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Author/s:

Heemels, WPMH;Nesic, D;Teel, AR;van de Wouw, N

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Networked and Quantized Control Systems with Communication Delays

W.P.M.H. Heemels, D. Nešić, A.R. Teel, N. van de Wouw

Abstract—There are many communication imperfections in networked control systems (NCSs) such as varying sampling/transmission intervals, varying delays, possible packet loss, communication constraints and quantization effects. Most of the available literature on NCSs focuses on only some of these phenomena, while ignoring the others, although recently some papers appeared that consider at least three of these phenomena. In one paper time-varying delays, time-varying transmission intervals and communication constraints are considered, while in another time-varying transmission intervals, communication constraints and quantization effects are studied. As both approaches are based on the same underlying hybrid modeling framework, it will be shown here that the models can be combined in a unifying hybrid model including the five mentioned network phenomena under some restrictions. On the basis of this model, stability will be analyzed of the closed-loop system in which the controller is obtained using an emulation approach. The analysis provides tradeoffs between the maximally allowable transmission interval (MATI), the maximally allowable delay (MAD) and the quantization parameters, while still guaranteeing closed-loop stability.

I. INTRODUCTION

Networked control systems (NCSs) have received considerable attention in recent years, see e.g. the overview papers [11], [23], [26], [27]. At present, there is a search for control algorithms that can deal with the various communication imperfections that are introduced by the presence of communication networks:

- (i) Quantization errors in the signals transmitted over the network due to the finite word length of the packets;
- (ii) Packet dropouts caused by the unreliability of the network;
- (iii) Variable sampling/transmission intervals;
- (iv) Variable communication delays;
- (v) Communication constraints caused by the sharing of the network by multiple nodes and the fact that only one node is allowed to transmit its packet per transmission.

It is well known that the presence of these network phenomena can degrade the performance of the control loop significantly.

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Maurice Heemels and Nathan van de Wouw are with the Dept. of Mechanical Eng., Eindhoven University of Technology, Eindhoven, The Netherlands, m.heemels@tue.nl

Andy Teel is with the Electrical and Computer Engineering Dept., the University of California, Santa Barbara, CA, USA

Dragan Nešić is with Department of Electrical and Electronic Engineering, The University of Melbourne, Australia.

cantly and can even lead to instability, see e.g. [3], [4] for an illustrative example. Therefore, it is important to understand how these phenomena influence the closed-loop stability and performance properties, preferably in a quantitative manner. However, up to now, much of the available literature on NCS considers only some of above mentioned types of network phenomena and there is no framework that incorporates all of them. The closest ones (that consider more than 2 of these imperfections) are [20], which consider imperfections of type (i), (iii), (v), [12], [18], [19], which study simultaneously type (ii), (iii), (iv), [21], which focusses on type (ii), (iii), (v) and [2], [5], [9], [10] that consider type (iii), (iv) and (v). Note that some of the mentioned approaches that study varying transmission intervals and/or varying communication delays can be extended to include type (ii) phenomena as well by modeling dropouts as prolongations of the maximal admissible transmission interval (cf. also Remark 2 below).

In this paper we will provide a unifying modeling framework incorporating all these five types of networked-induced effects based upon uniting the models adopted in [9], [10] on one hand and [20] on the other. Both [9], [10] and [20] are based on the same underlying modeling framework being hybrid inclusions [7]. We exploit this commonality in obtaining a unifying NCS model including the five different network phenomena (under some restrictions, e.g. the delays are smaller than the sampling interval). On the basis of this model results for stability analysis will be presented for the closed-loop NCS in which the controllers are obtained through an emulation approach [1], [9], [10], [20], [21], [24], [25]. This stability analysis will lead to tradeoff curves between the maximally allowable transmission interval (MATI), the maximally allowable delay (MAD) and the quantization parameters, while still guaranteeing closed-loop stability. The benchmark example of the batch reactor will be used to demonstrate the complete design procedure.

II. NCS MODEL AND PROBLEM STATEMENT

We first introduce the model that will describe NCSs including quantization, communication constraints, varying transmission intervals and delays and we will discuss how dropouts can be included as well (see Remark 2). This model forms an extension of the NCS models used in [21] (without quantization and delays), [9], [10] (without quantization) and [20] (without delays) that were motivated by the initial work in [25]. We consider the continuous-time plant

$$\dot{x}_p = f_p(x_p, \hat{u}, w), \quad y = g_p(x_p) \quad (1)$$

that is sampled. Here, $x_p \in \mathbb{R}^{n_p}$ denotes the state, $\hat{u} \in \mathbb{R}^{n_u}$ denotes the most recent control values available at the plant,

$w \in \mathbb{R}^{n_w}$ is the disturbance and $y \in \mathbb{R}^{n_y}$ is the output. The controller is given by

$$\dot{x}_c = f_c(x_c, \hat{y}, w), \quad u = g_c(x_c), \quad (2)$$

where the variable $x_c \in \mathbb{R}^{n_c}$ is the state of the controller, $\hat{y} \in \mathbb{R}^{n_y}$ is the most recent output measurement of the plant that is available at the controller and $u \in \mathbb{R}^{n_u}$ denotes the control input. The functions f_p , f_c are assumed to be continuous and g_p and g_c are assumed to be continuously differentiable. At times t_i , $i \in \mathbb{N}$, parts of the input u at the controller and/or the output y at the plant are sampled and sent over the network. The transmission times satisfy $0 \leq t_0 < t_1 < t_2 < \dots$ and there exists a $\delta > 0$ such that the transmission intervals $t_{i+1} - t_i$ satisfy $\delta \leq t_{i+1} - t_i \leq h_{mati}$ for all $i \in \mathbb{N}$, where h_{mati} denotes the maximally allowable transmission interval (MATI). At each transmission time t_i , $i \in \mathbb{N}$, the protocol determines which of the nodes $j \in \{1, 2, \dots, l\}$ is granted access to the network. Each node corresponds to a collection of sensors or actuators. The sensors/actuators corresponding to the node that is granted access at time t_i , denoted by $S_i \in \{1, \dots, l\}$, collect their values as obtained from a quantized measurement of the corresponding entries in $y(t_i)$ or $u(t_i)$ that will be sent over the communication network. They will arrive after a transmission delay of τ_i time units at the controller or actuator. This results in updates of the corresponding entries in $\hat{y}$ or $\hat{u}$ at times $t_i + \tau_i$, $i \in \mathbb{N}$. As said, the values corresponding to a node that are sent over the network are obtained by a quantizer. Each node has a quantizer $q_j : \mathbb{R}^{n_j} \rightarrow \mathcal{Q}_j \subseteq \mathbb{R}^{n_j}$, $j = 1, \dots, l$, where each $\mathcal{Q}_j$ consists of a finite or countable collection of quantization points. The integer n_j denotes the number of signals corresponding to node j .

