Semiglobal Practical Stability of a Class of Parameterized Networked Control Systems

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***Abstract*— This paper studies a class of parameterized networked control systems that are designed via an emulation procedure. In the first step, a controller is designed ignoring network so that semiglobal practical stability is achieved for the closed-loop. In the second step, it is shown that if the same controller is emulated and implemented over a large class of networks, then the networked control system is also semiglobally practically asymptotically stable; in this case, the controller parameter needs to be sufficiently small and communication bandwidth sufficiently high.**

I. INTRODUCTION

Networked control systems (NCSs) is an emerging technology because of its numerous benefits such as cheaper and lighter systems that are easier to maintain and install [1], [2]. In this context, multiple sensor and actuator nodes need to communicate over a network with a limited bandwidth in order to control a given plant. The communication bandwidth limitations typically deteriorate performance if not taken into account in design. On the other hand, the design that considers all these issues is much harder than in classical control systems.

An emulation-like approach for controller design of NCSs was proposed by Walsh and his coworkers in [1], [2] and further extended in [3], [4]. The idea is to first design a controller for a given plant ignoring the network and then implement the same controller over the network with sufficiently fast transmission rate; in other words, the so called maximal allowable transmission interval (MATI) needs to be sufficiently small. A similar approach was pursued for wireless NCSs in [1] and extended in [5]. Some stochastic results were investigated using a similar approach in [6] and references cited therein.

In [7], it is proved that semiglobal practical asymptotic stability of NCSs can be guaranteed via the emulation approach if the system without network is *globally asymptotically stable*. We note that there may exist some systems where global stability of the non-netorked system is not achievable while semiglobal practical stability is; in other words, the domain of attraction can be made arbitrarily large and the ultimate bound on trajectories arbitrarily small if a controller tuning parameter ϵ is sufficiently reduced. Such situation arises, for instance, if we use high gain or low gain controllers, controllers based on high gain observers or continuous sliding mode control ([8], [9], [10]). We note that in some situations such as singularly perturbed systems or pulse width modulated systems where averaging is used to design a controller we also may obtain semiglobal practical stability as the outcome of the design ([11], [12]).

It is the purpose of this paper to extend the emulation approach to NCSs to the situation where non-networked system is *semiglobally practically asymptotically stable*. In particular, we show that if the closed-loop non-networked systems is semiglobally practically asymptotically stable in a controller parameter ϵ, then the NCS with the emulated controller will be semiglobally practically asymptotically stable in the parameter ϵ and MATI. Our problem setting and proofs are based heavily on [7] but we believe that the extension is significant since it extends the emulation approach for NCSs to a large class of new controllers.

The paper is organized as follows. Section II and III give the preliminaries and introduce the hybrid model for NCSs. Section IV presents the main results to answer the posed question. Section VII concludes the paper.

II. PRELIMINARIES

We denote by R and Z the sets of real and integer numbers respectively. Also $R_{\geq 0}$ and $Z_{\geq 0}$ positive real and positive integer numbers. The Euclidean norm is denoted by $|\cdot|$. A function $\alpha : R_{\geq 0} \to R_{\geq 0}$ is said to be of class K if it is continuous, zero at zero and strictly increasing. It is said to be of K_∞ if it is of class K and it is unbounded. A function $\gamma : R_{\geq 0} \to R_{\geq 0}$ is said to be of class L if it is continuous and decreasing to zero. A function $\beta : R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is said to be of class KL if $\beta(\cdot, t)$ is of class K for each $t \geq 0$ and $\beta(s, \cdot)$ is of class L for each $s \geq 0$. A function $\beta : R_{\geq 0} \times R_{\geq 0} \times R_{\geq 0} \to R_{\geq 0}$ is said to be of class KLL if, for each $r \geq 0$, $\beta(\cdot, r, \cdot)$ and $\beta(\cdot, \cdot, r)$ belong to class KL. To simplify notation, we sometimes write $(x, e) := [x^\top e^\top]^\top$ for two vectors x and e.

We recall definitions given in [13] that we will use to develop a hybrid model of a NCS. The reader should refer to [13] for the motivation and more details on these definitions.

