The Asymptotic Distribution of the Determinant of a Random Correlation Matrix

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Abstract

Our main result gives the asymptotic distribution of the determinant of a random correlation matrix sampled in a particular way from the space of $d \times d$ correlation matrices. Several spin-off results are proven along the way, and an interesting connection with the law of the determinant of general random matrices, proven by Vu and Nguyen (2012) is investigated. Since different methods for generating random correlation matrices are proposed in the literature, one application of our result is that in can be employed to differentiate between those.

Keywords: random correlation matrix, determinant, beta distribution, random matrix, regular vines

1 Introduction

Random correlation matrices are studied for both theoretical interestingness and importance for applications. Holmes (1991) is interested in their interpretation as covariance matrices of purely random signals, Qiu (2004) employs them in the generation of random clusters for studying clustering methods, whereas Johnson and Welcht (1980) use them for studying subset selection in multiple regression, etc. The determinant of a matrix is one of the most basic and important matrix functions, and this makes studying the distribution of the determinant of a random correlation matrix important. Holmes (1991), Joe (2006) and Lewandowski, Kurowicka and Joe (2009), amongst others, have studied extensively the problem of generating random correlation matrices. However, their approaches are fundamentally different, i.e. Holmes (1991) uses the eigenvalues of a random correlation matrix, whereas Joe (2006) employs the properties of partial correlations. Consequently, this leads to distinct joint distributions of the marginal correlations. In this respect, studying the determinant of random correlation matrices can be used to differentiate between distinct methods for generating random correlation matrices. Lewandowski, Kurowicka and Joe (2009) present two methods for generating random correlation matrices, building on the approach described in Joe (2006). Their approach allows generating matrices "from a uniform distribution" defined in a particular sense discussed further. Joe (2006) has shown that since a $d-$dimensional positive definite correlation matrix $R = (\rho_{ij})_{i,j=1,\ldots,d}$ can be parametrised in terms of the correlations $\rho_{i,i+1}$ and partial correlations $\rho_{i,j;i+1,\ldots,j-1}$ for $(j-i) \geq 2$, and these parameters can independently take values in the interval $(-1, 1)$, one can generate a random correlation matrix by choosing independent distributions F_{ij}, $1 \leq i < j \leq d$ for these parameters. Appropriate choices for F_{ij} lead to a joint density for $\rho_{ij:1\leq i<j\leq d}$ that is proportional to $det(R)^{\eta-1}$, where $\eta > 0$. Lewandowski, Kurowicka and Joe (2009) have shown that in this case the joint density is invariant to the order of

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indexing of variables for partial correlations, and each ρ_{ij} marginally has a *Beta* $\left(\eta - 1 + \frac{d}{2}, \eta - 1 + \frac{d}{2}\right)$ distribution on $(-1, 1)$. The uniform joint density for $\rho_{ij:1 \leq i < j \leq d}$ is obtained when $\eta = 1$. The proof of this result is formulated using the notion of the partial correlation regular vines. Vines were introduced by Cooke (1997) and by Bedford and Cooke (2002). A vine on d variables is a nested set of trees. The edges of the j^{th} tree are the nodes of the $(j+1)^{th}$ tree. A *regular* vine on d variables is a vine in which two edges in tree j are joined by an edge in tree $j+1$ only if these edges share a common node. More formally

Definition 1. *$\mathcal{V}$ is called a regular vine on d elements if*

1. *$\mathcal{V} = (T_1, \ldots, T_{d-1})$;*
2. *T_1 is a tree with nodes $N_1 = \{1, \ldots, d\}$, and edges E_1 and for $i = 2, \ldots, d-1$, T_i is a tree with nodes $N_i = E_{i-1}$;*
3. *For $i = 2, \ldots, d-1$, $a, b \in E_i$, $\#a \triangle b = 2$, where $\triangle$ denotes the symmetric difference. In other words if a and b are nodes of T_i connected by an edge in T_i, where $a = \{a_1, a_2\}$, $b = \{b_1, b_2\}$, then exactly one of the a_i equals one of the b_i.*

For each edge of the vine we distinguish a *constraint*, a *conditioning*, and a *conditioned* set. Variables reachable from an edge, via the membership relation, form its constraint set. If two edges are joined by an edge in the next tree the intersection and symmetric difference of their constraint sets give the conditioning and conditioned sets, respectively.

Each regular vine edge may be associated with a partial correlation. A *complete partial correlation vine specification* is a regular vine with a partial correlation specified for each edge. A partial correlation vine specification does not uniquely specify a joint distribution; moreover a given set of marginal distributions may not be consistent with a given set of partial correlations. Nonetheless, there is a joint distribution satisfying the specified information, as shown by Bedford and Cooke (2002).
The property of vines that plays a crucial role in the starting point of this paper is given in the next theorem by Kurowicka and Cooke (2006).

Theorem. *Let D_d be the determinant of a $d \times d$ correlation matrix, R, of variables $X_1, \cdots, X_d$, with $D_d > 0$. For any partial correlation vine*

$$D_d = \prod_{e \in E(\mathcal{V})} \left(1 - \rho^2_{e_1, e_2; \mathcal{C}_e}\right), \tag{1}$$

where $E(\mathcal{V})$ is the set of edges of the vine $\mathcal{V}$, $\mathcal{C}_e$ denotes the conditioning set associated with edge e, and $\{e_1, e_2\}$ is the conditioned set of e.

Vines provide a way of factorising the determinant of the correlation matrix in terms of partial correlations. As mentioned earlier, the uniform density over the set of positive definite correlation matrices in the sense defined above is invariant to the order of indexing of variables for the partial correlations, and each ρ_{ij} marginally follows a *Beta* $\left(\frac{d}{2}, \frac{d}{2}\right)$ distribution on $(-1, 1)$.

Remark 1. *It is worth mentioning that if the ρ_{ij}'s would be independent, then they would individually approach 0, as the dimension approaches infinity. As a consequence the determinant of the correlation matrix would then approach 1. The main theorem from Section 2 reveals a completely different behaviour of the determinant of a random correlation matrix.*

It is shown that when each partial correlation $\rho_{i,j;K}$ from factorisation (1) has a *Beta* $\left(\frac{d-k}{2}, \frac{d-k}{2}\right)$ distribution on $(-1, 1)$, where k is the cardinality of the conditioning set K, it follows that each

$(\rho_{i,j;K})^2$ has a $Beta\left(\frac{1}{2}, \frac{d-k}{2}\right)$ distribution on $(0,1)$ and each $\left(1-\rho^2_{i,j;K}\right)$ is distributed according to a $Beta\left(\frac{d-k}{2}, \frac{1}{2}\right)$ distribution. Rearranging the terms from factorisation (1), we find that D_d has the same distribution as a product of independent $Beta\left(\frac{d-k}{2}, \frac{1}{2}\right)$ variables on $(0,1)$

$$D_d \overset{d}{=} \prod_{j=0}^{d-2}\prod_{i=j}^{d-2} B_{ij}, \tag{2}$$

where $B_{ij} \sim Beta\left(1+\frac{i}{2}, \frac{1}{2}\right)$, for $i = j, \ldots, d-2$ and $j = 0, \ldots, d-2$, and $\overset{d}{=}$ denotes equality in distribution.

