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The second largest eigenvalue of Cayley graphs on symmetric groups

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Abstract

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The spectral gap of a regular graph is defined as the difference between the two largest eigenvalues of its adjacency matrix. It is a significant algebraic parameter reflecting the geometric properties of the graph, such as connectivity and expansion properties. The spectral gap is also a key index indicating the convergence rates of random processes on the graph.

Cayley graphs are ideal candidates for constructing expanders due to their regularity and excellent symmetry. Thus, investigating the second largest eigenvalue of Cayley graphs is of great importance. One of the most famous results on this topic is Aldous' spectral gap conjecture, proposed by Aldous in 1992 and completely confirmed by Caputo, Liggett, and Richthammer in 2010. It states that for any finite connected graph Γ of order n, two continuous-time Markov Chains on Γ, the interchange process and the random walk, have the same spectral gap. In terms of algebraic graph theory, Aldous' spectral gap conjecture is equivalent to stating that the Cayley graph on S_n generated by transpositions corresponding to edges of Γ has the same spectral gap as Γ. Another equivalent form is that the second largest eigenvalue of any connected Cayley graph on S_n generated by transpositions is achieved by the standard representation of S_n.

The purpose of this thesis is to identify more general families of Cayley graphs on symmetric groups S_n possessing the Aldous property, that is, their second largest eigenvalues are achieved by the standard representation of S_n. In Chapter 3, three families of normal Cayley graphs are proved to possess the Aldous property. In Chapter 4, the second largest eigenvalue of normal Cayley graphs on S_n generated by cycles is considered. As a corollary, a recent conjecture about the Aldous property of the Cayley graph on S_n generated by cycles of a fixed length k $(2 \leq k \leq n-2)$ is confirmed. Chapter 5 provides the solution to a conjecture on the Aldous property of a nonnormal family of Cayley graphs generated by cycles. The final chapter outlines unsolved problems from Chapters 3 – 5 and potential directions for future research.

Declaration of Authorship

I, Yuxuan Li, declare that this thesis titled "The second largest eigenvalue of Cayley graphs on symmetric groups" and the work presented in it are my own. I confirm that:

- The thesis comprises only my original work towards the PhD except where indicated in the preface;
- due acknowledgement has been made in the text to all other material used; and
- the thesis is fewer than the maximum word limit in length, exclusive of tables, maps, bibliographies and appendices as approved by the Research Higher Degrees Committee.

Signed:

Date:

Preface

This work was completed under the supervision of Sanming Zhou and Binzhou Xia (University of Melbourne).

- The work in Chapters 3, 4 and 5 of this thesis is based on the following three papers carried out in collaboration with others, respectively. For each of the three papers, I am the primary author and my contribution to it is more than 50%.

    1. Yuxuan Li, Binzhou Xia and Sanming Zhou, Aldous' spectral gap property for normal Cayley graphs on symmetric groups. European Journal of Combinatorics, 110:103657, 2023.
    2. Yuxuan Li, Binzhou Xia and Sanming Zhou, The second largest eigenvalue of normal Cayley graphs on symmetric groups generated by cycles. Journal of Combinatorial Theory, Series A, 206:105885, 2024.
    3. Yuxuan Li, Binzhou Xia and Sanming Zhou, The second largest eigenvalue of some nonnormal Cayley graphs on symmetric groups. Submitted on 4 February 2024 (see arXiv preprint (2024), `https://doi.org/10.48550/arXiv.2402.02427`)

    Aside from the previously mentioned chapters, I am the sole contributor to all other parts of this thesis

- This thesis contains no material submitted for other qualifications or carried out prior to enrolment in the degree.

- No third party editorial assistance was provided in preparation of the thesis.

- The MAGMA computer algebra system [17] was used for testing theories and conducting experiments.

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Acknowledgements

> Mathematics is the language with which God has written the universise.
>
> Galileo Galilei

As I near the completion of this journey, I recall the day I began learning mathematics—12 years ago. Completing a PhD after 12 continuous years of study has been incredibly challenging. I now realize that without the help of others, I could never have done this alone.

First, I would like to thank my parents, who provided me with the opportunity to receive an education. They started my mathematics journey by suggesting I study mathematics when I had no idea what I liked. There were many times when I lacked confidence and struggled with the complexity of mathematics. I am deeply grateful to my parents for their consistent support and love, which allowed me to discover the beauty of mathematics and my passion for research.

I would also like to thank my principal supervisor, Prof. Sanming Zhou, for agreeing to be my supervisor and supporting my application for a PhD position at the University of Melbourne. At that time, I had little confidence in my potential and ability in mathematics. However, he trusted me, and encouraged me to pursue a PhD for the second time after my first attempt was unsuccessful. He revived my academic journey and gave me the chance to retry. Coming to Melbourne to pursue my PhD was the best decision of my life. I am truly thankful to Prof. Sanming Zhou for rebuilding my academic career.

I also want to express my gratitude to both Prof. Sanming Zhou and my co-supervisor, Dr. Binzhou Xia, for their excellent ideas during our regular meetings, their efficient feedback on every draft, their encouragement during my discouraged times, and their suggestions for my future development. Additionally, I am thankful to the chair of my advisory committee, Assoc. Prof. Lawrence Reeves, for his kind help and professional guidance over the past four years. They made the entire PhD process a delightful experience, worth remembering for a lifetime.

I would also like to thank my friends from HCGF, my flatmates, and my friends from our department for their care and friendship. Thank you for making my life in Melbourne colorful and sweet. Lastly, I want to express my love and gratitude to my partner. Without his love and support, I would never have made it through.

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Abbreviations

CTMC	**C**ontinuous-**T**ime **M**arkov **C**hain
RW	**R**andom **W**alk
IP	**I**nterchange **P**rocess
EP	Symmetric **E**xclusion **P**rocess
CEP	**C**olored **E**xclusion **P**rocess
NGEP	**N**ormal **G**eneralized **E**xclusion **P**rocess

Symbols

$\mathbb{N}$	set of nonnegative integers
$\mathbb{N}^+$	set of positive integers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_{\geq 0}$	set of nonnegative real numbers
$\subseteq$	subset symbol
$\subset$	proper subset symbol
S_n	symmetric group of degree n
A_n	alternating group of degree n
I_n	identity matrix of order n
$\mathrm{Cay}(G, S)$	Cayley graph on a group G with respect to a set S
$\mathrm{Spec}(\cdot)$	spectrum of a graph or a matrix
$\mathbb{C}S_n$	group algebra of S_n over $\mathbb{C}$
$\widehat{S_n}$	a complete list of pairwise inequivalent irreducible matrix representations of S_n
$M^{(n-1,1)}$	natural permutation module of S_n
$\rho_{(n)}$	trivial representation of S_n
$\rho_{(1^n)}$	sign representation of S_n
$\rho_{(n-1,1)}$	standard representation of S_n

$\zeta, \gamma \vdash n$	partitions ζ, γ of n
S^{ζ}	Specht module of S_n with respect to a partition ζ
ρ_{ζ}	matrix representation of S_n with respect to ζ
χ_{ζ}	character of ρ_{ζ}
$\tilde{\chi}_{\zeta}$	normalized character of ρ_{ζ}
d_{ζ}	dimension of ρ_{ζ}
ζ'	conjugate or transpose of a partition ζ

$c_i(\gamma)$	number of terms in a partition γ that are equal to i
$C(S_n, \gamma)$	conjugacy class of S_n corresponding to a partition γ
$\mathrm{sgn}(\gamma)$	sign of permutations in $C(S_n, \gamma)$
$\chi_\zeta(\gamma)$	character value of ρ_ζ on $C(S_n, \gamma)$
$\tilde{\chi}_\zeta(\gamma)$	normalized character value of ρ_ζ on $C(S_n, \gamma)$

Chapter 1

Introduction

This chapter commences with an introduction on the background and motivation behind the topic of this thesis, followed by a comprehensive literature review. The final section will then encapsulate the principal results of this thesis.

1.1 Background and motivation

Graph theory, a discipline within discrete mathematics, delves into the study of graphs, which are mathematical structures modeling relations between objects. The paper on *Seven Bridge Königsberg* written by L. Euler and published in 1736 [43] is widely regarded as the beginning of graph theory. After centuries of development, graph theory has become a fascinating area of study with a wide range of theoretical and practical implications. Numerous practical problems spanning physics [42], chemistry [12, 99, 100], biology [82], computer science [38] and social systems [53] can be effectively represented by graphs. In mathematics, graph theory and certain parts of geometry and topology mutually fertilize each other, and ultimately common development was brought about in these areas. For more information on graph theory and its applications, see [5, 14–16, 26, 50, 54, 102].

Algebraic graph theory constitutes a specialized domain of graph theory that employs algebraic methodologies to address graph-related problems, encompassing techniques from linear algebra and group theory, alongside exploration of graph invariants. One prominent branch of algebraic graph theory is called the spectral graph theory, which studies the correlation between structural properties of graphs and the eigenvalues of matrices associated with graphs. Spectral graph theory emerged in the 1950's as a result of the practical demands stemming from chemistry and physics. The first instance where graph spectra were introduced is the 1931 paper [59] by Hückel, and [29] is the first mathematical paper on graph spectra. Nowadays spectral graph theory has evolved into a systematic and integral framework, and it has become a powerful tool in numerous areas of science. See [19, 31, 32] for excellent textbooks on spectral graph theory and [30] for a survey of literature on application of graph spectra.

In the following, we provide an introductory overview of the motivations driving the issues explored in this thesis: the second largest eigenvalue (with respect to the adjacency matrix) of Cayley graphs on symmetric groups. Across the subsequent three subsections, it will become evident that the second largest eigenvalue of a graph serves as an indicator of the graph's connectivity, its expander properties, and the convergence rates of certain random processes occurring on the graph. Additionally, given the pivotal role of Cayley graphs as potential constructions for generating expanders, we introduce a tool for handling the eigenvalues of Cayley graphs, which is essential for comprehending the subject matter of this thesis. The selection of symmetric groups as the underlying group is closely tied to the well-known Aldous' spectral gap conjecture, which is throughly expounded upon in Subsection 1.1.3.

1.1.1 Cheeger inequality

One of the most important theorems in spectral graph theory is the renowned Cheeger inequality, which vividly illustrates the intricate interplay between Riemannian geometry and graph theory. The original isoperimetric constant is defined on a compact Riemannian manifold and measures how "thin" the bottleneck regions are in the manifold, and the Cheeger inequality [25], named after Jeff Cheeger, establishes a bound for the measure in terms of the spectral properties of the Laplace-Beltrami operator. This proved to be an influential idea for understanding geometric and topological properties of Riemannian manifolds. The subsequent discrete analogue of the Cheeger inequality offers valuable insights into how the spectral properties of a graph relate to its overall structure and connectivity.

We only consider finite simple undirected graphs in this thesis. Suppose that $\Gamma = (V(\Gamma), E(\Gamma))$ is such a graph, where $V(\Gamma)$ is the vertex set and $E(\Gamma) \subseteq \{\{x, y\} \mid x, y \in V(\Gamma), x \neq y\}$ is the edge set. Suppose that the *order* of Γ, that is, the cardinality of $V(\Gamma)$, is $n \in \mathbb{N}^+$. Denote the n vertices in $V(\Gamma)$ by $v_1, v_2, \ldots, v_n$. If $\{v_i, v_j\}$ is an edge of Γ, then we say that the two vertices v_i and v_j are *adjacent* to each other, and that the edge $\{v_i, v_j\}$ is *incident* to v_i and v_j. If any two vertices of Γ are adjacent, then Γ is said to be *complete*.

A *path* of length r from v_i to v_j in Γ is a sequence of $r+1$ distinct vertices starting with v_i and ending with v_j, where each pair of consecutive vertices is adjacent. If there is a path between any two distinct vertices in Γ, then Γ is said to be *connected*; otherwise it is *disconnected*. A graph $\Gamma' = (V(\Gamma'), E(\Gamma'))$ is a *subgraph* of Γ if $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$. A maximal connected subgraph of Γ is called a *connected component* of Γ, often simply referred to as a "component".

A *cycle* in Γ is defined as a closed path where the first and the last vertices are the same. A graph is called a *tree* if it is connected and contains no cycles as subgraphs. A graph Γ is *multipartite* if its vertex set can be divided into $s \geq 2$ parts, with every edge of Γ incident to vertices from different parts. Specifically, when $s = 2$, the graph Γ is termed *bipartite*. If any two vertices from different parts in a multipartite graph Γ are adjacent, then Γ is called a *complete multipartite graph*. A *star* is a complete bipartite graph where one part contains only a single vertex.

The *adjacency matrix* of Γ, denoted by $A(\Gamma)$, is an $n \times n$ matrix whose rows and columns are indexed by vertices of Γ. The (i, j)-entry a_{ij} of $A(\Gamma)$ is equal to 1 if $\{v_i, v_j\}$ is an edge of Γ, and a_{ij} equals 0 otherwise. As $A(\Gamma)$ is a real symmetric matrix, its eigenvalues are all real, and they are called the *eigenvalues* of Γ. (Given that adjacency matrices constructed with varying vertex orders exhibit similarity, the eigenvalues serve as invariants of G, irrespective of the vertex ordering.) We typically arrange them in a non-increasing manner, represented as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

and identify the distinct values among them as

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m.$$

Whenever we want to stress the dependence of the i-th largest eigenvalue (strictly i-th largest eigenvalue, respectively) of Γ or a real symmetric matrix M, we write $\lambda_i(\Gamma)$ or $\lambda_i(M)$ ($\alpha_i(\Gamma)$ or $\alpha_i(M)$, respectively) in place of λ_i (α_i, respectively). The *spectrum* of M, written $\mathrm{Spec}(M)$, is the collection of the distinct eigenvalues of M along with their multiplicities.

When Γ is connected, the largest eigenvalue λ_1 is of multiplicity one by Perron-Frobenious Theorem [48, Theorem 8.8.1], and $\lambda_1 - \lambda_2$ is known as the *spectral gap* of Γ. The *degree* d_i of the vertex v_i is the number of vertices which are adjacent to v_i. The graph Γ is *k-regular* if each of its vertices has the same degree k. By [13, Proposition 3.1], when Γ is k-regular, the largest eigenvalue λ_1 is exactly k with multiplicity equal to the number of connected components of Γ.

The *Laplacian matrix* of Γ is defined as $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix with the degree d_i of v_i as the (i, i)-entry. According to [13, Proposition 4.8], $L(\Gamma)$ is positive semi-definite, and 0 is an eigenvalue of $L(\Gamma)$ with multiplicity equal to the number of connected components of Γ by [13, 4e]. Denote the eigenvalues of $L(\Gamma)$ by $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. If Γ is k-regular, then $L(\Gamma) = kI_n - A(\Gamma)$ and $\mu_i = k - \lambda_i$, where I_n denotes the identity matrix of order n.

