i-1})$ of the cash flow increment that security *S* pays to investors when the cash flow rises from y_{i-1} to y_i. It is easy to see that the portfolio consisting of q_i^S shares of each tranche *i* has the same payout function as the security *S*. And since *S* is monotone, the increment $S(y_i) - S(y_{i-1})$ in the security's payout lies between zero and $y_i - y_{i-1}$, inclusive, whence each quantity q_i^S lies in $[0,1] = \left[0, a_i^*\right]$ and is thus feasible.

2. Conversely, let *q* be a portfolio of tranches in the given Asset Sale game. This portfolio has the same payout function as the single security $S^q(Y) = qF^*(Y)$. Clearly, $S^q(Y)$ is zero at $Y = 0$, is nondecreasing in the cash flow *Y* as the tranches have this property, and rises no more quickly than $a^*F^*(Y) = Y$ which has a slope of at most one in *Y*. Thus, S^q is a monotone security as claimed.[62]

By this correspondence and PROPOSITION 1**,** there is a unique intuitive outcome of the EPSD game, which can be computed using RLP:

---

[62] A similar construction appears in Hart and Moore (1995), although they use it for a different purpose.

PROPOSITION 7. Suppose the issuer is restricted to monotone securities $S \in M$ in the EPSD game and the issuer's information satisfies First-Order Stochastic Dominance: for any cutoff y and types $t > s$, $\Pr\left(Y \le y \mid s\right) \ge \Pr\left(Y \le y \mid t\right)$. Then in any intuitive equilibrium of the EPSD game, the issuer's optimal security $S_t^*(Y)$ equals $q^*(t)F^*(Y)$ where $q^*(t)$ solves the RECURSIVE LINEAR PROGRAM with $\left(a^*, f^*\right)$ substituted for $\left(a, f\right)$.

PROOF: Online appendix. ◆

3.4. The Optimality of Debt

While PROPOSITION 7 shows how to compute the issuer's unique optimal security, it does not reveal much about the security's structure. Under HRO, we can say much more: this security is standard debt. And in order to signal optimism, the issuer chooses a lower face value of this debt, thus retaining a larger portion of her cash flow.[63]

A rough argument is as follows. One can easily verify that the tranches F^* are PRIORITIZED ASSETS (Section 2.5), where the seniority of tranche *i* is declining in the index *i*.[64] Hence, under HRO, PROPOSITION 3 implies that the issuer uses a hurdle class strategy: she sells all shares of tranches i with indices below some threshold *c*, no shares of tranches $i > c$, and some $q_c^*\left(t\right)$ shares of tranche *c*. Her corresponding monotone security in the EPSD game is thus

$$\begin{aligned} S^q(Y) = qF^*(Y) &= q_c^*(t)(y_c - y_{c-1}) \times 1\left[Y \ge y_c\right] + \sum_{i=1}^{c-1}(y_i - y_{i-1}) \times 1\left[Y \ge y_i\right] \\ &= \min(D, Y) \end{aligned}$$

[63] Near the end of this section we show, by example, that the issuer may sometimes sell equity if HRO is violated. This may be relevant to young firms, which tend to use equity funding. While a general result in this direction seems elusive, it might be an interesting avenue for future research.

[64] For any two tranches $i < j$, $F_j^*\left(Y\right) = F_i^*\left(Y\right) r\left(Y\right)$ where $r\left(Y\right)$ equals zero if $Y < y_j$ and $\left(y_j - y_{j-1}\right) / \left(y_i - y_{i-1}\right)$ otherwise. Thus, tranche *i* is senior to tranche *j*.

where $D = y_{c-1} + q_c^*(t)(y_c - y_{c-1})$. But this is just a standard debt contract with face value D. By PROPOSITION 3, $q^*(t)$ is nonincreasing in t, whence higher-type issuers choose a weakly lower face value D.[65]

Prior results can now be used to determine the face value D for any type t. Let $v(D,t) = E[\min\{Y, D\} \mid t]$ denote the expected payout of debt with face value D conditional on the issuer's type t. Let D_t^* denote the equilibrium face value chosen by an investor of type t. If an investor of type t imitates a type t+1 investor by choosing face value D_{t+1}^*, investors must assign the value $v\left(D_{t+1}^*, t+1\right)$ to her security by fair pricing. Since her valuation of the security is $\delta v\left(D_{t+1}^*, t\right)$, her payoff from the deviation is the difference $v\left(D_{t+1}^*, t+1\right) - \delta v\left(D_{t+1}^*, t\right)$. On the other hand, if she chooses her equilibrium face value of D_t^*, investors will pay her $v\left(D_t^*, t\right)$ for a security that she values at $\delta v\left(D_t^*, t\right)$, so her payoff is $(1-\delta) v\left(D_t^*, t\right)$. Since, by PROPOSITION 4**,** each type $t = 0, \ldots, T-1$ is just willing not to imitate type $t+1$, the face value D_{t+1}^* of type $t+1$ is uniquely[66] given by

$$v\left(D_{t+1}^*, t+1\right) - \delta v\left(D_{t+1}^*, t\right) = (1-\delta) v\left(D_t^*, t\right). \tag{11}$$

The most pessimistic issuer (type zero) sells her entire cash flow, which is equivalent to choosing debt with face value D_0^* equal the maximum cash flow y_n. This yields the following characterization for the security design game:

> **PROPOSITION 8.** Suppose the issuer's information satisfies HRO and the issuer is restricted to monotone securities. Then in the unique intuitive equilibrium of the Ex-Post Security Design game, the optimal security design is standard debt with

[65] We will show that this face value is, in fact, strictly decreasing in the issuer's type.

[66] The left side of (11) is strictly increasing in $D \in (0, y_n]$. Because $v(D, t+1) > V(D,t)$ and $v(0,t) = 0$, a unique solution D_{t+1}^* to (11) exists, which lies in $\left(0, D_t^*\right)$.

face value given by (11) with initial condition $D_0^* = y_n$. The face value D_t^* is positive and strictly decreasing in the issuer's type *t*.

PROOF: Online appendix. ◆

We end this section with an example in which HRO does not hold. In this example, the intermediate type of issuer sells equity rather than debt, in contrast with PROPOSITION 8.

EXAMPLE 6. The issuer owns a stochastic cash flow *Y* that takes on three possible values, 0, 5, or 10, with type-contingent probabilities given by the following table.

Type *t*	$\Pr(Y=0\,\vert\,t)$	$\Pr(Y=5\,\vert\,t)$	$\Pr(Y=10\,\vert\,t)$
0	0.8	0	0.2
1	0.2	0.4	0.4
2	0	0	1

HRO does not hold: conditional on *Y* being 5 or 10, type 0 thinks *Y* equals 10 for sure while type 1 thinks 5 and 10 are equally likely.[67]

Applying (9), any monotone security derived from the cash flow *Y* is equivalent to a portfolio of two assets: asset 1 ("debt") pays 5 if the cash flow *Y* is at least 5 while asset 2 ("equity") pays 5 if the cash flow is at least 10. Using the prior table, one can verify that the conditional expected values of these assets are as in EXAMPLE 1: $f(0)=(1,1)$, $f(1)=(4,2)$, and $f(0)=(5,5)$. Hence, as shown in EXAMPLE 1, the issuer's optimal quantities to sell of these two assets are $q^*(0)=(1,1)$, $q^*(1)=(0,2/3)$, and $q^*(2)=(2/9,0)$. PROPOSITION 7 then yields the equilibrium of the EPSD game with cash flow *Y*: the low type sells her whole cash flow, the high type sells 2/3 shares of a debt security with face value of 5, and the intermediate type sells an equity claim to 2/9 of the residual that is received after the given debt security (which the issuer retains) is fully repaid.

Why does the intermediate type sell equity? In the information structure of EXAMPLE 6, increasing the issuer's type from 0 to 1 raises the chance of the intermediate realization of

[67] For $y' > y$ and $t > s$, HRO states that $\frac{\Pr(Y \ge y|s)}{\Pr(Y \ge y|t)} > \frac{\Pr(Y \ge y'|2)}{\Pr(Y \ge y'|t)}$ or equivalently that $\frac{\Pr(Y \ge y'|t)}{\Pr(Y \ge y|t)} > \frac{\Pr(Y \ge y'|2)}{\Pr(Y \ge y|s)}$: the probability that $Y \ge y'$ conditional on $Y \ge y$ is increasing in the issuer's type. As noted, this does not hold in the table for $y=5$, $y'=10$, $s=0$, and $t=1$.

the cash flow proportionally more than the chance of the high realization (which, as noted, violates HRO). As a result, it also raises the expected value of the debt tranche proportionally more than that of the equity tranche. Retaining the debt tranche is therefore a more efficient way for type 1 to separate from type 0. In contrast, under HRO, a type increase always raises the conditional expected value of a junior (equitylike) tranche proportionally more than that of a senior (debtlike) tranche: the issuer can always signal her optimism more efficiently by retaining a junior rather than a senior tranche.

This example suggests that in the absence of HRO, an issuer may sell equity rather than debt. The issuer may be a firm that wishes to sell claims to its assets to raise funds. In this case, the firm's optimal capital structure may depend on the type of information that the firm has about the value of its underlying assets. Are new firms, which favor equity, more likely to have information structures that violate HRO? This might be an interesting area of future research.

3.5. Continuous Security Design

The preceding results assume discrete distributions of the issuer's type *t* and the cash flow *Y*. In the applications in DeMarzo (2005) and Frankel and Jin (2015), these distributions are continuous. We will show that in the continuous model, the optimal face value of the issuer's debt is given by a simple differential equation.

3.5.1. Characterizing the Face Value

Recall that $v(D,t)=E\left[\min\{Y,D\}\,|\,t\right]$ denotes the conditional expected payout of simple debt with face value *D*. If the increment between types is Δ rather than one, the incentive compatibility condition (11) becomes

$$v\left(D_{t+\Delta},t+\Delta\right)-\delta v\left(D_{t+\Delta},t\right)=\left(1-\delta\right)v\left(D_t,t\right). \qquad (12)$$

Taking the derivative of (12) with respect to the increment Δ at $\Delta=0$, we have

$$(1-\delta)v_1\left(D_t,t\right)\frac{dD_t}{dt}+v_2\left(D_t,t\right)=0.$$

Letting $\pi(D,t)$ be the conditional probability $\Pr(Y < D \mid t)$ of default, and $\bar{y}$ be the maximum value of the cash flow *Y*, we can use the fact that $v_1(D,t) = 1 - \pi(D,t)$ to obtain the following characterization of the equilibrium debt issuance:

$$D_0 = \bar{y} \quad \text{and} \quad \frac{dD_t}{dt} = -\frac{v_2(D_t,t)}{(1-\delta)\left[1-\pi(D_t,t)\right]} < 0, \tag{13}$$

Specifically, consistent with PROPOSITION 8, the most pessimistic issuer sells her entire cash flow, and higher types choose a lower face value and borrow less to signal their optimism. The rate $\left| dD_t/dt \right|$ at which the face value falls as the type rises is the issuer's benefit-cost ratio of lowering the face value by one small unit in order to pretend to be more optimistic. Her benefit from this deviation is the increase $v_2(D_t,t)$ in securitization revenue that results from raising investors' beliefs. The cost is the surplus that is lost, holding investors' beliefs fixed: the probability $1-\pi(D_t,t)$ that the face value will be repaid, times the surplus $1-\delta$ that is lost by retaining the additional unit of cash flow. The larger is this benefit/cost ratio, the faster the face value must fall in order to prevent such deviations.

If this initial value problem has a unique solution D_t^∞, then it is a separating equilibrium of the continuous model:[68]

> **PROPOSITION 9.** Assume HRO and the existence of a unique, continuous solution D_t^∞ to (13). Then there is a separating equilibrium of the continuous model of the security design game in which, given her type *t*, the issuer announces a security whose payout function is $S^*(Y) = \min(D_t^\infty, Y)$.
>
> **PROOF:** Online appendix. ◆

We now illustrate the continuous solution with a numerical example, depicted in Figure 7 below.

[68] Existence and uniqueness are discussed below in section 3.5.2.

EXAMPLE 7. Suppose the cash flows Y are lognormally distributed conditional on the issuer's information. Let the issuer's information equal the mean of ln(Y), so that $Y = 100e^{t-\sigma^2/2+\sigma Z}$ where Z is standard normal.[69] Suppose $t \in [0, .25]$ and $\delta = 0.95$. Figure 7 shows the issuer's optimal face value as a function of t for different volatilities σ. As shown, the face value falls in t, while it is nonmonotone in the volatility σ.

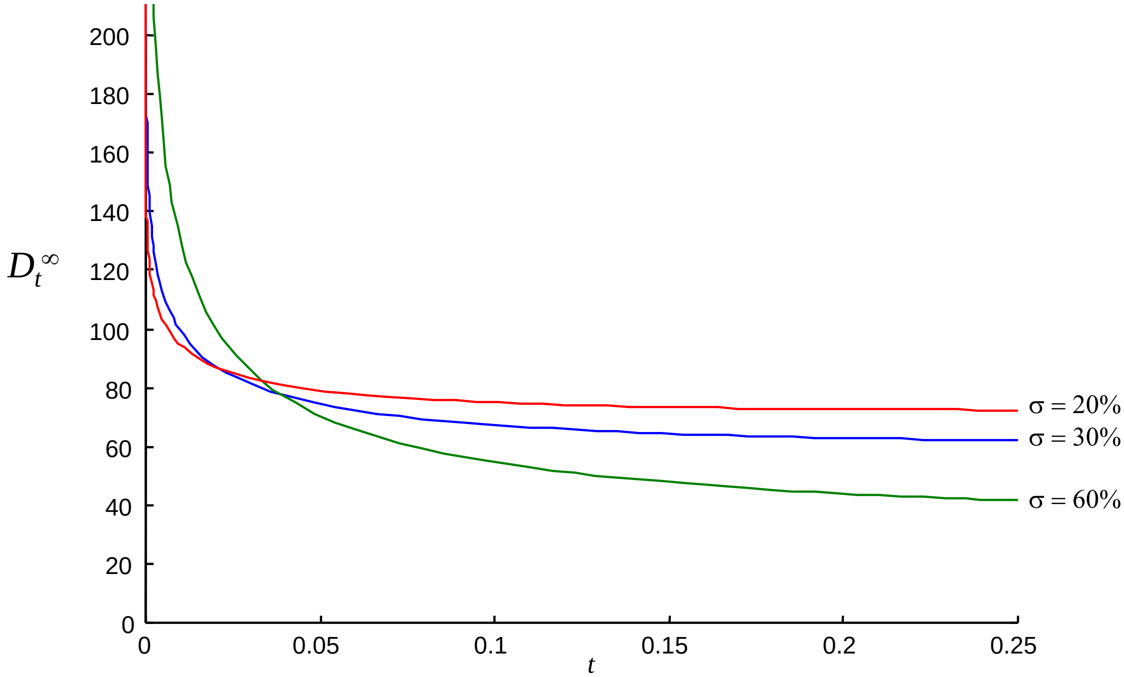

Figure 7: Optimal Face Value of Debt as a Function of Issuer Type and Asset Volatility

Since the quality of the debt depends upon *both* face value and volatility, it is more meaningful to compare the issuance revenue $E[\min(D_t^\infty, Y) \mid t]$ in these same three cases. This comparison appears in Figure 8. As shown, issuance revenue is decreasing in the volatility of the cash flows. Intuitively, when the volatility σ is large, the "cash flow surprise" $Y - E[Y \mid t]$ is more likely to take on large negative values which cause the security to default. This makes the issuer's information more critical in evaluating the expected payout of her security. Thus, convincing investors that her type is higher causes

[69] The convexity correction $-\sigma^2/2$ implies that the conditional expected value $E[Y \mid t]$ equals $100e^t$ and thus is unaffected by changes in the volatility σ.

a larger increase in the security's price when volatility is high. In order to deter imitation by lower types, the issuer must therefore send a more costly signal: she must give up more issuance revenue. Hence issuance revenue declines more quickly in the issuer's type when volatility σ is higher in Figure 6.

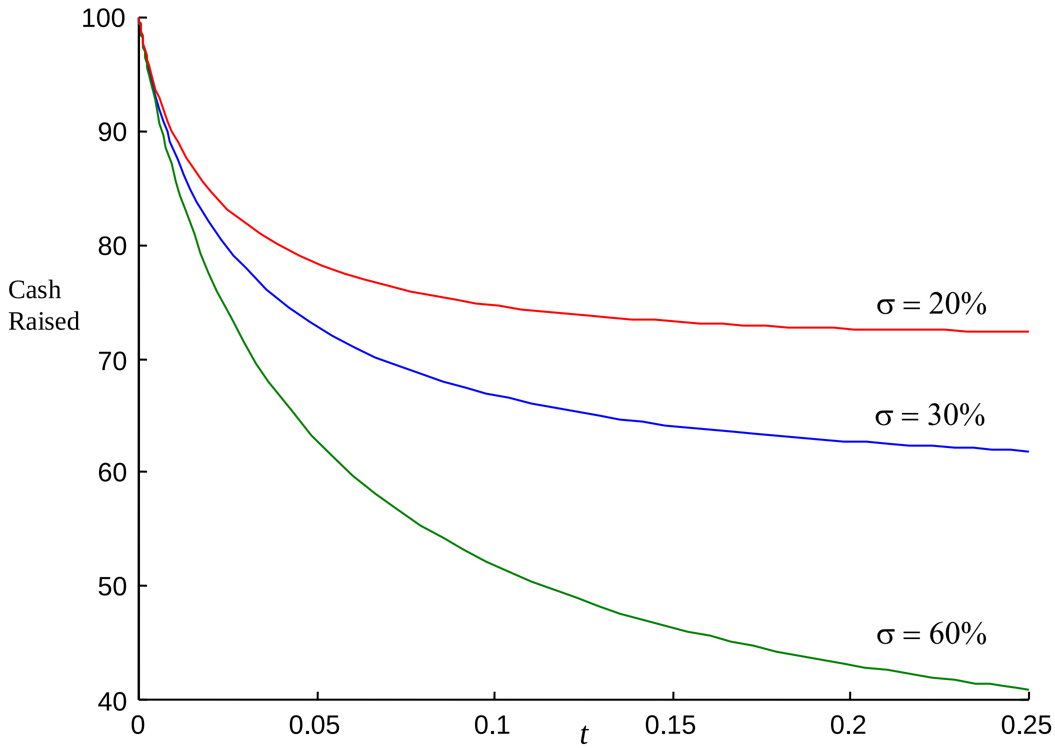

Figure 8: Issuance Revenue as a Function of Issuer Type and Asset Volatility

After a suitable rescaling, the curves in Figure 6 can be reinterpreted as the issuer's conditional expected payoffs.[70] Thus, the incentive to sell assets with minimal informational sensitivity leads the issuer to prefer a less volatile underlying cash flow. This "implied risk aversion" holds even though all participants are risk neutral and creates the incentive for pooling in DeMarzo (2005).

[70] In this rescaling, the numbers on the vertical axis are multiplied by the difference $1-\delta$ in discount factors.

3.5.2. Existence, Uniqueness, and Convergence (Technical)

PROPOSITION 9 assumes the existence of a unique solution to equation (13). In our online appendix, we verify this property. We assume the issuer's type $t \sim G$ has full support [0,1] and the final cash flow Y has conditional and unconditional support $[0, \bar{y}]$ and conditional distribution function $H(\cdot \mid t)$. The function H satisfies the Hazard Rate Ordering property and the following regularity condition:

> **LIPSCHITZ-*H* (L-*H*).** The conditional distribution function H is continuously differentiable, has no atoms, and has some minimal sensitivity to its arguments: there are constants k_0, $k_1 \in (0,\infty)$ such that for all t in [0,1] and y in $[0, \bar{y}]$, the derivative $H_1(y \mid t)$ is in (k_0,k_1) and $-H_2(y \mid t)$ lies in $[k_0 y(\bar{y} - y), k_1)$.[71,72] Further, both partial derivatives of H are Lipschitz continuous in the type t: there is a constant k_2 in $(0,\infty)$ such that for all y in $[0, \bar{y}]$ and t', t'' in [0,1], both $\left|H_1(y \mid t') - H_1(y \mid t'')\right|$ and $\left|H_2(y \mid t') - H_2(y \mid t'')\right|$ are less than $k_2 \left|t' - t''\right|$.

We show that, in this setting, there exists a unique solution to equation (13), which is the limit of the solutions to the discrete problem (12) as the gap Δ between types shrinks to zero. Hence, the unique solution to equation (13) is a good approximation of the unique intuitive equilibrium in the discrete model. This strengthens the case for the solution to using equation (13) to study the continuous model.

In addition, a researcher may wish to embed the security design problem in a setting in which the joint distribution of types and shocks depends on the actions chosen by the issuer and other players in a prior "pregame" period (e.g., Frankel and Jin 2015). In our online appendix, we also show that as the gaps between types and shocks shrink to zero, the gap

---

[71] A higher type t is good news about the shock and thus lowers $H(y \mid t)$. Accordingly, we state the bounds in terms of $-H_2(y \mid t)$ which is nonnegative.

[72] As $H(0 \mid t)$ and $H(\bar{y} \mid t)$ are identically zero and one, respectively, we cannot require that $H_2(y \mid t)$ be sensitive to the type t for each y. However, we can require that as y moves away from its highest and lowest values, this sensitivity rises at least linearly in y. This is ensured by the factor $y(\bar{y} - y)$ in the lower bound on $-H_2(y \mid t)$.

between profits in the discrete and continuous model also shrinks to zero, uniformly in the joint distribution of types and shocks (and thus in the pregame action profile). This result may be used in applications to show that that the issuer's optimal action in the discrete case converges to that of the continuous model.[73] This robustness property further strengthens the case for using the solution to (13) rather than any alternative equilibria.

3.6. Empirical Predictions

Begley and Purnanandam (2017) find that when the equity (retained) tranche of a pool of residential mortgage-backed securities makes up a larger proportion of the pool's face value, the loans in the pool have lower subsequent delinquency rates conditional on observables and the securities that are sold fetch higher prices conditional on their credit ratings. This is implied by equation (13): the issuer's face value D_t is lower (and so her equity tranche is higher) when her type *t* is higher (and so she is more optimistic about the repayment probabilities of the underlying loan pool).

Begley and Purnanandam also find that issuers retain larger proportions of the face value of RMBS pools that contain a higher proportion of no-documentation loans. This is also predicted by equation (13), by the following result.

> **PROPOSITION 10**. Assume HRO and Lipschitz-*H* holds for conditional distribution functions $\hat{H}$ and $\tilde{H}$. Suppose that $\tilde{H}(y|1) \geq \hat{H}(y|1)$ and $\tilde{H}_2(y|t) \leq \hat{H}_2(y|t)$ for all $t \in [0,1]$ and $y \in [0, \bar{y}]$. Let the face value functions $\hat{D}_t$ and $\tilde{D}_t$ solve equation (13) for H equal to $\hat{H}$ and $\tilde{H}$, respectively. Then $\hat{D}_t \geq \tilde{D}_t$ for all *t*: the issuer does not choose a lower face value under $\hat{H}$ than under $\tilde{H}$.
>
> **PROOF:** Online appendix. ◆

PROPOSITION 10 relates to Begley and Purnanandam as follows. Suppose a bank plans to issue loans with aggregate face value $\bar{y}$. Prior to lending, the bank flips a coin to determine whether to issue no-documentation or full-documentation loans. It then issues the loans

[73] The convergence of the issuer's optimal action will generally require additional assumptions, such as the strict quasiconcavity of the issuer's expected total profits as a function of the issuer's pregame action.

and privately observes a local macroeconomic shock *t* that affects the distribution of the aggregate repayment $Y \in [0, \bar{y}]$. Assume, plausibly, that the distribution of the macro shock *t* does not depend on the outcome of the prior coin flip.

Let $\tilde{H}(\cdot | t)$ and $\hat{H}(\cdot | t)$ denote the distributions of the aggregate repayment *Y* conditional on the macro shock *t*, in the no- and full-documentation cases respectively. The main effect of requiring documentation is to prevent loan applicants from exaggerating their financial strength. Thus, even if the shock *t* takes its highest value of one, the probability that the repayment *Y* falls below any fixed threshold will be higher when there are more no-documentation loans in the pool. This is the first inequality assumed in PROPOSITION 10. Moreover, as no-documentation borrowers tend to be more financially fragile conditional on observables, the repayment *Y* will be more sensitive to the shock *t*. Thus, the probability that *Y* falls below any given threshold will be more sensitive to the macro shock *t* in the case of no-documentation loans. This is captured by the second inequality assumed in PROPOSITION 10.

PROPOSITION 10 then predicts that if the bank chooses to issue no-documentation loans, the face value of its security will be lower for any given shock *t*: it will retain a higher portion of its loans' face value, as Begley and Purnanandam find. An intuition appears above in section 1.

4. Relaxing Monotonicity

ASSUMPTION A includes a monotonicity property: each asset's expected value is nondecreasing in the seller's type. In our online appendix we show that, for generic parameters, this property can be omitted in the case of two assets and two types. Intuitively, one type must expect her portfolio to have a higher total payout than the other. Swapping indices if needed, we can assume this is type 2. Moreover, an increase in the seller's type from 1 to 2 must raise the expected value of one asset proportionally more than that of the other. Again swapping indices if needed, we can assume this is asset 2. Hence, Increasing Informational Sensitivity (IIS) holds automatically in the 2x2 case: an increase in the seller's type from 1 to 2 raises the expected value of asset 2 proportionally more than that of asset 1. We show, further, that IIS can be used in place of monotonicity to prove that there is a unique intuitive equilibrium in the 2x2 case.

As in our base model under IIS, the equilibrium displays the Pecking Order property: the seller will not retain any shares of asset 1 unless she retains asset 2 in its entirety. However, a stronger prediction emerges in the special case in which asset 1 is worth *less* to type 2 than to type 1 - a case that ASSUMPTION A rules out. In this situation, the seller uses only asset 2 to signal her type: she never retains any shares of asset 1.

To illustrate, consider debt and equity securities, and suppose that both the mean and volatility of the cash flow are higher for type 2 than for type 1. With a large enough difference in volatility, debt will be worth less to type 2 than to type 1. In this case, both types sell the debt in its entirety: the seller retains only equity in order to signal her type. Intuitively, type 1 gains more from retaining debt than type 2 does. Hence, if type 2 were to retain some debt, then she would have to retain even more equity in order to signal credibly that she is not type 1. Such inefficient signaling is ruled out by the Intuitive Criterion. Although we do not pursue this point further, the feature that securities whose value declines with type will not be retained should extend to more general settings.

5. Concluding Remarks

In this paper we study the problem of an informed seller who wishes to sell securities to raise cash. Using the Intuitive Criterion, we identify the unique equilibrium when the seller has a fixed set of assets whose expected payouts are nondecreasing in her information. We show by example that the usual monotonicity property can fail: a more pessimistic seller may sell fewer shares of a given asset. This occurs when the assets alternate in their relative sensitivity to the seller's information.

In the rest of the paper we assume the assets can be ordered according to their informational sensitivity. In this case, the seller sells her least informationally sensitive assets first. If her information satisfies the Hazard Rate Ordering property and her assets securities can be ranked by seniority (using a novel and weak definition), senior assets are less informationally sensitive. In this case, the seller will retain all securities with seniority below a threshold, where this threshold increases with her optimism.

We also consider a general setting in which, before and/or after seeing her information, an issuer can design new assets that are secured by her initial assets. Under the Hazard Rate Ordering, she has two equivalent, optimal strategies. First, she may pool her initial assets, see her information, and issue a single debt security whose face value is decreasing in her degree of optimism. Second, she may pool her initial assets, design a maximal set of prioritized tranches, see her information, and sell those tranches whose seniority exceeds a threshold that is increasing in her information. The latter behavior, in particular, mirrors the way loan pools are commonly securitized. When cash flows are continuously distributed, the level of debt issued (or, equivalently, the most junior tranche sold) is given by a simple differential equation.

The model has empirical predictions that have been confirmed by Begley and Purnanandam (2017) in the case of residential mortgage-backed securities. First, controlling for observables (including the proportion of no-documentation loans), loans in securitization deals with larger equity (retained) tranches have lower delinquency rates. Second, loan pools with more no-documentation loans have larger equity tranches.

5.1. Avenues of Future Research

DeMarzo (2005) studies a setting in which a seller has private information about the payoffs of each of *n* initial assets. As the information is independent across assets, there is no cross-signaling: the amount the seller sells of one asset does not signal her information about any other asset. In contrast, cross-signaling is a key focus of the present paper. We study it in the simplest possible setting: each asset's payoff depends on the same one-dimensional information. In a more realistic model, the seller might see multiple imperfectly correlated signals of her assets' payoffs. This extension could be an interesting topic for future research.

Such a realistic extension may also help shed light on optimal firm boundaries. That is, it may yield conditions under which it is better to combine assets into one entity and issue claims on their pooled cash flow. In this way, it might generate testable hypotheses about when mergers and spinoffs can enhance firm value.

In our setting, the ability to design securities improves an issuer's payoff. But as the lowest type always realizes all of the gain from trade, it is the higher types who benefit more from this flexibility. Thus, security design should raise the incentive for the production of high quality assets – e.g., by exerting more effort to screen loan applicants.[74] Exploring these and related implications might be a potential area of future empirical research.

Finally, we assume an issuer can signal quality only through quantity choices or security design. Williams (2015) instead adapts the competitive search framework of Guerrieri, Shimer, and Wright (2010) to demonstrate that it is also possible to achieve a similar equilibrium outcome in which issuers signal quality by choosing the liquidity of the market in which they choose to trade, and shows that debt is the optimal monotone security in that context as well. An interesting extension might be to consider equilibrium liquidity choices when the issuer can sell multiple securities as in our paper. Another possibility, in a dynamic context, might be to allow the issuer to signal by delaying her trades as in Varas (2014).

6. References

Biais, B., and T. Mariotti. 2005. Strategic liquidity supply and security design, *Review of Economic Studies* 72:615-649.

Begley, T.A., and A. Purnanandam. 2017. Design of financial securities: Empirical evidence from private-label RMBS deals, *Review of Financial Studies* 30:120–161.

Billingsley, P. 1986. *Probability and measure* (*2nd ed.*), New York: Wiley.

Brown, D.B., B.I. Carlin, and M.S. Lobo. 2010. Optimal portfolio liquidation with distress risk, *Management Science* 56:1997-2014.

Chemla, G., and C.A. Hennessy. 2014. Skin in the game and moral hazard, *Journal of Finance* 69:1597-1641.

[74] Relatedly, Chemla and Hennessy (2014) and Vanasco (2017) show that mandating a minimum level of retention (a.k.a. "skin in the game") can also improve incentives by lowering the payoff of low types, who place the lowest value on the retained cash flow.

Cho, I. K., and D. Kreps. 1987. Signaling games and stable equilibria, *Quarterly Journal of Economics* 102:179-221.

Cho, I. K., and J. Sobel. 1990. Strategic stability and uniqueness in signaling games, *Journal of Economic Theory* 50:381-413.

Dang, T.V., G. Gorton, and B. Holmström. 2015. Ignorance, debt, and financial crises. Mimeo.

DeMarzo, P.M. 2003. Portfolio liquidation and security design with private information. Working paper, Stanford University.

DeMarzo, P.M. 2005. The pooling and tranching of securities: A model of informed intermediation, *Review of Financial Studies* 18:1-35.

DeMarzo, P.M., and D. Duffie. 1999. A liquidity based model of security design, *Econometrica* 67:65-99.

DeMarzo, P.M., and D.M. Frankel. 2020. Mechanism design with an informed principal: extensions and generalizations. Mimeo.

DeMarzo, P.M., D.M. Frankel, and Y. Jin. 2018. Online appendix to "Portfolio liquidity and security design with private information." Mimeo.

Fama, E.F., and K.R. French. 2002. Testing trade-off and pecking order predictions about dividends and debt, *Review of Financial Studies* 15:1-33.

Flannery, M.J., and K.P. Rangan. 2006. Partial adjustment toward target capital structures, *Journal of Financial Economics* 79:469-506.

Franke, G., and J.P. Krahnen. 2007. "Default risk sharing between banks and markets: the contribution of collateralized debt obligations." In Carey, Mark and René M. Stulz, eds., *The Risks of Financial Institutions*, Chicago: University of Chicago Press.

Frankel, D.M., and Y. Jin. 2015. Securitization and lending competition, *Review of Economic Studies* 82: 1383-1408.

Guerrieri, V., R. Shimer, and R. Wright. 2010. Adverse selection in competitive search equilibrium, *Econometrica* 78:1823-1862.

Hart, O., and J. Moore. 1995. Debt and seniority: An analysis of the role of hard claims in constraining management, *American Economic Review* 85:567-585.

He, Z. 2009. The sale of multiple assets with private information, *Review of Financial Studies* 22:4787-4820.

Leland, H., and D. Pyle. 1977. Information asymmetries, financial structure and financial intermediaries, *Journal of Finance* 32:371-387.

Manelli, A.M. 1996. The convergence of equilibrium strategies of approximate signaling games, *Economic Theory* 7:323-335.

Manelli, A.M. 1997. The never-a-weak-best-response test in infinite signaling games, *Journal of Economic Theory* 74:152-173.

Maskin, E., and J. Tirole. 1992. The principal-agent relationship with an informed principal, II: common values, *Econometrica* 60:1-42.

Matthews, S. 2001. Renegotiating moral hazard contracts under limited liability and monotonicity, *Journal of Economic Theory* 97:1-29.

Myers, S. 1984. The capital structure puzzle, *Journal of Finance* 39:575-592.

Myers, S., and N. Majluf. 1984. Corporate financing and investment when firms have information shareholders do not have, *Journal of Financial Economics* 13:197-221.

Nachman, D., and T. Noe. 1994. Optimal design of securities under asymmetric information, *Review of Financial Studies* 7:1-44.

Opler, T., L. Pinkowitz, R. Stulz, and R. Williamson. 1999. The determinants and implications of corporate cash holdings, *Journal of Financial Economics* 52:3-46.

