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Author/s:

Zhang, J;Chen, P;Jin, Z;Li, S

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OPEN-LOOP EQUILIBRIUM STRATEGY FOR MEAN-VARIANCE PORTFOLIO SELECTION: A LOG-RETURN MODEL

JIANNAN ZHANG, PING CHEN*, ZHUO JIN AND SHUANMING LI

Centre for Actuarial Studies, Department of Economics
The University of Melbourne
VIC, 3010, Australia

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ABSTRACT. This paper investigates a continuous-time mean-variance portfolio selection problem based on a log-return model. The financial market is composed of one risk-free asset and multiple risky assets whose prices are modelled by geometric Brownian motions. We derive a sufficient condition for open-loop equilibrium strategies via forward backward stochastic differential equations (FBSDEs). An equilibrium strategy is derived by solving the system. To illustrate our result, we consider a special case where the interest rate process is described by the Vasicek model. In this case, we also derive the closed-loop equilibrium strategy through the dynamic programming approach.

1. Introduction. Mean-variance portfolio selection problem can be date back to [15], where a single-period model is investigated. The main idea is to find the optimal portfolio weights among assets to achieve a trade-off between the mean and variance of the portfolio return. Thereafter, the work is extended to various multi-period and continuous-time models. To name a few, in [13] the authors firstly formulate and solve a multi-period mean-variance portfolio selection problem through an embedding technique. In [20], the explicit expressions for the efficient portfolio and efficient frontier are obtained by using an indefinite stochastic linear-quadratic control approach in the continuous-time setting. Since then, the model in [20] has been extended in a number of papers. The author investigates a complete market with random coefficients in [14]. A spread between the interest rates of lending and borrowing is considered in [8]. The mean-variance principle is also applied to asset-liability management in [16, 19].

However, in the multi-period and continuous-time setting, all the papers mentioned above refer to the so-called “precommitted” problem, where the “optimal” strategy is only optimal from the point of view of time zero. That is, the problem is time-inconsistent in the sense that the Bellman optimality principle does not hold. As a result, an optimal strategy viewed from today will not be optimal viewed from tomorrow. A precommitted strategy may only be economically meaningful in certain circumstances, which is argued in [12].

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* Corresponding author: Ping Chen.

The first step to solve time-inconsistent problems is to formulate a series of objective functions according to a progression of times. Then we have a family of optimization problems instead of a single one which is only formulated for time zero in the precommitted case. As a consequence, equilibrium strategies are attempted instead of an optimal strategy. In the literature, basically there are two methods to define a system of equilibrium strategies: closed-loop and open-loop. Following the idea in [16], the closed-loop equilibrium strategy is proposed in [4, 3], where authors view the time-inconsistent problem within a game theoretic framework in order to find the subgame perfect Nash equilibrium strategy. They derive an extension of the standard HJB equation in the discrete-time and continuous-time settings for a time-inconsistent stochastic control problem and also prove the associated verification theorem. The open-loop equilibrium strategy is provided in [10], where they derive a sufficient condition via a series of FBSDEs, and then they find an explicit equilibrium strategy for the time-inconsistent stochastic linear-quadratic control problem. The uniqueness of equilibrium open-loop controls for two special cases is also proved in [11].

As one of the further developments of the time-consistent portfolio selection problems, a mean-variance formulation is considered in [1] by writing the standard iterated expectation expression through a constant risk aversion. The result implies that the dollar amount invested in risky assets is a constant regardless of the investor's initial wealth, which is not economically reasonable. Then the portfolio optimization problem with a wealth-dependent risk aversion is reformulated in [5]. In [18], the authors obtain an equilibrium strategy for the mean-variance asset-liability management problem with regime switching. In [10, 17], the authors give an open-loop equilibrium strategy for the mean-variance asset-liability problem with random coefficients. By doing those extensions, they present that the optimal allocation follows the practical investment wisdom through deterministic, random or regime-switching coefficients.

In addition to the point that the optimal allocation is dependent on the wealth level, there are two other investment wisdoms. Firstly, the investor who lives longer should invest more in risky assets, which suggests that the younger investors are supposed to take more risks than the older within a finite investment horizon, see, for example, [9]. Unfortunately, the result in [1] violates this rule. Secondly, short-selling should be prohibited in the long run for time-consistent portfolio selection problems, see [2]. However, they point out that the previously existing results in [5, 10, 17] lead to a negative wealth process and result in an unbounded value function which is economically unsound.

To overcome the economic weaknesses of the existing solutions, In [7] the authors propose a dynamic portfolio choice model under the mean-variance criterion based on the portfolio's log-return. The mean-variance objective function is described by the expected logarithm of return and the variance of the logarithm of return, which leads to an exponential growth of the wealth level and allows us to easily identify the risk preference of investors. This approach is brought closer to standard economic theory by generalizing it to a criterion based on a trade-off between the expected logarithm of return and the variance of the logarithm of return. The model in [7] yields analytical and time-consistent optimal portfolio policies and can be used for robo-advising. The equilibrium strategy derived in their model is in a closed-loop setting, then it is natural to consider the problem in an open-loop setting. Hence the contribution of this paper is that we consider the open-loop equilibrium

strategy for the mean-variance portfolio selection through the log-return instead of the wealth level in a continuous-time setting. We find that our solutions also meet the conventional investment wisdom, that is, rich and young people invest more in risky assets and there is no short sale in the long run. To be specific, our aim is to look for the optimal percentage allocation in the risky assets under the mean-variance criteria, which results that the dollar amount invested is dependent on the wealth level.

Similar to [17], we adopt the approach motivated by [10]. Firstly, after taking the logarithm of the wealth process, we derive a series of FBSDEs to describe a sufficient condition for the equilibrium strategy. Then we introduce a series of BSDEs as the solutions of the previous FBSDEs. Under the sufficient condition, we prove that the strategy derived is indeed an equilibrium strategy for the problem. As mentioned before, the equilibrium strategy defined as [10] is an open-loop equilibrium strategy which is different from the closed-loop equilibrium strategy inspired from [3]. Therefore, we also derive the closed-loop equilibrium strategy through a system of extended HJB equations. In contrast to [17], the authors derive the optimal amount invested in the risky asset, we give the optimal proportion. Solutions in this paper are still economically reasonable when those coefficients degenerate to the constant case. By taking the logarithm, the dynamics of wealth process are changed, the investor is guided by a mean-variance criteria over the log-return and short-selling is not allowed in this paper.

