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Author/s:
Wang, Y;Puig, V;Cembrano, G

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RESEARCH ARTICLE

Robust fault estimation based on zonotopic Kalman filter for discrete-time descriptor systems

Ye Wang*[1] | Vicenç Puig[1] | Gabriela Cembrano[1,2]

[1]Institut de Robòtica i Informàtica Industrial, CSIC-UPC, Universitat Politècnica de Catalunya-BarcelonaTech, Barcelona, Spain
[2]Cetaqua, Water Technology Centre, Barcelona, Spain

Correspondence
*Ye Wang, Institut de Robòtica i Informàtica Industrial, CSIC-UPC, Automatic Control Department, Universitat Politècnica de Catalunya-BarcelonaTech, 08028 Barcelona, Spain. Email: ywang@iri.upc.edu

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Summary

This paper proposes a set-based approach for robust fault estimation of discrete-time descriptor systems. The considered descriptor systems are subject to unknown-but-bounded uncertainties (state disturbances and measurement noise) in predefined zonotopes, and additive actuator faults. The zonotopic fault estimation filter for descriptor systems is built based on fault detectability indices and matrix to estimate fault magnitude in a deterministic set. The zonotopic fault estimation filter gain is designed in a parametrised form. Within a set-based framework, following the zonotopic Kalman filter, the optimal filter gain is computed by minimising the size of the corresponding zonotopes to achieve robustness against uncertainties and the identification of occurred actuator faults. Besides, boundedness of the proposed zonotopic fault estimation is analysed, which proves that the size of obtained fault estimation bounds is not growing in time. Finally, the simulation results with two application examples are provided to show the effectiveness of the proposed approach.

KEYWORDS:
robust fault estimation, zonotopic filter, optimal filter gain, boundedness, discrete-time descriptor systems

1 | INTRODUCTION

Fault estimation, as a significant stage of fault diagnosis, aims to estimate the magnitude of occurred faults in a system. The problem of fault estimation has been studied using a large amount of approaches during the past decades, see e.g. [1,2,3]. A suitable fault estimation with robust performance against system uncertainties is very useful for implementing an active fault-tolerant control system [4,5,6]. Based on the robust control techniques, robust fault estimations are implemented in a variety of systems as e.g. [7,8,9], where the effects of uncertainties are bounded and as a result, fault estimation results are obtained with the minimum estimation error.

The model-based fault diagnosis for physical systems relies on making use of the mathematical model to describe the system dynamics by means of differential or difference equations. In terms of large-scale complex systems, such as cyber-physical systems and critical infrastructures, additionally to system dynamics, system variables are also constrained by static relations, for instance mass and energy balances. Therefore, these static relations are modelled using algebraic equations. In this case, the systems, including not only differential/difference equations but also algebraic equations, are called *descriptor* systems (also known as *singular*, *differential-algebraic* systems) [10]. The descriptor system model can be used in a large amount of applications, such as drinking-water distribution networks [11,12], chemical processes [13], electrical systems [14,15] and aircraft [16]. For a critical system, when some components are malfunctioning, in order to maintain the system functioning, a fault-tolerant control strategy

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with system reconfiguration is required. To limit the performance degradation, a robust fault estimation plays an important role in system design.

In literature, several fault estimation approaches for different types of descriptor systems have been investigated. In[4], a Lyapunov-based robust fault estimation approach is developed for Lipschitz non-linear descriptor systems. Robust fault estimation approaches for linear descriptor systems can be found in[8,17,18]. Besides, the fault estimation approaches have also studied for linear parameter-varying systems[19,20,21,22] and switched descriptor systems[23]. Among these approaches, the estimated fault results are obtained as punctual values. Alternatively, set-based approaches have been established for state estimation[24,27], fault diagnosis and fault-tolerant control[25]. Considering system uncertainties bounded in a pre-defined set, the uncertain variables are propagated by operating these sets. Regarding the application to robust fault estimation, under the set-based framework, fault estimation results are characterised in a deterministic set. The robustness against uncertainties can be achieved by shrinking the size of these sets. A preliminary result of the zonotopic fault estimation filter for descriptor systems was reported in[26]. The filter design is based on the combination of the filter design and the zonotopic set-membership approach. However, the structure of a zonotopic fault estimation filter is not explicitly formulated within a set-based framework in[26]. In this paper, we systematically propose a structure of zonotopic fault estimation filter for discrete-time descriptor systems. In the faulty-free case, the guaranteed zonotope can be used for over-bounding uncertain states. Therefore, this state bounding zonotope is used in the construction of the fault estimation zonotope. Besides, we analyse the boundedness of the generated zonotopic bounds of fault estimation.

The main contribution of this paper is to propose a robust fault estimation based on zonotopic Kalman filter for discrete-time descriptor systems subject to unknown-but-bounded uncertainties and additive actuator faults. The fault estimation results provide not only a punctual value but also a deterministic set bounding the propagated uncertainties. Following the set-based framework for descriptor systems proposed in[27], we first define the structure of the zonotopic fault estimation filter based on fault detectability indices and matrix proposed in[28]. The zonotopic fault estimation filter gain is formulated in a parametrised form. Within a set-based framework, the optimal filter gain is designed to obtain a zonotopic fault estimation with the smallest size compatible with the uncertainty bounds. Furthermore, we discuss the boundedness of the propagated zonotopic fault estimation. Finally, we apply the proposed robust fault estimation approach to a numerical example and a power system of a machine infinite bus to show its effectiveness.

The remainder of this paper is structured as follows. The problem statement is formulated in Section 2. Some preliminary results including definitions and properties that will be used in this paper are addressed in Section 3. The main results including the structure of zonotopic fault estimation filter, the design of optimal filter gain as well as the analysis of boundedness of zonotopic fault estimation are presented in Section 4. Simulation results obtained with two application examples are provided to show the effectiveness of the proposed approach in Section 5. Finally, some conclusions are drawn in Section 6.

Notation.

We use I_r to denote an identity matrix with dimension r. Note that the dimension of I may be dropped when it can be implied in the context. For a matrix X, we use $\mathrm{tr}(X)$, $\mathrm{rank}(X)$ and $X^\top$ to denote the trace, the rank and the transpose of X and we also use $X^\dagger$ to denote the Moore-Penrose pseudo-inverse matrix of X. If X is positive definite, we denote it as $X \succ 0$. Given a weighting matrix $W = W^\top \succ 0$, the weighted Frobenius norm of X is denoted by $\|X\|_{F,W} = \sqrt{\mathrm{tr}(X^\top W X)}$, and $\|X\|_F$, obtained with $W = I$, denotes the non-weighted Frobenius norm. For a vector z, $\|z\|_2$ denotes the 2-norm of z. Besides, we denote the Minkowski sum and the linear image as $\oplus$ and $\odot$.

2 | PROBLEM STATEMENT

Consider the discrete-time descriptor linear time-invariant (LTI) system with additive actuator faults as follows:

$$Ex(k+1) = Ax(k) + Bu(k) + D_w w(k) + Ff(k), \tag{1a}$$

$$y(k) = Cx(k) + D_v v(k), \tag{1b}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ denote the system state and the known input vectors, $w \in \mathbb{R}^{m_w}$ and $v \in \mathbb{R}^{m_v}$ denote the state disturbance vector and measurement noise vector, $y \in \mathbb{R}^p$ denotes the measurement output vector, $f \in \mathbb{R}^q$ denotes the actuator fault vector. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D_w \in \mathbb{R}^{n\times m_w}$ and $D_v \in \mathbb{R}^{p\times m_v}$ are the system matrices. Besides, $F \in \mathbb{R}^{n\times q}$ denotes the fault distribution matrix describing the directions of the fault vector. For the descriptor system (1), the matrix $E \in \mathbb{R}^{n\times n}$ might be singular, that is $\mathrm{rank}(E) \leq n$.