When Γ is connected and non-complete, it is well known that $0 < \mu_2 \leq \kappa(\Gamma) \leq \kappa'(\Gamma) \leq \delta(\Gamma)$, where $\kappa(\Gamma)$, $\kappa'(\Gamma)$, and $\delta(\Gamma)$ are the vertex connectivity, edge connectivity, and minimum degree of Γ, respectively (see [33, 44]). Due to this fact, μ_2 is called the *algebraic connectivity* of Γ by M. Fiedler. If in addition Γ is k-regular, its algebraic connectivity $\mu_2 = k - \lambda_2$ coincides with its spectral gap, and thus the second largest eigenvalue λ_2 is related to the connectivity property of Γ.

The *isoperimetric number* $h(\Gamma)$ of Γ is defined as

$$h(\Gamma) = \min_{\emptyset \neq S \subset V(\Gamma)} \frac{|\partial S|}{\min\{|S|, |\bar{S}|\}},$$

where $\bar{S} = V(\Gamma) \setminus S$ and $\partial S = \{\{x, y\} \in E(\Gamma) \mid x \in S, y \in \bar{S}\}$. This parameter is used to measure the expansion of a graph, which requires the graph to be simultaneously sparse and highly connected. This property is significant in many mathematical and computational contexts. For example, in [55] expansion is related to the convergence rates of Markov Chains. It is well known that graphs with large isoperimetric numbers always possess good expansion properties. A family of *expanders* is an infinite family of finite regular graphs of increasing orders with a common degree and isoperimetric numbers bounded below by a positive constant. Expanders are instrumental in designing highly efficient communication networks. See [79] for applications of expanders.

When Γ is connected and k-regular, its spectral gap $k - \lambda_2$ provides an estimate for the isoperimetric number $h(\Gamma)$ through the following discrete Cheeger inequality:

$$\frac{k-\lambda_2}{2} \leq h(\Gamma) \leq \sqrt{2k(k-\lambda_2)}.$$

This result was proved by Dodziuk [41] and independently by Alon-Milman [8] and by Alon [7]. A better upper bound given by Bojan Mohar [83] says that $h(\Gamma) \leq \sqrt{k^2 - \lambda_2^2}$. Thus for a connected regular graph, its second largest eigenvalue is linked to the expansion property of the graph, and this connection is important in the theory of expander graphs and the theory of random walks [4].

1.1.2 Cayley graphs and their eigenvalues

Cayley graphs serve as a fundamental link between graph theory and group theory. Given any group, Cayley graphs can be built on the group with respect to various sets of generators for the group. Those Cayley graphs have distinguished symmetry properties and encode the abstract structure of the group in a visual way. Group theorists can gain insights into the algebraic structure of groups by examining the geometric properties of Cayley graphs. Moreover, the structure and symmetry of Cayley graphs render them good candidates for constructing expander graphs [36, 78, 80], and thus extensive research has been conducted on the spectra of Cayley graphs, particularly focusing on the second largest eigenvalue of their adjacency matrix. We refer the reader to [76] for a collection of results on eigenvalues of Cayley graphs.

Let G be a finite group with identity element $\mathbf{1}$, and let S be an inverse-closed subset of $G \setminus \{\mathbf{1}\}$, that is, $S = S^{-1} = \{s^{-1} \mid s \in S\}$. The *Cayley graph* on G with respect to S, denoted by $\mathrm{Cay}(G, S)$, is the $|S|$-regular graph with vertex set G and edge set $\{\{g, gs\} \mid g \in G, s \in S\}$. It is readily seen that $\mathrm{Cay}(G, S)$ is connected if and only if its *connection set* S is a generating subset of G. We say that $\mathrm{Cay}(G, S)$ is a *normal* [1]

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[1] The definition is different from the following notion defined in [101]: A Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is called a normal Cayley graph if the right regular representation of G is normal in $\mathrm{Aut}(\Gamma)$.

Cayley graph if S is closed under conjugation; otherwise, $\mathrm{Cay}(G,S)$ is called a *nonnormal* Cayley graph.

The representation theory of finite groups plays a critical role in determining eigenvalues of Cayley graphs, as will be seen shortly in Propositions 1.1 and 1.2. For basic concepts and properties of representations and characters of groups, the reader is referred to [60, 64, 95, 98]. In what follows we use

$$\widehat{G} = \{\rho_1, \rho_2, \ldots, \rho_k\}$$

to denote a complete list of pairwise inequivalent (complex) irreducible matrix representations of G, with the assumption that ρ_1 is the trivial representation. For any $\rho_i \in \widehat{G}$, the map

$$\chi_i : g \mapsto \mathrm{Trace}(\rho_i(g)),\ \ g \in G$$

is the *character* of ρ_i, and the ratio

$$\tilde{\chi}_i(g) := \frac{\chi_i(g)}{\chi_i(\mathbf{1})}$$

is known as the *normalized character* of ρ_i on $g \in G$, where $\chi_i(\mathbf{1})$ equals the dimension $\dim \rho_i$ of ρ_i. Note that $\dim \rho_1 = 1$ for the trivial representation ρ_1.

It is known [9, 39, 77] that the adjacency matrix of $\mathrm{Cay}(G,S)$ equals $\sum_{s\in S} R_{\mathrm{reg}}(s)$, where reg is the right regular representation of G and $R_{\mathrm{reg}}(s)$ is the permutation matrix depicting the multiplication on G from the right by s^{-1} (see also [70]). To be specific, the (g,h)-entry of $R_{\mathrm{reg}}(s)$ equals 1 if $g = hs^{-1}$; otherwise, the (g,h)-entry of $R_{\mathrm{reg}}(s)$ equals 0. It is well known [92, Proposition 1.10.1] that the regular representation of G decomposes as a direct sum of all irreducible representations of G, each appearing with multiplicity identical to its dimension. Therefore, we have the following proposition, where

$$\rho_i(S) := \sum_{s\in S} \rho_i(s)$$

and $\oplus$ denotes the direct sum of matrices.

Proposition 1.1. [70, Proposition 7.1] *The adjacency matrix of* $\mathrm{Cay}(G,S)$ *is similar to*

$$d_1\rho_1(S) \oplus d_2\rho_2(S) \oplus \cdots \oplus d_k\rho_k(S),$$

where d_i is the dimension of $\rho_i \in \widehat{G}$, and

$$d_i\rho_i(S) = \underbrace{\rho_i(S) \oplus \rho_i(S) \oplus \cdots \oplus \rho_i(S)}_{d_i}.$$

This implies that the multiset of eigenvalues of $\mathrm{Cay}(G,S)$ is the union of d_i multisets of eigenvalues of $\rho_i(S)$ for $1 \leq i \leq k$. If a fixed number λ is an eigenvalue of each $\rho_i(S)$ with multiplicity m_i, which could be 0, then as an eigenvalue of $\mathrm{Cay}(G,S)$ its multiplicity equals $\sum_{i=1}^{k} d_i \cdot m_i$ (see also [39, Theorem 3]). In the case that $\mathrm{Cay}(G,S)$ is normal, by Schur's Lemma (see [92, Theorem 1.6.5]), all $\rho_i(S)$'s are scalar matrices [39, Lemma 5] and the eigenvalues of $\mathrm{Cay}(G,S)$ can be expressed in terms of the irreducible characters of G in the following way.

Proposition 1.2 ([39, 104])**.** *Let $\{\chi_1, \chi_2, \ldots, \chi_k\}$ be a complete set of inequivalent irreducible characters of G. Then the eigenvalues of any normal Cayley graph* $\mathrm{Cay}(G,S)$ *on G are given by*

$$\lambda_j = \frac{1}{\chi_j(\mathbf{1})}\sum_{s\in S}\chi_j(s) = \sum_{s\in S}\tilde{\chi}_j(s), \;\; j = 1, 2, \ldots, k.$$

Moreover, the multiplicity of λ_j is equal to $\sum_{1\leq i\leq k, \lambda_i=\lambda_j}\chi_i(\mathbf{1})^2$.

We say that the strictly second largest eigenvalue of $\mathrm{Cay}(G,S)$ is *attained* or *achieved* by $\rho_i \in \widehat{G}$ if

$$\alpha_2(\mathrm{Cay}(G,S)) \in \mathrm{Spec}(\rho_i(S));$$

when $\mathrm{Cay}(G,S)$ is normal, this is equivalent to

$$\alpha_2(\mathrm{Cay}(G,S)) = \sum_{s\in S}\tilde{\chi}_i(s).$$

We use $\langle S\rangle$ to denote the subgroup of S_n generated by S, and let $c = [G : \langle S\rangle]$ be the index of $\langle S\rangle$ in G. Then $\mathrm{Cay}(G,S)$ is the union of c copies of the connected Cayley graph $\mathrm{Cay}(\langle S\rangle, S)$ with degree $|S|$. Thus the largest eigenvalue $|S|$ of $\mathrm{Cay}(G,S)$ has multiplicity c, and the strictly second largest eigenvalue $\alpha_2(\mathrm{Cay}(G,S))$ just equals $\lambda_{c+1}(\mathrm{Cay}(G,S))$.

Proposition 1.2 illustrates that addressing the eigenvalues of normal Cayley graphs is in general simpler than tackling those of nonnormal Cayley graphs. For a nonnormal Cayley graph $\mathrm{Cay}(G,S)$, the matrices $\rho_i(S)$, where $\rho_i \in \widehat{G}$, are not scalar matrices, and each may possess multiple distinct eigenvalues. Consequently, character theory cannot be directly employed to compute the eigenvalues of nonnormal Cayley graphs.

1.1.3 Aldous' spectral gap conjecture

Although there is a large body of research on Cayley graphs, few results have successfully determined the exact value of the strictly second largest eigenvalue of some classes of Cayley graphs so far. Even for the normal case, it is still hard to figure out which eigenvalue is the second largest one in Proposition 1.2. One of the most noteworthy results in this area is the Aldous' spectral gap conjecture, which was made by David Aldous [2] in 1992 and was completely confirmed in 2010 [20]. In the following, we state Aldous' spectral gap conjecture from three distinct points of view.

Aldous' spectral gap conjecture originates in a probabilistic background and it is about the convergence rates of two different continuous-time Markov chains. Prior to delving into the original version of Aldous' spectral gap conjecture, we provide an overview of fundamental concepts in continuous-time Markov chains. We recommend two textbooks by Norris [86] and Grimmett and Stirzaker [49] for a clear treatment of the basic theory of Markov chains.

Let I be a finite set. Each $i \in I$ is called a *state* and I is called the *state space*. We say $\tau = (\tau_i : i \in I)$ is a *distribution* on I if $0 \leq \tau_i < \infty$ for all $i \in I$ and $\sum_{i \in I} \tau_i$ equals 1. A Q-*matrix* on I is a matrix $Q = (q_{ij})_{i,j \in I}$ satisfying the following conditions: (i) $0 \leq -q_{ii} < \infty$ for all $i \in I$; (ii) $q_{ij} \geq 0$ for all $i \neq j$; (iii) $\sum_{j \in I} q_{ij} = 0$ for all $i \in I$. Let $(X_t)_{t \geq 0}$ be a family of random variables taking values in I. Then $(X_t)_{t \geq 0}$ is the (homogeneous) *continuous-time Markov chain* (CTMC) with respect to some *initial distribution* τ and some Q-matrix Q, denoted by Markov(τ, Q), if the distribution of X_0 is τ, and for all $n = 0, 1, \ldots$, all times $0 \leq t_0 \leq \ldots \leq t_{n+1}$, and all states $i_0, \ldots, i_{n+1}$,

$$\mathbb{P}(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n) = p_{i_n, i_{n+1}}(t_{n+1} - t_n),$$

where $p_{i,j}(t)$ is the (i, j)-entry in e^{tQ}. The Q-matrix Q is known as the *generator matrix* of $(X_t)_{t \geq 0}$ and determines how the process evolves from the initial distribution. The (i, j)-entry q_{ij} of Q is called the *rate* of going from i to j. The value $p_{i,j}(t)$ is the *transition probability* form i to j in time t, which is equal to $\mathbb{P}(X_t = j \mid X_0 = i)$. The matrix $P(t) = e^{tQ}$ is called the *transition matrix.*

A Markov chain Markov(τ, Q) is *irreducible* if for any pair of states $i, j \in I$, there is some t such that $p_{i,j}(t) > 0$. Note from [86, Theorem 3.2.1] that if Markov(τ, Q) is irreducible, then $p_{i,j}(t) > 0$ for any pair of states $i, j \in I$ and any time $t \geq 0$. A distribution π is a *stationary distribution* of Markov(τ, Q) if $\pi = \pi P(t)$ for all $t \geq 0$. Furthermore, π is a stationary distribution of Markov(τ, Q) if and only if $\pi Q = 0$. It is well known that an irreducible CTMC with a finite state space has a unique stationary distribution π and $p_{i,j}(t) \to \pi_j$, $t \to \infty$ for all $i, j \in I$. We say Markov(τ, Q) is *reversible* if, for all $T > 0$, $(X_{T-t})_{0 \leq t \leq T}$ is also Markov(τ, Q). A necessary and sufficient condition for Markov(τ, Q) to be reversible is that $\tau_i q_{ij} = \tau_j q_{ji}$ for all $i, j \in I$, and this implies that τ is the unique stationary distribution for Markov(τ, Q) as $(\tau Q)_i = \sum_{j\in I} \tau_j q_{ji} = \sum_{j \in I} \tau_i q_{ij} = \tau_i \sum_{j \in I} q_{ij} = 0$.

In the following, we always suppose that Markov(τ, Q) is an irreducible Markov chain with its generator matrix Q satisfying $q_{ij} = q_{ji}$ for all $i \neq j$. Denote the size of its state space I by n. In this case, the uniform distribution π is the unique stationary distribution of Markov(τ, Q). Thus 0 is a simple eigenvalue of Q with π the unique left eigenvector. Note that $-Q$ is positive semi-definite. In fact, for any vector $(f_i)_{i\in I}$, the inner product $\langle Qf, f\rangle = \sum_{i\in I} \sum_{j \in I} q_{ij} f_j f_i = \sum_i \sum_{j\neq i} q_{ij}(f_j f_i - f_i^2) = \sum_{i \neq j} -q_{ij}(f_i - f_j)^2 \leq 0$. Denote the eigenvalues of $-Q$ by

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n.$$

The significance of μ_2, known as the *spectral gap* of Markov(τ, Q), is its interpretation as "the asymptotic rate of convergence to the stationary distribution" [4]:

$$p_{i,j}(t) - \tau_j = c_{ij} e^{-\mu_2 t} + o\left(e^{-\mu_2 t}\right) \qquad \text{as } t \to \infty,$$

where typically $c_{ij} \neq 0$ (and $c_{ii} > 0$ for some i). For this reason, $1/\mu_2$ is referred to as the *relaxation time* of Markov(τ, Q) by David Aldous in [4]. We say that Markov(τ_2, Q_2) with state space I' is a *subprocess* of Markov(τ, Q) with state space I if there is a surjective map $h : I \to I'$ such that

$$Q(f \circ h) = (Q_2 f) \circ h \quad \textit{for all } f : I' \to \mathbb{R}. \tag{1.1}$$

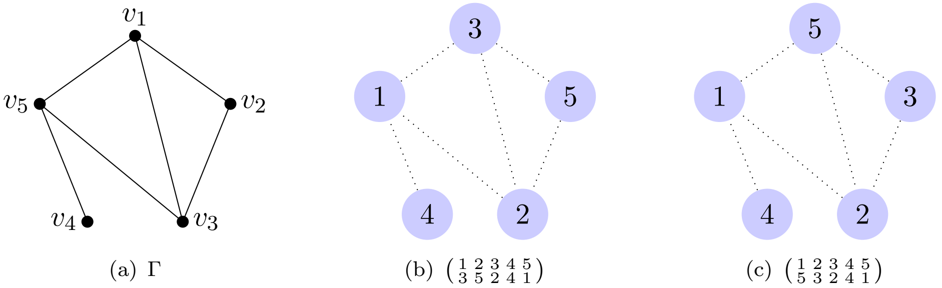
If f is an eigenfunction of $-Q_2$ with eigenvalue $\lambda \neq 0$, then $-Q(f \circ h) = (-Q_2 f) \circ h = \lambda f \circ h$ and $f \circ h \neq 0$ for $f \neq 0$, which implies that $f \circ h$ is an eigenfunction of $-Q$ with the same eigenvalue λ. Thus we get

$$\operatorname{Spec}(-Q_2) \subset \operatorname{Spec}(-Q),$$

and in particular, $\mu_2(-Q) \leq \mu_2(-Q_2)$.