Throughout the paper, we will denote by B_i Beta distributed random variables on $(0,1)$. Although determined by i(or j), the parameter of the Beta variables can differ. This abuse of notation will be compensated by an increased clarity of the exposition.

2 Main Results

Theorem 1. *If the marginal distribution of each correlation is Beta* $\left(\frac{d}{2}, \frac{d}{2}\right)$ *on* $(-1,1)$*, the joint distribution of the marginal correlations from a* $d \times d$ *correlation matrix is uniform. Take* R *to be such a correlation matrix and let* D_d *be its determinant. Then* D_d *has the same distribution as a product of* $(d-1)$ *independent Beta distributed random variables*

$$D_d \overset{d}{=} \prod_{j=1}^{d-1} B_j, \tag{3}$$

where $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right), j = 1, \ldots, d-1.$

Proof. Let D_d be represented as in equation (2). If $A_i \sim Beta\left(1+\frac{i}{2}, \frac{1}{2}\right)$ are independent, for $i = 1, \ldots, n$ with $a_{i+1} = a_i + b_i$, then Nadarajah and Gupta (2004) prove that for such variables,

$$\prod_i A_i \sim Beta\left(a_i, \sum_i b_i\right).$$

Using the above property we reduce the second product in (2) to one variable with known distribution

$$\prod_{i=j}^{d-2} B_{ij} \sim Beta\left(1+\frac{j}{2}, \sum_{i=j}^{d-2}\frac{1}{2}\right) = Beta\left(1+\frac{j}{2}, \frac{d-j-1}{2}\right).$$

Then

$$D_d \overset{d}{=} \prod_{j=1}^{d-1} A_j = \prod_{j=1}^{d-1} B_j,$$

where $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right)$.

□

Remark 2. *Observe that the sum of the Beta distribution parameters in Theorem 1 is constant and equal to* $S = \frac{d+1}{2}$*; then the expression of the determinant in equation* (3) *can be written as the product of* $B_j \sim Beta\left(S - \frac{j}{2}, \frac{j}{2}\right)$ *distributed variables,* $j = 1, \ldots, d-1$.

Rearranging the terms and looking separately at odd and even values of d we obtain

$$D_d \stackrel{d}{=} \begin{cases} B_d \cdot \prod_{j=1}^{\lfloor \frac{d-2}{2} \rfloor} \left(B_j \cdot \widetilde{B_j} \right) & , d = 2k \\ B_d \cdot B_k \cdot \prod_{j=1}^{\lfloor \frac{d-2}{2} \rfloor} \left(B_j \cdot \widetilde{B_j} \right) & , d = 2k+1, \end{cases}$$

where $B_d \sim Beta\left(\frac{d}{2}, \frac{1}{2}\right), B_k \sim Beta\left(\frac{k+1}{2}, \frac{k+1}{2}\right), B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right)$ and $\widetilde{B_j} \sim Beta\left(\frac{d-j}{2}, \frac{j+1}{2}\right)$.

Parameter η determines the joint distribution of the marginal correlation. When $\eta = 1$, this joint distribution departs from uniformity. Moreover when $\eta > 1$, the individual correlations are shrunk probabilistically towards 0 and when $0 < \eta < 1$, the individual correlations are expanded probabilistically from 0. We are interested to study the behaviour of the random determinant for different values of η, i.e. for different correlation structures.

Corollary 1. *Consider the $d \times d$ correlation matrices such that each ρ_{ij} marginally has a Beta distribution on $(-1, 1)$, with both parameters equal to $\left(\eta - 1 + \frac{d}{2}\right)$, for $\eta > 1$. Then the determinant D_d of such correlation matrix has the same distribution as the product of $(d-1)$ independent Beta distributed random variables B_j, i.e. $D_d \stackrel{d}{=} \prod_{j=1}^{d-1} B_j$, where $B_j \sim Beta\left(\eta + \frac{j-1}{2}, \frac{d-j}{2}\right)$.*

The following theorem provides the convergence in probability of $D_d^{1/d}$.

Theorem 2 (Main Theorem)**.** *For $d > 0$, consider the $d \times d$ correlation matrices with a uniform joint distribution of the marginal correlations. Let R_d be such a random correlation matrix and let D_d be its determinant. Then, as $d \to \infty$,*

$$D_d^{1/d} \xrightarrow{p} \frac{1}{e}.$$

Proof. Consider the expression for D_d given in equation (3), where $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right)$. The variables B_j are independent, so the expectation of $D_d^{1/d}$ can be calculated as a product of expectations of *Beta* variables raised to the power $\frac{1}{d}$.

Let $\alpha_j = \frac{j+1}{2}$, $\beta_j = \frac{d-j}{2}$ and $S = \alpha_j + \beta_j = \frac{d+1}{2}$. Let $B(\alpha_j, \beta_j)$ denote the Beta function, and $\Gamma(\cdot)$ denote the Gamma function. Then

$$\begin{aligned} E\left(D_d^{1/d}\right) &= E\left(\prod_{j=1}^{d-1} B_j^{1/d}\right) \\ &= \prod_{j=1}^{d-1} E\left(B_j^{1/d}\right) \\ &= \prod_{j=1}^{d-1} \frac{B(\alpha_j + \frac{1}{d}, \beta_j)}{B(\alpha_j, \beta_j)} = \prod_{j=1}^{d-1} \frac{\Gamma(\alpha_j + \frac{1}{d}) \cdot \Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j) \cdot \Gamma(\alpha_j + \beta_j + \frac{1}{d})} \\ &= \left[\frac{\Gamma(S)}{\Gamma(S + \frac{1}{d})}\right]^{d-1} \cdot \prod_{j=1}^{d-1} \frac{\Gamma(\frac{j+1}{2} + \frac{1}{d})}{\Gamma(\frac{j+1}{2})} \\ &\equiv A \cdot B. \end{aligned}$$

Using results originally formulated in Erdélyi and Tricomi (1951) and refined in Natalini and Laforgia (2012), we can write

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left[1+O\left(z^{-1}\right)\right], \tag{4}$$

for $a, b \geq 0$ and $z \to \infty$. Then,

$$A = \left(S^{-\frac{1}{d}}\right)^{d-1}\left[1+O\left(d^{-1}\right)\right].$$

The term B is a product of ratios of gamma functions and, for small positive integers j, a different approximation is required. For fixed x and ε,

$$\frac{\Gamma(x+\varepsilon)}{\Gamma(x)} = 1+\psi(x)\varepsilon + O(\ln^2 x)\cdot\varepsilon^2,$$

by Taylor expansion. Furthermore,

$$\psi(x) = \ln(x) + O(x^{-1}),$$

from the Euler-Maclaurin formula. This gives

$$\frac{\Gamma(x+\frac{1}{d})}{\Gamma(x)} = 1+\frac{1}{d}\ln x + \frac{1}{d}O(x^{-1}) + \frac{1}{d^2}O(\ln^2 x),$$

for $\varepsilon = \frac{1}{d}$. Finally, using that $e^x = x+1+O(x^2)$, we can write

$$x+1 = e^x + O(x^2),$$

which re-writes as

$$\frac{1}{d}\ln x + 1 = x^{\frac{1}{d}} + O\left(\frac{\ln^2 x}{d^2}\right).$$

We conclude that

$$\frac{\Gamma(x+\frac{1}{d})}{\Gamma(x)} = x^{\frac{1}{d}} + \frac{1}{d}O(x^{-1}) + \frac{1}{d^2}O(\ln^2 x).$$

