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Household Lifetime Strategies under a Self-Contagious Market

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Abstract

In this paper, we consider the optimal strategies in asset allocation, consumption, and life insurance for a household with an exogenous stochastic income under a self-contagious market which is modeled by bivariate self-exciting Hawkes jump processes. By using the Hawkes process, jump intensities of the risky asset depend on the history path of that asset. In addition to the financial risk, the household is also subject to an uncertain lifetime and a fixed retirement date. A lump-sum payment will be paid as a heritage, if the wage earner dies before the retirement date. Under the dynamic programming principle, explicit solutions of the optimal controls are obtained when asset prices follow special jump distributions. For more general cases, we apply the Feynman-Kac formula and develop an iterative numerical scheme to derive the optimal strategies. We also prove the existence and uniqueness of the solution to the fixed point equation and the convergence of an iterative numerical algorithm. Numerical examples are presented to show the effect of jump intensities on the optimal controls.

Key Words. Dynamic programming, self-contagious market, stochastic labor income, investment and consumption, life insurance

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1 Introduction

Recent financial markets have shown increasingly obvious sudden shocks and jumps of asset prices, especially during the financial crisis in 2007-2009. One traditional method to explore the impact of those abrupt events in investment and portfolio management is to introduce Lévy processes into the dynamics of the asset prices, such as the compound Poisson process. Studies on the impact of jumps on the portfolio management can be traced back to Merton (1969), who initially explores the asset allocation and consumption problem in a continuous-time model. Then, instead of considering continuous underlying processes, Merton (1976) introduces Poisson jumps into the stock dynamics. Thereafter, many researchers have investigated the abrupt price jumps and extended it to follow more general Lévy processes. Early works include Aase (1984), Das and Uppal (2004), and Cont and Tankov (2004), which follow the assumption of serially independent jump increments. More recently, in addition to considering jumps purely in the asset price, Liu et al. (2003) investigate the asset allocation problem with jumps in the volatility process as well. The analytic solution suggests that jumps in the volatility process have large effects on the investors' willingness to take leverage or short positions. Furthermore, Branger et al. (2017) explore a multi-asset Wishart-model where return variances and correlations can jump. They find that the optimal controls are critically changed by the jumps in the second moments. Aït-Sahalia and Matthys (2019) consider a robust consumption and portfolio management problem where asset prices follow Lévy processes. Although previous studies have widely explored the dramatic impact of jumps on the asset allocation problem, they have largely ignored the cluster and propagation properties of the jump processes.

Boswijk et al. (2018) observe that a significant decline in real equity market triggers multiple drops in close succession over a short period. The phenomenon is called self excitation that allows a realized jump to have positive or negative effects on its own stochastic jump intensities. In addition, Aït-Sahalia et al. (2015) develop an estimation method to test the propagation phenomenon based on the method of moments, which highly supports the existence of self-excitation and mutual-excitation in real market. Feinstein (2020) consider the impacts of financial contagion caused by extreme negative events of one bank or firm on other banks or firms. Standard Lévy processes assume independent increments, which cannot produce any type of clusters or propagation of jumps. Therefore, to better model and reproduce the investment environment, we need to consider more versatile jump processes that can capture the propagation phenomenon described above.

One of the processes that can address this problem is the Hawkes process introduced by Hawkes (1971), where the arrival of one jump changes its own or other jump intensities and the intensities revert to a long-tern mean level in the absence of jumps. Branger et al. (2014) compare the Hawkes jump process with a regime switching model to show that only the Hawkes process induces the clustering property. Callegaro et al. (2019) observe that Hawkes processes can capture the jumps clustering features in Forward prices of Power markets. Du and Luo (2019) apply a two-factor

Hawkes jump-diffusion model to examine the S&P 500 index and observe severe but short lived propagation effect . Additionally, Cui et al. (2019) investigate the valuation of options under time-changed Markov processes, including integral of CIR process, Lévy Subordinators, and Markovian Hawkes process.

To the best of our knowledge, Aït-Sahalia and Hurd (2015) first include the Hawkes process into the asset allocation problem and figure out the explicit optimal controls for the investor with log-utilities. They consider a class of multidimensional assets in which each price jump-down in one asset price increases the probability of future jumps in that asset and other assets. For the value function, they investigate the expected discounted utilities from life-time consumption. Under certain assumptions, the value function and the optimal controls are simplified to be time independent. Kokholm (2016) analyzes the impact of propagation on the derivative market by including the Hawkes process in the dynamics of the indexes' log-returns and finds that this model generates results that are consistent with empirical evidences and fit option volatility surfaces of the four candidate indexes over an extended period of time. Hainaut (2017) analyzes the impact of the contagion phenomenon between the financial and insurance market. The optimal allocation, dividend, and reinsurance policies are obtained by assuming that the value function follows an exponential affine structure. Dassios and Zhao (2017) extend the standard Hawkes process by considering CIR-type intensities and introduce a very efficient simulation scheme.

Literature on the optimal consumption, life insurance, and portfolio management of an investor with uncertain death time can be traced back to early works of Yarri (1965) and Richard (1975). Recently, Pliska and Ye (2007) assume that the investor's lifetime is bounded by a fixed time horizon and derive closed-form solutions for the investor with a deterministic income stream. Kwak et al. (2011) extend the model from an individual investor to a family where the parents receive deterministic labor income until a fixed time. If the parents die before the fixed time, the children with no income have to choose optimal allocation and consumption policies with the remaining wealth and life insurance benefit. Duarte et al. (2014) provide solutions for the optimal consumption, investment, and life insurance polices in a market comprised of a risk-free bond and an arbitrary number of risky securities driven by multi-dimensional Brownian motion. In stead of considering deterministic labor income, Wang (2009) investigates an optimization problem for an investor with a stochastic wage process. From then on, researches on this topic become much more popular. Recent works include Zeng et at. (2015), Wang et al. (2016), and Hou et al. (2018).

In this paper, we mainly follow the framework of Aït-Sahalia and Hurd (2015) and Hainaut (2017) and discuss the impact of self-contagion on the optimal allocation, consumption, and life insurance policies for a wage earner with exponential utilities and uncertain lifetime. The wage earner's family members will receive a lump-sum payment if the wage earner dies before the retirement date. In stead of considering a self-financing wealth process, we follow the work of Wang

(2009) and assume the investor receives an exogenous stream of mean-reverting stochastic labor income. Under those complex settings, closed-form solutions of the optimal controls are not available for general cases as in Aït-Sahalia and Hurd (2015). However, we can apply the exponential affine structure of Hainaut (2017) to derive explicit expressions under special jump distributions and use numerical methods to solve more general jump models.

To prove the convergence of the numerical method, we follow the idea of Delong and Klüppelberg (2008) and define an operator for the fixed point equation. We derive a theorem verifying that the operator is a contraction mapping. Then, by using Banach fixed point theorem, we find that the fixed point equation has a unique solution and the iterative method will converge to that unique solution. Delong and Klüppelberg (2008) consider a Black-Scholes market with coefficients driven by an external Ornstein-Uhlenbeck process and have to obtain relevant upper bounds for the external stochastic factor. In this paper, we consider a contagion market with underlying assets following bivariate self-exciting Hawkes jump processes. Then, due to the jumps in the assets price proesses, the optimal investment strategy depends on the specific price jump distributions. Hence, it becomes more difficult to find an upper bound for a function of the investment control, which is the key step to prove the operator is a contraction mapping.

The main contributions of our paper follow the three ways. First, for the Hawkes jump process, only consider the effect of negative price jumps on the optimal portfolio selection and Fan (2017) further assumes that the sizes for the price jump-downs are all negative constants, which indeed can largely simplify the computation task. However, in real market, the effect of price jump-ups cannot be the same as the effect of price jump-downs. To describe the asymmetric properties of jump intensities, we follow the work of Jang and Dassios (2013) and apply a bivariate self-exciting process where we introduce two jump intensities for a single asset: one for price jump-up and the other for price jump-down. The arrival of price jump-up increases the jump-up intensity and decreases the jump-down intensity. The arrival of price jump-down decreases the jump-up intensity and increases the jump-down intensity. The idea of separating the specific effect of price jumps can be easily extended to higher dimensional models. By incorporating the bivariate self-exciting process, the HJB equation will be too complex to have closed-form solutions for general jump distributions. Moreover, we provide the rigorous convergence proof of the numerical algorithm by using the Banach fixed point theorem for given exponential jump distribution.

Second, Aït-Sahalia and Hurd (2015) consider the utilities of instantaneous consumption, where the value function can be much simplified. We study a life-time optimization problem for a household and further generalize this problem by maximizing the expected discounted utilities from both the instantaneous consumption and the death-time heritage. In addition, we consider the market process until the minimum of the wage earner's death time and the retirement date. Due to the random death time and the fixed retirement date, the value function cannot be converted to be time independent, which makes it much more complicated to derive closed-form solutions of the value function and the optimal controls. To solve the value function, we follow the work of Hainaut (2017) and consider a wage earner with exponential utilities. Applying the exponential affine structure, we can obtain closed-form solutions under special jump size distributions by solving a system of ordinary differential equations. For more general cases, we apply a numerical method according to a contraction mapping and the Feynman-Kac formula.

Third, we generalize the work of Zeng et at. (2015) by adopting a linear combination of exponential distributions to model the random death time of the investor, as a linear combination of exponential distributions can provide an arbitrarily close approximation to nonnegative distributions including exponential-tail and power-tail distributions in the sense of weak convergence.

The rest of this paper is organized as follows. In Section 2, we briefly introduce the bivariate self-excitation jump processes and characterize the dynamics of different assets. A comprehensive description of the formulation is stated. Section 3 develops the value function and the HJB equation. In Section 4, we apply the exponential affine structure to simplify the HJB equation and present a numerical method to solve the value function under the Feyman-Kac formula. In Section 5, numerical examples are given to illustrate the effects of the stochastic income and jump intensities on the optimal controls. Section 6 concludes this paper.

2 Market Formulation

2.1 Bivariate Self-Excitation Jumps

In this paper, we follow the works of Aït-Sahalia et al. (2014) and Hainaut (2017) and introduce a bivariate self-excitation jump process for the asset prices. According to Hawkes (1971), self-excitation jump processes are special cases of path-dependent point processes and the intensities of those processes depend on the paths of that point process. The jump intensities are stochastic and obey the following mean-reverting dynamics:

$$
\begin{aligned}
d\lambda_t^U &= \alpha^U(\bar{\lambda}^U - \lambda_t^U)dt + \eta_{11} dN_t^U + \eta_{12} dN_t^D, \\
d\lambda_t^D &= \alpha^D(\bar{\lambda}^D - \lambda_t^D)dt + \eta_{21} dN_t^U + \eta_{22} dN_t^D,
\end{aligned} \tag{2.1}
$$

where λ_t^U and λ_t^D are the intensities of jump processes N_t^U and N_t^D, respectively. The intensities revert at speed α^U and α^D to $\bar{\lambda}^U$ and $\bar{\lambda}^D$, respectively. The coefficients are assumed to satisfy the following conditions:

$$
\eta_{11} > 0, \eta_{12} < 0, \eta_{21} < 0, \eta_{22} > 0.
$$

For example, when a price jump-up happens (a jump of N_t^U), the intensity λ_t^U of N_t^U increases by a positive amount η_{11}, resulting a shorter expected waiting time for the next price jump-up. Simultaneously, the appearance of price jump-ups motivates investors to participate in the financial market, which will increase the demand side. Hence, $\eta_{21} < 0$, leading to a longer expected waiting time to observe the next price jump-down. The other coefficients can be verified by similar arguments. Then the jump process is defined by the intensity processes in (2.1). We use N_t^U as an example to demonstrate the process as follows:

$$\begin{cases} \mathbb{P}(N_{t+\Delta t}^U - N_t^U = 0|\mathcal{F}_t) = 1 - \lambda_t^U \Delta t + o(\Delta t), \\ \mathbb{P}(N_{t+\Delta t}^U - N_t^U = 1|\mathcal{F}_t) = \lambda_t^U \Delta t + o(\Delta t), \\ \mathbb{P}(N_{t+\Delta t}^U - N_t^U > 1|\mathcal{F}_t) = o(\Delta t). \end{cases} \tag{2.2}$$

2.2 Life Uncertainty Risk

In this paper, we consider a wage earner under Merton's framework: the wage earner maximizes the expected discounted utilities from instantaneous consumption and the terminal heritage by choosing optimal consumption, investment, and life insurance policies, where the stopping time is the minimum of the random death time (τ_d) and the fixed retirement date (T).

For the random variable τ_d, we introduce a linear combination of exponential distributions $\bar{F}(s,t)$ to represent the conditional survival probability at time s, given that the wage earner is alive at time t for $s \geq t \geq 0$. Then, $\bar{F}(s,t) = \mathbb{P}\{\tau_d > s|\tau_d > t\}$ and the conditional death probability at time s is $F(s,t) = 1 - \bar{F}(s,t) = \mathbb{P}\{\tau_d \leq s|\tau_d > t\}$. Then, the conditional probability density function for the death probability is $f(s,t) = \frac{dF(s,t)}{ds}$. We have

$$\bar{F}(s,t) = \frac{\mathbb{P}\{\tau_d > s\}}{\mathbb{P}\{\tau_d > t\}} = \frac{\sum_{i=1}^n p_i \exp\left(-\int_0^s \lambda_i^m(u)du\right)}{\sum_{i=1}^n p_i \exp\left(-\int_0^t \lambda_i^m(u)du\right)}, \tag{2.3}$$

and the probability density function has the form

$$f(s,t) = \frac{\sum_{i=1}^n p_i \lambda_i^m(s) \exp\left(-\int_0^s \lambda_i^m(u)du\right)}{\sum_{i=1}^n p_i \exp\left(-\int_0^t \lambda_i^m(u)du\right)}, \tag{2.4}$$

with $\sum_{i=1}^n p_i = 1, i = 1,\ldots,n, n \geq 1$ and $\lambda_i^m(t) > 0, \forall t > 0$.