Now we introduce the two continuous-time Markov chains involved in the original Aldous' spectral gap conjecture, the random walk and the interchange process on a graph. Let $\Gamma = (V(\Gamma), E(\Gamma)))$ be a connected graph with n vertices denoted by $v_1, v_2, \ldots, v_n$. Take n particles labeled $1, 2, \ldots, n$.

In the *interchange process* (IP), a state is an assignment of the n particles to the vertices of Γ such that each particle occupies exactly one vertex, and for each edge $\{v_i, v_j\}$ of Γ, at rate 1 the particles at v_i and v_j are interchanged. Figure 1.1 illustrates a transition from state $\left(\begin{smallmatrix}1&2&3&4&5\\3&5&2&4&1\end{smallmatrix}\right)$ to state $\left(\begin{smallmatrix}1&2&3&4&5\\5&3&2&4&1\end{smallmatrix}\right)$ in the interchange process on the leftmost graph Γ. Note that in the notation of a state the first row refers to the labels of the vertices of Γ.



(a) Γ (b) $\left(\begin{smallmatrix}1&2&3&4&5\\3&5&2&4&1\end{smallmatrix}\right)$ (c) $\left(\begin{smallmatrix}1&2&3&4&5\\5&3&2&4&1\end{smallmatrix}\right)$

FIGURE 1.1: An illustration of a transition in the interchange process

Thus the state space of the interchange process comprises all permutations of $[n] := \{1, 2, \ldots, n\}$, forming the *symmetric group* S_n of degree n with function composition as the group operation. In this paper, we adhere to the convention of composing two permutations in S_n from right to left. For instance, $(1, 2, 3) \circ (1, 3)$ yields $(2, 3)$. Let

$$T = \{(i, j) \in S_n \mid \{v_i, v_j\} \in E(\Gamma)\}.$$

The Q-matrix $Q^{\text{IP}} = (q_{\sigma,\sigma'})_{\sigma,\sigma' \in S_n}$ of the interchange process is

$$q_{\sigma,\sigma'} = \begin{cases} 1, & \text{if } \sigma' = \sigma \circ \tau \text{ for some } \tau \in T; \\ -|T|, & \text{if } \sigma' = \sigma; \\ 0, & \text{otherwise,} \end{cases}$$

which corresponds precisely to the negation of the Laplacian matrix of $\text{Cay}(S_n, T)$. As by [48, Lemma 3.10.1] T generates S_n when Γ is connected, $\text{Cay}(S_n, T)$ is connected and the spectral gap of the interchange process on Γ, denoted by $\lambda^{\text{IP}}(\Gamma)$, is positive owing to

$$\lambda^{\text{IP}}(\Gamma) = \mu_2(L(\text{Cay}(S_n, T))) > 0.$$

The (1-particle) *random walk* (RW) on Γ is the Markov chain in which if $\{v_i, v_j\} \in E(\Gamma)$, the single particle jumps from v_i to v_j at rate 1. Figure 1.2 shows a transition from state 1 to state 2 in the random walk on Γ. The state space of the random walk on Γ is $[n]$ and the Q-matrix is $Q^{\text{RW}} = -L(\Gamma)$, the negation of the Laplacian matrix of Γ. Since Γ is connected, the spectral gap of the random walk on Γ, denoted by $\lambda^{\text{RW}}(\Gamma)$, is positive owing to

$$\lambda^{\text{RW}}(\Gamma) = \mu_2(L(\Gamma)) > 0.$$

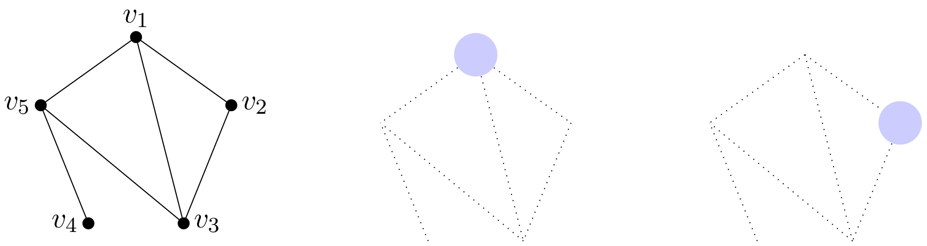


FIGURE 1.2: An illustration of a transition in the random walk

The random walk is a subprocess of the interchange process, achieved by disregarding all particles except for the one labeled as 1. Let $h : S_n \to [n]$, $\sigma \mapsto \sigma^{-1}(1)$, that is, h sends σ to the label of the vertex occupied by the particle labeled as 1. For example, in Figure 1.1 the map h sends the state in (b) to 5, as the particle labeled as 1 is at vertex

v_5. Then one can verify that

$$L(\mathrm{Cay}(S_n, T))(f \circ h) = (L(\Gamma)f) \circ h \quad \text{for all } f : [n] \to \mathbb{R}.$$

Thus the random walk on Γ is a subprocess of the interchange process on Γ, which implies

$$\mathrm{Spec}(L(\Gamma)) \subset \mathrm{Spec}(L(\mathrm{Cay}(S_n, T)))$$

and in particular

$$\mu_2(L(\mathrm{Cay}(S_n, T))) \leq \mu_2(L(\Gamma)). \tag{1.2}$$

This means that the spectral gap of the interchange process on Γ is no larger than that of the random walk on Γ. In 1992, David Aldous [2] proposed the following conjecture (see [4, Open Problem 14.29] for an explicit statement), asserting that equality holds in the preceding inequality (1.2).

Aldous' spectral gap conjecture (v.1) Let Γ be a finite simple connected graph. The random walk and the interchange process on Γ have the same spectral gap.

When examining Aldous' spectral gap conjecture through the lens of algebraic graph theory, it becomes evident that it revolves around the algebraic connectivity of Cayley graphs on S_n, as elucidated in (1.2). In presenting the second version of Aldous' spectral gap conjecture, we leverage transposition graphs. Let T be a set of transpositions of S_n. The *transposition graph* $\mathrm{Tra}(T)$ is the simple graph with vertex set $[n]$ such that $\{s, t\}$ is an edge in $\mathrm{Tra}(T)$ if and only if the transposition (s, t) is in T. Note that $\mathrm{Tra}(T)$ is connected whenever T generates S_n (see [48, Lemma 3.10.1]). It is clear that every simple graph Γ with order n is isomorphic to some transposition graph $\mathrm{Tra}(T)$ with $T \subset S_n$. Recall that Aldous' spectral gap conjecture is equivalent to the assertion that equality is achieved in (1.2). Now, we can express Aldous' spectral gap conjecture in an alternative equivalent form.

Aldous' spectral gap conjecture (v.2) If T is a generating subset of S_n composed of transpositions, then $\mathrm{Cay}(S_n, T)$ and $\mathrm{Tra}(T)$ have the same algebraic connectivity.

This version also appears in [46] with an extra condition that $\mathrm{Tra}(T)$ is a bipartite graph. The inequality (1.2) can also be readily derived within the framework of algebraic graph theory. Specifically, the Laplacian matrix of $\mathrm{Tra}(T)$ is identified as that of the Schreier coset graph $\mathrm{Sch}(S_n, G_i, T)$, where G_i is the stabilizer subgroup of S_n with respect to any $i \in [n]$, and the adjacency matrix of $\mathrm{Sch}(S_n, G_i, T)$ is equal to the quotient matrix of $\mathrm{Cay}(S_n, T)$ with respect to the equitable partition $\{G_{j,i}\}_{j\in[n]}$, which comprises the right cosets of G_i in S_n. For more details, see Section 2.3.

The representation theory of symmetric groups illuminates key insights into comprehending the essence of Aldous' spectral gap conjecture from the perspective of algebraic graph theory. A *partition* of a positive integer n is a sequence of positive integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ satisfying $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$ and $n = \gamma_1 + \gamma_2 + \cdots + \gamma_m$. We use $\gamma \vdash n$ to indicate that γ is a partition of n. Recall from the representation theory of symmetric groups [92] that for each partition of n, we can construct an irreducible module of S_n known as the Specht module. It is well known that all the Specht modules corresponding to partitions of n form a complete list $\widehat{S_n}$ of inequivalent irreducible modules of S_n. For any subset H of S_n, define

$$H^+ = \sum_{h\in H} h \in \mathbb{C}S_n.$$

Lemma 1.3. [96, Lemma 6.3] *Let $G = S_n$ and let L be an irreducible G-module. Then there is a basis of L such that the matrices of $g + g^{-1}$ on L are symmetric for any $g \in G$. Consequently, if $H \subset G$ is a subset such that $H = H^{-1}$, then the matrix of H^+ on L is symmetric.*

For $\zeta \vdash n$, we use S^ζ to denote the corresponding Specht module of S_n and ρ_ζ the matrix representation of S^ζ, with dimension d_ζ, under the basis such that $\rho_\zeta(H)$[2] is symmetric and thus has real eigenvalues whenever $H \subset S_n$ is closed under inverse. The existence of this basis is guaranteed by Lemma 1.3. The representations $\rho_{(n)}$ and $\rho_{(1^n)}$ are just the *trivial* and the *sign representations* of S_n, respectively. The *standard representation* of S_n refers to $\rho_{(n-1,1)}$. We use the notations χ_ζ and $\tilde{\chi}_\zeta$ to indicate the character and normalized character of ρ_ζ, respectively.

---

[2] We use the notation $\rho(H)$ to represent $\rho(H^+)$, as established before Proposition 1.1.

Recall that T is any generating subset of S_n composed of transpositions. By Proposition 1.1, we see that the multiset of eigenvalues of $\mathrm{Cay}(S_n, T)$ is the union of d_ζ multisets of eigenvalues of $\rho_\zeta(T)$ for all partitions ζ of n. Since $\mathrm{Cay}(S_n, T)$ is $|T|$-regular, its algebraic connectivity $\mu_2(L(\mathrm{Cay}(S_n, T)))$ equals $|T| - \lambda_2(\mathrm{Cay}(S_n, T))$.

Let I_n be the identity matrix of order n and def the *defining representation* of S_n. The matrix assigned to $\sigma \in S_n$ by def is the permutation matrix $P_{\mathrm{def}}(\sigma)$, whose (i, j)-entry $p_{ij}(\sigma)$ equals 1 if σ maps j to i, and $p_{ij}(\sigma) = 0$ otherwise. Then one can verify that the Laplacian matrix of $\mathrm{Tra}(T)$ is equal to $|T|I_n - \sum_{(s,t)\in T} P_{\mathrm{def}}((s, t))$. Note that the defining representation decomposes as the direct sum of one standard representation and one trivial representation. Thus there exists some matrix M such that for any $\sigma \in S_n$,

$$M \cdot P_{\mathrm{def}}(\sigma) \cdot M^{-1} = \rho_{(n)}(\sigma) \oplus \rho_{(n-1,1)}(\sigma),$$

and $L(\mathrm{Tra}(T))$ is similar to

$$|T|I_n - \rho_{(n)}(T) \oplus \rho_{(n-1,1)}(T).$$

Since $\rho_{(n)}(T) = |T|$, the trivial representation $\rho_{(n)}$ gives the smallest eigenvalue 0 of $L(\mathrm{Tra}(T))$ with multiplicity 1, and the algebraic connectivity of $\mathrm{Tra}(T)$ is given by the standard representation $\rho_{(n-1,1)}$, that is,

$$\mu_2(L(\mathrm{Tra}(T))) = |T| - \lambda_1(\rho_{(n-1,1)}(T)).$$

Aldous' spectral gap conjecture asserts that

$$|T| - \lambda_2(\mathrm{Cay}(S_n, T)) = |T| - \lambda_1(\rho_{(n-1,1)}(T)),$$

which is equivalent to

$$\lambda_2(\mathrm{Cay}(S_n, T)) = \lambda_1(\rho_{(n-1,1)}(T)).$$

Therefore, Aldous' spectral gap conjecture is equivalently formulated in its third version of the statement.

Aldous' spectral gap conjecture (v.3) If T is a generating subset of S_n composed of transpositions, then the second largest eigenvalue of $\mathrm{Cay}(S_n, T)$ is achieved by the standard representation of S_n, that is,

$$\lambda_2(\mathrm{Cay}(S_n, T)) = \lambda_1(\rho_{(n-1,1)}(T)). \tag{1.3}$$

Note that Proposition 1.1 indicates that $\lambda_2(\mathrm{Cay}(S_n, T)) \geq \lambda_1(\rho_{(n-1,1)}(T))$.

This thesis endeavors to generalize the Aldous' spectral gap conjecture by identifying some broader families of Cayley graphs on symmetric groups that fulfill the condition (1.3). It is important to note that when T does not generate S_n, the equation (1.3) still holds with $\lambda_2(\mathrm{Cay}(S_n, T))$ replaced by $\alpha_2(\mathrm{Cay}(S_n, T))$. When generalizing Aldous' spectral gap conjecture, we do not require the graph $\mathrm{Cay}(S_n, S)$ that we are studying to be connected. The advantage of fixing the underlying group as S_n rather than defining the Cayley graph on $\langle S \rangle$, the subgroup of S_n generated by S, is that we can utilize the representation theory of symmetric groups to handle the eigenvalues of $\mathrm{Cay}(S_n, S)$, and that there are systematic algorithms and techniques for constructing and studying representations of symmetric groups.