Note that the first error term is the dominating error term for large d. Moreover, for finite x and large d,

$$\frac{\Gamma(x+\frac{1}{d})}{\Gamma(x)} = x^{\frac{1}{d}}[1+O(d^{-1})]. \tag{5}$$

Concluding,

$$B = \prod_{j=1}^{d-1}\left(\frac{j+1}{2}\right)^{\frac{1}{d}}\cdot\left[1+O\left(d^{-1}\right)\right]^{d-1},$$

and

$$\begin{aligned} E\left(D_d^{1/d}\right) &= \left(S^{-\frac{1}{d}}\right)^{d-1}\cdot\prod_{j=1}^{d-1}\left(\frac{j+1}{2}\right)^{\frac{1}{d}}\cdot\left[1+O\left(d^{-1}\right)\right]^{d} \\ &= \frac{2^{\frac{d-1}{d}}}{(d+1)^{\frac{d-1}{d}}}\cdot\frac{(d!)^{\frac{1}{d}}}{2^{\frac{d-1}{d}}}\cdot\left[1+O\left(d^{-1}\right)\right]^{d}. \end{aligned}$$

Furthermore, by using Stirling's approximation,

$$d! = \sqrt{2\pi d}\left(\frac{d}{e}\right)^d \left[1 + O\left(d^{-1}\right)\right], \tag{6}$$

then

$$\begin{aligned} E\left(D_d^{1/d}\right) &= \frac{d^{\frac{1}{2d}+1}}{(d+1)^{1-\frac{1}{d}}} \cdot \left(\sqrt{2\pi}\right)^{\frac{1}{d}} \cdot \frac{1}{e} \cdot \left[1 + O\left(d^{-1}\right)\right] \\ &= \frac{d}{d+1} \cdot d^{\frac{1}{2d}} \cdot (d+1)^{\frac{1}{d}} \cdot \left(\sqrt{2\pi}\right)^{\frac{1}{d}} \cdot \frac{1}{e} \cdot \left[1 + O\left(d^{-1}\right)\right]. \end{aligned} \tag{7}$$

It follows immediately that

$$\lim_{d\to\infty} E\left(D_d^{1/d}\right) = \frac{1}{e}. \tag{8}$$

Similarly,

$$\lim_{d\to\infty} E\left[\left(D_d^{1/d}\right)^2\right] = \lim_{d\to\infty} E\left(D_d^{2/d}\right) = \frac{1}{e^2}. \tag{9}$$

By (8) and (9),

$$\lim_{d\to\infty} var\left(D_d^{1/d}\right) = \lim_{d\to\infty}\left[E\left(D_d^{1/d}\right)\right]^2 - \lim_{d\to\infty} E\left[\left(D_d^{1/d}\right)^2\right] = 0. \tag{10}$$

By (8) and (10), and by using Chebyshev's inequality, the proof is complete.

□

Since convergence in probability implies convergence in distribution, then $D_d^{1/d} \xrightarrow{d} \frac{1}{e}$.

The result of Theorem 2 also holds for the space of correlation matrices where each element marginally distributed according to a $Beta\left(\eta - 1 + \frac{d}{2}, \eta - 1 + \frac{d}{2}\right)$ distribution on $(-1, 1)$. This is stated in the corollary below and the proof can be found in the Appendix.

Corollary 2. *Consider the $d \times d$ correlation matrices such that each ρ_{ij} marginally follows a Beta distribution on $(-1, 1)$, with both parameters equal to $\left(\eta - 1 + \frac{d}{2}\right)$, for $\eta > 0$. Let R_d be such a matrix and D_d its determinant. Then $D_d^{1/d} \to \frac{1}{e}$, as $d \to \infty$, in probability.*

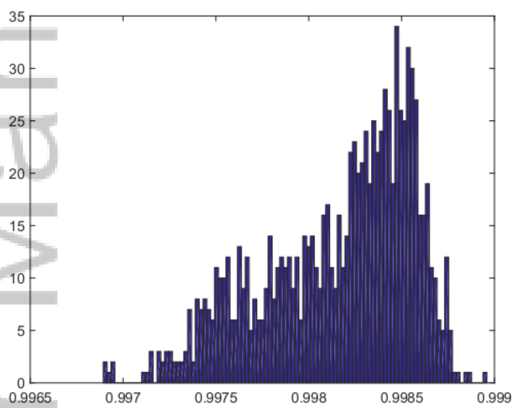
The above corollary shows that $D_d^{1/d}$ converges in probability to $\frac{1}{e}$ for a general value of $\eta > 0$, when the correlation matrices are generated with the procedure described in this paper (and in Lewandowski, Kurowicka and Joe (2009)). However, the speed of convergence seems to be affected by different choices of η. Moreover, several simulation exercises have been performed using two approaches for generating correlation matrices described in Lewandowski, Kurowicka and Joe (2009), for different values of d and η. These were contrasted with the results obtained by using the approach described in Holmes (1991). Table 1 summarizes the results of the simulations performed for dimensions 300, 400 and 500, for three approaches called in this table: *Holmes*, the *Onion* and the *Vine* method[1]. The simulations indicate a different asymptotic behaviour for the determinant of random correlation matrices when simulated with the *Holmes* and *Onion* methods. While using the *Onion* method results in only slightly larger values of the mean of the scaled determinant, which could indicate a slower convergence to the same value, the results obtained using the *Holmes* approach generate hugely different (larger) values of the mean of the scaled determinant. The standard deviations are comparable across methods, with the

[1]The simulations were performed with the R Package "clusterGeneration", Version 1.3.4. 1000 correlation matrices were simulated with each method and for each value of d.

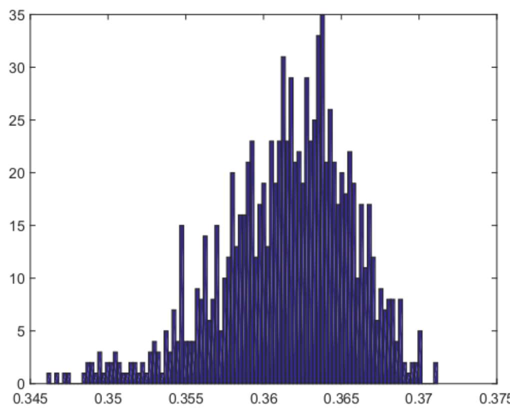
Table 1: Estimates of the mean and standard deviation of the empirical distribution of the scaled determinant of correlation matrices generated with different methods (Holmes, Onion, Vine), for different dimensions (d).