Assumption 1 There exist strictly positive numbers M_j and Δ_j , $j = 1, \dots, l$, such that for all $j = 1, \dots, l$ and all $z_j \in \mathbb{R}^{n_j}$ it holds that

$$\begin{aligned} |z_j| \leq M_j &\Rightarrow |q_j(z_j) - z_j| \leq \Delta_j \\ |z_j| > M_j &\Rightarrow |q_j(z_j)| > M_j - \Delta_j \end{aligned}$$

where $|\cdot|$ denotes the standard Euclidean norm.

The variable Δ_j is related to the resolution and M_j is associated with the range of the j th quantizer. These type of conditions for quantizers were introduced in [14]. Note that the first condition gives a bound on the quantization error when the quantizer does not saturate (the variable z_j is in range). The second condition provides a way to detect the possibility of saturation. Each quantizer q_j has also a “zoom” parameter μ_j , which can be adjusted in order to increase or decrease the resolution and the range of the quantizer. This leads to the quantizer (with a slight abuse of notation)

$$q_j(z_j, \mu_j) := \mu_j q_j \left(\frac{z_j}{\mu_j} \right),$$

where $z_j \in \mathbb{R}^{n_j}$ contains the values of $z := (y, u)$ corresponding to node j . We focus here on these so-called “zoom quantizers,” which require conditions such as Assumption 1,

although some other quantizers can also be included in this framework as long as the stated assumptions remain true. By re-ordering we can have that $z = (z_1, \dots, z_l)$. Similarly, we denote the “networked” version of z (consisting of the latest available information) by $\hat{z} := (\hat{y}, \hat{u})$ and $\hat{z} = (\hat{z}_1, \dots, \hat{z}_l)$. Hence, z and $\hat{z} \in \mathbb{R}^{n_z}$ with $n_z = n_y + n_u$.

If we now take $\mu := (\mu_1, \mu_2, \dots, \mu_l)$, we define the overall quantizer as

$$q(z, \mu) = (q_1(z_1, \mu_1), q_2(z_2, \mu_2), \dots, q_l(z_l, \mu_l)).$$

Using this terminology, we have now that at transmission time t_i the values $q_{S_i}(z_{S_i}(t_i), \mu_{S_i}(t_i))$ corresponding to node S_i are collected and transmitted over the network and they arrive at their destination after a delay of τ_i time units. Moreover, at transmission time t_i (after coding the message) the “zooming” parameter μ is updated according to

$$\mu(t_i^+) = \Omega_{\text{zoom}} \mu(t_i) \quad (3)$$

with

$$\Omega_{\text{zoom}} = \text{diag}(\Omega_1, \dots, \Omega_l)$$

and $\Omega_j \in (0, 1)$ for each j . In between transmission times the zooming parameter μ remains constant and hence, we have that

$$\dot{\mu} = 0. \quad (4)$$

Actually, the assumption of $\dot{\mu} = 0$ does not hold for all quantizers such as the “box” quantizers [15], [16]. Handling these quantizers is a topic for future research.

Remark 1 In principle the networked system has zoom parameters μ_c and μ_d corresponding to the coder and the decoder side, respectively, of the communication channel. Essentially, we have that at t_i it holds that

$$\mu_c(t_i^+) = \Omega_{\text{zoom}} \mu_c(t_i) \text{ and } \mu_d(t_i^+) = \mu_d(t_i),$$

where we assume that the value of $q_{S_i}(z_{S_i}(t_i), \mu_{S_i}(t_i))$ is collected before the encoder zoom parameter μ_c is updated. At $t_i + \tau_i$ the message arrives and we have the updates

$$\mu_c((t_i + \tau_i)^+) = \mu_c(t_i + \tau_i); \quad \mu_d((t_i + \tau_i)^+) = \Omega_{\text{zoom}} \mu_d(t_i + \tau_i),$$

where we assume that the decoding action takes place before the decoder zoom parameter μ_d is updated. This guarantees that the decoding is performed with the same zoom parameter with which the coding took place provided that both zoom parameters were initialized at the same value, i.e. $\mu_d(0) = \mu_c(0)$. Indeed, under the latter condition, we would have that $\mu_c(t_i) = \mu_d(t_i + \tau_i)$ and $\mu_c(t_{i+1}) = \mu_d(t_{i+1}) = \mu_c((t_i + \tau_i)^+) = \mu_d((t_i + \tau_i)^+)$ for all $i \in \mathbb{N}$ due to (4). Although the practical implementation of the networked control system requires two zoom parameters, in the mathematical model that we derive here, it suffices to use only one zoom parameter μ (which is actually equal to μ_c). For discussions on the case of mismatch between coder/decoder initializations, we refer to [13].

We adopt the following assumption on the quantizer, which was also used in [20].

Assumption 2 The initial states $x_p(0)$, $x_c(0)$, the disturbance inputs w and $\mu(0)$ are such that for all $j \in \{1, \dots, l\}$ we have that

$$\frac{|z_j(t_i)|}{\mu_j(t_i)} \leq M_j \text{ for all } i \in \mathbb{N}.$$

If $M_j = \infty$ for all $j = 1, \dots, l$, meaning that all quantizers have infinite range, this assumption always holds. For finite-range quantizers, it is not difficult to satisfy this assumption, at least for linear systems (see e.g. [14], [20]).

Regarding the communication delays, it is assumed that there are bounds on the maximal delay in the sense that $\tau_i \in [0, \tau_{mad}]$, $i \in \mathbb{N}$, where $0 \leq \tau_{mad} \leq h_{mati}$ is the maximally allowable delay (MAD). To be more precise, we adopt the following assumption.