Definition 1.4. We say that a Cayley graph $\mathrm{Cay}(S_n, S)$ on S_n has the *Aldous property* if its strictly second largest eigenvalue is attained by the standard representation of S_n, that is,

$$\alpha_2(\mathrm{Cay}(S_n, S)) \in \mathrm{Spec}(\rho_{(n-1,1)}(S)). \tag{1.4}$$

We refer the reader to a weighted version[3] of this definition in next subsection, which is called property (A) as proposed by Cesi in [21]. Let $c = [S_n : \langle S \rangle]$ be the index of $\langle S \rangle$ in S_n. When $\mathrm{Cay}(S_n, S)$ is disconnected, the largest eigenvalue $|S|$ of $\mathrm{Cay}(S_n, S)$ has multiplicity c and might be achieved by $\rho_{(n-1,1)}$. Hence in equation (1.4), we only require $\alpha_2(\mathrm{Cay}(S_n, S))$ to appear in the spectrum of $\rho_{(n-1,1)}(S)$. Note also that in this case $\lambda_{c+1}(\mathrm{Cay}(S_n, S))$ is indeed the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, S)$, that is, $\alpha_2(\mathrm{Cay}(S_n, S)) = \lambda_{c+1}(\mathrm{Cay}(S_n, S))$. In view of Proposition 1.2, when $\mathrm{Cay}(S_n, S)$ is normal, it has the Aldous property if and only if

$$\alpha_2(\mathrm{Cay}(S_n, S)) = \sum_{\sigma \in S} \tilde{\chi}_{(n-1,1)}(\sigma). \tag{1.5}$$

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[3] In [21], property (A) is defined on connected weighted Cayley graphs on symmetric groups.

1.1.4 Weighted version of Aldous' spectral gap conjecture

Although this thesis primarily addresses unweighted graphs, as introduced in Subsection 1.1.1, we present the weighted version of Aldous' spectral gap conjecture in this subsection for the sake of completeness.

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simple connected graph with vertices $v_1, v_2, \ldots, v_n$. A nonnegative *weight function* is a function $w : V(\Gamma) \times V(\Gamma) \longrightarrow \mathbb{R}_{\geq 0}$ such that $w(v_i, v_j) = w(v_j, v_i)$ and $w(v_i, v_j) > 0$ if and only if there is an edge joining v_i and v_j. The (i, j)-entry of the adjacency matrix $A(\Gamma)$ of Γ is defined as the weight $w(v_i, v_j)$. The degree of v_i is defined as $d_i := \sum_{j=1}^{n} w(v_i, v_j)$, and the Laplacian matrix of Γ is $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $D(\Gamma)$ is the diagonal matrix $\operatorname{diag}\{d_1, d_2, \ldots, d_n\}$.

Recall that in Subsection 1.1.3, we define the interchange process and the random walk on a finite graph $\Gamma = (V(\Gamma), E(\Gamma))$, where every edge has a rate of 1. Now, we extend this to allow arbitrary positive non-constant numbers as rates for the edges of Γ, denoting the rate of the edge $\{v_i, v_j\}$ as $q_{i,j} > 0$. Here, Γ can be seen as a nonnegative weighted graph, with $q_{i,j}$ representing the weight of the edge $\{v_i, v_j\}$. The following general weighted version of Aldous' spectral gap conjecture was proposed and verified by Caputo, Liggett, and Richthammer [20]. See Subsection 1.2.1 and Theorem 1.5 for more details of their ingenious proof.

Weighted version of Aldous' spectral gap conjecture (v.1) For any nonnegative weighted graph Γ, the random walk and the interchange process on Γ have the same spectral gap.

To introduce the other two equivalent versions of weighted Aldous' spectral gap conjecture, we define the weighted transposition graph and the corresponding weighted Cayley graph. Let T be a set of transpositions of S_n, and each element of T is associated with a positive weight. Denote the weight of the transposition (i, j) by $q_{i,j}$. The weighted transposition graph $\mathrm{Tra}(T)$ has vertex set $[n]$, and $\{i, j\}$ is an edge with weight $q_{i,j}$ if and only if $(i, j) \in T$. The weighted Cayley graph $\mathrm{Cay}(S_n, T)$ has vertex set S_n and edge set $\{\{\sigma, \sigma\tau\} \mid \forall \sigma \in S_n, \tau \in T\}$, where the edge $\{\sigma, \sigma\tau\}$ has weight the same as $\tau \in T$. (This definition generalizes to $\mathrm{Cay}(S_n, S)$ with S being any weighted subset of S_n.) According to the discussions in Subsection 1.1.3, the equivalence of spectral gaps between the random walk and interchange process on Tra(T) suggests that the two weighted graphs, Tra(T) and Cay(S_n, T), possess identical algebraic connectivity. This leads us to the following equivalent version of the weighted Aldous' spectral gap conjecture.

Weighted version of Aldous' spectral gap conjecture (v.2) If T is a positive weighted generating subset of S_n composed of transpositions, then Cay(S_n, T) and Tra(T) have the same algebraic connectivity.

Any matrix representation ρ of S_n extends to a matrix representation of the complex group algebra $\mathbb{C}S_n$. For an element $\omega = \sum_{\sigma\in S_n} w_\sigma \sigma$ of $\mathbb{C}S_n$, the matrix of ρ on ω is

$$\rho(\omega) = \sum_{\sigma\in S_n} w_\sigma \rho(\sigma).$$

The *support* of ω is defined as $\mathrm{supp}(\omega) = \{\sigma \in S_n \mid w_\sigma \neq 0\}$. We call $\omega \in \mathbb{R}S_n$ *symmetric* if $w_{\sigma^{-1}} = w_\sigma$ for every $\sigma \in S_n$ and *nonnegative* if $w_\sigma \geq 0$ for every $\sigma \in S_n$. By Lemma 1.3, when $\omega \in \mathbb{R}S_n$ is symmetric, the matrix $\rho_\zeta(\omega)$, given by any irreducible representation ρ_ζ of S_n, is symmetric, where $\zeta \vdash n$. We call $\omega \in \mathbb{R}S_n$ *normal* if the coefficients $w_\sigma \in \mathbb{R}$ are constant on every conjugacy class of S_n.

Let S be any weighted subset of S_n with $w_\sigma \in \mathbb{R}$ the weight of $\sigma \in S$. We define $S^+ = \sum_{\sigma\in S} w_\sigma \sigma \in \mathbb{R}S_n$ and $|S^+| = \sum_{\sigma\in S} w_\sigma \in \mathbb{R}$. When S^+ is symmetric and nonnegative, the adjacency matrix of the weighted Cayley graph Cay(S_n, S) is $R_{\mathrm{reg}}(S)$[4], which is symmetric and nonnegative[5]. Here reg is the right regular representation of S_n. If further $\mathrm{supp}(S^+)$ generates S_n, by Perron-Frobenious Theorem the spectrum of Cay(S_n, S) is contained in the interval $[-|S^+|, |S^+|]$ with $|S^+|$ a simple eigenvalue. Moreover, the Laplacian matrix of Cay(S_n, S) is

$$L(\mathrm{Cay}(S_n, S)) = |S^+| I_{n!} - R_{\mathrm{reg}}(S),$$

which is symmetric positive semi-definite.

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[4] Similar to the unweighted case, we use $\rho(S)$ to denote $\rho(S^+)$ for any representation ρ of S_n.

[5] A matrix is nonnegative if every entry in this matrix is nonnegative.

Let us return to the second version of weighted Aldous' spectral gap conjecture. The Laplacian matrix of the weighted transposition graph $\mathrm{Tra}(T)$ is similar to

$$|T^+|I_n - \rho_{(n)}(T) \oplus \rho_{(n-1,1)}(T),$$

where $\rho_{(n)}(T) = |T^+|$. Thus, the algebraic connectivity of $\mathrm{Tra}(T)$ is $|T^+| - \lambda_1(\rho_{(n-1,1)}(T))$. Clearly, the algebraic connectivity of $\mathrm{Cay}(S_n, T)$ equals $|T^+| - \lambda_2(\mathrm{Cay}(S_n, T))$. We can now state the weighted version of Aldous' spectral gap conjecture in a third way.

Weighted version of Aldous' spectral gap conjecture (v.3) If T is a positive weighted generating subset of S_n composed of transpositions, then the second largest eigenvalue of $\mathrm{Cay}(S_n, T)$ is achieved by the standard representation of S_n, that is,

$$\lambda_2(\mathrm{Cay}(S_n, T)) = \lambda_1(\rho_{(n-1,1)}(T)).$$

Note that according to Proposition 1.1, the adjacency matrix $R_{\mathrm{reg}}(T)$ of the $\mathrm{Cay}(S_n, T)$ is similar to

$$\bigoplus_{\zeta \vdash n} d_\zeta \rho_\zeta(T),$$

where $d_{(n)}\rho_{(n)}(T) = |T^+|$ gives the simple largest eigenvalue of $\mathrm{Cay}(S_n, T)$. Therefore, $\rho_{(n-1,1)}$ achieving $\lambda_2(\mathrm{Cay}(S_n, T))$ is equivalent to the following condition:

$$\lambda_1(\rho_\zeta(T)) \leq \lambda_1(\rho_{(n-1,1)}(T)), \quad \forall \zeta \vdash n, \zeta \neq (n).$$

For any weighted subset S of S_n such that S^+ is symmetric and nonnegative with $\mathrm{supp}(S^+)$ generating S_n, Cesi defined in [21] that the weighted Cayley graph $\mathrm{Cay}(S_n, S)$ has *property (A)* if one of the following two equivalent conditions holds:

(i) If $\zeta \vdash n$ and $\zeta \neq (n)$, then $\lambda_1(\rho_\zeta(S)) \leq \lambda_1(\rho_{(n-1,1)}(S))$;

(ii) $\lambda_2(\mathrm{Cay}(S_n, S)) = \lambda_1(\rho_{(n-1,1)}(S))$.

This can be viewed as the general version of Definition 1.4 for connected weighted Cayley graphs.

1.2 Literature review

In this section, we undertake a comprehensive review of the literature pertaining to Aldous' spectral gap conjecture and its generalizations, as well as the investigation of (strictly) second largest eigenvalues of Cayley graphs on symmetric groups and other relevant groups.

1.2.1 Proofs of Aldous' spectral gap conjecture

According to David Aldous' webpage [2], the initial version of Aldous' spectral gap conjecture (referenced on Page 12) emerged around 1992 during discussions with Persi Diaconis, and was subsequently articulated explicitly as an open problem in [4].

Supporting evidence for this conjecture can be found from certain earlier literature. With a motivation from shuffling problems (investigating how fast a deck of cards approaches a uniform distribution when shuffled using a particular method, such as transpositions), Diaconis and Shahshahani [39] determined in 1981 the second largest eigenvalue of $\mathrm{Cay}(S_n, T)$ when T is the set of all transpositions of S_n. In this case, $\mathrm{Tra}(T)$ is the complete graph and T is a conjugacy class of S_n. The key ingredients used by Diaconis and Shahshahani is Proposition 1.2 and a monotonicity property [39, Lemma 10] of normalized irreducible characters for transpositions. From Corollary 4 and Lemma 10 in [39], one can finally deduce that when $\mathrm{Tra}(T)$ is the complete graph, the second largest eigenvalue of $\mathrm{Cay}(S_n, T)$ is achieved by the standard representation $\rho_{(n-1,1)}$, thus validating Aldous' spectral gap conjecture for this specific case. With a shared motivation from shuffling problems, when $\mathrm{Tra}(T)$ is a star with n as its central vertex, Flatto, Odlyzko and Wales in 1985 [45, Theorem 3.7] leveraged Branching Rule from the representation theory of symmetric groups to express all eigenvalues of $\rho_\zeta(T)$ for every $\zeta \vdash n$ in terms of coordinates of certain boxes in the Young diagram of ζ. While $T = \{(i, n) \in S_n \mid 1 \leq i \leq n-1\}$ is not closed under conjugation in S_n, it is normalized by S_{n-1}, which serves as a pivotal aspect in proving Theorem 3.7 in [45]; furthermore, from Theorem 3.7 in [45], we derive that the standard representation $\rho_{(n-1,1)}$ attains the second largest eigenvalue of $\mathrm{Cay}(S_n, T)$, thereby confirming Aldous' conjecture in the case where $\mathrm{Tra}(T)$ is a star. In 2010, Cesi's remarkable contributions have advanced this algebraic approach forward by employing the Littlewood-Richardson

Rule from representation theory of symmetric groups, thereby leading to the resolution of Aldous' spectral gap conjecture for all complete multipartite graphs [22].

Handjani and Jungreis [52] achieved the first significant breakthrough in Aldous' spectral gap conjecture in 1996. They confirmed that the interchange process and the random walk on any tree with any positive edge-rates, which are allowed to be non-constant on edges, have the same spectral gap. They discovered in [52, Theorem 1] that augmenting a graph Γ with a new vertex along with corresponding edges does not diminish the spectral gap of the interchange process. Consequently, they inferred that if, in addition, the spectral gap of the random walk does not increase upon the addition of the vertex, then the spectral gaps of the two processes remain equal after introducing the new vertex if such equality holds on the original Γ. This insight extends to an inductive proof of Aldous' spectral gap conjecture for all trees in [52]. Specifically, adding a new leaf and its associated edge to a graph does not increase the spectral gap of the random walk, as proved in [52, Corollary 1]. Moreover, for a tree consisting of only one edge, it is evident that the two processes exhibit identical spectral gaps. Handjani and Jungreis also provided an inductive proof in [52, Corollary 2] reaffirming that on complete graphs with constant edge-rates, the interchange process and the random walk share the same spectral gap.

Let us state the inductive idea utilized by Handjani and Jungreis in [52] formally. Suppose there is a sequence of graphs $G_2, G_3, \ldots$ such that the order of G_2 is 2 and G_{i+1} is obtained from G_i by adding one new vertex and corresponding edges. If the spectral gaps of random walks are non-increasing:

$$\lambda^{\mathrm{RW}}(G_2) \geq \lambda^{\mathrm{RW}}(G_3) \geq \cdots,$$

then the two continuous-time Markov chains IP and RW have the same spectral gap on every G_i owing to [52, Theorem 1] and (1.2). This induction argument was independently rediscovered by Koma and Nachtergaele within a distinct context [69], and was further employed in [84, Lemma 2] and [97, Corollary 3.4] to establish that the spectral gaps of IP and RW on the d-dimensional discrete hypercube of side-length n are asymptotically equivalent as n tends to infinity. This finding provides compelling support for Aldous' spectral gap conjecture. Here the d-dimensional discrete hypercube of side-length n

denotes the graph with the vertex set

$$R_n^d = \{(x_1, \dots, x_d) \in \mathbb{Z}^d \mid 1 \leq x_1, \dots, x_d \leq n\},$$

and two vertices $(x_1, x_2, \dots, x_d)$ and $(y_1, y_2, \dots, y_d)$ are adjacent if and only if $\sum_{i=1}^{d} |x_i - y_i| = 1$. Subsequently, we shall observe that this recursive methodology was further developed and extended in [21, Lemma 3.1], addressing certain extensions of the Aldous' spectral gap conjecture.