Ramey, G. 1996. D1 signaling equilibria with multiple signals and a continuum of types, *Journal of Economic Theory* 69:508-531.

Shaked, M., and J.G. Shanthikumar. 2007), *Stochastic orders*, New York: Springer.

Shyam-Sunder, L., and S. C. Myers. 1999. Testing static tradeoff against pecking order models of capital structure, *Journal of Financial Economics* 51:219-244.

Vanasco, V. 2017. The downside of asset screening for market liquidity, *Journal of Finance* 72:1937-1981.

Varas, F. 2014. Trading dynamics of financial assets under asymmetric information. Working Paper, Duke University.

Williams, B. 2015. Search, liquidity, and retention: Signaling multidimensional private information. Working Paper, NYU.

7. Appendix

This appendix contains the omitted proofs of the results in section 2. Omitted proofs from section 3 appear in our online appendix: DeMarzo, Frankel, and Jin (2020).

PROOF OF PROPOSITION 1. The uniqueness of ρ^* follows from the uniqueness of $u^*(t)=(1-\delta)\rho^*(t)$. The claims in parts 2-4 regarding the revenue function ρ in any fairly priced equilibrium follow from the corresponding properties of the seller's payoff function u in such an equilibrium, together with the following result.

> **CLAIM 1**. In any fairly priced equilibrium in which a type-t seller receives payoff $u(t)$, her issuance revenue $\rho(t)$ must equal $u(t)/(1-\delta)$.
>
> **PROOF.** By fair pricing, revenue $\rho(t)$ equals the value $q(t)f(t)$ of the assets sold, so the seller's payoff $u(t)=\rho(t)-\delta q(t)f(t)$ must equal $(1-\delta)\rho(t)$.

We now turn to the remaining claims of the proposition.

Part 1. The proof is by induction. Let us say that $u^*(t)$ has property X if $u^*(t)$ exists, is unique, and is positive. First, $u^*(0)$ has property X by ASSUMPTION A. Now suppose $u^*(s)$ has property X for all $s < t$. Then for type t, the set

$$Q_t=\left\{q\in\Re_+^n:\ q\le a \text{ and, for all } s<t,\, u^*(s)\ge q\left[f(t)-\delta f(s)\right]\right\}$$

is nonempty as it includes the zero vector; it is compact as it is defined using weak inequalities. Thus, by the Extreme Value Theorem, a maximizer $q^*(t)$ exists and the maximand $u^*(t)$ exists and is unique. Finally, $u^*(t)$ is positive: for small enough $\varepsilon > 0$, εa

lies in Q_t, whence $u^*(t) \geq (1-\delta)\varepsilon af(t) \geq (1-\delta)\varepsilon af(0) > 0$. Thus, $u^*(t)$ has property X as claimed.

It remains to show that u^* is nonincreasing in *t*. By construction, the payoff $u^*(s)$ that RLP assigns to any type *s* is not less than the payoff $q^*(t)\left[f(t)-\delta f(s)\right]$ that type *s* would get by imitating some type $t > s$. This imitation payoff, in turn, cannot be less than the payoff $u^*(t) = (1-\delta)q^*(t)f(t)$ that RLP assigns to type *t* since the sales revenue $q^*(t)f(t)$ is the same and, by ASSUMPTION A, the opportunity cost of selling the assets is no higher for *s* than for *t*: $\delta q^*(t)f(s) \leq \delta q^*(t)f(t)$.

<u>Part 2.</u> Investors must break even in any fairly priced equilibrium. This implies our base case: the payoff of a seller of type 0 is bounded above by the potential gains from trade $u^*(0)$ for this type. As for types $t > 0$, if type *t*'s equilibrium quantity vector *q* does not lie in Q_t, then there is a type $s < t$ which, by choosing *q*, gets a payoff $q\left[f(t)-\delta f(s)\right]$ (by fair pricing for type *t*), which exceeds the upper bound $u^*(s)$ on her equilibrium payoff: type *s* will deviate. Hence type *t* must choose a quantity vector in Q_t, whence $u^*(t)$ is an upper bound on *t*'s payoff as well by fair pricing.[75] The tightness of the bound will follow from part 3.

<u>Part 3.</u> We will verify that the profile $e^* = (q^*, \overline{p}^*, p^*, \mu^*)$ defined in the text is a fairly priced equilibrium with outcome u^*. This will imply that it is an intuitive equilibrium since, as shown in the text, the beliefs μ^* are intuitive. By construction, the beliefs function μ^* is given by Bayes's rule on the equilibrium path, whence Rational Updating holds in the definition of an Asset Sale Equilibrium. Since the price function p^* is the result of substituting μ^* for μ in (1), Competitive Pricing holds. To verify that the profile is an equilibrium, it thus remains to verify Payoff Maximization, which we will do shortly.

---

[75] The definition of u^* does not rule out pooling. If $q^*(t)f(s)$ equals $q^*(t)f(t)$, then type *s* could pool with type *t* in equilibrium: $q^*(s)$ might equal $q^*(t)$.

We claim that the profile $e^* = \left(q^*, \bar{p}^*, p^*, \mu^*\right)$ satisfies fair pricing and yields the payoff function u^* for the seller. Why? For any possible choice $(q, \bar{p})$ of the seller, let $T(q, \bar{p})$ denote the set $\left\{t \mid (q, \bar{p}) \text{ equals } (q^*(t), \bar{p}^*(t))\right\}$ of types who make this choice in equilibrium. If $T(q, \bar{p})$ is a singleton $\{t\}$, then $p^*(q, \bar{p}) = f(t)$ by (1) as the price caps $\bar{p}^*(t)$ are nonbinding by construction. Else let the types $s < t$ lie in $T(q, \bar{p})$, whence $q^*(s) = q^*(t) = q$ so, by RLP, $u^*(s) = (1-\delta)qf(s) \geq q\left[f(t) - \delta f(s)\right]$ or, equivalently, $q\left[f(t) - f(s)\right] \leq 0$. Since $f(t) \geq f(s)$ by ASSUMPTION A, if $q_i > 0$ then $f_i(t) = f_i(s)$. Since this is true for all types in $T(q, \bar{p})$, $q_i > 0$ implies $p_i^*(q, \bar{p}) = f_i(t)$ for all $t \in T(q, \bar{p})$ by (1): the equilibrium is fairly priced. Thus $q^*(t)\, p^*(q^*(t), \bar{p}^*(t)) = q^*(t)\, f(t)$, whence the payoff of a type-t seller is $u^*(t) = (1-\delta) q^*(t)\, f(t)$ as claimed.

We now show Payoff Maximization. We first show that no type gains by imitating another type. Since e^* is fairly priced and the payoff of a type t seller is $u^*(t)$, RLP ensures that no type can gain from imitating a higher type in e^*. Suppose some type gains from imitating a lower type; let t be the lowest such type, and suppose she gains from imitating type $s < t$:

$$q^*(s)\left[f(s) - \delta f(t)\right] > u^*(t). \tag{14}$$

By construction, each type $s' < t$ is willing not to imitate type s: $u^*(s') \geq q^*(s)\left[f(s) - \delta f(s')\right]$. By ASSUMPTION A,

$$\begin{aligned} q^*(s)\left[f(s) - \delta f(t)\right] &= q^*(s)\left[f(s) - f(t)\right] + (1-\delta) q^*(s)\, f(t) \\ &\leq (1-\delta) q^*(s)\, f(t). \end{aligned}$$

That is, type t's payoff from imitating type s is bounded above by the joint surplus yielded by this imitation. (The investors may also get some of the surplus as they pay only $q^*(s)\, f(s)$ for a portfolio that is worth $q^*(s)\, f(t)$.) Hence, $q^*(s)$ cannot lie in Q_t: if it did, type t's joint surplus $(1-\delta) q^*(s)\, f(t)$ from imitating type s could not exceed type t's payoff $u^*(t)$ and thus type t could not gain from imitating type s (contrary to hypothesis).

Let $\alpha \in [0,1)$ be the largest scalar such that $\alpha q^*(s)$ lies in Q_t. (We know that $\alpha \geq 0$ since Q_t contains the zero vector.) Then for some $s' < t$,

$$\alpha q^*(s)\left[f(t) - \delta f(s')\right] = u^*(s') \geq q^*(s)\left[f(s) - \delta f(s')\right]$$

where the equality holds by choice of α and the inequality by choice of *t*. That is, some type $s' < t$ (who, as stated above, is willing *not* to imitate *s*) must be willing to sell the portfolio $\alpha q^*(s)$ if this leads investors to take her for type *t*. Rearranging, and using the fact that $\alpha < 1$ and the definition of u^*, yields an upper bound on type *t*'s payoff from imitating type *s*:

$$\begin{aligned}
q^*(s)\left[f(s) - \delta f(t)\right] &= q^*(s)\left[f(s) - \delta f(s')\right] + \delta q^*(s)\left[f(s') - f(t)\right] \\
&\leq \alpha q^*(s)\left[f(t) - \delta f(s')\right] + \delta q^*(s)\left[f(s') - f(t)\right] \\
&= (1-\delta)\alpha q^*(s) f(t) + \delta \alpha q^*(s)\left[f(t) - f(s')\right] + \delta q^*(s)\left[f(s') - f(t)\right] \\
&\leq (1-\delta)\alpha q^*(s) f(t) \text{ since } \alpha < 1 \text{ and by ASSUMPTION A} \\
&\leq u^*(t)
\end{aligned}$$

which contradicts (14): no type *t* gains from imitating any lower type.

We next show that no type will deviate to any choice $(q, \bar{p})$ that lies off the equilibrium path: for which $T(q, \bar{p}) = \varnothing$. As shown in the text, if type $\tau^*(q)$ does not gain from deviating to $(q, \bar{p})$, then no other type gains from deviating to $(q, \bar{p})$. It thus suffices to show that type τ^* = $\tau^*(q)$ will not deviate to $(q, \bar{p})$. Given beliefs μ^*, the price vector following the deviation is $p^*(q, \bar{p}) = \bar{p} \wedge f(\tau^*) \leq f(\tau^*)$ by (1). Thus, for type τ^* not to gain by deviating to $(q, \bar{p})$, it suffices that $(1-\delta) q f(\tau^*) \leq u^*(\tau^*)$. If *q* is in Q_{τ^*}, we are done. If not, let α be the largest scalar such that αq is in Q_{τ^*}. Then $\alpha < 1$ and by continuity we must have $\alpha q\,[f(\tau^*) - \delta f(s)] = u^*(s)$ for some $s < \tau^*$. Therefore, by (4),

$$\alpha q[f(\tau^*) - \delta f(s)] = u^*(s) = u^*(s) + \delta q f(s) - \delta q f(s) > u^*(\tau^*) + \delta q f(\tau^*) - \delta q f(s).$$

(The inequality is strict by (4) and the fact that $s < \tau^*$.) This can be rearranged to yield

$$u^*(\tau^*) < (1-\delta)\alpha q f(\tau^*) + \delta q(1-\alpha)(f(s) - f(\tau^*)) \leq (1-\delta)\alpha q f(\tau^*),$$

where the last inequality follows since $\alpha < 1$ and $f(s) \leq f(\tau^*)$. But this implies that αq, which lies in Q_{τ^*}, is strictly better than $q^*(\tau^*)$ is for type $\tau^*(q)$ – a contradiction. Having now verified Payoff Maximization, we conclude that the profile e^* is an Asset Sale Equilibrium.

Part 4. Let $e = (q, \bar{p}, p, \mu)$ be an intuitive equilibrium with outcome u and revenue function ρ. We first show that e is fairly priced. Suppose that with positive probability in e, type t makes asset sale decision $(q, \bar{p})$. Assume investors respond with price vector p. First say $qp < qf(t)$: the issue is underpriced. Let s be the largest type such that $qf(s) < qf(t)$.[76] Then define $\lambda \in [0,1)$ such that

$$\frac{q(p-\delta f(t))}{q(f(t)-\delta f(t))} < \lambda < \frac{q(p-\delta f(s))}{q(f(t)-\delta f(s))}. \tag{15}$$

(Such a λ must exist as the second ratio is decreasing in $qf(s)$.) Consider the feasible deviation $(\lambda q, f(t))$ for type t. First, by ASSUMPTION A, for all $s' \leq s$, $qf(s') \leq qf(s) < qf(t)$ and so from (15), $\lambda q[f(t) - \delta f(s')] < q[p - \delta f(s')] \leq u(s')$, where the last inequality follows from the incentive constraint for type s'. Thus, no type $s' \leq s$ has an incentive to deviate to $(\lambda q, f(t))$.

On the other hand, from (15), $\lambda q[f(t) - \delta f(t)] > q[p - \delta f(t)] = u(t)$, so type t could gain from the deviation if the realized price p equals $f(t)$. But then the Intuitive Criterion implies that $\mu(s' \mid \lambda q, f(t)) = 0$ for all $s' \leq s$. That is, investor beliefs must put weight only on types t' such that $qf(t') \geq qf(t)$. By ASSUMPTION A, for these types $f_i(t') \geq f_i(t)$ for all securities i such that $q_i > 0$. Hence, for each such security i,

$$p_i(\lambda q, f(t)) = f_i(t) \wedge \textstyle\sum_{t'} f_i(t')\, \mu(t' \mid \lambda q, f(t)) = f_i(t).$$

Accordingly, type t gains from the deviation to $(\lambda q, f(t))$ - a contradiction. Thus, it must be the case that $qp \geq qf(t)$ for all types t that make sale decision $(q, \bar{p})$ in equilibrium; that is, there is no equilibrium underpricing. But from (1),

[76] If there is no such type s then, for all types s, $f_i(t) \leq f_i(s)$ whenever q_i is positive. Thus, if type t deviates to $(q, f(t))$, investors will pay $qf(t)$ which, by assumption, exceeds $pf(t)$: type t will deviate.

$$qp = q\left[\bar{p} \wedge \sum_{t'} f(t)\mu(t \mid q, \bar{p})\right] \leq \sum_{t'} qf(t)\,\mu(t \mid q, \bar{p})\,.$$

That is, qp is bounded above by a convex combination of $qf(t)$ for all types t that make sale decision $(q, \bar{p})$ in equilibrium. But as just shown, qp is also bounded below by $qf(t)$ for any such t. Thus, qp must equal $qf(t)$ for all types t that ever choose $(q, \bar{p})$: without underpricing there can be no overpricing.

Further, if t, t' make sale decision $(q, \bar{p})$ in equilibrium and $t > t'$, then $f(t) \geq f(t')$ by ASSUMPTION A. Since $qf(t) = qf(t')$, it must be that $f_i(t) = f_i(t')$ if $q_i > 0$. Hence, $p_i(q, \bar{p}) = f_i(t)$ if $q_i > 0$, whence the equilibrium e is fairly priced.

We next show that the outcome u of e equals u^* and the equilibrium asset sale function $q(\cdot)$ of e solves RLP. Let q^* solve RLP. Since e is fairly priced, $u \leq u^*$ by part 2. Let t be the smallest type such that either (a) $u(t) < u^*(t)$ or (b) $q(t)$ does not solve RLP. As $u(s) = u^*(s)$ for all $s < t$, the payoff $u(t)$ equals its maximum value of $u^*(t)$ if and only if $q(t)$ solves RLP. Hence, conditions (a) and (b) are equivalent: they *both* must hold for type t. Since $u(t) < u^*(t)$, it follows that $u(t) + \delta q^*(t)f(t)$ is less than $u^*(t) + \delta q^*(t)f(t)$, which equals $q^*(t)f(t)$ by RLP. Also by RLP, for all $s < t$, $q^*(t)f(t)$ does not exceed $u^*(s) + \delta q^*(t)f(s)$, which equals $u(s) + \delta q^*(t)f(s)$ by hypothesis. Hence, there exists a price cap vector $\bar{p}$ close to but less than $f(t)$ such that for any type $s < t$,

$$u(t) + \delta q^*(t)\,f(t) < q^*(t)\,\bar{p} < q^*(t)\,f(t) \leq u(s) + \delta q^*(t)\,f(s)\,.$$

But then intuitive beliefs put no weight on types $s < t$ if $(q^*(t), \bar{p})$ is observed. Hence, $p(q^*(t), \bar{p}) = \bar{p}$. But then because $u(t) < q^*(t)(\bar{p} - \delta f(t))$, type t could gain by deviating to $(q^*(t), \bar{p})$ – a contradiction. It follows that there is no such minimum type t: for all types t, $u(t) = u^*(t)$ as claimed and q solves RLP.

<u>Part 5.</u> We require the following lemma.

LEMMA. For any issuance choice $(q, \bar{p})$, whether or not it occurs in equilibrium, beliefs are concentrated on the set $\arg\min_{t'}\left[u^*(t') + \delta qf(t')\right]$.

PROOF OF LEMMA: If no type chooses $(q, \bar{p})$ in equilibrium, the claim holds by equations (3) and (4) which define μ^* in this case. If instead some type t chooses $(q, \bar{p})$ in equilibrium then, by fair pricing, the issuance revenue $p(q, \bar{p})q$ must equal $qf(t)$. Hence, the IC constraint for each type $s \neq t$ states that $u^*(s)+\delta qf(s)$ is no less than the issuance revenue $qf(t)$ from $(q, \bar{p})$ which, in turn, equals $u^*(t)+\delta qf(t)$ as $u^*(t)$ is type t's payoff. Accordingly, $t \in \arg\min_{t'}\left[u^*(t') + \delta qf(t')\right]$ which implies the claim by Rational Updating. ◆

Now consider a fixed price cap vector $\bar{p}$ and quantity vectors $q' \geq q$. Let τ be in the set of beliefs following $(q, \bar{p})$ and τ' be in the set of beliefs following $(q', \bar{p})$. Then by the preceding Lemma, $\tau \in \arg\min_t\left[u^*(t) + \delta qf(t)\right]$ and $\tau' \in \arg\min_t\left[u^*(t) + \delta q'f(t)\right]$. It follows that $u^*(\tau) + \delta qf(\tau) \leq u^*(\tau') + \delta qf(\tau')$ and $u^*(\tau) + \delta q'f(\tau) \geq u^*(\tau') + \delta q'f(\tau')$. Subtracting the first inequality from the second and simplifying we obtain $(q'-q)\left[f(\tau) - f(\tau')\right] \geq 0$ which implies $\tau \geq \tau'$ by ASSUMPTION A. This, in turn, yields $p^*(q', \bar{p}) \leq p^*(q, \bar{p})$ by Rational Updating and ASSUMPTION A. ◆

PROOF OF PROPOSITION 2: Since $q_j^*(0) = a_j$ for all j, we may assume $t > 0$. Omitting the constraint $0 \leq q \leq a$ in RLP and terms that are independent of q, the Lagrangian is $qf(t) - \sum_{s<t} \lambda(s) q\left[f(t) - \delta f(s)\right]$. The derivative with respect to q_i can then be written as $\delta \sum_{s<t} \lambda(s) f_i(s) - f_i(t)k$ where $k = \sum_{s<t} \lambda(s) - 1$. Thus, $q_j^*(t) < a_j$ implies $\sum_{s<t} \lambda(s)\left[f_j(s)/f_j(t)\right] \leq k/\delta$. But the Lagrange multipliers $\lambda(s)$ are all nonnegative

and, as $t>0$, positive for at least one $s<t$.[77] Since $f_i(s)/f_i(t)<f_j(s)/f_j(t)$, it follows that $\sum_{s<t}\lambda(s)\left[f_i(s)/f_i(t)\right]<k/\delta$, and thus $q_i^*(t)=0$ as claimed. ◆

PROOF OF PROPOSITION 3: The fact that $q^*(t)$ is of the given form follows from IIS and PROPOSITION 2. The existence of a hurdle class implies that q^* is ordered: either $q^*(t)\le q^*(s)$ or $q^*(t)\ge q^*(s)$. Furthermore, PROPOSITION **1** implies $u^*(t)\le u^*(s)$ and thus $q^*(t)\,f(t)\le q^*(s)\,f(s)$. Since $f(t)\ge f(s)$ by ASSUMPTION A, it follows that $q^*(t)\ge q^*(s)$ as claimed. ◆

PROOF OF PROPOSITION 4. First, $q^*(0) = a$ by RLP and ASSUMPTION B. For $t > 0$, since $f(t) - \delta f(t-1) > 0$ by ASSUMPTION A and ASSUMPTION B, a unique hurdle class vector satisfies equation (5), which is the incentive constraint in RLP for $s = t-1$. It remains to show that this constraint binds when $s = t-1$. Suppose instead that

$$q^*(t)[f(t) - \delta f(t-1)] < u^*(t-1) = (1-\delta)q^*(t-1)f(t-1).$$

From the definition of $q^*(t-1)$ in RLP, for any $s < t-1$, $q^*(t-1)\,(\,f(t-1) - \delta\, f(s)\,) \le u^*(s)$. Combining these two yields:

$$q^*(t)[f(t) - \delta f(t-1)] - (1-\delta)q^*(t-1)f(t-1) + q^*(t-1)\,(\,f(t-1) - \delta\, f(s)\,) < u^*(s)$$

or equivalently, $q^*(t)\,(\,f(t) - \delta\, f(s)\,) + \delta\,(\,q^*(t-1) - q^*(t)\,)\,(\,f(t-1) - f(s)\,) < u^*(s)$. From PROPOSITION **1**, $q^*(t-1) \ge q^*(t)$, and from ASSUMPTION A, $f(t-1) \ge f(s)$. Thus,

$$q^*(t)\,(\,f(t) - \delta\, f(s)\,) < u^*(s),$$

which implies that none of the incentive constraints in RLP bind for $q^*(t)$. This contradicts the optimality of $q^*(t)$ unless $q^*(t)\,f(t) = a\,f(t)$. But by the initial supposition,

$$q^*(t)\,f(t) < u^*(t-1) + \delta\, q^*(t)\,f(t-1) \le a\,f(t-1) \le a\,f(t).$$

[77] If instead none of the incentive compatibility constraints bind, then the equilibrium payoff $u^*(s)=(1-\delta)q^*(s)\,f(s)$ of each type $s<t$ must exceed $q^*(t)\left[f(t)-\delta f(s)\right]$ which in turn is not less than $(1-\delta)q^*(t)\,f(s)$ by ASSUMPTION A. But this can hold only if $q_\ell^*(t)<q_\ell^*(s)$ for at least one asset ℓ. Hence it is possible to raise $q_\ell^*(t)$ and thus type t's payoff $u^*(t)$ without violating any constraints – which contradicts the optimality of $q^*(t)$ in RLP.

Hence, the incentive constraint for t–1 must bind. ◆

PROOF OF PROPOSITION 5. The proposition follows from PROPOSITION **1** as long as we can show that if *j* is junior to *i* then it is more informationally sensitive; that for any $t > s$,

$$f_j(t)/f_j(s) > f_i(t)/f_i(s).$$

Fix types $t > s$. As both informational sensitivity and priority are invariant to an arbitrary rescaling of each security's payout, w.l.o.g. we can let $f_j(t) = f_i(t) = 1$. Hence, we need to show that for $s < t$, $f_j(s) < f_i(s)$.

For convenience, let Y_t denote a random variable whose unconditional distribution equals the posterior distribution of the cash flow *Y* given the type *t*: $\Pr(Y_t \le z) = \Pr(Y \le z \mid t)$. Because asset *j* is junior to *i*, $F_j(Y) = r(Y)F_i(Y)$ where, by the above normalization,

$$\begin{aligned} 0 &= f_j(t) - f_i(t) = E[F_j(Y_t) - F_i(Y_t)] = E[(r(Y_t)-1)F_i(Y_t)] \\ &= E[(r(Y_t)-1)F_i(Y_t) \mid F_i(Y_t) > 0]. \end{aligned}$$

Thus, because *r* is nondegenerate on the set $F_i(Y_t) > 0$, it must lie strictly below 1 and strictly above 1 with positive probability on this set.

Next we make use of the following lemma.

> **LEMMA:** Suppose *Y* satisfies HRO. Then there exists a random variable *Z* with the same support as, and independent of, Y_t, such that Y_s and $Y_t \wedge Z$ have the same distributions.
>
> **PROOF:** Let $\bar{y}$ denote the essential supremum of *Y*: $\bar{y} = \inf\{y : \Pr(Y \ge y) = 0\}$. Fix $s < t$, and let $R(y) = \dfrac{\Pr(Y_s \ge y)}{\Pr(Y_t \ge y)}$ for $y < \bar{y}$ and zero for $y > \bar{y}$, with $R(\bar{y})$ defined so that R is left-continuous. Moreover, $R(0) = 1$ and R is strictly decreasing on the support of *Y* and constant elsewhere. Thus we can define a new,

independent random variable Z with distribution $Pr(Z \geq y) = R(y)$ that has the same support as Y.[78] Finally,

$$\Pr(Y_t \wedge Z \geq y) = \Pr(Y_t \geq y, Z \geq y) = \Pr(Y_t \geq y)\Pr(Z \geq y) = \Pr(Y_t \geq y)\frac{\Pr(Y_s \geq y)}{\Pr(Y_t \geq y)}$$

which equals $\Pr(Y_s \geq y)$ as claimed. ◆

Using the lemma, together with the monotonicity of the securities, we have

$$f_i(s) = E\left[F_i(Y) \mid s\right] = E\left[F_i(Y_s)\right] = E\left[F_i(Y_t \wedge Z)\right],$$

and similarly for f_j. If $r(z) < 1$, then

$$E\left[F_j(Y_t \wedge z)\right] = E\left[r(Y_t \wedge z)F_i(Y_t \wedge z)\right] \leq E\left[F_i(Y_t \wedge z)\right],$$

where the inequality is strict unless $F_i(z) = 0$. Conversely, if $r(z) \geq 1$ then

$$\begin{aligned} E\left[F_j(Y_t \wedge z)\right] &= E\left[F_j(Y_t) - \left(F_j(Y_t) - F_j(z)\right)^+\right] \\ &= 1 - E\left[\left(r(Y_t)F_i(Y_t) - r(z)F_i(z)\right)^+\right] \\ &\leq 1 - E\left[\left(r(z)F_i(Y_t) - r(z)F_i(z)\right)^+\right] \\ &\leq 1 - E\left[\left(F_i(Y_t) - F_i(z)\right)^+\right] = E\left[F_i(Y_t \wedge z)\right]. \end{aligned}$$

Thus, $f_j(s) = E\left[F_j(Y_t \wedge Z)\right] < E\left[F_i(Y_t \wedge Z)\right] = f_i(s)$ where the inequality is strict because there is a positive probability that $F_i(Z) > 0$ and $r(Z) < 1$. ◆

[78] See e.g. Theorem 12.4 of Billingsley (1986).

Online Appendix to "Portfolio Liquidity and Security Design with Private Information"†

Peter M. DeMarzo (*Stanford University*)

David M. Frankel (*Melbourne Business School and Iowa State University*)

Yu Jin (*Shanghai University of Finance and Economics*)

This Revision: November 5, 2020

1. Introduction

This document is the online appendix to "Portfolio Liquidity and Security Design with Private Information" by DeMarzo, Frankel, and Jin (2020), henceforth "DFJ". It is organized as follows. Section 2 formally states the technical results that are previewed in section 3.5.2 of DFJ. Section 3 proves these results as well as the results that are stated without proof in section 3 of DFJ. Finally, section 4 relaxes ASSUMPTION A (monotonicity) in the case of 2 assets and 2 seller types.

2. Technical Results

We first formally state the technical results that we discuss in section 3.5.2 of DFJ. We begin by specifying the continuous and discrete models. The continuous model, denoted "model ∞", is as follows. The issuer's type $t \sim G$ has full support [0,1] and the final asset value Y has conditional and unconditional support $[0, \bar{y}]$ and conditional distribution function H that satisfies the Hazard Rate Ordering property and Lipschitz-H (defined in DFJ, section 3.5.2).

We now define a sequence of discrete models i = 1,2,... that converge to model ∞. Let $(N_i)_{i=1}^{\infty}$ and $(N_i')_{i=1}^{\infty}$ be any two increasing sequences of positive integers. In model i, let

† DeMarzo: Stanford, CA 94305-5015; pdemarzo@stanford.edu. Frankel: 200 Leicester St., Carlton, VIC 3053, Australia; d.frankel@mbs.edu. Sadly, our coauthor Yu Jin died in 2018.

the gaps between adjacent types t and shocks Z be $\Delta_i = 1/N_i$ and $\Delta_i' = \bar{y}/N_i'$, respectively. That is, t lies in $S_i = \{0, \Delta_i, \ldots, 1-\Delta_i, 1\}$ and Z lies in $S_i' = \{0, \Delta_i', \ldots, \bar{y}-\Delta_i', \bar{y}\}$. By construction, both gaps Δ_i and Δ_i' converge to zero as i goes to infinity. Let the conditional distribution of Y in model i be the restriction of the continuous distribution function H to types in S_i and shocks in S_i'. Similarly, the distribution of t in model i is the restriction of G to types t in S_i.[79]

Let E^i and E^∞ denote the expectations operators in models i and ∞, respectively. Let $v^i(D,t) = E^i[\min(D,Y) \mid t]$ denote the expected payout of simple debt with face value D in model i given a type $t \in S_i$. Let $v^\infty(D,t) = E^\infty[\min(D,Y) \mid t]$ denote the expected payout of the same security in model ∞ given a type $t \in [0,1]$.

Fix equilibria in models i and ∞ in which the issuer's security is simple debt. Let D_t^i and D_t^∞ be the equilibrium face values of these securities for a given type t. The equilibrium price of this security in model i, denoted $p^i(t)$, is simply the security's expected payout $v^i(D_t^i, t)$. And the issuer's expected issuance profit, denoted $u^i(t)$, is simply the gains from trade $(1-\delta)v^i(D_t^i, t)$ as competition drives investors' payoffs to zero. Similarly, in the continuous model the price $p^\infty(t)$ of the security equals the expected payout $v^\infty(D_t^\infty, t)$ and the issuer's profit $u^\infty(t)$ equals the expected gains from trade $(1-\delta)v^\infty(D_t^\infty, t)$.

In the present notation, equation (13) is written as:[80]

[79] That is, in model i, the probability that the type does not exceed some t in S_i is $G(t)$, while the probability, conditional on a type t, that the cash flow Y does not exceed some y in S_i' is $H(y \mid t)$.

[80] Integrating by parts, $v(D,t) = \int_{y=0}^{\bar{y}} \min\{y, D\}\, dH(y \mid t)$ equals $D - \int_{y=0}^{D} H(y \mid t)\, dy$ whence $v_2(D,t) = -\int_{y=0}^{D} H_2(y \mid t)\, dy$. As $H(D \mid t) = \pi(D,t)$ by Lipschitz-H, CIVP is equivalent to (13).

The Continuous Initial Value Problem (CIVP).

$$D_0 = \bar{y} \text{ and, for each } t, \quad \frac{dD_t}{dt} = \frac{1}{1-\delta} \frac{\int_{y=0}^{D_t} H_2(y \mid t)\, dy}{1 - H(D_t \mid t)} < 0.$$

Our first result shows that CIVP and its discrete analog have unique solutions:

Proposition 11. Assume Hazard Rate Ordering and Lipschitz-*H*.

1. There exists a unique function D^∞ that satisfies CIVP for $v = v^\infty$. This function is decreasing and differentiable in the type *t*, and takes values in $(0, \bar{y}]$. The associated price and profit functions, p^∞ and u^∞, are decreasing and continuous in the type *t*.
2. For each discrete model *i*=1,2,..., there exists a unique, decreasing function D^i with $D_0^i = \bar{y}$ and satisfying (12) with $v = v^i$ for all $t \in S_i$ and $\Delta = \Delta_i$.

The proof for CIVP runs roughly as follows. The Picard-Lindelöf theorem is the usual tool for proving the existence and uniqueness of the solution to a differential equation. Unfortunately, we cannot apply this theorem directly because the differential equation in CIVP is not Lipschitz continuous in D^∞: it approaches negative infinity as D^∞ approaches $\bar{y}$.

We sidestep this difficulty in the following way. We define upper and lower bounds on D^∞ using a modification of CIVP that is Lipschitz continuous with constant *k*. We then show that as *k* grows, these upper and lower bounds approach the same limit, which satisfies CIVP and thus must be its unique solution D^∞.

Having shown the existence of unique face value functions in both models, we now show that the face value function in the discrete model converges to that of the continuous model uniformly in the issuer's type *t*. As the equilibrium of the discrete model uniquely satisfies the intuitive criterion, this supports the use of the continuous equilibrium when convenient.

> **PROPOSITION 12.** Assume HRO and Lipschitz-H. For any $\varepsilon > 0$ there is an i^* such that for all models $i > i^*$ and all types t in [0,1], $\left|D_t^i - D_t^\infty\right| < \varepsilon$.[81]

The idea of the proof is as follows. For any model i = 1,2,... and constant $k > 0$, we first show that any solution D^i must lie between fixed upper and lower bounds $\bar{D}_k^i$ and $\underline{D}_k^i$, where these bounds are Lipschitz continuous with constant k. Moreover, these bounds converge to the aforementioned upper and lower bounds on D^∞ as i grows. By the prior intuition, these bounds on D^∞ converge in turn to D^∞ as k grows. Thus, by taking i and k to infinity simultaneously, $\bar{D}_k^i$ and $\underline{D}_k^i$ - and thus D^i which lies between them – must converge to the unique solution D^∞ of model ∞.

Our next result gives conditions under which the convergence of the discrete model to the continuous model is uniform in various parameters. This property can be useful in applications in which the security design game is preceded by some interaction in which the issuer also chooses an optimal action, as in Frankel and Jin (2015). In such settings, the result can help establish that the issuer's optimal choices in the discrete model are well-approximated by her optimal choice in the continuous model.

Henceforth, we assume the distribution of types G is continuous with a strictly positive density g that satisfies the following technical condition:

> **LIPSCHITZ-*G* (L-*G*).** There are constants $k_3, k_4 \in (0,\infty)$ such that for all types t and t' in [0,1], $g(t) \le k_3$ and $\left|g(t) - g(t')\right| \le k_4 \left|t - t'\right|$.