Definition 1: A compact hybrid time domain is a set $D \subset R_{\geq 0} \times Z_{\geq 0}$ given by $D = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$, where $J \in Z_{\geq 0}$ and $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. A hybrid time domain $D \subset R_{\geq 0} \times Z_{\geq 0}$ is a set such that, for each $(T, J) \in D$, $D \bigcap ([0, T] \times \{0, \cdots, J\})$ is a compact hybrid time domain.

Definition 2: A hybrid trajectory is a pair (dom ξ, ξ) consisting of hybrid time domain dom ξ and a function ξ defined on dom ξ that is continuously differentiable in t on (dom ξ) $\bigcap (R_{\geq 0} \times \{j\})$ for each $j \in Z \geq 0$.

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Definition 3: For the hybrid system H given by the open state space $O \subseteq R^n$ and the data (F, G, C, D) where $F : O \to R^n$ is continuous, $G : O \to O$ is locally bounded, and C and D are subsets of O, a hybrid trajectory $\xi : \text{dom}\,\xi \to O$ is a solution to H if

1) For all $j \in Z_{\geq 0}$ and for almost all $t \in I_j$ where I_j is such that $I_j \times \{j\} := \text{dom}\,\xi \bigcap (R_{\geq 0} \times \{j\})$, we have $\xi(t, j) \in C$ and $\dot{\xi}(t, j) = F(\xi(t, j))$.
2) For all $(t, j) \in \text{dom}\,\xi$ such that $(t, j+1) \in \text{dom}\,\xi$, we have $\xi(t, j) \in D$ and $\xi(t, j+1) = G(\xi(t, j))$.

The hybrid system models that are considered in [13] and related references are of the form:

$$\begin{aligned} \dot{\xi}(t, j) &= F(\xi(t, j)), \quad \xi(t, j) \in C \\ \xi(t_{j+1}, j+1) &= G(\xi(t_{j+1}, j)), \quad \xi(t_{j+1}, j) \in D \end{aligned} \tag{1}$$

We sometimes omit the time arguments and write:

$$\begin{aligned} \dot{\xi} &= F(\xi), \quad \xi \in C \\ \xi^+ &= G(\xi), \quad \xi \in D \end{aligned} \tag{2}$$

where we denote $\xi(t_{j+1}, j+1)$ as ξ^+ in the last equation. We also note that typically $C \bigcap D \neq \emptyset$ and, in this case, if $\xi(0, 0) \in C \bigcap D$ we have that either a jump or flow is possible, the latter only if flowing keeps the state in C. Hence, such hybrid models may have non-unique solutions. In this paper, we consider parameterized families of hybrid systems of the form[1]:

$$\begin{aligned} \dot{\xi} &= F(\xi, \epsilon), \quad \xi \in C \\ \xi^+ &= G(\xi, \epsilon), \quad \xi \in D\ . \end{aligned} \tag{3}$$

If we think of the hybrid system as a closed-loop system consisting of a plant and controller, then ϵ is typically a controller parameter that the designer can adjust to achieve desirable performance.

III. A Hybrid Model of Networked Control Systems

Model presented in this section extends models given in [3] to the case when controller depends on a parameter ϵ that can be tuned; for more details refer to [3], [4], [7].

Let the sequence $t_j, j \in Z_{\geq 0}$ of monotonically increasing transmission times satisfy $\mu \leq t_{j+1} - t_j \leq \tau_{MATI}$ for all $j \in Z_{\geq 0}$ and some fixed $\mu, \tau_{MATI} > 0$. We assume that μ is arbitrary; in practice μ maybe lower bounded and determined by the fastest sampling frequency achievable. In any case, as μ is positive, this prevents Zeno solutions in the hybrid model. At each t_j, the protocol gives access to the network to one of the nodes $i \in \{1, 2, \cdots, l\}$. Now we introduce the general parameterized nonlinear NCS in the following form:

$$\begin{aligned} \dot{x}_P &= f_P(x_P, \hat{u}) \quad t \in [t_{j-1}, t_j] && (4) \\ y &= g_P(x_P) && (5) \\ \dot{x}_C &= f_C(x_C, \hat{y}, \epsilon) \quad t \in [t_{j-1}, t_j] && (6) \\ u &= g_C(x_C, \hat{y}, \epsilon) && (7) \\ \dot{\hat{y}} &= \hat{f}_P(x_P, x_C, \hat{y}, \hat{u}) \quad t \in [t_{j-1}, t_j] && (8) \\ \dot{\hat{u}} &= \hat{f}_C(x_P, x_C, \hat{y}, \hat{u}) \quad t \in [t_{j-1}, t_j] && (9) \\ \hat{y}(t_j^+) &= y(t_j) + h_y(j, e(t_j)) && (10) \\ \hat{u}(t_j^+) &= u(t_j) + h_u(j, e(t_j)) && (11) \end{aligned}$$

where x_P and x_C are states of the plant and the controller respectively; y is the plant output and u is the controller output; $\hat{y}$ and $\hat{u}$ are the vectors of most recently transmitted plant and controller output values via the network. ϵ is the tuning parameter, which is independent of t, to stabilize (4). Note that in most cases, ϵ appears in either (6) or (7). Here, for generality, we let ϵ appear in both (6) and (7). For the same purpose, we also let $\hat{y}$ appear in (7) to include static output feedback controllers in the model. We assume that f_P, f_C, $\hat{f}_P$ and $\hat{f}_C$ are locally Lipschitz continuous and zero at zero. It is often assumed that $\hat{f}_P = 0$ and $\hat{f}_C = 0$ which means that the network operates in the zero-and-hold (ZOH) fashion. We assume that g_P and g_C are continuously differentiable in all their arguments. Combine the controller and plant states into a vector $x := (x_P, x_C)$, and define the network induced error e as

$$e(t) := \begin{pmatrix} \hat{y}(t) - y(t) \\ \hat{u}(t) - u(t) \end{pmatrix} = \begin{pmatrix} e_y \\ e_u \end{pmatrix} \tag{12}$$

we can rewrite (4) as

$$\begin{aligned} \dot{x} &= f(x, e, \epsilon) \quad \forall t \in [t_{j-1}, t_j] && (13) \\ \dot{e} &= g(x, e, \epsilon) \quad \forall t \in [t_{j-1}, t_j] && (14) \\ e(t_j^+) &= h(j, e(t_j)) && (15) \end{aligned}$$

where $x \in R^{n_x}$ and $e \in R^{n_e}$. f, g and h are obtained from (4):

$$f(x, e, \epsilon) := \begin{pmatrix} f_P\Big(x_P, g_C(x_C, \hat{y}, \epsilon) + e_u\Big) \\ f_C\Big(x_C, g_P(x_P) + e_y, \epsilon\Big) \end{pmatrix} \tag{16}$$

$$g(x, e, \epsilon) := \begin{pmatrix} \hat{f}_P\Big(x_P, x_C, g_P(x_P) + e_y, g_C(x_C, \hat{y}, \epsilon) + e_u\Big) \\ \hat{f}_C\Big(x_P, x_C, g_P(x_P) + e_y, g_C(x_C, \hat{y}, \epsilon) + e_u\Big) \\ -\frac{\partial g_P}{\partial x_P}(x_P) f_P\Big(x_P, g_C(x_C, \epsilon) + e_u\Big) \\ -\frac{\partial g_C}{\partial x_C}(x_C, \hat{y}, \epsilon) f_C\Big(x_C, g_P(x_P) + e_y, \epsilon\Big) \dots \\ \dots - \frac{\partial g_C}{\partial \hat{y}}(x_C, \hat{y}, \epsilon) \hat{f}_P(x_P, x_C, \hat{y}, \hat{u}) \end{pmatrix} \tag{17}$$

$$h(j, e(t_j)) := \begin{pmatrix} h_y(j, e(t_j)) \\ h_u(j, e(t_j)) \end{pmatrix} \tag{18}$$

We refer to (15) as a protocol which determines the algorithm by which access to the network is assigned to different nodes

[1]Sometimes it maybe useful to consider the situation when the flow set C and jump set D also depend on the parameter; however, in our applications this does not happen.

in the system. Please note that the model (4) considers the dynamic output feedback. When it comes to static output feedback, controller state x_C is eliminated and the consequent model can be deemed as a special case of (4). We assume that f and g are Lipschitz continuous in their first two arguments.