In 2010, Aldous' spectral gap conjecture was completely verified in a more general form. In their work [20], Caputo, Liggett, and Richthammer established the following weighted version of Aldous' spectral gap conjecture.

Theorem 1.5. *On any weighted graph Γ, the interchange process and the random walk have the same spectral gap:*

$$\lambda^{\mathrm{IP}}(\Gamma) = \lambda^{\mathrm{RW}}(\Gamma).$$

Caputo, Liggett, and Richthammer developed a general recursive approach grounded in the concept of network reduction, which is effective for all weighted graphs with positive weights. Given the weighted graph Γ in question and a vertex $v_i \in V(\Gamma)$, the *reduced network* obtained by removing the vertex v_i is a new graph Γ_i with vertex set $V(\Gamma_i) = V(\Gamma) \setminus \{v_i\}$, edge set $E(\Gamma_i) = \{\{v_k, v_j\} \in E(\Gamma) \mid k, j \neq i\}$, and the weight $\tilde{q}_{k,j}$ of $\{v_k, v_j\}$ defined as follows:

$$\tilde{q}_{k,j} := q_{k,j} + q_{k,j}^{\star,i} \geq q_{k,j}, \quad q_{k,j}^{\star,i} := \frac{q_{i,k} q_{i,j}}{\sum_{w \neq i} q_{i,w}}.$$

Caputo, Liggett, and Richthammer first established in [20, Proposition 2.1] that

$$\lambda^{\mathrm{RW}}(\Gamma_i) \geq \lambda^{\mathrm{RW}}(\Gamma), \tag{1.6}$$

thus extending the findings of [52, Corollary 1], which addressed the particular case where v_i is a leaf of Γ. Then they reduce the proof of Theorem 1.5 to the proof of the following comparison inequality, named the *octopus inequality*, for any function $g : S_n \to \mathbb{R}$

$$\sum_{\sigma \in S_n} \sum_{\substack{j \neq i \\ \{v_i, v_j\} \in E(\Gamma)}} q_{ij} [g(\sigma) - g(\sigma(i,j))]^2 \geq \sum_{\sigma \in S_n} \sum_{\{v_k, v_j\} \in E(\Gamma_i)} q_{k,j}^{\star,i} [g(\sigma) - g(\sigma(k,j))]^2.$$

The octopus inequality is the main technical ingredient used in [20], and in broad terms, it helps to contrast the spectral gaps between the interchange processes on Γ_i and Γ.

The reduced network Γ_i and the octopus inequality emerged independently and simultaneously from a different perspective in [40], where Dieker provided proofs for some particular cases of the octopus inequality. Moreover, Dieker proved that the eigenvalues of the Q-matrix governing the random walk on Γ_i interlace those on Γ, thereby implying (1.6). Additionally, Dieker offered a representation-theoretic interpretation of the octopus inequality. In [23], Cesi offered a clearer and more transparent proof of the octopus inequality from an algebraic perspective, leveraging advanced results from the representation theory of symmetric groups.

1.2.2 Algebraic generalizations of Aldous' spectral gap conjecture

The Aldous' spectral gap conjecture has ignited significant interest in its generalization from various perspectives. One avenue for expanding the conjecture involves identifying Cayley graphs on symmetric groups, where the connection set is formed not solely by transpositions but also encompasses other elements, while still exhibiting the Aldous property (see Definition 1.4). Furthermore, one can extend Aldous' spectral gap conjecture to Cayley graphs on different groups.

Besides, the (strictly) second largest eigenvalue of Cayley graphs is interesting in its own right because Cayley graphs serve as excellent candidates for constructing expanders. Note that according to the Cheeger inequality, an infinite family of finite connected regular graphs $\{\Gamma_i\}_{i\geq 1}$ of increasing orders with a common degree k is a family of expanders if and only if their spectral gap $k-\lambda_2(\Gamma_i)$ is uniformly bounded away from zero [36]. Thus another pertinent research direction is to determine the (strictly) second largest eigenvalue of Cayley graphs on symmetric groups or other groups and to uncover expander families.

In the sequel, we provide a summary of findings regarding the (strictly) second largest eigenvalues of Cayley graphs on symmetric groups or other relevant groups, their expansion properties, and Aldous properties.

Recall that the second largest eigenvalue of $\mathrm{Cay}(S_n, T_n)$ with $T_n = \{(1,n),(2,n),\ldots,(n-1,n)\}$ was determined to be $n-2$ in [45]. When $T_n = \{(1,2),(2,3),\ldots,(n-1,n)\}$ is the

Coxeter generators for S_n, the graph $\mathrm{Cay}(S_n, T_n)$ is known as the *bubble sort graph*, and its entire spectrum was determined in [11] with $n-3+2\cos(\pi/n)$ the second largest one. Friedman [46] finally proved that if T is any set of $n-1$ transpositions which generates S_n, then $\lambda_2(\mathrm{Cay}(S_n, T)) \geq n-2$, with equality achieved if and only if $T = \{(i,j) \mid j \neq i\}$ for some fixed $i \in [n]$. (Here one derives that if T is a set of $n-1$ transpositions which generates S_n, and $T \neq \{(i,j) \mid j \neq i\}$ for some fixed $i \in [n]$, then $\mathrm{Cay}(S_n, T)$ is never integral[6]. Refer to [76, Section 3.3] for a comprehensive review of integral Cayley graphs on symmetric groups.)

In 1995, Lubotzky [78] raised the question of whether expanders could be constructed from Cayley graphs on symmetric groups. When limiting the connection set to transpositions, this endeavor proves unfeasible due to the necessity of at least $n-1$ transpositions to generate S_n, resulting in unbounded degrees for such a family. In [85], the case where the connection set consists solely of reversals is explored, given that S_n can be generated by just three reversals. Here an involution τ in S_n is a *reversal* if there exist i, j with $1 \leq i < j \leq n$ such that

$$\tau(p) = \begin{cases} p, & \text{if } p < i \text{ or } p > j; \\ j+i-p, & \text{if } i \leq p \leq j. \end{cases}$$

In other words, τ reverses the interval from i to j and fixes all the other numbers. Nash [85] gave a negative answer to Lubotzky's question among Cayley graphs on S_n generated by a set T of reversals with $|T| = o(n)$. Actually, Nash proved a stronger result in [85]. It is well known that for any subgroup H of a group G, the spectrum of the Schreier coset graph $\mathrm{Sch}(G, H, S)$ is contained within the spectrum of the Cayley graph $\mathrm{Cay}(G, S)$ as $\mathrm{Sch}(G, H, S)$ is a quotient graph of $\mathrm{Cay}(G, S)$ with respect to an equitable partition. Consequently, if a family of Cayley graphs constitutes an expanding family, then any corresponding collection of Schreier coset graphs does as well. In [85], Nash examines the Schreier coset graphs associated with $H_n = S_2 \times S_{n-2} \leq S_n$, and shows that if T_n is a set of reversals such that $|T_n| = o(n)$, then the spectral gaps of Schreier coset graphs $\mathrm{Sch}(S_n, H_n, T_n)$ converge to 0 as $n \to \infty$. The condition $|T_n| = o(n)$ is in a sense optimal, as demonstrated by Gunnells, Scott and Walden [51], who established that the spectral gap of $\mathrm{Sch}(S_n, H_n, T_n)$ equals 1 with T_n comprising the n *initial reversals* $w_1, w_2, \ldots, w_n$, where w_i reverses the initial interval from 1 to i. In [66], an explicit construction of an

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[6] A graph is called integral if its eigenvalues are all integers.

expander family of Cayley graphs on symmetric groups is provided, offering a positive answer to Lubotzky's inquiry.

In [51], Gunnells, Scott and Walden reported that empirical evidence suggests that when T_n consists of the n initial reversals $w_1, w_2, \ldots, w_n$, the spectral gap of $\mathrm{Cay}(S_n, T_n)$ remains 1. This observation was subsequently validated by Cesi [21] using algebraic techniques, which extended the approach employed in earlier studies such as [52], [69], and [84]. In conjunction with a finding from [51], Cesi affirmed that the second largest eigenvalue of $\mathrm{Cay}(S_n, T_n)$, where $T_n = \{w_1, w_2, \ldots, w_n\}$, is achieved by the standard representation of S_n. Consequently, this establishes the first family of Cayley graphs on S_n exhibit the Aldous property, wherein the connection sets are formed by permutations other than transpositions.

An initial reversal is termed a prefix reversal in [27], where the Cayley graph on S_n generated by the $n-1$ prefix reversals $w_2, w_3, \ldots, w_n$ is referred to as the *pancake graph* P_n. Chung and Tobin [27] introduced a family $\mathcal{F}_n$ of graphs that generalizes the pancake graph, and showed that every graph in $\mathcal{F}_n$, including P_n, has spectral gap equal to one. Their methodology showcases remarkable ingenuity, heavily relying on graph coverings, a concept introduced earlier in [28], which can be understood as an extension of equitable partitions. For a graph covering, the counterpart of a quotient graph arising from an equitable partition is referred to as a projection. By identifying a common projection for each graph in $\mathcal{F}_n$, they demonstrated that the spectral gap of any graph within $\mathcal{F}_n$ does not exceed that of the projection. Employing inductive reasoning, the authors conclusively established that every graph within $\mathcal{F}_n$ shares the same spectral gap as the projection, which is precisely 1. Furthermore, utilizing this methodology, Chung and Tobin also analyzed the spectral gap of the *reversal graph* R_n, defined as the Cayley graph on S_n generated by all reversals in S_n. They constructed an appropriate projection of R_n, whose spectral gap serves as an upper bound for the spectral gap of R_n. Subsequently, through induction, they demonstrated that the spectral gap of the projection precisely matches the actual spectral gap of R_n. An important observation in the induction process is the decomposition of R_n into one pancake graph P_n and $n-1$ copies of R_{n-1}, thereby establishing the spectral gap of R_n based on that of P_n.

It can be verified that the projection of R_n constructed by Chung and Tobin in [27, Lemma 10] possesses the adjacency matrix equal to $P_{\mathrm{def}}(\Sigma)$, where P_{def} represents the permutation matrix of the defining representation of S_n, and Σ denotes the set comprising all reversals of S_n. Consequently, R_n having the same spectral gap with this projection implies that the second largest eigenvalue of R_n is attained by the standard representation of S_n. Hence the reversal graph R_n constitutes the second family of Cayley graphs on S_n exhibiting the Aldous property, with its connection set formed by permutations other than transpositions.

The methodology utilized by Chung and Tobin [27] was subsequently employed in [57] to determine the second largest eigenvalue of three Cayley graphs on the alternating group, generated by cycles of length 3. Let A_n denote the alternating group of degree n with $n \geq 3$. The *alternating group graph* AG_n, *extended alternating group graph* EAG_n and *complete alternating group graph* CAG_n are defined as the Cayley graphs $\mathrm{Cay}(A_n, T_1)$, $\mathrm{Cay}(A_n, T_2)$ and $\mathrm{Cay}(A_n, T_3)$, respectively, where $T_1 = \{(1,2,i),(1,i,2) \mid 3 \leq i \leq n\}$, $T_2 = \{(1,i,j),(1,j,i) \mid 2 \leq i < j \leq n\}$ and $T_3 = \{(i,j,k),(i,k,j) \mid 1 \leq i < j < k \leq n\}$. Huang and Huang [57] devised proper equitable partitions for AG_n, EAG_n, and CAG_n to effectively bound their second largest eigenvalues from below. Then they demonstrated by induction that these lower bounds precisely coincide with the second largest eigenvalues of these three graphs. During the induction process, EAG_n are decomposed into one AG_n and n copies of EAG_{n-1}, while CAG_n are decomposed into one EAG_n and n copies of CAG_{n-1}.

We now turn our attention to the three Cayley graphs on S_n generated by T_1, T_2 and T_3. As A_n has index 2 in S_n, each graph $\mathrm{Cay}(S_n, T_i)$, where i ranges over $\{1,2,3\}$, comprises two connected components, both isomorphic to $\mathrm{Cay}(A_n, T_i)$, and hence

$$\alpha_2(\mathrm{Cay}(S_n, T_i)) = \lambda_2(\mathrm{Cay}(S_n, T_i)).$$

Huang and Huang's results [57] imply that for each i in $\{1,2,3\}$

$$\alpha_2(\mathrm{Cay}(S_n, T_i)) = \lambda_1(\rho_{(n-1,1)}(T_i)),$$

thus indicating that these three Cayley graphs on S_n generated by T_1, T_2 and T_3 exhibit the Aldous property.

In their paper [58], Huang, Huang, and Cioabă defined the edge set of $\Gamma = \mathrm{Cay}(G, T)$ as $\{\{g, tg\} \mid g \in G, t \in T\}$, where $T = T^{-1}$. They found that when G is a finite group acting transitively on $[n]$, the left coset decomposition of G with respect to the stabilizer subgroup G_i for each $i \in [n]$ forms an equitable partition of Γ. All these equitable partitions share the same quotient matrix B_Π, and the spectrum of B_Π is contained within that of Γ. In [58, Theorem 7], they proved that the eigenvalues of Γ not belonging to B_Π can be bounded above by the sum of the second largest eigenvalues of certain subgraphs of Γ. Theorem 7 in [58] plays a critical role in Chapter 5 of this paper, so the method developed in [58] will be introduced in detail in Section 2.3.

Building on the above results, they further assumed that Γ is connected and normal, and that the action of G on $[n]$ is highly transitive. Under these conditions, they developed a recursive method in [58, Theorem 14] that reduces the problem of proving $\lambda_2(\Gamma) = \lambda_2(B_\Pi)$, an equivalent condition for Γ to possess the Aldous property, to verifying this for some smaller graphs. Finally, as applications of [58, Theorem 14], Huang, Huang, and Cioabă identified 41 out of 56 families of connected normal Cayley graphs on S_n generated by permutations moving at most 5 points which possess the Aldous property in [58, Theorem 15].

Significant progress in generalizing Aldous' spectral gap conjecture to normal Cayley graphs on S_n was achieved by Parzanchevski and Puder [88]. They demonstrated the existence of a positive integer N_0, conjectured to be 17, such that for every $n \geq N_0$, if S is a single conjugacy class of S_n, then the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, S)$ is achieved by one of the following eight irreducible representations of S_n:

$$\rho_{(n-1,1)},\ \rho_{(n-1,1)'},\ \rho_{(n-2,2)},\ \rho_{(n-2,2)'},\ \rho_{(n-2,1,1)'},\ \rho_{(n-3,3)},\ \rho_{(n-3,2,1)},\ \rho_{(n-4,4)},$$

where ζ' denotes the transpose of the partition of $\zeta \vdash n$ (see the definition in Section 2.1). The key ingredient employed in proving this result is [73, Theorem 1.3] (see Lemma 2.5), which is also a principal tool of Chapter 3.