The remaining paper is organized as follows. In Section 2, we introduce the formulation of the mean-variance portfolio selection problem. In Section 3, a sufficient condition is developed for the open-loop equilibrium strategy. Section 4 is an application of our results to a special case when the interest rate process is modelled by the Vasicek model. The closed-loop equilibrium strategy is also derived within the game theoretic framework. Finally, Section 5 concludes this paper.

2. Problem formulation. Throughout the paper, we use boldface letters to denote the vectors and matrices. For a matrix $\mathbf{M}$, we define $\mathbf{M}'$ to be the transpose matrix and $|\mathbf{M}| = \sqrt{\sum_{i,j} m_{i,j}^2}$ to be the Frobenius norm. Let $T > 0$ be a finite time horizon and $\mathbf{B}(\cdot) \equiv (B_1(\cdot), \dots, B_m(\cdot))'$ be an m -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote $\mathcal{F}_t$ to be the augmented filtration generated by B_t and $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ to be the augmented filtration generated by $\mathbf{B}(\cdot)$.

For $p \geq 1$, $\mathbf{H} = \mathbb{R}^m$ or $\mathbb{R}^{m \times m}$, we use the following notation:

$$L_{\mathcal{F}_t}^p(\Omega; \mathbf{H}) = \left\{ \mathbf{X} : \Omega \rightarrow \mathbf{H} \mid \mathbf{X}(\cdot) \text{ is } \mathcal{F}_t\text{-measurable, } \mathbb{E}[|\mathbf{X}|^p] < \infty \right\},$$

$$L_{\mathbb{F}}^2(s, t; \mathbf{H}) = \left\{ \mathbf{X} : [s, t] \times \Omega \rightarrow \mathbf{H} \mid \mathbf{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted,} \right. \\ \left. \mathbb{E} \left[\int_s^t |\mathbf{X}(\nu)|^2 d\nu \right] < \infty \right\},$$

$$L_{\mathbb{F}}^p(\Omega; L^2(s, t; \mathbf{H})) = \left\{ \mathbf{X} : [s, t] \times \Omega \rightarrow \mathbf{H} \mid \mathbf{X}(\cdot) \text{ is } \mathbb{F}\text{-adapted,} \right. \\ \left. \mathbb{E} \left[\left(\int_s^t |\mathbf{X}(\nu)|^2 d\nu \right)^p \right] < \infty \right\},$$

$$L_{\mathbb{F}}^\infty(\Omega; C([s, t]; \mathbf{H})) = \left\{ \mathbf{X} : [s, t] \times \Omega \rightarrow \mathbf{H} \mid \mathbf{X}(\cdot) \text{ is bounded } \mathbb{F}\text{-adapted,} \right.$$

has continuous paths and $\mathbb{E} \left[\sup_{\nu \in [s, t]} |\mathbf{X}(\nu)|^p \right] < \infty \}$.

We consider a financial market consists of a bank account and m stocks within the time horizon $[0, T]$. The bank account $A(\cdot)$ evolves as

$$\begin{cases} dA(s) = r(s)A(s)ds, & s \in [0, T], \\ A(0) = a_0 > 0, \end{cases} \quad (1)$$

where the interest rate $r(\cdot) > 0$ is a bounded continuous $\mathbb{F}$ -adapted process. For $i = 1, 2, \dots, m$, the price of the i th stock $S_i(\cdot)$ is given by

$$\begin{cases} dS_i(s) = S_i(s) \left[\mu_i(s)ds + \sum_{j=1}^m \sigma_{ij}(s)dB_j(s) \right], & s \in [0, T], \\ S_i(0) = s_0 > 0, \end{cases} \quad (2)$$

where $\boldsymbol{\mu}(\cdot) = (\mu_1(\cdot), \mu_2(\cdot), \dots, \mu_m(\cdot))' \in L_{\mathbb{F}}^{\infty}(\Omega; C([0, T]; \mathbb{R}^m))$ and $\boldsymbol{\sigma}(\cdot) = (\sigma_{ij}(\cdot))_{1 \leq i, j \leq m} \in L_{\mathbb{F}}^{\infty}(\Omega; C([0, T]; \mathbb{R}^{m \times m}))$ are the expected return rate vector and the corresponding volatility process of stocks, respectively. We assume that throughout the paper $\mu_i(\cdot) > r(\cdot)$, $\boldsymbol{\sigma}(\cdot)\boldsymbol{\sigma}(\cdot)' \geq \delta I$ for a constant $\delta > 0$.

Let $\pi_i(\cdot)$ be the proportion of the investment amount in the i th stock. Then the dynamics of the wealth process $W(\cdot)$ is given by

$$\begin{cases} \frac{dW(s)}{W(s)} = [r(s) + \boldsymbol{\theta}'(s)\boldsymbol{\pi}(s)]ds + \boldsymbol{\pi}(s)' \boldsymbol{\sigma}(s) d\mathbf{B}(s), & s \in [0, T], \\ W(0) = w_0 > 0, \end{cases} \quad (3)$$

where $\boldsymbol{\pi}(\cdot) = (\pi_1(\cdot), \dots, \pi_m(\cdot))'$ and $\boldsymbol{\theta}(\cdot) = (\mu_1(\cdot) - r(\cdot), \dots, (\mu_m(\cdot) - r(\cdot)))'$.

