Author/s:
Artemov, G

Title:
Integer game with delay

Date:
2020-03-01

Citation:
Artemov, G. (2020). Integer game with delay. Economics Letters, 188, https://doi.org/10.1016/j.econlet.2020.108968.

Persistent Link:
https://hdl.handle.net/11343/253869

Integer Game with Delay*

Georgy Artemov

January 16, 2020

In this note, we study a version of a war of attrition, in which the players pick delays and the player with the longest delay wins. Unlike the war of attrition, *all* players have to experience the *longest* delay before the consumption takes place. We show that the game has no mixed strategy Nash equilibria. The game can be seen as a re-interpretation of the integer game, which is one of the most important and most criticized constructions in the full implementation literature. Unlike the integer game, it has a well-defined best response against any mixed strategy.

JEL classification: D78, D71

Keywords: full implementation, integer game, war of attrition, best response property, delay

*The Department of Economics, The University of Melbourne, 3010, Australia; e-mail: gartemov@unimelb.edu.au; ORCiD: 0000-0001-7246-0573.
I gratefully acknowledge support through National Science Foundations grant SES-0133113 and Australian Research Council grant DP160101350.

The integer game, in which a player who announces the highest integer receives her most preferred outcome, is a staple of implementation literature. It facilitates the construction of the implementing mechanisms because it has no Nash equilibria. It is also one of the literature's most criticized features. The most damaging is the observation that the best response does not exist against a mixed strategy which assigns a positive probability to any arbitrarily large integer. If the best response does not exist, Nash equilibrium is not an appropriate solution concept (Jackson 1992). Furthermore, such a mixed strategy cannot be dismissed as an irrelevant theoretical construct, as, intuitively, a player's main concern is that an opponent would come up with an unimaginably large number. Hence, she would try to find the best response against exactly that mixed strategy.

In this note, we propose to add delay to the integer game. The proposed "integer game with delay" has a well-defined best response. It expands the outcome space to allow for an undesirable delay, but otherwise can be used in any existing mechanism instead of the integer game.[1]

The integer game with a delay is more realistic than the standard integer game. It resembles a well-studied war of attrition, in which participants who discount future try to out-wait each other (see, e.g., Park and Smith (2008) and Siegel (2009)). The only change is the waiting time: the players have to endure the winner's entire chosen waiting time, even if the opponent's time – unknown to the winner – is shorter. As players receive no information as time passes, the game can be analyzed as a simultaneous-move game in which players pick their waiting times at the beginning of the game.

For notational convenience, we restrict attention to the game between two players who have two different most-preferred outcomes and assume exponential discounting. The game can be readily extended to include any finite number of players, as long as at least two differ in their most-preferred outcome, and any time preferences, as long as delay is costly.

There are two agents, 1 and 2, and two "physical" outcomes, a and b. The outcomes can be delivered at time $t \in \mathbb{Z}_+$. We denote an outcome $z \in \{a, b\}$ delivered at time t by (z, t).

The preferences of two agents are given by utility function

$$u_i(z,t) = \delta^t u_i(z),$$

[1] Artemov (2015) obtains necessary and sufficient conditions for full implementation in the environment with time. He relies on standard canonical mechanisms.

where $\delta < 1$, and, together with the unit of time measurement, is chosen so that the following conditions hold:

$$\delta u_1(a) > u_1(b) \geq 0, \delta u_2(b) > u_2(a) \geq 0. \tag{1}$$

The game $\Gamma = \{\mathbb{Z}_+, \mathbb{Z}_+, g\}$, is described by an action set for each player, $\mathbb{Z}_+$, and the outcome function $g : \mathbb{Z}_+ \times \mathbb{Z}_+ \mapsto \{a, b\} \times \mathbb{Z}_+$, defined as:

$$g(m_1, m_2) = \begin{cases} (a, m_1) & \text{if } m_1 \geq m_2 \\ (b, m_2) & \text{if } m_2 > m_1. \end{cases}$$

A mixed strategy of player i, σ_i, is a probability distribution on the simplex $\Delta(\mathbb{Z}_+)$. We denote a probability assigned to $k \in \mathbb{Z}_+$ in strategy σ_i by p_i^k. We denote the support of σ_i by $\hat{S}_i = \{k \in \mathbb{Z}_+ : p_i^k > 0\}$. We denote the lowest element in $\hat{S}_i \subseteq \mathbb{Z}_+$ by $\underline{s}_i$.