In addition, we assume that the wage earner can purchase life insurance by paying the insurance premium P_t at time t. If the wage earner dies at time t, his family can receive a lump-sum death benefit $\frac{P_t}{\eta_t}$, where η_t is the premium-insurance ratio determined by insurance companies. To simplify computation, we assume that insurance companies do not require a risk loading, which means $\eta_t = f(t,t)$.

2.3 Asset Return Dynamics

We assume that there are only two tradable assets: a risk-free bond B_t and a risky stock S_t. We characterize the processes of those assets as follows:

1. The risk-free bond B_t appreciates at a constant rate r:

$$dB_t = rB_t dt. \tag{2.5}$$

2. The risky stock S_t follows a Hawkes jump-diffusion model:

$$dS_t = \mu S_{t^-} dt + \sigma S_{t^-} dW_t^S + J_t^U S_{t^-} dN_t^U + J_t^D S_{t^-} dN_t^D, \tag{2.6}$$

   where μ is the drift term, σ is the volatility term, and N_t^U and N_t^D denote the jumps or crashes of stock prices. J_t^U and J_t^D capture the size of jump-up and jump-down, respectively. To simplify the notation scheme, the distribution functions of J_t^U and J_t^D will be given later.

2.4 Wealth Dynamics

Let X_t denote the total wealth of the wage earner and π_t denote the allocation policy, which represents the actual amount invested in the risky stock at time t, then $X_t - \pi_t$ is the amount invested in the risk-free bond. C_t denotes the instantaneous consumption amount. P_t is the life insurance premium purchased by the wage earner and his family will receive a lump-sum payment at his death time, which depends on the life insurance premium and the mortality rate at that time.

Additionally, we suppose the wage earner receives an exogenous stream of stochastic income I_t at time t. Following the work of Wang (2009), we assume that the labor income satisfies a mean-reverting process where θ denotes income growth parameter, k denotes the persistence of the income, and $\tilde{\sigma}$ measures the volatility of the income process. The evolution processes for the total wealth and income are as follows:

$$\begin{aligned} dX_t =& (X_t - \pi_t)\frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_t} - C_t dt - P_t dt, \\ =& [rX_t + \pi_t(\mu - r) + I_t - C_t - P_t]dt + \pi_t \sigma dW_t^S + \pi_t J_t^U dN_t^U + \pi_t J_t^D dN_t^D, \end{aligned} \tag{2.7}$$

and

$$dI_t = (\theta - kI_t)dt + \tilde{\sigma} dW_t^I. \tag{2.8}$$

And we assume the correlation coefficient between the two Brownian motions W_t^S and W_t^I is ρ.

3 Optimization Problem and The HJB Equation

Now we demonstrate the wage earner's objective function. Following the work of Pliska and Ye (2007), we consider three sources of utilities: continuous consumption between initial time t and exit time $\min(\tau_d, T)$; terminal wealth if the wage earner survives at time T; and the legacy if he dies before the retirement date T. Therefore, in a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions of right-continuity and completeness, the target function V is as follows:

$$\begin{aligned} V(t, X_t, I_t, \lambda_t^U, \lambda_t^D) = \max_{\{C,\pi,P,\}} \mathbb{E}\Big[\int_t^{\tau_d \wedge T} & e^{-\beta(s-t)} U(C_s) ds + e^{-\beta(T-t)} U(X_T) \mathbb{I}\{\tau_d \geq T\} \\ & + e^{-\beta(\tau_d - t)} U(X_{\tau_d}, P_{\tau_d}) \mathbb{I}\{\tau_d < T\} | \mathcal{F}_t \Big], \end{aligned} \tag{3.1}$$

where $\mathbb{E}[\cdot|\mathcal{F}_t]$ is the conditional expectation given the information at time t. X_t stands for the initial total wealth of the wage earner, λ_t^U and λ_t^D is the initial value of jump intensities, β is the subjective discount rate, and $\mathbb{I}$ is the identity function. $U(\cdot)$ stands for the utility function, with

$$U(X_{\tau_d}, P_{\tau_d}) = U\left(X_{\tau_d} + \frac{P_{\tau_d}}{f(\tau_d, \tau_d)} \right). \tag{3.2}$$

To solve this optimization problem, we need to express the target function in a dynamic programming form and then use stochastic dynamic programming principle to derive the HJB equation and get formulas for the optimal control policies.

Lemma 3.1. *The target function V can be restated as follows:*

$$\begin{aligned} V(t, X_t, I_t, \lambda_t^U, \lambda_t^D) = \max_{\{C,\pi,P\}} \mathbb{E}\Big[\int_t^T & \big(\bar{F}(s,t) e^{-\beta(s-t)} U(C_s) + f(s,t) e^{-\beta(s-t)} U(X_s, P_s) \big) ds \\ & + \bar{F}(T,t) e^{-\beta(T-t)} U(X_T) | \mathcal{F}_t \Big], \end{aligned} \tag{3.3}$$

with $\bar{F}(s,t)$ being the survival probability up to time s, given that the wage earner is alive at time t as mentioned before, and $f(s,t)$ being the probability density function w.r.t. $F(s,t)$.

The proofs of Lemma 3.1 are provided in Appendix A. Now, we use the dynamic programming principle to derive the HJB equation of this optimization problem.

Lemma 3.2. *The target function satisfies the following recursive relationship for some $u \in (t, T)$:*

$$\begin{aligned} V(t, X_t, I_t, \lambda_t^U, \lambda_t^D) = \max_{\{C,\pi,P\}} \mathbb{E}\Big[& \bar{F}(u,t) e^{-\beta(u-t)} V(u, X_u, I_u, \lambda_u^U, \lambda_u^D) \\ & + \int_t^u (\bar{F}(s,t) e^{-\beta(s-t)} U(C_s) + f(s,t) e^{-\beta(s-t)} U(X_s, P_s)) ds \Big| \mathcal{F}_t]. \end{aligned} \tag{3.4}$$

The proofs of Lemma 3.2 are provided in Appendix B. Now, we state that the HJB equation that characterizes the optimal controls of this Merton's problem is as follows:

$$\begin{aligned} 0 = & -(f(t,t) + \beta) V + V_t + (r X_t + I_t) V_X + (\theta - k I_t) V_I \\ & + \alpha^U (\bar{\lambda}^U - \lambda_t^U) V_{\lambda^U} + \alpha^D (\bar{\lambda}^D - \lambda_t^D) V_{\lambda^D} + 0.5 \tilde{\sigma}^2 V_{II} \\ & + \max_\pi \left[\pi_t (\mu - r) V_X + 0.5 \pi_t^2 \sigma^2 V_{XX} + \rho \pi_t \sigma \tilde{\sigma} V_{XI} + \lambda_t^U \mathbb{E}(V^U - V) + \lambda_t^D \mathbb{E}(V^D - V) \right] \\ & + \max_C \left[-C_t V_X + U(C_t) \right] \\ & + \max_P \left[-P_t V_X + f(t,t) U(X_t, P_t) \right], \end{aligned} \tag{3.5}$$

with the terminal boundary condition

$$V(T, X_T, I_T, \lambda_T^U, \lambda_T^D) = U(X_T).$$

The derivation of the above HJB equation is included in Appendix C. For the optimal controls π^*, C^*, and P^*, we use the first-order condition to (3.5) to derive the following expressions in terms of the target function $V(t, \cdot)$:

$$\begin{aligned} \pi_t^* &= \arg\max_{\pi} \left\{ \pi(\mu - r)V_X + 0.5\pi^2\sigma^2 V_{XX} + \rho\sigma\tilde{\sigma}\pi V_{XI} + \lambda^U \mathbb{E}(V^U - V) + \lambda^D \mathbb{E}(V^D - V) \right\}, \\ C_t^* &= U_C^{-1}(V_X), \\ P_t^* &= U_D^{-1}\left(\frac{V_X}{f(t,t)}\right), \end{aligned} \tag{3.6}$$

where U_C^{-1} and U_D^{-1} are the inverse functions of partial derivatives of U_C and U_D. Furthermore, V^U and V^D are the value functions right after the price jumps, with $V^U = V(t, X_t + \pi_t J_t^U, I_t, \lambda_t^U + \eta_{11}, \lambda_t^D + \eta_{21})$ and $V^D = V(t, X_t + \pi_t J_t^D, I_t, \lambda_t^U + \eta_{12}, \lambda_t^D + \eta_{22})$, respectively.

In order to get the explicit formulas for the optimal controls and the target function, we need more assumptions on the utility function.

We consider a wage earner with the standard exponential utility $U(X) = -\frac{e^{-\gamma X}}{\gamma}$ with $\gamma > 0$. Following the work of Duffie et al. (2000), Aït-Sahalia and Hurd (2015), and Fan (2017), we know that if the jump intensities follows (2.1), the value function $V(t, X_t, I_t, \lambda^U, \lambda^D)$ follows an exponential affine structure. Then, we separate the value function as

$$V(t, X_t, I_t, \lambda^U, \lambda^D) = \frac{-1}{\gamma} e^{A(t,T)I_t + D(t,T)X_t} g(t, \lambda_t^U, \lambda_t^D), \tag{3.7}$$

where $g(t, \lambda_t^U, \lambda_t^D)$ is an implicit function for the affine structure. More specific, we have

$$g(t, \lambda_t^U, \lambda_t^D) = e^{E(t,T) + B^U(t,T)\lambda_t^U + B^D(t,T)\lambda_t^D}, \tag{3.8}$$

where $E(\cdot), B^U(\cdot), B^D(\cdot), A(\cdot)$, and $D(\cdot)$ are functions to be determined. Explicit solutions for the five unknown functions can be obtained when the price jumps follow some special distributions. We will present some examples with special jump distributions later.

To solve for the cases with general jump distributions, we can only find explicit solutions for $A(t,T)$ and $D(t,T)$ and have to apply Feynman-Kac formula and an iterative scheme to solve g numerically, which is covered in Section 4.

4 Analysis and Numerical Algorithm

For general cases, we suppose that the price jump-up sizes and the price jump-down sizes satisfy general distributions. We follow (3.7) to introduce an implicit function $g(t, \lambda_t^U, \lambda_t^D)$ for the affine structure. In Section 4.1, we find the closed-form solutions for $A(t,T)$ and $D(t,T)$ and introduce an iterative scheme to solve g. In section 4.2, we prove the existence and uniqueness of the solution to this iterative scheme. Convergence of the algorithm is shown accordingly. Finally, in Section 4.3, we provide the details of the numerical algorithm.

4.1 Analysis of Controls and Value Function

In this section, we follow the structure in (3.7) and omit the parameters to reduce notations. Then, the partial derivatives become

$$\begin{aligned} &V_t = \frac{-1}{\gamma} e^{AI_t + DX_t} \left[(\frac{dA}{dt} I_t + \frac{dD}{dt} X_t) g + \frac{\partial g}{\partial t} \right], \\ &V_X = DV, \qquad V_{XX} = D^2 V, \qquad V_I = AV, \qquad V_{II} = A^2 V, \\ &V_{XI} = ADV, \qquad V_{\lambda^U} = \frac{-1}{\gamma} e^{AI_t + DX_t} \frac{\partial g}{\partial \lambda^U}, \qquad V_{\lambda^D} = \frac{-1}{\gamma} e^{AI_t + DX_t} \frac{\partial g}{\partial \lambda^D}, \end{aligned} \tag{4.1}$$

and the value functions right after price jumps become

$$\begin{aligned} V^U &= \frac{-1}{\gamma} e^{AI_t + D(X_t + \pi_t J_t^U)} g(t, \lambda_t^U + \eta_{11}, \lambda_t^D + \eta_{21}), \\ V^D &= \frac{-1}{\gamma} e^{AI_t + D(X_t + \pi_t J_t^D)} g(t, \lambda_t^U + \eta_{12}, \lambda_t^D + \eta_{22}). \end{aligned} \tag{4.2}$$

Therefore, the HJB equation (3.5) can be rewritten as

$$\begin{aligned} 0 = &-(f(t,t) + \beta) V + \frac{-1}{\gamma} e^{AI_t + DX_t} \left[(\frac{dA}{dt} I_t + \frac{dD}{dt} X_t) g + \frac{\partial g}{\partial t} \right] + (rX_t + I_t) DV + (\theta - kI_t) AV \\ &+ \alpha^U (\bar{\lambda}^U - \lambda_t^U) \frac{-1}{\gamma} e^{AI_t + DX_t} \frac{\partial g}{\partial \lambda^U} + \alpha^D (\bar{\lambda}^D - \lambda_t^D) \frac{-1}{\gamma} e^{AI_t + DX_t} \frac{\partial g}{\partial \lambda^D} + 0.5 \tilde{\sigma}^2 A^2 V \\ &+ \max_{\pi} \pi_t (\mu - r) DV + 0.5 \sigma^2 \pi_t^2 D^2 V + \rho \pi_t \sigma \tilde{\sigma} ADV \\ &\quad + \lambda_t^U \mathbb{E} \left[\frac{-1}{\gamma} e^{AI_t + D(X_t + \pi_t J_t^U)} g(t, \lambda_t^U + \eta_{11}, \lambda_t^D + \eta_{21}) - V \right] \\ &\quad + \lambda_t^D \mathbb{E} \left[\frac{-1}{\gamma} e^{AI_t + D(X_t + \pi_t J_t^D)} g(t, \lambda_t^U + \eta_{12}, \lambda_t^D + \eta_{22}) - V \right] \\ &+ \max_{C} -C_t DV + \frac{-1}{\gamma} e^{-\gamma C_t} \\ &+ \max_{P} -P_t DV + f(t,t) \frac{-1}{\gamma} e^{-\gamma (X_t + \frac{P_t}{f(t,t)})}. \end{aligned} \tag{4.3}$$

By the first condition, we can rewrite (4.1) as

$$\begin{aligned} C_t^* &= -\frac{1}{\gamma} \ln(DV), \\ P_t^* &= -f(t,t) \left(\frac{\ln(DV)}{\gamma} + X_t \right), \\ \pi_t^* &= \arg\min_{\pi} H(\pi_t, g(t, \lambda_t^U, \lambda_t^D)), \end{aligned} \tag{4.4}$$

where

$$H(\pi_t, g(t,\lambda_t^U,\lambda_t^D)) = \pi_t(\mu - r)D + 0.5\sigma^2\pi_t^2 D^2 + \rho\pi_t\sigma\tilde{\sigma}AD + \lambda_t^U \frac{g(t,\lambda_t^U+\eta_{11},\lambda_t^D+\eta_{21})}{g(t,\lambda_t^U,\lambda_t^D)}\mathbb{E}\left[e^{D\pi_t J_t^U}-1\right] + \lambda_t^D \frac{g(t,\lambda_t^U+\eta_{12},\lambda_t^D+\eta_{22})}{g(t,\lambda_t^U,\lambda_t^D)}\mathbb{E}\left[e^{D\pi_t J_t^D}-1\right]. \tag{4.5}$$

Remark 4.1. Note that in (4.5), $H(\pi_t, g(t,\lambda_t^U,\lambda_t^D))$ depends on the moment generating function of J_t^U and J_t^D. For simple jump distributions, we can apply the affine structure in (3.8) to obtain explicit formulas for the value function and the optimal controls. We include some examples in the Appendix D.