By using Itô's formula, we derive the dynamics of log-return $R(s)$ as

$$\begin{cases} dR(s) = d(\ln W(s)) = [r(s) + \boldsymbol{\theta}'(s)\boldsymbol{\pi}(s) - \frac{1}{2}\boldsymbol{\pi}(s)' \boldsymbol{\sigma}(s) \boldsymbol{\sigma}(s)' \boldsymbol{\pi}(s)]ds \\ \quad + \boldsymbol{\pi}(s)' \boldsymbol{\sigma}(s) d\mathbf{B}(s), & s \in [0, T], \\ R(0) = \ln w_0. \end{cases} \quad (4)$$

Definition 2.1. A strategy $\boldsymbol{\pi}(\cdot)$ is said to be admissible if $\boldsymbol{\pi}(\cdot) \in L_{\mathbb{F}}^2(0, T; \mathbb{R}^m)$ such that the stochastic differential equation (SDE) in (4) admits a unique solution $R(\cdot) \in L_{\mathbb{F}}^2(\Omega; C([0, T]; \mathbb{R}))$.

For any initial state (t, R_t) , we assume that the investor is guided by a mean-variance criteria over the log-return $R(T)$. Our objective is to find the trade-off between the expectation and variance of $R(T)$, namely, to minimize

$$J(t, R_t; \boldsymbol{\pi}(\cdot)) = \frac{1}{2} \text{Var}_t[R(T)] - \lambda \mathbb{E}_t[R(T)], \quad t \in [0, T], \quad (5)$$

where $\lambda > 0$ is a constant and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$. In the next section, we will investigate the open-loop equilibrium strategy of this problem through a sufficient condition.

3. The existence of the equilibrium strategy.

3.1. A sufficient condition for the equilibrium strategy. Problem (5) is time-inconsistent due to the variance term in the sense that it does not follow the Bellman optimality principle. Instead of an optimal strategy, we try to find the equilibrium strategy which is usually defined in two ways: an open-loop strategy inspired from [10] and a closed-loop strategy developed in [3]. The authors investigate the closed-loop case in [7], our paper concentrates on the open-loop strategy. Firstly, we present a general sufficient condition for the open-loop equilibrium strategy.

For any $t \in [0, T]$, $\varepsilon > 0$ and $\nu \in L^2_{\mathcal{F}_t}(\Omega; R^m)$, given a control $\bar{\pi}(\cdot)$ we define

$$\pi^{t,\varepsilon,\nu}(s) = \bar{\pi}(s) + \nu I_{s \in [t, t+\varepsilon]}(s), \quad s \in [t, T]. \quad (6)$$

Definition 3.1. Let $\bar{\pi}(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ be a given strategy and $\bar{R}(\cdot)$ be the corresponding log-return process in (4). The control $\bar{\pi}(\cdot)$ is called an equilibrium if

$$\liminf_{\varepsilon \downarrow 0} \frac{J(t, \bar{R}(t); \pi^{t,\varepsilon,\nu}(\cdot)) - J(t, \bar{R}(t); \bar{\pi}(\cdot))}{\varepsilon} \geq 0.$$

Theorem 3.2. $\bar{\pi}(\cdot) \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^m)$ is an open-loop equilibrium strategy if for any $t \in [0, T]$ there exists a pair of processes $(Y(\cdot; t), \mathbf{Z}(\cdot; t)) \in L^2_{\mathbb{F}}(\Omega; (C[t, T]; \mathbb{R})) \times L^2_{\mathbb{F}}(\Omega; L^2(t, T; \mathbb{R}^m))$ that solves the system of FBSDE

$$\begin{cases} dR(s) = [r(s) + \theta'(s)\pi(s) - \frac{1}{2}\pi(s)'\sigma(s)\sigma(s)'\pi(s)]ds \\ \quad + \pi(s)'\sigma(s)d\mathbf{B}(s), \quad s \in [0, T], \\ R(0) = R_0, \\ dY(s; t) = \mathbf{Z}'(s; t)d\mathbf{B}(s), \quad s \in [t, T], \\ Y(T; t) = \bar{R}(T) - E_t[\bar{R}(T)] - \lambda, \end{cases} \quad (7)$$

and it holds condition

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E_t \left[\int_t^{t+\varepsilon} \Lambda'(s; t) ds \right] = \mathbf{0}, \quad a.s., \quad (8)$$

where $\Lambda(s; t) = Y(s; t)\theta(s) + \sigma(s)\mathbf{Z}(s; t) - Y(s; t)\pi'(s)\sigma(s)\sigma'(s)\pi(s)$ for $s \in [t, T]$.

Proof. Let $\bar{\pi}(\cdot)$ satisfy condition (8). Consider the strategy $\pi^{t,\varepsilon,\nu}(\cdot)$ given by (6), we define $R_1^{t,\varepsilon,\nu}(\cdot) = R^{t,\varepsilon,\nu}(\cdot) - \bar{R}(\cdot)$, where $\bar{R}(\cdot)$ and $R^{t,\varepsilon,\nu}(\cdot)$ are the log-return rate processes associated with $\bar{\pi}(\cdot)$ and $\pi^{t,\varepsilon,\nu}(\cdot)$, respectively. Then the dynamics of $R_1^{t,\varepsilon,\nu}(\cdot)$ is shown as

$$\begin{cases} dR_1^{t,\varepsilon,\nu}(s) = [\theta(s)'\nu I_{[t, t+\varepsilon]}(s) - \frac{1}{2}\nu'\sigma(s)\sigma'(s)\nu I_{[t, t+\varepsilon]}(s) \\ \quad - \pi'(s)\sigma(s)\sigma'(s)\nu I_{[t, t+\varepsilon]}(s)]ds \\ \quad + \nu'\sigma(s)I_{[t, t+\varepsilon]}(s)d\mathbf{B}(s), \quad s \in [t, T], \\ R_1^{t,\varepsilon,\nu}(t) = 0, \end{cases} \quad (9)$$

and $R_1^{t,\varepsilon,\nu}(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([t, T]; \mathbb{R}))$.