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Assume that the disturbance vector w and the noise vector v are unknown but bounded by the centred zonotopes $w(k) \in \langle 0, I_{m_w} \rangle$ and $v(k) \in \langle 0, I_{m_v} \rangle$, $\forall k \in \mathbb{N}$ and the initial state $x(0)$ is constrained in the zonotope $x(0) \in \langle c(0), H(0) \rangle$. Based on[28], we assume $\text{rank}(C) = p$ and $\text{rank}(F) = q$ with $q \leq p$. Besides, we assume that the descriptor system (1) is R-observable, that is matrices E and C satisfy[10]

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n. \tag{2}$$

Since that the rank condition (2) holds, there always exist two non-empty matrices $T \in \mathbb{R}^{n \times n}$ and $N \in \mathbb{R}^{n \times p}$ such that[8]

$$TE + NC = I_n. \tag{3}$$

Then, the general solution of T and N satisfying (3) is given by

$$T = \Theta \tau_1, \quad N = \Theta \tau_2, \tag{4}$$

with

$$\Theta = \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger} + S \left(I - \begin{bmatrix} E \\ C \end{bmatrix} \begin{bmatrix} E \\ C \end{bmatrix}^{\dagger} \right), \quad \tau_1 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \quad \tau_2 = \begin{bmatrix} 0 \\ I_p \end{bmatrix},$$

where $S \in \mathbb{R}^{n \times (n+p)}$ is an arbitrary matrix such that the matrix T is non-singular.

In this paper, we design a set-based robust fault estimation filter for the discrete-time descriptor system (1) to estimate the actuator fault magnitude f. The fault estimation filter is built in a zonotopic framework considering unknown-but-bounded disturbances and measurement noise. Using this framework, the robustness against uncertainties can be achieved by minimising the size of the zonotope bounding estimation errors, disturbances and noise. The fault estimation results are bounded using a zonotopic set.

3 | PRELIMINARY RESULTS

In this section, we introduce some preliminary results including some set definitions and properties that will be used in this paper.

3.1 | Matrix calculus

Let X, A, B and C be matrices of appropriate dimensions. The following well-known results will be used in this paper[24]

$$\frac{\partial}{\partial X} \text{tr}(AX^{\top}B) = A^{\top}B^{\top}, \tag{5a}$$

$$\frac{\partial}{\partial X} \text{tr}(AXBX^{\top}C) = BX^{\top}CA + B^{\top}X^{\top}A^{\top}C^{\top}. \tag{5b}$$

3.2 | Zonotopes

Zonotopes, as a special class of polytopes, are symmetric and its definition and properties are introduced as follows[29,24].

Definition 1 (Zonotope[29]). A r-order zonotope $\mathcal{Z} \subset \mathbb{R}^n$ in n-dimensional space is defined with its centre $c \in \mathbb{R}^n$ and the segment matrix $H \in \mathbb{R}^{n \times r}$ as

$$\mathcal{Z} = \langle c, H \rangle = \{ c + Hz, z \in \mathbf{B}^r \},$$

where $\mathbf{B}^r = [-1, +1]^r \subset \mathbb{R}^r$ is an r-order hypercube. With the Minskowski sum, the zonotope can be also defined as

$$\mathcal{Z} = c \oplus H\mathbf{B}^r.$$

Besides, the following properties related to zonotopes hold:

$$\langle c_1, H_1 \rangle \oplus \langle c_2, H_2 \rangle = \langle c_1 + c_2, \begin{bmatrix} H_1, & H_2 \end{bmatrix} \rangle, \tag{6a}$$

$$L \odot \langle c, H \rangle = \langle Lc, LH \rangle, \tag{6b}$$

where L is an arbitrary matrix of appropriate dimension.

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Definition 2 (Interval hull[29]). Given a zonotope $\mathcal{Z} = \langle c, H \rangle \subset \mathbb{R}^n$, the interval hull $rs(H) \in \mathbb{R}^{n\times n}$ is defined as an aligned minimum box such that the inclusion property holds: $\langle c, H \rangle \subset \langle c, rs(H) \rangle$, where $rs(H)$ is a diagonal matrix with diagonal elements of $rs(H)_{i,i} = \sum_{j=1}^{r} \left| H_{i,j} \right|, i = 1, 2, \dots, n$.

Definition 3 (F_W-radius[24]). Given a zonotope $\mathcal{Z} = \langle c, H \rangle \subset \mathbb{R}^n$ and a symmetric and positive definite matrix $W \in \mathbb{R}^{n\times n}$, the F_W-radius of $\mathcal{Z}$ is defined using the weighted Frobenius norm of H as $\|H\|_{F,W}$.

Definition 4 (Covariation[24]). Given a zonotope $\mathcal{Z} = \langle c, H \rangle \subset \mathbb{R}^n$, the covariation of $\mathcal{Z}$ is defined by $P = HH^\top$.

For a zonotope $\mathcal{Z} = \langle c, H \rangle \subset \mathbb{R}^n$, the weighted reduction operator proposed in[24] is denoted as $\downarrow_{\ell,W}(H)$, where ℓ specifies the maximum number of columns of H and $W \in \mathbb{R}^{n\times n}$, $W = W^\top \succ 0$ is a weighting matrix. The inclusion property also holds:

$$\langle c, H \rangle \subset \langle c, \downarrow_{\ell,W}(H) \rangle.$$

The operator $\downarrow_{\ell,W}(H)$ can be obtained by the following procedure:

- Sort the columns of segment matrix H on decreasing order: $\downarrow_W(H) = [h_1, h_2, \dots, h_r]$, $\|h_j\|_W^2 \geq \|h_{j+1}\|_W^2$, where $\left\|h_j\right\|_W$ is the weighted 2-norm of h_j.
- Take the first ℓ-columns of $\downarrow_W(H)$ and enclose a set $H_<$ generated by rest columns into a smallest box (interval hull) as follows:

$$\begin{aligned}
&\textbf{If } r \leq \ell, \textbf{ then } \downarrow_{\ell,W}(H) = \downarrow_W(H),\\
&\textbf{Else } \downarrow_{\ell,W}(H) = \left[H_>, rs(H_<)\right] \in \mathbb{R}^{n\times \ell},\\
&\quad H_> = \left[h_1, \dots, h_\ell\right],\ H_< = \left[h_{\ell+1}, \dots, h_r\right].
\end{aligned}$$

3.3 | Fault detectability indices and matrix

Denote the fault distribution matrix $F = \left[F_1, \cdots, F_q\right]$ and the fault vector $f(k) = \left[f_1(k), \cdots, f_q(k)\right]^\top$, $\forall k \in \mathbb{N}$, where F_i is the i-th column of F and $f_i(k)$ is the i-th element of $f(k)$ for $i = 1, \dots, q$, $\forall k \in \mathbb{N}$. We recall definitions of fault detectability indices and matrix first introduced in[28,30] and extended for descriptor systems with a non-singular matrix T in[8] as follows.

Definition 5 (Fault detectability indices[8]). The discrete-time descriptor system (1) is said to have fault detectability indices $\rho = \left\{\rho_1, \rho_2, \dots, \rho_q\right\}$ if

$$\rho_i = \min\left\{\sigma \mid C(TA)^{\sigma-1}TF_i \neq 0, i = 1, 2, \dots\right\}. \tag{7}$$

and $s = \max\left\{\rho_1, \rho_2, \dots, \rho_q\right\}$ denotes the maximum of fault detectability indices.

Definition 6 (Fault detectability matrix[8]). With the fault detectability indices of the descriptor system (1) defined as $\rho = \left\{\rho_1, \rho_2, \dots, \rho_q\right\}$, the fault detectability matrix is given by

$$\Upsilon = C\Psi, \tag{8}$$

with

$$\Psi = \left[(TA)^{\rho_1-1}TF_1,\ (TA)^{\rho_2-1}TF_2,\ \cdots,\ (TA)^{\rho_q-1}TF_q\right]. \tag{9}$$

Remark 1. According to[8], due to the chosen matrix T is non-singular, the condition $\operatorname{rank}(\Upsilon) = q$ holds.

4 | MAIN RESULTS

In this section, we propose a zonotopic fault estimation filter for the descriptor system (1). By means of fault detectability indices and matrix, we analyse and construct the fault estimation zonotope to estimate occurred actuator faults. Therefore, the optimal fault estimation filter gain is computed. Besides, we discuss boundedness of zonotopic fault estimation.