They further established that when S is any general weighted subset of S_n such that $S^+ \in \mathbb{R}S_n$ is nonnegative and normal, no finite set of irreducible representations suffices to capture the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, S)$. However, for any such S, they proved that the gap between $|S^+|$ and $\alpha_2(\mathrm{Cay}(S_n, S))$ is bounded by $|S^+| - \lambda_1(\rho_{(n-1,1)}(S))$ multiplied by a decaying multiplicative factor:

$$|S^+| - \alpha_2(\mathrm{Cay}(S_n, S)) \geq [|S^+| - \lambda_1(\rho_{(n-1,1)}(S))](1 - o_n(1)).$$

Recently, Siemons and Zalesski [96] considered the normal Cayley graph generated by the conjugacy class $C(n,k)$ of k-cycles in S_n. They determined the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,k))$ for $k = n$ and $n-1$. Since $\mathrm{Cay}(S_n, C(n,k))$ is normal, by Proposition 1.2 its eigenvalues can be expressed in terms of the irreducible characters of k-cycles. Their results indicate that when k is n or $n-1$, the normal Cayley graph $\mathrm{Cay}(S_n, C(n,k))$ does not possess the Aldous property. However, Siemons and Zalesski anticipated that $\mathrm{Cay}(S_n, C(n,k))$ would exhibit the Aldous property for every $2 \leq k \leq n-2$ and every $n \geq 5$, and this was formally proposed by Maleki and Razafimahatratra as Conjecture 1.5 in [81]. Proposition 2.3 in [88] supports this conjecture for sufficiently large n. Furthermore, Maleki and Razafimahatratra [81] proved this conjecture to be true when $n-k$ is relatively small compared to n and when $k = 5$. The cases $k = 2$, 3 and 4 are covered by earlier works [39, 57] and [58], respectively. In Chapter 4, we provide a complete proof for this conjecture along with other results.

In [96], Siemons and Zalesski also considered a family of nonnormal Cayley graphs on S_n. For $1 \leq r < k < n$, they use $C(n,k;r)$ to denote the subset of $C(n,k)$ consisting of k-cycles of S_n that move every point from 1 to r. Note that $C(n,k;r)$ is not closed under conjugation and that $\mathrm{Cay}(S_n, C(n,k;r))$ is nonnormal. They successfully determined all the eigenvalues of the matrix $\rho_{(n-1,1)}(C(n,k;r))$, which are the eigenvalues of $\mathrm{Cay}(S_n, C(n,k;r))$ afforded by the standard representation $\rho_{(n-1,1)}$ of S_n. They further conjectured that

$$\alpha_2(\mathrm{Cay}(S_n, C(n,k;r))) = \lambda_1(\rho_{(n-1,1)}(C(n,k;r))),$$

which, if true, would indicate that the nonnormal Cayley graph $\mathrm{Cay}(S_n, C(n,k;r))$ has the Aldous property. They confirmed the conjecture for the case $k = r+1$. Results from [57] also provide supportive evidence for the case $k = 3$ of this conjecture. In Chapter 5, we provide a complete solution to this conjecture.

Cesi [24] extended Aldous' spectral gap conjecture to the Weyl group $W(B_n)$ following the strategy described in [23]. To be specific, Cesi proved an analogous result by showing that for a class of weighted generating subsets of $W(B_n)$, the spectral gap of the Laplacian of the Cayley graph on $W(B_n)$ is attained by a $2n$-dimensional permutation representation of W_n, denoted by $\mathbf{P}_n$. In the original Aldous' spectral gap conjecture, the standard representation is the unique irreducible representation of S_n containing the trivial representation when restricted to S_{n-1}. Similarly, the permutation representation $\mathbf{P}_n$ of $W(B_n)$ consists of all irreducible representations of $W(B_n)$ which, when restricted to $W(B_{n-1})$, contain the trivial representation. Furthermore, Cesi demonstrated that the result in [24] cannot be strengthened by replacing $\mathbf{P_n}$ with a subrepresentation.

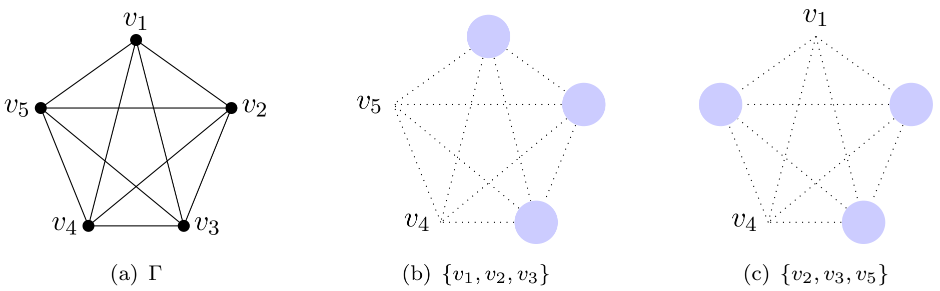
In [67, Theorem 6.1], it is proved that if (G, S) is a finite Coxeter system, both the spectral gap and the Kazhdan constant are determined by the defining representation. This result encompasses every finite Coxeter group. Specifically, if G is $W(B_n)$ and S is a Coxeter generator for G, the defining representation being a subrepresentation of $\mathbf{P_n}$ makes this result stronger than Cesi's for this particular S. The approach in [67, Theorem 6.1] differs significantly from the strategies used in [23, 24]. For an explanation of why the method in [67, Theorem 6.1] is unlikely to be effective for dealing with more general elements rather than a Coxeter generator, refer to [23, Section 5]. For further research on eigenvalues of $\mathrm{Cay}(G, S)$ with (G, S) a finite Coxeter system, see [1, 61].

1.2.3 Probabilistic generalization of Aldous' spectral gap conjecture

In this subsection, we explore the interconnections between spectral gaps within various stochastic processes defined on graphs, highlighting the analogous form of Aldous' spectral gap conjecture.

As corollaries of Theorem 1.5, several stochastic processes that are sub-processes of the interchange process are discussed in [20]. In the following, we suppose $\Gamma = (V(\Gamma), E(\Gamma))$ is the complete graph on n vertices, denoted by $v_1, v_2, \ldots, v_n$, with $q_{i,j} \geq 0$ as the weight of the edge $\{v_i, v_j\}$. We further assume that the corresponding skeleton graph, which is the spanning subgraph of Γ with positive weight edges, is connected.

Symmetric exclusion process (EP) In the k-particle exclusion process, a state is an assignment of k indistinguishable particles to k of the n vertices of Γ. For an edge $\{v_i, v_j\}$, if one of the positions v_i, v_j is occupied and the other is empty, then at rate $q_{i,j}$ the particle at the occupied position jumps to the empty position and this is one transition. See Figure 1.3 for a transition from the state $\{v_1, v_2, v_3\}$ to the state $\{v_2, v_3, v_5\}$. Note that the random walk process defined in Subsection 1.1.3 is the 1-particle exclusion process.



(a) Γ (b) $\{v_1, v_2, v_3\}$ (c) $\{v_2, v_3, v_5\}$

FIGURE 1.3: An illustration of a transition in the symmetric exclusion process

Formally, the k-particle exclusion process is the continuous-time Markov chain with state space $S^{EP} = \{X \subset V(\Gamma) \mid |X| = k\}$ and Q-matrix $Q^{\mathrm{EP}} = (q_{X,Y})_{X,Y \in S^{\mathrm{EP}}}$, where

$$q_{X,Y} = \begin{cases} q_{i,j}, & \text{if } X \triangle Y = \{v_i, v_j\}; \\ -\sum\limits_{|X \cap \{v_i,v_j\}|=1} q_{ij}, & \text{if } X = Y; \\ 0, & \text{otherwise.} \end{cases}$$

Here $\triangle$ denotes the symmetric difference. It can be obtained as a subprocess of the interchange process by the following surjective map

$$\pi : S_n \to S^{\mathrm{EP}}, \ \pi(\sigma) = \{\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(k)\}$$

in the sense of (1.1), which gives $\mathrm{Spec}(-Q^{\mathrm{EP}}) \subset \mathrm{Spec}(-Q^{\mathrm{IP}})$. On the other hand, if $f : V(\Gamma) \to \mathbb{R}$ is an eigenfunction of Q^{RW} with eigenvalue λ, then $g : S^{\mathrm{EP}} \to \mathbb{R}$ with $g(X) = \sum_{v_i \in X} f(v_i)$ is also an eigenfunction of Q^{EP} with eigenvalue λ. This gives

$$\mathrm{Spec}(-Q^{\mathrm{RW}}) \subset \mathrm{Spec}(-Q^{\mathrm{EP}}) \subset \mathrm{Spec}(-Q^{\mathrm{IP}}).$$

As a corollary of Theorem 1.5, it is deduced in [20] that for an arbitrary number $k \in \{1, 2, \ldots, n-1\}$, for every weighted graph Γ, the k-particle exclusion process on Γ has the same spectral gap as the random walk on Γ, that is,

$$\lambda^{\mathrm{EP}}(\Gamma) = \lambda^{\mathrm{RW}}(\Gamma).$$

Colored exclusion process (CEP) In the colored exclusion process, there are $r \geq 2$ types of particles, with $n_i \geq 1$ of type i such that $n_1 + n_2 + \ldots + n_r = n$, where particles of the same type are indistinguishable. A state is an assignment of these particles to the vertices of Γ such that every vertex is occupied by exactly one particle. In a transition, at rate $q_{i,j}$, the particles at v_i and v_j interchange their positions; see Figure 1.4.

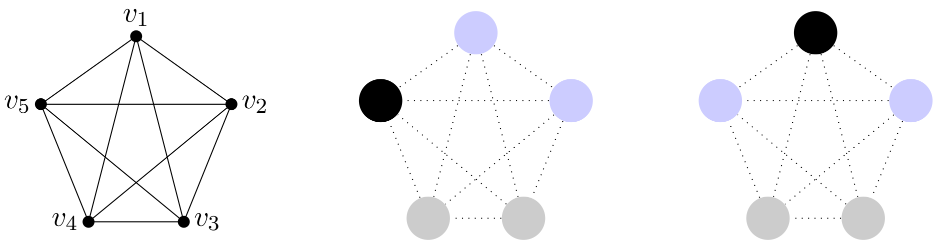


FIGURE 1.4: An illustration of a transition in the colored exclusion process

By ignoring all sites occupied by particles of types $2, 3, \dots, r$, we obtain the n_1-exclusion process as a subprocess of the colored exclusion process. This colored exclusion process, in turn, becomes a subprocess of the interchange process if we declare particles $1, 2, \dots, n_1$ to be of type 1, …, and particles $n - n_r + 1, \dots, n$ to be of type r. Thus as a corollary of Theorem 1.5, it is shown in [20] that for any choice of the parameters $r, n_1, \dots, n_r$, for any weighted graph Γ, the colored exclusion process on Γ has the same spectral gap as the random walk on Γ, that is,

$$\lambda^{\mathrm{CEP}}(\Gamma) = \lambda^{\mathrm{RW}}(\Gamma).$$

Normal generalized exclusion process (NGEP) We fix $k \in \mathbb{N}$, referred to as the *maximal occupancy*, and $l \in \mathbb{N}^+$, referred to as the *number of particles*. We assume that $l < N$, where $N = kn$. A state of the *generalized exclusion process* is an assignment of l indistinguishable particles to n vertices, ensuring that at most k particles are allocated to any $v_i \in V(\Gamma)$. We use S^{GEP} to denote the state space of the generalized exclusion process, that is,

$$S^{\mathrm{GEP}} := \{\pi : V(\Gamma) \to \{0, 1, \dots, k\} \mid \sum_{i=1}^{n} \pi(v_i) = l\}.$$

For a state $\pi \in S^{\text{GEP}}$ and $v_i, v_j \in V(\Gamma)$, if $\pi(v_i) > 0$ and $\pi(v_j) < k$, we define π^{ij} as the state obtained from π through v_i to v_j as follows:

$$\pi^{ij}(v_s) = \begin{cases} \pi(v_i) - 1, & \text{if } s = i; \\ \pi(v_j) + 1, & \text{if } s = j; \\ \pi(v_s), & \text{otherwise.} \end{cases}$$

See Figure 1.5 for an example. For any $v_i, v_j \in V(\Gamma)$, we assume the transition from a state $\pi \in S^{\text{GEP}}$ to $\pi^{ij} \in S^{\text{GEP}}$ occurs at rate $\mu(\pi, v_i, v_j)q_{ij}$ where $\mu : S^{\text{GEP}} \times V \times V \to \mathbb{R}_{\geq 0}$. When $\mu(\pi, v_i, v_j) = \pi(v_i)(k - \pi(v_j))$ for $v_i, v_j \in V(\Gamma)$ and $\pi \in S^{\text{GEP}}$, the generalized exclusion process with this transition rate μ is called the *normal generalized exclusion process* (NGEP). The main result of [65] states that the spectral gap of the normal generalized exclusion process on Γ is equal to k times that of the random walk on Γ, that is,

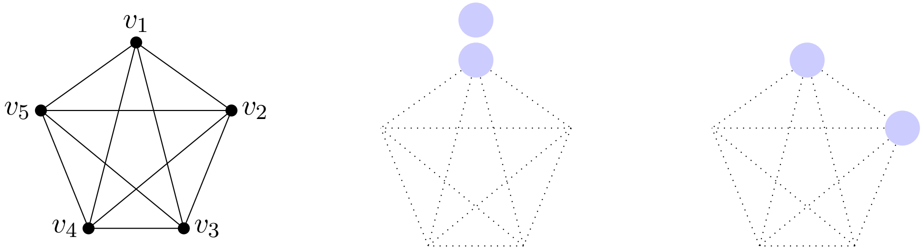


FIGURE 1.5: Generalized exclusion process with $l = 2$ and $k = 2$

$$\lambda^{\text{NGEP}}(\Gamma) = k \cdot \lambda^{\text{RW}}(\Gamma).$$

In particular, λ^{NGEP} is independent of the number of particles l.

For more recent works on the analogous of Aldous' spectral gap conjecture of other continuous-time Markov chains, see [18, 68, 90, 93, 94].

1.3 Thesis structure and main results

In this paper, we focus on the strictly second largest eigenvalue of Cayley graphs on the symmetric group S_n and investigate which partitions of n are responsible for this eigenvalue. Our aim is to identify families of Cayley graphs on S_n with the Aldous property, where the strictly second largest eigenvalue is achieved by the standard representation $\rho_{(n-1,1)}$ of S_n.

In Chapter 2, we introduce the techniques used in this thesis, including the fundamentals of representation theory of symmetric groups and various methods in algebraic graph theory, such as the Weyl inequalities and the equitable partition of graphs.