We now show that the key functions of the model converge uniformly in the distributions G and H, the type $t \in [0,1]$, and the cash flow parameter $\bar{y}$.[82] These key functions are the face value function D^i, price function p^i, and the conditional profit function u^i. We also show uniform convergence of the issuer's unconditional expected issuance profits $Eu^i = E^i\left[u^i(t)\right]$ to its continuous analogue, $Eu^\infty = E^\infty[u^\infty(t)]$. Finally, let $\Pi^i(t)$ denote

[81] Technically, the function D^i is defined only for types t in the discrete set S_i. We extend it to all types t in [0,1] by evaluating it at the highest type in S_i that does not exceed t.

[82] Uniformity in t does not apply to the expected profit functions Eu and EΠ, as they do not depend on t.

the issuer's conditional *total* profits in model i: the sum of issuance profits $u^i(t)$ and the conditional expected portfolio return $E^i[Y \mid t]$. Let $E\Pi^i = E^i\left[\Pi^i(t)\right]$ denote unconditional expected total profits.[83] We show that these converge uniformly to their continuous counterparts, $\Pi^\infty(t) = u^\infty(t) + E^\infty[Y \mid t]$ and $E\Pi^\infty = E^\infty\left[\Pi^\infty(t)\right]$.

> **PROPOSITION 13.** Fix constants k_0, k_1, k_2, k_3, k_4, and $\mathbf{y}$, all in $(0,\infty)$. Let $\mathbf{H}$ be the set of conditional distribution functions $H(z \mid t)$ that satisfy Hazard Rate Ordering and Lipschitz-H with constants k_0, k_1, and k_2. Let $\mathbf{G}$ be the set of distribution functions G that satisfy Lipschitz-G with constants k_3 and k_4. For all $\varepsilon > 0$ there is an i^* such that for all models $i > i^*$, G in $\mathbf{G}$, H in $\mathbf{H}$, $\bar{y}$ in $[0,\mathbf{y}]$, and t in $[0,1]$, $|\omega^i(t)-\omega^\infty(t)|$ is less than ε for ω equal to D, p, u, and Π; and $|E\omega^i-E\omega^\infty|$ is less than ε for ω equal to u and Π.[84]

We next show a useful comparative statics property: the issuer chooses a higher face value when the rate of change function in (13) (which is negative) is smaller in absolute value. PROPOSITION 10 is essentially a special case of this result (and its proof relies on this result).

> **PROPOSITION 14**. Let the discount factors $\hat{\delta}$ and $\tilde{\delta}$ lie in $(0,1)$. Assume the conditional distribution functions $\hat{H}$ and $\tilde{H}$ satisfy HRO and Lipschitz-H. suppose that for any given D and t, the rate of change function $\frac{1}{1-\delta}\frac{\int_{y=0}^{D} H_2(y \mid t)\,dy}{1-H(D|t)}$ in (13) is no larger in absolute value when (δ, H) equals $(\hat{\delta},\hat{H})$ than when it equals $(\tilde{\delta},\tilde{H})$. Let the face value functions $\hat{D}_t$ and $\tilde{D}_t$ solve equation (13) for (δ, H) equal to $(\hat{\delta},\hat{H})$ and $(\tilde{\delta},\tilde{H})$, respectively. Then for all t,

[83] This last quantity is especially important in applications: if there is a pregame period, the issuer will act so as to maximize the sum of $E\Pi^i$ (perhaps multiplied by a discount factor) and any pregame payoff.

[84] As in Proposition 14, we extend these functions to all types t in [0,1] by evaluating them at the highest type in S_i that does not exceed t.

$\hat{D}_t \geq \tilde{D}_t$: the issuer does not choose a lower face value under $(\hat{\delta}, \hat{H})$ than under $(\tilde{\delta}, \tilde{H})$.

Finally, we show a homogeneity property that can simplify the analysis of models in which security design is embedded (e.g., Frankel and Jin 2015).

COROLLARY 15. The face value functions D^i and D^∞, the price functions p^i and p^∞, the profit functions u^i, Π^i, u^∞, and Π^∞, and the issuer's expected profits Eu^i, $E\Pi^i$, Eu^∞, and $E\Pi^\infty$, defined above, are all homogeneous of degree one in the cash flow parameter $\bar{y}$.

3. Proofs

We now give the omitted proofs from section 3 of DFJ, as well as the proofs of the results of section 2 of this online appendix.

PROOF OF PROPOSITION 6. We first define the Intuitive Criterion in the GSD game. Consider an equilibrium and any interim asset vector *I*. If type *t* sticks to her continuation strategy, she will get $u(t \mid I) \overset{d}{=} U(I, P_t(I), \bar{\rho}_t(I) \mid t, p)$. If instead she chooses some other *ex-post* action $(P, \bar{\rho})$, for her to lose from this deviation it suffices that her equilibrium payoff $u(t \mid I)$ exceeds her maximum payoff $\bar{\rho} - \delta E\left[W_P^I(Y) \mid t\right]$ from the deviation – or, equivalently, that

$$\bar{\rho} < u(t \mid I) + \delta E\left[W_P^I(Y) \mid t\right]. \tag{16}$$

The right hand side of (16) is thus the minimum revenue that type *t* requires to be willing to deviate to $(P, \bar{\rho})$.

The Intuitive Criterion for the GSD game states that on seeing the deviation $(P, \bar{\rho})$ following *I*, investors must put zero probability on type *t* if she is never willing to choose $(P, \bar{\rho})$ following *I* but some other type might be: if condition (16) holds for *t* but fails for some other type *s*. That is:

THE INTUITIVE CRITERION (GSD GAME). An equilibrium of the GSD game with posterior belief function μ and outcome u is *intuitive* if, on seeing any action $(I, P, \bar{\rho})$, investors' posterior probability $\mu(t \mid I, P, \bar{\rho})$ is zero for any type t that satisfies (16) as long as there is some type s for which the inequality is reversed: for which $\bar{\rho} \geq u(s \mid I) + \delta E\left[W_P^I(Y) \mid s\right]$.

Let $E = \left(\hat{I}, P_{\cdot}(\bullet), \bar{\rho}_{\cdot}(\bullet), p, \mu\right)$ be any equilibrium of the GSD game. We will show that the following equilibrium $\tilde{E} = \left(\tilde{I}, \tilde{P}_{\cdot}(\bullet), \tilde{\rho}_{\cdot}(\bullet), \tilde{p}, \tilde{\mu}\right)$ is also an equilibrium, and is intuitive if E is. Let $\#(V)$ denote the length of a vector V.

1. Securitization. In $\tilde{E}$, the issuer's interim asset vector $\tilde{I} = \left(\tilde{I}^1\right)$ consists of a single security whose payout equals her cash flow: $\tilde{I}^1(Y) = Y$. She then chooses an ex-post action as follows for any given t and interim asset vector I. First, if she deviated in the prior stage (so that $I \neq \tilde{I}$), she chooses the ex-post action $\left(P_t(I), \bar{\rho}_t(I)\right)$ that she would choose in E after choosing I. Else she issues an ex-post action that consists of her equilibrium revenue cap $\bar{\rho}_t\left(\hat{I}\right)$ in E together with an ex-post asset vector consisting of a single security whose payout equals the aggregate payout $\sum_{j=1}^{\#\left(P_t(\hat{I})\right)} P_t^j\left(\hat{I}\right)\left(\hat{I}(Y)\right)$ of her equilibrium ex-post security vector $P_t\left(\hat{I}\right)$ in E.

2. Beliefs. Let $\tilde{T}\left(P, \bar{\rho} \mid I\right)$ denote the set of types whose strategies in $\tilde{E}$ instruct them to choose the ex-post action $(P, \bar{\rho})$ conditional on having chosen some interim asset vector I. Upon seeing an action $\left(\tilde{I}, P, \bar{\rho}\right)$ that is expected in $\tilde{E}$, the investors' posterior probability $\mu(t \mid \tilde{I}, P, \bar{\rho})$ equals the probability that the issuer's type is t conditional on her type being in $\tilde{T}\left(P, \bar{\rho} \mid \tilde{I}\right)$.[85] That is, it equals

[85] An action $\left(\tilde{I}, P, \bar{\rho}\right)$ is *expected* in an equilibrium if it is selected with positive probability in that equilibrium.

$g(t)\left[\sum_{s\in\tilde{T}(P,\overline{\rho}|\tilde{I})} g(s)\right]^{-1}$ if $t\in\tilde{T}(P,\overline{\rho}\,|\,\tilde{I})$ and zero otherwise. Upon seeing an action $(I,P,\overline{\rho})$ that is unexpected in $\tilde{E}$, $\tilde{\mu}(t\,|\,I,P,\overline{\rho})$ equals $\mu(t\,|\,I,P,\overline{\rho})$: its value in *E*.

3. Pricing. Prices are given by equation (8) with $\tilde{\mu}$ substituted for μ: for any action $(I,P,\overline{\rho})$, the resulting price vector is $\tilde{p}(I,P,\overline{\rho})=\sum_t E[P(I(Y))\,|\,t]\;\tilde{\mu}(t\,|\,I,P,\overline{\rho})$.

Pricing and Beliefs ensure that $\tilde{E}$ satisfies Competitive Pricing and Rational Updating. As for Payoff Maximization, the payoff of an issuer of type *t* from taking an action $(I,P,\overline{\rho})$ that is unexpected in $\tilde{E}$ equals her payoff from taking this action in *E*, since, by Pricing and Beliefs, investors respond with the same price vector. As for expected actions in $\tilde{E}$, we rely on the following claim, whose proof appears below.

> **Claim.** Fix an action $(\tilde{I},P,\overline{\rho})$ that is expected in $\tilde{E}$. Let Σ denote the set of equilibrium actions $(\hat{I},P',\overline{\rho})$ that are taken in *E* by any type whose prescribed action in $\tilde{E}$ is $(\tilde{I},P,\overline{\rho})$.[86] Then an issuer of any given type gets the same payoff in $\tilde{E}$ from taking the action $(\tilde{I},P,\overline{\rho})$ as she gets in *E* from taking any action $(\hat{I},P',\overline{\rho})$ in Σ.

By this Claim, no action in $\tilde{E}$ affords an issuer of a given type more than her equilibrium payoff in *E*; and taking her prescribed action in $\tilde{E}$ gives her the same payoff that she gets, in equilibrium, in *E*. Thus she will take her prescribed action in $\tilde{E}$: Payoff Maximization holds, whence $\tilde{E}$ is an equilibrium as claimed.

> **Proof of Claim.** Let $(\hat{I},P',\overline{\rho})$ be any action in Σ. Note first that $(\hat{I},P',\overline{\rho})$ has the same aggregate payout function $W(Y)=P^1(\tilde{I}^1(Y))$ as $(\tilde{I},P,\overline{\rho})$ and thus the

[86] By construction of $\tilde{E}$, a given type *t* chooses the same equilibrium revenue cap $\overline{\rho}_t(\hat{I})$ in $\tilde{E}$ as in *E*. Thus, every equilibrium action in Σ must involve the same revenue cap that is prescribed in $\tilde{E}$.

same aggregate expected payout conditional on the issuer's type, $E\left[W(Y)\,|\,t\right]$. Hence the Claim holds if $\left(\hat{I}, P', \bar{\rho}\right)$ raises the same revenue in E as $\left(\tilde{I}, P, \bar{\rho}\right)$ raises in $\tilde{E}$. We first show that all actions in Σ raise the same revenue in E:

LEMMA. In E, following the equilibrium interim asset choice $\hat{I}$, if there are two ex-post actions that are expected given $\hat{I}$ and that have the same revenue cap and aggregate payout function, they must raise the same revenue.

PROOF OF LEMMA. Suppose not: there are types t and t' (possibly equal) who, after choosing $\hat{I}$ and seeing their types, choose ex-post actions $(P, \bar{\rho})$ and $(P', \bar{\rho})$, respectively, that have the same revenue cap and aggregate payout function, such that $(P, \bar{\rho})$ yields less revenue than $(P', \bar{\rho})$: $\bar{\rho} \wedge \sum_{j=1}^{\#(P)} p^j\left(I, P, \bar{\rho}\right) < \bar{\rho} \wedge \sum_{j=1}^{\#(P')} p^j\left(I, P', \bar{\rho}\right)$. Since the aggregate payout functions are identical, their conditional expectations are the same conditional on the issuer's type being t: $E\left[W_P^{\hat{I}}(Y)\,|\,t\right] = E\left[W_{P'}^{\hat{I}}(Y)\,|\,t\right]$. But then by (7), type t's payoff $U\left(\hat{I}, P', \bar{\rho}\,|\,t, p\right)$ from $(P', \bar{\rho})$ exceeds her payoff $U\left(\hat{I}, P, \bar{\rho}\,|\,t, p\right)$ from $(P, \bar{\rho})$ which she is therefore unwilling to choose - a contradiction. ◆

By this Lemma, each action in Σ raises the same issuance revenue R_Σ in E. And if, in E, the issuer chooses action $\left(\hat{I}, P', \bar{\rho}\right)$ in Σ then, by (6) and (8), investors' willingness to pay $WTP_E\left(\hat{I}, P', \bar{\rho}\right) \overset{d}{=} \sum_{j=1}^{\#(P')} p^j\left(\hat{I}, P', \bar{\rho}\right)$ for her ex-post assets equals their estimate $\sum_t E\left[W(Y)\,|\,t\right]\,\mu(t\,|\,\hat{I}, P', \bar{\rho})$ of the aggregate payout given what the action $\left(\hat{I}, P', \bar{\rho}\right)$ reveals about the issuer's type. The issuance revenue R_Σ in E is then the minimum of $WTP_E\left(\hat{I}, P', \bar{\rho}\right)$ and the cap ρ. Since, by the above Lemma, this issuance revenue is the same for any action $\left(\hat{I}, P', \bar{\rho}\right)$ in Σ, it follows that either (i) $WTP_E\left(\hat{I}, P', \bar{\rho}\right) > \bar{\rho}$ for every such action or (ii) $WTP_E\left(\hat{I}, P', \bar{\rho}\right) \leq \bar{\rho}$ for every such action. Moreover, in case (ii), $WTP_E\left(\hat{I},P',\bar{\rho}\right)$ must take a common value $WTP_E\left(\Sigma\right)$ for any action $\left(\hat{I},P',\bar{\rho}\right)$ in Σ.

Likewise, if the issuer chooses action $\left(\tilde{I},P,\bar{\rho}\right)$ in $\tilde{E}$ then the amount $\tilde{p}^1\left(\tilde{I},P,\bar{\rho}\right)$ that investors are willing to pay for her ex-post security is just their estimate $\sum_t E\left[W\left(Y\right)|\,t\right]\tilde{\mu}(t\,|\,\tilde{I},P,\bar{\rho})$ of this security's payout conditional on the action $\left(\tilde{I},P,\bar{\rho}\right)$; the resulting issuance revenue is the minimum of $\tilde{p}^1\left(\tilde{I},P,\bar{\rho}\right)$ and the cap $\bar{\rho}$.

But for each *t*, by Beliefs, $\tilde{\mu}(t\,|\,\tilde{I},P,\bar{\rho})$ can be written as $\dfrac{g\left(t\right)1_{t\in\tilde{T}\left(P,\bar{\rho}|\tilde{I}\right)}}{\sum_s g\left(s\right)1_{s\in\tilde{T}\left(P,\bar{\rho}|\tilde{I}\right)}}$. And the sets $T\left(P',\bar{\rho}\,|\,\hat{I}\right)$ of types who, in *E*, take actions $\left(\hat{I},P',\bar{\rho}\right)\in\Sigma$, is a partition of the set $\tilde{T}\left(P,\bar{\rho}\,|\,\tilde{I}\right)$ of types who take the action $\left(\tilde{I},P,\bar{\rho}\right)$ in $\tilde{E}$. Thus we can rewrite $\tilde{\mu}(t\,|\,\tilde{I},P,\bar{\rho})$ as a weighted sum $\sum_{\left(\hat{I},P',\bar{\rho}\right)\in\Sigma}\mu\left(t\,|\,\hat{I},P',\bar{\rho}\right)w_{\left(\hat{I},P',\bar{\rho}\right)}$ where the weight $w_{\left(\hat{I},P',\bar{\rho}\right)}=\dfrac{\sum_s g\left(s\right)1_{s\in T\left(P',\bar{\rho}|\hat{I}\right)}}{\sum_s g\left(s\right)1_{s\in\tilde{T}\left(P,\bar{\rho}|\tilde{I}\right)}}$ is the probability that the issuer would have chosen $\left(\hat{I},P',\bar{\rho}\right)$ in *E* conditional on her having chosen $\left(\tilde{I},P,\bar{\rho}\right)$ in $\tilde{E}$. Hence we can rewrite $\tilde{p}^1\left(\tilde{I},P,\bar{\rho}\right)$ as $\sum_{\left(\hat{I},P',\bar{\rho}\right)\in\Sigma}w_{\left(\hat{I},P',\bar{\rho}\right)}\sum_t E\left[W\left(Y\right)|\,t\right]\mu\left(t\,|\,\hat{I},P',\bar{\rho}\right)$ or, equivalently, as $\sum_{\left(\hat{I},P',\bar{\rho}\right)\in\Sigma}w_{\left(\hat{I},P',\bar{\rho}\right)}R_E\left(\hat{I},P',\bar{\rho}\right)$. Since, the weights $w_{\left(\hat{I},P',\bar{\rho}\right)}$ sum to one, the revenue $\bar{\rho}\wedge R_{\tilde{E}}\left(\tilde{I},P,\bar{\rho}\right)$ raised by the action $\left(\tilde{I},P,\bar{\rho}\right)$ in $\tilde{E}$ equals $\bar{\rho}$ in case (i) above (where the revenue raised in *E* by each $\left(\hat{I},P',\bar{\rho}\right)$ in Σ is also $\bar{\rho}$) and the common value $WTP_E\left(\Sigma\right)$ in case (ii) (when the revenue raised in *E* by each $\left(\hat{I},P',\bar{\rho}\right)$ in Σ is also $WTP_E\left(\Sigma\right)$). This proves the Claim. ◆

It remains to show that if E is intuitive, so is $\tilde{E}$. Suppose E is intuitive. Let $u(\cdot \mid I)$ and $\tilde{u}(\cdot \mid I)$ denote the equilibrium payoffs in E and $\tilde{E}$, respectively, of an issuer of type t who chooses the interim asset vector I and then follows her continuation strategy in the given equilibrium. Consider any action $(I, P, \bar{\rho})$ for which there are types t and s satisfying:

$$\bar{\rho} < \tilde{u}(t \mid I) + \delta E\left[W_P^I(Y) \mid t\right] \text{ and } \bar{\rho} \geq \tilde{u}(s \mid I) + \delta E\left[W_P^I(Y) \mid s\right]. \tag{17}$$

First assume $(I, P, \bar{\rho})$ is unexpected in $\tilde{E}$. As previously noted, the payoff of an issuer of type t from taking the action $(I, P, \bar{\rho})$ in $\tilde{E}$ equals her payoff from taking this action in E. It follows that $\tilde{u}(v \mid I) = u(v \mid I)$ for $v = s, t$. Hence $\bar{\rho} < u(t \mid I) + \delta E\left[W_P^I(Y) \mid t\right]$ and $\bar{\rho} \geq u(s \mid I) + \delta E\left[W_P^I(Y) \mid s\right]$ whence, since E is intuitive, $\mu(t \mid I, P, \bar{\rho})$ is zero. But since $(I, P, \bar{\rho})$ is unexpected in $\tilde{E}$, Beliefs implies that $\mu(t \mid I, P, \bar{\rho})$ equals $\tilde{\mu}(t \mid I, P, \bar{\rho})$ which thus also is zero as required.

Now suppose instead that $(I, P, \bar{\rho})$ is expected in $\tilde{E}$: $\tilde{T}(P, \bar{\rho} \mid I)$ is nonempty. By (17), $\tilde{u}(t \mid I)$ exceeds $\bar{\rho} - \delta E\left[W_P^I(Y) \mid t\right]$ which is the highest payoff type t can expect from $(I, P, \bar{\rho})$. Hence t is not in $\tilde{T}(P, \bar{\rho} \mid I)$ whence, by Beliefs, $\tilde{\mu}(t \mid I, P, \bar{\rho})$ is zero. We conclude that $\tilde{E}$ is intuitive as claimed. ◆

PROOF OF PROPOSITION 7. We first define the notions of equilibrium, intuitive beliefs, and fair pricing for an EPSD game. Each definition is the natural restriction of the analogous definition for a GSD game.

> **EPSD EQUILIBRIUM.** A perfect Bayesian equilibrium of the EPSD game is a security design $S_t \in M$ and price cap $\bar{\rho}_t \in \Re_+$ for each type t, as well as a price function $\hat{p}(S, \bar{\rho})$ and belief function $\hat{\mu}(t \mid S, \bar{\rho})$, with the following properties:
>
> 1. Payoff Maximization: for all t, the issuer's choice $(S_t, \bar{\rho}_t)$ solves $\max_{S,\bar{\rho}} \left\{\hat{p}(S, \bar{\rho}) - \delta E\left[S(Y) \mid t\right]\right\}$ subject to $S \in$ M.

2. Competitive Pricing: for any monotone security $S \in$ M and price cap $\bar{\rho}$, the price function $\hat{p}(S,\bar{\rho})$ equals $\min\left\{\bar{\rho},\sum_t E[S(Y)\,|\,t]\ \hat{\mu}(t\,|\,S,\bar{\rho})\right\}$.

3. Rational Updating: the investors' belief function $\hat{\mu}(t\,|\,S,\bar{\rho})$ follows Bayes's rule when applicable.

Fair Pricing (EPSD Game). An equilibrium $\left(\left(S_t,\bar{\rho}_t\right)_{t=0}^T,\hat{p},\hat{\mu}\right)$ of the GSD game is *fairly priced* if, for each type *t*, the price of ex-post security S_t equals its expected value conditional on the issuer's type: $\hat{p}(S_t,\bar{\rho}_t)=E\left[S_t\left(Y\right)\middle|t\right]$.

The *outcome* of the EPSD game is the function $\hat{u}\left(t\right)=\hat{p}(S_t,\bar{\rho}_t)-\delta E\left[S_t(Y)\,|\,t\right]$ giving the securitization payoff of each type *t*.

The Intuitive Criterion (EPSD Game). A perfect Bayesian equilibrium $\left(\left(S_t,\bar{\rho}_t\right)_{t=0}^T,\hat{p}\left(\cdot,\cdot\right),\hat{\mu}\left(\cdot\,|\,\cdot,\cdot\right)\right)$ of the EPSD game, with outcome $\hat{u}\left(\cdot\right)$, is *intuitive* if, for any security $S\in M$ and revenue cap $\bar{\rho}\in\Re_+$ for which $\bar{\rho}\geq\hat{u}\left(t\right)+\delta E\left[S(Y)\,|\,t\right]$ for some type *t*, $\bar{\rho}<\hat{u}\left(s\right)+\delta E\left[S(Y)\,|\,s\right]$ implies $\hat{\mu}(s\,|\,S,\bar{\rho})=0$.

Define the following two sets.

1. I_{AS} is the set of intuitive Asset Sale Equilibria $\left(q(\cdot),\bar{p}(\cdot),p(\cdot,\cdot),\mu(\cdot\,|\,\cdot,\cdot)\right)$ of the asset sale game G_{AS} with portfolio (F^*, *a*) in which investor beliefs following any issuance choice $\left(q,\bar{p}\right)$ depend only on the quantity choices *q* and the issuer's maximum issuance revenue $\bar{\rho}=\bar{p}q$. Changing notation slightly, we now denote these beliefs as $\mu\left(\cdot\,|\,q,\bar{p}q\right)$ rather than $\mu\left(\cdot\,|\,q,\bar{p}\right)$.

2. I_{SD} is the set of intuitive equilibria $\left(\left(S_t,\bar{\rho}_t\right)_{t=0}^T,\hat{p}\left(\cdot,\cdot\right),\hat{\mu}\left(\cdot\,|\,\cdot,\cdot\right)\right)$ of the Ex Post Security Design game G_{SD}.

These two sets are equivalent in the following sense

LEMMA.

1. For any equilibrium e in I_{AS}, there is an equilibrium $\hat{e}$ in I_{SD} with the same outcome. If type t sells the quantities $q(t)$ in e, then this type issues the security $S_t(Y) = q(t)F^*(Y)$ in $\hat{e}$.

2. For any equilibrium $\hat{e}$ in I_{SD}, there is a set of equilibria e in I_{AS} with the same outcome. If type t issues the security S_t in $\hat{e}$, then this type sells the quantities q^{S_t} in any such equilibrium e.

PROOF OF LEMMA. Consider any equilibrium e in I_{AS}. Its outcome is $u(t) = q(t)\left[p\left(q(t), \bar{p}(t)\right) - \delta f^*(t)\right]$ and investors' price function is $p(q, \bar{p}) = \bar{p} \wedge \sum_t f^*(t)\,\mu(t \mid q, \bar{p}q)$. We build an equivalent Security Design Equilibrium $\hat{e}$ in I_{SD} as follows. Type t issues the monotone security $S_t(y) = q(t)F^*(y)$ with maximum revenue $\bar{\rho}_t$ equal to $\bar{p}(t)q(t)$. This implies, in particular, that $q_i^{S_t} = q(t)\left[F^*(y_i) - F^*(y_{i-1})\right] / (y_i - y_{i-1}) = q_i(t)$; collecting terms, $q^{S_t} = q(t)$. Given any security design choice $(S, \bar{\rho})$, investors respond with beliefs $\hat{\mu}(\cdot \mid S, \bar{\rho}) = \mu(\cdot \mid q^S, \bar{\rho})$ and associated price

$$\hat{p}(S, \bar{\rho}) = \max\left\{\bar{\rho}, \sum_t E\left[S(Y) \mid t\right]\hat{\mu}(t \mid S, \bar{\rho})\right\},$$

whence Competitive Pricing holds in $\hat{e}$. A type t issuer's payoff from the choice $(S, \bar{\rho})$ is $\hat{p}(S, \bar{\rho}) - \delta E\left[S(Y) \mid t\right] = \max\left\{\bar{\rho}, q^S \sum_t f^*(t)\,\mu(t \mid q^S, \bar{\rho})\right\} - \delta q^S f^*(t)$, which is identical to her payoff $q^S\left[p\left(q^S, \bar{p}\right) - \delta f^*(t)\right]$ from the issuance choice $\left(q^S, \bar{p}\right)$ in e where $\bar{p}$ is any n-vector of asset price caps that gives the same maximum revenue: $\bar{\rho} = \bar{p}q^S$. Moreover, all such asset price cap vectors give the same payoff in e because – by assumption - investor beliefs and thus issuance revenue depend only on the quantity vector q and maximum revenue $q\bar{p}$. Hence, since $\left(q(t), \bar{p}(t)\right)$ maximizes the payoff of type t in e, it follows that $\left(S_t, \bar{\rho}_t\right)$ maximizes the payoff of type t in $\hat{e}$:

Payoff Maximization holds in $\hat{e}$. By restricting to choices made in equilibrium, the preceding also implies that the outcome $\hat{u}(t)=\hat{p}(S_t,\rho_t)-\delta E[S_t(Y)|t]$ of $\hat{e}$ equals the outcome $u(t)=q(t)[p(q(t),\overline{p}(t))-\delta f^*(t)]$ of *e*. By construction, $\hat{\mu}(\cdot|S_t,\overline{\rho}_t)=\mu(\cdot|q^{S_t},\overline{\rho}_t)=\mu(\cdot|q(t),\overline{p}(t))$ and the latter is given by Bayes's Rule whenever possible. Since the issuer's behavior is also the same in the two cases, Rational Updating holds: $\hat{e}$ is a Security Design Equilibrium. Finally, consider any security design choice $(S,\overline{\rho})$ in $\hat{e}$ such that, for some type *t*,

$$\overline{\rho}\geq\hat{u}(t)+\delta E[S(Y)|t]=u(t)+\delta q^S f^*(t).$$

Then following the issuance choice $(q^S,\overline{p})$ in *e*, investors put zero weight on any type *s* for which $\overline{\rho}$ is less than $u(s)+\delta q^S f^*(s)$ which equals $\hat{u}(s)+\delta E[S(Y)|s]$. Hence $\hat{\mu}(s|S,\overline{\rho})=\mu(s|q^S,\overline{\rho})=0$: $\hat{e}$ is intuitive.

Consider any equilibrium $\hat{e}$ in I_{SD}. Its outcome is $\hat{u}(t)=\hat{p}(S_t,\overline{\rho}_t)-\delta E[S_t(Y)|t]$ and the investors' price function is $\hat{p}(S,\overline{\rho})=\max\{\overline{\rho},\sum_t E[S(Y)|t]\hat{\mu}(t|S,\overline{\rho})\}$. We build an equivalent Asset Sale Equilibrium *e* in I_{AS} as follows. (As the price cap vector will not be unique, this defines a set of equilibria *e*, as indicated in the statement of this proposition.) Type *t* chooses the quantity vector $q(t)=q^{S_t}$ and *any* price cap vector $\overline{p}(t)$ that satisfies $\overline{p}(t)q(t)=\overline{\rho}_t$. In response to any asset sale choice $(q,\overline{p})$, investors' posterior beliefs are $\mu(\cdot|q,\overline{p})=\hat{\mu}(\cdot|qF^*,\overline{p}q)$.[87] The associated price vector is $p(q,\overline{p})=\overline{p}\wedge\sum_t f^*(t)\,\mu(t|q,\overline{p})$, which implies Competitive Pricing in *e*. The price functions in *e* and $\hat{e}$ are thus related by $p(q,\overline{p})q=\hat{p}(qF^*,\overline{p}q)$. Hence, a type *t* issuer's payoff $q[p(q,\overline{p})-\delta f^*(t)]$ in *e* from the issuance choice $(q,\overline{p})$ equals her payoff $\hat{p}(qF^*,\overline{p}q)-\delta E[qF^*(Y)|t]$ in $\hat{e}$ from the security design $(qF^*,\overline{p}q)$. Thus, since $(S_t,\overline{\rho}_t)$ maximizes the payoff of type *t* in $\hat{e}$, it follows that $(q^{S_t},\overline{p})$ maximizes

[87] The security $qF^*(Y)$ in G_{SD} promises the same realized payout to investors as the quantity vector *q* in G_{AS}.

the payoff of type t in e, where $\bar{p}$ is any price cap vector that satisfies $\bar{p}q^{S_t} = \bar{\rho}_t$: Payoff Maximization holds in e. By restricting to choices made in equilibrium, the preceding also implies that the outcome $u(t) = q(t)\left[p(q(t), \bar{p}(t)) - \delta f^*(t)\right]$ of e equals the outcome $\hat{u}(t) = \hat{p}(S_t, \rho_t) - \delta E\left[S_t(Y) \mid t\right]$ of $\hat{e}$. By construction,

$$\mu(\cdot \mid q(t), \bar{p}(t)) = \hat{\mu}(\cdot \mid q(t)F^*, \bar{p}(t)q(t)) = \hat{\mu}(\cdot \mid S_t, \bar{\rho}_t)$$

and the last is given by Bayes's Rule whenever possible. Since the issuer's behavior is also the same in the two settings, Rational Updating holds: e is an Asset Sale Equilibrium. Finally, consider any issuance choice $(q, \bar{p})$ in e such that, for some type t, $\bar{p}q \geq u(t) + \delta q f^*(t) = \hat{u}(t) + \delta E\left[S(Y) \mid t\right]$ where S denotes the security qF^*. Then, following the corresponding issuance choice $(S, \bar{p}q)$ in $\hat{e}$, investors put zero weight on any type s for which $\bar{p}q < \hat{u}(s) + \delta E\left[S(Y) \mid s\right] = u(s) + \delta q f^*(s)$. Hence, $\mu(s \mid q, \bar{p}) = \hat{\mu}(s \mid qF^*, \bar{p}q) = 0$: e is intuitive. ◆

Now let e^*_{AS} denote the intuitive equilibrium e^* of the Asset Sale game, specialized to the case in which the endowment (a, f) equals the endowment (a^*, f^*) of G_{AS}. We claim that e^*_{AS} lies in I_{AS}. Why? In e^*_{AS}, each type t can choose any price cap vector that is not less than $f^*(t)$ and investors ignore these caps. Thus, beliefs on the equilibrium path do not depend on the price cap vector. Following any out-of-equilibrium choice $(q, \bar{p})$, investors' beliefs are given by (3) and (4), which are also independent of the price cap vector. So investor beliefs in e^*_{AS} do not depend on the price cap vector at all. Thus, as e^*_{AS} is intuitive by PROPOSITION **1**, it must lie in I_{AS}.