It is straightforward to check that

$$\begin{aligned} & J(t, \bar{R}(t); \pi^{t,\varepsilon,\nu}(\cdot)) - J(t, \bar{R}(t); \bar{\pi}(\cdot)) \\ &= \frac{1}{2} E_t[(R_1^{t,\varepsilon,\nu}(T) - E_t[R_1^{t,\varepsilon,\nu}(T)])R_1^{t,\varepsilon,\nu}(T)] \\ & \quad + E_t[(\bar{R}(T) - E_t[\bar{R}(T)] - \lambda)R_1^{t,\varepsilon,\nu}(T)] \\ &= J_1(t) + J_2(t), \end{aligned}$$

where

$$\begin{aligned} J_1(t) &= \frac{1}{2} E_t[(R_1^{t,\varepsilon,\nu}(T) - E_t[R_1^{t,\varepsilon,\nu}(T)])R_1^{t,\varepsilon,\nu}(T)], \\ J_2(t) &= E_t[(\bar{R}(T) - E_t[\bar{R}(T)] - \lambda)R_1^{t,\varepsilon,\nu}(T)]. \end{aligned}$$

Obviously, we have

$$\begin{aligned} J_1(t) &= \frac{1}{2} \left\{ E_t[(R_1^{t,\varepsilon,\nu}(T))^2] - (E_t[R_1^{t,\varepsilon,\nu}(T)])^2 \right\} \\ &= \frac{1}{2} \text{Var}_t[R_1^{t,\varepsilon,\nu}(T)] \geq 0. \end{aligned}$$

Recalling that the pair $(Y(\cdot; t), \mathbf{Z}(\cdot; t))$ solves the backward equation in (7). Applying Itô's formula to $(Y(\cdot; t)R_1^{t,\varepsilon,\nu}(\cdot))$, it yields that

$$\begin{aligned} d(Y(s; t)R_1^{t,\varepsilon,\nu}(s)) &= \left(\Lambda'(s; t)\nu I_{[t,t+\varepsilon]}(s) - \frac{1}{2}Y(s; t)\nu'\sigma(s)\sigma'(s)\nu I_{[t,t+\varepsilon]}(s) \right) ds \\ &\quad + \left(Y(s; t)\nu'\sigma(s)I_{[t,t+\varepsilon]}(s) + R_1^{t,\varepsilon,\nu}(s)\mathbf{Z}'(s; t) \right) d\mathbf{B}(s). \end{aligned}$$

By the integrability of $Y(\cdot; t)$ and $\mathbf{Z}(\cdot; t)$, we have

$$\begin{aligned} J_2(t) &= \mathbb{E}_t[(\bar{R}(T) - \mathbb{E}_t[\bar{R}(T)] - \lambda)R_1^{t,\varepsilon,\nu}(T)] \\ &= \mathbb{E}_t \int_t^T \left[\Lambda'(s; t)\nu I_{[t,t+\varepsilon]}(s) - \frac{1}{2}Y(s; t)\nu'\sigma(s)\sigma'(s)\nu I_{[t,t+\varepsilon]}(s) \right] ds, \end{aligned}$$

and

$$\begin{aligned} &\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [J(t, \bar{R}(t); \pi^{t,\varepsilon,\nu}(\cdot)) - J(t, \bar{R}(t); \bar{\pi}(\cdot))] \\ &= \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [J_1(t) + J_2(t)] \\ &\geq \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_t \int_t^{t+\varepsilon} \left[\Lambda'(s; t)\nu I_{[t,t+\varepsilon]}(s) - \frac{1}{2}Y(s; t)\nu'\sigma(s)\sigma'(s)\nu I_{[t,t+\varepsilon]}(s) \right] ds. \end{aligned}$$

Note that $Y(s; t)$ in (7) is a martingale, which means that $\mathbb{E}_t Y(s; t) = \mathbb{E}_t Y(T; t) = -\lambda \leq 0$, then we have

$$\mathbb{E}_t \int_t^{t+\varepsilon} \frac{1}{2} Y(s; t) \nu' \sigma(s) \sigma'(s) \nu I_{[t,t+\varepsilon]}(s) ds \leq 0.$$

By condition (8), we can see that

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathbb{E}_t \left[\int_t^{t+\varepsilon} \Lambda'(s; t) ds \right] \nu = 0.$$

Then we have $\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} [J(t, \bar{R}(t); \pi^{t,\varepsilon,\nu}(\cdot)) - J(t, \bar{R}(t); \bar{\pi}(\cdot))] \geq 0$. The proof is completed. $\square$

Remark 1. In terms of constructing an equilibrium strategy according to Theorem 3.2, we usually simplify the limit in condition (8) by letting

$$\Lambda(t; t) = 0. \quad (10)$$

That is, in the next subsection, we try to construct $Y(\cdot; t)$ and $\mathbf{Z}(\cdot; t)$ such that

$$0 = \Lambda(t; t) = Y(t; t)\theta(t) + \sigma(t)\mathbf{Z}(t; t) - Y(t; t)\pi'(t)\sigma(t)\sigma'(t).$$

3.2. An equilibrium strategy. In this subsection, an equilibrium strategy $\bar{\pi}(\cdot)$ will be constructed according to Theorem 3.2. We suppose that $Y(s; t)$ can be expressed as

$$Y(s; t) = \bar{R}(s) - \mathbb{E}_t[\bar{R}(s)] + \lambda \mathbb{E}_t[P_2(s)] - \lambda P_3(s) + \ln P_1(s),$$

where $(P_1(\cdot), \mathbf{Q}_1(\cdot))$ is the unique solution of BSDE

$$\begin{cases} dP_1(s) = -r(s)P_1(s)ds + \mathbf{Q}_1'(s)d\mathbf{B}(s), & s \in [0, T], \\ P_1(T) = 1, \end{cases} \quad (11)$$

and $(P_i(\cdot), \mathbf{Q}_i(\cdot))$, for $i = 2, 3$, are the unique solutions to the following BSDEs

$$\begin{cases} dP_i(s) = -f_i(s, P_i(s), \mathbf{Q}_i(s))ds + \mathbf{Q}_i'(s)d\mathbf{B}(s), & s \in [0, T], \\ P_i(T) = 0, \text{ if } i = 2, \text{ and } P_i(T) = 1, \text{ if } i = 3, \end{cases} \quad (12)$$