In Chapter 3, we prove three results, namely Theorems 3.1–3.3, regarding normal Cayley graphs on S_n that possess the Aldous property for sufficiently large n. Among these, Theorem 3.3 can be viewed as a generalization of the normal case of Aldous' spectral gap conjecture. The *support* of a permutation $\sigma \in S_n$ is defined as

$$\mathrm{supp}(\sigma) = \{i \in [n] \mid \sigma(i) \neq i\}.$$

For $\emptyset \neq I \subseteq \{2, 3, \ldots, n-1, n\}$ and $2 \leq k \leq n$, set

$$T(n, I) = \{\sigma \in S_n \mid |\mathrm{supp}(\sigma)| \in I\}$$

and

$$T(n, k) = \{\sigma \in S_n \mid 2 \leq |\mathrm{supp}(\sigma)| \leq k\}.$$

Theorem 3.1 indicates that when n is sufficiently large, for any conjugacy class S of S_n, the normal Cayley graph $\mathrm{Cay}(S_n, S)$ has the Aldous property if and only if $2 \leq |\mathrm{supp}(\sigma)| \leq n-2$ for some (and hence all) $\sigma \in S$. Theorem 3.2 shows that when n is sufficiently large and and $\emptyset \neq I \subset \{2, 3, \ldots, n\}$ with $|I \cap \{n-1, n\}| \neq 1$, the normal Cayley graph $\mathrm{Cay}(S_n, T(n, I))$ has the Aldous property if and only if $I \cap \{n-1, n\} = \emptyset$. Finally, as an extension of the normal case of the Aldous' spectral gap conjecture, we prove in Theorem 3.3 that when n is sufficiently large, for any $2 \leq k \leq n$, the connected normal Cayley graph $\mathrm{Cay}(S_n, T(n, k))$ has the Aldous property.

In Chapter 4, we study the normal Cayley graphs $\mathrm{Cay}(S_n, C(n, I))$, where $I \subseteq \{2, 3, \ldots, n\}$ and $C(n, I)$ is the set of all cycles in S_n with length in I. We prove that the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ can only be achieved by at most four irreducible representations of S_n. For some special $I \subseteq \{2, 3, \ldots, n\}$, we determine exactly the multiplicity of the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ by identifying all partitions of n whose corresponding irreducible representations achieve this eigenvalue. As a corollary, we prove the conjecture of Maleki and Razafimahatratra [81, Conjecture 1.4] (Corollary 4.7) that for any $n \geq 4$ and $2 \leq k \leq n-2$, the normal Cayley graph $\mathrm{Cay}(S_n, C(n, k))$ has the Aldous property, where $C(n, k) = C(n, \{k\})$, and thus solve an open problem posed by Siemons and Zalesski in [96]. A summary of our main results of Chapter 4 can be found in Table 4.1.

For $1 \leq r < k < n$, let $C(n,k;r)$ be the set of k-cycles of S_n moving every point in $\{1,2,\ldots,r\}$. Siemons and Zalesski [96] posed a conjecture (Conjecture 5.1) which is equivalent to saying that for any $n \geq 5$ and $1 \leq r < k < n$ the nonnormal Cayley graph $\mathrm{Cay}(S_n, C(n,k;r))$ on S_n with connection set $C(n,k;r)$ has the Aldous property. Solving this conjecture, we prove in Chapter 5 that all these graphs have the Aldous property except when (i) $(n,k,r) = (6,5,1)$ or (ii) n is odd, $k = n-1$, and $1 \leq r < \frac{n}{2}$. Along the way we determine all irreducible representations of S_n that can achieve the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-1;r))$ as well as the smallest eigenvalue of this graph.

This thesis concludes with several open problems related to Aldous' spectral gap conjecture, as presented in the final chapter.

Chapter 2

Preliminaries and methodologies

In this chapter, we introduce additional techniques from the representation theory of symmetric groups in Section 2.1, as well as methods from matrix theory and algebraic graph theory, such as equitable partitions, in Section 2.3. We also explore the connection between Aldous' spectral gap conjecture and equitable partitions or Schreier graphs. Specifically, we relate the second largest eigenvalue of a connected Cayley graph on S_n generated by transpositions to that of a quotient graph of the Cayley graph with respect to an equitable partition, or to that of a Schreier graph isomorphic to the quotient graph.

2.1 Representation theory of finite groups

All definitions in this section can be found in [62, 63, 92]. Representation theory can be expressed either through matrices or in terms of modules. A complex *matrix representation* ρ of a finite group G is a group homomorphism from G to the complex general linear group GL_d of degree d. The parameter d is called the *dimension* of this representation, denoted by $\dim\rho$. Every group has a *trivial representation*, which has the dimension of 1 and sends every group element to the matrix (1). For the symmetric group S_n, the *sign* of a permutation $\sigma \in S_n$, written $\operatorname{sgn}(\sigma)$, is defined to be 1 if σ is even and -1 if σ is odd. The *sign representation* of S_n has the dimension of 1 and maps every permutation in S_n to its sign.

All vector spaces in this thesis are over the complex numbers and of finite dimensions. A vector space V, with dimension $\dim V$, is a *G-module* if there is a group homomorphism from G to $GL(V)$, which is the *general linear group of* V consisting of all invertible linear transformations of V to itself. We can switch between matrix representations of G and G-modules, as the two definitions are equivalent. Given a matrix representation ρ of G with dimension d, the vector space $\mathbb{C}^d$ of all column vectors over $\mathbb{C}$ of length d is a G-module with $g \in G$ the following linear transformation

$$g\mathbf{v} = \rho(g)\mathbf{v}, \quad \mathbf{v} \in \mathbb{C}^d,$$

where the operation on the right is matrix multiplication. Conversely, if V is a G-module, then by taking a basis $\mathcal{B}$ for V, the map ρ sending $g \in G$ to the matrix of the linear transformation g in this basis $\mathcal{B}$ is a matrix representation of G.

Two G-modules V and W are *equivalent* if there is a bijective linear transformation $\theta : V \to W$ such that

$$\theta(g\mathbf{v}) = g\theta(\mathbf{v})$$

for all $g \in G$ and $\mathbf{v} \in V$. In matrix terms, two matrix representations ρ_1 and ρ_2 of G are *equivalent*, denoted by $\rho_1 \cong \rho_2$, if there is a fixed invertible matrix T such that

$$T\rho_1(g)T^{-1} = \rho_2(g)$$

for all $g \in G$. The *character* of a matrix representation ρ of G is the map $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \mathrm{Trace}(\rho(g)), \quad g \in G,$$

where $\mathrm{Trace}(\cdot)$ represents the trace of a matrix. Clearly, the character of ρ on the identity element $\mathbf{1}$ of G gives the dimension of ρ, that is,

$$\chi_\rho(\mathbf{1}) = \mathrm{dim}\rho.$$

Two representations of G are equivalent if and only if they have the same character. The *normalized character* of ρ on $g \in G$ is defined as

$$\tilde{\chi}_\rho(g) = \frac{\chi_\rho(g)}{\chi_\rho(\mathbf{1})}.$$

Given a G-module V, a subspace W of V is called a *nontrivial submodule* of V if W is a

G-module in its own right and $0 < \dim W < \dim V$. A nonzero G-module V is *reducible* if it contains a nontrivial submodule; otherwise, V is said to be *irreducible.* Equivalently, V is reducible if it has a basis $\mathcal{B}$ in which every $g \in G$ is assigned a block matrix of the form

$$X(g) = \left(\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right),$$

where the $A(g)$ are square matrices, all of the same size, and 0 is a nonempty matrix of zeros.

Let V be a vector space with two subspaces U and W. We say V is the *internal direct sum of* U *and* W, written $V = U \oplus W$, if every $\mathbf{v} \in V$ can be written uniquely as a sum

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad \mathbf{u} \in U, \ \mathbf{w} \in W.$$

If X is a matrix, then X is the *direct sum of matrices* A *and* B, written $X = A \oplus B$, if X has the block diagonal form

$$X = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right).$$

Maschke's Theorem states that for a finite group G, every nonzero G-module V is a direct sum of irreducible G-submodules of V. Equivalently, every positive-dimensional representation of a finite group G is equivalent to a direct sum of irreducible representations of G.

If $G = \{g_1, g_2, \ldots, g_n\}$, then G acts on itself by right multiplication: the action of g_i on g_j is defined as the usual product $g_j g_i^{-1}$ in G. Then we have the following G-module $\mathbb{C}G$, called the *group algebra*[1] of G,

$$\mathbb{C}G = \{c_1\mathbf{g_1} + c_2\mathbf{g_2} + \ldots + c_n\mathbf{g_n} \mid c_i \in \mathbb{C} \text{ for all } i\}.$$

Vector addition and scalar multiplication in $\mathbb{C}G$ are defined by

$$\begin{aligned}&(c_1\mathbf{g_1} + c_2\mathbf{g_2} + \ldots + c_n\mathbf{g_n}) + (d_1\mathbf{g_1} + d_2\mathbf{g_2} + \ldots + d_n\mathbf{g_n})\\ =&(c_1 + d_1)\mathbf{g_1} + (c_2 + d_2)\mathbf{g_2} + \ldots + (c_n + d_n)\mathbf{g_n}\end{aligned}$$

---

[1] We put the elements of G in boldface print when they are being considered as vectors.

and

$$c(c_1\mathbf{g_1}+c_2\mathbf{g_2}+\ldots+c_n\mathbf{g_n}) = cc_1\mathbf{g_1}+cc_2\mathbf{g_2}+\ldots+cc_n\mathbf{g_n},$$

respectively. Let $\mathbf{g_j g_i^{-1}} = \mathbf{g_k}$ in $\mathbb{C}G$ if $g_j g_i^{-1} = g_k$ in G. Then the action of $g \in G$ on $\mathbb{C}G$ can be expressed as

$$g(c_1\mathbf{g_1}+c_2\mathbf{g_2}+\ldots+c_n\mathbf{g_n}) = c_1(\mathbf{g_1 g^{-1}})+c_2(\mathbf{g_2 g^{-1}})+\cdots+c_n(\mathbf{g_n g^{-1}}).$$

Denote by R_{reg} the matrix representation corresponding to $\mathbb{C}G$ in the natural basis $\{\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_n}\}$, and R_{reg} is called the *right regular representation* of G. When writing R_{reg} as a direct sum of irreducible representations

$$R_{\text{reg}} \cong \bigoplus_{i=1}^{t} d_i \rho_i,$$

we have that $\{\rho_i\}_{i=1}^{t}$ forms a complete list of all pairwise inequivalent irreducible representations of G. Furthermore, the multiplicity d_i of ρ_i equals $\dim\rho_i$ in the above direct sum, that is,

$$d_i\rho_i := \underbrace{\rho_i \oplus \rho_i \oplus \cdots \oplus \rho_i}_{\dim\rho_i}.$$

In the following, we use $\widehat{G} = \{\rho_1, \rho_2, \ldots, \rho_t\}$ to denote a complete list of pairwise inequivalent irreducible representations of G with d_i the dimension of ρ_i.

2.2 Representation theory of symmetric groups

A *partition* of a positive integer n is a sequence of positive integers $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ satisfying $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m$ and $n = \gamma_1+\gamma_2+\cdots+\gamma_m$. We use $\gamma \vdash n$ to indicate that γ is a partition of n and use $c_i(\gamma)$ to denote the number of terms in γ which are equal to i. We say that γ is a *hook* if $\gamma = (n-m, 1^m)$ with $0 \leq m \leq n-1$, and $\gamma = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$ is called a *near hook*.

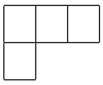
Every permutation $\sigma \in S_n$ has a decomposition into disjoint cycles. The *cycle type* of σ is the partition of n whose parts are the lengths of the cycles in its decomposition. It is widely known that two elements of S_n are conjugates if and only if they have the same cycle type. This means that the conjugacy classes of S_n are characterized by the cycle types and thus correspond to the partitions of n. Denote by $C(S_n, \gamma)$ the conjugacy class of S_n that corresponds to the partition $\gamma \vdash n$. For each $\gamma \vdash n$, we define $\operatorname{sgn}(\gamma) = 1$ if all permutations in $C(S_n, \gamma)$ are even and $\operatorname{sgn}(\gamma) = -1$ otherwise.

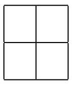
Recall from the representation theory of symmetric groups [92] that for each partition of n, we can construct an irreducible S_n-module, known as the Specht module. It is well known that all the Specht modules corresponding to partitions of n form a complete list of inequivalent irreducible modules of S_n. As in Subsection 1.1.3, for $\zeta \vdash n$, we use S^ζ to denote the corresponding Specht module of S_n and ρ_ζ the matrix representation of S^ζ, with dimension d_ζ, under the basis such that $\rho_\zeta(H)$ is symmetric and thus has real eigenvalues whenever $H \subset S_n$ is closed under inverse. The existence of this basis is guaranteed by Lemma 1.3. The notation $\widehat{S_n}$ denotes the complete list $\{\rho_\zeta\}_{\zeta \vdash n}$ of pairwise inequivalent irreducible representations of S_n. The representations $\rho_{(n)}$ and $\rho_{(1^n)}$ are just the trivial and the sign representations of S_n, respectively. The *standard representation* of S_n refers to $\rho_{(n-1,1)}$. Sometimes we say ζ, instead of ρ_ζ, attains the strictly second largest eigenvalue of some Cayley graph on S_n.

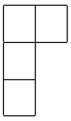
We use $\chi_\zeta(\cdot)$ and $\tilde{\chi}_\zeta(\cdot)$ to denote the character and normalized character of ρ_ζ for any $\zeta \vdash n$, respectively. It is well known that $\chi_{(n)}(\sigma) = \tilde{\chi}_{(n)}(\sigma) = 1$ and $\chi_{(1^n)}(\sigma) = \tilde{\chi}_{(1^n)}(\sigma) = \operatorname{sgn}(\sigma)$ for any $\sigma \in S_n$. Since $\chi_\zeta(\cdot)$ (respectively, $\tilde{\chi}_\zeta(\cdot)$) is a class function on S_n, we use $\chi_\zeta(\gamma)$ (respectively, $\tilde{\chi}_\zeta(\gamma)$) to denote the value of $\chi_\zeta(\cdot)$ (respectively, $\tilde{\chi}_\zeta(\cdot)$) on the conjugacy class $C(S_n, \gamma)$. We always use $\mathbf{1}$ to denote the identity element of S_n. Note that $\chi_\zeta(\mathbf{1})$ equals the dimension d_ζ of $\rho_\zeta \in \widehat{S_n}$ for any $\zeta \vdash n$.

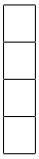
A *Young diagram* is a finite collection of boxes arranged in left-justified rows, with the row sizes weakly decreasing. The Young diagram associated to the partition $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)$ is the one that has m rows and γ_i boxes on the i-th row. For instance, the following are the Young diagrams corresponding to all partitions of 4. Since there is a clear one-to-one correspondence between partitions and Young diagrams, we use the two terms interchangeably, and we always use Greek letters γ and ζ to denote them.