By the above Lemma, then, there exists an equilibrium e^*_{SD} in I_{SD} with the same outcome as e^*_{AS}, in which each type t issues the security $S^*_t(Y) = q^*(t)F^*(Y)$ where $q^*(t)$ is chosen by type t in e^*_{AS} and solves RLP. Finally, FOSD together with the fact that F^* is monotone implies that ASSUMPTION A holds. Hence, by PROPOSITION **1**, the quantity choice function in any equilibrium in I_{AS} must be a solution $q^*(\cdot)$ to RLP. Thus, by the above Lemma, in *any* equilibrium in I_{SD}, the issuer's optimal security $S_t^*(Y)$ must equal $q^*(t)F^*(Y)$ where $q^*(t)$ solves RLP. ◆

PROOF OF PROPOSITION 8. From PROPOSITION 7, $S_t^*\left(Y\right)=q^*\left(t\right)F^*\left(Y\right)$. By PROPOSITION 5, IIS holds so that from PROPOSITION 3, $q_i^*\left(t\right)=1$ for $i < h(t)$ and $q_i^*\left(t\right)=0$ for $i > h(t)$. Therefore $S_t^*\left(y_i\right)=q^*\left(t\right)F^*\left(y_i\right)=y_i$ for $i < h(t)$ and

$$S_t^*\left(y_i\right)=q^*\left(t\right)F^*\left(y_i\right)=y_{h(t)-1}+q_{h(t)}^*\left(t\right)\left(y_{h(t)}-y_{h(t)-1}\right)$$

for $i \geq h(t)$. Thus, $S_t^*\left(Y\right)=\min\left(D_t^*,Y\right)$ where $D_t^*=y_{h(t-1)}+q_{h(t)}^*\left(t\right)\left(y_{h(t)}-y_{h(t)-1}\right)$. Finally, since q^* and h are decreasing in t by PROPOSITION 3, D_t^* is decreasing in t. That D_t^* satisfies (11) follows from PROPOSITION 4. Finally, as the face value function is strictly decreasing, the equilibrium is fully revealing; hence, $u^*(t)=(1-\delta)v(D_t,t)$. Off-equilibrium, suppose debt $D\in(D_{t+1}^*,D_t^*)$ is issued. The price p that would make type t willing to deviate satisfies $p=u^*(t)+\delta v(D,t)=(1-\delta)v(D_t^*,t)+\delta v(D,t)$. For any type $s<t$, the incentive constraint for s implies $v(D_t^*,t)-\delta v(D_t^*,s)\leq u^*(s)$. Combining these, we have $p+\delta\underbrace{\left[v(D_t^*,t)-v(D,t)-\left(v(D_t^*,s)-v(D,s)\right)\right]}_{>0}\leq u^*(s)+\delta v(D,s)$, where the term in square brackets is positive: the value of junior debt with face value D_t^*-D is higher for type t than for type $s<t$. Therefore, types $s<t$ would not find it profitable to deviate given price p. The beliefs specified in (3) must therefore concentrate on t, and so the market price following a deviation to D is $v(D,t)$. ◆

PROOF OF PROPOSITION 9. Let the range of D^{∞} be $(\underline{D},\bar{y}]$. Define $U\left(t,\widehat{t},D\right)$ to be the payoff $E\left[\min(D,Y)\,|\widehat{t}\right]-\delta E\left[\min(D,Y)\,|t\right]$ of an issuer with type t when she issues debt with face value D and the investors believe her type is $\widehat{t}$. Then

$$\frac{U_D}{U_{\widehat{t}}}=\frac{\Pr\left(Y>D|\widehat{t}\right)-\delta\Pr\left(Y>D|t\right)}{\frac{\partial}{\partial\widehat{t}}E\left[\min(D,Y)\,|\widehat{t}\right]}$$

is nonincreasing in t by First-Order Stochastic Dominance (FOSD). Since by assumption D_t is decreasing in t and since $U_2\left(t,\widehat{t},D\right)\geq 0$ by FOSD, part 1 of Theorem 6 in Mailath and von Thadden (2013) implies that an issuer of type t will not imitate any other type: she will not sell debt with face value $D^{\infty}_{\widehat{t}}$ for any $\widehat{t}\neq t$. It remains to consider whether she will sell debt with a face value not in the range of D^{∞}, as well as other types of monotone securities. We may assume that investors respond to any such deviation with the the most pessimistic beliefs: that her type is zero.

First consider a deviation of a type t to debt with a face value D that is not in the range $[\underline{D},\bar{y}]$ of D^{∞}. Assume this deviation makes type t strictly better off than sticking to the equilibrium. First, suppose $D>\bar{y}$. This security has the same payout for any Y as debt with face value $\bar{y}$, and both lead investors to believe that $t=0$. By the preceding result, no type strictly prefers such a deviation. Now consider $D<\underline{D}$. Investor beliefs cannot be more optimistic than those that result from debt with face value $\underline{D}$, since these are the beliefs that the type is one. Moreover, there are gains from trade, so a higher face value is more profitable for the issuer (holding investor beliefs constant). Hence, any $D<\underline{D}$ is worse for the issuer than $D=\underline{D}$.

Finally, we consider deviations of type t to a general monotone security $\widehat{S}$. The issuer's securitization profit from $\widehat{S}$ equals the price $E\left[\widehat{S}(Y)\,|0\right]$ assigned by investors less the discounted expected security payout $\delta E\left[\widehat{S}(Y)\,|t\right]$. This profit can be written as

$$\int_{Y=0}^{\bar{y}}\widehat{S}(Y)\,d\left[H(Y|0)-\delta H(Y|t)\right]$$

where H denotes the conditional distribution of Y given t. The security $\widehat{S}$ is monotone if and only if $\widehat{S}(0)=0$ and, for all Y, the control variable $c(Y)=\widehat{S}'(Y)$ lies in $[0,1]$. For all

$Y \in [0, \bar{y}]$, define the Hamiltonian

$$\mathscr{H} = \lambda(Y)c(Y) + \widehat{S}(Y)\left[H'(Y|0) - \delta H'(Y|t)\right] + \mu_0(Y)c(Y) + \mu_1(Y)\left[1 - c(Y)\right]$$

where $\mu_0, \mu_1 \geq 0$ are Lagrange multipliers that capture the constraints on c and $\lambda(Y)$ is a costate variable. By Pontryagin's maximization principle (Pontryagin *et al* 1962), the optimal control c must maximize the Hamiltonian $\mathscr{H}$ while the multipliers $\mu_0, \mu_1 \geq 0$ must minimize it. As $\mathscr{H}$ is linear in c, this implies the first order condition

$$0 = \frac{\partial}{\partial c(Y)}\mathscr{H} = \lambda(Y) + \mu_0(Y) - \mu_1(Y) \tag{18}$$

as well as the Kuhn-Tucker (complementary slackness) conditions:

$$\mu_0(Y)c(Y) = 0 \text{ and } \mu_1(Y)\left[1 - c(Y)\right] = 0. \tag{19}$$

Additionally, for all Y, the costate equation

$$\lambda'(Y) = -\mathscr{H}_{\widehat{S}} = -H'(Y|0) + \delta H'(Y|t) \tag{20}$$

must be satisfied. Finally, as the final state $\widehat{S}(\bar{y})$ is not fixed, the terminal costate must be zero:

$$\lambda(\bar{y}) = 0. \tag{21}$$

Solving (20) subject to (21), we obtain $\lambda(Y) = [1 - H(Y|0)]\left[1 - \delta\frac{1-H(Y|t)}{1-H(Y|0)}\right]$. By HRO, $\frac{1-H(Y|t)}{1-H(Y|0)}$ is increasing in Y since $t > 0$. Hence, by (18), (19), and the nonnegativity of μ_0 and μ_1, there exists an $D \in \Re$ such that $c(Y) = 1$ for all $Y < D$ and $c(Y) = 0$ for all $Y > D$: the optimal security is debt with face value D and thus, as shown above, $D = D_t^\infty$. Q.E.D.$_{\text{PROPOSITION 9}}$

PROOF OF PROPOSITION 10: First, $1 - H(y|t)$ can be written as $1 - H(y|1) + \int_{s=t}^{1} H_2(y|s)\,ds$, which is smaller for $H = \widetilde{H}$ than for $H = \widehat{H}$. Hence, the rate of change function in (1) is no larger in absolute value when H equals $\widehat{H}$ than when it equals $\widetilde{H}$. The result then follows from PROPOSITION 14. Q.E.D.$_{\text{PROPOSITION 10}}$

PROOF OF PROPOSITIONS 11-14, AND COROLLARY 15: In this proof, we will write $D^{\infty}(t)$ and $D^{i}(t)$ in place of D_t^{∞} and D_t^{i}. We will also treat $\bar{y}$ as a parameter and the relative cash flow $Z = Y/\bar{y}$ as an exogenous random variable with support $[0,1]$ and conditional distribution function $\widehat{H}$ defined by $\widehat{H}(z|t) = H(z\bar{y}|t)$ for z in $[0,1]$. Lipschitz-H then entails the following property.[1]

LIPSCHITZ-$\widehat{H}$. There are constants $\widehat{k}_0, \widehat{k}_1, \widehat{k}_2 \in (0,\infty)$ such that for all $z, t, t', t'' \in [0,1]$, $\frac{\partial \widehat{H}(z|t)}{\partial z} \in \left(\widehat{k}_0, \widehat{k}_1\right)$, $-\frac{\partial \widehat{H}(z|t)}{\partial t} \in \left[\widehat{k}_0 z(1-z), \widehat{k}_1\right)$, $\left|\frac{\partial \widehat{H}(z|t')}{\partial z} - \frac{\partial \widehat{H}(z|t'')}{\partial z}\right| < \widehat{k}_2 |t'-t''|$, and

$$\left| \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t''} \right| < \widehat{k}_2 \left|t'-t''\right|.$$

Henceforth we rename $\widehat{H}$ to H and each $\widehat{k}_n$ to k_n (for $n = 0,1,2$), and refer to Lipschitz-$\widehat{H}$ as Lipschitz-H or L-H.

Recall that E^i and E^{∞} denote the expectation operators in models i and ∞. For any $D \in \mathfrak{R}$, let

$$v^{H\bar{y}i}(D,t) = E^i\left[\min\{D, \bar{y}Z\} |t\right] = \sum_{c=1}^{1/\Delta_i'} \min\left\{D, \bar{y}c\Delta_i'\right\} \left[H\left(c\Delta_i'|t\right) - H\left((c-1)\Delta_i'|t\right)\right] \tag{22}$$

[1] Lipschitz-H implies that there are constants $k_0, k_1, k_2 \in (0,\infty)$ such that for all $z, t, t', t'' \in [0,1]$,

$$\frac{\partial \widehat{H}(z|t)}{\partial z} = \frac{\partial H(z\bar{y}|t)}{\partial z} = \frac{\partial H(z\bar{y}|t)}{\partial (z\bar{y})}\frac{d(z\bar{y})}{dz} \in (k_0\bar{y}, k_1\bar{y}),$$

$$-\frac{\partial \widehat{H}(z|t)}{\partial t} = -\frac{\partial H(z\bar{y}|t)}{\partial t} \in [k_0 y(\bar{y}-y), k_1) = \left[\bar{y}^2 k_0 z(1-z), k_1\right),$$

$$\begin{aligned}\left|\frac{\partial \widehat{H}(z|t')}{\partial z} - \frac{\partial \widehat{H}(z|t'')}{\partial z}\right| &= \left|\frac{\partial H(z\bar{y}|t')}{\partial (z\bar{y})} - \frac{\partial H(z\bar{y}|t'')}{\partial (z\bar{y})}\right| \frac{d(z\bar{y})}{dz} \\ &= \left|\frac{\partial H(y|t')}{\partial y} - \frac{\partial H(y|t'')}{\partial y}\right| \bar{y} < k_2\bar{y}\left|t'-t''\right|,\end{aligned}$$

and

$$\begin{aligned}\left| \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial \widehat{H}(z|t)}{\partial t}\right|_{t=t''} \right| &= \left| \left.\frac{\partial H(z\bar{y}|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial H(z\bar{y}|t)}{\partial t}\right|_{t=t''} \right| \\ &= \left| \left.\frac{\partial H(y|t)}{\partial t}\right|_{t=t'} - \left.\frac{\partial H(y|t)}{\partial t}\right|_{t=t''} \right| < k_2\left|t'-t''\right|.\end{aligned}$$

This implies Lipschitz-$\widehat{H}$ with constants $\widehat{k}_0 = k_0 \min\left\{\bar{y}, \bar{y}^2\right\}$, $\widehat{k}_1 = k_1 \max\{\bar{y}, 1\}$, and $\widehat{k}_2 = k_2 \max\{\bar{y}, 1\}$.

and

$$v^{H\bar{y}}(D,t) = E^{\infty}\left[\min\{D,\bar{y}Z\}|t\right] = \int_{z=0}^{1} \min\{D,\bar{y}z\}\, dH(z|t) \tag{23}$$

denote the expected payout of debt with face value D in models i and ∞, respectively, conditional on the issuer's type t. Integrating by parts yields

$$v^{H\bar{y}}(D,t) = D - \bar{y}\int_{z=0}^{D/\bar{y}} H(z|t)\, dz. \tag{24}$$

Given the above notation, the incentive compatibility condition in model i can be stated as the following initial value problem (henceforth "IVP"):

DISCRETE IVP WITH PARAMETERS $H,\delta,\bar{y}$ (DP$^{H\delta\bar{y}}$). The condition

$$\begin{aligned} & v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t+\Delta_i), t+\Delta_i\right) - \delta v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t+\Delta_i), t\right) \\ & = (1-\delta)\, v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), t\right) \end{aligned} \tag{25}$$

with $D^{H\delta\bar{y}i} : S_i \to \Re$, together with the initial value $D^{H\delta\bar{y}i}(0) = \bar{y}$.

Define the function

$$f^{H\delta\bar{y}}(D,t) = -\frac{1}{1-\delta}\frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} = \frac{\bar{y}}{1-\delta}\frac{\int_{z=0}^{\frac{D}{\bar{y}}} \frac{\partial H(z|t)}{\partial t} dz}{1-H\left(\frac{D}{\bar{y}}|t\right)} \leq 0, \tag{26}$$

where the second equality follows from (24). A restatement of CP in model ∞ is as follows.

CONTINUOUS IVP WITH PARAMETERS $H,\delta,\bar{y}$ (CP$^{H\delta\bar{y}}$). The differential equation

$$\frac{dD^{H\delta\bar{y}}}{dt} = f^{H\delta\bar{y}}\left(D^{H\delta\bar{y}}, t\right), \tag{27}$$

with $D^{H\delta\bar{y}} : [0,1] \to \Re$, together with the initial value $D^{H\delta\bar{y}}(0) = \bar{y}$.

In model i, let $p^{H\delta\bar{y}i}(t)$ denote $v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), t\right)$: the price and equilibrium expected payout of a standard debt contract, conditional on the type t. Let $u^{H\delta\bar{y}i}(t)$ denote $(1-\delta)\, p^{H\delta\bar{y}i}(t)$: the issuer's profit and expected gains from trade from such a contract conditional on the type t. Let $Eu^{GH\delta\bar{y}i} = E^i\left[u^{H\delta\bar{y}i}(t)\right]$ denote the unconditional expected gains from trade and issuer's profit. Analogously, in the continuous model let $p^{H\delta\bar{y}}(t)$ denote $v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t),t\right)$, let $u^{H\delta\bar{y}}(t)$ denote $(1-\delta)\,p^{H\delta\bar{y}}(t)$, and let $Eu^{GH\delta\bar{y}}$ denote $E^{\infty}\left[u^{H\delta\bar{y}}(t)\right]$. Let $\Pi^{GH\delta\bar{y}i}(t) = u^{GH\delta\bar{y}i}(t) + E^{i}[\bar{y}Z|t]$ denote the issuer's conditional (on t) expected total profits in model i and let $E\Pi^{GH\delta\bar{y}i} = E^{i}\left[\Pi^{GH\delta\bar{y}i}(t)\right]$ denote her unconditional expected total profits. In the continuous model, define the analogous quantities $\Pi^{GH\delta\bar{y}}(t) = u^{GH\delta\bar{y}}(t) + E^{\infty}[\bar{y}Z|t]$ and $E\Pi^{GH\delta\bar{y}} = E^{\infty}\left[\Pi^{GH\delta\bar{y}}(t)\right]$. As noted, we extend the functions in model i to any type $t \in [0,1]$ by evaluating them them at the highest type in S_i that does not exceed t.

With this notation, Propositions 11-14, together with Corollary 15, are combined as follows.

Theorem 1. *Fix constants k_0, k_1, k_2, k_3, k_4, and $\mathbf{y}$, all in $(0,\infty)$. Let $\mathscr{G}$ be the set of distribution functions G that satisfy Lipschitz-G with constants k_3 and k_4. Let $\mathscr{H}$ be the set of conditional distribution functions H that satisfy Lipschitz-H with constants k_0, k_1, and k_2, and Hazard Rate Ordering. For any distribution function G in $\mathscr{G}$, conditional distribution function H in $\mathscr{H}$, parameter $\bar{y}$ in $(0,\mathbf{y}]$, and discount factor δ in $(0,1)$:*

1. *There exists a unique function $D^{H\delta\bar{y}}$ that satisfies $CP^{H\delta\bar{y}}$. This function is decreasing and differentiable, and takes values in $(0,\bar{y}]$. The associated price and profit functions, $p^{H\delta\bar{y}}$ and $u^{H\delta\bar{y}}$, are continuous and decreasing in the type t as well.*

2. *For each discrete model $i = 1,2,...$, there exists a unique, decreasing function $D^{H\delta\bar{y}i}$ that satisfies $DP^{H\delta\bar{y}i}$.*

3. *The sequences of face value functions, price functions, conditional and unconditional expected securitization profit functions, and conditional and unconditional expected total profit functions in the discrete models converge to their continuous counterparts as i grows, uniformly in the distributions G and H, the parameter y, and (except in the case of the unconditional expected profit functions which do not depend on the type) the type $t \in [0,1]$. More precisely:*

    - *For all $\varepsilon > 0$ there is an i^* such that for all models $i > i^*$, G in $\mathscr{G}$, H in $\mathscr{H}$, $\bar{y}$*

in $(0,\mathbf{y}]$*, and* $t\in[0,1]$*,* $\left|\omega^{H\delta\bar{y}i}(t)-\omega^{H\delta\bar{y}}(t)\right|<\varepsilon$ *for each* $\omega=D,p,u,\Pi$*, and* $\left|E\omega^{H\delta\bar{y}i}-E\omega^{H\delta\bar{y}}\right|<\varepsilon$ *for each* $\omega=u,\Pi$*.*

4. *All of the functions defined above are homogeneous of degree one in the parameter* $\bar{y}$*:* $\omega^{H\delta\bar{y}}=\bar{y}\omega^{H\delta 1}$ *and* $\omega^{H\delta\bar{y}i}=\bar{y}\omega^{H\delta 1i}$ *for each* $\omega=D,p,u,Eu,\Pi,E\Pi$*.*

5. *Let* $\widetilde{H},\widehat{H}\in\mathscr{H}$ *and* $\widetilde{\delta},\widehat{\delta}\in(0,1)$ *satisfy* $f^{\widetilde{H}\widetilde{\delta}\bar{y}}\le f^{\widehat{H}\widehat{\delta}\bar{y}}\le 0$*. Then* $D^{\widetilde{H}\widetilde{\delta}\bar{y}}(t)\le D^{\widehat{H}\widehat{\delta}\bar{y}}(t)$ *for all* $t\in[0,1]$*.*

We now prove Theorem 1. Without loss of generality, we restrict to face values D that do not exceed the maximum final asset value $\bar{y}$.[2] First, we define an integration by parts formula for the function $v^{H\bar{y}i}$ defined in (22).

Claim 1. For any face value $D\in[0,\bar{y}]$ and type t,

$$v^{H\bar{y}i}(D,t)=D-\bar{y}\int_{z=0}^{\frac{D}{\bar{y}}}H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right)dz. \tag{28}$$

Proof of Claim 1. One can easily verify (using $H(1|t)=1$ and $H(0|t)=0$) that for $D\in[0,\bar{y}]$, $v^{H\bar{y}i}(D,t)=\min\{D,\bar{y}\}-\sum_{c=1}^{\frac{1}{\Delta_i'}-1}H(c\Delta_i'|t)\left[\min\{D,\bar{y}(c+1)\Delta_i'\}-\min\{D,\bar{y}c\Delta_i'\}\right]$ by (22). As c is an integer, $c\le x$ iff $c\le\lfloor x\rfloor$. Thus,

$$\min\left\{D,\bar{y}(c+1)\Delta_i'\right\}-\min\left\{D,\bar{y}c\Delta_i'\right\}=\begin{cases}\bar{y}\Delta_i' & \text{if } c\le\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor-1\\ D-\bar{y}c\Delta_i' & \text{if } c=\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\\ 0 & \text{if } c>\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\end{cases},$$

whence, as $D\le\bar{y}$,

$$v^{H\bar{y}i}(D,t)=D-\bar{y}\Delta_i'\left[\begin{array}{c}\sum_{c=1}^{\lfloor D/(\bar{y}\Delta_i')\rfloor-1}H(c\Delta_i'|t)\\ +H\left(\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\Delta_i'|t\right)\left(\frac{D}{\bar{y}\Delta_i'}-\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\right)1\left(\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor\le\frac{1}{\Delta_i'}-1\right)\end{array}\right]. \tag{29}$$

[2] Choosing a higher face value is equivalent to choosing $\bar{y}$ since the underlying assets cannot be worth more than $\bar{y}$.

If $z \in [c\Delta_i', (c+1)\Delta_i')$, then $\lfloor z/\Delta_i' \rfloor = c$. So

$$\begin{aligned}
\sum_{c=1}^{\lfloor D/(\bar{y}\Delta_i')\rfloor - 1} H\left(c\Delta_i'|t\right) &= \sum_{c=1}^{\lfloor D/(\bar{y}\Delta_i')\rfloor - 1} \frac{1}{\Delta_i'} \int_{z=c\Delta_i'}^{(c+1)\Delta_i'} H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |t\right) dz \\
&= \frac{1}{\Delta_i'} \int_{z=\Delta_i'}^{\Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor} H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |t\right) 1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right) dz \qquad (30)
\end{aligned}$$

since $1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right)$ equals one for all $z < \Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor$. Moreover,

$$\begin{aligned}
& H\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \Delta_i'|t\right) \left(\frac{D}{\bar{y}\Delta_i'} - \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor\right) 1\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \le \frac{1}{\Delta_i'} - 1\right) \\
&= \frac{1}{\Delta_i'} \int_{z=\Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor}^{\frac{D}{\bar{y}}} H\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \Delta_i'|t\right) 1\left(\left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor \le \frac{1}{\Delta_i'} - 1\right) dz \qquad (31) \\
&= \frac{1}{\Delta_i'} \int_{z=\Delta_i' \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor}^{\frac{D}{\bar{y}}} H\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor \Delta_i'|t\right) 1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right) dz \qquad (32)
\end{aligned}$$

where the last equality holds for two reasons. First, $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor = \left\lfloor \frac{D}{\bar{y}\Delta_i'} \right\rfloor$ in the interval of integration in line (31). Second, since $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor$ and $\frac{1}{\Delta_i'}$ are both integers, $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}$ if and only if $\left\lfloor \frac{z}{\Delta_i'} \right\rfloor \le \frac{1}{\Delta_i'} - 1$. Combining (29), (30), and (32), we obtain

$$v^{H\bar{y}i}(D,t) = D - \bar{y} \int_{z=\Delta_i'}^{\frac{D}{\bar{y}}} H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |t\right) 1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right) dz.$$

As $1\left(\left\lfloor \frac{z}{\Delta_i'} \right\rfloor < \frac{1}{\Delta_i'}\right)$ equals one except possibly at the upper endpoint of the integral, it can be omitted. Finally, the lower limit of integration can be reduced to zero since $H(0|t) = 0$. Q.E.D.$_{\text{Claim 1}}$

We next show that the first derivatives with respect to D and t of the expected payout $v^{H\bar{y}i}(D,t)$ in model i are bounded above and below (parts 1 and 2) and converge uniformly to the respective first derivatives of the expected payout $v^{H\bar{y}}(D,t)$ in the continuous case (parts 3 and 4):

Claim 2. Assume L-H. Define

$$\Omega^{H\bar{y}i}\left(t',t'',D',D''\right) = v^{H\bar{y}i}\left(D',t'\right) - v^{H\bar{y}i}\left(D'',t''\right) - \left[v^{H\bar{y}}\left(D',t'\right) - v^{H\bar{y}}\left(D'',t''\right)\right].$$

1. For all $D \in (\bar{y}\Delta_i', \bar{y}]$, and all t', t'' in S_i such that $t' > t''$,

$$\max\left\{0, k_0 \frac{(D')^2 (3\bar{y} - 2D')}{6\bar{y}^2}\right\} < \frac{v^{H\bar{y}i}(D,t') - v^{H\bar{y}i}(D,t'')}{t' - t''} < \bar{y}k_1 \min\left\{\frac{D}{\bar{y}}, 1 - \Delta_i'\right\},$$

where $D' = D - 2\bar{y}\Delta_i'$.

2. For all D', D'' in $[0, \bar{y}]$ such that $D' > D''$ and for all $t \in S_i$,

$$k_1\left(1 - \frac{D''}{\bar{y}} + \Delta_i\right) > \frac{v^{H\bar{y}i}(D',t) - v^{H\bar{y}i}(D'',t)}{D' - D''}$$
$$> k_0\left(1 - \Delta_i'\left\lceil\frac{D'}{\bar{y}\Delta_i'}\right\rceil + \Delta_i'\right) \geq k_0\left(1 - \frac{D'}{\bar{y}}\right).$$

3. For all $\varepsilon > 0$ there exists an i^* such that if $i > i^*$, then for all $G \in \mathscr{G}$, $H \in \mathscr{H}$, $\bar{y} \in (0, \mathbf{y}]$, D in $[0, \bar{y}]$ and t', t'' in S_i such that $t' > t''$, $\left|\Omega^{H\bar{y}i}(t', t'', D, D)\right| < \varepsilon(t' - t'')$.

4. For all $\varepsilon > 0$ there exists an i^* such that if $i > i^*$, then for all $G \in \mathscr{G}$, $H \in \mathscr{H}$, $\bar{y} \in (0, \mathbf{y}]$, $t \in S_i$, and D', D'' in $[0, \bar{y}]$ such that $D' > D''$, $\left|\Omega^{H\bar{y}i}(t, t, D', D'')\right| < \varepsilon(D' - D'')$.

Proof of Claim 2. Part 1. By (28),

$$v^{H\bar{y}i}(D,t') - v^{H\bar{y}i}(D,t'') = \bar{y}\int_{z=0}^{\frac{D}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t''\right) - H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t'\right)\right]dz$$

which by L-H is less than $\bar{y}k_1(t' - t'')\left(\frac{D}{\bar{y}} - \Delta_i'\right)$ and at least

$$\bar{y}k_0(t' - t'')(\Delta_i')^2\int_{z=0}^{\frac{D}{\bar{y}}}\left\lfloor\frac{z}{\Delta_i'}\right\rfloor\left(\frac{1}{\Delta_i'} - \left\lfloor\frac{z}{\Delta_i'}\right\rfloor\right)dz, \tag{33}$$

which is zero if $D \leq \bar{y}\Delta_i'$ and positive otherwise. Let $c = \left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor$ where $D' = D - 2\bar{y}\Delta_i'$. Recall $N_i' = 1/\Delta_i'$. Hence, for $D > \bar{y}\Delta_i'$, the integral in (33) is at least

$$\int_{z=0}^{c\Delta_i'}\left\lfloor\frac{z}{\Delta_i'}\right\rfloor\left(N_i' - \left\lfloor\frac{z}{\Delta_i'}\right\rfloor\right)dz = \Delta_i'\sum_{n=1}^{c-1}n(N_i' - n) = \Delta_i'\frac{c(c-1)(3N_i' - 2c + 1)}{6}$$
$$\geq \Delta_i'\frac{\left(\frac{D'}{\bar{y}\Delta_i'} + 1\right)\frac{D'}{\bar{y}\Delta_i'}\left(3N_i' - 2\frac{D'}{\bar{y}\Delta_i'} - 1\right)}{6} \text{ as } c \geq \frac{D'}{\bar{y}\Delta_i'} + 1$$
$$> \frac{(D')^2(3\bar{y} - 2D')}{6\bar{y}^3(\Delta_i')^2} \text{ as } D' \leq \bar{y}(1 - 2\Delta_i').$$

This proves the result.

Part 2. By L-H and (22),

$$
\begin{aligned}
& v^{H\overline{y}i}\left(D',t\right)-v^{H\overline{y}i}\left(D'',t\right) \\
& =\sum_{c=1}^{1/\Delta_i'}\left[\min\left\{D',\overline{y}c\Delta_i'\right\}-\min\left\{D'',\overline{y}c\Delta_i'\right\}\right]\left[H\left(c\Delta_i'|t\right)-H\left(\left(c-1\right)\Delta_i'|t\right)\right]
\end{aligned}
$$

but

$$
\begin{aligned}
\min\left\{D',\overline{y}c\Delta_i'\right\}-\min\left\{D'',\overline{y}c\Delta_i'\right\} &= \begin{cases} 0 & \text{if} \quad c\leq\frac{D''}{\overline{y}\Delta_i'} \\ \overline{y}c\Delta_i'-D'' & \text{if} \quad \frac{D''}{\overline{y}\Delta_i'}<c<\frac{D'}{\overline{y}\Delta_i'} \\ D'-D'' & \text{if} \quad c\geq\frac{D'}{\overline{y}\Delta_i'} \end{cases} \\
&= \begin{cases} 0 & \text{if} \quad c\leq\left\lfloor\frac{D''}{\overline{y}\Delta_i'}\right\rfloor \\ \overline{y}c\Delta_i'-D'' & \text{if} \quad \left\lfloor\frac{D''}{\overline{y}\Delta_i'}\right\rfloor<c<\left\lceil\frac{D'}{\overline{y}\Delta_i'}\right\rceil \\ D'-D'' & \text{if} \quad c\geq\left\lceil\frac{D'}{\overline{y}\Delta_i'}\right\rceil \end{cases}
\end{aligned}
$$

so

$$
\begin{aligned}
v^{H\overline{y}i}\left(D',t\right)-v^{H\overline{y}i}\left(D'',t\right) &\geq \sum_{c=\left\lceil\frac{D'}{\overline{y}\Delta_i'}\right\rceil}^{1/\Delta_i'}\left(D'-D''\right)\left[H\left(c\Delta_i'|t\right)-H\left(\left(c-1\right)\Delta_i'|t\right)\right] \\
&> \left(D'-D''\right)k_0\Delta_i'\left(\frac{1}{\Delta_i'}-\left\lceil\frac{D'}{\overline{y}\Delta_i'}\right\rceil+1\right) \\
&\geq \left(D'-D''\right)k_0\left(1-D'/\overline{y}\right).
\end{aligned}
$$

Moreover, $v^{H\overline{y}i}\left(D',t\right)-v^{H\overline{y}i}\left(D'',t\right)$ is at most

$$
\begin{aligned}
& \sum_{c=\left\lceil\frac{D''}{\overline{y}\Delta_i'}\right\rceil}^{1/\Delta_i'}\left(D'-D''\right)\left[H\left(c\Delta_i'|t\right)-H\left(\left(c-1\right)\Delta_i'|t\right)\right] \\
& <\left(D'-D''\right)k_1\Delta_i'\left(\frac{1}{\Delta_i'}-\left\lceil\frac{D''}{\overline{y}\Delta_i'}\right\rceil+1\right)\leq\left(D'-D''\right)k_1\left(1-D''/\overline{y}+\Delta_i\right).
\end{aligned}
$$

Part 3. Let i be large enough that $\mathbf{y}k_2\Delta_i' < \varepsilon$. Then

$$\begin{aligned}\left|\Delta\left(t',t'',D,D\right)\right| &= \left|\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\left[\min\left(D,\bar{y}c\Delta_i'\right)-\min\left(D,\bar{y}z\right)\right]d\left[H\left(z|t'\right)-H\left(z|t''\right)\right]\right| \\ &\leq \sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\left|\min\left(D,\bar{y}c\Delta_i'\right)-\min\left(D,\bar{y}z\right)\right|d\left[H\left(z|t'\right)-H\left(z|t''\right)\right] \\ &\leq \sum_{c=1}^{1/\Delta_i'}\bar{y}k_2\left(\Delta_i'\right)^2\left|t'-t''\right| = \bar{y}k_2\Delta_i'\left|t'-t''\right| < \varepsilon\left|t'-t''\right|\end{aligned}$$

since $|\min(D,\bar{y}c\Delta_i') - \min(D,\bar{y}z)| < \bar{y}\Delta_i'$ and by L-H, $\left|\int_{z=(c-1)\Delta_i'}^{c\Delta_i'} d\left[H(z|t') - H(z|t'')\right]\right| \leq k_2\Delta_i'|t'-t''|$.

Part 4. As $\bar{y} > 0$, $\left|\Omega^{H\bar{y}i}(t,t,D',D'')\right| = \left|\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH(z|t)\right|$ where $\eta(\zeta',\zeta'',\zeta_0,\zeta_1) = \max\{\zeta_0,\min\{\zeta_1,\zeta'\}\} - \max\{\zeta_0,\min\{\zeta_1,\zeta''\}\}$.

Remark 1. $\eta(\zeta',\zeta'',\zeta_0,\zeta_1)$ lies in $[0,\zeta'-\zeta'']$ if $\zeta e' \geq \zeta e''$ and is zero if either

$$\max\left\{\zeta',\zeta''\right\} \leq \zeta e_0$$

or $\min\{\zeta',\zeta''\} \geq \zeta e_1$.

Since z lies in $[(c-1)\Delta_i', c\Delta_i']$, the integrand $\eta(\bar{y}c\Delta_i',\bar{y}z,D'',D')$ is zero if either $c \leq \frac{D''}{\bar{y}\Delta_i'}$ or $c \geq \frac{D'}{\bar{y}\Delta_i'}+1$. Hence,

$$\begin{aligned}&\sum_{c=1}^{1/\Delta_i'}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH(z|t) \\ &= \sum_{c=\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil}^{\min\left\{1,\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\right\}}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH(z|t).\end{aligned} \tag{34}$$

Since $c\Delta_i' \geq z$ in each integral, the integrands $\eta(\bar{y}c\Delta_i',\bar{y}z,D'',D')$ are all nonnegative, so we may dispense with the absolute value signs. The first summand, which corresponds to

$c=\lceil D''/\bar{y}\Delta_i'\rceil$, is

$$\begin{aligned}
&\int_{z=\left(\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-1\right)\Delta_i'}^{\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil\Delta_i'}\eta\left(\bar{y}\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil\Delta_i',\bar{y}z,D'',D'\right)dH(z|t)\\
&=\left[\begin{array}{c}\eta\left(\bar{y}\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil\Delta_i',D'',D'',D'\right)\left[H\left(\frac{D''}{\bar{y}}|t\right)-H\left(\left(\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-1\right)\Delta_i'|t\right)\right]\\
+\int_{z=D''/\bar{y}}^{\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil\Delta_i'}\eta\left(\bar{y}\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil\Delta_i',\bar{y}z,D'',D'\right)dH(z|t)\end{array}\right]\\
&\leq\left[\begin{array}{c}\bar{y}k_1(\Delta_i')^2\left(\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-\frac{D''}{\bar{y}\Delta_i'}\right)\left[\frac{D''}{\bar{y}\Delta_i'}-\left(\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-1\right)\right]\\
+\bar{y}k_1(\Delta_i')^2\left(\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-\frac{D''}{\bar{y}\Delta_i'}\right)^2\end{array}\right]=\bar{y}k_1\left(\Delta_i'\right)^2\left(\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-\frac{D''}{\bar{y}\Delta_i'}\right),
\end{aligned}$$

by L-H. There are now two cases.