	Holmes		Onion ($\eta = 5$)		Vine (η=0.2)		Vine (η =0.5)	
d	mean	std	mean	std	mean	std	mean	std
300	0.9970	0.0007	0.4081	0.0024	0.3575	0.0030	0.3657	0.0036
400	0.9977	0.0005	0.3999	0.0024	0.3602	0.0024	0.3662	0.0036
500	0.9982	0.0004	0.3948	0.0019	0.3615	0.0042	0.3663	0.0029

Holmes method generating standard deviations which are one order of magnitude smaller that the rest. As mentioned before, a possible application of the results proved in this paper is to use the asymptotic distribution of the determinant of random correlation matrices to distinguish between the different methods of generating correlation matrices. Figure 1 (a) shows the empirical distribution of the scaled determinant, when matrices are generated with the approach from Holmes (1991). Both, the shape of the distribution, and the values of the scaled determinant are different than those obtained when using the *Vine* method for $eta = 0.2$, showed in Figure 1 (b).



(a) The *Holmes* method



(b) The *Vine* method with $\eta = 0.2$

Figure 1: The empirical distribution of the scaled determinant of a 500 by 500 random correlation matrix simulated with (a) the *Holmes* method, and (b) the *Vine* method for $\eta = 0.2$.

Remark 3. *Each correlation in the correlation matrix marginally has a Beta* $\left(\frac{d}{2}, \frac{d}{2}\right)$ *distribution on* $(-1, 1)$*. When* $d \to \infty$*, each Beta* $\left(\frac{d}{2}, \frac{d}{2}\right)$ *distribution will approach* 0*. In other words, if the entries in the correlation matrix would be independent, the determinant would approach* 1 *as the dimension increases. The above theorem reveals an opposite behaviour, namely the convergence of the determinant towards its other bound, namely* 0*. This behaviour is induced by the dependence amongst the entries of the correlation matrix.*

3 Spin-off Results

The asymptotic behaviour of the d^{th} root of the first two moments of D_d is formulated in the following proposition. The proof of this proposition can be found in the Appendix.

Proposition 1. *For the $d \times d$ correlation matrices with the marginal distribution of each correlation following a Beta $\left(\frac{d}{2}, \frac{d}{2}\right)$ distribution on $(-1,1)$, the determinant D_d of such correlation matrix follows the expression from equation (3). Then*

1. $\lim_{d \longrightarrow \infty} \left[E\left(D_d\right)\right]^{\frac{1}{d}} = \frac{1}{e} = \lim_{d \longrightarrow \infty} E\left(D_d^{1/d}\right)$

2. $\lim_{d \longrightarrow \infty} \left[var\left(D_d\right)\right]^{\frac{1}{d}} = \frac{1}{e^2} = \lim_{d \longrightarrow \infty} \left[E\left(D_d\right)^{\frac{1}{d}}\right]^2.$

The first equality from the above proposition suggests a linear behaviour caused by the degenerate distribution of the scaled determinant. The same behaviour will be later observed in a different context.

Proposition 2. *For the $d \times d$ correlation matrices with the marginal distribution of each correlation following a Beta $\left(\frac{d}{2}, \frac{d}{2}\right)$ distribution on $(-1,1)$, the determinant D_d of such correlation matrix follows the expression from equation (3). Then*

1. $E(D_d) = \frac{d!}{(d+1)^{d-1}}$

2. $var(D_d) = \frac{1}{6} \cdot \frac{(d+1)!}{(d+1)^d} \cdot \frac{(d+3)!}{(d+3)^d} - \left[\frac{(d+1)!}{(d+1)^d}\right]^2.$

Form a completely different perspective, it is interesting to notice that $E(D_d)$ can be expressed as an elementary norm, see Hardy, Littlewood and Polya (1988).

Remark 4. *It is easy to show that $E(D_d)$ can be approximated to $[E(D_d^{1/d})]^d$. This approximation is the continuous version of the elementary r-norm, where $r = \frac{1}{d}$. Using Theorem 187 from Hardy, Littlewood and Polya (1988), one can prove*

$$E(D_d) = e^{E[\ln(D_d)]} + O\left(d^{-1}\right).$$

More results about the logarithm of the determinant of the correlation matrix are investigated in the following section.

4 The Logarithm of the Determinant of a Random Correlation Matrix

Since sums of independent random variables play a more important role than products, and they have been studied more thoroughly, it would be convenient to investigate the behavior of the logarithm of the determinant of the correlation matrix, rather than of the determinant itself. Let us denote $Y_d = \ln D_d$ and look at the first two moments of this new random variable.

Consider $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right)$, where $d = 2k^2$, ψ the digamma function, and γ the Euler - Mascheroni constant.

$$\begin{aligned} E(Y_d) &= \sum_{j=1}^{d-1} E\left(\ln B_j\right) = \sum_{j=1}^{d-1} \left[\psi\left(\frac{j+1}{2}\right) - \psi\left(\frac{d+1}{2}\right)\right] \\ &= \psi(1) + \cdots + \psi(k) + \psi\left(1 + \frac{1}{2}\right) + \cdots + \psi\left((k-1) + \frac{1}{2}\right) - (d-1)\psi\left(\frac{d+1}{2}\right) \\ &= -\gamma(d-1) - \ln(2)(d-2) + \sum_{i=1}^{k-1} \left[\frac{4i-1}{(2i-1)i}(k-i)\right] - (d-1)\psi\left(k + \frac{1}{2}\right). \end{aligned}$$

[2] The calculations will follow the same lines for an odd dimension.

Rewriting $\psi\left(k+\frac{1}{2}\right)$ as $\gamma - 2\ln(2) + \sum_{i=1}^{k} \frac{2}{2i-1}$ and rearranging and cancelling terms we obtain

$$E(Y_d) = -d\left[\sum_{i=1}^{k}\left(\frac{1}{2i-1} - \frac{1}{2i}\right) - \ln(2) + 1\right] + \sum_{i=1}^{k} \frac{1}{2i-1}. \tag{11}$$

We will further provide lower and upper bounds for $E(Y_d)$. Firstly, note the the first sum is a partial sum of an alternating series that converges to $\ln(2)$

$$\sum_{i=1}^{k}\left(\frac{1}{2i-1} - \frac{1}{2i}\right) = \sum_{i=1}^{2k} \frac{(-1)^{k-1}}{k} \equiv S_{2k}.$$

The partial sum S_{2k} approximates $\ln(2)$ with an error that can be bounded be the next term in the series, that is

$$|S_{2k} - \ln(2)| \leq \frac{1}{2k+1}. \tag{12}$$

For the second sum in (11), we use that, for any $N > 1$

$$\ln(N+1) < \sum_{i=1}^{N} \frac{1}{i} \leq 1 + \ln(N).$$

Then

$$\sum_{i=1}^{k} \frac{1}{2i-1} > 1 + \sum_{i=1}^{k} \frac{1}{2i} > 1 + \frac{1}{2}\ln(k+1).$$

Similarly,

$$\sum_{i=1}^{k} \frac{1}{2i-1} = \sum_{i=0}^{k-1} \frac{1}{2i+1} < 1 + \sum_{i=1}^{k-1} \frac{1}{2i} \leq 1 + \frac{1}{2}[1 + \ln(k-1)].$$