Assumption 3 The transmission times satisfy $\delta \leq t_{i+1} - t_i < h_{mati}$, $i \in \mathbb{N}$ and the delays satisfy $0 \leq \tau_i \leq \min\{\tau_{mad}, t_{i+1} - t_i\}$, $i \in \mathbb{N}$, where $\delta \in (0, h_{mati}]$ can be taken arbitrarily small.

The latter condition implies that each transmitted packet arrives before the next sample is taken. This assumption indicates that we are considering the so-called small delay case as opposed to the large delay case, where delays can be larger than the transmission interval. The inequalities $\tau_i \leq t_{i+1} - t_i$ and $\tau_{mad} \leq h_{mati}$ can be taken non-strict with the understanding that in case the update instant $t_i + \tau_i$ coincides with the next transmission instant t_{i+1} , the update is performed before the next sample is taken. At the update instant $t_i + \tau_i$ the value of the networked version $\hat{z}_{S_i}$ is updated to $q_{S_i}(z_{S_i}(t_i), \mu_{S_i}(t_i))$, while the values of $\hat{z}_j$ for $j \neq S_i$ remain the same and thus equal to $\hat{z}_j(t_i + \tau_i)$. This can actually be conveniently rephrased by utilizing the Kronecker delta δ_{ij} , which is equal to 1 when $i = j$ and equal to 0 when $i \neq j$. We define now $\Psi(S) := \text{diag}(\delta_{1S}, \delta_{2S}, \dots, \delta_{lS})$ for $S \in \{1, \dots, l\}$. The update of $\hat{z}$ at $t_i + \tau_i$ is given by

$$\hat{z}((t_i + \tau_i)^+) = \Psi(S_i)q(z(t_i), \mu(t_i)) + (I - \Psi(S_i))\hat{z}(t_i + \tau_i). \quad (5)$$

In between the update times of $\hat{y}$ and $\hat{u}$, the network is assumed to operate in a zero order hold (ZOH) fashion, meaning that the values of $\hat{z} = (\hat{y}, \hat{u})$ remain constant between $t_i + \tau_i$ and $t_{i+1} + \tau_{i+1}$ for all $i \in \mathbb{N}$:

$$\dot{\hat{z}} = 0. \quad (6)$$

The network error $e := \hat{z} - z \in \mathbb{R}^{n_e}$ with $n_e = n_z$ undergoes resets at the update times $t_i + \tau_i$ given by:

$$\begin{aligned} e((t_i + \tau_i)^+) &= \hat{z}((t_i + \tau_i)^+) - z((t_i + \tau_i)^+) \\ &= \Psi(S_i)q(z(t_i), \mu(t_i)) + (I - \Psi(S_i))\hat{z}(t_i + \tau_i) - z(t_i + \tau_i) \\ &= \Psi(S_i)q(z(t_i), \mu(t_i)) - \Psi(S_i)\hat{z}(t_i + \tau_i) + e(t_i + \tau_i) \\ &= \Psi(S_i)q(z(t_i), \mu(t_i)) - \Psi(S_i)[e(t_i) + z(t_i)] + e(t_i + \tau_i) \\ &= e(t_i + \tau_i) - e(t_i) \\ &\quad + \underbrace{\Psi(S_i)[q(z(t_i), \mu(t_i)) - z(t_i)] + (I - \Psi(S_i))e(t_i)}_{=: \tilde{h}(S_i, z(t_i), e(t_i), \mu(t_i))}. \end{aligned} \quad (7)$$

In the second equality we used that z is continuous and in the fourth equality we used that $\hat{z}(t_i + \tau_i) = \hat{z}(t_i) = z(t_i) + e(t_i)$ due to the zero order hold character of the network and the definition of e . We also implicitly employed Assumption 3 as we used that there always occurs an update before the next sample is taken ($t_i + \tau_i \leq t_{i+1}$). We split the expression (7) in two parts of which one is given by a function $\tilde{h} : \{1, \dots, l\} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_e} \times \mathbb{R}^l \rightarrow \mathbb{R}^{n_e}$, which is related to the resets of the networked and quantized control systems (NQCS) *without* delays as studied in [20]. By writing the update of e in this form, we can exploit specific results in [20], as we will show later. As $z = (y, u)$ is a function of x_p and x_c due to (1)-(2), we introduce $x = (x_p, x_c) \in \mathbb{R}^{n_x}$ with $n_x = n_p + n_c$ and rewrite (7) in terms of S_i , $x(t_i)$, $e(t_i)$ and $\mu(t_i)$. In addition, we will also directly introduce the scheduling mechanism that determines which node S_i obtains access to the network at time t_i . In general this is done on the basis of i , $x(t_i)$, $e(t_i)$ and $\mu(t_i)$ and therefore we have

$$S_i = S(i, x(t_i), e(t_i), \mu(t_i)) \in \{1, \dots, l\}. \quad (8)$$

Later we will provide particular instances of the scheduling function $S : \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^l \rightarrow \{1, \dots, l\}$ such as the well-known round-robin (RR) and try-once-discard (TOD) scheduling protocols. The update of e in (7) can now be rewritten as

$$\begin{aligned} e((t_i + \tau_i)^+) &= e(t_i + \tau_i) - e(t_i) + \\ &\quad + \tilde{h}(S_i, (g_p(x_p(t_i)), g_c(x_c(t_i))), e(t_i), \mu(t_i)) \\ &=: e(t_i + \tau_i) - e(t_i) + h(i, x(t_i), e(t_i), \mu(t_i)) \end{aligned} \quad (9)$$

for a new function $h : \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}^l \rightarrow \mathbb{R}^{n_e}$, which is often referred to as the overall protocol including the scheduling function and the quantizer.

Combining the above derivations, we obtain the following model for the NCS with delays and quantization:

$$\left. \begin{aligned} \dot{x}(t) &= f(x(t), e(t), w(t)) \\ \dot{e}(t) &= g(x(t), e(t), w(t)) \\ \dot{\mu}(t) &= 0 \end{aligned} \right\} \quad t \neq t_i \wedge t \neq t_i + \tau_i, \quad (10a)$$

$$e((t_i + \tau_i)^+) = h(i, x(t_i), e(t_i), \mu(t_i)) + e(t_i + \tau_i) - e(t_i), \quad (10b)$$

$$\mu(t_i^+) = \Omega_{\text{zoom}} \mu(t_i), \quad (10c)$$

where f , g are appropriately defined functions depending on f_p , g_p , f_c and g_c . See [21] for the explicit expressions of f and g , which also reveal how we use the differentiability conditions on g_c and g_p imposed earlier.