Case 1: $\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\leq\frac{1}{\Delta_i'}$. The last summand in (34), which corresponds to $c=\lfloor D'/\bar{y}\Delta_i'+1\rfloor$, is[3]

$$\begin{aligned}
&\int_{z=\left(\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor-1\right)\Delta_i'}^{\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\Delta_i'}\eta\left(\bar{y}\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\Delta_i',\bar{y}z,D'',D'\right)dH(z|t)\\
&=\int_{z=\left(\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor-1\right)\Delta_i'}^{D'/\bar{y}}\eta\left(\bar{y}\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\Delta_i',\bar{y}z,D'',D'\right)dH(z|t)\\
&+\eta\left(\bar{y}\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\Delta_i',D',D'',D'\right)\left[H\left(\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\Delta_i'|t\right)-H\left(\frac{D'}{\bar{y}}|t\right)\right] &(35)\\
&\leq\bar{y}k_1\left(\Delta_i'\right)^2\left[\frac{D'}{\bar{y}\Delta_i'}+1-\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\right]. &(36)
\end{aligned}$$

The remainder of the sum in (34) is

$$\sum_{c=\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil+1}^{\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor-1}\int_{z=(c-1)\Delta_i'}^{c\Delta_i'}\eta\left(\bar{y}c\Delta_i',\bar{y}z,D'',D'\right)dH(z|t)\leq\bar{y}k_1\left(\Delta_i'\right)^2\left(\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor-\left\lceil\frac{D''}{\bar{y}\Delta_i'}\right\rceil-1\right).$$

Collecting terms, $\left|\Omega^{H\bar{y}i}(t,t,D',D'')\right|\leq k_1\Delta_i'(D'-D'')$. Now take i^* large enough that $k_1\Delta_{i^*}'<\varepsilon$.

[3]By Remark 1, line (35) is zero as $\bar{y}\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor\Delta_i'-D'=\bar{y}\Delta_i'\left(\left\lfloor\frac{D'}{\bar{y}\Delta_i'}+1\right\rfloor-\frac{D'}{\bar{y}\Delta_i'}\right)>0$. The inequality in line (36) then follows from Lipschitz-H.

Case 2: $\left\lfloor \frac{D'}{\bar{y}\Delta_i'} + 1 \right\rfloor > \frac{1}{\Delta_i'}$ or, equivalently, $\left\lfloor \frac{D'}{\bar{y}\Delta_i'} \right\rfloor > \frac{1}{\Delta_i'} - 1$. The final sum on the right hand side of (34) then corresponds to $c = 1/\Delta_i'$. Moreover,

$$\sum_{c=\left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil + 1}^{1/\Delta_i'} \int_{z=(c-1)\Delta_i'}^{c\Delta_i'} \eta\left(\bar{y}c\Delta_i', \bar{y}z, D'', D'\right) dH(z|t) \leq \bar{y}k_1\left(\Delta_i'\right)^2 \left[\frac{1}{\Delta_i'} - \left\lceil \frac{D''}{\bar{y}\Delta_i'} \right\rceil - 1\right].$$

Thus,

$$\begin{aligned} \left|\Omega^{H\bar{y}i}\left(t,t,D',D''\right)\right| &\leq \bar{y}k_1\left(\Delta_i'\right)^2 \left[\frac{1}{\Delta_i'} - 1 - \frac{D''}{\bar{y}\Delta_i'}\right] < \bar{y}k_1\left(\Delta_i'\right)^2 \left[\left\lfloor \frac{D'}{\bar{y}\Delta_i'} \right\rfloor - \frac{D''}{\bar{y}\Delta_i'}\right] \\ &\leq \bar{y}k_1\left(\Delta_i'\right)^2 \left[\frac{D'}{\bar{y}\Delta_i'} - \frac{D''}{\bar{y}\Delta_i'}\right] = k_1\Delta_i'\left(D' - D''\right) \end{aligned}$$

as before. Q.E.D.$_{\text{Claim 2}}$

For all $D, D' \in [0, \bar{y}]$ and all $t \in S_i \backslash \{1\}$, define the difference quotients of $v^{H\bar{y}i}(D,t)$ with respect to D and t:

$$\Delta_1^{H\bar{y}i}\left(D,D',t\right) = \frac{v^{H\bar{y}i}(D,t) - v^{H\bar{y}i}(D',t)}{D - D'} \text{ and} \tag{37}$$

$$\Delta_2^{H\bar{y}i}(D,t) = \frac{v^{H\bar{y}i}(D,t+\Delta_i) - v^{H\bar{y}i}(D,t)}{\Delta_i}. \tag{38}$$

By parts 1 and 2 of Claim 2, if $D < D'$, then

$$\Delta_1^{H\bar{y}i}\left(D,D',t\right) \in \left(k_0\left(1 - \Delta_i'\left\lceil \frac{D'}{\bar{y}\Delta_i'} \right\rceil + \Delta_i'\right), k_1\left(1 - \frac{D}{\bar{y}} + \Delta_i\right)\right) \subset (0,\infty) \tag{39}$$

while if $D > \bar{y}\Delta_i'$,

$$\Delta_2^{H\bar{y}i}(D,t) \in \left(0, k_1 \min\left\{D, \bar{y}\left(1 - \Delta_i'\right)\right\}\right). \tag{40}$$

By (24), for any $D \in [0,\bar{y}]$, the partial derivatives of $v^{H\bar{y}}(D,t)$ are given by

$$v_2^{H\bar{y}}(D,t) = -\bar{y}\int_{z=0}^{D/\bar{y}} \frac{\partial H(z|t)}{\partial t} dz \text{ and} \tag{41}$$

$$v_1^{H\bar{y}}(D,t) = 1 - H\left(\frac{D}{\bar{y}}|t\right) = \int_{z=D/\bar{y}}^{1} \frac{\partial H(z|t)}{\partial z} dz. \tag{42}$$

For all $D \in [0,\bar{y}]$ and $t \in [0,1]$,

$$\frac{\partial v_2^{H\bar{y}}(D,t)}{\partial D} = -\frac{\partial}{\partial t} H\left(\frac{D}{\bar{y}}|t\right) \in \left[k_0 \frac{D}{\bar{y}}\left(1 - \frac{D}{\bar{y}}\right), k_1\right) \tag{43}$$

by (41) and L-H;

$$\left|\frac{\partial v_2^{H\bar{y}}(D,t)}{\partial t}\right| \leq k_2 D \tag{44}$$

by (41) and L-H;

$$\frac{\partial v_1^{H\bar{y}}(D,t)}{\partial D} = -\frac{\partial}{\partial D}H\left(\frac{D}{\bar{y}}|t\right) \in \left(-\frac{k_1}{\bar{y}}, -\frac{k_0}{\bar{y}}\right) \tag{45}$$

by (42) and L-H;

$$\frac{k_0 D^2 [3\bar{y}-2D]}{6\bar{y}^2} < v_2^{H\bar{y}}(D,t) < k_1 D \tag{46}$$

by (41) and L-H;

$$k_0\left(1-\frac{D}{\bar{y}}\right) < v_1^{H\bar{y}}(D,t) < k_1\left(1-\frac{D}{\bar{y}}\right) \tag{47}$$

by (42) and L-H; for all $t' \in [0,1]$,

$$\left|v_1^{H\bar{y}}(D,t) - v_1^{H\bar{y}}(D,t')\right| = \left|H\left(\frac{D}{\bar{y}}|t\right) - H\left(\frac{D}{\bar{y}}|t'\right)\right| \leq k_1 \left|t-t'\right| \tag{48}$$

by (42) and L-H; for all $D' \in [0,\bar{y}]$,

$$\left|v_1^{H\bar{y}}(D,t) - v_1^{H\bar{y}}(D',t)\right| = \left|H\left(\frac{D}{\bar{y}}|t\right) - H\left(\frac{D'}{\bar{y}}|t\right)\right| < \frac{k_1}{\bar{y}}\left|D-D'\right| \tag{49}$$

by L-H. By (46) and (47),

$$0 \leq \frac{k_0 D^2(3\bar{y}-2D)}{6\bar{y}k_1(\bar{y}-D)} < \frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} < \frac{k_1\bar{y}}{k_0}\left(\frac{D}{\bar{y}-D}\right), \tag{50}$$

and

$$\frac{6\bar{y}k_1(\bar{y}-D)}{k_0 D^2(3\bar{y}-2D)} > \frac{v_1^{H\bar{y}}(D,t)}{v_2^{H\bar{y}}(D,t)} > \frac{k_0}{k_1\bar{y}}\left(\frac{\bar{y}-D}{D}\right). \tag{51}$$

By (43), (45), (46), (47), and (51),

$$\frac{\partial}{\partial D}\left(\frac{v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)}\right) = \frac{\frac{\partial}{\partial D}v_2^{H\bar{y}}(D,t)}{v_1^{H\bar{y}}(D,t)} - \frac{v_2^{H\bar{y}}(D,t)\,\frac{\partial}{\partial D}v_1^{H\bar{y}}(D,t)}{\left[v_1^{H\bar{y}}(D,t)\right]^2} \in \left[\gamma_1^{\bar{y}}(D), \gamma_2^{\bar{y}}(D)\right] \tag{52}$$

where

$$\gamma_1^{\bar{y}}(D) = \frac{k_0 D}{k_1\bar{y}}\left[1+\frac{Dk_0(3\bar{y}-2D)}{6k_1(\bar{y}-D)^2}\right] \geq 0 \tag{53}$$

and

$$\gamma_2^{\bar{y}}(D) = \frac{k_1 \bar{y}}{k_0(\bar{y} - D)} \left[1 + \frac{D k_1}{k_0(\bar{y} - D)} \right] > 0. \tag{54}$$

Note that $\gamma_1^{\bar{y}}(D)$ lies in $(0,\infty)$ if $D \in (0,\bar{y})$, is zero if $D = 0$, and is ∞ if $D = \bar{y}$. Moreover, $\gamma_2^{\bar{y}}(D)$ is increasing in D, lies in $(0,\infty)$ if $D \in [0,\bar{y})$, and is ∞ if $D = \bar{y}$. Finally, for $0 \le a \le b$,

$$\max_{D \in [a,b]} \gamma_2^{\bar{y}}(D) \le \frac{k_1 \bar{y}}{k_0(\bar{y} - b)} \left[1 + \frac{b k_1}{k_0(\bar{y} - b)} \right], \tag{55}$$

which is positive and, if $b < \bar{y}$, finite.

For any real number ℓ, let $(\ell,\infty]$ and $[\ell,\infty]$ denote the sets $(\ell,\infty) \cup \{\infty\}$ and $[\ell,\infty) \cup \{\infty\}$, respectively. Recall that $f^{H\delta\bar{y}}(D,t)$ is defined in (26). For $\underline{t} \in [0,1]$ and $a \in (0,\bar{y}]$, and $k \in (0,\infty]$, we define the following modification of $\mathrm{CP}^{H\delta\bar{y}}$:

> **Continuous IVP with Parameters** $H, \delta, \bar{y}, \underline{t}, a, k$ ($\mathrm{CP}_{\underline{t}ak}^{H\delta\bar{y}}$). The differential equation
>
> $$\frac{dD_{\underline{t}ak}^{H\delta\bar{y}}}{dt} = \max\left\{ f^{H\delta\bar{y}}\left(D_{\underline{t}ak}^{H\delta\bar{y}}(t), t \right), -k \right\} \tag{56}$$
>
> with $D_{\underline{t}ak}^{H\delta\bar{y}} : [\underline{t}, 1] \to \Re$, together with the initial value $D_{\underline{t}ak}^{H\delta\bar{y}}(\underline{t}) = a$.

Clearly, any $D_{0\bar{y}\infty}^{H\delta\bar{y}}$ that solves $\mathrm{CP}_{0\bar{y}\infty}^{H\delta\bar{y}}$ must also be a solution $D^{H\delta\bar{y}}$ to $\mathrm{CP}^{H\delta\bar{y}}$ and vice-versa.

Claim 3. Consider any $\underline{t} \in [0,1]$, H in $\mathscr{H}$, $a \in (0,\bar{y}]$, and $k \in (0,\infty]$.

1. If either $a < \bar{y}$ or $k < \infty$ (or both), then there exists a unique solution to $\mathrm{CP}_{\underline{t}ak}^{H\delta\bar{y}}$, which is decreasing and differentiable in t and takes values in $(0,a]$.

2. Let $\widetilde{\delta}, \widehat{\delta} \in (0,1)$ and $\widetilde{H}, \widehat{H} \in \mathscr{H}$ satisfy $f^{\widetilde{H}\widetilde{\delta}\bar{y}} \le f^{\widehat{H}\widehat{\delta}\bar{y}}$, and let $a' \in (0,a]$ and $k' \in [k,\infty]$. Suppose there exist (possibly nonunique) solutions $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\bar{y}}$ and $D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\bar{y}}$ to $\mathrm{CP}_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\bar{y}}$ and $\mathrm{CP}_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\bar{y}}$, respectively. Then $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\bar{y}}(t) \ge D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\bar{y}}(t)$ for all $t \in [\underline{t}, 1]$.

3. If either $a < \bar{y}$ or $k < \infty$ (or both), then the function $D_{\underline{t}ak}^{H\delta\bar{y}}$ is Lipschitz continuous in t with Lipschitz constant $\min\{k, k_a\}$ where

$$k_a = \frac{k_1 \bar{y} a}{(1-\delta) k_0 (\bar{y} - a)}. \tag{57}$$

4. If $a < \bar{y}$, then for all $t \in [\underline{t}, 1]$, $D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}}_{\underline{t}a\infty}(t) \in [0, \bar{y} - a]$.

PROOF OF CLAIM 3. Part 1. We first show that $\max\left\{f^{H\delta\bar{y}}(D,t), -k\right\}$ is (a) continuous in $t \in [0,1]$ and (b) Lipschitz continuous in $D \in [0,a]$. Since max is a Lipschitz-continuous function, it suffices to show that $f^{H\delta\bar{y}}(D,t)$ has properties (a) and (b) whenever $f^{H\delta\bar{y}}(D,t) \geq -k$. If $k = \infty$, then $D \leq a < \bar{y}$. If instead $k < \infty$ and $f^{H\delta\bar{y}}(D,t) \geq -k$ then by (50), $\frac{k_0 D^2(3\bar{y}-2D)}{6\bar{y}k_1(\bar{y}-D)} < (1-\delta)k$; rearranging, $\frac{D^2}{\bar{y}-D} < \frac{6\bar{y}k_1(1-\delta)k}{k_0(3\bar{y}-2D)} \leq \frac{6k_1(1-\delta)k}{k_0}$ (since $D \leq \bar{y}$). Since $\frac{D^2}{\bar{y}-D}$ is continuous and increasing in D and approaches ∞ as $D \uparrow \bar{y}$, there is a constant $a' < \bar{y}$ such that $D \leq a'$ for any D and t satisfying $f^{H\delta\bar{y}}(D,t) \geq -k$. Collecting cases, $D \leq b = \max\{a, a'\} < \bar{y}$. Hence $1 - H\left(\frac{D}{\bar{y}}|t\right) \geq 1 - H\left(\frac{b}{\bar{y}}|t\right) > 0$; as H is continuously differentiable in t, $f^{H\delta\bar{y}}(D,t)$ is continuous in t and thus satisfies (a). And by (52), (53), and (55), $\left|\frac{\partial}{\partial D} f^{H\delta\bar{y}}(D,t)\right|$ is at most $\frac{k_1\bar{y}}{k_0(\bar{y}-b)(1-\delta)}\left[1 + \frac{bk_1}{k_0(\bar{y}-b)}\right]$, whence $f^{H\delta\bar{y}}(D,t)$ satisfies (b). By the Picard-Lindelöf theorem, there thus exists a unique solution $D^{H\delta\bar{y}}_{\underline{t}ak}$ to $\mathrm{CP}^{H\delta\bar{y}}_{\underline{t}ak}$. The solution $D^{H\delta\bar{y}}_{\underline{t}ak}$ is differentiable in t since the right hand side of (56) is finite. Finally, $f^{H\delta\bar{y}}(D,t) < 0$ for all $D \in (0, \bar{y}]$ and $f^{H\delta\bar{y}}(0,t) = 0$. Hence, $D^{H\delta\bar{y}}_{\underline{t}ak}(t)$ is decreasing in t until and unless it hits zero, where it remains. Thus, by (50), $D^{H\delta\bar{y}}_{\underline{t}ak}(t) \leq a$ for all $t \in [\underline{t}, 1]$. Finally, $D^{H\delta\bar{y}}_{\underline{t}ak}(t) > 0$ by the following lemma. Let $D^{H\delta\bar{y}}_{\underline{t}ak}(t)$ first reach its minimum value of $\underline{D} \geq 0$ at $t = \bar{t} > \underline{t}$.

Lemma 1. *The minimum face value $\underline{D}$ is nonzero.*

PROOF OF LEMMA 1: For $t \in [\underline{t}, \bar{t}]$, the function $D^{H\delta\bar{y}}_{\underline{t}ak}(t)$ has a strictly decreasing inverse $t^{H\delta\bar{y}}_{\underline{t}ak}$ that satisfies the following inverse problem:

> **INVERSE CONTINUOUS IVP WITH PARAMETERS** $H, \bar{y}, \underline{t}, a, k$ ($\mathrm{ICP}^{H\delta\bar{y}}_{\underline{t}ak}$). The differential equation
>
> $$\frac{dt^{H\delta\bar{y}}_{\underline{t}ak}}{dD} = -\max\left\{(1-\delta)\frac{v_1^{H\bar{y}}\left(D, t^{H\delta\bar{y}}_{\underline{t}ak}(D)\right)}{v_2^{H\bar{y}}\left(D, t^{H\delta\bar{y}}_{\underline{t}ak}(D)\right)}, \frac{1}{k}\right\} \tag{58}$$
>
> with $t^{H\delta\bar{y}}_{\underline{t}ak} : [\underline{D}, a] \to [\underline{t}, \bar{t}]$, together with the terminal value $t^{H\delta\bar{y}}_{\underline{t}ak}(a) = \underline{t}$.

By (51), $\frac{dt_{\underline{t}ak}^{H\delta\bar{y}}}{dD} \leq -(1-\delta)\frac{k_0}{k_1\bar{y}}\left(\frac{\bar{y}-D}{D}\right)$. Hence, as $t_{\underline{t}ak}^{H\delta\bar{y}}(a) = \underline{t}$,

$$\begin{aligned}
\bar{t} = t_{\underline{t}ak}^{H\delta\bar{y}}(\underline{D}) &= t_{\underline{t}ak}^{H\delta\bar{y}}(a) - \int_{D=\underline{D}}^{a} \frac{dt_{\underline{t}ak}^{H\delta\bar{y}}}{dD} dD \\
&\geq \underline{t} + \frac{(1-\delta)k_0}{k_4\bar{y}} \int_{D=\underline{D}}^{\bar{y}} \left(\frac{\bar{y}-D}{D}\right) dD = \underline{t} + \frac{(1-\delta)k_0}{k_4} F\left(\frac{\underline{D}}{\bar{y}}\right),
\end{aligned}$$

where $F(x)$ denotes $x - \ln x - 1$, which is finite and differentiable for all finite $x > 0$. F is decreasing in $x \in (0,1)$: $F'(x) = 1 - 1/x < 0$. Thus, $F\left(\frac{\underline{D}}{\bar{y}}\right) < \frac{k_4}{(1-\delta)k_0}(\bar{t} - \underline{t})$, whence $\underline{D} > \bar{y}F^{-1}\left(\frac{k_4}{(1-\delta)k_0}(\bar{t}-\underline{t})\right)$. Moreover, $F(1) = 0$ and $\lim_{x\downarrow 0} F(x) = \infty$, so the inverse F^{-1} is decreasing in $x \in (0,\infty)$ and satisfies $F^{-1}(0) = 1$ and $\lim_{x\to\infty} F^{-1}(x) = 0$. Hence, as $\bar{t} - \underline{t} > 0$, $\underline{D} > 0$. Q.E.D.$_{\text{Lemma 1}}$

Part 2. Suppose not. Since $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\widehat{y}}(\underline{t}) = a \geq a' = D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\widetilde{y}}(\underline{t})$ and both solutions are continuous in t, there must be $\underline{t} \leq t_0 < t_1 \leq 1$ such that $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\widehat{y}}(t_0) = D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\widetilde{y}}(t_0)$ and, for all $t \in (t_0, t_1]$, $D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\widehat{y}}(t) < D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\widetilde{y}}(t)$. Hence, by (56),

$$0 < D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\widetilde{y}}(t_1) - D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\widehat{y}}(t_1) = \int_{t=t_0}^{t_1} \left[\begin{array}{c} \max\left\{ f^{\widetilde{H}\widetilde{\delta}\widetilde{y}}\left(D_{\underline{t}a'k'}^{\widetilde{H}\widetilde{\delta}\widetilde{y}}(t), t\right), -k' \right\} \\ -\max\left\{ f^{\widehat{H}\widehat{\delta}\widehat{y}}\left(D_{\underline{t}ak}^{\widehat{H}\widehat{\delta}\widehat{y}}(t), t\right), -k \right\} \end{array} \right] dt$$

which is impossible: since $f^{H\delta\bar{y}}(D,t)$ is decreasing in D and $k' \geq k$, the integrand is nonpositive for all t in $(t_0, t_1]$.

Part 3 follows from (50) and the fact that $D_{\underline{t}ak}^{H\delta\bar{y}} \leq a$.

Part 4. By part 2, $\Gamma_a(t) \stackrel{d}{=} D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \geq 0$. By (26),

$$\Gamma_a'(t) = \max\left\{ \begin{array}{c} \left[f^{H\delta\bar{y}}\left(D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t), t\right) - f^{H\delta\bar{y}}\left(D_{\underline{t}a\infty}^{H\delta\bar{y}}(t), t\right)\right], \\ -k_a - f^{H\delta\bar{y}}\left(D_{\underline{t}a\infty}^{H\delta\bar{y}}(t), t\right) \end{array} \right\}.$$

Both entries in the max are nonpositive by (50), (26), and the fact that $D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) \leq a$. Accordingly, for all $t \in [\underline{t}, 1]$, $\left|D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right| \leq \Gamma_a(\underline{t}) = \bar{y} - a$. Q.E.D.$_{\text{Claim 3}}$

Before addressing the discrete case, we prove some useful bounds:

Claim 4. Let $w, w', \zeta e, \zeta e'$ be in $(0, \bar{y}]$ and satisfy $w' \geq w$, $\zeta e' \geq \zeta e$, $w > \zeta e$, and $w' > \zeta e'$. Then

$$0 \leq \Delta_2^{H\bar{y}i}\left(\zeta e', t\right) - \Delta_2^{H\bar{y}i}\left(\zeta e, t\right) \leq k_1\left(\zeta e' - \zeta e\right). \tag{59}$$

Moreover, if $\min\{w, w', \zeta, \zeta'\} > \bar{y}\Delta_i'$ then

$$0 \geq \Delta_1^{H\bar{y}i}\left(\zeta e', w', t\right) - \Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right) \geq -k_1\left[\frac{\max\{w' - w, \zeta e' - \zeta e\}}{\bar{y}} + \Delta_i'\right] \tag{60}$$

and

$$\frac{\Delta_2^{H\bar{y}i}\left(\zeta e, t\right)}{\Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right)} \in \left(0, \frac{k_1 \zeta e}{k_0\left(1 - \Delta_i'\left\lceil \frac{w}{\bar{y}\Delta_i'}\right\rceil + \Delta_i'\right)}\right) \subset (0, \infty). \tag{61}$$

PROOF OF CLAIM 4. By (28), for any $\zeta e \in [0, \bar{y}]$ and $t \in S_i \setminus \{1\}$,

$$\begin{aligned} &v^{H\bar{y}i}\left(\zeta e, t + \Delta_i\right) - v^{H\bar{y}i}\left(\zeta e, t\right) \\ &= \bar{y}\int_{z=0}^{\frac{\zeta e}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t\right) - H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t + \Delta_i\right)\right] dz. \end{aligned} \tag{62}$$

As the integrand is nonnegative, (62) is nondecreasing in ζe, so $\Delta_2^{H\bar{y}i}\left(\zeta', t\right) - \Delta_2^{H\bar{y}i}\left(\zeta, t\right) \geq 0$. By L-$H$, for any $z \in [0, \bar{y}]$, $H(z|t) - H(z|t + \Delta_i) < k_1\Delta_i$. Equation (59) then follows from (38) and (62).

By (28),

$$\Delta_1^{H\bar{y}i}\left(\zeta e', w', t\right) - \Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right) = \frac{v^{H\bar{y}i}\left(\zeta e', t\right) - v^{H\bar{y}i}\left(w', t\right)}{\zeta e' - w'} - \frac{v^{H\bar{y}i}\left(\zeta e, t\right) - v^{H\bar{y}i}\left(w, t\right)}{\zeta e - w}$$

$$= \frac{\bar{y}}{w - \zeta e}\int_{z=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t\right) dz - \frac{\bar{y}}{w' - \zeta e'}\int_{z=\frac{\zeta e'}{\bar{y}}}^{\frac{w'}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t\right) dz.$$

Define the change of variables $z' = \frac{\zeta}{\bar{y}} + \left(\frac{w-\zeta}{w'-\zeta'}\right)\left(z - \frac{\zeta'}{\bar{y}}\right)$. When $z = \frac{\zeta'}{\bar{y}}$, $z' = \frac{\zeta}{\bar{y}}$, and when $z = \frac{w'}{\bar{y}}$, $z' = \frac{w}{\bar{y}}$. Moreover, $dz = \frac{w'-\zeta'}{w-\zeta}dz'$ and $z = \frac{\zeta'}{\bar{y}} + \left(\frac{w'-\zeta'}{w-\zeta}\right)\left(z' - \frac{\zeta}{\bar{y}}\right)$ which we denote $\psi(z')$. So $\int_{z=\frac{\zeta'}{\bar{y}}}^{\frac{w'}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t\right) dz = \int_{z'=\frac{\zeta}{\bar{y}}}^{\frac{w}{\bar{y}}} H\left(\Delta_i'\left\lfloor\frac{\psi(z')}{\Delta_i'}\right\rfloor | t\right)\frac{w'-\zeta'}{w-\zeta}dz'$. Renaming z' to z and simplifying,

$$\Delta_1^{H\bar{y}i}\left(\zeta e', w', t\right) - \Delta_1^{H\bar{y}i}\left(\zeta e, w, t\right) = \frac{\bar{y}}{w - \zeta e}\int_{z=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor | t\right) - H\left(\Delta_i'\left\lfloor\frac{\psi(z)}{\Delta_i'}\right\rfloor | t\right)\right] dz. \tag{63}$$

We can write

$$z-\psi(z)=\frac{1}{w-\zeta e}\left(\left(z-\frac{\zeta e'}{\bar{y}}\right)[w-\zeta e]-\left(z-\frac{\zeta e}{\bar{y}}\right)\left[w'-\zeta e'\right]\right). \tag{64}$$

As the right hand side is linear in z, it reaches its maximum and minimum at the endpoints of the interval of integration. At the lower endpoint (at $z=\frac{\zeta}{\bar{y}}$), the right hand side of (64) equals $\frac{\zeta-\zeta'}{\bar{y}}$, while at the upper endpoint (at $z=\frac{w}{\bar{y}}$), it equals $\frac{w-w'}{\bar{y}}$. Thus, $-\overline{w}\leq z-\psi(z)\leq -\underline{w}$ where $\underline{w}=\bar{y}^{-1}\min\{w'-w,\zeta'-\zeta\}$ and $\overline{w}=\bar{y}^{-1}\max\{w'-w,\zeta'-\zeta\}$. As $\underline{w}$ and $\overline{w}$ are both nonnegative, $z\leq\psi(z)$, which by (63) establishes the first inequality in (60). Finally, by (63) and L-H,

$$\begin{aligned}
&\Delta_1^{H\bar{y}i}\left(\zeta e',w',t\right)-\Delta_1^{H\bar{y}i}(\zeta e,w,t)\\
&\geq\frac{\bar{y}}{w-\zeta e}\int_{z'=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{\psi(z)-\overline{w}}{\Delta_i'}\right\rfloor|t\right)-H\left(\Delta_i'\left\lfloor\frac{\psi(z)}{\Delta_i'}\right\rfloor|t\right)\right]dz\\
&\geq\frac{\bar{y}k_1}{w-\zeta e}\int_{z'=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[\Delta_i'\left\lfloor\frac{\psi(z)-\overline{w}}{\Delta_i'}\right\rfloor-\Delta_i'\left\lfloor\frac{\psi(z)}{\Delta_i'}\right\rfloor\right]dz\\
&\geq-\frac{\bar{y}k_1}{w-\zeta e}\int_{z'=\frac{\zeta e}{\bar{y}}}^{\frac{w}{\bar{y}}}\left[\overline{w}+\Delta_i'\right]dz=-k_1\left[\overline{w}+\Delta_i'\right].
\end{aligned}$$

This establishes the second inequality in (60). Finally, (61) follows from (39) and (40). Q.E.D.$_{\text{Claim 4}}$

Claim 5. Fix H, $\bar{y}$, and i. Define

$$\phi(D,t)=v^{H\bar{y}i}(D,t+\Delta_i)-\delta v^{H\bar{y}i}(D,t). \tag{65}$$

For any $D\in(\bar{y}\Delta_i',\bar{y}]$ and any $t\in S_i$ there exists a unique solution $D^*=D^*(D)$, which lies in $(\bar{y}\Delta_i',D)$, to

$$\phi(D^*,t)=(1-\delta)v^{H\bar{y}i}(D,t). \tag{66}$$

Moreover, $D^*(D)$ is increasing in D and $D-D^*(D)$ is nondecreasing in D. Finally,

$$\frac{\partial\phi}{\partial D}\geq k_0(1-\delta)\left(1-\frac{D}{\bar{y}}\right). \tag{67}$$

PROOF OF CLAIM 5. We first show three properties.

1. $\phi(\bar{y}\Delta_i', t) < (1-\delta) v^{H\bar{y}i}(D,t)$. Proof: by (28), for all t in S_i, $v^{H\bar{y}i}(\bar{y}\Delta_i', t) = \bar{y}\Delta_i'$. Hence, $\phi(\bar{y}\Delta_i', t) = (1-\delta)\bar{y}\Delta_i'$. Moreover, $v^{H\bar{y}i}(D,t)$ is strictly increasing in $D \in [0,\bar{y}]$ by part 2 of Claim 2, so $v^{H\bar{y}i}(D,t) > \bar{y}\Delta_i'$. The result then follows from (65) and (66).

2. $\phi(D,t) > (1-\delta) v^{H\bar{y}i}(D,t)$. Proof: for $D \in (\bar{y}\Delta_i', \bar{y}]$, $v^{H\bar{y}i}(D,t)$ is increasing in t by part 1 of Claim 2. The result then follows from (65) and (66).

3. ϕ is continuous and increasing in $D \in [0,\bar{y}]$. Proof: $v^{H\bar{y}i}(D,t)$ is continuous in D by (22), so $\phi(D,t)$ also is continuous in D. By (28),

$$\phi(D,t) = (1-\delta)D - \bar{y}\left[\int_{z=0}^{\frac{D}{\bar{y}}}\left[H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t+\Delta_i\right) - \delta H\left(\Delta_i'\left\lfloor\frac{z}{\Delta_i'}\right\rfloor|t\right)\right]dz\right].$$

By L-H, $1-H(z|t) > k_0(1-z)$ for all $z \in [0,1]$ (and thus, substituting $z=0$, $k_0 < 1$), so letting $z_0 = \Delta_i'\left\lfloor\frac{D}{\bar{y}\Delta_i'}\right\rfloor \leq \frac{D}{\bar{y}}$,

$$\begin{aligned}\frac{\partial\phi}{\partial D} &= 1-\delta - [H(z_0|t+\Delta_i) - \delta H(z_0|t)] \geq (1-\delta)[1-H(z_0|t)] \\ &\geq (1-\delta)k_0(1-z_0) \geq k_0(1-\delta)\left(1-\frac{D}{\bar{y}}\right).\end{aligned}$$

This establishes the result as well as equation (67).

Facts 1-3 imply that for any $D \in (\bar{y}\Delta_i', \bar{y}]$, there exists a unique $D^* = D^*(D)$ satisfying (66), and that it lies in $(\bar{y}\Delta_i', D)$. Moreover, $D^*(D)$ is increasing in D.