This gives that

$$1 + \frac{1}{2}\ln(k+1) < \sum_{i=1}^{k} \frac{1}{2i-1} \leq 1 + \frac{1}{2}[1 + \ln(k-1)]. \tag{13}$$

By (12),

$$-\frac{1}{2k+1} + 1 \leq S_{2k} - \ln(2) + 1 \leq \frac{1}{2k+1} + 1,$$

which gives

$$-d\left(\frac{1}{d+1} + 1\right) \leq -d[S_{2k} - \ln(2) + 1] \leq -d\left(-\frac{1}{d+1} + 1\right).$$

Together with (13), we get

$$-d\left(\frac{1}{d+1} + 1\right) + 1 + \frac{1}{2}\ln\left(\frac{d}{2} + 1\right) < E(Y_d) \leq -d\left(-\frac{1}{d+1} + 1\right) + 1 + \frac{1}{2}\left[1 + \ln\left(\frac{d}{2} - 1\right)\right]. \tag{14}$$

To get an approximation for $E(Y_d)$, we can use the result in Theorem 3.2 in Apostol (1976)

$$\sum_{i=1}^{n} \frac{1}{i} = \ln(n) + \gamma + O\left(n^{-1}\right). \tag{15}$$

This gives

$$E(Y_d) = -d + \frac{1}{2}\ln\left(\frac{d}{2}\right) + 1 + \frac{\gamma}{2} + o(1). \tag{16}$$

Remark 5. *Using Remark 4 and the above result we obtain* $E(D_d) = e^{E(Y_d)} + O\left(d^{-1}\right) \to 0$*, as* $d \to \infty$.

To calculate the variance of Y_d we need the first derivative of the digamma function, also known as the trigamma function, denoted by ψ_1. Then

$$\begin{aligned} var(Y_d) &= \sum_{j=1}^{d-1} var\left(\ln B_j\right) = \sum_{j=1}^{d-1}\left[\psi_1\left(\frac{j+1}{2}\right) - \psi_1\left(\frac{d+1}{2}\right)\right] \\ &= k\frac{\pi^2}{6} + (k-1)\frac{\pi^2}{2} - \sum_{i=1}^{k}\left[(k-i)\left(\frac{1}{i^2} + \frac{4}{(2i-1)^2}\right)\right] - (d-1)\psi_1\left(\frac{d+1}{2}\right). \end{aligned}$$

By rewriting $\psi_1\left(\frac{d+1}{2}\right)$ as $\psi_1\left(\frac{1}{2}\right) - 4\sum_{i=1}^{k}\frac{1}{(2i-1)^2}$, substituting for the known values of ψ_1, reducing and manipulating remaining terms, we obtain

$$var(Y_d) = \ln k + \gamma + \frac{1}{2k} + \frac{\pi^2}{4} + 2\sum_{i=1}^{k}\frac{1}{2i-1}.$$

By using (13), we obtain the lower and upper bounds for $var(Y_d)$

$$\ln\left(\frac{d}{2}\right) + \gamma + \frac{1}{d} + \frac{\pi^2}{4} + 2 + \ln\left(\frac{d}{2}+1\right) < var(Y_d) \le \ln\left(\frac{d}{2}\right) + \gamma + \frac{1}{d} + \frac{\pi^2}{4} + 3 + \ln\left(\frac{d}{2}-1\right). \tag{17}$$

It can be easily shown that $var(Y_d) = 2\ln\left(\frac{d}{2}\right) + o(1)$.

Even though the expression of the first two moments of Y_d do not look particularly promising, they will be employed in proving the central limit theorem for independent, but not necessarily identically distributed, random variables. However, before diving into the proof, we present a simulation based confirmation of such a result, which provides an interesting link with results for the determinant of random correlation matrices in general.

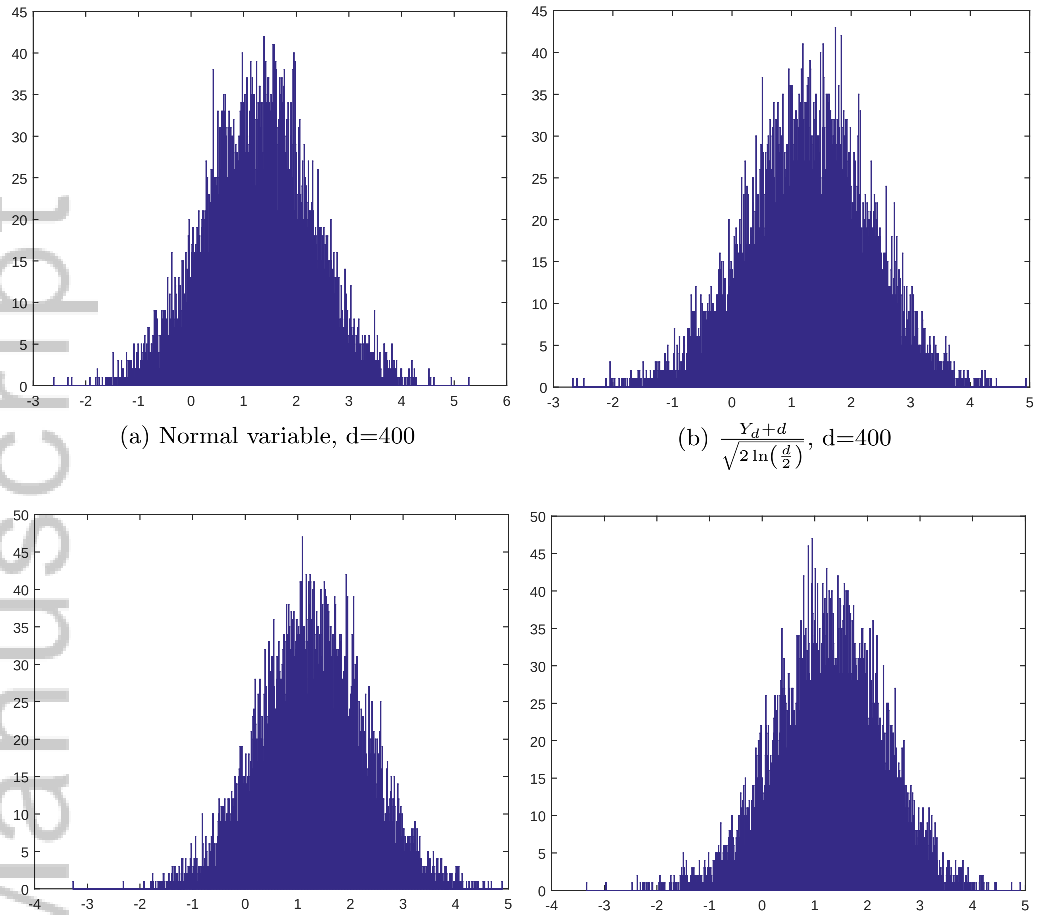
4.1 The law of the determinant of random matrices

Vu and Nguyen (2012) proved that for a $d-$dimensional random matrix A_d whose entries a_{ij} are independent real random variables with mean zero and variance one, the logarithm of $|det(A_d)|$ satisfies the central limit theorem. More precisely

$$\sup_{x\in R}|P\left(\frac{\log(|detA_d|) - 1/2\log(d-1)!}{\sqrt{1/2\log d}} \le x\right) - \Phi(x)| \le \log^{-1/3+o(1)} d.$$

Since the entries of a random correlation matrix are definitely not independent, we cannot use this result as it is. So far, we only have an indication of the order of the first two moments of Y_d.