where

$$\begin{aligned}
f_2(s, P_2, \mathbf{Q}_2) &= \frac{1}{\lambda} \left\{ -r - \frac{1}{\ln P_1 + \lambda(P_2 - P_3) - 1} \theta'(\sigma')^{-1} [(\ln P_1 + \lambda(P_2 - P_3)) \right. \\
&\quad \left. \theta'(\sigma')^{-1} - \lambda \mathbf{Q}_3 + \frac{\mathbf{Q}_1}{P_1}] + \frac{1}{(\ln P_1 + \lambda(P_2 - P_3) - 1)^2} [(\ln P_1 \right. \\
&\quad \left. + \lambda(P_2 - P_3)) \theta'(\sigma')^{-1} - \lambda \mathbf{Q}_3 + \frac{\mathbf{Q}_1}{P_1}]^2 \right\}, \\
f_3(s, P_3, \mathbf{Q}_3) &= \frac{1}{\lambda} \left\{ -r - \frac{1}{\ln P_1 + \lambda(P_2 - P_3) - 1} \theta'(\sigma')^{-1} [(\ln P_1 + \lambda(P_2 - P_3)) \right. \\
&\quad \left. \theta'(\sigma')^{-1} - \lambda \mathbf{Q}_3 + \frac{\mathbf{Q}_1}{P_1}] + \frac{1}{(\ln P_1 + \lambda(P_2 - P_3) - 1)^2} [(\ln P_1 \right. \\
&\quad \left. + \lambda(P_2 - P_3)) \theta'(\sigma')^{-1} - \lambda \mathbf{Q}_3 + \frac{\mathbf{Q}_1}{P_1}]^2 \right\} + \frac{1}{\lambda} \left(r + \frac{1}{2} \frac{\mathbf{Q}_1' \mathbf{Q}_1}{P_1^2} \right).
\end{aligned}$$

Thanks to proposition 3.5 in [17], for $i = 1, 2, 3$, $P_i(\cdot)$ are bounded and $\int_0^\cdot \mathbf{Q}_i(s) dB(s)$ is a BMO-martingale. The existence and uniqueness of the solutions to (12) can be proved. Here, we omit the details.

Theorem 3.3. *For any constant $p > 1$, $(P_i(\cdot), \mathbf{Q}_i(\cdot)) \in L^p(\Omega; C([0, T]; \mathbb{R})) \times L^p(\Omega; L^2(0, T; \mathbb{R}^m))$, for $i = 2, 3$, are the unique solutions to equation (11) and (12). The strategy given by*

$$\begin{aligned}
\bar{\pi}(\cdot) &= \frac{(\sigma(\cdot))^{-1}}{\ln P_1(\cdot) + \lambda(P_2(\cdot) - P_3(\cdot)) - 1} \left[(\ln P_1(\cdot) + \frac{\mathbf{Q}_1(\cdot)}{P_1(\cdot)} \right. \\
&\quad \left. + \lambda(P_2(\cdot) - P_3(\cdot))) \sigma^{-1}(\cdot) \theta(\cdot) - \lambda \mathbf{Q}_3(\cdot) \right],
\end{aligned} \tag{13}$$

is an equilibrium control. If

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\mathbf{Q}_1(t)|^2 \right] < \infty. \tag{14}$$

Proof. Firstly, we show the existence of the solution to the backward equation in (7). By using Itô's formula, we have (suppressing the variable s)

$$\begin{aligned}
dY(s; t) &= \left[r + \theta' \bar{\pi} - \frac{1}{2} \bar{\pi}' \sigma \sigma' \bar{\pi} + \lambda f_3(s, P_3, \mathbf{Q}_3) - r - \frac{1}{2} \frac{\mathbf{Q}_1' \mathbf{Q}_1}{P_1^2} \right] ds \\
&\quad - \mathbb{E}_t \left[r + \theta' \bar{\pi} - \frac{1}{2} \bar{\pi}' \sigma \sigma' \bar{\pi} + \lambda f_2(s, P_2, \mathbf{Q}_2) \right] ds \\
&\quad + (\bar{\pi}' \sigma - \lambda \mathbf{Q}_3' + \frac{\mathbf{Q}_1'}{P_1}) dB(s).
\end{aligned}$$

Plugging the repressions of $f_2(s, P_2, \mathbf{Q}_2)$, $f_3(s, P_3, \mathbf{Q}_3)$ and $\bar{\pi}(\cdot)$, we have

$$\begin{aligned}
0 &= r + \theta' \bar{\pi} - \frac{1}{2} \bar{\pi}' \sigma \sigma' \bar{\pi} + \lambda f_2(s, P_2, \mathbf{Q}_2) \\
&= r + \frac{1}{\ln P_1 + \lambda(P_2 - P_3) - 1} \theta'(\sigma')^{-1} [(\ln P_1 + \lambda(P_2 - P_3)) \theta'(\sigma')^{-1} - \lambda \mathbf{Q}_3 + \frac{\mathbf{Q}_1}{P_1}] \\
&\quad - \frac{1}{(\ln P_1 + \lambda(P_2 - P_3) - 1)^2} [(\ln P_1 + \lambda(P_2 - P_3)) \theta'(\sigma')^{-1} - \lambda \mathbf{Q}_3 + \frac{\mathbf{Q}_1}{P_1}]^2 \\
&\quad + \lambda f_2(s, P_2, \mathbf{Q}_2), \\
0 &= r + \theta' \bar{\pi} - \frac{1}{2} \bar{\pi}' \sigma \sigma' \bar{\pi} + \lambda f_3(s, P_3, \mathbf{Q}_3) - r - \frac{1}{2} \frac{\mathbf{Q}_1' \mathbf{Q}_1}{P_1^2}.
\end{aligned}$$

Thus, by comparing the coefficients, we get

$$\begin{aligned} Z(s; t) &= \sigma' \bar{\pi} - \lambda Q_3 + \frac{Q_1}{P_1}, \\ Y(s; t) &= \bar{R} - E_t[\bar{R}] + \lambda E_t[P_2] - \lambda P_3 + \ln P_1. \end{aligned} \quad (15)$$

It is easy to see that, for any $t \in [0, T]$, $Y(s; t)$ and $Z(s; t)$ solve the backward equation in (7).