Finally, let $\widehat{D}_0 > D$ and let $\widehat{D}^* = D^*\left(\widehat{D}_0\right)$. To show that $D - D^*(D)$ is nondecreasing in D, we must show that $\widehat{D}_0 - \widehat{D}^* \geq D - D^*$ or, equivalently, that $\widehat{D}_0 - D \geq \widehat{D}^* - D^*$. By (65),

$$\phi(D,t) = v^{H\bar{y}i}(D,t+\Delta_i) - \delta v^{H\bar{y}i}(D,t) = (1-\delta)v^{H\bar{y}i}(D,t) + v^{H\bar{y}i}(D,t+\Delta_i) - v^{H\bar{y}i}(D,t).$$

Hence, by (38), (59), and (66), and since $\widehat{D}^* > D^*$,

$$\begin{aligned}(1-\delta)\left[v^{H\bar{y}i}\left(\widehat{D}_0,t\right) - v^{H\bar{y}i}(D,t)\right] &= \phi\left(\widehat{D}^*,t\right) - \phi(D^*,t) \\ &\geq (1-\delta)\left[v^{H\bar{y}i}\left(\widehat{D}^*,t\right) - v^{H\bar{y}i}(D^*,t)\right],\end{aligned}$$

whence by part 2 of Claim 2,

$$\frac{\widehat{D}^{*}-D^{*}}{\widehat{D}_{0}-D}\leq\frac{\frac{v^{H\bar{y}i}\left(\widehat{D}_{0},t\right)-v^{H\bar{y}i}(D,t)}{\widehat{D}_{0}-D}}{\frac{v^{H\bar{y}i}\left(\widehat{D}^{*},t\right)-v^{H\bar{y}i}(D^{*},t)}{\widehat{D}^{*}-D^{*}}},$$

which is in $(0,1]$ by (37) and (60). Thus, $\widehat{D}^{*}-D^{*}\leq\widehat{D}_{0}-D$ as claimed. Q.E.D.$_{\text{Claim 5}}$

For any real number ℓ, let $(\ell,\infty]$ and $[\ell,\infty]$ denote the sets $(\ell,\infty)\cup\{\infty\}$ and $[\ell,\infty)\cup\{\infty\}$, respectively. For any constants $\underline{t}\in S_i$, $a\in(0,\bar{y}]$, and $k\in(0,\infty]$, consider the following initial value problem, where $S_i^{\underline{t}}$ denotes the set of types $t\geq\underline{t}$ in S_i:

DISCRETE IVP WITH PARAMETERS $H,\bar{y},\underline{t},a,k$ (DP$_{\underline{t}ak}^{H\delta\bar{y}i}$). The condition

$$D_{\underline{t}ak}^{H\delta\bar{y}i}(t+\Delta_i)=\max\left\{D_{\underline{t}ak}^{H\delta\bar{y}i*}(t+\Delta_i),D_{\underline{t}ak}^{H\delta\bar{y}i}(t)-k\Delta_i\right\} \tag{68}$$

with $D_{\underline{t}ak}^{H\delta\bar{y}i}:S_i^{\underline{t}}\to\Re$, where $D_{\underline{t}ak}^{H\delta\bar{y}i*}(t+\Delta_i)$ is the (by Claim 5) unique solution $D^{*}\in\left(\bar{y}\Delta_i',D_{\underline{t}ak}^{H\delta\bar{y}i}(t)\right)$ to

$$v^{H\bar{y}i}(D^{*},t+\Delta_i)-\delta v^{H\bar{y}i}(D^{*},t)=(1-\delta)\,v^{H\bar{y}i}\left(D_{\underline{t}ak}^{H\bar{y}i}(t),t\right), \tag{69}$$

together with the initial value $D_{\underline{t}ak}^{H\delta\bar{y}i}(\underline{t})=a>\bar{y}\Delta_i'$.

Clearly, any $D_{0\bar{y}\infty}^{H\delta\bar{y}i}$ that solves DP$_{0\bar{y}\infty}^{H\delta\bar{y}i}$ must also be a solution $D^{H\delta\bar{y}i}$ to DP$^{H\delta\bar{y}i}$ and vice-versa.

Claim 6. For any $\underline{t}\in S_i$, $a\in(\bar{y}\Delta_i',\bar{y}]$, and $k\in(0,\infty]$:

1. There exists a unique solution $D_{\underline{t}ak}^{H\delta\bar{y}i}$ to DP$_{\underline{t}ak}^{H\delta\bar{y}i}$. This function is decreasing in $t\in S_i^{\underline{t}}$ and takes values in $(\bar{y}\Delta_i',a]$.

2. Let $a'\in(\bar{y}\Delta_i',a]$ and $k'\in[k,\infty]$. Then $D_{\underline{t}ak}^{H\delta\bar{y}i}(t)\geq D_{\underline{t}a'k'}^{H\delta\bar{y}i}(t)$ for all types $t\in S_i^{\underline{t}}$.

3. Let $k_a^i=\frac{k_1\bar{y}(\bar{y}+a)}{k_0(1-\delta)\left(\bar{y}\left(1-\Delta_i'\right)-a\right)}$. For all types $t\in S_i^{\underline{t}}$, $0>D_{\underline{t}ak}^{H\delta\bar{y}i*}(t+\Delta_i)-D_{\underline{t}ak}^{H\delta\bar{y}i}(t)\geq -\Delta_i k_a$ and hence $0>D_{\underline{t}ak}^{H\delta\bar{y}i}(t+\Delta_i)-D_{\underline{t}ak}^{H\delta\bar{y}i}(t)\geq-\Delta_i\min\{k,k_a\}$.

4. For all $t\in S_i^{\underline{t}}$, $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t)-D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t)\in[0,\bar{y}-a]$.

PROOF OF CLAIM 6. Part 1. Follows from Claim 5.

Part 2. Clearly, $D^{H\delta\bar{y}i}_{\underline{t}ak}(\underline{t}) - D^{H\delta\bar{y}i}_{\underline{t}a'k'}(\underline{t}) = a - a' \geq 0$. And if, for some $t \in S^{\underline{t}}_i \backslash \{1\}$, $D^{H\delta\bar{y}i}_{\underline{t}ak}(t) \geq D^{H\delta\bar{y}i}_{\underline{t}a'k'}(t)$, then $D^{H\delta\bar{y}i}_{\underline{t}ak}(t) - k\Delta_i \geq D^{H\delta\bar{y}i}_{\underline{t}a'k'}(t) - k'\Delta_i$ and, by Claim 5, $D^{H\delta\bar{y}i*}_{\underline{t}ak}(t+\Delta_i) \geq D^{H\delta\bar{y}i*}_{\underline{t}a'k'}(t+\Delta_i)$, so $D^{H\delta\bar{y}i}_{\underline{t}ak}(t+\Delta_i) \geq D^{H\delta\bar{y}i}_{\underline{t}a'k'}(t+\Delta_i)$.

Part 3. Let $D' = D^{H\delta\bar{y}i}_{\underline{t}ak}(t)$, $D'' = D^{H\delta\bar{y}i}_{\underline{t}ak}(t+\Delta_i)$, and $D^* = D^{H\delta\bar{y}i*}_{\underline{t}ak}(t+\Delta_i)$. By Claim 5, $D^* - D' < 0$ and $\min\{D', D'', D^*\} > \bar{y}\Delta'_i$. We will show that $D^* - D' \geq -\Delta_i k_a$ which, by (68), implies $0 > D'' - D' \geq -\Delta_i \min\{k, k_a\}$. The result is trivial when $a = \bar{y}$ since $k_{\bar{y}} = \infty$. Suppose $a < \bar{y}$. By (38), (40), (65), and (66),

$$\begin{aligned}\phi\left(D',t\right) - \phi\left(D^*,t\right) &= \phi\left(D',t\right) - (1-\delta)\, v^{H\bar{y}i}\left(D',t\right) \\ &= v^{H\bar{y}i}\left(D',t+\Delta_i\right) - v^{H\bar{y}i}\left(D',t\right) \leq k_1 D'\Delta_i \leq k_1 a\Delta_i.\end{aligned}$$

But by (67), $\phi\left(D',t\right) - \phi\left(D^*,t\right) \geq \left[D' - D^*\right] k_0 (1-\delta)\left(1 - \frac{a}{\bar{y}}\right)$. Combining the two inequalities and using (57) yields the result.

Part 4. Fix $\underline{t} \in [0,1]$ and $a \in (0,\bar{y})$, whence $k_a \in (0,\infty)$. By part 2, $D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) \geq D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)$ for all types $t \in S^{\underline{t}}_i$. Let $\Gamma_a(t) = D^{H\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t) \geq 0$. As $D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t+\Delta_i) = D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i)$,

$$\Gamma_a(t+\Delta_i) - \Gamma_a(t) = \max\left\{\begin{array}{c} D^{H\delta\bar{y}i*}_{\underline{t}\bar{y}k_a}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - \left[D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right], \\ -k_a\Delta_i - \left[D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right] \end{array}\right\}$$

for any $t < 1$ in $S^{\underline{t}}_i$ by (68). By Claim 5, $D' - D^*(D')$ is nondecreasing in D', so

$$D^{H\delta\bar{y}i*}_{\underline{t}\bar{y}k_a}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - \left[D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right] \leq 0,$$

and by part 3, $D^{H\delta\bar{y}i*}_{\underline{t}a\infty}(t+\Delta_i) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t) \geq -\Delta_i k_a$. Hence, $\Gamma_a(t+\Delta_i) \in [0, \Gamma_a(t)]$, so for all $t \in S^{\underline{t}}_i$, $\left|D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) - D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)\right| \leq \Gamma_a(\underline{t}) = \bar{y} - a$. Q.E.D.$_{\text{Claim 6}}$

As already noted, we extend any function defined on S_i to any $t \in [0,1]$ by evaluating it at $\tau^i_t = \Delta_i \lfloor t/\Delta_i \rfloor$.

Claim 7. Fix $\underline{t} \in [0,1)$, $a \in (\bar{y}\Delta'_i, \bar{y}]$, and $k \in (0,\infty]$ such that either $a < \bar{y}$ or $k < \infty$ (or both). For any $a_0, a_1 \in (0,a]$ and any type $t \in [\underline{t},1]$, $D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t)$ converges to zero as $i \to \infty$ and $|a_0 - a_1| \to 0$ (in either order), uniformly in $H \in \mathscr{H}$, $\bar{y} \in (0,\mathbf{y}]$, and t.

Proof of Claim 7. By (69), for any $t \in [0,1]$,

$$\left(1-\delta\right)\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right), \tau_t^i\right)\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t\right)+\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), \tau_t^i\right)=0 \tag{70}$$

where we define

$$\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t\right)=\frac{D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right)-D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)}{\Delta_i}. \tag{71}$$

Equation (70) can be rewritten

$$\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t\right)=-\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right), \tau_t^i\right)},$$

whence $D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right)=D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)-\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right), \tau_t^i\right)}\Delta_i$ and thus, by (68),

$$D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right)=D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)-\Delta_i\min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right), \tau_t^i\right)}, k\right\},$$

and so

$$\begin{aligned}\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right) &\stackrel{d}{=} \frac{D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right)-D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)}{\Delta_i}\\ &=-\min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right), D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right), \tau_t^i\right)}, k\right\}.\end{aligned}$$

Lemma 2. *In the above formula for $\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)$, we may replace $D_{\underline{t}a_0k}^{H\delta\bar{y}i*}$ by $D_{\underline{t}a_0k}^{H\delta\bar{y}i}$. That is,*

$$\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)=-\min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right), \tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right), D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right), \tau_t^i\right)}, k\right\}. \tag{72}$$

Proof of Lemma 2. Let $w=w'=D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t\right)$. Let $\zeta e=D_{\underline{t}a_0k}^{H\delta\bar{y}i*}\left(t+\Delta_i\right)$ and $\zeta e'=D_{\underline{t}a_0k}^{H\delta\bar{y}i}\left(t+\Delta_i\right)$. By Claim 5 and part 1 of Claim 6, $\min\{w,\zeta,w',\zeta'\}>\bar{y}\Delta_i'$. Hence, by Claim 4, $\Delta_1^{H\bar{y}i}\left(\zeta',w',\tau_t^i\right)\leq\Delta_1^{H\bar{y}i}\left(\zeta,w,\tau_t^i\right)$ and $\Delta_2^{H\bar{y}i}\left(\zeta',\tau_t^i\right)\geq\Delta_2^{H\bar{y}i}\left(\zeta,\tau_t^i\right)$. Thus,

$$\frac{\Delta_2^{H\bar{y}i}\left(\zeta e',\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(\zeta e',w',\tau_t^i\right)}\geq\frac{\Delta_2^{H\bar{y}i}\left(\zeta e,\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(\zeta e,w,\tau_t^i\right)}, \tag{73}$$

with equality when $\zeta e' = \zeta e$. Accordingly, there are two cases. If $\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}(\zeta,\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta,w,\tau_t^i)} \leq k$, then $\zeta e' = \zeta e$, which implies (72). If $\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}(\zeta,\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta,w,\tau_t^i)} \geq k$, then $\Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) = -k$ but by (73), $\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}(\zeta',\tau_t^i)}{\Delta_1^{H\bar{y}i}(\zeta',w',\tau_t^i)} \geq k$ as well, so (72) holds. Q.E.D.$_{\text{Lemma 2}}$

For all $t' \in [\underline{t},1]$, since $D_{\underline{t}a_0k}^{H\delta\bar{y}i}(\underline{t}) = a_0$,

$$\begin{aligned}
D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t') &= a_0 + \Delta_i \sum_{t\in S_i^{\underline{t}},t<\tau_{t'}^i} \Delta D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) \\
&= a_0 - \Delta_i \sum_{t\in S_i^{\underline{t}},t<\tau_{t'}^i} \min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),t\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),t\right)},k\right\} \\
&= a_0 - \int_{t=\underline{t}}^{\tau_{t'}^i} \min\left\{\frac{1}{1-\delta}\frac{\Delta_2^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),\tau_t^i\right)}{\Delta_1^{H\bar{y}i}\left(D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i),D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t),\tau_t^i\right)},k\right\} dt, \qquad (74)
\end{aligned}$$

By (56) and (26), for all $t' \in [\underline{t},1]$,

$$D_{\underline{t}a_1k}^{H\delta\bar{y}}(t') = a_1 - \int_{t=\underline{t}}^{t'} \min\left\{\frac{1}{1-\delta}\frac{v_t\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t),t\right)}{v_D\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}(t),t\right)},k\right\} dt. \qquad (75)$$

Define the analogous quantities to $\Delta_1^{H\bar{y}i}$ and $\Delta_2^{H\bar{y}i}$ with $v^{H\bar{y}i}$ replaced by $v^{H\bar{y}}$: for any $D',D'' \in [0,\bar{y}]$, let

$$\begin{aligned}
\Delta_1^{H\bar{y}}(D',D'',t) &= \frac{v^{H\bar{y}}(D',t) - v^{H\bar{y}}(D'',t)}{D'-D''} \\
&\in \left(k_0\left(1-\frac{\max\{D',D''\}}{\bar{y}}\right),k_1\left(1-\frac{\min\{D',D''\}}{\bar{y}}\right)\right) \text{ and} \qquad (76) \\
\Delta_2^{H\bar{y}}(D',t,\Delta_i) &= \frac{v^{H\bar{y}}(D',t+\Delta_i) - v^{H\bar{y}}(D',t)}{\Delta_i} \in \left(\frac{k_0(D')^2[3\bar{y}-2D']}{6\bar{y}^2},k_1D'\right), \qquad (77)
\end{aligned}$$

where the bounds follow from (46) and (47) and imply that

$$\frac{\Delta_2^{H\bar{y}}(D',t,\Delta_i)}{\Delta_1^{H\bar{y}}(D',D'',t)} \in \left(\frac{k_0(D')^2[3\bar{y}-2D']}{6\bar{y}k_1(\bar{y}-\min\{D',D''\})},\frac{\bar{y}k_1D'}{k_0(\bar{y}-\max\{D',D''\})}\right). \qquad (78)$$

If $a<\bar{y}$ then by (61), (78), (50), and (57), for all $D,D' \in (\bar{y}\Delta_i',a]$ and all $t\in[0,1]$, the ratios $\frac{\Delta_2^{H\bar{y}i}(D',\tau_t^i)}{\Delta_1^{H\bar{y}i}(D',D'',\tau_t^i)}$, $\frac{\Delta_2^{H\bar{y}}(D',t,\Delta_i)}{\Delta_1^{H\bar{y}}(D',t,\tau_t^i)}$, $\frac{v_t^{H\bar{y}}(D',\tau_t^i)}{v_D^{H\bar{y}}(D',\tau_t^i)}$, and $\frac{v_t^{H\bar{y}}(D',t)}{v_D^{H\bar{y}}(D',t)}$ are all at most $(1-\delta)k_a$. Let

$$\kappa = (1-\delta)\min\{k,k_a\} < \infty. \qquad (79)$$

It follows that by (74), (75) and the triangle inequality that for all $t' \in [\underline{t}, 1]$,

$$(1-\delta)\left|D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t') - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t')\right| = \left|\begin{array}{c} \int_{t=\underline{t}}^{\tau^i_{t'}} \min\left\{\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\}dt \\ -\int_{t=\underline{t}}^{t'} \min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),t\right)},\kappa\right\}dt + a_1 - a_0 \end{array}\right|$$

$$\leq |a_1 - a_0| + A_1 + A_2 + A_3 + A_4 + A_5 + A_6$$

where

$$A_1 = \left|\int_{t=\tau^i_{t'}}^{t'} \min\left\{\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\}dt\right|,$$

$$A_2 = \int_{t=\underline{t}}^{t'}\left|\begin{array}{c} \min\left\{\frac{\Delta_2^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t\right)}{\Delta_1^{H\bar{y}i}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\} \\ -\min\left\{\frac{\Delta_2^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t,\Delta_i\right)}{\Delta_1^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\}\end{array}\right|dt,$$

$$A_3 = \int_{t=\underline{t}}^{t'}\left|\begin{array}{c} \min\left\{\frac{\Delta_2^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t,\Delta_i\right)}{\Delta_1^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\} \\ -\min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\}\end{array}\right|dt,$$

$$A_4 = \int_{t=\underline{t}}^{t'}\left|\min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t+\Delta_i),\tau^i_t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\} - \min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\}\right|dt,$$

$$A_5 = \int_{t=\underline{t}}^{t'}\left|\min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t),\tau^i_t\right)},\kappa\right\} - \min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),\tau^i_t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),\tau^i_t\right)},\kappa\right\}\right|dt,$$

and

$$A_6 = \int_{t=\underline{t}}^{t'}\left|\min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),\tau^i_t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),\tau^i_t\right)},\kappa\right\} - \min\left\{\frac{v_t^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),t\right)}{v_D^{H\bar{y}}\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t),t\right)},\kappa\right\}\right|dt.$$

Clearly, $A_1 \leq \kappa\Delta_i$. For A_2, A_3, A_4, and A_5, we require the following claim.

Lemma 3. *For any* $a,b,c,d \geq 0$, $\max\{|a-b|,|c-d|\}$ *is an upper bound on both*

$$|\min\{a,c\} - \min\{b,d\}|$$

and $|\max\{a,c\} - \max\{b,d\}|$.

Proof of Lemma 3. We prove the result for $|\min\{a,c\} - \min\{b,d\}|$; the proof for $|\max\{a,c\} - \max\{b,d\}|$ is identical with the keyword min replaced max throughout. First, assume $a \geq b$ and $c \geq d$. Then $|\min\{a,c\} - \min\{b,d\}| = \min\{a,c\} - \min\{b,d\}$. And $\max\{|a-b|,|c-d|\} = \max\{a-b,c-d\}$. But

$$\min\{a,c\} - \min\{b,d\} \leq \max\{a-b,c-d\}$$

since

$$\begin{aligned}\min\{a,c\} &\leq \min\{\max\{a,b+c-d\},\max\{c,d+a-b\}\}\\ &= \min\{b+\max\{a-b,c-d\},d+\max\{a-b,c-d\}\}\\ &= \min\{b,d\} + \max\{a-b,c-d\}.\end{aligned}$$

The other cases (in which $b > a$ or $c > d$ or both) are analogous. Q.E.D.$_{\text{Lemma 3}}$

This Lemma leads to the following useful bound.

Lemma 4. *For any* $a,b,\kappa \in (0,\infty)$ *and* $c,d \in [0,\infty)$,

$$\left|\min\left\{\frac{a}{c},\kappa\right\} - \min\left\{\frac{b}{d},\kappa\right\}\right| \leq \left|\frac{a-b}{\max\{c,a/\kappa\}}\right| + \frac{b}{\max\{d,b/\kappa\}}\frac{\max\{|c-d|,|a-b|/\kappa\}}{\max\{c,a/\kappa\}}. \tag{80}$$

Proof of Lemma 4. For any $a,b \geq 0$ and $c,d > 0$,

$$\left|\frac{a}{c} - \frac{b}{d}\right| = \left|\frac{ad-bc}{cd}\right| \leq \left|\frac{ad-bd}{cd}\right| + \left|\frac{bd-bc}{cd}\right| = \left|\frac{a-b}{c}\right| + \frac{b}{d}\left|\frac{d-c}{c}\right|. \tag{81}$$

Moreover, for any $a > 0$ and $c,\kappa \geq 0$,

$$\min\left\{\frac{a}{c},\kappa\right\} = \kappa\min\left\{\frac{a}{c\kappa},1\right\} = \kappa\min\left\{\frac{a}{c\kappa},\frac{a}{a}\right\} = \frac{a\kappa}{\max\{c\kappa,a\}} = \frac{a}{\max\{c,a/\kappa\}}. \tag{82}$$

By (81) and (82) and using Lemma (in that order),

$$\begin{aligned}\left|\min\left\{\frac{a}{c},\kappa\right\}-\min\left\{\frac{b}{d},\kappa\right\}\right| &= \left|\frac{a}{\max\{c,a/\kappa\}}-\frac{b}{\max\{d,b/\kappa\}}\right| \\ &\leq \left|\frac{a-b}{\max\{c,a/\kappa\}}\right|+\frac{b}{\max\{d,b/\kappa\}}\left|\frac{\max\{d,b/\kappa\}-\max\{c,a/\kappa\}}{\max\{c,a/\kappa\}}\right| \\ &\leq \left|\frac{a-b}{\max\{c,a/\kappa\}}\right|+\frac{b}{\max\{d,b/\kappa\}}\frac{\max\{|c-d|,|a-b|/\kappa\}}{\max\{c,a/\kappa\}}.\end{aligned}$$

Q.E.D.$_{\text{Lemma 4}}$

Let $D_t' = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t+\Delta_i) \in (\bar{y}\Delta_i', a_0)$, $D_t'' = D_{\underline{t}a_0k}^{H\delta\bar{y}i}(t) \in (\bar{y}\Delta_i', a_0]$, $a_t = \Delta_2^{H\bar{y}i}\left(D_t', \tau_t^i\right)$, $b_t = \Delta_2^{H\bar{y}}\left(D_t', \tau_t^i, \Delta_i\right)$, $c_t = \Delta_1^{H\bar{y}i}\left(D_t', D_t'', \tau_t^i\right)$, and $d_t = \Delta_1^{H\bar{y}}\left(D_t', D_t'', \tau_t^i\right)$. By (80),

$$A_2 \leq \int_{t=\underline{t}}^{t'}\left|\frac{a_t-b_t}{\max\{c_t,a_t/\kappa\}}\right|dt+\int_{t=\underline{t}}^{t'}\left(\frac{b_t}{\max\{d_t,b_t/\kappa\}}\frac{\max\{|c_t-d_t|,|a_t-b_t|/\kappa\}}{\max\{c_t,a_t/\kappa\}}\right)dt,$$

By (37), (38), (76), (77), and parts 3 and 4 of Claim 2, for any $\varepsilon > 0$ there is an $i^* < \infty$ such that if $i > i^*$, $\max\{|a_t-b_t|,|c_t-d_t|\} < \varepsilon$. By (77), $b_t \in \left[\frac{k_0(D_t')^2}{6\bar{y}}, k_1\bar{y}\right]$. By part 1 of Claim 2, $a_t > k_0\frac{(D')^2(3\bar{y}-2D')}{6\bar{y}^2} > k_0\frac{(D')^2}{6\bar{y}}$ where $D' = D_t' - 2\bar{y}\Delta_i' < \bar{y}$. Hence, $\min\{a_t,b_t\} > \frac{k_0\left(D_t'-2\bar{y}\Delta_i'\right)^2}{6\bar{y}}$. By (37), (76), and part 2 of Claim 2, $\min\{c_t,d_t\} \geq k_0\left(1-\frac{D_t''}{\bar{y}}\right)$ so since $D_t' \geq D_t'' - k\Delta_i$, both $\max\{c_t,a_t/\kappa\}$ and $\max\{d_t,b_t/\kappa\}$ are at least

$$k_0\max\left\{1-\frac{D_t''}{\bar{y}},\frac{(D_t'-2\bar{y}\Delta_i')^2}{6\bar{y}\kappa}\right\} \geq k_0\max\left\{1-\frac{D_t''}{\bar{y}},\frac{(D_t''-k\Delta_i-2\bar{y}\Delta_i')^2}{6\bar{y}\kappa}\right\},$$

which is bounded below by a strictly positive constant κ' for large enough i as $\bar{y} > 0$. Collecting these bounds, $A_2 \leq \varepsilon\kappa_2[t'-\underline{t}]$ where $\kappa_2 = \frac{1}{\kappa'}\left[1+\frac{k_1\bar{y}}{\kappa'}\max\{1,1/\kappa\}\right] \in (0,\infty)$.

By (80), redefining $a_t = \Delta_2^{H\bar{y}}\left(D_t', \tau_t^i, \Delta_i\right)$, $b_t = v_t^{H\bar{y}}\left(D_t', \tau_t^i\right)$, $c_t = \Delta_1^{H\bar{y}}\left(D_t', D_t'', \tau_t^i\right)$, and $d_t = v_D^{H\bar{y}}\left(D_t'', \tau_t^i\right)$,

$$A_3 \leq \int_{t=\underline{t}}^{t'}\left|\frac{a_t-b_t}{\max\{c_t,a_t/\kappa\}}\right|dt+\int_{t=\underline{t}}^{t'}\left(\frac{b_t}{\max\{d_t,b_t/\kappa\}}\frac{\max\{|c_t-d_t|,|a_t-b_t|/\kappa\}}{\max\{c_t,a_t/\kappa\}}\right)dt.$$

By (46), $b_t \leq k_1\bar{y}$. By the Mean Value Theorem, there is a $t \in \left[\tau_t^i, \tau_t^i+\Delta_i\right]$ such that $v_t^{H\bar{y}}\left(D_t', t\right) = a_t$. Thus, by (44),

$$|a_t-b_t| = \left|v_t^{H\bar{y}}\left(D_t',t\right)-b_t\right| = \left|v_t^{H\bar{y}}\left(D_t',t\right)-v_t^{H\bar{y}}\left(D_t',\tau_t^i\right)\right| \leq k_2\bar{y}\Delta_i.$$

Also by the Mean Value Theorem, there is a $D \in [D'_t, D''_t]$ such that $v_D^{H\bar{y}}\left(D, \tau_t^i\right) = c_t$. By part 3 of Claim 6 and (79),

$$\left|D''_t - D'_t\right| \leq \Delta_i \min\{k, k_a\} = \frac{\kappa}{(1-\delta)}\Delta_i. \tag{83}$$

Hence, $|c_t - d_t| = \left|v_D^{H\bar{y}}\left(D, \tau_t^i\right) - v_D^{H\bar{y}}\left(D''_t, \tau_t^i\right)\right| \leq \frac{k_1}{\bar{y}}\left|D''_t - D'_t\right| \leq \frac{k_1\kappa}{\bar{y}(1-\delta)}\Delta_i$. By (46), (47), (76), and (77), $\min\{a_t, b_t\} \geq \frac{k_0 (D'_t)^2}{6\bar{y}}$, and $\min\{c_t, d_t\} \geq k_0\left(1 - \frac{D''_t}{\bar{y}}\right)$ so as shown in the prior paragraph, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds, $A_3 \leq \Delta_i \kappa_3 [t' - \underline{t}]$ where $\kappa_3 = \frac{\bar{y}}{\kappa'}\left[k_2 + \frac{k_1}{\kappa'}\max\left\{\frac{k_1\kappa}{\bar{y}(1-\delta)}, \frac{k_2\bar{y}}{\kappa}\right\}\right] \in (0, \infty)$.

By (80), redefining $a_t = v_t^{H\bar{y}}\left(D'_t, \tau_t^i\right)$, $b_t = v_t^{H\bar{y}}\left(D''_t, \tau_t^i\right)$, and $c_t = d_t = v_D^{H\bar{y}}\left(D''_t, \tau_t^i\right)$,

$$A_4 \leq \int_{t=\underline{t}}^{t'} \left|\frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}}\right| dt + \int_{t=\underline{t}}^{t'} \left(\frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{|a_t - b_t|/\kappa}{\max\{c_t, a_t/\kappa\}}\right) dt,$$

By (46), $b_t \leq k_1\bar{y}$. By (49) and (83),

$$|a_t - b_t| = \left|v_t^{H\bar{y}}\left(D'_t, \tau_t^i\right) - v_t^{H\bar{y}}\left(D''_t, \tau_t^i\right)\right| \leq \frac{k_1\kappa}{(1-\delta)\bar{y}}\Delta_i.$$

By (46) and (47), $\min\{a_t, b_t\} \geq \frac{k_0 (D'_t)^2}{6\bar{y}}$, and $\min\{c_t, d_t\} \geq k_0\left(1 - \frac{D''_t}{\bar{y}}\right)$ so as shown in the prior paragraph, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds, $A_4 \leq \Delta_i \kappa_4 [t' - \underline{t}]$, where $\kappa_4 = \frac{k_1}{\kappa'(1-\delta)}\left[\frac{\kappa}{\bar{y}} + \frac{k_1}{\kappa'}\right] \in (0, \infty)$.

By (80), redefining $a_t = v_t^{H\bar{y}}\left(D''_t, \tau_t^i\right)$, $b_t = v_t^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right), \tau_t^i\right)$, $c_t = v_D^{H\bar{y}}\left(D''_t, \tau_t^i\right)$, and $d_t = v_D^{H\bar{y}}\left(D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right), \tau_t^i\right)$,

$$A_5 \leq \int_{t=\underline{t}}^{t'} \left|\frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}}\right| dt + \int_{t=\underline{t}}^{t'} \left(\frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}}\right) dt.$$

By (46), $b_t \leq k_1\bar{y}$. By (44), $|a_t - b_t| \leq k_2\bar{y}\left|D''_t - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right)\right|$. By (49),

$$|c_t - d_t| \leq \frac{k_1}{\bar{y}}\left|D''_t - D_{\underline{t}a_1k}^{H\delta\bar{y}}\left(\tau_t^i\right)\right|.$$

By (46), (47), $\min\{a_t, b_t\} \geq \frac{k_0 (D'_t)^2}{6\bar{y}}$ and $\min\{c_t, d_t\} \geq k_0\left(1 - \frac{D''_t}{\bar{y}}\right)$ so as shown above, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds and using

$D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\tau^i_t)$,

$$A_5 \le \kappa_5 \int_{t=\underline{t}}^{t'} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\tau^i_t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(\tau^i_t) \right| dt \le \kappa_5 \left[t' - \underline{t}\right] \max_{t\in[\underline{t},t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\tau^i_t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(\tau^i_t) \right|$$
$$\le \kappa_5 \left[t' - \underline{t}\right] \max_{t\in[\underline{t},t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right|,$$

where $\kappa_5 = \frac{\bar{y}}{\kappa'}\left[k_2 + \frac{k_1}{\kappa'}\max\left\{\frac{k_1}{\bar{y}}, \frac{k_2\bar{y}}{\kappa}\right\}\right] \in (0,\infty)$.

Now redefine $a_t = v^{H\bar{y}}_t\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t), \tau^i_t\right)$, $b_t = v^{H\bar{y}}_t\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t), t\right)$, $c_t = v^{H\bar{y}}_D\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t), \tau^i_t\right)$, and $d_t = v^{H\bar{y}}_D\left(D^{H\delta\bar{y}}_{\underline{t}a_1k}(t), t\right)$. By (80),

$$A_6 \le \int_{t=\underline{t}}^{t'} \left| \frac{a_t - b_t}{\max\{c_t, a_t/\kappa\}} \right| dt + \int_{t=\underline{t}}^{t'} \left(\frac{b_t}{\max\{d_t, b_t/\kappa\}} \frac{\max\{|c_t - d_t|, |a_t - b_t|/\kappa\}}{\max\{c_t, a_t/\kappa\}} \right) dt.$$

By (46), $b_t \le k_1\bar{y}$. By (44), $|a_t - b_t| \le k_2\bar{y}\Delta_i$. By (48), $|c_t - d_t| \le k_1\Delta_i$. By (46) and (47), $\min\{a_t, b_t\} \ge \frac{k_0(D'_t)^2}{6\bar{y}}$ and $\min\{c_t, d_t\} \ge k_0\left(1 - \frac{D''_t}{\bar{y}}\right)$ so as shown above, both $\max\{c_t, a_t/\kappa\}$ and $\max\{d_t, b_t/\kappa\}$ are at least $\kappa' > 0$. Collecting these bounds, $A_6 \le \Delta_i\kappa_6\left[t' - \underline{t}\right]$ where $\kappa_6 = \frac{\bar{y}}{\kappa'}\left[k_2 + \frac{k_1}{\kappa'}\max\left\{k_1, \frac{k_2\bar{y}}{\kappa}\right\}\right] \in (0,\infty)$.