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4.1 | Zonotopic fault estimation filter

When the condition (2) is fulfilled, there exists matrices T and N satisfying (3). We consider a state estimation filter for the discrete-time descriptor system (1) as

$$\begin{cases} z(k+1) &= TA\hat{x}(k) + TBu(k) + G(k)\left(y(k) - C\hat{x}(k)\right) \\ \hat{x}(k) &= z(k) + Ny(k), \end{cases} \tag{10}$$

where $\hat{x} \in \mathbb{R}^n$ denotes the estimated state vector and $z \in \mathbb{R}^n$ denotes the filter state vector.

Let us define the state estimation error $e(k) = x(k) - \hat{x}(k)$ and the output estimation error $\varepsilon(k) = y(k) - C\hat{x}(k)$. Then, the error dynamics of e and ε can be written as follows:

$$\begin{cases} e(k+1) &= (TA - G(k)C)e(k) + TFf(k) + TD_w w(k) \\ & \quad -G(k)D_v v(k) - ND_v v(k+1), \\ \varepsilon(k) &= Ce(k) + D_v v(k). \end{cases}$$

In order to analyse the effects of uncertainties and faults, we split e and ε into two parts: $e = e_f + e_w$ and $\varepsilon = \varepsilon_f + \varepsilon_w$, where e_f and ε_f are the errors only affected by actuator faults ($w(k) = 0$ and $v(k) = 0$, $\forall k \in \mathbb{N}$), and e_w and ε_w are the errors only affected by disturbances and noise ($f(k) = 0$, $\forall k \in \mathbb{N}$).

$$\begin{cases} e_f(k+1) &= (TA - G(k)C)e_f(k) + TFf(k), \\ \varepsilon_f(k) &= Ce_f(k), \end{cases} \tag{11}$$

and

$$\begin{cases} e_w(k+1) &= (TA - G(k)C)e_w(k) + TD_w w(k) \\ & \quad -G(k)D_v v(k) - ND_v v(k+1), \\ \varepsilon_w(k) &= Ce_w(k) + D_v v(k), \end{cases} \tag{12}$$

with the following initial conditions $e_f(k) = 0$ and $e_w(0) = e(0)$. Therefore, we know $\varepsilon_f(k) = 0$, $\forall k \in \mathbb{N}$.

We now analyse the effects of occurred actuator faults and uncertainties in the estimation errors using the fault detectability indices and matrix in Definition 5 and Definition 6 in the following theorem.

Theorem 1 (Fault estimation condition)**.** Consider the descriptor system (1). If there exists the filter gain $G(k) \in \mathbb{R}^{n\times p}$ such that

$$(TA - G(k)C)\Psi = 0, \tag{13}$$

then the effect of the faults on $\varepsilon(k)$ can be expressed as

$$\varepsilon(k) = C\Psi\left[f_1(k-\rho_1),\ \cdots,\ f_q(k-\rho_q)\right]^\top + \varepsilon_w(k). \tag{14}$$

Proof. By merging (11), we can derive from the time instant $k = 0$ to k that

$$\varepsilon_f(k) = C\Phi^k e_f(0) + C\Phi^{k-1}TFf(0) + \cdots + C\Phi_1 TFf(k-1), \tag{15}$$

where $\Phi^k = \prod_{j=1}^{k}(TA - G_jC)$. According to[8, Theorem 1], we obtain

$$C\Phi_j TF_i = \begin{cases} C(TA)^{\rho_i - 1}TF_i, & j = \rho_i, \\ 0, & j \neq \rho_i. \end{cases} \tag{16}$$

Substituting (16) into (15) yields

$$\begin{aligned} \varepsilon_f(k) &= C\Phi^k e_f(0) + C(TA)^{\rho_1-1}TF_1 f_1(k-\rho_1) \\ & \quad + \cdots + C(TA)^{\rho_q-1}TF_q f_q(k-\rho_q) \\ &= C\Phi^k e_f(0) + C\Psi\left[f_1(k-\rho_1),\ \cdots,\ f_q(k-\rho_q)\right]^\top. \end{aligned} \tag{17}$$

Since $e_f(0) = 0$, (17) becomes $\varepsilon_f(k) = C\Psi\left[f_1(k-\rho_1),\ \cdots,\ f_q(k-\rho_q)\right]^\top$. Therefore, from $\varepsilon(k) = \varepsilon_f(k) + \varepsilon_w(k)$, we obtain (14). □

From Theorem 1, we can see that the effects of faults and uncertainties can be separated in (14). Therefore, we define the zonotopic fault estimation filter for the descriptor system (1) in the following theorem.

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Theorem 2 (Zonotopic fault estimation filter for descriptor systems)**.** Given the descriptor system (1) with $w(k) \in \langle 0, I_{m_w} \rangle$ and $v(k) \in \langle 0, I_{m_v} \rangle$, $\forall k \in \mathbb{N}$, matrices $T \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times p}$ satisfying (3). Consider the state bounding zonotope $x_w(k-1) \in \langle c(k-1), H(k-1) \rangle \subseteq \langle c(k-1), \bar{H}(k-1) \rangle$ with $\bar{H}(k-1) = \downarrow_{\ell, W} (H(k-1))$, the state bounding zonotope $x_w(k) \in \langle c(k), H(k) \rangle$, $\forall k \in \mathbb{N}$ is recursively defined by

$$c(k) = (TA - G(k-1)C)\, c(k-1) + TBu(k-1) + G(k-1)y(k-1) + Ny(k), \tag{18a}$$

$$H(k) = \left[(TA - G(k-1)C)\, \bar{H}(k-1),\ TD_w,\ -G(k-1)D_v,\ -ND_v \right]. \tag{18b}$$

If there exist matrices $G(k-1) \in \mathbb{R}^{n \times p}$ satisfying (13) and $M \in \mathbb{R}^{q \times p}$ satisfying

$$M = (C\Psi)^{\dagger} = \Upsilon^{\dagger}, \tag{19}$$

then the actuator faults is bounded by $\hat{f}(k) = \left[f_1(k-\rho_1),\ \cdots,\ f_q(k-\rho_q) \right]^{\top} \in \langle c_f(k), H_f(k) \rangle$, where

$$c_f(k) = My(k) - MCc(k), \tag{20a}$$

$$H_f(k) = \left[-MCH(k),\ -MD_v \right]. \tag{20b}$$

Proof. From the analysis of effects of occurred actuator faults and uncertainties in (14), we can build state bounding zonotope and fault estimation zonotope in the following.

(*State bounding zonotope*) With a filter gain $G(k-1)$, from (10), we can derive

$$\hat{x}(k) = (TA + G(k-1)C)\, \hat{x}(k-1) + TBu(k-1) + G(k-1)y(k-1) + Ny(k).$$

For $x_w(k-1) \in \langle c(k-1), \bar{H}(k-1) \rangle$, we set $\hat{x}(k-1) = c(k-1)$ and we know $e_w(k-1) = x_w(k-1) - c(k-1) \in \langle 0, \bar{H}(k-1) \rangle$. From (12), with $w(k) \in \langle 0, I_{m_w} \rangle$, $v(k) \in \langle 0, I_{m_v} \rangle$, $\forall k \in \mathbb{N}$, we derive $x_w(k) = \hat{x}(k) + e_w(k)$ obtaining

$$\begin{aligned} x_w(k) &\in \langle c(k), H(k) \rangle \\ &= ((TA - G(k-1)C) \odot \langle c(k-1), 0 \rangle) \oplus (TB \odot \langle u(k-1), 0 \rangle) \oplus (G(k-1) \odot \langle y(k-1), 0 \rangle) \\ &\quad \oplus (N \odot \langle y(k), 0 \rangle) \oplus \left((TA - G(k-1)C) \odot \langle 0, \bar{H}(k-1) \rangle \right) \oplus \left(TD_w \odot \langle 0, I_{m_w} \rangle \right) \\ &\quad \oplus \left((-G(k-1)D_v) \odot \langle 0, I_{m_v} \rangle \right) \oplus \left((-ND_v) \odot \langle 0, I_{m_v} \rangle \right). \end{aligned}$$

By using the properties in (6), we obtain $c(k)$ and $H(k)$ in (18).