IV. Main Results – Uniform Semiglobal Practical Asymptotic Stability

We now embed the model (13)-(15) into a hybrid model in similar manner as in [7]

$$\left.\begin{array}{rcl} \dot{x} & = & f(x,e,\epsilon) \\ \dot{e} & = & g(x,e,\epsilon) \\ \dot{\tau} & = & 1 \\ \dot{k} & = & 0 \end{array}\right\} \tau \in [0, \tau_{MATI}]$$
$$\left.\begin{array}{rcl} x^+ & = & x \\ e^+ & = & h(k,e) \\ \tau^+ & = & 0 \\ k^+ & = & k+1 \end{array}\right\} \tau \in [\mu, \tau_{MATI}] \tag{19}$$

where μ can be arbitrarily small and $\tau_{MATI} > \mu$. The main difference with [7] is that f and g now depend on the controller parameter ϵ that can be adjusted to achieve appropriate stability properties of the NCS; note also that the above model is a special class of (3).

We also need a definition of uniformly globally asymptotically stable (UGAS) protocol, see [3], [4]. These references introduce an auxiliary discrete-time system induced by the $h(\cdot,\cdot)$ that defines the network protocol in (19):

$$e^+ = h(k,e)\ , \tag{20}$$

and use it in stating various results on emulation of NCS. We need the following:

Definition 4: We say that the protocol (20) is UGAS if there exist a function W: $Z_{\geq 0} \times R^{n_e} \to R_{\geq 0}$ and class-K_∞ functions $\underline{\alpha}_W$, $\bar{\alpha}_W$ such that, any $k \in Z_{\geq 0}$ and $e \in R^{n_e}$,

$$\underline{\alpha}_W(|e|) \leq W(k,e) \leq \bar{\alpha}_W(|e|) \tag{21}$$

There exist a positive real number σ and a class-K_∞ function α_W, such that for any $k \in Z_{\geq 0}$ and $e \in R^{n_e}$:

$$W(k+1, h(k,e)) \leq e^{-\sigma} W(k,e) \tag{22}$$

and for all $k \in Z_{>0}$ and almost all $e \in R^{n_e}$:

$$\left|\frac{\partial W(k,e)}{\partial e}\right| \leq \alpha_W(|e|). \tag{23}$$

Note that

$$\dot{x} = f(x,0,\epsilon) \tag{24}$$

represents the closed-loop system (4)-(7) without the network; in other words $u = \hat{u}$ and $y = \hat{y}$.

Our goal is to show that emulation method works in the case when the closed loop system (24) satisfies an appropriate semiglobal practical asymptotic stability property. Note that in [3], [7] only non-parameterized systems of the form (2) were considered and global stability of (24) assumed. Here, we consider (3) and assume a weaker notion of semiglobal practical stability for (24). Lyapunov conditions for uniform semiglobal practical asymptotic stability (USPAS) w. r. t. ϵ of the system (24) are given below.

Definition 5: (24) is said to satisfy the USPAS Lyapunov condition if there exist class K_∞ functions $\underline{\alpha}_V(\cdot)$, $\bar{\alpha}_V(\cdot)$, $\alpha_V(\cdot)$, $\check{\alpha}_V(\cdot)$ such that given any $\Delta > 0$, $\nu > 0$, there exist $\epsilon^* = \epsilon^*(\nu, \Delta)$ and a continuously differentiable function V : $R^n \times (0, \epsilon^*) \to R$ such that for all $\epsilon \in (0, \epsilon^*)$ and all $|x| \leq \Delta$,

$$\underline{\alpha}_V(|x|) \leq V(x,\epsilon) \leq \bar{\alpha}_V(|x|) \tag{25}$$

$$\frac{\partial V}{\partial x} f(x,0,\epsilon) \leq -\alpha_V(|x|) + \nu \tag{26}$$

$$\left|\frac{\partial V}{\partial x}\right| \leq \check{\alpha}_V(|x|) \tag{27}$$

The property USPAS tells us that the domain of attraction can be arbitrarily large and the ultimate bound on solutions can be arbitrarily reduced by adjusting ϵ. Such situation is common in many robust control techniques.

Now, we present the main results of the paper.

Theorem 1: Consider the hybrid NCS (19) and suppose that the following conditions hold:

1) The protocol (20) is UGAS.

2) The system (24) satisfies USPAS Lyapunov condition.