Assumption 4 f and g are continuous and h is locally bounded. ■

Observe that the system

$$\dot{x} = f(x, 0, w) \quad (11)$$

is the closed-loop system (1)-(2) without the network (i.e. $y(t) = \hat{y}(t)$ and $u(t) = \hat{u}(t)$ in (1)-(2)).

The problem that we consider here is the stability analysis of the NCS using a controller (2) that is obtained through an emulation approach [1], [9], [10], [20], [21], [24], [25].

Problem 1 Suppose that the controller (2) was designed for the plant (1) rendering the closed-loop (1)-(2) (or equivalently, (11)) stable in some sense. Determine the value of h_{mati} and τ_{mad} so that the NCS given by (10) is stable as well when the transmission intervals and delays satisfy Assumption 3. ■

Remark 2 The inclusion of packet dropouts can be realized by modeling them as prolongations of the transmission interval. If we assume that there is a bound $\bar{\delta} \in \mathbb{N}$ on the maximum number of successive dropouts, the stability bounds derived below are still valid for the MATI given by $h'_{mati} := \frac{h_{mati}}{\bar{\delta}+1}$, where h_{mati} is the obtained value for the dropout-free case. ■

III. REFORMULATION IN A HYBRID SYSTEM FRAMEWORK

To facilitate the stability analysis, we transform the above NCS model into the hybrid system framework discussed in [8]. To do so, we introduce the auxiliary variables $s \in \mathbb{R}^n$, $\kappa \in \mathbb{N}$, $\tau \in \mathbb{R}_{\geq 0}$ and $\ell \in \{0, 1\}$ to reformulate the model in terms of flow equations and reset equations. The variable s is an auxiliary variable containing the memory in (10b) storing the value $h(i, x(t_i), e(t_i), \mu(t_i)) - e(t_i)$ for the update of e at the update instant $t_i + \tau_i$, κ is a counter keeping track of the transmission, τ is a timer to constrain both the transmission interval as well as the transmission delay and ℓ is a Boolean keeping track whether the next event is a transmission event or an update event. The hybrid system $\mathcal{H}_{NCS}$ is now given by the flow equations

$$\left. \begin{aligned} \dot{x} &= f(x, e, w) \\ \dot{e} &= g(x, e, w) \\ \dot{\mu} &= 0 \\ \dot{s} &= 0 \\ \dot{\tau} &= 1 \\ \dot{\kappa} &= 0 \\ \dot{\ell} &= 0 \end{aligned} \right\} \begin{aligned} &(\ell = 0 \wedge \tau \in [0, h_{mati}]) \vee \\ &(\ell = 1 \wedge \tau \in [0, \tau_{mad}]) \end{aligned}, \quad (12)$$

and the reset equations are obtained by combining the “transmission reset relations,” active at the transmission instants $\{t_i\}_{i \in \mathbb{N}}$, and the “update reset relations,” active at the update instants $\{t_i + \tau_i\}_{i \in \mathbb{N}}$, given by

$$\begin{aligned} (x^+, e^+, \mu^+, s^+, \tau^+, \kappa^+, \ell^+) &= G(x, e, \mu, s, \tau, \kappa, \ell), \\ \text{if } (\ell = 0 \wedge \tau \in [\delta, h_{mati}]) \vee (\ell = 1 \wedge \tau \in [0, \tau_{mad}]) \end{aligned} \quad (13)$$

with G given by the transmission resets (when $\ell = 0$) at t_i , $i \in \mathbb{N}$

$$G(x, e, \mu, s, \tau, \kappa, 0) = (x, e, \Omega_{\text{zoom}}\mu, h(\kappa, x, e, \mu) - e, 0, \kappa+1, 1) \quad (14)$$

and the update resets (when $\ell = 1$) at $t_i + \tau_i$, $i \in \mathbb{N}$

$$G(x, e, \mu, s, \tau, \kappa, 1) = (x, s + e, \mu, 0, \tau, \kappa, 0). \quad (15)$$

The constant $\delta > 0$ can be chosen arbitrarily small and it is included to prevent certain Zeno behavior (an infinite number of reset events in a finite length time interval) in the model. For general modeling purposes, we include disturbance signals in the framework, but next we will focus on asymptotic stability for zero disturbance input, i.e. $w = 0$. See [10, Def. IV.1] or [9] for the exact definition of uniform global asymptotic stability (UGAS) of the set $\mathcal{E} := \{(x, e, \mu, s, \tau, \kappa, \ell) \mid x = 0, e = s = 0\}$ that we adopt here for the hybrid system $\mathcal{H}_{NCS}$ with $w = 0$.

IV. STABILITY ANALYSIS

We are going to construct a Lyapunov function for $\mathcal{H}_{NCS}$ based on the following conditions for the reset part (13) (the protocol) and the flow part (12) of the system.

Conditions on the reset part

For the delay-free case, one considers in [1], [20], [21] protocols satisfying the following condition:

Condition 1 The protocol given by h is UGES (uniformly globally exponentially stable), meaning that there exists a function $W : \mathbb{N} \times \mathbb{R}^{n_e} \times \mathbb{R}^l \rightarrow \mathbb{R}_{\geq 0}$ that is locally Lipschitz such that for all $\kappa \in \mathbb{N}$, $e \in \mathbb{R}^{n_e}$, $\mu \in \mathbb{R}^l$ and $x \in \mathbb{R}^{n_x}$

$$\underline{\alpha}_W |(e, \mu)| \leq W(\kappa, e, \mu) \leq \bar{\alpha}_W |(e, \mu)| \quad (16a)$$

$$W(\kappa + 1, h(\kappa, x, e, \mu), \Omega_{\text{zoom}}\mu) \leq \lambda W(\kappa, e, \mu) \quad (16b)$$

for constants $0 < \underline{\alpha}_W \leq \bar{\alpha}_W$ and $0 < \lambda < 1$. ■

Additionally we assume here that

$$W(\kappa + 1, e, \Omega_{\text{zoom}}\mu) \leq \lambda_W W(\kappa, e, \mu) \quad (17)$$

for some constant $\lambda_W \geq 1$ and that for almost all $e \in \mathbb{R}^{n_e}$ and all $\kappa \in \mathbb{N}$

$$\left| \frac{\partial W}{\partial e}(\kappa, e, \mu) \right| \leq M_1 \quad (18)$$

for some constant $M_1 > 0$. In Lemma 1 below, we specify appropriate values for these constants in case of the often used Round Robin (RR) and the Try-Once-Discard (TOD) protocols.