(*Fault estimation zonotope*) From $x_w(k) \in \langle c(k), H(k) \rangle$ and $\hat{x}(k) = c(k)$, we know $e_w(k) \in \langle 0, H(k) \rangle$. By definition, we also have the output estimation error $\varepsilon(k) = y(k) - Cc(k)$. On the other hand, by pre-multiplying $M \in \mathbb{R}^{q \times p}$ on both sides of (14), we obtain

$$M\varepsilon(k) = MC\Psi \left[f_1(k-\rho_1),\ \cdots,\ f_q(k-\rho_q) \right]^{\top} + M\varepsilon_w(k). \tag{21}$$

Denote $\hat{f}(k) = \left[f_1(k-\rho_1),\ \cdots,\ f_q(k-\rho_q) \right]^{\top}$. Taking into account M satisfying (19), we know $MC\Psi = I$. Therefore, from (21), we obtain

$$\begin{aligned} \hat{f}(k) &= M\varepsilon(k) - M\varepsilon_w(k) \\ &= M\varepsilon(k) - M\left(Ce_w(k) + D_v v(k) \right). \end{aligned} \tag{22}$$

Recall $\varepsilon(k) = y(k) - Cc(k)$, $e_w(k) \in \langle 0, H(k) \rangle$ and $v(k) \in \langle 0, I_{m_v} \rangle$. From (22), we can derive

$$\begin{aligned} \hat{f}(k) &\in \langle c_f(k), H(k) \rangle \\ &= (M \odot \langle y(k) - Cc(k), 0 \rangle) \oplus (-MC \odot \langle 0, H(k) \rangle) \oplus \left(-MD_v \odot \langle 0, I_{m_v} \rangle \right). \end{aligned}$$

Again, by using the properties in (6), we obtain $c_f(k)$ and $H_f(k)$ as in (20). □

Remark 2. From the structure of the zonotopic fault estimation filter proposed in Theorem 2, it is clear that the estimated fault $\hat{f}(k)$ has delays for each element and the delays are determined by the fault detectability indices ρ_i for $i = 1, \ldots, q$.

4.2 | Optimal fault estimation filter gain

We now present the results of optimal fault estimation filter gain. For designing the filter gain for fault estimation, the following criteria are taken into account:

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- $G(k), \forall k \in \mathbb{N}$ satisfies the algebraic condition (13);
- $G(k), \forall k \in \mathbb{N}$ minimises the estimation error $e_w(k+1)$, that reduces the size of the zonotope $\langle c(k+1), H(k+1)\rangle$.

Inspired by the zonotopic Kalman filter proposed in[24, Theorem 5], the size of a zonotope can be measured by the F_W-radius (see Definition 3). From (18b), we can also obtain $H(k+1)$. The objective of the zonotope minimisation can be defined by $J = \mathrm{tr}(WP(k+1))$ with a weighting matrix $W = W^\top \succ 0$ and the covariation

$$P(k+1) = H(k+1)H(k+1)^\top. \tag{23}$$

Theorem 3 (Optimal fault estimation filter gain)**.** Given $H(k+1)$, a weighting matrix $W \in \mathbb{R}^{n\times n}$ with $W = W^\top \succ 0$, the fault detectability matrix Υ in (8) with $\mathrm{rank}(\Upsilon) = q$. The optimal filter gain $G^*(k)$ can be computed by the parametrised form:

$$G^*(k) = \Phi M + \bar{G}^*(k)\Omega, \tag{24}$$

with

$$\Phi = TA\Psi,\ M = \Upsilon^\dagger,\ \Omega = \alpha\left(I_m - \Upsilon M\right), \tag{25}$$

where $\alpha \in \mathbb{R}^{(p-q)\times p}$ is an arbitrary matrix guaranteeing that Ω has full-row rank and $\bar{G}(k) \in \mathbb{R}^{n\times(p-q)}$. Besides, $\bar{G}(k) = \bar{G}^*(k)$ minimises $J = \mathrm{tr}(WP(k+1))$ with $P(k+1)$ in (23), which is computed through the following procedure

$$\bar{G}^*(k) = \tilde{L}(k)\tilde{S}(k)^{-1}, \tag{26}$$

$$\tilde{L}(k) = (TA - \Phi MC)\bar{P}(k)C^\top\Omega^\top - \Phi MV\Omega^\top, \tag{27}$$

$$\tilde{S}(k) = \Omega\left(C\bar{P}(k)C^\top + V\right)\Omega^\top, \tag{28}$$

with $\bar{P}(k) = \bar{H}(k)\bar{H}(k)^\top$ and $V = D_v D_v^\top$.

Proof. From $M = \Upsilon^\dagger$ and $\mathrm{rank}(\Upsilon) = q$, we have $M\Upsilon = I_q$. Since $\mathrm{rank}(\Upsilon) = q$, we can obtain a matrix $\Omega \in \mathbb{R}^{(p-q)\times p}$ such that $\Omega\Upsilon = 0$.

Therefore, with $G(k)$ defined in (24), we derive

$$\begin{aligned}(TA - G(k)C)\Psi &= \left(TA - \left(\Phi M + \bar{G}(k)\Omega\right)C\right)\Psi \\ &= TA\Psi - TA\Psi MC\Psi - \bar{G}(k)\Omega C\Psi \\ &= TA\Psi - TA\Psi M\Upsilon - \bar{G}(k)\Omega\Upsilon.\end{aligned}$$

Since $M\Upsilon = I_q$ and $\Omega\Upsilon = 0$, the above equation leads to $TA\Psi - TA\Psi M\Upsilon - \bar{G}(k)\Omega\Upsilon = 0$. Thus, (13) is satisfied with $G(k)$ parametrised as in (24).

Then, the problem is converted to find $\bar{G}(k)$ minimising $J = \mathrm{tr}(WP(k+1))$. By definition, J is convex with respect to $\bar{G}(k)$. Thus, $\bar{G}^*(k)$ is a value of $\bar{G}(k)$ such that $\frac{\partial J}{\partial \bar{G}(k)} = 0$.

Set $\tilde{L}(k)$ and $\tilde{S}(k)$ as in (27) and (28). Evaluating $\frac{\partial J}{\partial \bar{G}(k)} = 0$, we have

$$\frac{\partial \mathrm{tr}}{\partial \bar{G}(k)}\left(W\bar{G}(k)\tilde{S}(k)\bar{G}(k)^\top\right) - 2\frac{\partial \mathrm{tr}}{\partial \bar{G}(k)}\left(W\tilde{L}(k)\bar{G}(k)^\top\right) = 0. \tag{29}$$

By means of the matrix calculus in (5), (29) can be simplified as

$$W\tilde{S}(k)\bar{G}(k)^\top + W\tilde{S}(k)^\top\bar{G}(k)^\top - 2W\tilde{L}(k)^\top = 0.$$

Because $\tilde{S}(k)$ is also a symmetric matrix, we thus obtain $\bar{G}(k)$ as in (26). □

From the proof of Theorem 3, we can see the independence of $\bar{G}^*(k)$ with respect to the weighting matrix W. Thus, W can be set as a free matrix, for instance $W = I_n$. Besides, time-varying weighting matrix $W(k)$ will be used for proving boundedness of the proposed zonotopic fault estimation for descriptor systems in the next subsection.