Now we can substitute the optimal controls in (4.4) into the HJB equation (4.3) and match the coefficient of X_t. Then, we can obtain

$$\frac{dD}{dt} + rD + f(t,t)D + \frac{f(t,t)+1}{\gamma}D^2 = 0, \tag{4.6}$$

with the terminal condition $D(T,T) = -\gamma$. It is straightforward to check that $D(t,T)$ admits the closed-form solution:

$$D(t,T) = \left[-\frac{f(t,t)+1}{\gamma}\frac{1}{r+f(t,t)}\left(1-e^{-\int_t^T r+f(u,u)du}\right) - \frac{1}{\gamma}e^{-\int_t^T r+\lambda^m(u))du}\right]^{-1}. \tag{4.7}$$

Similarly, matching the coefficient of I_t, we have

$$\frac{dA}{dt} + D - kA + \frac{1+f(t,t)}{\gamma}AD = 0, \tag{4.8}$$

with the terminal condition $A(T,T) = 0$. Therefore, $A(t,T)$ has the closed-form solution:

$$A(t,T) = \int_t^T D(s,T)e^{\int_t^s\left(\frac{f(u,u)+1}{\gamma}D(u,T)-k\right)du}ds. \tag{4.9}$$

After applying the closed-form solutions of $A(t,T)$ in (4.7) and $D(t,T)$ in (4.9) to reduce the terms of X_t and I_t, we can simplify the HJB equation (4.3) as:

$$\begin{aligned} 0 = &-(f(t,t)+\beta)g + \frac{\partial g}{\partial t} + \theta Ag + 0.5\tilde{\sigma}^2A^2g \\ &+ \alpha^U(\bar{\lambda}^U - \lambda_t^U)\frac{\partial g}{\partial\lambda^U} + \alpha^D(\bar{\lambda}^D - \lambda_t^D)\frac{\partial g}{\partial\lambda^D} \\ &+ \lambda_t^U(g(t,\lambda_t^U+\eta_{11},\lambda_t^D+\eta_{21}) - g) + \lambda_t^D(g(t,\lambda_t^U+\eta_{12},\lambda_t^D+\eta_{22}) - g) \\ &+ H(\pi_t^*, g(t,\lambda_t^U,\lambda_t^D))g + \frac{f(t,t)+1}{\gamma}Dg\left(\ln\left(\frac{-D}{\gamma}\right) + \ln g - 1\right). \end{aligned} \tag{4.10}$$

Then we can define the infinitesimal generator $\mathcal{A}$ for the jump intensity processes by

$$\begin{aligned} \{\mathcal{A}g\}(t,\lambda_t^U,\lambda_t^D) = &\frac{\partial g}{\partial t} + \alpha^U(\bar{\lambda}^U - \lambda_t^U)\frac{\partial g}{\partial\lambda^U} + \alpha^D(\bar{\lambda}^D - \lambda_t^D)\frac{\partial g}{\partial\lambda^D} \\ &+ \lambda_t^U(g(t,\lambda_t^U+\eta_{11},\lambda_t^D+\eta_{21}) - g) + \lambda_t^D(g(t,\lambda_t^U+\eta_{12},\lambda_t^D+\eta_{22}) - g), \end{aligned} \tag{4.11}$$

and define $Q(t,T) = \theta A(t,T) - f(t,t) - \beta + 0.5\tilde{\sigma}^2 A(t,T)^2 + \frac{f(t,t)+1}{\gamma} D(t,T)(\ln\left(\frac{-D(t,T)}{\gamma}\right) - 1)$. We know the function $g(t, \lambda_t^U, \lambda_t^D)$ must satisfy

$$\begin{aligned}&\{\mathcal{A}g\}(t, \lambda_t^U, \lambda_t^D) + Q(t,T) g(t, \lambda_t^U, \lambda_t^D) + J(t, g(t, \lambda_t^U, \lambda_t^D)) = 0, \\ &J(t, g(t, \lambda_t^U, \lambda_t^D)) = g(t, \lambda_t^U, \lambda_t^D)\left(\frac{f(t,t)+1}{\gamma} D(t,T) \ln g(t, \lambda_t^U, \lambda_t^D) + H(\pi_t^*, g(t, \lambda_t^U, \lambda_t^D))\right).\end{aligned} \tag{4.12}$$

By Feynman-Kac formula, we can express $g(t, \lambda_t^U, \lambda_t^D)$ as the following conditional expectation,

$$g(t, \lambda_t^U, \lambda_t^D) = \mathbb{E}\left[\int_t^T e^{\int_t^s Q(u,T)du} J(s, g(s, \lambda_s^U, \lambda_s^D)) ds + e^{\int_t^T Q(s,T)ds} | \mathcal{F}_t\right]. \tag{4.13}$$

Note that in (4.13), we have an implicit solution of $g(t, \lambda_t^U, \lambda_t^D)$ and need a numerical scheme to solve it. We follow the work of Delong and Klüppelberg (2008) and Aït-Sahalia and Hurd (2015), and write the solution of $g(t, \lambda_t^U, \lambda_t^D)$ as an iterative scheme $\{g^{(n)}\}$ for $n = 0, 1, \cdots$, $g^{(0)} = 1$, and $g^{(n+1)} = \mathbb{E}\left[\int_t^T e^{\int_t^s Q(u,T)du} J(s, g^{(n)}(s, \lambda_s^U, \lambda_s^D)) ds + e^{\int_t^T Q(s,T)ds} | \mathcal{F}_t\right]$. We provide the proof of the existence and uniqueness of the solution to (4.13) in the next section and we will discuss this numerical scheme in more details later.

4.2 Convergence of Iterations

We introduce an operator $\mathcal{L}$ acting on function g in (4.13) with

$$(\mathcal{L}g)(t, \lambda_t^U, \lambda_t^D) = \mathbb{E}\left[\int_t^T e^{\int_t^s Q(u,T)du} J(s, g(s, \lambda_s^U, \lambda_s^D)) ds + e^{\int_t^T Q(s,T)ds} | \mathcal{F}_t\right]. \tag{4.14}$$

Then, the solution to (4.13) becomes the solution to the following fixed point equation

$$(\mathcal{L}g)(t, \lambda_t^U, \lambda_t^D) = g(t, \lambda_t^U, \lambda_t^D). \tag{4.15}$$

Based on Banach Fixed Point Theorem, to prove the existence and uniqueness of the solution to (4.13), we only need to prove that the operator $\mathcal{L}$ is a contraction mapping. Then, the iterative method introduced in Section 4.3 will converge. According to (4.5), we know $H(\cdot, \cdot)$ depends on the jump size distributions. Therefore, in this proof, we suppose that the jump sizes follow *i.i.d.* exponential distributions and for other distributions, one can attempt to verify that the operator $\mathcal{L}$ is also a contraction mapping and we do not attempt this here.

Lemma 4.2. *The operator $\mathcal{L}$ is bounded with*

$$\begin{aligned}&(\mathcal{L}g)(t, \lambda_t^U, \lambda_t^D) < \alpha_1, \\ &(\mathcal{L}g)(t, \lambda_t^U, \lambda_t^D) > \alpha_2,\end{aligned} \tag{4.16}$$

where α_1 and α_2 are positive constant numbers.

The proof of Lemma 4.2 is included Appendix E.

Now, we define the space of continuous functions ζ on $[0,T]\times(0,\infty)\times(0,\infty)$ by $\mathfrak{C}_e([0,T]\times(0,\infty)\times(0,\infty))$ with $\alpha_2<\zeta(t,\lambda_t^U,\lambda_t^D)<\alpha_1$ and $\zeta(t,\lambda_t^U,\lambda_t^D)=e^{\tilde{C}(t,T)+\tilde{B}^U(t,T)\lambda_t^U+\tilde{B}^D(t,T)\lambda_t^D}$ for some bounded functions $\tilde{C}(t,T)$, $\tilde{B}^U(t,T)$, and $\tilde{B}^D(t,T)$. And we define a metric on $\mathfrak{C}_e([0,T]\times(0,\infty)\times(0,\infty))$ by

$$d(f,h)=\sup_{(t,\lambda_t^U,\lambda_t^D)\in[0,T]\times(0,\infty)\times(0,\infty)}\left|e^{-\alpha(T-t)}\left(f(t,\lambda_t^U,\lambda_t^D)-h(t,\lambda_t^U,\lambda_t^D)\right)\right|, \tag{4.17}$$

for some positive α to be specified later.

Then, our objective is to prove $d(\mathcal{L}f,\mathcal{L}h)\le\phi d(f,h)$ for any two functions $f,h\in\mathfrak{C}_e([0,T]\times(0,\infty)\times(0,\infty))$ and for some $\phi\in[0,1)$.

$$\begin{aligned}
d(\mathcal{L}f,\mathcal{L}h)&=\sup_{(t,\lambda_t^U,\lambda_t^D)}\left|e^{-\alpha(T-t)}\left(\mathcal{L}f-\mathcal{L}h\right)\right|\\
&=\sup_{(t,\lambda_t^U,\lambda_t^D)}\left|e^{-\alpha(T-t)}\mathbb{E}\left[\int_t^T e^{\int_t^s Q(v,T)dv}\left(J(s,f(s,\lambda_s^U,\lambda_s^D))-J(s,h(s,\lambda_s^U,\lambda_s^D))\right)ds|\mathcal{F}_t\right]\right|\\
&=\sup_{(t,\lambda_t^U,\lambda_t^D)}\left|e^{-\alpha(T-t)})\mathbb{E}\left[\int_t^T e^{\int_t^s Q(v,T)dv}\left(\frac{f(t,t)+1}{\gamma}D(t,T)(f\ln f-h\ln h)\right.\right.\right.\\
&\qquad\left.\left.\left.+fH(\pi_t^1,f)-hH(\pi_t^2,h)\right)ds|\mathcal{F}_t\right]\right|,
\end{aligned} \tag{4.18}$$

where we use π_t^1 and π_t^2 to denote the optimal controls corresponding to f and h, respectively.

Lemma 4.3. *For any two functions $f,h\in\mathfrak{C}_e([0,T]\times(0,\infty)\times(0,\infty))$, let π_t^1 and π_t^2 be the optimal controls in (4.4) corresponding to f and h, respectively. We have*

$$fH(\pi_t^1,f)-hH(\pi_t^2,h)=(f-h)\Phi(x), \tag{4.19}$$

where $\Phi(x)$ is a polynomial of degree three and $|\Phi(x)|$ is bounded.

The proof of Lemma 4.3 is included in Appendix F.

Theorem 4.4. *The mapping $\mathcal{L}$: $[0,T]\times(0,\infty)\times(0,\infty)\to\mathfrak{C}_e([0,T]\times(0,\infty)\times(0,\infty))$ is a contraction mapping with respect to the metric (4.17) with $\alpha>\alpha_3+\bar{Q}(t,T)$, where $\bar{Q}(t,T)=\max_{v\in(t,T)}Q(v,T)$ and α_3 is a positive constant number.*

The proof of Theorem 4.4 is included in the Appendix G.

Now, according to Banach fixed point theorem, we prove the existence and uniqueness of the solution to (4.13). Then, we can apply a convergent iterative method to derive the solution, which is included in the next section.

4.3 Numerical Scheme

We provide a numerical scheme to solve $g(t, \lambda_t^U, \lambda_t^D)$ according to the contraction mapping

$$\begin{aligned} g^{(n+1)} &= \mathcal{G}(g^{(n)}), \\ \mathcal{I}(g) &= \int_t^T e^{\int_t^s Q(u,T)du} J(s, g(s, \lambda_s^U, \lambda_s^D))ds + e^{\int_t^T Q(s,T)ds}, \\ \mathcal{G}(g) &= \mathbb{E}[\mathcal{I}(g)|\mathcal{F}_t], \end{aligned} \tag{4.20}$$

with the initial guess $g^{(0)} = 1$ for all t, λ_t^U, and λ_t^D.

1. Approximate the range of the three variables with grid points $(t_i, \lambda^{Uj}, \lambda^{Dk})$ with dimension $N \times MU \times MD$ and set $g^{(0)} = 1$ for all the grid points.

2. Using each grid point as initial values, we follow *Ogata's modified thinning algorithm* (Embrechts et al., 2011) to generate M groups of the jump intensities process $\lambda_{t_m}^U$, $\lambda_{t_m}^D$ at each time node.

3. According to the closed-form solutions of $A(t, T)$ and $D(t, T)$, compute their values at the mth time node for the lth simulation, and denote them as $A^l(t_m, T)$ and $D^l(t_m, T)$. Then, we can get $Q^l(t_m, T)$.

4. Given the value for $\lambda_{t_m}^U, \lambda_{t_m}^U, A^l(t_m, T)$, $D^l(t_m, T)$, and $g^{(n)}$ at the mth time node of the lth simulation and according to (4.4) and (4.5), we can use a simple numerical method to get the corresponding optimal control π^*. Then, with that optimal control π^*, we can get $J_m^l(t, g^{(n)}(t, \lambda^U, \lambda_t^D))$ at the mth time node of the lth simulation.