Then, there exist K_∞ functions $\underline{\alpha}_U(\cdot)$, $\bar{\alpha}_U(\cdot)$, $\alpha_U(\cdot)$ such that given any $\bar{\Delta} > 0$, $\bar{\nu} > 0$, there exists $\epsilon^* > 0$, such that for any $\epsilon \in (0, \epsilon^*)$, there exists $\tau_{MATI} = \tau_{MATI}(\epsilon)$ and a Lyapunov function $U(\xi, \epsilon)$ such for all ξ with $\tau \in [0, \tau_{MATI}]$ we have

$$\underline{\alpha}_U(|(x,e)|) \leq U(\xi,\epsilon) \leq \bar{\alpha}_U(|(x,e)|)\ , \tag{28}$$

for all ξ in the set $\Lambda = \{(x,e,\tau,k) : \tau \in [\mu, \tau_{MATI}], k \in Z_{\geq 0}\}$, we have

$$U(\xi^+, \epsilon) \leq U(\xi, \epsilon)\ , \tag{29}$$

and, finally, for almost all ξ in the set $\bar{\Lambda} = \{(x,e,\tau,k) : |(x,e)| \leq \bar{\Delta}, \tau \in [0, \tau_{MATI}], k \in Z_{\geq 0}\}$, we have

$$\frac{\partial U(\xi,\epsilon)}{\partial \xi} F(\xi,\epsilon) \leq -\alpha_U(U(\xi,\epsilon)) + \bar{\nu}\ , \tag{30}$$

where $\xi := (x,e,\tau,k)$ and $F(\xi,\epsilon) := (f(x,e,\epsilon), g(x,e,\epsilon), 1, 0)$.

Note that the existence of Lyapunov function satisfying (28), (29), (30) implies that there exists $\beta \in KLL$ such that for any positive $\Delta > \nu$ there exists ϵ^* and for any $\epsilon \in (0, \epsilon^*)$ there exists τ_{MATI} such that for all $\tau \in (0, \tau_{MATI})$ and all $|(x,e)| \leq \Delta$ we have

$$|(x(t,j), e(t,j))| \leq \beta(|(x(0,0), e(0,0))|, t, j) + \nu$$

for all (t,j) in the solution's domain. In other words, NCS is uniformly semiglobally practically asymptotically stable.

Note that our first condition of Theorem 1 can be satisfied by selecting an appropriate network protocol. A range of UGAS protocols were proposed and discussed in the literature, see for instance [3], [4]. The second condition of the theorem can be satisfied by designing a controller for the plant without network so that Lyapunov USPAS property holds. There are numerous techniques in the literature where such controllers are presented for classes of plants, such as high gain control, low gain control or continuous sliding mode control.

Remark 1: The function W in Theorem 1 which satisfies (22) can be guaranteed by the UGAS protocol $h(k,e)$. That is, there exists a continuous, positive definite function ϱ such that for W which satisfies (21), and for any $k \in Z_{\geq 0}$ and $e \in R^{n_e}$:

$$W(k+1, h(k,e)) \leq W(k,e) - \varrho(e) \tag{31}$$

In fact, from (21) and (31), [12] and [14] lead to the existence of a continuously differentiable function $\rho \in K_\infty$ and $\sigma > 0$, such that $\hat{W}(k,e) = \rho(W(k,e))$ satisfies

$$\hat{W}(k+1, h(k,e)) \leq e^{-\sigma}\hat{W}(k,e) \tag{32}$$

V. Conclusion

In this paper, a class of parameterized networked control systems are studied. Assuming that the controller was designed ignoring the network and that it achieves semiglobal practical stability of the closed-loop in a controller parameter ϵ, it was shown that if the same controller is emulated and implemented over a network with a uniformly globally stable protocol, then the networked control system would preserve semiglobal practical stability property; in this case, both the controller parameter ϵ and MATI need to be reduced sufficiently to achieve a desired domain of attraction and the ultimate bound on the trajectories.