The best response correspondence is

$$B_i(\sigma_{-i}) = \{\sigma_i \in \Delta(\mathbb{Z}_+) : U_i(\sigma_i, \sigma_{-i}) \geq U_i(\sigma_i', \sigma_{-i}) \text{ for all } \sigma_i' \in \Delta(\mathbb{Z}_+)\},$$

where $U_i(\sigma)$ is i's expected payoff when the strategy profile is σ and outcome is given by $g(\cdot)$:

$$U_1(\sigma) = p_2^0 \sum_{k=0}^{\infty} p_1^k \delta^k u_1(a) + \sum_{k=1}^{\infty} p_2^k \left(\delta^k u_1(b) \sum_{l=0}^{k-1} p_1^l + \sum_{l=k}^{\infty} p_1^l \delta^l u_1(a) \right)$$
$$U_2(\sigma) = \sum_{k=0}^{\infty} p_1^k \left(\delta^k u_2(a) \sum_{l=0}^{k} p_2^l + \sum_{l=k+1}^{\infty} p_2^l \delta^l u_2(b) \right)$$

Definition 1. The strategy profile σ^* is a mixed strategy Nash equilibrium if $\sigma_i^* = B_i(\sigma_{-i}^*)$ for $i \in \{1, 2\}$.

Claim 1. For any player $i \in \{1, 2\}$ and any mixed strategy $\sigma_{-i} \in \Delta(\mathbb{Z}_+)$, the best response correspondence $B_i(\sigma_{-i})$ is well-defined.

Proof. For notational convenience, the proof below is for $i = 1$; even though the outcome function $g(\cdot)$ is not symmetric, the proof for $i = 2$ is analogous.

Suppose that the support of strategy σ_2, $\hat{S}_2$, is finite; denote the maximum element by $\bar{s} = \max\{\hat{S}_2\}$. Then any strategy σ_1 such that $p_1^k > 0$ for some $k > \bar{s} + 1$ is strictly dominated by σ_1', which is the same as σ_1 but $(p_1^{\bar{s}+1})' = p_1^{\bar{s}+1} + p_1^k$ and $(p_1^k)' = 0$. Thus, we can restrict attention to strategies σ_1 with finite support $\{0, \dots, \bar{s}+1\}$ that maximize linear function $U_1(\sigma_1, \sigma_2)$; at least one such strategy exists.

Suppose that $\hat{S}_2$ is infinite and consider strategy σ_1 with an infinite support. There exists L such that $\delta^L < p_2^{\underline{s}_2} \delta^{\underline{s}_2}$, as RHS does not depend on L. Consider strategy σ_1' such that

$$\left(p_1^{\underline{s}_2}\right)' = p_1^{\underline{s}_2} + \sum_{l \geq L} p_1^l$$

and $\left(p_1^l\right)' = 0$ for all $l \geq L$ (for player 2, the probability weight would be shifted to $\underline{s}_2 + 1$). Then

$$\begin{aligned} &U_1(\sigma_1', \sigma_2) - U_1(\sigma_1, \sigma_2) \\ &\geq p_2^{\underline{s}_2} \sum_{l \geq L} p_1^l \delta^{\underline{s}_2} u_1(a) - \sum_{l \geq L} p_1^l \delta^l u_1(a) = u_1(a) \sum_{l \geq L} p_1^l \left(p_2^{\underline{s}_2} \delta^{\underline{s}_2} - \delta^l\right) > 0 \end{aligned}$$

Therefore, we can restrict attention to mixed strategies with support $\hat{S}_1 \subseteq \{0, \dots, L\}$. We have argued above that, among these strategies, there exists at least one that maximizes $U_1(\sigma_1, \sigma_2)$. □

Claim 2. There is no mixed strategy Nash equilibrium of Γ.

Proof. Suppose equilibrium strategy profile σ^* exists.

Consider first the case where $\underline{s}_1 \geq \underline{s}_2$. Then player 2's number $\underline{s}_2$ never wins and strategy σ_2' such that $\left(p_2^{\underline{s}_1+1}\right)' = p_2^{\underline{s}_1+1} + p_2^{\underline{s}_2}$ and $p_2^{\underline{s}_2} = 0$ is a profitable deviation from σ_2^* because, by assumption, $\delta u_2(b) > u_2(a)$.

Consider next the case where $\underline{s}_1 < \underline{s}_2$. Then player 1's number $\underline{s}_1$ never wins and player 1 can shift the probability weight to $\underline{s}_2$, increasing her payoff. □

Acknowledgements

I thank Roberto Serrano for insightful discussions and anonymous referee for comments and suggestions.

References

Artemov, G. (2015): "Time and Nash implementation," *Games and Economic Behavior*, 91, 229–236.

Jackson, M. (1992): "Implementation in undominated strategies: A look at bounded mechanisms," *The Review of Economic Studies*, 59(4), 757–775.

Park, A., and L. Smith (2008): "Caller Number Five and related timing games," *Theoretical Economics*, 3(2), 231–256.

Siegel, R. (2009): "All-Pay Contests," *Econometrica*, 77(1), 71–92.