The entries in our correlation matrix $R = (\rho_{ij})_{i,j=1,\ldots,d}$ are $Beta\left(\frac{d}{2},\frac{d}{2}\right)$ distributed on $(-1,1)$. That is to say, they are random variables with mean 0 and variance $\frac{1}{d+1}$. Then the variables $\sqrt{d+1}\cdot\rho_{ij}$ have mean zero and variance one. Ignoring for a moment that the $\rho'_{ij}s$ are not independent,



(a) Normal variable, d=400

(b) $\frac{Y_d+d}{\sqrt{2\ln\left(\frac{d}{2}\right)}}$, d=400

(c) Normal variable, d=500

(d) $\frac{Y_d+d}{\sqrt{2\ln\left(\frac{d}{2}\right)}}$, d=500

Figure 2: Simulations of normally distributed random variables (a,c) and scaled logarithm of the determinant of a random correlation matrix (b,d), for d=400 and d=500.

taking $a_{ij} = \sqrt{d+1} \cdot \rho_{ij}$, and applying the result above, we obtain that Y_d can be approximated by $\left(\sqrt{O\left(2\ln\left(\frac{d}{2}\right)\right)}\right) Z + O(-d)$, where Z is a standard normal variable.

The above result is somewhat in line with the previous calculations for the mean and variance of Y_d. Moreover, simulating $\frac{Y_d+d}{\sqrt{2\ln\left(\frac{d}{2}\right)}}$, for dimensions 400, and 500, we obtain a distribution that is undistinguishable from a normal distribution (when performing a two-sample Kolmogorov-Smirnov goodness-of-fit hypothesis test) with mean 1.27, and standard deviation 1.00. As shown above, the mean of Y_d is not $O(-d)$, but $O\left(-d + \frac{1}{2}\ln\left(\frac{d}{2}\right) + 1 + \frac{\gamma}{2}\right)$. The following section includes the formal proof of the asymptotic normality of Y_d.

4.2 The asymptotic normality of Y_d

In this section we will prove the Lyapunov central limit theorem for the logarithm of the determinant of a random correlation matrix

$$Y_d = \ln D_d = \sum_{j=1}^{d-1} \ln B_j, \tag{18}$$

where $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right)$ are independent random variables. Note that Y_d involves a sequence of sequences, thus a central limit theorem for triangular arrays is provided below.

Theorem 3 (Asymptotic normality)**.** *For $d > 0$, consider the $d \times d$ correlation matrices with a uniform joint distribution of the marginal correlations. Let R_d be a random correlation matrix and let D_d be its determinant. Moreover, let Y_d be the logarithm of D_d. Then, as $d \to \infty$,*

$$\frac{Y_d + d + \frac{1}{2}\ln\left(\frac{d}{2}\right) + \frac{\gamma}{2} + 1}{\sqrt{2\ln\left(\frac{d}{2}\right)}} \xrightarrow{d} N(0,1).$$

Proof. Let

$$\mu_j = E(\ln B_j) = \psi\left(\frac{j+1}{2}\right) - \psi\left(\frac{d+1}{2}\right),$$

and

$$\sigma_j^2 = Var(\ln B_j) = \psi_1\left(\frac{j+1}{2}\right) - \psi_1\left(\frac{d+1}{2}\right).$$

where ψ is the digamma function and ψ_1 is the trigamma function. The following inequality from Mitrinovic (1970),

$$\frac{1}{x} < \psi_1(x) < \frac{1}{x-1}, \tag{19}$$

for any $x > 1$, implies that

$$\sigma_j^2 < \frac{2}{j-1} - \frac{2}{d+1} < \infty.$$

For $j = 1$, $\psi_1(1) = \frac{\pi^2}{6}$ and

$$\sigma_1^2 < \frac{\pi^2}{6} - \frac{2}{d+1} < \infty.$$

Thus $\sigma_j^2 < \infty$, for all $j = 1, \ldots, d-1$. Let

$$s_{d-1}^2 = \sum_{j=1}^{d-1} \sigma_j^2.$$

It is immediate that the sequence $\frac{Y_d - E(Y_d)}{s_{d-1}}$ has mean 0 and variance 1. To prove its asymptotic normality, we will show that the Lyapunov condition is satisfied, that is, there exists $\delta > 0$ such that

$$\frac{1}{s_{d-1}^{2+\delta}} \sum_{j=1}^{d-1} E\left(|\ln B_j - \mu_j|^{2+\delta}\right) \to 0, \tag{20}$$

for $d \to \infty$.

We will prove (20) for $\delta = 2$, hence investigating the fourth central moment of the random variable $\ln B_j$, for $j = 1, \ldots, d-1$,

$$E\left[(\ln B_j - \mu_j)^4\right] = k_4 + 3E\left[(\ln B_j - \mu_j)^2\right],$$

where k_4 is the fourth cumulant of the distribution of $\ln B_j$, with $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right)$. By letting ψ_{n-1} denote the polygamma functions, then

$$k_n = \psi_{n-1}\left(\frac{j+1}{2}\right) - \psi_{n-1}\left(\frac{d+1}{2}\right),$$

and we can write

$$E\left((\ln B_j - \mu_j)^4\right) = \psi_3\left(\frac{j+1}{2}\right) - \psi_3\left(\frac{d+1}{2}\right) + 3\psi_1\left(\frac{j+1}{2}\right) - 3\psi_1\left(\frac{d+1}{2}\right).$$

Guo et al. (2015) have shown that, for any $x > 0$,

$$\frac{(n-1)!}{(x+1)^n} + \frac{n!}{x^{n+1}} < |\psi_n(x)| < \frac{(n-1)!}{(x+\frac{1}{2})^n} + \frac{n!}{x^{n+1}}.$$

By using this inequality, it follows that

$$\begin{aligned}\sum_{j=1}^{d-1} E\left((\ln B_j - \mu_j)^4\right) < &\sum_{j=1}^{d-1}\left[\frac{16}{(j+2)^3} + \frac{96}{(j+1)^4} + \frac{16}{(d+2)^3} + \frac{96}{(d+1)^4}\right.\\ &\left. + \frac{6}{j+2} + \frac{12}{(j+1)^2} + \frac{6}{d+2} + \frac{12}{(d+1)^2}\right]\\ &\equiv A_d.\end{aligned}$$