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VI. Appendix

A. Proof of Theorem 1

Suppose all conditions of the theorem hold. Let arbitrary $\bar{\Delta} > 0$, $\bar{\nu} > 0$ be given. Let ϵ^* be generated using the second condition of the theorem with $\Delta = \bar{\Delta}$ and $\nu = \frac{\bar{\nu}}{2}$. Let σ, $\underline{\alpha}_V(\cdot)$, $\bar{\alpha}_V(\cdot)$, $\alpha_V(\cdot)$, $\check{\alpha}_V(\cdot)$, $\underline{\alpha}_W$, $\bar{\alpha}_W$, $\alpha_w(\cdot)$ come from conditions of the theorem. Define the following function[2]:

$$\begin{aligned}\varphi(x,e,\epsilon) :=& \check{\alpha}_V(|x|)|f(x,e,\epsilon) - f(x,0,\epsilon)| \\ &+ \alpha_W(|e|)|g(x,e,\epsilon)| \ ,\end{aligned}$$

and note that for any $\epsilon \in (0, \epsilon^*)$ this is a continuous function with $\varphi(x,0,\epsilon) = 0$. Consider any such fixed ϵ. Then, using arguments similar to [7, Theorem 2], we conclude that there exists $\tau_{MATI} = \tau_{MATI}(\epsilon)$ such that for any $|(x,e)| \leq \bar{\Delta}$ we have that

$$\begin{aligned}-0.5\alpha_V(|x|) - 0.5\underline{\alpha}_W(|e|) + \frac{\bar{\nu}}{2} \geq& \\ -\alpha_V(|x|) - \frac{\sigma e^{-\sigma}}{\tau_{MATI}}\underline{\alpha}_W(|e|) + \varphi(x,e,\epsilon) \ .&\end{aligned} \tag{33}$$

Define

$$U(\xi,\epsilon) = V(x,\epsilon) + e^{-\sigma\tau/\tau_{MATI}}W(k,e) \tag{34}$$

From (21), (25), we have

$$\underline{\alpha}_V(|x|) + e^{-\sigma}\underline{\alpha}_W(|e|) \leq U(\xi,\epsilon) \leq \bar{\alpha}_V(|x|) + \bar{\alpha}_W(|e|) \ . \tag{35}$$

It follows that there exist K_∞ functions $\underline{\alpha}_U$ and $\bar{\alpha}_U$ such that (28) holds. On the other hand, using (19), (22) and (34), we get

$$U(\xi^+,\epsilon) \leq V(x,\epsilon) + e^{-\sigma}W(k,e) \leq U(\xi,\epsilon) \tag{36}$$

for all $\xi \in \Lambda$, which shows that (29) holds. Below, by abuse of notation, we consider the quantity $\frac{\partial U}{\partial \xi}F(\xi,\epsilon)$ even though U is not differentiable with respect to k. This is justified since the component in $F(\xi,\epsilon)$ corresponding to k is zero. Consider $\xi \in \bar{\Lambda}$ and we can write using our choice of ϵ and τ_{MATI}, the definition of φ, (21), (26) and (33) that the

[2]Note that if the function φ can be bounded by another function $\tilde{\varphi}$ that is independent of ϵ for all $\epsilon \in (0,\epsilon^*)$, then $\tau_M ATI$ in the theorem will not depend on ϵ.

following holds:

$$\begin{aligned}\frac{\partial U}{\partial \xi}F(\xi,\epsilon) =& \frac{\partial V}{\partial x}f(x,e,\epsilon) + e^{-\sigma\tau/\tau_{MATI}}\frac{\partial W}{\partial e}g(x,e,\epsilon) \\ &- \frac{\sigma}{\tau_{MATI}}e^{-\sigma\tau/\tau_{MATI}}W(k,e) \\ \leq& \frac{\partial V}{\partial x}f(x,0,\epsilon) - \frac{\sigma}{\tau_{MATI}}e^{-\sigma\tau/\tau_{MATI}}W(k,e) \\ &+ \varphi(x,e,\epsilon) \\ \leq& -\alpha_V(|x|) + \frac{\bar{\nu}}{2} - \frac{\sigma e^{-\sigma}}{\tau_{MATI}}\underline{\alpha}_W(|e|) \\ &+ \varphi(x,e,\epsilon) \\ \leq& -0.5\alpha_V(|x|) - 0.5\underline{\alpha}_W(|e|) + \bar{\nu}\ ,\end{aligned}$$

which shows that there exists $\alpha_U(\cdot)$ such that (30) holds. This completes the proof.