Conditions on the flow part

We assume the growth condition

$$|g(x, e, 0)| \leq m_x(x) + M_e|e| \quad (19)$$

on the NCS model (12), where $m_x : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ and we will also use the following condition that is only slightly modified with respect to the *delay-free conditions* in [1]:

Condition 2 There is a locally Lipschitz continuous function $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ satisfying the bounds

$$\underline{\alpha}_V(|x|) \leq V(x) \leq \bar{\alpha}_V(|x|) \quad (20)$$

for some $\mathcal{K}_\infty$ -functions¹ $\underline{\alpha}_V$ and $\overline{\alpha}_V$, and

$$\langle \nabla V(x), f(x, e, 0) \rangle \leq -m_x^2(x) - \rho(|x|) + (\gamma^2 - \varepsilon)W^2(\kappa, e, \mu) \quad (21)$$

for almost all $x \in \mathbb{R}^{n_x}$ and all $e \in \mathbb{R}^{n_e}$ with $\rho \in \mathcal{K}_\infty$, where the constants in (21) satisfy $0 < \varepsilon < \max\{\gamma^2, 1\}$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^{n_x}$.

The constant $\varepsilon > 0$ is typically chosen sufficiently small.

Lumping the above parameters into four new ones, namely

$$L_0 = \frac{M_1 M_e}{\underline{\alpha}_W}; \quad L_1 = \frac{M_1 M_e \lambda_W}{\lambda \underline{\alpha}_W}; \quad \gamma_0 = M_1 \gamma; \quad \gamma_1 = \frac{M_1 \gamma \lambda_W}{\lambda} \quad (22)$$

we can determine MAD and MATI that guarantee stability of $\mathcal{H}_{NCS}$ based on the differential equations

$$\dot{\phi}_0 = -2L_0\phi_0 - \gamma_0(\phi_0^2 + 1) \quad (23a)$$

$$\dot{\phi}_1 = -2L_1\phi_1 - \gamma_0(\phi_1^2 + \frac{\gamma_1^2}{\gamma_0^2}). \quad (23b)$$

Observe that the solutions to these differential equations are strictly decreasing as long as $\phi_\ell(\tau) \geq 0$, $\ell = 0, 1$.

Theorem 1 *Let Assumptions 1, 2, 3 and 4 be true. Consider the system $\mathcal{H}_{NCS}$ with $w = 0$ that satisfies Condition 1 with (17) and (18), and Condition 2. Suppose $h_{mati} \geq \tau_{mad} \geq 0$ satisfy*

$$\phi_0(\tau) \geq \lambda^2 \phi_1(0) \text{ for all } 0 \leq \tau \leq h_{mati} \quad (24a)$$

$$\phi_1(\tau) \geq \phi_0(\tau) \text{ for all } 0 \leq \tau \leq \tau_{mad} \quad (24b)$$

for solutions ϕ_0 and ϕ_1 of (23) corresponding to certain chosen initial conditions $\phi_\ell(0) > 0$, $\ell = 0, 1$, with $\phi_1(0) \geq \phi_0(0) \geq \lambda^2 \phi_1(0) \geq 0$ and $\phi_0(h_{mati}) > 0$. Then for the system $\mathcal{H}_{NCS}$ with $w = 0$ the set $\mathcal{E}$ is UGAS. ■

The proof is omitted due to space limitations, but is based on the construction of Lyapunov functions as in [9]. From the above theorem quantitative numbers for h_{mati} and τ_{mad} can be obtained by constructing the solutions to (23) for certain initial conditions. By computing the τ value of the intersection of ϕ_0 and the constant line $\lambda^2 \phi_1(0)$ provides h_{mati} according to (24a), while the intersection of ϕ_0 and ϕ_1 gives a value for τ_{mad} due to (24b). Different values of the initial conditions $\phi_0(0)$ and $\phi_1(0)$ lead, of course, to different solutions ϕ_0 and ϕ_1 of the differential equations (23) and thus too different h_{mati} and τ_{mad} . As a result, tradeoff curves between h_{mati} and τ_{mad} can be obtained that indicate when stability of the NCS is still guaranteed. This will be illustrated in Section V. See [10] for a systematic procedure for stability analysis based on the above ideas.

To apply the above theorem for a given protocol we need to establish the values λ , M_1 , λ_W , $\underline{\alpha}_W$ and $\overline{\alpha}_W$. We give these constants for the RR and TOD protocols. For the RR protocol the scheduling function S as in (8) is given by

$$S(i, x, e, \mu) = S_{RR}(i) = j, \text{ when } i = j + kl \text{ for some } k \in \mathbb{N} \quad (25)$$

¹A function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$ and to class $\mathcal{K}_\infty$ if additionally $\varphi(s) \rightarrow \infty$ as $s \rightarrow \infty$.

and for the TOD protocol it is given by

$$S(i, x, e, \mu) = S_{TOD}(e) = \arg \max_j |e_j|. \quad (26)$$

Hence, in the RR protocol the node j transmits periodically with a period l and in the TOD protocol the node j obtains access to the network for which the networked induced error $|e_j|$ is the largest. If these scheduling functions are combined with the quantizers as introduced earlier, we obtain the overall protocol h as in (9), which is given by

$$h_{RR, zoom}(i, x, e, \mu) := \tilde{h}(S_{RR}(i), (g_p(x_p), g_c(x_c)), e, \mu) \quad (27)$$

in case of the RR protocol, and by

$$h_{TOD, zoom}(i, x, e, \mu) := \tilde{h}(S_{TOD}(e), (g_p(x_p), g_c(x_c)), e, \mu) \quad (28)$$

in case of the TOD protocol.