To bound the variance term in (20), we employ another inequality provided in Guo et al. (2015)

$$\frac{1}{x-\frac{1}{2}} - \frac{1}{12(x-\frac{1}{2})^2} < \psi_1(x) < \frac{1}{x-\frac{1}{2}},$$

for $x > \frac{1}{2}$. we have that

$$\psi_1\left(\frac{j+1}{2}\right) - \psi_1\left(\frac{d+1}{2}\right) > \frac{2}{j} - \frac{1}{3j^2} - \frac{2}{d},$$

which gives that

$$s_{d-1}^4 > \left[\sum_{j=1}^{d-1}\left(\frac{2}{j} - \frac{1}{3j^2} - \frac{2}{d}\right)\right]^2 \equiv B_d.$$

We have that

$$\frac{1}{s_{d-1}^4}\sum_{j=1}^{d-1} E\left(|\ln B_j - \mu_j|^4\right) < \frac{A_d}{B_d} < \frac{A'_d}{B_d},$$

where

$$A'_d = \sum_{j=1}^{d-1}\left[\frac{32}{(j+2)^3} + \frac{192}{(j+1)^4} + \frac{12}{j+2} + \frac{24}{(j+1)^2}\right]$$

Proving that $\frac{A'_b}{B_d} \to 0$ for $d \to \infty$ completes then the proof of the theorem. Note that, when letting $d \to \infty$, the denominator and numerator contain convergent p-series. The dominant terms are the harmonic series, and by (15), they can be approximated by $\ln(d-1) + \gamma$. The squared sum in the denominator gets the limit to zero. □

4.3 Y_d's moment generating function

The moment generating function of Y_d can be calculated as follows

$$M_{Y_d}(t) = E\left(e^{t\cdot\sum_{j=1}^{d-1}\ln B_j}\right) = \prod_{j=1}^{d-1} \frac{B\left(t+\frac{j+1}{2}, \frac{d-j}{2}\right)}{B\left(\frac{j+1}{2}, \frac{d-j}{2}\right)}, \tag{21}$$

where $B(\alpha, \beta)$ is a Beta function.
Using (4) and (5), with $S = \frac{d+1}{2}$, and Proposition 2 we obtain[3]

$$\begin{aligned} M_{Y_d}(t) &= \prod_{j=1}^{d-1} \frac{\Gamma\left(\frac{j+1}{2}+t\right)}{\Gamma\left(\frac{j+1}{2}\right)} \cdot \left[\frac{\Gamma(S)}{\Gamma(t+S)}\right]^{d-1} \\ &\approx \prod_{j=1}^{d-1}\left(\frac{j+1}{2}\right)^t \cdot \frac{2^{t(d-1)}}{(d+1)^{t(d-1)}} \\ &= \left[\frac{d!}{(d+1)^{d-1}}\right]^t = \left[E(D_d)\right]^t. \end{aligned}$$

In this case, the first derivative of $M_{Y_d}(t)$ evaluated at $t = 0$ equals $\ln E(D_d)$, hence $\ln E(D_d) \approx E(\ln D_d)$ as $d \to \infty$. This suggests again an approximate linear behaviour caused by the degenerate distribution of the scaled determinant.

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5 Appendix

The proof of Corollary 2 follows the same lines as the proof of Theorem 2.

Corollary 2. *Consider the $d \times d$ correlation matrices such that each ρ_{ij} marginally follows a Beta distribution on $(-1, 1)$, with both parameters equal to $\left(\eta - 1 + \frac{d}{2}\right)$, for $\eta > 0$. Let R_d be such a matrix and D_d its determinant. Then $D_d^{1/d} \to \frac{1}{e}$, as $d \to \infty$, in probability.*

Proof. Consider the expression for D_d given in Corollary 1. Let B_j denote the *Beta* variables from the product; B_j's are independent, so the expectation of $D_d^{1/d}$ can be calculated as a product of expectations of *Beta* variables raised to the power $\frac{1}{d}$.

Let $\alpha_j = \eta + \frac{j-1}{2}$, $\beta_j = \frac{d-j}{2}$ and $S = \alpha_j + \beta_j = \eta + \frac{d-1}{2}$. Let $B(\alpha_j, \beta_j)$ denote the Beta function,

[3] The product is dominated by the higher order terms.

and $\Gamma(\cdot)$ denote the Gamma function. Then

$$
\begin{aligned}
E\left(D_d^{1/d}\right) &= E\left(\prod_{j=1}^{d-1} B_j^{1/d}\right) \\
&= \prod_{j=1}^{d-1} E\left(B_j^{1/d}\right) \\
&= \prod_{j=1}^{d-1} \frac{B(\alpha_j + \frac{1}{d}, \beta_j)}{B(\alpha_j, \beta_j)} = \prod_{j=1}^{d-1} \frac{\Gamma(\alpha_j + \frac{1}{d}) \cdot \Gamma(\alpha_j + \beta_j)}{\Gamma(\alpha_j) \cdot \Gamma(\alpha_j + \beta_j + \frac{1}{d})} \\
&= \left[\frac{\Gamma(S)}{\Gamma(S + \frac{1}{d})}\right]^{d-1} \cdot \prod_{j=1}^{d-1} \frac{\Gamma(\eta + \frac{j-1}{2} + \frac{1}{d})}{\Gamma(\eta + \frac{j-1}{2})}.
\end{aligned}
$$

By (4) and (5),

$$
\begin{aligned}
E\left(D_d^{1/d}\right) &= \left(S^{-\frac{1}{d}}\right)^{d-1} \cdot \prod_{j=1}^{d-1} \left(\frac{2\eta + j - 1}{2}\right)^{\frac{1}{d}} \cdot [1 + O(d^{-1})]^d \\
&= \frac{2^{\frac{d-1}{d}}}{(2\eta + d - 1)^{\frac{d-1}{d}}} \cdot \frac{[(2\eta + d - 2)!]^{\frac{1}{d}}}{[(2\eta - 1)!]^{\frac{1}{d}}} \cdot \frac{1}{2^{\frac{d-1}{d}}} \cdot [1 + O(d^{-1})]^d.
\end{aligned}
$$

Using (6) and rearranging the terms gives

$$
\begin{aligned}
E\left(D_d^{1/d}\right) = &\frac{1}{[(2\eta - 1)!]^{\frac{1}{d}}} \cdot \frac{1}{e^{\frac{2\eta + d - 2}{d}}} \cdot \left(\sqrt{2\pi}\right)^{\frac{1}{d}} \cdot (2\eta + d - 2)^{\frac{1}{2d}} \\
&\cdot \frac{2\eta + d - 2}{2\eta + d - 1} \cdot (2\eta + d - 2)^{\frac{2\eta - 2}{d}} \cdot (2\eta + d - 1)^{\frac{1}{d}} \cdot [1 + O(d^{-1})]^d.
\end{aligned}
$$

Then $\lim\limits_{d \longrightarrow \infty} E\left(D_d^{1/d}\right) = \dfrac{1}{e}$.

In the same fashion we show that $\lim\limits_{d \longrightarrow \infty} E(D_d^{2/d}) = \dfrac{1}{e^2}$. To summarize we have

$$
\begin{aligned}
\lim_{d \longrightarrow \infty} E\left(D_d^{1/d}\right) &= \frac{1}{e} \\
\lim_{d \longrightarrow \infty} \left[E\left(D_d^{1/d}\right)\right]^2 &= \frac{1}{e^2} \\
\lim_{d \longrightarrow \infty} E\left(\left(D_d^{1/d}\right)^2\right) &= \frac{1}{e^2}.
\end{aligned}
$$

The above equalities prove $\lim\limits_{d \longrightarrow \infty} var\left(D_d^{1/d}\right) = 0$ and the Corollary. □

Let us now prove Propositions 1 and 2.