5. Compute the integration in (4.20) for each simulation l by

$$\mathcal{I}^l(g^{(n)}) = \tilde{t} \sum_{m=i}^{N} \left[e^{\tilde{t} \sum_{n=1}^{i} Q^l(t_n,T)} J_m^l(t, g^{(n)}(t, \lambda^U, \lambda_t^D)) \right] + e^{\tilde{t} \sum_{m=i}^{N} Q^l(t_m T)}, \tag{4.21}$$

   where $\tilde{t}$ is the step size between each time node. Then, by the law of large numbers, the expectation operator in (4.20) can be calculated as the mean of all M simulations,

$$g^{(n+1)} = \mathcal{G}(g^{(n)}) = \frac{\sum_{l=1}^{M} \mathcal{I}(g^{(n)})}{M}. \tag{4.22}$$

6. Repeat Steps 2 to 5 to update $g^{(n+1)}$ for all the grid points until the convergence of $g^{(n)}$ for each grid point.

5 Numerical Results

In this section, we present numerical results to gain more insights of the optimal controls under different parameters. First, we assume that the price jump-up sizes and price jump-down sizes both follow exponential distributions and apply most of the parameter settings in Hainaut (2017), such as those of Hawkes processes, bond and stock dynamics. Second, we assume that the long-term mean level, the speed of reversion and the instantaneous volatility of the unknown income process to be 0.06, 0.02, and 0.15. Third, to reflect the effect of the stochastic income, we set the initial wealth level and income level to be 1 unit. Finally, we assume that the wage earner will retire in 10 years, with $T = 10$. And to simplify computation, we suppose that there are 4 factors that dominate the hazard function and the mortality rate for τ_d, with $p_i = 0.25$, for $i = 1, 2, 3, 4$ and $\lambda_1^m = 0.05, \lambda_2^m = 0.03, \lambda_3^m = 0.02$, and $\lambda_4^m = 0.015$. Other parameters are reported in Table 5.1.

$\bar{\lambda}^U$	$\bar{\lambda}^D$	α^U	α^D	η_{11}	η_{12}	η_{21}	η_{22}	β	λ
0.48	0.48	19.91	19.91	0.1	-0.1	-0.1	0.1	0.8	20
r	μ	σ	θ	k	$\tilde{\sigma}$	ρ	T	γ	
0.02	0.05	0.2	0.06	0.02	0.15	0	10	8	

Table 5.1: Parameters and Values

In addition, following Section 4, we assume that the jump size satisfies independent exponential distributions, which can produce general but not too complex results. Let u and d denote the magnitudes of price jump-up and price jump-down sizes, with $u > 0$ and $d < 0$. Then, we have

$$\begin{aligned} &\mathbb{P}\{J^U \leq u\} = 1 - e^{-\lambda u}, \text{with } u \in (0, \infty), \\ &\mathbb{P}\{|J^D| \leq |d|\} = 1 - e^{\lambda d}, \text{with } d \in (-\infty, 0). \end{aligned} \tag{5.1}$$

Then, the moment generating functions can be easily found, with $\mathbb{E}[e^{J^U t}] = \frac{\lambda}{\lambda - t}$, for $t < \lambda$ and $\mathbb{E}[e^{J^D s}] = \frac{\lambda}{\lambda + s}$, for $s > -\lambda$. As $\mathbb{E}[J^U] = \frac{1}{\lambda}$ and $\mathbb{E}[J^D] = -\frac{1}{\lambda}$, we know the average jump-up and jump-down size should be 5% and -5% of the stock price, respectively, if we set $\lambda = 20$.

5.1 Value Function

Under our assumptions and the parameters in Table 5.1, we present behaviors of the value function in the following figures. In Figure 5.1, we show how the value function varies with different initial jump-up and jump-down intensities. First, when the jump-down intensity is relatively large, the value function decreases as the jump-up intensity λ_t^U increases, for example, $\lambda_t^D = 4$. This is because when the probability of price jump-down is high, the wage earner can short the risky stock to obtain higher returns. However, as the intensity of jump-up increases, the probability of price jump-up increases. The wage earner hesitates to short the risky stock, which means he holds a large

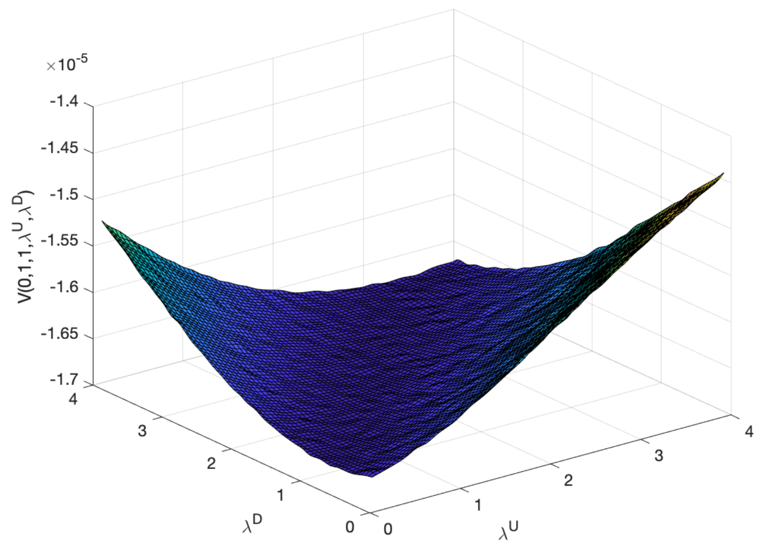

Figure 5.1: Value Function with Different Jump Intensities.

proportion of wealth in the risk-free bond. Similar results hold when λ_t^U is relatively small. The value function decreases as λ_t^D decreases. Second, the wage earner obtains large value functions at two extreme cases. The first is when $\lambda_t^U = 0$ and $\lambda_t^D = 4$, the wage earner can take advantage of the short position of the risky stock. The second is when $\lambda_t^U = 4$ and $\lambda_t^D = 0$, the wage earner can take advantage of the long position of the risky stock.

In Figure 5.2, we present how the value function varies according to different initial time t in the cases of three different groups of jump intensities and the case without price jumps. First, we notice that the value function deceases as t increases. The reason is that we fix the initial wealth level and income level to be 1 unit. As t increases to the fixed retirement time $T = 10$, there is less time for the income and wealth level to grow up. Second, the value function without price jumps is very similar with the value function with very small jump-up and jump-down intensities, shown by the green curve and the red curve. And the red curve is slightly higher than the green curve. Third, the value function presented by the blue curve with a larger initial jump-up intensity is always bigger than that presented by the red curve for any $t \in [0, 10]$. And value function of the black curve with a larger initial jump-down intensity is also bigger than that of the red curve, but the difference between the black curve and the red curve is smaller than that between the blue curve and the red curve. In other words, when the probability of price jump-up is high, the wage earner can allocate more wealth into the risky stock to obtain better expected utilities. When the probability of price jump-down is high, the wage earner can also short the risky stock to get better expected utilities. But the effect of price jump-up is much stronger .

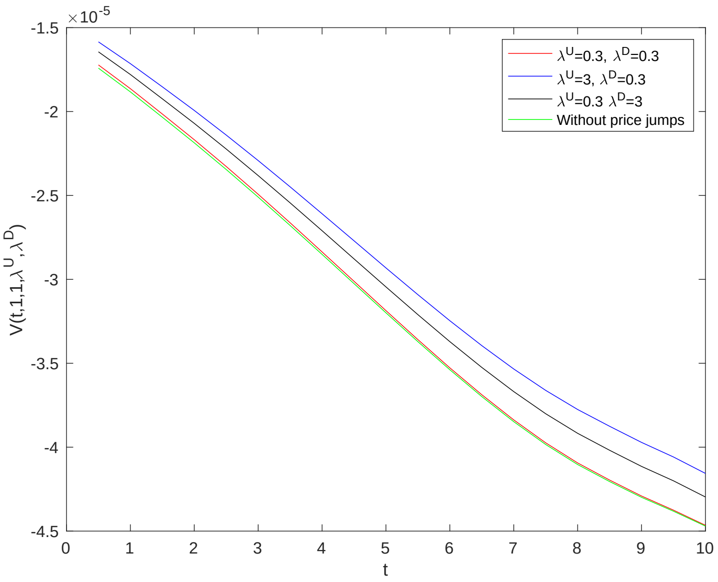

Figure 5.2: Value Function with Different Initial time.

5.2 Optimal Controls

Next, we present the behavior of the optimal control policies and examine the effect of different jump intensities to those policies. It is well documented that for the exponential utility with risk aversion parameter $\gamma > 0$, the wage earner can short any assets and consume at a negative rate to obtain higher utilities.

In Figure 5.3, we plot the optimal allocation policy at time 0 with respect to different jump-up and jump-down intensities. The optimal allocation amount in the risky stock increases as λ_t^D decreases or λ_t^U increases. In other words, the higher probability that the next price jump-up happens before the next price jump-down, the more wealth the wage earner will allocate into the risky stock.

In Figure 5.4, we examine the effect of the time to the optimal allocation policy. We consider three cases with different jump-up and jump-down intensities and one case without price jumps. First, even small probabilities of price jumps motivate the wage earner to invest more wealth into the risky stock, shown by the red curve and the green curve. However, this effect diminishes over time. Second, when the jump-up intensity is relatively large, the wage earner shorts the risk-free bond to long much more risky stock. But the amount allocated in the risky stock is reduced over time, shown by the blue curve. Finally, when the jump-down intensity is relatively large, the wage earner shorts the risky stock to long much more bonds. But the amount allocated in the risky stock increases with time, shown by the black curve. Therefore, the jump intensities have significant effects on the

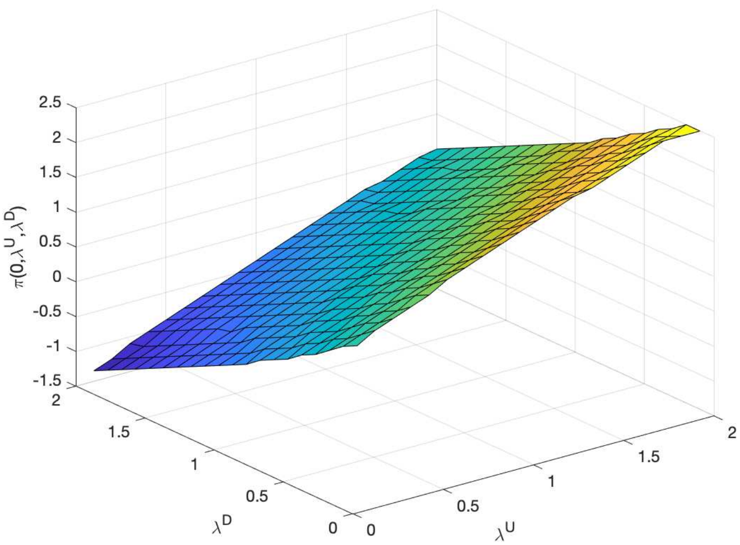

Figure 5.3: Optimal Allocation Policy at Time 0 with Different Jump Intensities.

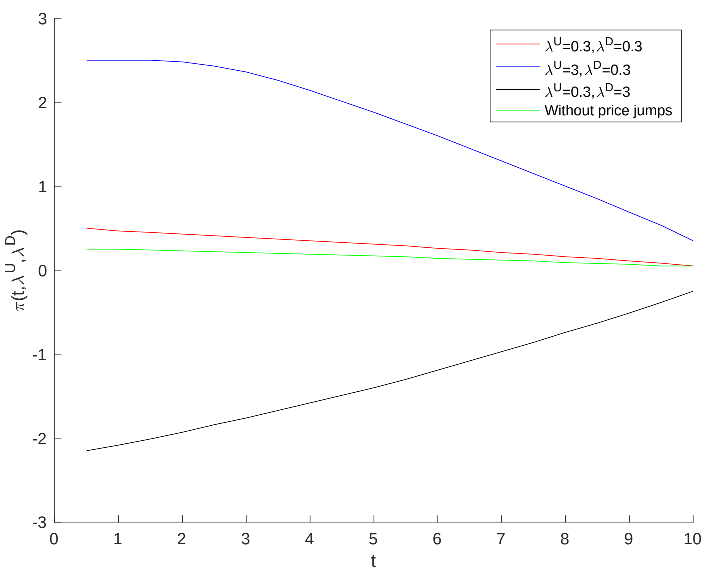

Figure 5.4: Optimal Allocation Policy with Different Initial Time.

optimal allocation policy and those effects diminish as time goes to the fixed retirement date.

In Figure 5.5, we fix the income level and the jump intensities to present the behavior of the optimal consumption policy with respect to different time and wealth levels. When t is small, the optimal consumption amount does not vary much with wealth levels. However, as time moves to the retirement date, the differences become much more significant. First, when the wealth level is small, the optimal consumption amount decreases as time goes to the fixed retirement date. This is because initially the wage earner consumes more than his total wealth, which means he will use his future labor income to finance current consumption. But this financing ability decreases as time goes to the retirement date. Second, when the wealth level is large, the optimal consumption amount increases as time goes to the fixed retirement date. Especially, when $t = 10$, the fixed retirement date, the corresponding optimal consumption policy is equal to the wealth level, which can be easily proved by plugging the terminal conditions into (4.4). This is because we set the same utility function for the intermediate consumption and the heritage. Therefore, at the retirement date, consuming all assets or leaving all assets as heritage will result in the same utility. We will explore the case with different risk-aversion constants for the intermediate consumption and the heritage in our future research.

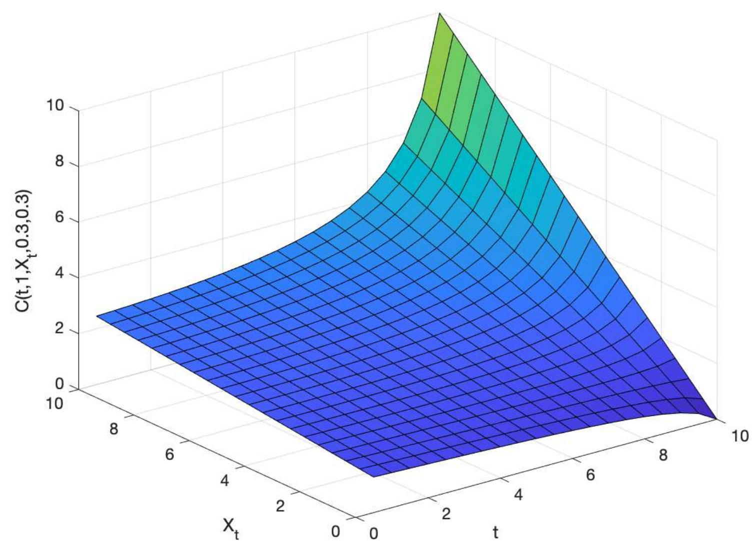

Figure 5.5: Optimal Consumption Policy with Different Time and Wealth Level.