Remark 3. For the proposed zonotopic fault estimation filter in Theorem 2, G that satisfies the condition $(TA - GC)\Psi = 0$ is a stabilizing filter gain if there exist matrices $W \in \mathbb{R}^{n\times n}$ with $W = W^\top \succ 0$, and Y

$$\begin{bmatrix} W & (WTA - W\Phi MC - Y\Omega C)^\top \\ WTA - W\Phi MC - Y\Omega C & W \end{bmatrix} \succ 0, \tag{30}$$

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Algorithm 1 Zonotopic fault estimation algorithm for descriptor systems

Given the system matrices E, A, B, C, D_w, D_v and F and the initial state bounded in $x_0 \in \langle c_0, H_0 \rangle$;
Solve the equation (3) by using to obtain T and N;
Compute the fault detectability indices ρ_i, $i = 1, \dots, q$;
Compute the fault detectability matrix $\Upsilon = C\Psi$;
$\Phi \leftarrow TA\Psi$;
$M \leftarrow \Upsilon^{\dagger}$;
$\Omega \leftarrow \alpha \left(I_m - \Upsilon M \right)$;
while $k > 0$ **do**
  Compute $\bar{G}^*(k-1)$ according to the procedure in (26)-(28);
  Obtain the optimal filter gain $G^*(k-1)$ following (24) with $\bar{G}^*(k-1)$ following (26)-(28);
  Compute the state bounding zonotope $\langle c(k), H(k) \rangle$ by using (18);
  Compute the fault estimation zonotope $\langle c_f(k), H_f(k) \rangle$ by using (20);
  Obtain the fault estimation $\hat{f}(k) = c_f(k)$ with its bounds $\hat{f}_i(k) \in \left[\underline{f}_i(k), \overline{f}_i(k) \right]$, $i = 1, \dots, q$ with

$$\overline{f}_i(k) = c_{f_i}(k) + rs(H_f(k))_{i,i},$$
$$\underline{f}_i(k) = c_{f_i}(k) - rs(H_f(k))_{i,i},$$

  where $c_f = \begin{bmatrix} c_{f_1} & \dots & c_{f_i} & \dots & c_{f_q} \end{bmatrix}^\top$.
end while

then the solutions give $G = \Phi M - W^{-1} Y \Omega$. Note that the condition (30) can be found by the Lyapunov stability condition and the parametrised filter gain as in (24).

With the zonotopic fault estimation filter defined in Theorem 2 and the optimal filter gain in Theorem 3, we summarise the fault estimation algorithm in Algorithm 1.

4.3 | Boundedness of zonotopic fault estimation

In this subsection, we prove the boundedness of zonotopic fault estimation filter by implementing Theorem 2 with the optimal filter gain obtained using Theorem 3. First, we introduce an auxiliary result that will be used for the proof of boundedness.

Proposition 1. Given the descriptor system $Ex(k+1) = Ax(k)$ with a measurement output $y(k) = Cx(k)$, matrices T and N satisfying (3), and $\gamma \in (0, 1)$. The filter $\hat{x}(k+1) = TA\hat{x}(k) + G(k)(y(k) - C\hat{x}(k)) + Ny(k+1)$ is γ-stable (stable with a decay rate γ) if there exist matrices $G(k) \in \mathbb{R}^{n \times p}$ and $W(k) \in \mathbb{R}^{n \times n}$ with $W(k) = W(k)^\top \succ 0$, $\forall k \geq 0$ such that

$$\begin{bmatrix} \gamma W(k) & (TA - G(k)C)^\top W(k+1)^\top \\ W(k+1)(TA - G(k)C) & W(k+1) \end{bmatrix} \succ 0. \tag{31}$$

Proof. With matrices T and N satisfying (3), we reformulate the system dynamics to be $x(k+1) = TAx(k) + Ny(k+1)$. Define the state estimation error $e(k) = x(k) - \hat{x}(k)$. Therefore, we have the error dynamics

$$e(k+1) = x(k+1) - \hat{x}(k+1) = (TA - G(k)C)e(k).$$

With a sequence of matrices $W(k) = W(k)^\top \succ 0$, $\forall k \geq 0$, we consider the Lyapunov candidate function as $V(k) = e(k)^\top W(k) e(k)$. Given $\gamma \in (0, 1)$, we have

$$\begin{aligned} \Delta V(k) = V(k+1) - V(k) &= e(k+1)^\top W(k+1) e(k+1) - e(k)^\top W(k) e(k) \\ &= e(k)^\top \left((TA - G(k)C)^\top W(k+1)(TA - G(k)C) - \gamma W(k) \right) e(k). \end{aligned}$$

For any $e(k) \neq 0$, $\Delta V(k) < 0$ implies $\gamma W(k) - (TA - G(k)C)^\top W(k+1)(TA - G(k)C) \succ 0$. By applying the Schur complement lemma with $\gamma W(k) \succ 0$, we thus obtain (31). □

Since the zonotope reduction operator $\downarrow_{\ell,W}(\cdot)$ is used in the proposed zonotopic fault estimation filter, we also introduce the following lemma to describe the boundedness of the use of $\downarrow_{\ell,W}(\cdot)$.

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Lemma 1 ([24])**.** Consider $H \in \mathbb{R}^{n\times r}$ as the generator matrix of a zonotope $\langle c, H\rangle \subset \mathbb{R}^n$, a weighting matrix $W \in \mathbb{R}^{n\times n}$, $W = W^\top \succ 0$ with all its eigenvalues in $\left[\underline{\lambda}, \overline{\lambda}\right] \subset \mathbb{R}$. By means of the reduction operator $\bar{H} =\downarrow_{\ell,W}(H)$ with $n \le \ell < r$, $\langle c, \bar{H}\rangle$ is a reduced zonotope such that $\langle c, H\rangle \subseteq \langle c, \bar{H}\rangle$. Let $\mu = \left(\frac{\overline{\lambda}(n+r-\ell)}{\underline{\lambda}} - 1\right)(n + r - \ell)$ and $\beta = 1 + \frac{\mu}{r}$. Then, it holds:

$$\left\|\bar{H}\right\|_{F,W}^2 \le \beta \left\|H\right\|_{F,W}^2 . \tag{32}$$

From the structure of the proposed zonotopic fault estimation filter in Theorem 2, due to that $\langle c_f(k), H_f(k)\rangle$ is a linear projection of $\langle c(k), H(k)\rangle$, $\forall k \in \mathbb{N}$, the filter dynamics is bounded by $\langle c(k), H(k)\rangle$ as defined in (18). Based on presented results above, we now discuss the boundedness of zonotopic fault estimation for descriptor systems in the following theorem.

Theorem 4 (Boundedness of zonotopic fault estimation)**.** Consider the zonotopic fault estimation filter $\langle c_f(k), H_f(k)\rangle$ in (20) with $\langle c(k), H(k)\rangle$ in (18) and the optimal filter gain $G^*(k)$ in (24), $W(k) = W(k)^\top \succ 0$, $\forall k \in \mathbb{N}$ and $\gamma \in (0, 1)$ satisfying (31). If there exists a bounded sequence $\psi(k)$ such that

$$\left\|TD_w\right\|_{F,W(k+1)}^2 + \left\|G(k)D_v\right\|_{F,W(k+1)}^2 + \left\|ND_v\right\|_{F,W(k+1)}^2 \le \psi(k), \quad \forall k \in \mathbb{N}, \tag{33}$$

and when $k \to \infty$, $\bar{\psi}$ is the upper bound of $\psi(k)$, then the F_W-radius of $\langle c(k), H(k)\rangle$ is bounded by

$$\left\|H(k+1)\right\|_{F,W(k+1)}^2 \le \bar{\gamma} \left\|H(k)\right\|_{F,W(k)}^2 + \psi(k), \quad \forall k \in \mathbb{N}, \tag{34}$$

with $\bar{\gamma} = \gamma\beta < 1$. Moreover, when $k \to \infty$, the upper bound $\left\|H(\infty)\right\|_{F,W(\infty)}^2$ is given by

$$\left\|H(\infty)\right\|_{F,W(\infty)}^2 \le \frac{\bar{\psi}}{1-\bar{\gamma}}. \tag{35}$$

Proof. Considering $H(k+1)$ and the optimal filter gain $G^*(k)$, the F_W-radius of $\langle c(k+1), H(k+1)\rangle$ is expressed as

$$\left\|H(k+1)\right\|_{F,W(k+1)}^2 = \left\|\left[(TA - G^*(k)C)\,\bar{H}(k),\ TD_w,\ -G^*(k)D_v,\ -ND_v\right]\right\|_{F,W(k+1)}^2 .$$