Then, we verify that condition (8) is satisfied. It follows from (15) that

$$\begin{aligned} \Lambda(s; t) &= Y(s; t)\theta(s) + \sigma(s)Z(s; t) - Y(s; t)\pi'(s)\sigma(s)\sigma'(s) \\ &= \Gamma(s; t) \left[-\lambda\sigma(s)Q_3(s) + \sigma(s)Q_1(s) + \theta(s) \right], \end{aligned}$$

where

$$\Gamma(s; t) = \frac{1}{(\ln P_1 + \lambda(P_2 - P_3) - 1)^2} [\lambda + Y(s; t)].$$

Noting that $\Gamma(t; t) = 0$, and by Hölder's inequality, we have

$$\begin{aligned} & \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} E_t \left[\int_t^{t+\varepsilon} |\Gamma(s; t)\sigma(s)Q_1(s)| ds \right] \\ & \leq \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left\{ E_t \left[\int_t^{t+\varepsilon} |\Gamma(s; t)\sigma(s)|^2 ds \right] E_t \left[\int_t^{t+\varepsilon} |Q_1(s)|^2 ds \right] \right\}^{\frac{1}{2}} \\ & \leq \liminf_{\varepsilon \downarrow 0} \left\{ \frac{1}{\varepsilon} E_t \left[\int_t^{t+\varepsilon} |\Gamma(s; t)\sigma(s)|^2 ds \right] \right\}^{\frac{1}{2}} \times \left\{ E_t \left[\sup_{s \in [0, T]} |Q_1(s)|^2 \right] \right\}^{\frac{1}{2}} = 0. \end{aligned}$$

According to Theorem 3.2, $\bar{\pi}(\cdot)$ is an equilibrium strategy. The proof is completed. $\square$

4. An application to the stochastic interest rate. As we have claimed before, the investment strategy derived in this paper satisfies the practical investment wisdom. In this section, to show how it works, we derive both the open-loop and closed-loop equilibrium strategies for a special case where the interest rate process follows the Vasicek model. As we know, the open-loop equilibrium strategy depends on the initial state, but the closed-loop equilibrium strategy does not depend on it. For equilibrium strategies, the existence of closed-loop strategy does not imply the existence of open-loop strategy, as they are essentially different. In their definitions, the local optimality conditions are different.

According to the Vasicek model, the interest rate process is described as

$$\begin{cases} dr(s) = (\zeta - \xi r(s))ds + \rho dB(s), & s \in [0, T], \\ r(0) = r_0, \end{cases} \quad (16)$$

where $\zeta, \xi > 0$ and $\rho \in \mathbb{R}$. To simplify the computations, we assume that there is only one risky asset, that is, $m = 1$, and $\sigma(\cdot)$ and $\theta(\cdot)$ are both bounded deterministic functions.

4.1. Open-loop equilibrium strategy. In this subsection, we apply the results in Section 3 to the Vasicek model.

Proposition 1. *Let $r(\cdot)$ be given by (16) and we assume that the solutions to the system of BSDEs (11) and (12) are given by*

$$(P_1(t), Q_1(t)) = (G_1(t)e^{g(t)r(t)}, \rho g(t)G_1(t)e^{g(t)r(t)}),$$

$$\begin{aligned}(P_2(t), Q_2(t)) &= (G_2(t), 0), \\ (P_3(t), Q_3(t)) &= (G_3(t), 0),\end{aligned}$$

where $g(t) = \frac{1}{\xi}(1 - e^{\xi(T-t)})$. Then

$$\begin{aligned}G_1(t) &= \exp \left\{ \int_t^T g(s) \left[\xi + \frac{1}{2} \rho^2 g(s) \right] ds \right\}, \\ G_2(t) &= \int_t^T \frac{1}{\lambda} \left[r - \frac{1}{\lambda G_4(s) + \ln G_1(s) + g(s)r - 1} \theta(s) \sigma^{-1}(s) [(\lambda G_4(s) + \ln G_1(s) \right. \\ &\quad \left. + g(s)r) \theta(s) \sigma^{-1}(s) - \rho g(s)] - \frac{\lambda^2}{(\lambda G_4(s) + \ln G_1(s) + g(s)r - 1)^2} \right. \\ &\quad \left. [(\lambda G_4(s) + \ln G_1(s) + g(s)r) \theta(s) \sigma^{-1}(s) - \rho g(s)]^2 \right] ds, \\ G_3(t) &= G_2(t) - G_4(t), \\ G_4(t) &= - \int_t^T \frac{1}{\lambda} \left[r + \frac{1}{2} \rho^2 g(s)^2 \right] ds - 1,\end{aligned}$$

and the open-loop equilibrium percentage allocation $\bar{\pi}(\cdot)$ can be simplified as

$$\bar{\pi}(t) = \frac{(\sigma(t))^{-1}}{G(t) - 1} [G(t) \sigma^{-1}(t) \theta(t) + \rho g(t)], \quad (17)$$

where $G(t) = \ln G_1(t) + g(t)r(t) + \lambda G_4(t)$.