In Figure 5.6 and 5.7, we fix the wealth level and the income level to be 1 unit and analyze the behaviors of the optimal consumption policy and the life insurance policy with respect to different time and jump intensities. We notice that the differences between the policies under the three groups of jump intensities are very small, which means that the jump intensities have very limited effect on the optimal consumption and life insurance policies.

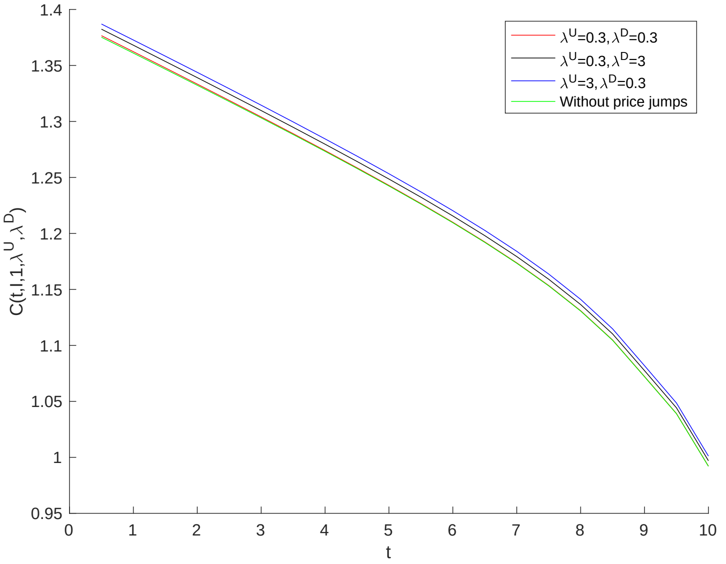

Figure 5.6: Optimal Consumption Policy with Different Initial Time.

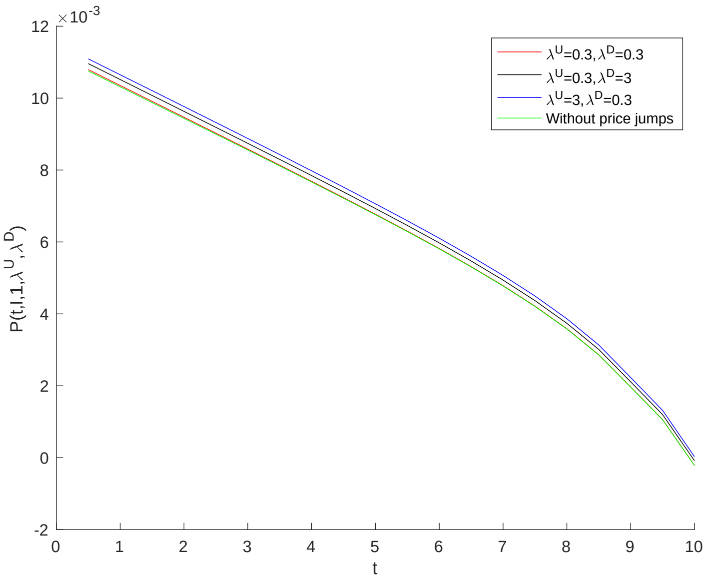

Figure 5.7: Optimal Life Insurance Policy with Different Initial Time.

5.3 Impact of Correlation between Wealth and Income

In above sections, we assume zero correlation between the stochastic income process and the dynamics of the risky stock. To be more realistic, we recalculate our model under another two situations, where the income process and the stock process are positively or negatively correlated.

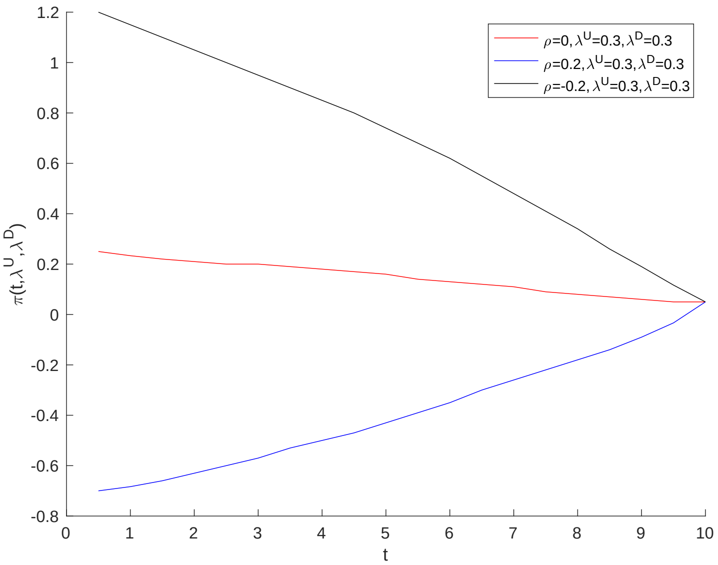

Figure 5.8: Optimal Allocation Policy with Different ρ.

In Figure 5.8, we present the optimal allocation policy under the different situations. We find that the correlation coefficient has critical influences on the optimal allocation policy. First, when the stochastic income process is independent with the wealth process, the optimal allocation policy is relatively stable, reflected by the red curve. Second, in a negatively correlated market, even though the probability of price jump-up and jump-down is small, the wage earner is willing to short the risk-free bond to purchase about 1.2 units stock initially. And the optimal allocation amount decreases a lot as time goes to the fixed retirement date, presented by the black curve. Finally, in a positively correlated market, the wage earner initially short the risky stock. And the optimal allocation amount increases a lot as time goes to the fixed retirement date, presented by the blue curve. This is because the stochastic income and the gains from the financial market are the only two sources for the wage earner to consume and purchase life insurance. In a positively correlated market, it is possible to see the two sources to drop together. Then, the wage earner has to decrease his total risks by reducing his stock holdings.

In Figure 5.9 and 5.10, we demonstrate the optimal consumption and life insurance purchase policies under the three kinds of markets. The correlation coefficient has very limited effect on those

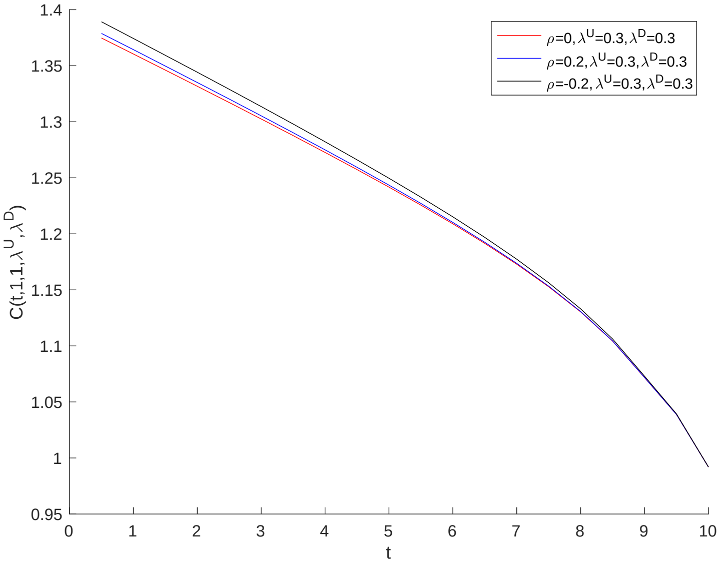

Figure 5.9: Optimal Consumption Policy with Different ρ.

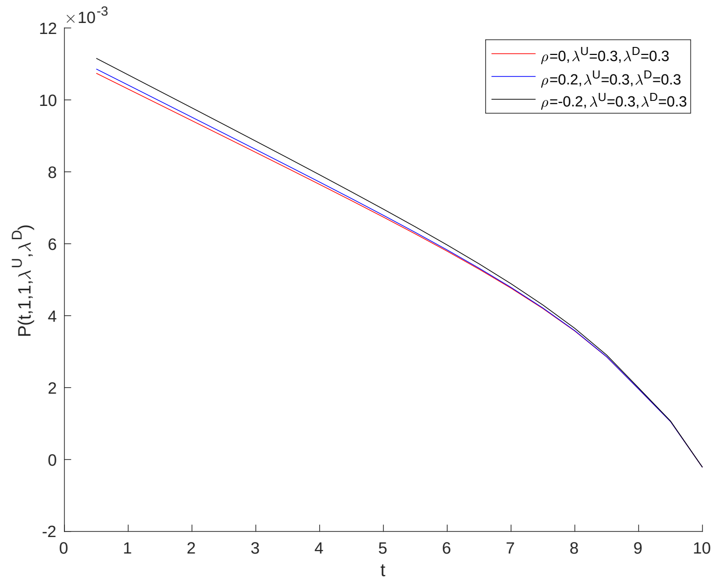

Figure 5.10: Optimal Life Insurance Policy with Different ρ.

two policies. In both the positively and negatively correlated market, the wage earner increases the consumption amount a little bit. But the magnitude is larger in the negatively correlated market than that in the positively correlated market. Similar results hold for the life insurance policy.

6 Concluding Remarks

This study extends the Hawkes jump model in Aït-Sahalia and Hurd (2015) to investigate the impact of the contagion phenomenon and the stochastic income on the asset allocation, consumption, and life insurance purchase policies. This framework presents several interesting findings. First, we construct the target function of a wage earner subject to a random death time and a fixed retirement date from both the life-time consumption and the heritage, where a general mortality distribution is presented to describe the randomness of the death time. Closed-form solutions of optimal controls are obtained in special cases. Second, the numerical results demonstrate that the jump intensities have a significant impact on the asset allocation policy but small effect on the optimal consumption and life insurance policies with given wealth and income levels. Finally, the correlation coefficient between the stock dynamics and the income process has critical impacts on the optimal controls. In a negatively correlated market, the wage earner allocates much more wealth into the risky stock and consumes a little bit more. However, in a positively correlated market, the wage earner dramatically decreases the risky holdings but also increases the consumption amount a little bit.

Recent financial market witnesses the value of investing in illiquid assets, such as real estate, private company interests, and some types of arts. The liquidity of those assets is highly influenced by event risks. Ang et al. (2014) introduced trading opportunities depending on the arrival of a randomly occurring events to consider the liquidity risk of investing in those illiquid assets. In our previous work, Jin et al. (2020) incorporated the liquidity risk with price jumps in a regime-switching model. Instead of modeling the liquidity risk by an independent Poisson process, we will apply the Hawkes process to describe the the price jumps and the trading opportunities in our future work. Then, we will consider an optimal control problem under a mutual-exciting market, which means the arrival of a trading opportunity and the occurrence of price jumps will influence each other. The mutual exciting process will make the dynamic wealth process much more complicated. However, adding the real estate into a family's asset portfolio is more realistic and versatile than the traditional two-asset model for a household's wealth management.

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Appendix

A Proof of Lemma 3.1

Following the work of Pliska and Ye (2007), the target function (3.3) can be rewritten as

$$\begin{aligned}
V(t, X_t, I_t, \lambda_t^U, \lambda_t^D) = &\max_{\{C,\pi,P\}} \mathbb{E}\Big[\big(\int_t^{\tau_d} e^{-\beta(s-t)}U(C_s)ds + e^{-\beta(\tau_d-t)}U(X_{\tau_d}, P_{\tau_d})\big)F(T,t) \\
&+ \big(\int_t^T e^{-\beta(s-t)}U(C_s)ds + e^{-\beta(T-t)}U(X_T)\big)\bar{F}(T,t)\Big] \\
= &\max_{\{C,\pi,P\}} \mathbb{E}\Big[\int_t^T f(s,t)ds\big[\int_t^s e^{-\beta(u-t)}U(C_u)du + e^{\beta(s-t)}U(X_s,P_s)\big] \\
&+ \bar{F}(T,t)\big[\int_t^T e^{-\beta(s-t)}U(C_s) + e^{-\beta(T-t)}U(X_T)\big]\Big].
\end{aligned} \tag{A.1}$$

Then, consider the first term of equation (A.1), we have

$$\begin{aligned}
\int_t^T f(s,t)ds\big[e^{-\beta(s-t)}U(C_s)ds + e^{-\beta(s-t)}U(X_s,P_s)\big] = \\
\int_t^T \int_t^s f(s,t)e^{-\beta(u-t)}U(C_u)duds \\
+ \int_t^T f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds.
\end{aligned} \tag{A.2}$$

After changing the order of integration, equation (A.2) becomes

$$\begin{aligned}
&\int_t^T \int_u^T f(s,t)e^{-\beta(u-t)}U(C_u)dsdu + \int_t^T f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds \\
= &\int_t^T (\bar{F}(u,t) - \bar{F}(T,t))e^{-\beta(u-t)}U(C_u)du + \int_t^T f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds.
\end{aligned} \tag{A.3}$$

We can directly put equation (A.3) into the target function (A.1), and replace u by s, we will complete the proof of Lemma 3.1. □

B Proof of Lemma 3.2

According to the dynamics of X_t and I_t and apply *Ito*'s lemma to the target function, we have:

$$\begin{aligned}
dV(t,\cdot) = &V_t dt + V_X dX_t + V_I dI_t + V_{\lambda^U} d\lambda^U + V_{\lambda^D} d\lambda^D + 0.5V_{xx}(dX_t)^2 \\
&+ 0.5V_{II}(dI_t)^2 + V_{XI}dX_t dI_t + [V^U - V]dN_t^U + [V^D - V]dN_t^D,
\end{aligned} \tag{B.1}$$

with V^U being the value function after price jump-up of risky stocks S_t, and V^D being the value function after price jump-down.