Summarizing our findings and since $\lim_{i\to\infty}\Delta_i = 0$ and $t' \in [\underline{t}, 1]$, for all $\varepsilon > 0$ there is an $i^* < \infty$ such that if $i > i^*$, then

$$(1-\delta)\left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t') - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t') \right| \le |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t' - \underline{t}\right] \max_{t\in[\underline{t},t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right|$$

where $\kappa'' = \kappa + (\kappa_2 + \kappa_3 + \kappa_4 + \kappa_6)\left[1 - \underline{t}\right]$. So for any $t'' \in [\underline{t}, t']$,

$$(1-\delta)\left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t'') - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t'') \right|$$
$$\le |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t'' - \underline{t}\right] \max_{t\in[\underline{t},t'']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right|$$
$$\le |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t' - \underline{t}\right] \max_{t\in[\underline{t},t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right|$$

and therefore,

$$(1-\delta)\max_{t\in[\underline{t},t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right|$$
$$\le |a_1 - a_0| + \kappa''\varepsilon + \kappa_5\left[t' - \underline{t}\right] \max_{t\in[\underline{t},t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right|.$$

Now for $t' \in [\underline{t}, \underline{t}+b]$ where $b = \frac{1-\delta}{2\kappa_5} > 0$, $(1-\delta) - \kappa_5 [t' - \underline{t}] \geq \frac{1-\delta}{2}$, so

$$\max_{t \in [\underline{t}, t']} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right| \leq \frac{2}{1-\delta} \left(|a_1 - a_0| + \kappa'' \varepsilon \right),$$

whence $\max_{t \in [\underline{t}, \underline{t}+b]} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right| \leq \frac{2}{1-\delta} (|a_1 - a_0| + \kappa'' \varepsilon)$. In particular,

$$\left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\underline{t}+b) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(\underline{t}+b) \right| \leq \frac{2}{1-\delta} \left(|a_1 - a_0| + \kappa'' \varepsilon \right).$$

Let $a_2 = D^{H\delta\bar{y}i}_{\underline{t}a_0k}(\underline{t}+b)$ and $a_3 = D^{H\delta\bar{y}}_{\underline{t}a_1k}(\underline{t}+b)$. Since $D^{H\delta\bar{y}i}_{\underline{t}a_0k}$ and $D^{H\delta\bar{y}}_{\underline{t}a_1k}$ are decreasing functions, $\max\{a_2, a_3\} < a$ so in the above reasoning we can use the same constant a and thus the same constant k_a and thus the same κ''. Accordingly,

$$\begin{aligned}
\max_{t \in [\underline{t}+b, \underline{t}+2b]} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right| &\leq \frac{2}{1-\delta} \left(|a_3 - a_2| + \kappa'' \varepsilon \right) \\
&\leq \frac{2}{1-\delta} \left(\frac{2}{1-\delta} \left(|a_1 - a_0| + \kappa'' \varepsilon \right) + \kappa'' \varepsilon \right) \\
&= \left(\frac{2}{1-\delta} \right)^2 |a_1 - a_0| + \left[\frac{2}{1-\delta} + \left(\frac{2}{1-\delta} \right)^2 \right] \kappa'' \varepsilon.
\end{aligned}$$

Iterating this reasoning $n = \left\lceil \frac{1-\underline{t}}{b} \right\rceil$ (which does not depend on i) times, we obtain

$$\max_{t \in [\underline{t}, 1]} \left| D^{H\delta\bar{y}i}_{\underline{t}a_0k}(t) - D^{H\delta\bar{y}}_{\underline{t}a_1k}(t) \right| \leq \left(\frac{2}{1-\delta} \right)^n |a_1 - a_0| + \kappa'' \varepsilon \sum_{i=1}^{n} \left[\left(\frac{2}{1-\delta} \right)^i \right].$$

Since the constants multiplying $|a_1 - a_0|$ and ε are independent of i, $t \in [\underline{t}, 1]$, $\bar{y} \in (0, \mathbf{y}]$, and $H \in \mathscr{H}$, the result follows. Q.E.D.$_{\text{Claim 7}}$

Claim 8. For all $\underline{t} \in [0,1)$, there exist unique solutions $D^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$ and $D^{H\delta\bar{y}i}_{\underline{t}\bar{y}\infty}$ to $\mathrm{CP}^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$ and $\mathrm{DP}^{H\delta\bar{y}i}_{\underline{t}\bar{y}\infty}$, respectively. They are decreasing in t and, in the case of $D^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$, continuous in $t \in [\underline{t}, 1]$.

Proof of Claim 8. Define, for all $t \in [\underline{t}, 1]$,

$$D^{H\delta\bar{y}}_{\underline{t}}(t) = \lim_{a \uparrow \bar{y}} D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t) = \lim_{a \uparrow \bar{y}} D^{H\delta\bar{y}}_{\underline{t}a\infty}(t), \tag{84}$$

and for all $t \in S^{\underline{t}}_i$, $D^{H\delta\bar{y}i}_{\underline{t}}(t) = \lim_{a \uparrow \bar{y}} D^{H\delta\bar{y}i}_{\underline{t}\bar{y}k_a}(t) = \lim_{a \uparrow \bar{y}} D^{H\delta\bar{y}i}_{\underline{t}a\infty}(t)$. The four limits exist by part 2 of Claims 3 and 6 and the Monotone Convergence Theorem. The pair of limits that appears in each equation are equal by part 4 of Claims 3 and 6. Hence, $D_{\underline{t}}^{H\delta\bar{y}}$ and $D_{\underline{t}}^{H\delta\bar{y}i}$ exist and are unique. Moreover, by part 2 of Claims 3 and 6, any solutions $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ and $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ to $\mathrm{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ and $\mathrm{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ must satisfy $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}} \geq D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}} \geq D_{\underline{t}a\infty}^{H\delta\bar{y}}$ and $D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i} \geq D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i} \geq D_{\underline{t}a\infty}^{H\delta\bar{y}i}$ for any $a \in (0,\bar{y}]$, and thus must coincide with $D_{\underline{t}}^{H\delta\bar{y}}$ and $D_{\underline{t}}^{H\delta\bar{y}i}$, respectively. It remains to show that $D_{\underline{t}}^{H\delta\bar{y}}$ and $D_{\underline{t}}^{H\delta\bar{y}i}$ solve $\mathrm{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ and $\mathrm{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$: that

$$D_{\underline{t}}^{H\delta\bar{y}}(\underline{t}) = \bar{y} \text{ and, for } t \in (\underline{t},1), \ \frac{dD_{\underline{t}}^{H\delta\bar{y}}(t)}{dt} = f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t),t\right) \tag{85}$$

and

$$D_{\underline{t}}^{H\delta\bar{y}i}(\underline{t}) = \bar{y} \text{ and, for } t \in S_i^{\underline{t}} \backslash \{1\}, D_{\underline{t}}^{H\delta\bar{y}i}(t+\Delta_i) = D_{\underline{t}}^{H\delta\bar{y}i*}(t+\Delta_i) \tag{86}$$

where $D_{\underline{t}}^{H\delta\bar{y}i*}(t+\Delta_i)$ is the (by Claim 5) unique solution $D^* \in \left(\bar{y}\Delta_i', D_{\underline{t}}^{H\delta\bar{y}i}(t)\right)$ to

$$v^{H\bar{y}i}(D^*,t+\Delta_i) - \delta v^{H\bar{y}i}(D^*,t) = (1-\delta)\, v^{H\bar{y}i}\left(D_{\underline{t}}^{H\delta\bar{y}i}(t),t\right). \tag{87}$$

First, $D_{\underline{t}}^{H\delta\bar{y}}(\underline{t}) = \lim_{a\uparrow\bar{y}} D_{\underline{t}a\infty}^{H\delta\bar{y}}(\underline{t}) = \lim_{a\uparrow\bar{y}} a = \bar{y}$. Second, for $t \in (\underline{t},1)$, we must show that $dD_{\underline{t}}^{H\delta\bar{y}}(t)/dt = f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t),t\right)$. By (52) and (26), $f^{H\delta\bar{y}}(\zeta,t)$ is continuous in t so, by (56), $f^{H\delta\bar{y}}\left(D_{\underline{t}}^{H\delta\bar{y}}(t),t\right)$ is the limit of the derivatives $dD_{\underline{t}a\infty}^{H\delta\bar{y}}(t)/dt$ as a goes to $\bar{y}$. We now invoke the following well known result.[4]

Theorem 2. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on $t \in [\underline{t},1]$. Suppose that they converge uniformly to some function f. Assume also that the sequence of derivatives $(f_n')_{n=1}^{\infty}$ converges, uniformly in $t \in [\underline{t},1]$, to some continuous function. Then f is differentiable and* $\lim_{n\to\infty} f_n'(t) = f'(t)$ *for all $t \in [\underline{t},1]$.*

Let $f_n = D_{\underline{t}a_n\infty}^{H\delta\bar{y}}$ where $(a_n)_{n=1}^{\infty}$ is an increasing sequence that converges to $\bar{y}$. By part 4 of Claim 3, $(f_n)_{n=1}^{\infty}$ converges to $f = D_{\underline{t}}^{H\delta\bar{y}}$, uniformly in $t \in [\underline{t},1]$. Fix $w \in (\underline{t},1)$. We will show that the sequence of derivatives $(f_n')_{n=1}^{\infty}$ converges, uniformly in $t \in [w,1]$, to f'. The result then follows by taking $w \to \underline{t}$. By (50) and (26), letting $b = \frac{1}{1-\delta}\frac{k_0}{6k_1}$ and $x = f_n(w)$,

$$\begin{aligned} x = f_n(\underline{t}) + \int_{t=\underline{t}}^{w} f_n'(t)\,dt = a_n + \int_{t=\underline{t}}^{w} f^{H\delta\bar{y}}(f_n(t),t)\,dt < a_n - \frac{b}{\bar{y}}\int_{t=\underline{t}}^{w} [f_n(t)]^2\,dt \\ < a_n - \frac{b}{\bar{y}}\int_{t=\underline{t}}^{w} x^2 dt = a_n - \frac{b}{\bar{y}} x^2 (w-\underline{t}) \end{aligned}$$

---

[4] See, e.g., Theorem 9.13 in Apostol (1981, p. 229).

so x lies below the higher root of $bx^2(w-\underline{t})+\bar{y}x-\bar{y}a_n=0$, which (since $a_n\le\bar{y}$) is at most $\frac{-\bar{y}+\sqrt{\bar{y}^2+4b(w-\underline{t})\bar{y}^2}}{2b(w-\underline{t})}$. If we set this equal to $\bar{y}(1-c)$ and solve for c, we obtain $\frac{c}{(1-c)^2}=b(w-\underline{t})$. Hence, for all $t\in[w,1]$,

$$\frac{f_n(t)}{\bar{y}}\le\frac{f_n(w)}{\bar{y}}<1-c \tag{88}$$

where c is positive and independent of n. Thus, for all $t\in[w,1]$ and all n,

$$\left|f_n'(t)\right|=-f^{H\delta\bar{y}}(f_n(t),t)<\frac{1}{1-\delta}\frac{k_1\bar{y}}{k_0}\frac{1-c}{c}$$

which is a finite constant that is independent of n. Thus, the functions $(f_n)_{n=1}^{\infty}$, as well as their limit (call it f_∞), are Lipschitz on $t\in[w,1]$ with the same Lipschitz constant. As $f^{H\delta\bar{y}}$ is continuous and f_∞ is Lipschitz continuous, the function

$$\lim_{n\to\infty}f_n'(t)=\lim_{n\to\infty}f^{H\delta\bar{y}}(f_n(t),t)=f^{H\delta\bar{y}}\left(\lim_{n\to\infty}f_n(t),t\right)=f^{H\delta\bar{y}}(f_\infty(t),t)$$

is continuous in $t\in[w,1]$. Moreover, convergence of $f_n'(t)$ is uniform since, for all $n,n'\ge1$, by (52), (26), and (88), letting $k_5=\frac{k_1}{k_0c(1-\delta)}\left[1+\frac{k_1(1-c)}{k_0c}\right]\in(0,\infty)$,

$$\begin{aligned}\left|f_n'(t)-f_{n'}'(t)\right|&=\left|f^{H\delta\bar{y}}(f_n(t),t)-f^{H\delta\bar{y}}(f_{n'}(t),t)\right|\le k_5\left|f_n(t)-f_{n'}(t)\right|\\&=k_5\left|D^{H\delta\bar{y}}_{\underline{t}a_n\infty}(\underline{t})-D^{H\delta\bar{y}}_{\underline{t}a_{n'}\infty}(\underline{t})\right|\le\bar{y}-\min\{a_n,a_{n'}\},\end{aligned}$$

where the last inequality is from part 4 of Claim 3. Hence, $(f_n')_{n=1}^{\infty}$ is a uniform (in $t\in[w,1]$) Cauchy sequence and thus converges uniformly. This proves existence and uniqueness.

By part 1 of claim 6, $D^{H\delta\bar{y}i}_{\underline{t}\bar{y}\infty}$ is decreasing in t. It remains to show that $D^{H\delta\bar{y}}_{\underline{t}\bar{y}\infty}$ is decreasing and continuous in $t\in[\underline{t},1]$. By L-H and (26), $f^{H\delta\bar{y}}(D,t)$ is finite and negative for all $D<\bar{y}$. Thus, by (85), $D^{H\delta\bar{y}}_{\underline{t}}$ is decreasing and continuous in $t\in(\underline{t},1]$. By (85), for continuity at $t=\underline{t}$ we must show that $\lim_{t\to\underline{t}}D^{H\delta\bar{y}}_{\underline{t}}(t)=\bar{y}$. By (84), $D^{H\delta\bar{y}}_{\underline{t}}(t)=\lim_{a\uparrow\bar{y}}D^{H\delta\bar{y}}_{\underline{t}\bar{y}k_a}(t)=\lim_{a\uparrow\bar{y}}D^{H\delta\bar{y}}_{\underline{t}a\infty}(t)$. By $\mathrm{CP}^{H\delta\bar{y}}_{\underline{t}ak}$, $D^{H\delta\bar{y}}_{\underline{t}ak}(\underline{t})=a$. By the triangle inequality, for any $a\in(0,\bar{y})$,

$$\left|D^{H\delta\bar{y}}_{\underline{t}}(t)-\bar{y}\right|=\left|D^{H\delta\bar{y}}_{\underline{t}}(t)-D^{H\delta\bar{y}}_{\underline{t}a\infty}(t)\right|+\left|D^{H\delta\bar{y}}_{\underline{t}a\infty}(t)-a\right|+\left|a-\bar{y}\right|.$$

By parts 2 and 4 of Claim 3, $\left|D_{\underline{t}}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right| \leq \left|D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right| \leq |\bar{y}-a|$. Finally, $\left|D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - a\right| = \left|D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(\underline{t})\right| \leq \frac{k_1\bar{y}a}{(1-\delta)k_0(\bar{y}-a)}|t-\underline{t}|$ by part 3 of Claim 3. Collecting terms and letting $a = \bar{y} - \sqrt{t-\underline{t}} \leq \bar{y}$, $\left|D_{\underline{t}}^{H\delta\bar{y}}(t) - \bar{y}\right| \leq \left[2 + \frac{k_1\bar{y}^2}{(1-\delta)k_0}\right]\sqrt{t-\underline{t}}$ which goes to zero as $t \downarrow \underline{t}$. Q.E.D.$_{\text{Claim 8}}$

In light of Claim 8, part 5 of Theorem 1 is implied by part 2 of Claim 3, setting $\underline{t} = 0$, $a = a' = \bar{y}$, and $k = k' = \infty$.

We now turn to the convergence of the discrete solutions to the continuous one. By part 2 of Claims 3 and 6, for all $a \in (0, \bar{y})$,

$$D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) \leq D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t) \leq D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)$$

and thus, by the triangle inequality,

$$\left|D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t)\right| \leq \max\left\{ \begin{array}{c} \left|D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right| + \left|D_{\underline{t}a\infty}^{H\delta\bar{y}}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t)\right|, \\ \left|D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t)\right| + \left|D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right| \end{array} \right\}. \quad (89)$$

Fix $\varepsilon > 0$. By Claim 7, there is an i^*, independent of $\bar{y}$, H, and t, such that for $i > i^*$, $\left|D_{\underline{t}a\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right|$ and $\left|D_{\underline{t}\bar{y}k}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}k}^{H\delta\bar{y}}(t)\right|$ are each less than $\varepsilon/2$. By part 4 of Claim 3, for all $a \in [\bar{y} - \varepsilon/2, \bar{y})$, $\left|D_{\underline{t}\bar{y}k_a}^{H\delta\bar{y}}(t) - D_{\underline{t}a\infty}^{H\delta\bar{y}}(t)\right| \leq \varepsilon/2$. But (89) holds for all $a \in (0, \bar{y})$, so it holds in particular for $a \in [\bar{y} - \varepsilon/2, \bar{y})$. Thus, for all $\varepsilon > 0$ there is an i^*, independent of $\bar{y}$, H, and t, such that for $i > i^*$, $\left|D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}(t) - D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t)\right| \leq \varepsilon$. We now set $\underline{t} = 0$ to obtain the desired result. Together with Claim 8, this proves that the unique solution $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ to $\mathrm{DP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}i}$ converges to the unique solution $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ to $\mathrm{CP}_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ as $i \to \infty$, uniformly in $H \in \mathscr{H}$, $\bar{y} \in [0, \mathbf{y}]$, and $t \in [0, 1]$.

We next prove that $p^{H\delta\bar{y}}$ and thus $u^{H\delta\bar{y}}$ is both continuous and decreasing in the type t. First, $p^{H\delta\bar{y}}(t) = v^{H\bar{y}}\left(D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}(t), t\right)$. By (24) and L-$H$, $v^{H\bar{y}}$ is continuous in both arguments. Since $D_{\underline{t}\bar{y}\infty}^{H\delta\bar{y}}$ is also continuous in t, so is $p^{H\delta\bar{y}}$. By (27),

$$\begin{aligned} \frac{d}{dt}\left[v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right] &= v_1^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\frac{dD^{H\delta\bar{y}}}{dt} + v_2^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right) \\ &= \delta v_1^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\frac{dD^{H\delta\bar{y}}}{dt} \end{aligned}$$

which is negative by (27), (46), and (47). Hence, $p^{H\delta\bar{y}}$ is decreasing in t.

We now turn to convergence of $p^{H\delta\bar{y}i}$ to $p^{H\delta\bar{y}}$ and thus of $u^{H\delta\bar{y}i}$ to $u^{H\delta\bar{y}}$. For any $t \in [0,1]$, recall that $\tau_t^i = \Delta_i \lfloor t/\Delta_i \rfloor$. As $D^{H\delta\bar{y}i}(t) = D^{H\delta\bar{y}i}\left(\tau_t^i\right)$,

$$\begin{aligned}\left|p^{H\delta\bar{y}i}(t) - p^{H\delta\bar{y}}(t)\right| &= \left|p^{H\delta\bar{y}i}\left(\tau_t^i\right) - p^{H\delta\bar{y}}(t)\right| \\ &= \left|v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right| \leq A_1' + A_2' + A_3'\end{aligned}$$

where

$$\begin{aligned}A_1' &= \left|v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right)\right|, \\ A_2' &= \left|v^{H\bar{y}}\left(D^{H\delta\bar{y}i}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), \tau_t^i\right)\right|, \text{ and} \\ A_3' &= \left|v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), \tau_t^i\right) - v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)\right|.\end{aligned}$$

By (24), (28), and since $H\left(0|\tau_t^i\right)$ is zero,

$$\frac{A_1'}{\bar{y}} \leq \int_{z=0}^{D^{H\delta\bar{y}i}(t)/\bar{y}} \left|H\left(z|\tau_t^i\right) - H\left(\Delta_i' \left\lfloor \frac{z}{\Delta_i'} \right\rfloor |\tau_t^i\right)\right| dz \leq k_1 \Delta_i',$$

where the second inequality is by L-H. By (24), $A_2' \leq 2\left|D^{H\delta\bar{y}i}(t) - D^{H\delta\bar{y}}(t)\right|$. By (24) and L-$H$,

$$A_3' \leq \bar{y} \int_{z=0}^{D^{H\delta\bar{y}}(t)/\bar{y}} \left|H(z|t) - H\left(z|\tau_t^i\right)\right| dz \leq \bar{y}k_1 \left|t - \tau_t^i\right| \leq \bar{y}k_1\Delta_i.$$

Hence, by the prior result, for all $\varepsilon > 0$ there is an $i^* < \infty$ such that if $i > i^*$,

$$\left|p^{H\delta\bar{y}i}(t) - p(t)\right| \leq \bar{y}k_1\Delta_i' + 2\left|D^{H\delta\bar{y}i}(t) - D^{H\delta\bar{y}}(t)\right| + \bar{y}k_1\Delta_i < \varepsilon,$$

for all $t \in [0,1]$, $\bar{y} \in (0, \mathbf{y}]$, and $H \in \mathscr{H}$, as claimed. Hence, $p^{H\delta\bar{y}i}$ converges uniformly to $p^{H\delta\bar{y}}$.

We now show uniform convergence of $\Pi^{H\delta\bar{y}i}$ to $\Pi^{H\delta\bar{y}}$:

$$\left|\Pi^{H\delta\bar{y}i}(t) - \Pi^{H\delta\bar{y}}(t)\right| \leq \left|E^i\left[\bar{y}Z|t\right] - E^\infty\left[\bar{y}Z|t\right]\right| + \left|u^{H\delta\bar{y}i}(t) - u^{H\delta\bar{y}}(t)\right|$$

and for $z \in [(c-1)\Delta_i', c\Delta_i')$, $c = \left\lfloor \frac{z}{\Delta_i'} \right\rfloor + 1$, so

$$\begin{aligned}
\left|E^i[\bar{y}Z|t] - E^\infty[\bar{y}Z|t]\right| &= \left|\sum_{c=1}^{1/\Delta_i'} \bar{y}c\Delta_i'\left[H\left(c\Delta_i'|t\right) - H\left((c-1)\Delta_i'|t\right)\right] - \int_{z=0}^{1} \bar{y}z dH(z|t)\right| \\
&= \bar{y}\left|\sum_{c=1}^{1/\Delta_i'} \int_{z=(c-1)\Delta_i'}^{c\Delta_i'} \left(\left\lfloor \frac{z}{\Delta_i'}\right\rfloor + 1\right)\Delta_i' dH(z|t) - \int_{z=0}^{1} z dH(z|t)\right| \\
&\leq \bar{y}\Delta_i' \int_{z=0}^{1} \left|\left\lfloor \frac{z}{\Delta_i'}\right\rfloor + 1 - \frac{z}{\Delta_i'}\right| dH(z|t) \leq \bar{y}\Delta_i'. \qquad (90)
\end{aligned}$$

Thus, $E^i[\bar{y}Z|t]$ converges uniformly to $E^\infty[\bar{y}Z|t]$ and hence $\Pi^{H\delta\bar{y}i}$ converges uniformly to $\Pi^{H\delta\bar{y}}$.

As there is no mention of G in the statement of the problems $\mathrm{DP}^{H\delta\bar{y}i}$ and $\mathrm{CP}^{H\delta\bar{y}}$, any solutions $D^{H\delta\bar{y}i}$ and $D^{H\delta\bar{y}}$ to these problems for one distribution G are also solutions for any other distribution $\widehat{G}$ that satisfies our assumptions.[5] Since, moreover, the solutions $D^{H\delta\bar{y}i}$ and $D^{H\delta\bar{y}}$ are unique by parts 1 and 2 of the theorem, they must be independent of G. Hence, convergence of $D^{H\delta\bar{y}i}$, $p^{H\delta\bar{y}i}$, $u^{H\delta\bar{y}i}$, and $\Pi^{H\delta\bar{y}i}$ is uniform in G as well.

We now show that $Eu^{GH\delta\bar{y}i}$ converges uniformly to $Eu^{GH\delta\bar{y}}$. Since G has no atoms, $G(0) = 0$; hence,

$$Eu^{GH\delta\bar{y}i} = \sum_{c=1}^{1/\Delta_i} u^{H\delta\bar{y}i}(c\Delta_i)\left[G(c\Delta_i) - G((c-1)\Delta_i)\right] = \int_{t=0}^{1} u^{H\delta\bar{y}i}\left(\tau_t^i + \Delta_i\right) dG(t)$$

and thus, $Eu^{GH\delta\bar{y}i} - Eu^{GH\delta\bar{y}} = A_1'' + A_2'' - A_3''$ where

$$A_1'' = \int_{t=0}^{1}\left[u^{H\delta\bar{y}i}\left(\tau_t^i + \Delta_i\right) - u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right)\right] dG(t),$$

$A_2'' = \int_{t=0}^{1} u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right) dG(t)$, and $A_3'' = \int_{t=0}^{1} u^{H\delta\bar{y}}(t)\, dG(t)$. As shown above, for all $\varepsilon > 0$ there is an i^* such that for all models $i > i^*$, parameters $\bar{y}$ in $(0, \mathbf{y}]$, and conditional distribution functions H in $\mathscr{H}$,

$$\left|A_1''\right| \leq \int_{t=0}^{1}\left|u^{H\delta\bar{y}i}\left(\tau_t^i + \Delta_i\right) - u^{H\delta\bar{y}}\left(\tau_t^i + \Delta_i\right)\right| dG(t) < \int_{t=0}^{1} \frac{\varepsilon}{2} dG(t) = \frac{\varepsilon}{2}.$$

[5]In particular, $\widehat{G}$ must have support $[0,1]$.

Moreover, since $u^{H\delta\bar{y}}$ is a nonincreasing function of t, it follows that

$$u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right) \leq u^{H\delta\bar{y}}(t) \leq u^{H\delta\bar{y}}\left(\tau_t^i\right),$$

so $A_2'' \leq A_3'' \leq A_4''$ where

$$A_4'' = \int_{t=0}^{1} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) = \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) + \int_{t=\Delta_i}^{1} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t).$$

But $\tau_{t+\Delta_i}^i = \tau_t^i + \Delta_i$. Hence, letting $t' = t - \Delta_i$ and renaming t' to t,

$$\int_{t=\Delta_i}^{1} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) = \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right) dG(t+\Delta_i).$$

Now let i^* also be large enough that $\Delta_{i^*} < \frac{\varepsilon}{2(1-\delta)\mathbf{y}[2k_3+k_4]}$. Then by Lipschitz-$G$ and since, by (23), $u^{H\delta\bar{y}} \in [0, (1-\delta)\bar{y}]$, for all $i > i^*$,

$$\begin{aligned}
&\left|A_2'' - A_3''\right| \leq \left|A_2'' - A_4''\right| \\
&= \left| \int_{t=0}^{1} u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right) dG(t) - \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) - \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right) dG(t+\Delta_i) \right| \\
&\qquad\qquad \leq \int_{t=0}^{\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i\right) dG(t) + \int_{t=1-\Delta_i}^{1} u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right) dG(t) \\
&\qquad\qquad\quad + \int_{t=0}^{1-\Delta_i} u^{H\delta\bar{y}}\left(\tau_t^i+\Delta_i\right) |\psi(t) - \psi(t+\Delta_i)|\, dt \\
&\qquad\qquad\qquad \leq (1-\delta)\bar{y}k_3\Delta_i + (1-\delta)\bar{y}k_3\Delta_i + (1-\delta)\bar{y}k_4\Delta_i < \frac{\varepsilon}{2}.
\end{aligned}$$

For all $i > i^*$, G in $\mathscr{G}$, H in $\mathscr{H}$, and $\bar{y} \in (0, \mathbf{y}]$, $\left|Eu^{GH\delta\bar{y}i} - Eu^{GH\delta\bar{y}}\right| < \varepsilon$ as claimed.

We now show that $E^i[\bar{y}Z]$ converges uniformly to $E^\infty[\bar{y}Z]$. Combined with the preceding result, this will show that $E\Pi^{GH\delta\bar{y}i}$ converges uniformly to $E\Pi^{GH\delta\bar{y}}$. First,

$$E^i[\bar{y}Z] = \sum_{c=1}^{1/\Delta_i} E^i[\bar{y}Z|t = c\Delta_i]\,[G(c\Delta_i) - G((c-1)\Delta_i)] = \int_{t=0}^{1} E^i\left[\bar{y}Z|\tau_t^i+\Delta_i\right] dG(t)$$

so by the triangle inequality, $\left|E^i[\bar{y}Z] - E^\infty[\bar{y}Z]\right| \leq A_1''' + A_2'''$ where

$$A_1''' = \int_{t=0}^{1} \left|E^i\left[\bar{y}Z|\tau_t^i+\Delta_i\right] - E^\infty\left[\bar{y}Z|\tau_t^i+\Delta_i\right]\right| dG(t)$$

and $A_2''' = \int_{t=0}^{1} \left| E^{\infty}\left[\bar{y}Z|\tau_t^i + \Delta_i\right] - E^{\infty}\left[\bar{y}Z|t\right]\right| dG(t)$. By (90), $A_1''' \leq \bar{y}\Delta_i'$. Integrating by parts, $E^{\infty}[Z|t] = 1 - \int_{z=0}^{1} H(z|t)\, dz$. Thus, by L-$H$,

$$\begin{aligned}\left|E^{\infty}\left[\bar{y}Z|t = \tau_t^i + \Delta_i\right] - E^{\infty}\left[\bar{y}Z|t\right]\right| &\leq \bar{y}\int_{z=0}^{1} \left|H\left(z|\tau_t^i + \Delta_i\right) - H(z|t)\right| dz \\ &\leq \bar{y}k_1 \int_{z=0}^{1} \left|\tau_t^i + \Delta_i - t\right| dz \leq \bar{y}k_1\Delta_i,\end{aligned}$$

so $A_2''' \leq \bar{y}k_1\Delta_i$. Hence, $E^i[\bar{y}Z]$ converges uniformly to $E^{\infty}[\bar{y}Z]$ as claimed.

As for homogeneity in $\bar{y}$, using (26), equation (27) for $\bar{y} = 1$ can be rewritten as

$$\frac{dD^{H\delta 1}}{dt} = \frac{1}{1-\delta} \frac{\int_{z=0}^{D^{H\delta 1}} \frac{\partial H(z|t)}{\partial t} dz}{1 - H\left(D^{H\delta 1}|t\right)}.$$

Hence, for any solution $D^{H\delta\bar{y}}$ to CP$^{H\delta\bar{y}}$, $D^{H\delta 1} = \bar{y}^{-1}D^{H\delta\bar{y}}$ is a solution to CP$^{H\delta 1}$. But both solutions $D^{H\delta\bar{y}}$ and $D^{H\delta 1}$ are unique as shown above. Hence, $D^{H\delta\bar{y}}$ must equal $\bar{y}D^{H\delta 1}$. Thus, the expected payout $p^{H\delta\bar{y}}(t) = v^{H\bar{y}}\left(D^{H\delta\bar{y}}(t), t\right)$ must equal $v^{H\bar{y}}\left(\bar{y}D^{H\delta 1}(t), t\right)$. But by (24), for any D, $v^{H\bar{y}}(D,t)$ equals $\bar{y}v^{H1}\left(\frac{D}{\bar{y}}, t\right)$, so $p^{H\delta\bar{y}}(t) = \bar{y}p^{H\delta 1}(t)$, $u^{H\delta\bar{y}}(t) = \bar{y}u^{H\delta 1}(t)$, and $Eu^{GH\delta\bar{y}} = \bar{y}Eu^{GH\delta 1}$ as claimed. These properties hold for $\Pi^{H\delta\bar{y}}$ and $E\Pi^{GH\delta\bar{y}}$ as well since $E^{\infty}[\bar{y}Z|t]$ is homogeneous of degree 1 in $\bar{y}$.

As for model i, by (22),

$$v^{H\bar{y}i}(D,t) = \bar{y}v^{H1i}\left(\frac{D}{\bar{y}}, t\right), \tag{91}$$

so if $D^{H\delta 1i}$ solves DP$^{H\delta 1i}$, then $\bar{y}^{-1}D^{H\delta 1i}$ solves DP$^{H\delta\bar{y}i}$. But both solutions $D^{H\delta\bar{y}i}$ and $D^{H\delta 1i}$ are unique as shown above. Hence, $D^{H\delta\bar{y}i}$ must equal $\bar{y}D^{H\delta 1i}$. Thus, by (91), the expected payout $p^{H\delta\bar{y}i}(t) = v^{H\bar{y}i}\left(D^{H\delta\bar{y}i}(t), t\right)$ must equal $\bar{y}v^{H1i}\left(D^{H\delta 1i}(t), t\right) = \bar{y}p^{H\delta 1i}(t)$ and so $u^{H\delta\bar{y}i}(t)$ equals $\bar{y}u^{H\delta 1i}(t)$ as claimed. These properties hold for $\Pi^{H\delta\bar{y}i}$ and $E\Pi^{GH\delta\bar{y}i}$ as well since $E^i[\bar{y}Z|t]$ is homogeneous of degree 1 in $\bar{y}$. This completes the proof of Theorem 1. Q.E.D.$_{\text{Theorem 1}}$

4 Relaxing Monotonicity

We now show that the monotonicity assumed in ASSUMPTION A is not necessary for our results, in the case of two assets and two types with generic parameters. Why? By genericity, one type must expect her portfolio to have a higher total payout than the other. Swapping indices if needed, we can assume this is type 2. Also by genericity, an increase in the seller's type from 1 to 2 must raise the expected value of one asset proportionally more than that of the other. Again swapping indices if needed, we can assume that asset 2 has this property. With respect to this labeling of types and assets, Increasing Informational Sensitivity (IIS) holds: an increase in the seller's type from 1 to 2 raises the expected value of asset 2 proportionally more than that of asset 1. We then show that IIS can be used as an alternative to monotonicity in establishing a unique intuitive equilibrium of the 2x2 model.

In this equilibrium, type 1 sells her entire portfolio, while type 2 retains first asset 2 and then, if needed, asset 1, until type 1 is just willing not to imitate her. Hence, as in our base model with the IIS assumption, the Pecking Order property holds: the asset whose value rises faster in the seller's type is retained first. However, there is one contrast. Without monotonicity, asset 1 may be worth *less* to type 2 than to type 1.[6] If so, the seller uses only asset 2 to signal her type: both types sell asset 1 in its entirety. Intuitively, type 1 gains more from retaining asset 1 than type 2 does. Hence, if type 2 retains some of asset 1, then she must retain even more of asset 2 in order to credibly signal her type. And such inefficient signalling is ruled out by the Intuitive Criterion, which we assume.

To take a concrete example, suppose a firm has both bonds and common stock to sell, and these are secured by a common underlying cash flow (such as the value of its real assets). Suppose when the firm's type is high, both the mean and the variance of its cash flow are higher. The latter effect raises the risk of bond default. If the default effect is strong enough, the high type can place a lower value on the bonds than the low type. In this case, the model predicts that the firm will always sell its bonds in their entirety: it will signal optimism by retaining stock alone.

[6] As shown below in Claim 9, this is not true of asset 2: it is worth more to type 2 than to type 1, as in the base model.

4.1 The Model

Let A2NM denote the generic 2x2 asset game without monotonicity, which is as follows. There are two assets $i = 1, 2$ and the seller has two possible types $t = 1, 2$.[7] Each type has positive prior probability. The seller owns a single unit of each asset: the endowment vector a is $(1, 1)$. Let

$$f_i(t) > 0 \tag{92}$$

denote the expected payout of asset $i = 1, 2$ conditional on the seller's type being $t = 1, 2$. We restrict to generic parameters so we can assume, w.l.o.g., that type 2 has a higher conditional expected portfolio value than type 1:

$$f_1(2) + f_2(2) > f_1(1) + f_2(1) > 0. \tag{93}$$

(If not, swap type indices.) Genericity also lets us assume, w.l.o.g., that

$$f_2(2)/f_2(1) > f_1(2)/f_1(1) > 0. \tag{94}$$

(If not, swap asset indices.) Intuitively, (94) says that an increase in the seller's type raises the expected value of asset 2 proportionally more than that of asset 1.