Since the optimal filter gain $G^*(k)$ is obtained by minimising $\left\|H(k+1)\right\|_{F,W(k+1)}^2$ with independence of $W(k+1)$, we thus have

$$\left\|H(k+1)\right\|_{F,W(k+1)}^2 \le \left\|\left[(TA - G(k)C)\,\bar{H}(k),\ TD_w,\ -G(k)D_v,\ -ND_v\right]\right\|_{F,W(k+1)}^2 ,$$

for any $G(k)$ instead of $G^*(k)$ satisfying (31). Then, considering the boundedness in (33), from above inequality, we obtain a sufficient condition

$$\left\|H(k+1)\right\|_{F,W(k+1)}^2 \le \left\|(TA - G(k)C)\,\bar{H}(k)\right\|_{F,W(k+1)}^2 + \psi(k). \tag{36}$$

Based on Proposition 1, with $W(k) = W(k)^\top \succ 0$, $\forall k \in \mathbb{N}$ and $\gamma \in (0,1)$ satisfying (31), $(TA - G(k)C)$ is γ-stable. By applying the Schur complement to (31), we obtain $\gamma W(k) - (TA - G(k)C)^\top W(k+1)(TA - G(k)C) \succ 0$. Since $\bar{H}(k) \neq 0$ and by the linearity of the operator $\mathrm{tr}(\cdot)$, we have

$$\mathrm{tr}\left(\bar{H}(k)^\top (TA - G(k)C)^\top W(k+1)(TA - G(k)C)\bar{H}(k)\right) < \gamma \mathrm{tr}\left(\bar{H}(k)^\top W(k)\bar{H}(k)\right).$$

By the F_W-radius definition, we obtain $\left\|(TA - G(k)C)\,\bar{H}(k)\right\|_{F,W(k+1)}^2 < \gamma \left\|\bar{H}(k)\right\|_{F,W(k)}^2$. Therefore, with (36), we have

$$\left\|H(k+1)\right\|_{F,W(k+1)}^2 \le \gamma \left\|\bar{H}(k)\right\|_{F,W(k)}^2 + \psi(k).$$

Based on the condition (32) in Lemma 1, we obtain

$$\left\|H(k+1)\right\|_{F,W(k+1)}^2 \le \gamma\beta \left\|H(k)\right\|_{F,W(k)}^2 + \psi(k).$$

Thus, with $\bar{\gamma} = \gamma\beta$, we obtain (34). Considering $\gamma \in (0,1)$, $\bar{\gamma} \in (0,1)$ can also hold.

Besides, when $k \to \infty$, with the upper bound $\psi(\infty) = \bar{\psi}$, (34) becomes

$$\left\|H(\infty)\right\|_{F,W(\infty)}^2 \le \bar{\gamma} \left\|H(\infty)\right\|_{F,W(\infty)}^2 + \bar{\psi},$$

which implies (35). □

According to Theorem 4, the boundedness of the state bounding zonotope $\langle c(k), H(k)\rangle$, $\forall k \in \mathbb{N}$ defined in (18) is provided by the boundedness condition. As a conclusion, ultimate boundedness of the proposed zonotopic fault estimation is obtained.

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5 | SIMULATION RESULTS

In this section, we first use a numerical example to show some comparison results for testing the performance of the proposed approach with the designed optimal filter gain. Then, the simulation of the machine infinite bus system used in [17,18] provides an insight on potential applications of the proposed approach.

5.1 | A numerical example

Consider a discrete-time descriptor system modelled by (1) with system matrices as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9 & 0.005 & -0.095 & 0 \\ 0.005 & 0.995 & 0.0997 & 0 \\ 0.095 & -0.0997 & 0.99 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad B = F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 1 & 1 \\ -0.1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_v = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

The initial state $x(0)$ is set as $x(0) = \begin{bmatrix} 0.5, & 1, & 0, & -0.5 \end{bmatrix}^\top$ and the initial state zonotope is given by $x(0) \in \langle c(0) = x(0), 0.1 I_4 \rangle$. Besides, $w(k) \in \langle 0, I_3 \rangle$ and $v(k) \in \langle 0, I_3 \rangle$, $\forall k \in \mathbb{N}$. The input signal is set as $u(k) = \begin{bmatrix} 2\sin(k), & 3\sin(k) \end{bmatrix}^\top$. From the general solution (4), we choose the matrix S as

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix},$$

and we obtain two non-empty matrices T and N satisfying the condition (3) as follows:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\mathrm{rank}(F) = \mathrm{rank}(CF) = 2$, we have $CTF_1 \neq 0$ and $CTF_2 \neq 0$. The fault detectability indexes are $\rho_1 = 1$ and $\rho_2 = 1$ and the fault detectability matrix is $\Upsilon = C\Psi = \begin{bmatrix} 0.5 & 0.5 \\ -0.05 & 0.5 \\ -1 & 0 \end{bmatrix}$ with $\Psi = \begin{bmatrix} Tf_1 & Tf_2 \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0.5 & 0.5 \\ -0.05 & 0.5 \\ -1 & 0 \end{bmatrix}$. Therefore, we obtain the matrices to obtain the optimal filter gain $G^*(k)$ as follows:

$$\Phi = \begin{bmatrix} 0.0973 & -0.045 \\ 0.2465 & 0.2737 \\ -0.0449 & 0.2226 \\ -0.95 & 0.5 \end{bmatrix}, \quad M = \begin{bmatrix} 0.2389 & -0.2389 & -0.8686 \\ 0.8925 & 1.1075 & 0.3909 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0.8686 & -0.8686 & 0.4777 \end{bmatrix}.$$

Therefore, the time-varying matrix $\bar{G}^*(k)$ can be obtained following (26)-(28) and we can find the optimal filter gain $G^*(k)$ in (24). Besides, as a comparison, according to Remark 3, by satisfying (30), we also obtain a stabilizing filter gain G as

$$G = \begin{bmatrix} 0.3283 & -0.4183 & 0.0878 \\ 0.4907 & 0.0566 & -0.0040 \\ -0.0279 & 0.4730 & 0.0073 \\ 0.3051 & 0.6949 & 1.0678 \end{bmatrix}.$$

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TABLE 1 The comparison result with $G^*(k)$ and G.

	MSE	$RMS(rs(H_f))$
$G^*(k)$	0.0389	1.3049
G	0.0641	2.0470

Consider the actuator faults are in the following scenarios:

$$f_1(k) = \begin{cases} 0 & k < 80 \\ 5 & k \geq 80 \end{cases}$$

$$f_2(k) = \begin{cases} 0 & k < 100 \\ 6\sin(0.1k) & k \geq 100 \end{cases}$$

As a result, the simulation has been carried out for $N_s = 200$ sampling steps and the robust fault estimation results are shown in Figure 1 with $G^*(k)$ and G. Note that due to $\rho_1 = 1$ and $\rho_2 = 1$, there is one-step delay in the estimation of the faults f_1 and f_2. In the figures, for allowing a better comparison, we plot the real faults delayed one sample, $f_i(k-1)$ with $i = 1, 2$. Using the proposed zonotopic fault estimation filter, the punctual values of estimated faults are obtained altogether with the worst-case bounds of estimated faults are also found in the estimation intervals under the assumption of unknown-but-bounded disturbances and measurement noise in given zonotopes.

The actual faults in red dashed lines are bounded by estimation intervals with $G^*(k)$ and G. From the Figure 1 , it is obvious that the bounds obtained with G are larger than the ones obtained with $G^*(k)$. For the comparison of the performance with $G^*(k)$ and G, the mean square error (MSE) between the actual faults and estimation faults (centres of fault estimation zonotopes) is computed by

$$MSE = \frac{1}{N_s} \sum_{k=1}^{N_s} \frac{1}{q} \left\| f(k) - c_f(k) \right\|,$$

and the root mean squared value of $rs(H_f(k)))$ for $k = 1, \ldots, N_s$ is computed, which is denoted by $RMS(rs(H_f))$. The computation result is shown in Table 1 . From the MSE results, the one obtained with $G^*(k)$ is close to zero and smaller than the other, which means that the estimation results with the optimal filter gain are more accurate than the ones obtained with the stabilizing filter gain G. Since the estimation errors of faults are bounded in the zonotopes, the obtained bounds with G are larger and the RMS result provides that the one with G is larger than the other.