Lemma 1 *Let l denote the number of nodes in the network and suppose that $0 < \Omega_j < 1$ for all $j = 1, \dots, l$. For the overall RR zoom protocol given by (27) and for any $\omega \in (0, \frac{1 - \max_j \Omega_j}{\sqrt{l} \max_j \Delta_j})$ there is a locally Lipschitz $W_{RR, zoom} : \mathbb{N} \times \mathbb{R}^{n_x} \times \mathbb{R}^l \rightarrow \mathbb{R}_{\geq 0}$ such that the conditions (16), (17) and (18) hold with*

$$\lambda_{RR, zoom} = \max \left\{ \sqrt{\frac{l-1}{l}}, \omega \sqrt{l} \max_j \Delta_j + \max_j \Omega_j \right\},$$

$$\underline{\alpha}_{W_{RR, zoom}} = \min\{1, \omega\}, \quad \overline{\alpha}_{W_{RR, zoom}} = 1 + \omega \sqrt{l},$$

$$\lambda_{W_{RR, zoom}} = \sqrt{l} \text{ and } M_{1, RR, zoom} = \omega \sqrt{l}.$$

For the overall TOD zoom protocol given by (28) and for any $\omega \in (0, \frac{1 - \max_j \Omega_j}{\max_j \Delta_j})$ $W_{TOD, zoom}(\kappa, e, \mu) = \omega |e| + |\mu|$ satisfies the conditions (16), (17) and (18) with

$$\lambda_{TOD, zoom} = \max \left\{ \sqrt{\frac{l-1}{l}}, \omega \max_j \Delta_j + \max_j \Omega_j \right\},$$

$$\underline{\alpha}_{W_{TOD, zoom}} = \min\{1, \omega\}, \quad \overline{\alpha}_{W_{TOD, zoom}} = 1 + \omega,$$

$$\lambda_{W_{TOD, zoom}} = 1 \text{ and } M_{1, TOD, zoom} = \omega. \quad \blacksquare$$

The proofs can be based on the results in [20].

V. CASE STUDY OF THE BATCH REACTOR

The case study of the batch reactor has developed over the years as a benchmark example for NCSs, see e.g. [1], [9], [10], [21], [25]. The functions in the NCS (10) for the batch reactor are linear and given by $f(x, e, 0) = A_{11}x + A_{12}e$ and $g(x, e, 0) = A_{21}x + A_{22}e$. The batch reactor, which is open-loop unstable, has $n_u = 2$ inputs, $n_y = 2$ outputs, $n_p = 4$ plant states and $n_c = 2$ controller states and $l = 2$ nodes (only the outputs are assumed to be sent over the network). See [21], [25] for the details and the numerical values of the matrices. The measurements are obtained through quantizers with the specifications $\max_j \Omega_j = 0.9$ and $\max_j \Delta_j = 1$. Focussing on the “TOD zoom protocol” first, Lemma 1 states

that $W_{\text{TOD},\text{zoom}}(\kappa, e, \mu) = \omega|e| + |\mu|$ is a Lyapunov function showing that the protocol is UGES for $\omega \in (0, 0.1)$. To apply the developed framework for stability analysis, we take $M_e = |A_{22}| := \sqrt{\lambda_{\max}(A_{22}^\top A_{22})}$ and $m_x(x) = |A_{21}x|$ in (19). To verify (21) we take $\rho(r) = \varepsilon r^2$ and consider a quadratic Lyapunov function $V(x) = x^\top P x$ to compute the $\mathcal{L}_2$ gain (or actually a value close to the $\mathcal{L}_2$ gain by selecting $\varepsilon > 0$ small) from $|e|$ (which is smaller than $\frac{1}{\omega} W_{\text{TOD},\text{zoom}}(\kappa, e, \mu)$) to $m_x(x)$ by minimizing $\bar{\gamma}$ subject to the following linear matrix inequalities (LMIs) in the matrix $P = P^\top \succ 0$:

$$\begin{pmatrix} A_{11}^\top P + P A_{11} + \varepsilon I + A_{21}^\top A_{21} & P A_{12} \\ A_{12}^\top P & (\varepsilon - \bar{\gamma}^2) I \end{pmatrix} \preceq 0. \quad (29)$$

Minimizing $\bar{\gamma}$ subject to the LMI (29) with $\varepsilon = 0.01$ using the SEDUMI solver [22] with the YALMIP interface [17] provides the minimal value of $\gamma^* = 15.9165$. As $W_{\text{TOD},\text{zoom}}(\kappa, e, \mu) = \omega|e| + |\mu|$ this gives that γ in (21) can be taken as $\gamma(\omega) = \frac{\gamma^*}{\omega}$ (depending on the choice of ω). From Lemma 1 and Theorem 1 we now obtain the values

$$\begin{aligned} L_0 &= |A_{22}|, \quad L_1 = \frac{|A_{22}|}{\omega + 0.9}, \quad \gamma_0 = \gamma^*, \\ \gamma_1 &= \frac{\gamma^*}{\omega + 0.9}, \quad \lambda_{\text{TOD},\text{zoom}} = \omega + 0.9. \end{aligned} \quad (30)$$

According to Theorem 1, MATI is only influenced by the choice of ω through $\lambda_{\text{TOD},\text{zoom}}$ via (24a) as L_0 and γ_0 are independent of ω . To obtain the largest value for MATI, $\omega \in (0, 0.1)$ should be minimized. Note that the effect of ω on the constants L_1 and γ_1 is relatively small and hence, MAD cannot be significantly influenced by ω . We take ω here equal to $\bar{\omega} = 0.005$ (at 5% of its maximal value). This results in the numerical values $L_0 = 15.7300$, $L_1 = 17.3812$, $\gamma_0 = 15.9165$, $\gamma_1 = 17.5872$ and $\lambda_{\text{TOD},\text{zoom}} = 0.905$.