Then, for any positive $u \in (t,T)$, we can restate the target function (3.3) as

$$
\begin{aligned}
V(t,\cdot) = \max_{\{C,\pi,P\}} \mathbb{E}\Big[& \int_t^u \bar{F}(s,t)e^{-\beta(s-t)}U(C_s) + f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds \\
& + \int_u^T \bar{F}(s,t)e^{-\beta(s-t)}U(C_s) + f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds \\
& + \bar{F}(T,t)e^{-\beta(T-t)}U(X_s)|\mathcal{F}_t\Big] \\
= \max_{\{C,\pi,P\}} \mathbb{E}\Big[& \int_t^u \bar{F}(s,t)e^{-\beta(s-t)}U(C_s) + f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds \\
& + \int_u^T \bar{F}(s,u)\bar{F}(u,t)e^{-\beta(s-u)-\beta(u-t)}U(C_s) \\
& + f(s,u)\bar{F}(u,t)e^{-\beta(s-u)-\beta(u-t)}U(X_s,P_s)ds \\
& + \bar{F}(T,u)\bar{F}(u,t)e^{-\beta(T-u)-\beta(u-t)}U(X_s)|\mathcal{F}_t\Big],
\end{aligned}
\tag{B.2}
$$

then, combine the common factor, we get

$$
\begin{aligned}
V(t,\cdot) = \max_{\{C,\pi,P\}} \mathbb{E}\Big[& \int_t^u \bar{F}(s,t)e^{-\beta(s-t)}U(C_s) + f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds \\
& + \bar{F}(u,t)e^{-\beta(u-t)}\big[\int_u^T \bar{F}(s,u)e^{-\beta(s-u)}U(C_s) + f(s,u)e^{-\beta(s-u)}U(X_s,P_s)ds \\
& + \bar{F}(T,u)e^{-\beta(T-u)}U(X_s)\big]|\mathcal{F}_t\Big].
\end{aligned}
\tag{B.3}
$$

Finally, compare with the target function (3.3), we easily complete the proof of Lemma 3.2. □

C Derivation of the HJB equation

For any positive time t and $t+h$, with $h>0$, by Lemma 3.2, we have

$$
\begin{aligned}
V(t,\cdot) = \max_{\{C,\pi,P\}} \mathbb{E}\Big[& e^{-\beta h}\bar{F}(t+h,t)V(t+h,\cdot) \\
& + \int_t^{t+h} \bar{F}(s,t)e^{-\beta(s-t)}U(C_s) + f(s,t)e^{-\beta(s-t)}U(X_s,P_s)ds|\mathcal{F}_t\Big].
\end{aligned}
\tag{C.1}
$$

Note that for h small enough,

$$
\begin{aligned}
e^{-\beta h}\bar{F}(t+h,t) &= \frac{e^{-\beta h}\sum_{i=1}^n p_i \exp\Big(-\int_0^{t+h}\lambda_i^m(u)du\Big)}{\sum_{i=1}^n p_i \exp\Big(-\int_0^t \lambda_i^m(u)du\Big)} \\
&= \frac{\sum_{i=1}^n p_i \exp\Big(-\int_0^t \lambda_i^m(u)du\Big)\exp\Big(-\int_t^{t+h}(\lambda_i^m(u)+\beta)du\Big)}{\sum_{i=1}^n p_i \exp\Big(-\int_0^t \lambda_i^m(u)du\Big)} \\
&= \frac{\sum_{i=1}^n p_i \exp\Big(-\int_0^t \lambda_i^m(u)du\Big)(1-(\lambda_i^m(t)+\beta)h+o(h^2))}{\sum_{i=1}^n p_i \exp\Big(-\int_0^t \lambda_i^m(u)du\Big)} \\
&= 1-(f(t,t)+\beta)h+0(h^2).
\end{aligned}
\tag{C.2}
$$

We have

$$
\begin{aligned}
V(t,\cdot) &= \max_{\{C,\pi,P\}} \mathbb{E}\Big[(1-(f(t,t)+\beta)h+o(h^2))V(t+h,\cdot) \\
&\quad + \int_t^{t+h} \bar{F}(s,t)e^{-\beta(s-t)}U(C_s) + f(s,t)e^{-\beta(s-t)}U(X_s,Z_s,D_s)ds\Big] \\
&= \max_{\{C,\pi,P\}} \mathbb{E}\Big[(1-(f(t,t)+\beta)h+o(h^2))[V(t,\cdot)+\int_t^{t+h} dV] \\
&\quad + \int_t^{t+h} (1-\beta(s-t)+o(s-t)^2)(\bar{F}(s,t)U(C_s)+f(s,t)U(X_s,P_s)ds|\mathcal{F}_t\Big].
\end{aligned}
\tag{C.3}
$$

Now, replace dV by (B.1) and use dynamics of dX_t and dI_t, the target function becomes:

$$
\begin{aligned}
V(t,\cdot) = \max_{\{C,\pi,P\}} \mathbb{E}\Big[&(1-(f(t,t)+\beta)h+o(h^2))V(t,\cdot) \\
&+ (1-(f(t,t)+\beta)h+o(h^2))\Big(\int_t^{t+h} [V_t + (rX_t + \pi_t(\mu^s - r) + I_t - C_t - P_t)V_X \\
&+ (\theta - kI_t)V_I + \alpha^U(\bar{\lambda}^U - \lambda_t^U)V_{\lambda^U} + \alpha^D(\bar{\lambda}^D - \lambda_t^D)V_{\lambda^D} \\
&+ 0.5\pi_t^2\sigma^2 V_{XX} + 0.5\tilde{\sigma}^2 V_{II} + \rho\pi_t\sigma\tilde{\sigma}V_{XI}]dt \\
&+ \int_t^{t+h} [\pi_t\sigma V_X]dW_t^S + \int_t^{t+h} [\tilde{\sigma}V_I]dW_t^I \\
&+ \int_t^{t+h} [V^U - V]dN_t^U + \int_t^{t+h} [V^D - V]dN_t^D\Big) \\
&+ \int_t^{t+h} (1-\beta(s-t)+o(s-t)^2)(\bar{F}(s,t)U(C_s)+f(s,t)U(X_s,P_s)ds|\mathcal{F}_t\Big].
\end{aligned}
\tag{C.4}
$$

Then, use mean value theorem to simplify the integration in (C.4), and multiply $\frac{1}{h}$ on both sides of the simplified equation and take limit as $h \to 0$, we can get the HJB equation in (3.5). □

D Some examples of Remark 4.1

In this section, we follow the structure in (3.7) and (3.8). First, we can simplify the partial derivatives as

$$
\begin{aligned}
&V_t = V\left[\frac{dE}{dt} + \frac{dB^U}{dt}\lambda_t^U + \frac{dB^D}{dt}\lambda_t^D + \frac{dA}{dt}I_t + \frac{dD}{dt}X_t\right], \\
&V_X = DV, \qquad V_{XX} = D^2V, \qquad V_I = AV, \qquad V_{II} = A^2V, \\
&V_{XI} = ADV, \qquad V_{\lambda^U} = B^UV, \qquad V_{\lambda^D} = B^DV.
\end{aligned}
\tag{D.1}
$$

Substituting (D.1) into the HJB equation (3.5), we have

$$
\begin{aligned}
0 = \max_{\{\pi_t, C, P\}} \Bigg[& -(f(t,t)+\beta)V + \left[\frac{dE}{dt} + \frac{dB^U}{dt}\lambda_t^U + \frac{dB^D}{dt}\lambda_t^D + \frac{dA}{dt} I_t + \frac{dD}{dt} X_t\right] V \\
& + [rX_t + \pi_t(\mu - r) + I_t - C_t - P_t] DV + 0.5\pi_t^2 \sigma^2 D^2 V + (\theta - kI_t) AV \\
& + 0.5\tilde{\sigma}^2 A^2 V + \rho\sigma\tilde{\sigma}\pi_t ADV + \alpha^U(\bar{\lambda}^U - \lambda_t^U) B^U V + \alpha^D(\bar{\lambda}^D - \lambda_t^D) B^D V \\
& + \lambda_t^U \mathbb{E}[e^{B^U\eta_{11} + B^D\eta_{21} + \pi_t D J_t^U} - 1] V + \lambda_t^D \mathbb{E}[e^{B^U\eta_{12} + B^D\eta_{22} + \pi_t D J_t^D} - 1] V \\
& + U(C_t) + f(t,t) U(X_t, P_t) \Bigg].
\end{aligned} \tag{D.2}
$$

By the first condition, we can rewrite (4.1) as

$$
\begin{aligned}
C_t^* &= -\frac{1}{\gamma}\ln(DV), \\
P_t^* &= -f(t,t)\left(\frac{\ln(DV)}{\gamma} + X_t\right), \\
\pi_t^* &= \arg\min_{\pi} H(\pi_t, \lambda_t^U, \lambda_t^D),
\end{aligned} \tag{D.3}
$$

where

$$
\begin{aligned}
H(\pi_t, \lambda_t^U, \lambda_t^D) = \big\{ & \pi_t(\mu - r)D + 0.5\pi_t^2\sigma^2 D^2 + \rho\sigma\tilde{\sigma}\pi_t AD \\
& + \lambda_t^U \mathbb{E}[e^{B^U\eta_{11} + B^D\eta_{21} + \pi_t D J_t^U} - 1] + \lambda_t^D \mathbb{E}[e^{B^U\eta_{12} + B^D\eta_{22} + \pi_t D J_t^D} - 1] \big\}.
\end{aligned} \tag{D.4}
$$

Substituting (D.3) into the HJB equation (D.2) and applying the closed-form solutions of $A(t,T)$ and $D(t,T)$ to reduce the terms of I_t and X_t, we can simplify the HJB equation (D.2) as

$$
\begin{aligned}
0 = & - (f(t,t)+\beta) + \frac{dE}{dt} + \frac{dB^U}{dt}\lambda_t^U + \frac{dB^D}{dt}\lambda_t^D + \theta A + 0.5\tilde{\sigma}^2 A^2 \\
& + \alpha^U(\bar{\lambda}^U - \lambda_t^U) B^U + \alpha^D(\bar{\lambda}^D - \lambda_t^D) B^D \\
& + H(\pi_t^*, \lambda_t^U, \lambda_t^D) + \frac{1 + f(t,t)}{\gamma}(\ln\left(\frac{-D}{\gamma}\right) + A + B^U\lambda_t^U + B^D\lambda_t^D - 1).
\end{aligned} \tag{D.5}
$$

Note that, to get the closed form solution of the target function $V(\cdot)$, we need to solve $E(t,T)$, $B^U(t,T)$, and $B^D(t,T)$ explicitly, which depends on whether we can get the explicit solution of the optimal control π_t^*.

Now, we investigate the behavior of π_t further. Applying the first condition to $H(\pi, \lambda_t^U, \lambda_t^D)$, we obtain

$$
0 = (\mu - r) + \rho\sigma\tilde{\sigma}A + \pi_t\sigma^2 D + \lambda_t^U e^{B^U\eta_{11} + B^D\eta_{21}} \mathbb{E}[J_t^U e^{DJ_t^U\pi_t}] + \lambda_t^D e^{B^U\eta_{12} + B^D\eta_{22}} \mathbb{E}[J_t^D e^{DJ_t^D\pi_t}]. \tag{D.6}
$$

Therefore, the two terms $\mathbb{E}[J_t^U e^{D\pi_t J_t^U}]$ and $\mathbb{E}[J_t^D e^{D\pi_t J_t^D}]$ are important and they are closely related to the moment generating function of J_t^U and J_t^D. Therefore, if we can solve (D.6) to obtain the closed-form solution of π_t^*, we can substitute π_t^* into (D.5) and match the coefficients of λ_t^U and λ_t^D to get a system of ODEs, which will provide a closed-form solution of the target function. In this section, we consider some special cases where closed-form solutions exist.

D.1 Case One: No Price Jumps

The simplest case is that there are no price jumps, *i.e.* $J_t^U = 0$ and $J_t^D = 0$ for any t. Then by (D.6), we know π^* only depends on t with

$$\pi_t^* = -\frac{\mu - r + \rho\sigma\tilde{\sigma}A(t,T)}{\sigma^2 D(t,T)}. \tag{D.7}$$

Plugging π_t^* into (D.4), we have

$$\begin{aligned} H(\pi_t^*, \lambda_t^U, \lambda_t^D) = &- \frac{(\mu - r + \rho\sigma\tilde{\sigma}A(t,T))^2}{2\sigma^2} \\ &+ \lambda_t^U (e^{B^U(t,T)\eta_{11} + B^D(t,T)\eta_{21}} - 1) + \lambda_t^D (e^{B^U(t,T)\eta_{12} + B^D(t,T)\eta_{22}} - 1). \end{aligned} \tag{D.8}$$

Then, after combining (D.5) and (D.8) and matching the coefficients of λ_t^U and λ_t^D, we can derive the following system of ODEs,

$$\begin{cases} \dfrac{dB^U(t,T)}{dt} = \left(\alpha^U - \dfrac{1+f(t,t)}{\gamma}\right) B^U(t,T) - (e^{B^U(t,T)\eta_{11} + B^D(t,T)\eta_{21}} - 1), \\ \dfrac{dB^D(t,T)}{dt} = \left(\alpha^D - \dfrac{1+f(t,t)}{\gamma}\right) B^D(t,T) - (e^{B^U(t,T)\eta_{12} + B^D(t,T)\eta_{22}} - 1), \\ \dfrac{dE(t,T)}{dt} = f(t,t) + \beta - \theta A(t,T) - 0.5\tilde{\sigma}^2 A(t,T)^2 \\ \qquad - \alpha^U \bar{\lambda}^U B^U(t,T) - \alpha^D \bar{\lambda}^D B^D(t,T) + \dfrac{(\mu - r + \rho\sigma\tilde{\sigma}A(t,T))^2}{2\sigma^2} \\ \qquad - \dfrac{1+f(t,t)}{\gamma}\left(\ln\left(\dfrac{-D(t,T)}{\gamma}\right) + E(t,T) - 1\right), \end{cases} \tag{D.9}$$

with the terminal conditions $E(T,T) = B^U(T,T) = B^D(T,T) = 0$. $A(t,T)$ and $D(t,T)$ are given by (4.7) and (4.9).