Proof. It is easy to see that

$$\lambda f_2(s, P_2, \mathbf{Q}_2) = \lambda f_3(s, P_3, \mathbf{Q}_3) - r - \frac{1}{2} \frac{\mathbf{Q}_1' \mathbf{Q}_1}{P_1^2},$$

then

$$\begin{cases} d(P_2(s) - P_3(s)) = [-f_2(s, P_2(s), \mathbf{Q}_2(s)) + f_3(s, P_3(s), \mathbf{Q}_3(s))] ds \\ \quad + (\mathbf{Q}_2(s) - \mathbf{Q}_3(s)) d\mathbf{B}(s), \quad s \in [0, T], \\ P_2(T) - P_3(T) = -1. \end{cases}$$

To simplify the notation, we let $(P_4(\cdot), \mathbf{Q}_4(\cdot)) = (P_2(\cdot) - P_3(\cdot), \mathbf{Q}_2(\cdot) - \mathbf{Q}_3(\cdot))$, and it satisfies the BSDE as follows:

$$\begin{cases} dP_4(s) = \frac{1}{\lambda} \left\{ r + \frac{1}{2} \frac{\mathbf{Q}_1' \mathbf{Q}_1}{P_1^2} \right\} ds + \mathbf{Q}_4(s) d\mathbf{B}(s), \quad s \in [0, T], \\ P_4(T) = -1. \end{cases}$$

Motivated by the generalized Feynman-Kac formulation, we consider the following system of partial differential equations (PDEs):

$$\begin{cases} F_{i,t}(t, r) + F_{i,r}(\zeta - \xi r) + \frac{1}{2} F_{i,rr}(t, r) \rho^2 \\ \quad + f_i(t, F_i(t, r), F_{i,r}(t, r) \rho) = 0, \quad t \in [0, T], \\ F_i(T, r) = P_i(T), \quad i = 1, 2, 3, 4. \end{cases} \quad (18)$$

If the unique solution to equation (18) exists, by Itô's formula we obtain

$$(P_i(\cdot), Q_i(\cdot)) = (F_i(\cdot, r(\cdot)), F_{i,r}(\cdot, r(\cdot) \rho)),$$

which solves the BSDEs in (11) and (12), for $i = 1, 2, 3, 4$. Consider the following ansatz:

$$\begin{aligned} F_1(t, r) &= G_1(t)e^{g(t)r}, \\ F_2(t, r) &= G_2(t), \\ F_3(t, r) &= G_2(t) - G_4(t), \\ F_4(t, r) &= G_4(t). \end{aligned}$$

Plugging these expressions into (18), we obtain

$$\begin{cases} G_{1,t}(t) + G_1(t)g(t)\left(\zeta + \frac{1}{2}g(t)\rho^2\right) = 0, & t \in [0, T], \\ G_1(T) = 1, \end{cases} \quad (19)$$

$$\begin{cases} G_{2,t}(t) + \frac{1}{\lambda} \left[r - \frac{1}{\lambda G_4(s) + \ln G_1(s) + g(s)r - 1} \theta(s)\sigma(s)^{-1} [(\lambda G_4(s) + \ln G_1(s) + g(s)r)\theta(s)\sigma(s)^{-1} - \rho g(s)] - \frac{\lambda^2}{(\lambda G_4(s) + \ln G_1(s) + g(s)r - 1)^2} [(\lambda G_4(s) + \ln G_1(s) + g(s)r)\theta(s)\sigma(s)^{-1} - \rho g(s)]^2 \right] = 0, & t \in [0, T], \\ G_2(T) = 0, \end{cases} \quad (20)$$

$$\begin{cases} G_{4,t}(t) - \frac{1}{\lambda} \left[r + \frac{1}{2}\rho^2 g(s)^2 \right] = 0, & t \in [0, T], \\ G_4(T) = -1. \end{cases} \quad (21)$$

Solving these ODEs, we obtain the expressions of $G_i(\cdot)$, for $i = 1, 2, 3, 4$, in Proposition 1. Plugging them into equation (13), then the open-loop equilibrium strategy becomes

$$\begin{aligned} \bar{\pi}(t) &= \frac{(\sigma(t))^{-1}}{\ln G_1(t) + g(t)r(t) + \lambda G_4(t) - 1} \left[(\ln G_1(t) + g(t)r(t) \right. \\ &\quad \left. + \lambda G_4(t))\sigma^{-1}(t)\theta(t) + \rho g(t) \right]. \end{aligned}$$

The proof is completed. $\square$

Remark 2. In a traditional financial market, the return rate from a risky asset is usually higher than the risk-free rate, that is, $\theta(t) = \mu(t) - r(t) > 0$. For the open-loop equilibrium strategy $\bar{\pi}(t)$ in (17), the coefficient is positive. It is obvious to see that as the expected return rate from the risky asset $\mu(t)$ increases, equation (17) leads to an increasing value of $\bar{\pi}(t)$, which means that a higher proportion should be invested in the risky asset. This is consistent with the conventional investment wisdom.

Remark 3. We also notice that in the expression of equilibrium strategy $\bar{\pi}(t)$ in (17), as the value of t decreases, the coefficient $g(t) = \frac{1}{\xi}(1 - e^{\xi(T-t)})$ also decreases, which leads to an increasing investment proportion $\bar{\pi}(t)$ in the risky asset. This is consistent with the conventional wisdom that the younger the investor, the higher proportion invested in the risky asset.

Remark 4. In addition, if we let $T - t$ go to infinity, then $g(t) = \frac{1}{\xi}(1 - e^{\xi(T-t)})$ goes to infinity and $\frac{G(t)}{G(t)-1}$ goes to 1. It is obvious to see in equation (17) that $\bar{\pi}(t)$ remains positive. That is, no short-selling is presented in the equilibrium strategy in the long run, which is another desired property of an investment strategy.

4.2. Closed-loop equilibrium strategy. In this subsection, we will present the closed-loop equilibrium strategy for the log-return mean-variance portfolio selection problem. Using the approach derived by [4], the optimal strategy is obtained by a system of extended HJB equation for a special case where the interest rate process follows the Vasicek model.