The above numerical values provide various combinations of $(h_{\text{mati}}, \tau_{\text{mad}})$ that yield stability of the NCS by varying the initial conditions $\phi_0(0)$ and $\phi_1(0)$. Hence, the initial conditions of both functions ϕ_0 and ϕ_1 can be used to make design tradeoffs. For instance, by taking $\phi_1(0)$ larger, the allowable delays become larger (as the ϕ_1 solution to (23) shifts upwards), while the maximum transmission interval becomes smaller as the value of $\lambda^2 \phi_1(0)$ will shift upwards as well causing its intersection with ϕ_0 to occur for a lower value of τ . For instance, by taking $\phi_0(0) = \phi_1(0) = \lambda_{\text{TOD},\text{zoom}}^{-1}$, we obtain the delay-free case with $\tau_{\text{mad}} = 0$ and $h_{\text{mati}} = 0.00315$. Following this procedure for various increasing values of $\phi_1(0)$, while keeping $\phi_0(0)$ equal to $\lambda_{\text{TOD},\text{zoom}}^{-1}$, the graph in Figure 1 is obtained.

Applying a similar procedure for the RR zoom protocol (where we exploit the special structure in the system just as was done in [21, Ex. 3] and [9], [10]), leads to the tradeoff curve between MATI and MAD for the RR zoom protocol as also given in Figure 1. These tradeoff curves can be used to impose conditions or select a suitable network with certain communication delay and bandwidth requirements (note that MATI is inversely proportional to the bandwidth).

Above we fixed the quantizer properties, but we could easily add a third (or fourth) axis to the tradeoff curves in

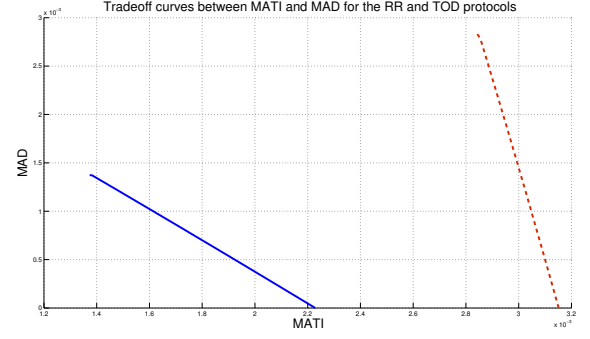


Fig. 1. Tradeoff curves between MATI and MAD for the TOD and RR zoom protocols.

Figure 1 showing the tradeoffs between MATI, MAD and the quantization properties ($\max_j \Omega_j$ and/or $\max_j \Delta_j$). For instance, in case of the TOD zoom protocol we would have for $q_1 := \max_j \Omega_j$ and $q_2 := \max_j \Delta_j$ that

$$\begin{aligned} \lambda &= \max\left\{\sqrt{\frac{1}{2}}, \omega q_2 + q_1\right\}, \quad \underline{\alpha}_W = \min(1, \omega); \quad \bar{\alpha}_W = 1 + \omega; \\ \lambda_W &= 1; \quad M_1 = \omega; \quad \gamma_0 = \gamma^*; \quad \gamma_1 = \frac{\gamma^*}{\max\left\{\sqrt{\frac{1}{2}}, \omega q_2 + q_1\right\}}; \\ L_0 &= \frac{\omega |A_{22}|}{\min\{1, \omega\}}; \quad L_1 = \frac{\omega |A_{22}|}{\min\{1, \omega\} \max\left\{\sqrt{\frac{1}{2}}, \omega q_2 + q_1\right\}} \end{aligned} \quad (31)$$

for $\omega \in (0, \frac{1-q_1}{q_2})$. We will take ω as $\bar{\omega} = 0.05 \frac{1-q_1}{q_2}$ (again at 5% of its maximum value as before) and assume for simplicity that $q_1 + 20q_2 \geq 1$, which implies that $\bar{\omega} \leq 1$. This gives $L_0 = |A_{22}|$; $L_1 = \frac{|A_{22}|}{\max\left\{\sqrt{\frac{1}{2}}, 0.05 + 0.95q_1\right\}}$, $\gamma_0 = \gamma^*$, $\gamma_1 = \frac{\gamma^*}{\max\left\{\sqrt{\frac{1}{2}}, 0.05 + 0.95q_1\right\}}$ and $\lambda = \max\left\{\sqrt{\frac{1}{2}}, 0.05 + 0.95q_1\right\}$.

Interestingly, the dependence on $q_2 = \max_j \Delta_j$ (the resolution of the quantizer) disappears and we only have a dependence on the “zooming rate” (assuming that $q_1 + 20q_2 \geq 1$) due to the choice of $\bar{\omega}$ in this example. Even more interestingly, when $\frac{q_1}{q_2} = \max_j \Omega_j \leq 0.6917$, we have that $0.05 + 0.95q_1 \leq \sqrt{\frac{1}{2}}$ and thus we obtain the values $L_0 = |A_{22}|$, $L_1 = |A_{22}|/\sqrt{2}$, $\gamma_0 = \gamma^*$, $\gamma_1 = \gamma^*/\sqrt{2}$ and $\lambda = \sqrt{\frac{1}{2}}$, which recover the *quantization-free* parameters and hence, the *quantization-free* MATI and MAD curve, see [9], [10]. Stated otherwise, if the zoom factor $q_1 = \max_j \Omega_j$ is small enough (smaller than 0.6917) it does not influence the stability any longer (at least based on the sufficient stability conditions as presented here). For various values of the zoom factor q_1 we can follow the above procedure and compute the corresponding tradeoff curves between MAD and MATI, which still guarantee UGAS of the NCS. This results in Fig. 2. It can indeed be observed that for $q_1 \leq 0.6917$ we recover the non-quantized curve as in [9], [10]. Note that also the curve corresponding to the TOD zoom protocol as given in Fig. 1 can be found in Fig. 2 for the value of $q_1 = 0.9$.

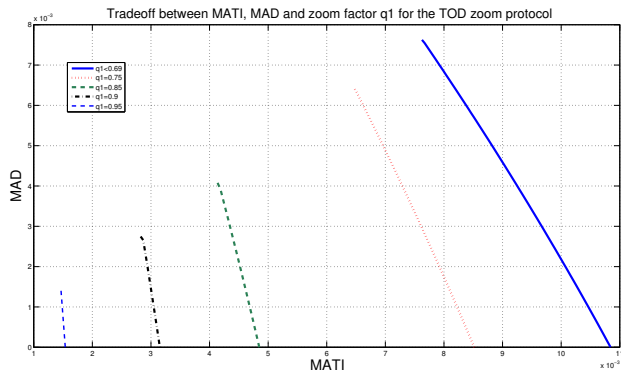


Fig. 2. Tradeoff curves between MATI, MAD and the zoom factor for the TOD zoom protocol.