Note that it is easy to check that the system of ODEs (D.9) admits the following closed-form solution,

$$\begin{aligned} B^U(t,T) &= B^D(t,T) = 0, \\ E(t,T) &= -\int_t^T G(s,T) e^{\int_t^s \frac{1+f(u,u)}{\gamma} du} ds, \end{aligned} \tag{D.10}$$

where

$$\begin{aligned} G(s,T) = &f(s,s) + \beta - \theta A(s,T) - 0.5\tilde{\sigma}^2 A(s,T)^2 - \alpha^U \bar{\lambda}^U B^U(t,T) - \alpha^D \bar{\lambda}^D B^D(t,T) \\ &+ \frac{(\mu - r + \rho\sigma\tilde{\sigma}A(s,T))^2}{2\sigma^2} - \frac{1+f(s,s)}{\gamma}\left(\ln\left(\frac{-D(s,T)}{\gamma}\right) - 1\right). \end{aligned} \tag{D.11}$$

Therefore, we can rewrite (D.3) to obtain the closed-form solutions for the other two optimal controls under the assumption with no price jumps.

$$\begin{aligned} C^*(t, I_t, X_t, \lambda_t^U, \lambda_t^D) &= -\frac{1}{\gamma} * \left(\ln\left(\frac{-D(t,T)}{\gamma}\right) + E(t,T) + A(t,T)I_t + D(t,T)X_t\right), \\ P^*(t, I_t, X_t, \lambda_t^U, \lambda_t^D) &= -f(t,t)\left(\frac{1}{\gamma} * \left(\ln\left(\frac{-D(t,T)}{\gamma}\right) + E(t,T) + A(t,T)I_t\right) + \left(1 + \frac{D(t,T)}{\gamma}\right) X_t\right), \end{aligned} \tag{D.12}$$

where $E(t,T), A(t,T)$, and $D(t,T)$ are given by (D.10), (4.6), and (4.8). Since the jump sizes are all equal to 0 in this simple case, it is meaningful to observe that the jump intensities λ_t^U and λ_t^D have no effect on the optimal control policies. Hence, we obtain an analytic solution which generalizes the work of Zeng et at. (2015), where the mortality rate follows a special exponential distribution.

In addition, the closed-form formula for the undetermined function $E(t,T)$ in (D.10) holds for general situations. We only need to determine the specific form of the optimal allocation policy with given jump-size distributions, then form the system of ODEs to obtain the formulas for $B^U(t,T)$ and $B^D(t,T)$. Following this scheme, we consider another two cases in the following sections.

D.2 Case Two: Exponential Jump-downs

Now, we consider another special jump distribution. Instead of assuming no price jumps, we suppose that the jump-down size satisfies an $i.i.d.$ exponential distributions and there are no price jump-ups in the market, with

$$\mathbb{P}\{|J^D| \le |d|\} = 1 - e^{-\lambda|d|} = 1 - e^{\lambda d}, \tag{D.13}$$

where $d \in (-\infty, 0)$ and λ is the parameter of the exponential distribution. Then, the term $\mathbb{E}[J_t^D e^{D\pi_t J_t^D}]$ becomes $-\frac{\lambda}{(\lambda+D\pi_t)^2}$ with $\lambda + D\pi_t > 0$. Therefore, (D.6) can be written as

$$\begin{aligned} 0 &= (\mu - r) + \rho\sigma\tilde{\sigma}A + \pi_t\sigma^2 D - \lambda_t^D e^{B^U\eta_{12}+B^D\eta_{22}} \frac{\lambda}{(\lambda + D\pi_t)^2} \\ &= \sigma^2 D^3 \pi_t^3 + (2\lambda\sigma^2 + \mu - r + \rho\sigma\tilde{\sigma}A) D^2 \pi_t^2 \\ &\quad + (2\lambda(\mu - r + \rho\sigma\tilde{\sigma}A) + \lambda^2\sigma^2) D\pi_t + \lambda^2(\mu - r + \rho\sigma\tilde{\sigma}A) - \lambda\lambda_t^D e^{B^U\eta_{12}+B^D\eta_{22}}. \end{aligned} \tag{D.14}$$

It is direct to check that the closed-form solution of the optimal allocation policy depends on t and λ_t^D and follows

$$\pi^*(t, \lambda_t^D) = W(t, \lambda_t^D) - \frac{2\lambda\sigma^2 + \mu - r + \rho\sigma\tilde{\sigma}A(t,T)}{3\sigma^2 D(t,T)} + \frac{(\lambda\sigma^2 + \mu - r + \rho\sigma\tilde{\sigma}A(t,T)}{9\sigma^4 D(t,T)^2 W(t,\lambda^D)}, \tag{D.15}$$

where

$$\begin{aligned} W(t,\lambda_t^D) &= \left[\left(\Delta(t,\lambda_t^D)^2 - \frac{(\lambda\sigma^2 + r - \mu - \rho\sigma\tilde{\sigma}A(t,T))^6}{729\sigma^{12}D(t,T)^6}\right)^{\frac{1}{2}} - \Delta(t,\lambda_t^U)\right]^{\frac{1}{3}}, \\ \Delta(t,\lambda_t^D) &= \frac{\lambda^2(r - \mu - \rho\sigma\tilde{\sigma}A(t,T)) + \lambda\lambda_t^D e^{B^U(t,T)\eta 12 + B^D(t,T)\eta_{22}})}{2\sigma^2 D(t,T)^3} \\ &\quad - \frac{(2\lambda\sigma^2 + \mu - r + \rho\sigma\tilde{\sigma}A(t,T))^3}{27\sigma^6 D(t,T)^3} \\ &\quad + \frac{\lambda(2\lambda\sigma^2 + \mu - r + \rho\sigma\tilde{\sigma}A(t,T))(\lambda\sigma^2 + 2\mu - 2r + 2\rho\sigma\tilde{\sigma}A(t,T))}{6\sigma^4 D(t,T)^3}. \end{aligned} \tag{D.16}$$

Even though we obtain the analytic solution for $\pi^*(t,\lambda_t^D)$, we cannot derive a concise system of ODEs , which means it is too complicated to derive the closed-form solutions of $B^U(t,T), B^D(t,T)$, and $E(t,T)$ as in section D.1.

D.3 Case Three: Gamma Jump-ups

In this section, following the work of Daly and Porporato (2010), we consider another special case where the jump-up size follows *i.i.d.* Gamma distributions $\Gamma(K,\Theta)$ and there are no jump-downs. Then, the average jump-up amount is $K\Theta$ of the stock price. We focus on Gamma distributions with the shape parameter $K=2$, because we only have explicit formulas for the general roots of polynomials up to degree four. Therefore, the interesting term $\mathbb{E}[J_t^U e^{D\pi_t J_t^U}]$ becomes $2\Theta(1-\Theta D\pi_t)^{-3}$. Then,(D.6) can be rewritten as

$$\begin{aligned}0 &= (\mu-r)+\rho\sigma\tilde{\sigma}A+\pi_t\sigma^2 D+\lambda_t^U e^{B^U\eta_{11}+B^D\eta_{21}}\frac{2\Theta}{(1-\Theta D\pi_t)^3}\\ &= -\sigma^2\Theta^3 D^4\pi_t^4+(3\sigma^2-(\mu-r)\Theta+\rho\sigma\tilde{\sigma}\Theta A)\Theta^2 D^3\pi_t^3-(3\sigma^2-3\Theta((\mu-r)+\rho\sigma\tilde{\sigma}A))\Theta D^2\pi_t^2\\ &+(\sigma^2-3\Theta((\mu-r)+\rho\sigma\tilde{\sigma}A))D\pi_t+(\mu-r)+\rho\sigma\tilde{\sigma}A+2\Theta\lambda_t^U e^{B^U\eta_{11}+B^D\eta_{21}}.\end{aligned} \tag{D.17}$$

As (D.17) is a quartic function of π_t, we can convert it to a depressed quartic function and find the Ferrari's solutions of this function. Since we consider a wage earner with exponential utilities, it is possible to obtain solutions indicating short-selling. After comparing the four roots of (D.17), we obtain the optimal allocation policy as follows

$$\pi^*(t,\lambda_t^U) = -\frac{a_2}{a_1}+S+0.5\sqrt{-4S^2-2p-\frac{q}{S}}, \tag{D.18}$$

where

$$\begin{cases} a_1 = -\sigma^2\Theta^3 D^4,\\ a_2 = (3\sigma^2-(\mu-r)\Theta+\rho\sigma\tilde{\sigma}\Theta A)\Theta^2 D^3,\\ a_3 = (3\sigma^2-3\Theta((\mu-r)+\rho\sigma\tilde{\sigma}A))\Theta D^2,\\ a_4 = (\sigma^2-3\Theta((\mu-r)+\rho\sigma\tilde{\sigma}A))D,\\ a_5 = (\mu-r)+\rho\sigma\tilde{\sigma}A+2\Theta\lambda_t^U e^{B^U\eta_{11}+B^D\eta_{21}},\\ p = \dfrac{8a_1a_3-3a_2^2}{8a_1^2},\\ q = \dfrac{a_2^3-4a_1a_2a_3+8a_1^2a_4}{8a_1^3},\\ S = 0.5\sqrt{-\dfrac{2p}{3}+\dfrac{1}{3a_1}\left(m+\dfrac{\Delta_0}{m}\right)},\\ m = \sqrt[3]{\dfrac{\Delta_1+\sqrt{\Delta_1^2-4\Delta_0^3}}{2}},\\ \Delta_0 = a_3^2-3a_2a_4+12a_1a_5,\\ \Delta_1 = 2a_3^3-9a_2a_3a_4+27a_2^2a_5+27a_1a_4^2-72a_1a_3a_5. \end{cases} \tag{D.19}$$

Similarly with the case in Section D.2, the explicit formula of $\pi^*(t,\lambda_t^U)$ in (D.18) is too complicated to derive the closed-form solutions of the target function and the other two optimal controls.

E Proof of Lemma 4.2

First, the value function in a market without price jumps should be smaller than the value function with price jump-ups and jump-downs. With price jumps, investors can obtain higher utilities by short-selling. Therefore, if we denote the value function obtained in Section D.1 by $V_1(t, X_t, I_t, \lambda_t^U, \lambda_T^D)$, then the value function with price jumps $V(t, X_t, I_t, \lambda_t^U, \lambda^D) > V_1(t, X_t, I_t, \lambda_t^U, \lambda_T^D)$. By (3.7), we know $g(t, \lambda_t^U, \lambda_t^D) < \alpha_1$, with $\alpha_1 = \frac{-\gamma V_1(t, X_t, I_t, \lambda_t^U, \lambda_T^D)}{e^{A(t,T)I_t + D(t,T)X_t}}$. And $\alpha_1 > 0$ can be easily checked. Since, $g(t, \lambda_t^U, \lambda_t^D)$ is the fixed-point of the operator, we know $(\mathcal{L}g)(t, \lambda_t^U, \lambda_t^D) < \alpha_1$.

Second, the value function with price jump-ups and jump-downs should be smaller than the value function with only price jumps-ups. Therefore, if we denote the value function obtained in Section D.2 by $V_2(t, X_t, I_t, \lambda_t^U, \lambda_T^D)$, we have $V(t, X_t, I_t, \lambda_t^U, \lambda^D) < V_2(t, X_t, I_t, \lambda_t^U, \lambda_T^D)$. By similar steps, we know the $(\mathcal{L}g)(t, \lambda_t^U, \lambda_t^D) > \alpha_2$, with $\alpha_2 = \frac{-\gamma V_2(t, X_t, I_t, \lambda_t^U, \lambda_T^D)}{e^{A(t,T)I_t + D(t,T)X_t}}$. We can also easily check the positivity of α_2. □

F Proof of Lemma 4.3

Applying the first condition to (4.5), we know that x_1 and x_2 satisfy the following equations

$$(\mu - r) + \rho\sigma\tilde{\sigma}A + \sigma^2 x_1 + \lambda_t^U \frac{f^U}{f} \frac{\lambda}{(\lambda - x_1)^2} - \lambda_t^D \frac{f^D}{f} \frac{\lambda}{(\lambda + x_1)^2} = 0, \tag{F.1}$$

$$(\mu - r) + \rho\sigma\tilde{\sigma}A + \sigma^2 x_2 + \lambda_t^U \frac{h^U}{h} \frac{\lambda}{(\lambda - x_2)^2} - \lambda_t^D \frac{h^D}{h} \frac{\lambda}{(\lambda + x_2)^2} = 0, \tag{F.2}$$

where $x_1 = \pi_t^1 D$, $x_2 = \pi_t^2 D$, and there exists $\epsilon_1 \in (0, \lambda)$ and $\epsilon_2 \in (0, \lambda)$ *s.t.* $x_1 \in (-\epsilon_1, \epsilon_1)$ and $x_2 \in (-\epsilon_2, \epsilon_2)$.