Proposition 1. *For the $d \times d$ correlation matrices with the marginal distribution of each correlation following a Beta $\left(\frac{d}{2}, \frac{d}{2}\right)$ distribution on $(-1, 1)$, the determinant D_d of such correlation matrix follows the expression from equation (3). Then*

1. $\lim\limits_{d \longrightarrow \infty} [E(D_d)]^{\frac{1}{d}} = \dfrac{1}{e} = \lim\limits_{d \longrightarrow \infty} E(D_d^{1/d})$

2. $\lim_{d\longrightarrow\infty}[var(D_d)]^{\frac{1}{d}} = \frac{1}{e^2} = \lim_{d\longrightarrow\infty}\left[E(D_d)^{\frac{1}{d}}\right]^2.$

Proof. The proof of the second equality in both statements can be found in the proof of Theorem 2. To prove the first equality in 1. calculate $E(D_d)$ as follows

$$E(D_d) = \prod_{j=1}^{d-1} \frac{\frac{j+1}{2}}{\frac{d+1}{2}} = \frac{d!}{(d+1)^{d-1}}. \tag{22}$$

By (6), we obtain

$$\begin{aligned}
[E(D_d)]^{\frac{1}{d}} &= \left[\frac{d!}{(d+1)^{d-1}}\right]^{\frac{1}{d}} \\
&= \frac{(2\pi)^{\frac{1}{d}} \cdot d^{\frac{1}{2}\cdot\frac{1}{d}} \cdot d^{d\frac{1}{d}}}{e^{d\cdot\frac{1}{d}} \cdot (d+1)^{\frac{d-1}{d}}} \cdot \left[1 + O\left(d^{-1}\right)\right]^{\frac{1}{d}} \\
&= (2\pi)^{\frac{1}{d}} \cdot \frac{d}{d+1} \cdot d^{\frac{1}{2d}} \cdot (d+1)^{\frac{1}{d}} \cdot \frac{1}{e} \cdot \left[1 + O\left(d^{-1}\right)\right]^{\frac{1}{d}}.
\end{aligned}$$

Then $\lim_{d\longrightarrow\infty}[E(D_d)]^{\frac{1}{d}} = \frac{1}{e}$.
Using (6) again, we can calculate

$$\begin{aligned}
[E(D_d)]^{\frac{2}{d}} &= \frac{(2\pi)^{\frac{2}{d}} \cdot d^{\frac{1}{2}\cdot\frac{2}{d}} \cdot d^{d\frac{2}{d}}}{e^{d\cdot\frac{2}{d}} \cdot (d+1)^{\frac{2(d-1)}{d}}} \cdot \left[1 + O\left(d^{-1}\right)\right]^{\frac{2}{d}} \\
&= (2\pi)^{\frac{2}{d}} \cdot \left(\frac{d}{d+1}\right)^2 \cdot d^{\frac{1}{d}} \cdot (d+1)^{\frac{2}{d}} \cdot \frac{1}{e^2} \cdot \left[1 + O\left(d^{-1}\right)\right]^{\frac{2}{d}}
\end{aligned}$$

which proves $\lim_{d\longrightarrow\infty}\left[E(D_d)^{\frac{1}{d}}\right]^2 = \frac{1}{e^2}$.
To prove Relation 2. from Proposition 1 we start from the expression of the determinant D_d given in equation (3) and consider $(d-1)$ independent, Beta distributed random variables, $B_j \sim Beta\left(\frac{j+1}{2}, \frac{d-j}{2}\right), j = 1, \dots, d-1$. Then

$$\begin{aligned}
var(D_d) &= var\left(\prod_{j=1}^{d-1} B_j\right) = \prod_{j=1}^{d-1} \frac{\frac{j+1}{2}\cdot\frac{j+3}{2}}{\frac{d+1}{2}\cdot\frac{d+3}{2}} - \prod_{j=1}^{d-1} \frac{\frac{(j+1)^2}{4}}{\frac{(d+1)^2}{4}} \\
&= \frac{1}{(d+1)^{d-1}\cdot(d+3)^{(d-1)}} \cdot \prod_{j=1}^{d-1}(j+1) \cdot \prod_{j=1}^{d-1}(j+3) - \frac{1}{(d+1)^{2(d-1)}} \cdot \prod_{j=1}^{d-1}(j+1)^2 \\
&= \frac{d!(d+2)!}{6(d+1)^{d-1}(d+3)^{d-1}} - \frac{(d!)^2}{(d+1)^{2(d-1)}} \\
&= \frac{d!}{(d+1)^{(d-1)}} \cdot \left[\frac{(d+2)!}{6(d+3)^{d-1}} - \frac{d!}{(d+1)^{d-1}}\right] \\
&= \frac{(d+1)!}{(d+1)^d} \cdot \left[\frac{(d+3)!}{6(d+3)^d} - \frac{(d+1)!}{(d+1)^d}\right].
\end{aligned} \tag{23}$$

By (6),

$$
\begin{aligned}
var(D_d) &= \frac{\sqrt{2\pi(d+1)}(d+1)^{d+1}}{e^{d+1}(d+1)^d} \cdot \left[\frac{\sqrt{2\pi(d+3)}(d+3)^{d+3}}{e^{d+3}6(d+3)^d} - \frac{\sqrt{2\pi(d+1)}(d+1)^{d+1}}{e^{d+1}(d+1)^d} \right] \cdot \left[1+O\left(d^{-1}\right)\right]^2 \\
&= \frac{\sqrt{2\pi(d+1)}(d+1)}{e^{d+1}} \cdot \left[\frac{\sqrt{2\pi(d+3)}(d+3)^3}{6\cdot e^{d+3}} - \frac{\sqrt{2\pi(d+1)}(d+1)}{e^{d+1}} \right] \cdot \left[1+O\left(d^{-1}\right)\right]^2 \\
&= \frac{2\pi\cdot(d+1)^{3/2}\cdot(d+1)^{1/2}}{e^{2(d+1)}} \cdot \left[\frac{\sqrt{\frac{d+3}{d+1}}}{6} \cdot \frac{(d+3)^3}{e^2} - (d+1) \right] \cdot \left[1+O\left(d^{-1}\right)\right]^2 \\
&= \frac{\pi\cdot(d+1)^{3/2}}{3e^{2(d+2)}} \left[(d+3)^{7/2} - 6e^2(d+1)^{3/2} \right] \cdot \left[1+O\left(d^{-1}\right)\right]^2 .
\end{aligned}
$$

By raising to the $1/d$ power and taking the limit when $d \to \infty$ we obtain $\lim_{d\longrightarrow\infty} [var(D)]^{\frac{1}{d}} = \frac{1}{e^2}$.

□

The relations from Proposition 2 find their derivations in the proof of Proposition 1 above.

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