5.2 | The machine infinite bus system

Consider a machine infinite bus system used in [17] and its linear continuous-time system with parameters described in [18] as follows:

$$\begin{aligned}
\dot{\delta}_1 &= \omega_1, \\
\dot{\delta}_2 &= \omega_2, \\
\dot{\delta}_3 &= \omega_3, \\
\dot{\omega}_4 &= \frac{1}{m_1}\left(p_1 - Y_{12}V_1V_2(\delta_1 - \delta_2)\right) - \frac{1}{m_1}\left(Y_{15}V_1V_2(\delta_1 - \delta_5) + c_1\omega_1\right), \\
\dot{\omega}_5 &= \frac{1}{m_2}\left(p_2 - Y_{21}V_2V_1(\delta_2 - \delta_1)\right) - \frac{1}{m_2}\left(Y_{25}V_2V_5(\delta_2 - \delta_5) + c_2\omega_2\right), \\
\dot{\omega}_6 &= \frac{1}{m_3}\left(p_3 - Y_{34}V_\infty\delta_3\right) - \frac{1}{m_3}\left(Y_{35}V_3V_5(\delta_3 - \delta_5) + c_3\omega_3\right), \\
0 &= P_{ch} - Y_{51}V_5V_1(\delta_5 - \delta_1) - Y_{52}V_5V_2(\delta_5 - \delta_2) - Y_{53}V_5V_3(\delta_5 - \delta_3) - Y_{54}V_5V_\infty\delta_5,
\end{aligned}$$

where δ_1, δ_2, δ_3 and δ_5 denote the phase angles of the generators, ω_1, ω_2 and ω_3 denote the speeds of the generators, p_1, p_2 and p_3 are the mechanical powers per unit that are set as $p_1 = 0.1$, $p_2 = 0.1$ and $p_3 = 0.1$, and P_{ch} is the unknown power load. From [18], the other parameters are chosen as follows: the inertia $m_1 = 0.014$, $m_2 = 0.026$ and $m_3 = 0.02$, the damping $c_1 = 0.057$, $c_2 = 0.15$ and $c_3 = 0.11$, the potential $V_1 = 1$, $V_2 = 1$, $V_3 = 1$, $V_\infty = 1$ and $V_5 = 1$, and the nominal admittance $Y_{15} = 0.5$,

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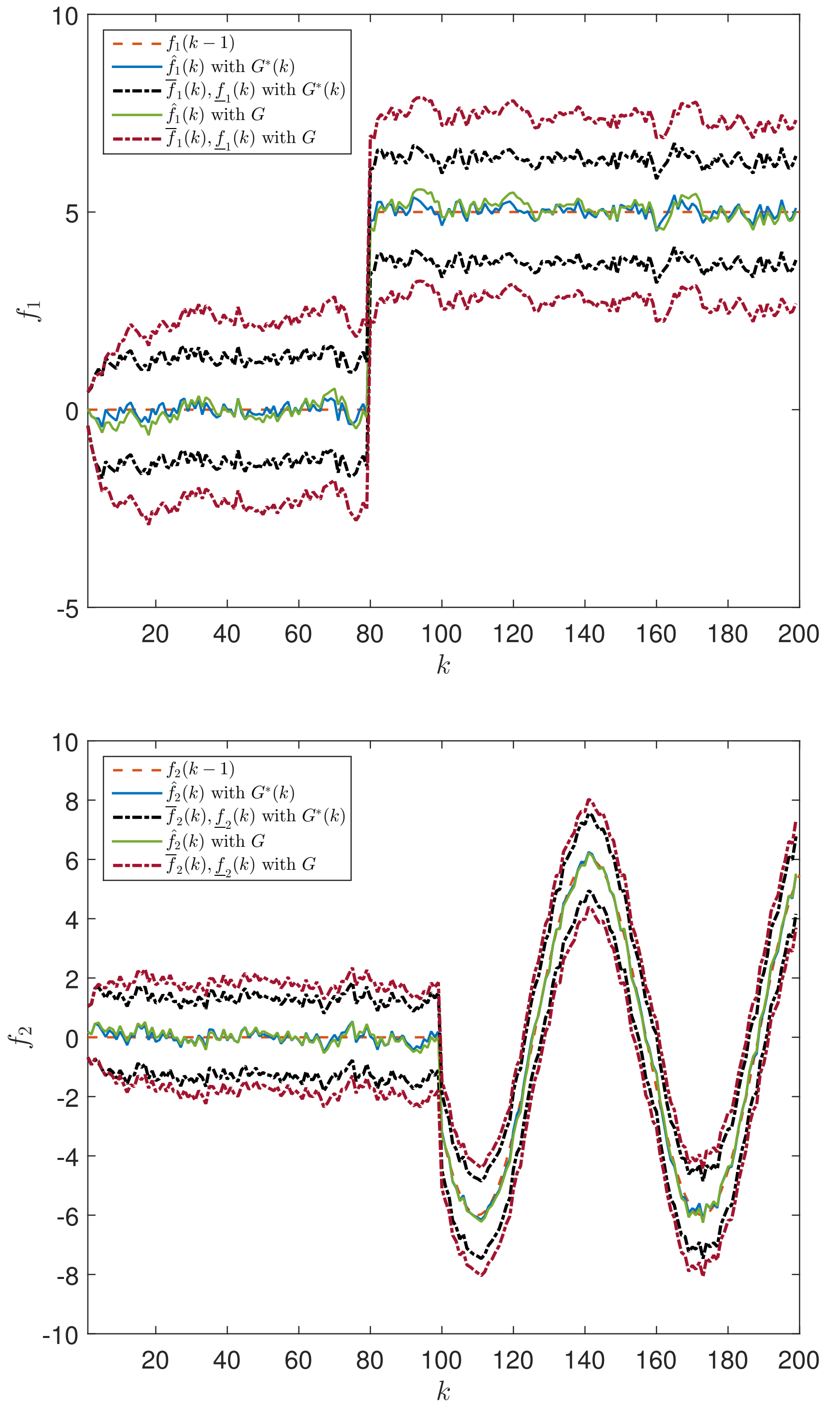

FIGURE 1 The actuator-fault estimation results with $G^*(k)$ and G.

$Y_{25} = 1.2$, $Y_{35} = 0.8$, $Y_{45} = 1$, $Y_{35} = 0.7$ and $Y_{12} = 1$. Besides, the uncertain part of the admittance is set in the state disturbances. Let us define

$$x = \left[\delta_1,\ \delta_2,\ \delta_3,\ \omega_1,\ \omega_2,\ \omega_3,\ \delta_5\right]^{\top},\quad u = \left[p_1,\ p_2,\ p_3\right]^{\top}.$$

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We use the Euler discretisation method with the sampling time $\Delta t = 0.05s$ to obtain the discrete-time descriptor model in the form of (1) with system matrices as follows:

$$E = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{bmatrix}, \quad A = \begin{bmatrix} 1&0&0&0.05&0&0&0\\ 0&1&0&0&0.05&0&0\\ 0&0&1&0&0&0.05&0\\ -5.3571&3.5714&0&0.7964&0&0&1.7857\\ 1.9231&-4.2308&0&0&0.7115&0&2.3077\\ 0&0&-3.75&0&0&0.725&2\\ 0.025&0.06&0.04&0&0&0&-0.175 \end{bmatrix},$$

$$B = F = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} = \begin{bmatrix} 0&0&0\\ 0&0&0\\ 0&0&0\\ 3.5714&0&0\\ 0&1.9231&0\\ 0&0&2.5\\ 0&0&0 \end{bmatrix}, \quad C = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$D_w = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0.3&0&0&0\\ 0&0.3&0&0\\ 0&0&0.3&0\\ 0&0&0&0.3 \end{bmatrix}, D_v = \begin{bmatrix} 0.025&0&0&0\\ 0&0.025&0&0\\ 0&0&0.025&0\\ 0&0&0&0.025 \end{bmatrix}.$$