Therefore, we have

$$\begin{aligned} &((\mu - r) + \rho\sigma\tilde{\sigma}A)(f - h) + \sigma^2(f x_1 - h x_2) \\ &\quad + \lambda_t^U \left(f^U \frac{\lambda}{(\lambda - x_1)^2} - h^U \frac{\lambda}{(\lambda - x_2)^2} \right) - \lambda_t^D \left(f^D \frac{\lambda}{(\lambda + x_1)^2} - h^D \frac{\lambda}{(\lambda + x_2)^2} \right) = 0. \end{aligned} \tag{F.3}$$

$$\begin{aligned} &f H(\pi_t^1, f) - h H(\pi_t^2, h) \\ &= (\mu - r + \rho\sigma\tilde{\sigma}A)(f x_1 - h x_2) + 0.5\sigma^2(f x_1^2 - h x_2^2) \\ &\quad + \lambda_t^U \left(f^U \frac{x_1}{\lambda - x_1} - h^U \frac{x_2}{\lambda - x_2} \right) - \lambda_t^D \left(f^D \frac{x_1}{\lambda + x_1} - h^D \frac{x_2}{\lambda + x_2} \right) \\ &= (\mu - r + \rho\sigma\tilde{\sigma}A)(f x_1 - h x_2) + 0.5\sigma^2(f x_1 - h x_2)(x_1 + x_2) - 0.5\sigma^2(f - h)x_1 x_2 \\ &\quad + \lambda_t^U \left(f^U \frac{x_1}{\lambda - x_1} - h^U \frac{x_2}{\lambda - x_2} \right) - \lambda_t^D \left(f^D \frac{x_1}{\lambda + x_1} - h^D \frac{x_2}{\lambda + x_2} \right). \end{aligned} \tag{F.4}$$

Now, combine (F.3) and (F.4), we have

$$
\begin{aligned}
&fH(\pi_t^1, f) - hH(\pi_t^2, g) \\
&= ((\mu - r + \rho\sigma\tilde{\sigma}A)\Delta - 0.5\sigma^2 x_1 x_2)(f - h) \\
&+ \lambda_t^U \Delta \left(f^U \frac{\lambda}{(\lambda - x_1)^2} - h^U \frac{\lambda}{(\lambda - x_2)^2} \right) - \lambda_t^D \Delta \left(f^D \frac{\lambda}{(\lambda + x_1)^2} - h^D \frac{\lambda}{(\lambda + x_2)^2} \right) \\
&+ \lambda_t^U \left(f^U \frac{x_1}{\lambda - x_1} - h^U \frac{x_2}{\lambda - x_2} \right) - \lambda_t^D \left(f^D \frac{x_1}{\lambda + x_1} - h^D \frac{x_2}{\lambda + x_2} \right),
\end{aligned}
\tag{F.5}
$$

where $\Delta = \frac{-\left(\mu - r + \rho\sigma\tilde{\sigma}A + 0.5\sigma^2(x_1 + x_2)\right)}{\sigma^2}$.

Then, combine (F.1), (F.2), and (F.5), we can further obtain

$$
\begin{aligned}
fH(\pi_t^1, f) - hH(\pi_t^2, h) &= ((\mu - r + \rho\sigma\tilde{\sigma}A)\Delta - 0.5\sigma^2 x_1 x_2)(f - h) \\
&+ \Delta(-(\mu - r + \rho\sigma\tilde{\sigma}A + \sigma^2 x_1)f + (\mu - r + \rho\sigma\tilde{\sigma}A + \sigma^2 x_2)h) \\
&+ \left(\lambda_t^U f^U \frac{x_1}{\lambda - x_1} - \lambda_t^D f^D \frac{x_1}{\lambda + x_1} \right) - \left(\lambda_t^U h^U \frac{x_2}{\lambda - x_2} - \lambda_t^D h^D \frac{x_2}{\lambda + x_2} \right).
\end{aligned}
\tag{F.6}
$$

Then, we can multiple $\lambda - x_1$ on both sides of (F.1) and multiple $\lambda + x_2$ on both sides of (F.1) to obtain

$$
\begin{aligned}
&\frac{(\mu - r) + \rho\sigma\tilde{\sigma}A + \sigma^2 x_1}{\lambda}(\lambda - x_1)f + \lambda_t^U f^U \frac{1}{\lambda - x_1} - \lambda_t^D f^D \frac{1}{\lambda + x_1} + 2\lambda^D f^D \frac{x_1}{(\lambda + x_1)^2} = 0, \\
&\frac{(\mu - r) + \rho\sigma\tilde{\sigma}A + \sigma^2 x_1}{\lambda}(\lambda + x_1)f + \lambda_t^U f^U \frac{1}{\lambda - x_1} - \lambda_t^D f^D \frac{1}{\lambda + x_1} + 2\lambda^U f^U \frac{x_1}{(\lambda - x_1)^2} = 0.
\end{aligned}
\tag{F.7}
$$

Therefore, we have

$$
\begin{aligned}
&\lambda_t^U f^U \frac{x_1}{\lambda - x_1} - \lambda_t^D f^D \frac{x_1}{\lambda + x_1} \\
&= -(\mu - r + \rho\sigma\tilde{\sigma}A + \sigma^2 x_1)x_1 f - \left(\lambda^U f^U \frac{x_1^2}{(\lambda - x_1)^2} + \lambda^D f^D \frac{x_1^2}{(\lambda + x_1)^2} \right).
\end{aligned}
\tag{F.8}
$$

Similarly,

$$
\begin{aligned}
&\lambda_t^U h^U \frac{x_2}{\lambda - x_2} - \lambda_t^D h^D \frac{x_2}{\lambda + x_2} \\
&= -(\mu - r + \rho\sigma\tilde{\sigma}A + \sigma^2 x_2)x_2 h - \left(\lambda^U h^U \frac{x_2^2}{(\lambda - x_2)^2} + \lambda^D h^D \frac{x_2^2}{(\lambda + x_2)^2} \right).
\end{aligned}
\tag{F.9}
$$

Then, we can simplify (F.6) by (F.8) and (F.9) as

$$
\begin{aligned}
fH(\pi_t^1, f) - hH(\pi_t^2, h) &= 0.5\sigma^2 x_2^2 h + \lambda^U h^U \frac{x_2^2}{(\lambda - x_2)^2} + \lambda^D h^D \frac{x_2^2}{(\lambda + x_2)^2} \\
&- \left(0.5\sigma^2 x_1^2 f + \lambda^U f^U \frac{x_1^2}{(\lambda - x_1)^2} + \lambda^D f^D \frac{x_1^2}{(\lambda + x_1)^2} \right).
\end{aligned}
\tag{F.10}
$$

Now, by the Mean Value Theorem, we can rewrite (F.10) as

$$
fH(\pi_t^1, f) - hH(\pi_t^2, h) = (f - h)\frac{d\zeta H\left(\pi(\zeta), \zeta\right)}{d\zeta}\Big|_{\zeta = \xi_1},
\tag{F.11}
$$

for some function ξ_1 between f and h. And

$$\zeta H\left(\pi(\zeta),\zeta\right) = -\left(0.5\sigma^2 x^2 \zeta + \lambda^U \zeta^U \frac{x^2}{(\lambda - x)^2} + \lambda^D \zeta^D \frac{x^2}{(\lambda + x)^2}\right), \tag{F.12}$$

where x is actually a function of ζ with

$$(\mu - r) + \rho\sigma\tilde{\sigma}A + \sigma^2 x + \lambda_t^U \frac{\zeta^U}{\zeta} \frac{\lambda}{(\lambda - x)^2} - \lambda_t^D \frac{\zeta^D}{\zeta} \frac{\lambda}{(\lambda + x)^2} = 0. \tag{F.13}$$

We apply Taylor series to estimate (F.13)

$$(\mu - r + \rho\sigma\tilde{\sigma}A)\zeta + \sigma^2 x\zeta + \lambda_t^U \zeta^U \left(\frac{1}{\lambda} + \frac{2}{\lambda^2}x + O(x^2)\right) - \lambda_t^D \zeta^D \left(\frac{1}{\lambda} - \frac{2}{\lambda^2}x + O(x^2)\right) = 0. \tag{F.14}$$

Since $\zeta(t, \lambda_t^U, \lambda_t^D) = e^{\tilde{C}(t,T) + \tilde{B}^U(t,T)\lambda_t^U + \tilde{B}^D(t,T)\lambda_t^D}$, we know $\zeta^U = e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}}\zeta$ and $\zeta^D = e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}}\zeta$. Then

$$\begin{aligned}(\mu - r + \rho\sigma\tilde{\sigma}A) + \sigma^2 x + \sigma^2 \zeta \frac{dx}{d\zeta} + \lambda_t^U e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} \left(\frac{1}{\lambda} + \frac{2}{\lambda^2}x\right) \\ + \lambda_t^U \zeta^U \frac{2}{\lambda^2}\frac{dx}{d\zeta} - \lambda_t^D e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}} \left(\frac{1}{\lambda} - \frac{2}{\lambda^2}x\right) + \lambda_t^D \zeta^D \frac{2}{\lambda^2}\frac{dx}{d\zeta} = 0.\end{aligned} \tag{F.15}$$

Therefore,

$$\zeta\frac{dx}{d\zeta} = \frac{\mathcal{K} - \lambda_t^D \zeta^D e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}} \left(\frac{1}{\lambda} - \frac{2}{\lambda^2}x\right)}{\sigma^2 + \frac{2\lambda_t^U}{\lambda^2} e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} + \frac{2\lambda_t^D}{\lambda^2} e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}}}, \tag{F.16}$$

where

$$\mathcal{K} = \mu - r + \rho\sigma\tilde{\sigma}A + \sigma^2 x + \lambda_t^U e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} \left(\frac{1}{\lambda} + \frac{2}{\lambda^2}x\right).$$

Then, according to (F.12), we can derive the derivative of $\zeta H\left(\pi(\zeta),\zeta\right)$,

$$\begin{aligned}\frac{d\zeta H\left(\pi(\zeta),\zeta\right)}{d\zeta} = &-\left[x^2 + 2x\zeta\frac{dx}{d\zeta}\right]\left[0.5\sigma^2 + \lambda_t^U e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} \left(\frac{1}{\lambda^2} + \frac{2}{\lambda^3}x\right)\right. \\ &\left. + \lambda_t^D e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}} \left(\frac{1}{\lambda^2} - \frac{2}{\lambda^3}x\right)\right] \\ &- x^2\zeta\left[\lambda_t^U e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} - \lambda_t^D e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}}\right]\frac{2}{\lambda^3}\frac{dx}{d\zeta}.\end{aligned} \tag{F.17}$$

Finally, we can obtain

$$fH(\pi_t^1, f) - hH(\pi_t^2, h) = (f - h)\Phi(x), \tag{F.18}$$

where $\Phi(x)$ is a polynomial of degree three.

$$\begin{aligned}&\Phi(x) \\ &= -x^2\left[0.5\sigma^2 + \lambda_t^U e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} \left(\frac{1}{\lambda^2} + \frac{2x}{\lambda^3}\right) + \lambda_t^D e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}} \left(\frac{1}{\lambda^2} - \frac{2x}{\lambda^3}\right)\right] \\ &\quad - \left[\sigma^2 x + \lambda_t^U e^{\tilde{B}^U(t,T)\eta_{11} + \tilde{B}^D(t,T)\eta_{21}} \left(\frac{2x}{\lambda^2} + \frac{6x^2}{\lambda^3}\right) + \lambda_t^D e^{\tilde{B}^U(t,T)\eta_{12} + \tilde{B}^D(t,T)\eta_{22}} \left(\frac{2x}{\lambda^2} - \frac{2x^2}{\lambda^3}\right)\right]\zeta\frac{dx}{d\zeta},\end{aligned} \tag{F.19}$$

where $\zeta\frac{dx}{d\zeta}$ is given by (F.16). When $\zeta = \xi_1$, we know x should be between x_1 and x_2. Therefore $\Phi(x)$ is bounded. □

G Proof of Theorem 4.4

Similarly, we know $f\ln f - h\ln h = (f-h)(\ln\xi_2 + 1)$ for some function ξ_2 between f and h.

The following inequality holds for all $(t, \lambda_t^U, \lambda_t^D) \in \mathfrak{C}_e([0,T]\times(0,\infty)\times(0,\infty))$. Therefore, it also holds for $d(\mathcal{L}f, \mathcal{L}h)$.

$$
\begin{aligned}
&d(\mathcal{L}f,\mathcal{L}h)\\
&= e^{-\alpha(T-t)}\left|\mathbb{E}_t\left[\int_t^T e^{\int_t^s Q(v,T)dv}\left(\frac{f(t,t)+1}{\gamma}D(t,T)(f\ln f - h\ln h) + fH(\pi_t^1,f) - hH(\pi_t^2,h)\right)ds\right]\right|\\
&= e^{-\alpha(T-t)}\left|\mathbb{E}_t\left[\int_t^T e^{\int_t^s Q(v,T)dv}\left(\frac{f(t,t)+1}{\gamma}D(t,T)(\ln\xi_2+1) + \Phi(x)\right)(f-h)ds\right]\right|\\
&\quad\le e^{-\alpha(T-t)}d(f,h)\left|\mathbb{E}_t\left[\int_t^T e^{\int_t^s Q(v,T)dv+\alpha(T-s)}\left(\frac{f(t,t)+1}{\gamma}D(t,T)(\ln\xi_2+1) + \Phi(x)\right)ds\right]\right|\\
&\quad\le e^{-\alpha(T-t)}d(f,h)\mathbb{E}_t\left[\int_t^T e^{\int_t^s Q(v,T)dv+\alpha(T-s)}\left|\left(\frac{f(t,t)+1}{\gamma}D(t,T)(\ln\xi_2+1) + \Phi(x)\right)\right|ds\right]\\
&= d(f,h)\mathbb{E}_t\left[\int_t^T e^{\int_t^s Q(v,T)dv-\alpha(s-t)}\left|\left(\frac{f(t,t)+1}{\gamma}D(t,T)(\ln\xi_2+1) + \Phi(x)\right)\right|ds\right]\\
&\quad\le d(f,h)\mathbb{E}_t\left[\int_t^T e^{\int_t^s Q(v,T)dv-\alpha(s-t)}\left(\left|\frac{f(t,t)+1}{\gamma}D(t,T)(\ln\xi_2+1)\right| + |\Phi(x)|\right)ds\right]\\
&\quad\le \frac{\left|\frac{f(t,t)+1}{\gamma}D(t,T)(\ln\xi_2+1)\right| + |\Phi(x)|}{\alpha - \bar{Q}(t,T)}d(f,h),
\end{aligned}
\tag{G.1}
$$

where $\mathbb{E}_t$ is the expectation conditioned on $\mathcal{F}_t$. Then according to Lemma 4.2 and Lemma 4.3, we know $\alpha_2 < \xi_2 < \alpha_1$ and $|\Phi(x)|$ is bounded. Therefore we have $\left|\frac{f(t,t)+1}{\gamma}D(t,T)(ln\xi_2+1)\right| + |\Phi(x)| < \alpha_3$, for some positive α_3. Therefore, if $\alpha > \alpha_3 + \bar{Q}(t,T)$, we find the ideal $\phi \in [0,1)$ *s.t.* $d(\mathcal{L}f,\mathcal{L}h) \le \phi d(f,h)$. □