Under our assumptions, asset 1 may be worth less to type 2 than it is to type 1: $f_1(2)$ may be less than $f_1(1)$. On the other hand, asset 2 must be worth more to type 2 than it is to type 1:

Claim 9. Assume (93) and (94). Then

$$f_2(2) > f_2(1). \tag{95}$$

Proof. If instead $f_2(1) \geq f_2(2)$ then $f_2(2)/f_2(1) \leq 1$ whence, by (94), $f_1(2)/f_1(1) < 1$ so $f_1(1) > f_1(2)$. Combining the first and last inequalities yields $f_1(1) + f_2(1) > f_1(2) + f_2(2)$, which contradicts (93). □

[7] We let the type numbers start at one rather than than at zero so as to mirror the asset numbering.

In all other respects, A2NM is simply the Asset Sale game specialized to the 2x2 case. On seeing her type t, the seller chooses quantities $q = (q_1, q_2) \in [0,1]^2$ to offer for sale as well as price caps $\overline{p} = (\overline{p}_1, \overline{p}_2)$. A competitive set of deep-pocketed, uninformed investors see this choice $(q, \overline{p})$ and form posterior beliefs $\mu(t|q, \overline{p})$ about the probability of each type t. The seller then sells q_i units of each asset $i = 1,2$ to the investors for some price $p_i \leq \overline{p}_i$; let $p = (p_1, p_2)$ denote the price vector. The payoff of a seller of type t is then $q[p - \delta f(t)]$: her issuance revenue pq less the present discounted value $\delta q f(t)$ of the assets she sells. This completes the specification of A2NM.

In A2NM, the Recursive Linear Program (RLP) takes the following simple form. For any $x \in [0,1]$, define

$$\Delta^x = (\Delta_1^x, \Delta_2^x) = f(2) - x f(1). \tag{96}$$

We will often write Δ and Δ_i as shorthand for Δ^δ and Δ_i^δ,respectively:

$$\Delta = (\Delta_1, \Delta_2) = f(2) - \delta f(1) \text{ where } \Delta_i = f_i(2) - \delta f_i(1). \tag{97}$$

By (95), Δ_2 is positive; however, Δ_1 may be of either sign.

RLP Let Q_1 equal the set $Q = [0,1]^2$ of all quantity vectors and let

$$Q_2 = \{q : q\Delta \leq u^*(1)\} \tag{98}$$

be the set of all such vectors that can be assigned to type 2 which, if accompanied by revenue $\rho = qf(2)$, do not tempt type 1 to imitate type 2. For $t = 1,2$, let

$$q^*(t) \in \arg\max_{q \in Q_t} [qf(t)], \tag{99}$$

$$\rho^*(t) = q^*(t) f(t), \text{ and} \tag{100}$$

$$u^*(t) = \rho^*(t) - \delta q^*(t) f(t) = (1-\delta) q^*(t) f(t). \tag{101}$$

Intuitively, $\rho^*(t)$ and $u^*(t)$ are the revenue and payoff functions of a type-t seller from the quantity vector $q^*(t)$ under fair pricing: when the resulting price vector is $f(t)$. We now characterize the solution to RLP with a sequence of results.

Claim 10. Any solution to RLP has the following properties.

1. Type 1 sells his whole portfolio:

$$q^*(1) = a = (1,1), \tag{102}$$

yielding revenue $\rho^*(1) = f_1(1) + f_2(1)$ and seller's payoff $u^*(1) = (1-\delta)[f_1(1) + f_2(1)]$.

2. Type 1's incentive compatibility (IC) constraint binds at type 2's quantity vector $q^*(2)$:

$$q^*(2)\Delta = u^*(1). \tag{103}$$

3. Type 2 sells all of asset 1 if she sells any of asset 2: if $q_2^*(2) > 0$, then $q_1^*(2) = 1$.

4. A type 2 seller retains a portion of asset 2: $q_2^*(2) < 1$.

5. The seller's revenue and payoff are decreasing in her type: $\rho^*(1) > \rho^*(2) > 0$ and $u^*(1) > u^*(2) > 0$.

The intuition for part 2 is simple: if type 1's IC constraint does not bind, then type 2 can sell a bit more of either asset without being mistaken for type 1, so she will do so. Parts 2 and 3 also yield an algorithm for finding type 2's optimal quantity vector. We start with $q = a$ and gradually lower q_2 until it reaches zero, and then begin lowering q_1. By part 2, we must stop when type 1 is just willing not to imitate and be mistaken for 2: at which equation (103) holds. Hence, a seller does not sell any of asset 2 until she has liquidated all of asset 1. Thus, the Pecking Order property carries over to the 2x2 case without monotonicity. The algorithm implied by parts 2 and 3 pins down the quantities $q^*(2)$ that type 2 sells:

Claim 11. In any solution to RLP, the quantities $q^*(2)$ sold by type 2 are unique and given by

$$q_1^*(2) = \min\left\{1, \frac{u^*(1)}{\Delta_1}\right\} \in (0,1] \tag{104}$$

and

$$q_2^*(2) = \max\left\{0, \frac{u^*(1) - \Delta_1}{\Delta_2}\right\} \in [0,1). \tag{105}$$

Claim 11 has an interesting implication. If the high-type seller does not value asset 1 more than the low type, then both types sell asset 1 in its entirety. This is the qualitative impact of discarding ASSUMPTION A. In particular, type 2 retains only asset 2 in order to signal her information. Intuitively, if type 2 does not value asset 1 more highly than type 1, then retaining it does not help her separate from type 1. As retention is costly, she will thus sell asset 1 in its entirety.

Corollary 1. *If the expected value of asset 1 is not increasing in the seller's type, then neither type retains any shares of this asset.*

Proof. By (102), type 1 sells asset 1 in its entirety. As for type 2,

$$\begin{aligned} u^*(1) - \Delta_1 &= (1-\delta)[f_1(1) + f_2(1)] - f_1(2) + \delta f_1(1) \\ &= f_1(1) - f_1(2) + (1-\delta) f_2(1). \end{aligned}$$

If $f_1(2) \leq f_1(1)$, the second line is positive and thus $q_1^*(2) = 1$ by (104). $\square$

The claim can also be proved without relying on (104). If type 2 retains part of asset 1 in equilibrium (if $q_1^*(2) < 1$) then she must also retain asset 2 in its entirety ($q_2^*(2) = 0$) by part 3 of Claim 10. Now suppose type 1 imitates type 2: he retains all of asset 2 as well as part of asset 1, without obtaining a higher price for asset 1 than he gets in equilibrium (since $f_1(2) \leq f_1(1)$). His payoff from this deviation must be less than $(1-\delta) f_1(1)$ which, in turn, is less than his equilibrium payoff of $u^*(1) = (1-\delta)[f_1(1) + f_2(1)]$. Since the deviation is strictly worse for type 1, his IC constraint does not bind, which contradicts part 1 of Claim 10.

The preceding results have the following useful implication.

Corollary 2. *In A2NM, RLP has a unique solution* (q^*, ρ^*, u^*).

Proof. By part 1 of Claim 10 and by Claim 11, the quantity function q^* is unique. Using (100) and (101), this then pins down the principal's revenue and payoff functions, ρ^* and u^*. $\square$

We now prove that the unique solution to RLP is also the unique intuitive outcome. A perfect Bayesian equilibrium (PBE) of A2NM is a profile

$$e = (q(\cdot), \overline{p}(\cdot), \mu(\cdot|\cdot,\cdot), p(\cdot,\cdot))$$

of strategies and beliefs that satisfies the following properties.[8]

Profit Maximization. The seller's behavior is optimal given the investors' price function: for each $t = 1, 2$, $(q(t), \overline{p}(t)) \in \arg\max_{q,\overline{p}} (q[p(q,\overline{p}) - \delta f(t)])$.

Competitive Pricing. Given the investors' beliefs, an asset's price equals its conditional expected value or the price cap, whichever is lower:

$$p(q,\overline{p}) = \overline{p} \wedge \left[\sum_{t=1}^{2} f(t)\, \mu(t|q,\overline{p}) \right]. \tag{106}$$

Rational Updating. Investors' beliefs $\mu(\cdot|q,\overline{p})$ are given by Bayes's Rule if some type chooses $(q,\overline{p})$ in equilibrium.

Clearly, any PBE e also induces a reduced-form price function $p(t) = p(q(t), \overline{p}(t))$, revenue function $\rho(t) = q(t)\, p(t)$, and payoff function $u(t) = \rho(t) - \delta q(t) f(t)$ of the seller.[9]

In this setting, the Intuitive Criterion is as follows. Fix a PBE e of A2NM with seller's payoff function $u(\cdot)$. Suppose a seller of type t deviates to $(q,\overline{p})$. The deviation must harm her if her equilibrium payoff $u(t)$ exceeds her maximum payoff $q[\overline{p} - \delta f(t)]$ from the deviation - or, equivalently, if her opportunity cost $u(t) + \delta q f(t)$ of deviating exceeds her maximum deviation revenue $q\overline{p}$. The Intuitive Criterion states that on seeing a deviation $(q,\overline{p})$, investors put zero weight on any type t who is definitely harmed by the deviation if there is some other type s who is not definitely harmed:

[8] The notation $V \wedge V'$ and $V \vee V'$ denotes componentwise minimum and maximum, respectively, for any vectors V and V' of equal length.

[9] We ignore the investors' payoff function $q(t)\, f(t) - \rho(t)$, which is identically zero in any intuitive equilibrium.

The Intuitive Criterion. A PBE e with seller's payoff function $u(\cdot)$ is intuitive if, for any deviation $(q, \overline{p})$ and any type t such that $u(t) + \delta q f(t) > \overline{p} q$, investors' posterior weight $\mu(t|q, \overline{p})$ on type t is zero if there exists a type $s \neq t$ for whom $u(s) + \delta q f(s) \leq \overline{p} q$.

We now embed the (by Corollary 2) unique quantity function q^* in a profile

$$e^* = (q^*(\cdot), \overline{p}^*(\cdot), \mu^*(\cdot|\cdot,\cdot), p^*(\cdot,\cdot)),$$

which we will show is an intuitive PBE; moreover, the outcome of any intuitive PBE is the same as that of e^*. Other than $q^*(\cdot)$, the components of e^* are as follows.

1. Any price caps may be chosen on the equilibrium path as long as they do not bind for any possible beliefs: $\overline{p}^*(t) \geq f(1) \vee f(2)$ for each type t.

2. The price function $p^*(\cdot,\cdot)$ is given by (106) with investors' beliefs μ^* replacing μ. Hence Competitive Pricing holds.

3. Investors' beliefs μ^* are given by Bayes's Rule in equilibrium: investors believe that the seller is of type 1 (resp., 2) for sure if they see the issuance choice $(q^*(1), \overline{p}^*(1))$ (resp., $(q^*(2), \overline{p}^*(2))$). Thus, Rational Updating holds. Since, moreover, the price caps do not bind, each equilibrium price $p_i^*(q^*(t), \overline{p}^*(t))$ equals the conditional expected payout $f_i(t)$ of asset i: the profile e^* is fairly priced. It follows from (100) and (101) that the type-t seller's revenue and payoff functions are $\rho^*(t)$ and the payoff $u^*(t)$ given by RLP. If the seller chooses a pair $(q, \overline{p})$ that never occurs in equilibrium, investors' beliefs $\mu^*(\cdot|q, \overline{p})$ put positive weight only on types t that have the lowest opportunity cost of giving up their equilibrium payoff u^*: those in the set $T(q) = \arg\min_{t'} [u^*(t') + \delta q f(t')]$. Hence, beliefs are intuitive. In particular, if the set $T(q)$ is a singleton $\{t\}$, then investors believe the seller is of type t. If $T(q)$ contains both types $t = 1, 2$, investors believe the seller is of that type t for which the deviation revenue $q[\overline{p} \wedge f(t)]$ under symmetric information is minimized. If this deviation revenue is the same for the two types, then investors' beliefs do not matter.

We can assume, e.g., that they believe that $t = 1$, although any other beliefs would give the same results.

The above specification and argument implies that, in e^*, a type-t seller sells the quantities $q^*(t)$ and gets the revenue $\rho^*(t)$ and payoff $u^*(t)$ that solve RLP. It remains only to show that e^* is the unique intuitive PBE. More precisely, (a) it is an intuitive PBE, and (b) any other intuitive PBE has identical equilibrium behavior and payoffs to e^*.[10]

Proposition 1. *The profile e^* is an intuitive PBE.*

Proposition 2. *If $e = (q(\cdot), \overline{p}(\cdot), \mu(\cdot|\cdot,\cdot), p(\cdot,\cdot))$ is an intuitive PBE of A2NM, with outcome $u(\cdot)$ and price function $p(\cdot)$, then the seller makes the same quantity choices and receives the same payoffs in e as in e^*: for each type t, $q(t) = q^*(t)$ and $u(t) = u^*(t)$. Moreover, if the quantity $q_i(t) = q_i^*(t)$ of an asset i sold by a type t is positive, the resulting price of asset i is the same in the two equilibria and equals the fair value of the asset: $p_i(t) = p_i^*(t) = f_i(t)$.*

4.2 Relaxing Monotonicity: Proofs

Proof of Claim 10. Part 1. First, $af(2) > af(1)$ by (93) whence $a \notin Q_2$ by (98). Hence either $q_1^*(2)$ or $q_2^*(2)$ is less than one; assume w.l.o.g. that $q_2^*(2) < 1$. Thus, if (103) does not hold then for ε in $(0, 1 - q_2^*(2)]$, the vector $q' = q^*(2) + (0, \varepsilon)$ is also in Q_2. By (98), q' offers type 2 a higher payoff under symmetric information than $q^*(2)$ does. Hence $q^*(2)$ is not optimal for type 2 in Q_2, a contradiction.

Part 2. Suppose not: $q_1^*(2) < 1$ and $q_2^*(2) > 0$. We will derive a contradiction. Let $\varepsilon > 0$ be small enough that $q_1^*(2) + \varepsilon < 1$ and $q_2^*(2) - \iota > 0$ where $\iota = \varepsilon\Delta_1/\Delta_2$, whose denominator Δ_2 is positive as noted above. Consider the alternative quantity vector $q =$

[10] The other PBE may differ in ways that are not payoff relevant. In particular, it may have different nonbinding price caps and different beliefs following deviations (as long as these beliefs are intuitive and serve to deter the deviations). Moreover, if a seller does not sell any shares of a given asset, then the price of this asset is indeterminate and hence may vary across equilibria.

(q_1, q_2) where $q_1 = q_1^*(2) + \varepsilon$ and q_2 equals the lesser of $q_2^*(2) - \iota$ and one. First, $u^*(1) - q\Delta$ can be written as the sum of $u^*(1) - q^*(2)\Delta$, which is nonnegative since $q^*(2)$ is in Q_2, and $[q^*(2) - q]\Delta$, which by (95) is bounded below by $-\varepsilon\Delta_1 + \iota\Delta_2$ which equals zero by definition of ι. Hence $u^*(1) \geq q\Delta$, whence q is in Q_2. Moreover, the effect on type 2's payoff from switching from $q^*(2)$ to q (under symmetric information) equals the difference $1 - \delta$ in discount factors times the change in revenue $qf(2) - q^*(2)f(2)$ which, in turn, equals $\min\{\varepsilon f_1(2) - \iota f_2(2), \varepsilon f_1(2) + 1 - q_2^*(2)\}$. But both elements in the min are positive: the first by (94)[11] and the second by (92) and since $\varepsilon > 0$ and $q_2^*(2) \leq 1$. We conclude that q, which is in Q_2, is better for 2 than $q^*(2)$ under symmetric information, which contradicts the definition of $q^*(2)$.

Part 3. This holds by part 2 and since, as shown in the proof of part 1, $q^*(2) \neq a$.

Part 4. Together with (92), part 3 implies $\delta q^*(1) f(1) = \delta a f(1) > \delta q^*(2) f(1)$ which, substituted into (103), implies $q^*(1) f(1) > q^*(2) f(2)$ as claimed. Multiplying each side by $1 - \delta$, we obtain $u^*(1) > u^*(2)$. Clearly, $u^*(2) \geq 0$ since a type 2 seller can always set $q = (0,0)$. Finally, let $\varepsilon = \frac{u^*(1)}{a\Delta}$. The denominator is positive by (93). The numerator is positive by (101) and (102) and less than the denominator by these equations together with (93). Hence, $\varepsilon \in (0,1)$ whence the quantity vector $(\varepsilon, \varepsilon)$ is feasible. Moreover, this vector is in Q_2 since $(\varepsilon, \varepsilon)\Delta = \varepsilon a\Delta = u^*(1)$ and by (98). It follows that $u^*(2)$ is not less than $(1 - \delta)(\varepsilon, \varepsilon) f(2)$, which is positive by (92). Hence $u^*(2)$ and $\frac{u^*(2)}{1-\delta} = q^*(2) f(2)$ are positive as claimed. Q.E.D.$_{\text{Claim 10}}$

PROOF OF CLAIM 11. Let $q_i = q_i^*(2)$ for assets $i = 1, 2$. By part 1 of Claim 10, type 1's IC constraint binds so

$$u^*(1) = q_1\Delta_1 + q_2\Delta_2. \tag{107}$$

[11] By (94), $\delta f_1(2) f_2(1) < \delta f_2(2) f_1(1)$, whence

$$f_1(2)\Delta_2 > f_2(2)\Delta_1$$

which can be rewritten as $f_1(2)\Delta_2^\delta > f_2(2)\Delta_1^\delta$. This, in turn, implies $\varepsilon f_1(2) > \varepsilon f_2(2)\Delta_1^\delta/\Delta_2^\delta = \iota f_2(2)$ as claimed.

We know from part 3 of Claim 10 that $q_2 < 1$. There are now three cases.

1. Say $q_1 < 1$. Then $q_2 = 0$ by part 3 of Claim 10. Thus, by (107), $u^*(1) = q_1\Delta_1$ whence $q_1 \neq 0$ and thus $q_1 > 0$ which implies $\Delta_1 > 0$. For $q_1 < 1$ we require $u^*(1) < \Delta_1$ (which implies $\Delta_1 > 0$).

2. Say $q_2 > 0$. Then $q_1 = 1$ by part 3 of Claim 10. Thus, by (107), $q_2\Delta_2 = u^*(1) - \Delta_1$ so, since $\Delta_2 > 0$ by (95), $u^*(1) > \Delta_1$.

3. Say that neither #1 nor #2 holds: $q_1 = 1$ and $q_2 = 0$. Then by (107), $u^*(1) = \Delta_1$.

As the above three conditions $u^*(1) \gtreqless \Delta_1$ are mutually exclusive and exhaustive, we can replace "if" in cases 1-3 by "if and only if". This implies the following three mutually exclusive possibilities, which corresponds respectively to cases 1-3 above.

1. If $\Delta_1 > u^*(1)$ then $q_1^*(2) = u^*(1)/\Delta_1 \in (0,1)$ and $q_2^*(2) = 0$.

2. If $\Delta_1 < u^*(1)$ then $q_1^*(2) = 1$ and $q_2^*(2) = \frac{u^*(1)-\Delta_1}{\Delta_2} \in (0,1)$.

3. If $\Delta_1 = u^*(1)$ then $q_1^*(2) = 1$ and $q_2^*(2) = 0$.

This result is stated more succinctly in equations (104) and (105). Q.E.D.$_{\text{Claim 11}}$

PROOF OF PROPOSITION 1. Beliefs are intuitive by construction and we verify Competitive Pricing and Rational Updating in the text. To show that e^* is an intuitive PBE, it thus remains only to verify Payoff Maximization: that neither type has a (strictly) profitable deviation. By construction, investors' beliefs μ^* following a deviation $(q, \overline{p})$ depend on q but not on $\overline{p}$. Hence, if a deviator chooses binding price caps, this can only lower his payoff from deviating. We can thus assume, w.l.o.g., that a deviator chooses price caps that can never bind (such as the caps $\overline{p}^*$).

We begin with type 1. Let

$$\Gamma(q) = q\Delta - u^*(1) \tag{108}$$

denote the change in type 1's payoff from deviating to q and being mistaken for type 2. Let

$$\Lambda(q) = [u^*(2) + \delta q f(2)] - [u^*(1) + \delta q f(1)] \tag{109}$$

denote the difference between type 2's and type 1's opportunity cost of deviating to q. By construction of μ^*, investors must believe the seller is of type 1 (resp., 2) when $\Lambda(q)$ is positive (negative). And when $\Lambda(q) = 0$, they think he is of type 1: beliefs are governed by deviation revenue $qf(t)$ which by (109) and part 3 of Claim 10, is lower for type 1 in this case. By (97), (101), and (103),

$$\Gamma(q^*(2)) = \Lambda(q^*(2)) = 0. \tag{110}$$

Now suppose type 1 deviates to $(q, \overline{p})$. If $\Lambda(q) \geq 0$, investors will think the seller's type is 1: he gets $(1-\delta) q f(1)$ which cannot exceed his equilibrium payoff $u^*(1)$ as $q \leq a$. So such a deviation is not profitable. If instead $\Lambda(q) < 0$, investors will think his type is 2 whence his payoff changes by $\Gamma(q)$, which is nonpositive by the following result.

Claim 12. For any q in $[0,1]^2$, if $\Lambda(q) < 0$ then $\Gamma(q) \leq 0$.

PROOF OF CLAIM 12. Suppose not: $\Lambda(q) < 0$ and $\Gamma(q) > 0$. We will derive a contradiction. For brevity let $\widehat{q}$ denote $q^*(2)$. Recall that Δ_i^x is defined in (96). By (92) and (95),

$$\Delta_2^x > 0 \tag{111}$$

for all such x, whence

$$\frac{d}{dx} \frac{\Delta_1^x}{\Delta_2^x} \propto f_1(2) f_2(1) - f_2(2) f_1(1) < 0 \tag{112}$$

by (94). We will rely on the following lemma.

Lemma 5. *Let $q' \in [0,1]^2$ be such that $\Gamma(q') \geq 0$ and $\Lambda(q') < 0$. Then $q'_1 > \widehat{q}_1$.*

Proof. By (110), $\Gamma(\widehat{q}) = 0 \leq \Gamma(q')$ whence by (108),

$$\left(\widehat{q}_1 - q'_1\right) \frac{\Delta_1^\delta}{\Delta_2^\delta} + \left(\widehat{q}_2 - q'_2\right) \leq 0. \tag{113}$$

Also by (110), $\Lambda(\widehat{q}) = 0 > \Lambda(q')$ whence, by (109),

$$\left(\widehat{q}_1 - q_1'\right) \frac{\Delta_1^1}{\Delta_2^1} + \left(\widehat{q}_2 - q_2'\right) > 0. \tag{114}$$

Subtracting (113) from (114) yields $\left(\widehat{q}_1 - q_1'\right)\left(\frac{\Delta_1^1}{\Delta_2^1} - \frac{\Delta_1^\delta}{\Delta_2^\delta}\right) > 0$, which by (112) yields $q_1' > \widehat{q}_1$. $\square$

Since $\Gamma(q) > 0$, $q\Delta^\delta > u^*(1) > 0$ by (108) and part 4 of Claim 10. Let $\alpha = u^*(1) / \left[q\Delta^\delta\right] \in (0,1)$ and $\widetilde{q} = \alpha q$. Since $\Lambda(q) < 0$ by assumption, $u^*(1) - u^*(2)$ (which is positive by part 4 of Claim 10) exceeds $\delta q \Delta^1$ and thus also exceeds $\delta \widetilde{q} \Delta^1$ since $\widetilde{q} = \alpha q$ and $\alpha \in (0,1)$. Rearranging, we obtain $\Lambda(\widetilde{q}) < 0$. Since, moreover, $\Gamma(\widetilde{q}) = 0$ by (108) and (96), Lemma 5 implies that $\widetilde{q}_1 > \widehat{q}_1$. Hence, by (112),

$$\left(\widetilde{q}_1 - \widehat{q}_1\right)\left(\frac{\Delta_1^0}{\Delta_2^0} - \frac{\Delta_1^\delta}{\Delta_2^\delta}\right) > 0. \tag{115}$$

We can also rearrange $\Gamma(\widehat{q}) = \Gamma(\widetilde{q}) = 0$ to obtain

$$\left(\widetilde{q}_1 - \widehat{q}_1\right) \frac{\Delta_1^\delta}{\Delta_2^\delta} + \widetilde{q}_2 - \widehat{q}_2 = 0. \tag{116}$$

Adding (115) and (116) yields $\left(\widetilde{q}_1 - \widehat{q}_1\right) \frac{\Delta_1^0}{\Delta_2^0} + \widetilde{q}_2 - \widehat{q}_2 > 0$. This can be rearranged (with the result multiplied by $1-\delta$) to yield $(1-\delta)\widetilde{q}\Delta^0 > (1-\delta)\widehat{q}\Delta^0$. Moreover, $\widetilde{q}$ is in Q_2 since $\Gamma(\widetilde{q}) = 0$. This contradicts the definition of $\widehat{q}$ as the vector in Q_2 that maximizes type 2's symmetric-information payoff. Q.E.D.$_{\text{Claim 12}}$

We have shown that a type 1 seller will not deviate. Let us now consider a deviation $(q, \overline{p})$ by a type 2 seller. There are two cases.

1. $\Lambda(q) \geq 0$. Then on seeing q, investors believe the seller's type is 1. Hence, the deviation is profitable for the type 2 seller only if

   $$0 < \Omega(q) = q\left[f(1) - \delta f(2)\right] - u^*(2).$$

   If this holds then, by (109), $0 < \Lambda(q) + \Omega(q) = (1-\delta) q f(1) - u^*(1)$ which is not possible by (92) and since $q \leq a$: the deviation is not profitable.

2. $\Lambda(q) < 0$. Then on seeing q, investors believe the seller's type is 2. Hence, type 2's change in payoffs from this deviation equals $\Gamma(q)$, which is nonpositive by Claim 12: the deviation is not profitable.

As this verifies Payoff Maximization, e^* is an intuitive PBE. Q.E.D.$_{\text{Proposition 1}}$

PROOF OF PROPOSITION 2. We first show that in any intuitive PBE, a seller of a given type is paid the expected value (given her type) of the portfolio that she sells. The precise property is as follows.

Fair Pricing. A PBE e with quantity function $q(\cdot)$ and price function $p(\cdot)$ is Fairly Priced if, for each type t, $p(t) f(t) = q(t) f(t)$.

Lemma 6. *Any intuitive PBE e is Fairly Priced.*

Proof. Let e be intuitive, and suppose type t chooses $(q, \overline{p})$ in equilibrium and that investors respond with price vector p. First suppose the issuance is underpriced: $pq < qf(t)$. Let S denote the set of types s for which $qf(s) < qf(t)$. There are two cases.

1. S is empty. Then by Competitive Pricing, if the type t seller deviates to $(q, f(t))$, her revenue cannot be less than $qf(t)$. As this would be a profitable deviation, this case is not possible.

2. S is nonempty. Let $s' \in S$ be a type s for which $qf(s)$ is at a maximum among all s in S: there is no type s in S for which $qf(s)$ exceeds $qf(s')$. Then $qf(s') < qf(t)$ whence, since $pq < qf(t)$, there is a $\lambda \in [0,1)$ such that

$$\frac{q[p - \delta f(t)]}{q[f(t) - \delta f(t)]} < \lambda < \frac{q[p - \delta f(s)]}{q[f(t) - \delta f(s)]}. \tag{117}$$

Let type t deviate to $(\lambda q, f(t))$. By (117), $\lambda q[f(t) - \delta f(t)]$ exceeds $q[p - \delta f(t)]$, which equals $u(t)$. And for each s in S, $\lambda q[f(t) - \delta f(s)]$ is less than $q[p - \delta f(s)]$ which, in turn, is not greater than $u(s)$. Accordingly, for each s in S,

$$u(t) + \delta(\lambda q) f(t) < \lambda q f(t) < u(s) + \delta(\lambda q) f(s)$$

and hence, since s is intuitive, $\mu(s|(\lambda q, f(t)))$ is zero for each s in S. Hence, by Competitive Pricing, the revenue that results from the deviation $(\lambda q, f(t))$ is $\lambda q f(t)$ whence type t's payoff from the deviation is $\lambda q[f(t) - \delta f(t)]$ which, as noted, exceeds $u(t)$: the deviation is profitable, which is not possible.

We have shown that in an intuitive PBE e, underpricing cannot occur (except perhaps off the equilibrium path). Hence, $pq \geq qf(t)$ for any type t who chooses $(q, \overline{p})$ in equilibrium. But by Competitive Pricing, $pq = q\left(\overline{p} \wedge \left[\sum_{t=1}^{2} f(t)\mu(t|q,\overline{p})\right]\right) \leq \sum_{t=1}^{2} qf(t)\mu(t|q,\overline{p})$. Thus, pq is bounded above by a convex combination of $qf(t)$ for those types t who choose $(q, \overline{p})$ in equilibrium. But by the prior result, pq is also bounded below by $qf(t)$ for each such type t, so it cannot be less than the highest such $qf(t)$. Hence all types who choose $(q, \overline{p})$ in equilibrium have the same value of $qf(t)$, and this also equals the revenue pq that they receive: the equilibrium is Fairly Priced. $\square$

Moreover, if a PBE is Fairly Priced, then each asset that is sold in equilibrium must be priced correctly.

Claim 13. Suppose a PBE e with quantity function $q(\cdot)$ and price function $p(\cdot)$ is Fairly Priced. Then for each type t, and each asset i for which $q_i(t) > 0$, we have $p_i(t) = f_i(t)$.

Proof. Suppose not. Then by Competitive Pricing, there is a type t who sells some shares of an asset i with a binding price cap: $q_i(t) > 0$ and $\overline{p}_i(t) < f_i(t)$. Thus, this type's revenue $p_i(t)q_i(t)$ from asset i is less than the value $q_i(t)f_i(t)$ of shares sold. And by Competitive Pricing, her revenue $p_j(t)q_j(t)$ from any other asset j cannot exceed its true value $q_j(t)f_j(t)$. Combining these facts, we find that her total issuance revenue $p(t)q(t)$ must be less than the total value $q(t)f(t)$ of shares she sells, which violates Fair Pricing. $\square$

Now let e be an intuitive PBE with outcome $u(\cdot)$ and price function $p(\cdot)$. By Lemma 6, e is Fairly Priced, whence

$$u(t) = (1-\delta)q(t)f(t) \text{ for each type } t. \tag{118}$$

Hence, $u(1) \le (1-\delta) af(1) = u^*(1)$. And as e is a PBE, type 1 cannot prefer to imitate type 2. So since e is Fairly Priced, type 2's choice $q(2)$ must satisfy $q(2)\Delta \le u(1)$. As $u(1) \le u^*(1)$, this implies that $q(2)$ lies in Q_2. But $u^*(2)$ is type 2's highest payoff $(1-\delta) qf(2)$ among all vectors q in Q_2, whence $u(2) \le u^*(2)$ as well.

Now suppose that in e, type 1 deviates to $(q, \overline{p})$ where $q = a$ and $\overline{p} = f(1) \vee f(2)$.[12] By Competitive Pricing and part 4 of Claim 10, his resulting revenue is $a\left[\sum_{t=1}^{2} f(t)\mu(t|q,\overline{p})\right] \ge af(1)$ whence his payoff from the deviation is at least $u^*(1)$. Thus, for 1 not to deviate, $u(1)$ must equal $u^*(1)$. By (92) and (118), this implies $q(1) = a = q^*(1)$ as claimed.

Now consider type 2. If $u(2) < u^*(2)$, suppose type 2 deviates to $(q, \overline{p})$ where $q = \lambda q^*(2)$ for some $\lambda \in \left(\frac{u(2)}{u^*(2)}, 1\right)$ and $\overline{p} = f(1) \vee f(2)$. By the prior result, $u(1) = u^*(1) = q^*(2)\Delta > \lambda q^*(2)\Delta$ and $u(2) < \lambda u^*(2) = (1-\delta)\lambda q^*(2) f(2)$. Thus, $u(1) + \delta\lambda q^*(2) f(1) > \lambda q^*(2) f(2) > u(2) + \delta\lambda q^*(2) f(2)$ whence, since e is intuitive, $\mu(2|q,\overline{p}) = 1$ and hence, by Competitive Pricing, type 2's payoff from the deviation is $\lambda u^*(2) > u(2)$. Thus, for type 2 not to deviate, her payoff $u(2)$ must equal $u^*(2)$. And since $u(1) = u^*(1)$ and by Fair Pricing, $q(2)$ must lie in Q_2 (else type 1 will deviate to $q(2)$). So $q = q(2)$ maximizes type 2's symmetric-information payoff $(1-\delta) qf(2)$ in Q_2 whence, by part 6 of Claim 10, $q(2)$ must equal $q^*(2)$ as claimed.

Finally, as $q(t)$ equals $q^*(t)$ for each type t, Claim 13 implies that the prices $p_i(t)$ and $p_i^*(t)$ each equal $f_i(t)$ for each asset i for which $q_i(t) = q_i^*(t)$ is positive. Q.E.D.$_{\text{Proposition 2}}$

References

[1] Apostol, T. M. 1981. *Mathematical Analysis*, 2nd ed. Reading, Mass.: Addison-Wesley.

[2] DeMarzo, P.M., D.M. Frankel, and Y. Jin. 2020. Portfolio Liquidity and Security Design with Private Information. Forthcoming, *Review of Financial Studies*.

[12] Recall that $\vee$ denotes the componentwise maximum.

[3] Frankel, D.M. and Y. Jin. 2015. Securitization and lending competition. *Review of Economic Studies* 82:1383-1408.

[4] Mailath, G., and E.-L. von Thadden. 2013. Incentive compatibility and differentiability: new results and classic applications. *Journal of Economic Theory* 148:1841–1861.

[5] Pontryagin, L.S., V.G. Boltyanskii, R.V. Gamkrelidze and E.F. Mishchenko. 1962. *The mathematical theory of optimal processes*. New York: Interscience.