The problem and model setting are same as our problem above. The dynamics of log-return $R(s)$ are given by

$$\begin{cases} dR(s) = [r(s) + \theta'(s)\pi(s) - \frac{1}{2}\pi(s)'\sigma(s)\sigma(s)'\pi(s)]ds \\ \quad + \pi(s)'\sigma(s)d\mathbf{B}(s), & s \in [0, T], \\ R(0) = \ln w_0. \end{cases}$$

For convenience, we rewrite the objective function (5) as

$$J(t, R_t; \pi(\cdot)) = E_t[F(R(T))] + G(E_t[R(T)]),$$

where $F(R) = \frac{1}{2}R^2 - \lambda R$, $G(R) = -\frac{1}{2}R^2$. As seen in [4], the term $G(E_t[R(T)])$ leads to a time-inconsistent problem. Our objective is to find the closed-loop equilibrium strategy $\hat{\pi}(\cdot)$.

By using the approach inspired from [4], we can get the extended HJB equations directly, which are given by

$$\begin{aligned} \inf \{ A^\pi V(t, R, r) - A^\pi G \circ h(t, R, r) + H^\pi(t, R, r) \} &= 0, \\ V(T, R, r) &= -\lambda R, \\ A^{\hat{\pi}} h(t, R, r) &= 0, \\ h(T, R, r) &= R, \end{aligned}$$

where $\hat{\pi}$ is the strategy which minimizes the first equation of HJB systems, $h(t, R, r) = E_t[R_T]$, $G \circ h(t, R, r) = G(h(t, R, r))$, $H^\pi h(t, R, r) = G_x(h(t, R, r))A^\pi h(t, R, r)$ and the operator A^π is defined as

$$\begin{aligned} A^\pi V(t, R, r) &= V_t(t, R, r) + [\theta(t)\pi + r(t) - \frac{1}{2}\pi^2\sigma^2(t)]V_R(t, R, r) \\ &\quad + \frac{1}{2}\pi^2\sigma^2(t)V_{RR}(t, R, r) + (\zeta - \xi r)V_r(t, R, r) + \frac{1}{2}\rho^2V_{rr}V_r(t, R, r) \\ &\quad + \pi\sigma(t)\rho V_{Rr}(t, R, r). \end{aligned}$$

Therefore, the extended HJB equations can be simplified as

$$\begin{aligned} \inf \{ A^\pi V(t, R, r) + \frac{1}{2}\pi^2\sigma^2(t)h_R^2(t, R, r) + \frac{1}{2}\rho^2h_r^2(t, R, r) \\ \quad + \pi\sigma(t)\rho h_R(t, R, r)h_r(t, R, r) \} &= 0, \\ V(T, R, r) &= -\lambda R, \\ A^{\hat{\pi}} h(t, R, r) &= 0, \\ h(T, R, r) &= R. \end{aligned}$$

Considering the ansatz

$$V(t, R, r) = A(t)R + g(t)r + B(t),$$

$$h(t, R, r) = H(t)R + g(t)r + I(t),$$

we have

$$\begin{aligned} A_t R + g_t r + B_t + [\theta(t)\pi + r - \frac{1}{2}\pi^2\sigma^2(t)]A(t) + (\zeta - \xi r)g(t) \\ + \frac{1}{2}\pi^2\sigma^2(t)H(t)^2 + \frac{1}{2}\rho^2g^2(t) + \pi\sigma(t)\rho H(t)g(t) = 0. \end{aligned}$$

Then according to the first order condition, we obtain the optimal control as

$$\hat{\pi} = \frac{\theta(t)A(t) + \rho\sigma(t)H(t)g(t)}{\sigma^2(t)A(t) - \sigma^2(t)H^2(t)}. \quad (22)$$

Plugging (22) into the extended HJB system and equating the coefficients, we obtain

$$\begin{cases} A_t(t) = 0, \\ A(T) = -\lambda, \\ H_t(t) = 0, \\ H(T) = 1, \end{cases} \quad (23)$$

which leads to the expression of the equilibrium strategy

$$\hat{\pi}(t) = \frac{1}{\sigma(t)} \left[\frac{\lambda}{\lambda+1} \frac{\theta(t)}{\sigma(t)} - \frac{\rho g(t)}{\lambda+1} \right].$$

Remark 5. (A comparison). Assume that the interest rate is a constant $r > 0$ which means $\zeta = \xi = \rho = 0$. In this case, $g(t) = T - t$. The open-loop strategy becomes

$$\bar{\pi}(t) = \frac{\lambda}{\lambda+1} \frac{\theta(t)}{\sigma^2(t)},$$

and the closed-loop strategy becomes

$$\hat{\pi}(t) = \frac{\lambda}{\lambda+1} \frac{\theta(t)}{\sigma^2(t)}.$$

If the interest rate is random, the open-loop strategy is shown as

$$\bar{\pi}(t) = \frac{(\sigma(t))^{-1}}{G(t)-1} \left[G(t)\sigma^{-1}(t)\theta(t) + \rho g(t) \right],$$

and the closed-loop strategy is

$$\hat{\pi}(t) = \frac{1}{1+\lambda} \frac{1}{\sigma(t)} \left[\lambda \frac{\theta(t)}{\sigma(t)} - \rho g(t) \right].$$

It is straightforward to see that if $\rho = 0$, the open-loop and closed-loop equilibrium strategies for this problem are the same. If $\rho \neq 0$, then the interest rate is random, the different coefficients indicate the difference between two kinds of equilibrium percentage allocations.

5. Concluding remarks. This paper studies the open-loop and closed-loop equilibrium strategies for a mean-variance portfolio selection problem based on a log-return approach. With the help of FBSDEs, we derive the explicit expression for the equilibrium percentage allocation, which suggests that the young and the rich should invest more money in risky assets and the short sale is prohibited in the long run.

Note that we only show the existence of the equilibrium strategy in this paper. The uniqueness of the equilibrium strategy would be proved in our future research work. It would be also interesting to extend our model to more concrete cases. For example, we can apply the approach to the investment-reinsurance problem

and investigate the economic property of the resulting equilibrium strategy. We can also generalize these problems with the incorporation of stochastic volatility or under constant elasticity variance (CEV) model.

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E-mail address: jiannanz2@student.unimelb.edu.au

E-mail address: pche@unimelb.edu.au

E-mail address: zhuo.jin@unimelb.edu.au

E-mail address: shli@unimelb.edu.au