Given the initial state $x(0) = 0$ and the initial state zonotope $x(0) \in \langle 0, 0.01 I_7 \rangle$, $w(k) \in \langle 0, I_4 \rangle$ and $v(k) \in \langle 0, I_4 \rangle$, $\forall k \in \mathbb{N}$. The input signal is set as $u(k) = \begin{bmatrix} 20, & 15, & 10 \end{bmatrix}^\top$, $\forall k \in \mathbb{N}$. From the general solution (4), we choose the matrix S as

$$S = \begin{bmatrix} 1&0&0&0&0&0&0&1&0&0&0\\ 0&1&0&0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix},$$

and we obtain two non-empty matrices T and N satisfying (3) and the matrix T is also non-singular as follows:

$$T = \begin{bmatrix} 0.5&0&0&0&0&0&0\\ 0&0.5&0&0&0&0&0\\ 0&0&0.5&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \ N = \begin{bmatrix} 0.5&0&0&0\\ 0&0.5&0&0\\ 0&0&0.5&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{bmatrix}.$$

Therefore, for the first actuator, we have $CTF_1 = 0$ and $C(TA)TF_1 \neq 0$. Hence, the fault detectability index for f_1 is $\rho_1 = 2$. Similarly, we have $\rho_2 = \rho_3 = 2$. Therefore, we have the fault detectability matrix Υ as

$$\Upsilon = C\Psi = \begin{bmatrix} 0.0893&0&0\\ 0&0.0481&0\\ 0&0&0.0625\\ 0&0&0 \end{bmatrix}.$$

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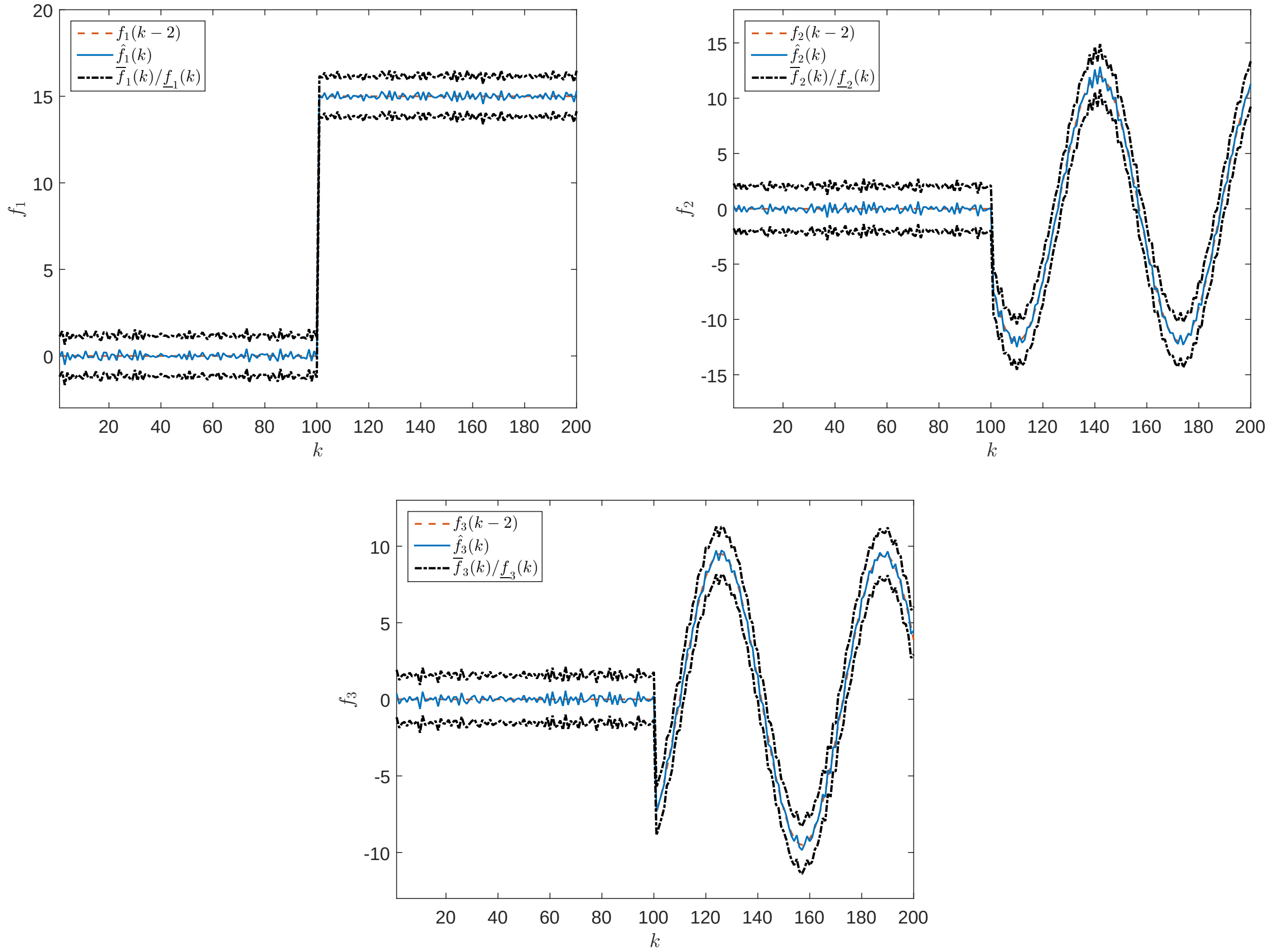

FIGURE 2 The actuator-fault estimation results of the machine infinite bus system.

Then, we can obtain the matrices for the optimal filter gain $G^*(k)$ as follows:

$$\Phi = \begin{bmatrix} 0.1158 & 0 & 0 \\ 0 & 0.0582 & 0 \\ 0 & 0 & 0.0766 \\ 1.7870 & 0.1717 & 0 \\ 0.1717 & 0.7702 & 0 \\ 0 & 0 & 1.0797 \\ 0.0022 & 0.0029 & 0.0025 \end{bmatrix}, \quad M = \begin{bmatrix} 11.2 & 0 & 0 & 0 \\ 0 & 20.8 & 0 & 0 \\ 0 & 0 & 16 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the simulation, consider the actuator fault $f(k)$ in the following

$$f(k) = \begin{cases} 0 & k \le 98 \\ \begin{bmatrix} 15, & 12\sin(0.1k), & 9.5\cos(0.1k) \end{bmatrix}^\top & k > 98 \end{cases}$$

The simulation has been carried out for $N_s = 200$ sampling time steps and the simulation results are shown in Figure 2 . Because of the fault detectability indices $\rho_1 = \rho_2 = \rho_3 = 2$, the fault $f(k)$ occurred at time k will be estimated in two samples. For different time-varying actuator faults, all the estimated results provide the satisfactory results including the punctual values and the worst-case bounds. By minimising the size of the filter zonotope bounding all the uncertainties and propagated estimation errors, the obtained optimal filter gain $G^*(k)$ reduces the estimation errors. Furthermore, during the propagations, the obtained fault estimation intervals (centres of fault estimation zonotopes and the worst-case bounds) are bounded.

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6 | CONCLUSION

In this paper, we have proposed a zonotopic fault estimation filter for discrete-time descriptor systems subject to unknown-but-bounded disturbances and measurement noise in given zonotopes. To achieve the robustness against system uncertainties and identification of occurred actuator faults, the filter gain has been formulated in a parametrised form and under the zonotopic Kalman filter framework, the optimal filter gain can be computed. With this optimal filter gain, the proposed approach guarantees the size of the fault estimation bounds is the smallest that can be obtained with the considered bounds of disturbances and measurement noise. Then, boundedness of the zonotopic fault estimation has been proved. Thus, the size of obtained fault estimation bounds is not growing in time. Finally, the proposed approach has been tested in two simulations. In the first simulation, the comparison results have been shown with a stabilizing filter gain. The obtained fault estimation result with the optimal filter gain has proved to be more accurate based on the mean squared error results and plots. In the second simulation, the proposed approach has been tested with the machine infinite bus system from which the results have shown its effectiveness.

Besides, the assumed rank condition from the ***R***-observability of descriptor systems may lead to restrictiveness. But this condition does not mean that all the states of descriptor systems are directly measured. As future research, a more relaxed rank condition together with different fault estimation conditions will be considered and the proposed fault estimation approach could also be linked with set-based fault isolation.

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