VI. CONCLUSIONS

For the first time a framework was presented for studying the stability of a NCS, which involves all networked-induced phenomena (communication constraints, varying transmission intervals, varying transmission delays, dropouts and quantization effects). Based on the newly developed model, that unites earlier works by the authors in which only a subset of the phenomena were included, a characterization of stability was provided for NCSs using deterministic bounds on delays (MAD), varying transmission intervals (MATI) and dropouts ($\bar{\delta}$ in Remark 2). The application of the results on a benchmark example showed how tradeoff curves between MATI, MAD and quantization properties can be computed providing designers of NCS with proper tools to support their design choices. As this is the first framework to analyse NCSs encompassing all networked-induced imperfections using deterministic bounds, various improvements are, of course, possible. In particular, topics of future research include tighter approximations of the true MATI and MAD (e.g. by exploiting the possible structure present in the model such as linearity, cf. [5], [6]), improving the way dropouts are currently tackled using prolongation of the MATI, including other classes of quantizers such as so-called box quantizers, treating the large-delay case (delays larger than the transmission interval), exploiting possible stochastic information on dropouts, delays and sampling intervals, etc.

REFERENCES

- [1] D. Carnevale, A.R. Teel, and D. Nešić. A Lyapunov proof of improved maximum allowable transfer interval for networked control systems. *IEEE Trans. Aut. Control*, 52:892–897, 2007.
- [2] A. Chaillet and A. Bicchi. Delay compensation in packet-switching networked controlled systems. In *IEEE Conf. Decision and Control*, pages 3620–3625, 2008.
- [3] M. Cloosterman, N. van de Wouw, W.P.M.H. Heemels, and H. Nijmeijer. Robust stability of networked control systems with time-varying network-induced delays. In *cdc*, pages 4980–4985, San Diego, USA, December 2006.
- [4] M.B.G. Cloosterman, N. van de Wouw, W.P.M.H. Heemels, and H. Nijmeijer. Stability of networked control systems with uncertain time-varying delays. *IEEE Trans. Autom. Control*, 54(7):1575–1580, 2009.

- [5] M.C.F. Donkers, W.P.M.H. Heemels, L. Hetel, N. van de Wouw, and M. Steinbuch. Stability analysis of networked control systems using a switched linear systems approach. *Submitted for journal publication*.
- [6] M.C.F. Donkers, L. Hetel, W.P.M.H. Heemels, N. van de Wouw, and M. Steinbuch. Stability analysis of networked control systems using a switched linear systems approach. In *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science, pages 150–164. Springer Verlag, 2009.
- [7] R. Goebel, R. Sanfelice, and A.R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.
- [8] R. Goebel and A.R. Teel. Solution to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica*, 42:573–587, 2006.
- [9] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, and D. Nešić. Networked control systems with communication constraints: Tradeoffs between sampling intervals, delays and performance. *submitted for journal publication*.
- [10] W.P.M.H. Heemels, A.R. Teel, N. van de Wouw, and D. Nešić. Networked control systems with communication constraints: tradeoffs between sampling intervals and delays. In *Proc. European Control Conference in Budapest, Hungary*, 2009.
- [11] J.P. Hespanha, P. Naghshtabrizi, and Y. Xu. A survey of recent results in networked control systems. *Proc. of the IEEE*, pages 138–162, 2007.
- [12] L. Hetel, M.B.G. Cloosterman, N. van de Wouw, W.P.M.H. Heemels, J. Daafouz, and H. Nijmeijer. Comparison of stability characterisations for networked control systems. In *Proc. Joint 48th IEEE Conference on Decision and Control (CDC) and 28th Chinese Control Conference, Shanghai, China*, 2009.
- [13] T. Kamenewa and D. Nešić. Robustness of quantized control systems with mismatch between coder/decoder initializations. *Automatica*, 2009, to appear.
- [14] D. Liberzon. Hybrid feedback stabilization of systems with quantized signals. *Automatica*, 39:1543–1554, 2003.
- [15] D. Liberzon. On stabilization of linear systems with limited information. *IEEE Trans. Aut. Control*, 48(2):304–307, 2003.
- [16] D. Liberzon and J. Hespanha. Stabilization of nonlinear systems with limited information feedback. *IEEE Trans. Autom. Control*, 50(6):910–915, 2005.
- [17] J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.
- [18] P. Naghshtabrizi and J.P. Hespanha. Designing an observer-based controller for a network control system. In *Proc. of the 44th Conference on Decision and Control, and the European Control Conference 2005*, pages 848–853, Seville, Spain, December 2005.
- [19] P. Naghshtabrizi and J.P. Hespanha. Stability of network control systems with variable sampling and delays. In *Proc. of the Forty-Fourth Annual Allerton Conf. on Communication, Control, and Computing*, 2006.
- [20] D. Nešić and D. Liberzon. A unified framework for design and analysis of networked and quantized control systems. *IEEE Trans. Autom. Control*, 54(4):732–747, 2009.
- [21] D. Nešić and A.R. Teel. Input-output stability properties of networked control systems. *IEEE Trans. Aut. Control*, 49(10):1650–1667, 2004.
- [22] J. Sturm. Using sedumi 1.02, a matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11-12, Special issue on Interior Point Methods:625–653, 1999.
- [23] Y. Tipsuwan and M.-Y. Chow. Control methodologies in networked control systems. *Control Engineering Practice*, 11:1099–1111, 2003.
- [24] G.C. Walsh, O. Belidman, and L.G. Bushnell. Asymptotic behavior of nonlinear networked control systems. *IEEE Trans. Automat. Contr.*, 46:1093–1097, 2001.
- [25] G.C. Walsh, O. Belidman, and L.G. Bushnell. Stability analysis of networked control systems. *IEEE Trans. Control Systems Technology*, 10:438–446, 2002.
- [26] T. C. Yang. Networked control system: a brief survey. *IEE Proc.-Control Theory Appl.*, 153(4):403–412, July 2006.
- [27] W. Zhang, M.S. Branicky, and S.M. Phillips. Stability of networked control systems. *IEEE Control Systems Magazine*, 21(1):84–99, 2001.