(4)


(3,1)


(2,2)


(2,1,1)


(1,1,1,1)

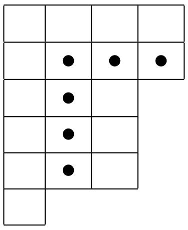
Denote by $a = (i, j)$ the box of a Young diagram ζ in the i-th row and j-th column. Then it has *hook*

$$H_a = H_{i,j} = \left\{ (i, j') \in \zeta : j' \geq j \right\} \cup \left\{ (i', j) \in \zeta : i' \geq i \right\}$$

with corresponding *hook length*

$$h_a = h_{i,j} = \left| H_{i,j} \right| .$$

To illustrate, if $\zeta = \left(4^2, 3^3, 1\right)$, then the dotted boxes in are the hook $H_{2,2}$ with hook length $h_{2,2} = 6$. The following theorem states the well-known Hook Length Formula for the dimension d_ζ of any irreducible representation $\rho_\zeta \in \widehat{S_n}$.



Theorem 2.1. (Hook Length Formula, [92, Theorem 3.10.2]) *If* $\zeta \vdash n$, *then*

$$d_\zeta = \frac{n!}{\prod_{(i,j)\in\zeta} h_{i,j}}.$$

It is known [47] that the character of any $\rho_\zeta \in \widehat{S_n}$ on any conjugacy class of S_n is an integer with absolute value at most the dimension of ρ_ζ. Hence $\tilde{\chi}_\zeta(\gamma)$ is a rational number in the interval $[-1, 1]$ for all $\zeta \vdash n$ and $\gamma \vdash n$. For the convenience of the reader and in order to provide self-contained proofs, we include the following Table 2.1 from [88], which gives the dimensions and characters of some irreducible representations of S_n.

The *conjugate* or *transpose* of a partition $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_m) \vdash n$ is defined as $\zeta' = (\zeta_1', \ \zeta_2', \ \ldots, \zeta_h') \vdash n$, where ζ_i' is the length of the i-th column of ζ. In other words, the Young diagram of ζ' is just the transpose of that of ζ. The relation between $\rho_\zeta(\cdot)$ (respectively, $\chi_\zeta(\cdot)$) and $\rho_{\zeta'}(\cdot)$ (respectively, $\chi_{\zeta'}(\cdot)$) is reflected in the following lemma.

$\boldsymbol{\zeta \vdash n}$	$\boldsymbol{d_\zeta = \chi_\zeta(1)}$	$\boldsymbol{\chi_\zeta(\gamma)}$ **with** $\boldsymbol{c_i(\gamma) = c_i}$
(n)	1	1
$(n-1,1)$	$n-1$	$c_1 - 1$
$(n-2,2)$	$\frac{n(n-3)}{2}$	$\frac{c_1(c_1-3)}{2} + c_2$
$(n-2,1^2)$	$\frac{(n-1)(n-2)}{2}$	$\frac{(c_1-1)(c_1-2)}{2} - c_2$
$(n-3,3)$	$\frac{n(n-1)(n-5)}{6}$	$\frac{c_1(c_1-1)(c_1-5)}{6} + (c_1-1)c_2 + c_3$
$(n-3,2,1)$	$\frac{n(n-2)(n-4)}{3}$	$\frac{c_1(c_1-2)(c_1-4)}{3} - c_3$
$(n-3,1^3)$	$\frac{(n-1)(n-2)(n-3)}{6}$	$\frac{(c_1-1)(c_1-2)(c_1-3)}{6} - (c_1-1)c_2 + c_3$
$(n-4,2,1^2)$	$\frac{n(n-2)(n-3)(n-5)}{8}$	$\frac{c_1(c_1-2)(c_1-3)(c_1-5)}{8} - \frac{(c_1^2-3c_1-1)c_2}{2} - \frac{c_2^2}{2} + c_4$

TABLE 2.1: Dimensions and characters of some irreducible representations of $\widehat{S_n}$

Lemma 2.2. [63, 2.1.8] *For any* $\zeta, \gamma \vdash n$*, we have*

$$\rho_{\zeta'}(\gamma) = \operatorname{sgn}(\gamma)\rho_\zeta(\gamma), \quad \chi_{\zeta'}(\gamma) = \operatorname{sgn}(\gamma) \cdot \chi_\zeta(\gamma).$$

In particular, we have $d_{\zeta'} = \chi_{\zeta'}(\mathbf{1}) = \chi_\zeta(\mathbf{1}) = d_\zeta$. That is, $\rho_{\zeta'}$ has the same dimension as ρ_ζ.

Let $E(n)$ and $O(n)$ be the numbers of even and odd derangements on $[n]$, respectively. The next two lemmas will be used in our proof of Theorem 3.2.

Lemma 2.3. [87] $E(n) - O(n) = (-1)^{(n-1)}(n-1)$.

Lemma 2.4. [88, Proposition 2.3] *There exists a positive integer* N_0 *such that for every* $n \geq N_0$ *and any* $\gamma \vdash n$ *with* $c_1(\gamma) \geq 2$*, we have*

$$\max_{\substack{\zeta \vdash n \\ \zeta \notin \{(n),(1^n)\}}} \tilde{\chi}_\zeta(\gamma) = \tilde{\chi}_{(n-1,1)}(\gamma).$$

Our main tool for proving the case $k = n-1$ in Theorem 3.3 is the following asymptotically sharp bound for the characters of S_n due to Larsen and Shalev [73].

Lemma 2.5. [73, Theorem 1.3] *Let γ be a partition of n and let $f = \max\{c_1(\gamma), 1\}$. Then for all $\zeta \vdash n$,*

$$|\chi_\zeta(\gamma)| \leq \chi_\zeta(\mathbf{1})^{1-\frac{\log(n/f)}{2\log n}+\varepsilon_n} = \chi_\zeta(\mathbf{1})^{\frac{1}{2}+\frac{\log f}{2\log n}+\varepsilon_n},$$

where ε_n is a real number tending to 0 as $n \to \infty$.

Another key ingredient for proving the case $k = n - 1$ in Theorem 3.3 is the following estimation of the dimensions of irreducible representations of S_n.

Lemma 2.6. [88, Lemma 2.6] *Let $n \geq 13$ and $\zeta \vdash n$ correspond to a Young diagram with at least three boxes outside the first row and at least three boxes outside the first column. Then the dimension d_ζ of $\rho_\zeta \in \widehat{S_n}$ satisfies*

$$d_\zeta \geq n^{2.05}.$$

The following two lemmas provide the characters on n-cycles and $(n-1)$-cycles, respectively, with their normalized characters derived using Hook Length Formula.

Lemma 2.7. [92, Lemma 4.10.3] *Suppose ζ and γ are two partitions of n. If $\gamma = (n)$ and $\sigma \in C(S_n, \gamma)$, then*

$$\chi_\zeta(\gamma) = \chi_\zeta(\sigma) = \begin{cases} (-1)^m, & \text{if } \zeta = (n-m, 1^m) \text{ with } 0 \leq m \leq n-1; \\ 0, & \text{otherwise}; \end{cases}$$

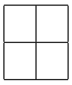
$$\tilde{\chi}_\zeta(\gamma) = \tilde{\chi}_\zeta(\sigma) = \begin{cases} \frac{(-1)^m n(n-m-1)!m!}{n!}, & \text{if } \zeta = (n-m, 1^m) \text{ with } 0 \leq m \leq n-1; \\ 0, & \text{otherwise}. \end{cases}$$

Lemma 2.8. [96, Lemma 4.3] *Suppose ζ and γ are two partitions of n. If $\gamma = (n-1, 1)$ and $\sigma \in C(S_n, \gamma)$, then*

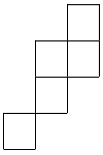
$$\chi_\zeta(\gamma) = \chi_\zeta(\sigma) = \begin{cases} 1, & \text{if } \zeta = (n); \\ (-1)^{n-2}, & \text{if } \zeta = (1^n); \\ (-1)^{m-1}, & \text{if } \zeta = (n-m, 2, 1^{m-2}) \text{ with } 2 \leq m \leq n-2; \\ 0, & \text{otherwise}; \end{cases}$$

$$\tilde{\chi}_\zeta(\gamma) = \tilde{\chi}_\zeta(\sigma) = \begin{cases} 1, & \textit{if } \zeta = (n); \\ (-1)^{n-2}, & \textit{if } \zeta = (1^n); \\ \frac{(-1)^{m-1}(n-1)(n-m)(n-m-2)!m(m-2)!}{n!}, & \textit{if } \zeta = (n-m, 2, 1^{m-2}) \\ & \quad \textit{with } 2 \leq m \leq n-2; \\ 0, & \textit{otherwise.} \end{cases}$$

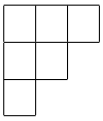
For any partition γ, let $\ell(\gamma)$ denote the number of parts of γ. Let m and n be positive integers. Given partitions μ and ζ of m and $m+n$, respectively, we say that μ is a *subpartition* of ζ, written $\mu \subseteq \zeta$, if $\ell(\mu) \leq \ell(\zeta)$ and $\mu_i \leq \zeta_i$ for $1 \leq i \leq \ell(\mu)$. The *skew diagram* ζ/μ is defined to be the set of boxes in ζ but not in μ. Denote by $\mathrm{ht}(\zeta/\mu)$ the number of nonempty rows of ζ/μ minus one. A skew diagram ζ/μ is called a *border strip* if it contains no subset that is a 2×2 box and the graph with vertices the boxes of ζ/μ and edges joining two neighbouring boxes in the same row or column is connected. For example, if $\zeta_0 = (3,3,2,1)$, $\mu_1 = (2,1,1)$ and $\mu_2 = (3)$, then we have the skew diagrams with $\mathrm{ht}(\zeta_0/\mu_1) = 3$ and $\mathrm{ht}(\zeta_0/\mu_2) = 2$. Note that these two skew diagrams are not border strips, while the following one is a border strip:

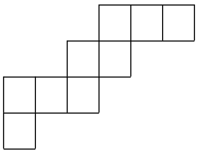


$\zeta_0/\mu_1 =$



$\zeta_0/\mu_2 =$





The main tool in Chapter 4 is as follows.

Theorem 2.9. (Murnaghan-Nakayama Rule, [92, Theorem 4.10.2]) *Given positive integers m and n, let $\rho \in S_{m+n}$ be an m-cycle and let π be a permutation of the remaining n elements of $[m+n]$. Then for any $\zeta \vdash m+n$,*

$$\chi_\zeta(\pi\rho) = \sum (-1)^{\mathrm{ht}(\zeta/\mu)} \chi_\mu(\pi),$$

where the sum is over all $\mu \vdash n$ *such that* $\mu \subset \zeta$ *and* ζ/μ *is a border strip.*

The special case of the Murnaghan-Nakayama Rule that ρ is just a 1-cycle, that is, a fixed point, is called the *Branching Rule.* To be specific, if ζ and γ are partitions of $n+1$ with $c_1(\gamma) \geq 1$, letting γ' be the partition of n with all $c_i(\gamma') = c_i(\gamma)$ except for $c_1(\gamma') = c_1(\gamma) - 1$, then

$$\chi_\zeta(\gamma) = \sum_{\zeta^-} \chi_{\zeta^-}(\gamma'),$$

where the sum is taken over all partitions of n which are obtained from ζ by removing one box.

In other words, Branching Rule describes how irreducible representations of S_n decompose into irreducible representations of S_{n-1}. We use the notation ζ^- to denote any partition of $n-1$ whose Young diagram is obtained by removing one box from that of $\zeta \vdash n$. When we restrict $\rho_\zeta \in \widehat{S_n}$ to S_{n-1}, then the restriction representation

$$\rho_\zeta \downarrow_{S_{n-1}} (\sigma) = \rho_\zeta(\sigma), \quad \sigma \in S_{n-1}$$

is not necessarily irreducible. In particular, we have the following theorem.

Theorem 2.10. (Branching Rule, [92, Theorem 2.8.3]) *If* $\zeta \vdash n$, *then* $\rho_\zeta \downarrow_{S_{n-1}} \cong \bigoplus_{\zeta^-} \rho_{\zeta^-}$.

2.3 Equitable partitions and other tools

Upper and lower bounds on eigenvalues of a symmetric matrix over the reals can be obtained from a theorem of Hermann Weyl [103]. See also [19, Theorem 2.8.1].

Theorem 2.11. (Weyl Inequalities) *Let* $C = A+B$ *be symmetric* $m \times m$ *matrices, and let* $\gamma_1 \geq \cdots \geq \gamma_m, \alpha_1 \geq \cdots \geq \alpha_m, \beta_1 \geq \cdots \geq \beta_m$ *be the eigenvalues of* C, A, B*, respectively. Then for* $i, j \in \{1, 2, \ldots, n\}$*, we have* $\gamma_{i+j-1} \leq \alpha_i + \beta_j$ *whenever* $i+j-1 \leq m$ *and* $\gamma_{i+j-m} \geq \alpha_i + \beta_j$ *whenever* $i+j-1 \geq m$*. In particular,*

$$\gamma_1 \leq \alpha_1 + \beta_1 \quad \textit{and} \quad \gamma_m \geq \alpha_m + \beta_m.$$

Remark 2.12. In Chapter 5, we may apply Weyl Inequalities to symmetric matrices A, B, C satisfying $C = A - B$. In this situation, the eigenvalues $\{\gamma_i\}_{i=1}^m$ of C are bounded by the eigenvalues $\{\alpha_i\}_{i=1}^m$ of A and the eigenvalues $\{\beta_i\}_{i=1}^m$ of B in the following way:

$$\gamma_{i+j-1} \leq \alpha_i - \beta_{m-j+1} \text{ whenever } i+j-1 \leq m,$$
$$\gamma_{i+j-m} \geq \alpha_i - \beta_{m-j+1} \text{ whenever } i+j-1 \geq m.$$

Let Γ be a finite simple graph with n vertices and adjacency matrix A. Consider a vertex partition $\Pi : V(\Gamma) = C_1 \cup C_2 \cup \cdots \cup C_r$ of Γ. The *characteristic matrix* P_Π of Π is a $|V(\Gamma)| \times r$ matrix whose columns are the characteristic vectors of the cells $C_1, C_2, \ldots, C_r$ of Π.

The partition Π is called an *equitable partition* of Γ if the number of neighbors in C_j of a vertex u in C_i is a constant b_{ij}, independent of u. The matrix $B_\Pi = (b_{ij})_{r\times r}$ is called the *quotient matrix* of Γ with respect to Π. The directed graph with the r cells of Π as its vertices and b_{ij} arcs from the i-th to the j-th cells of Π is called the *quotient graph* of Γ over the equitable partition Π, denoted by Γ/Π, whose adjacency matrix is exactly B_Π. In this case, it can be verified that $AP_\Pi = P_\Pi B_\Pi$, meaning the column space of P_Π is A-invariant. Indeed, a vertex partition of Γ with characteristic matrix P_Π is equitable if and only if the column space of P_Π is A-invariant [48, Lemma 9.3.2].

When we have an equitable partition Π of Γ, we can derive more information about the eigenvalues and eigenvectors of Γ from those of the quotient graph Γ/Π.

Lemma 2.13. [48] *Let Γ be a graph with adjacency matrix A, and $\Pi : V(\Gamma) = C_1 \cup C_2 \cup \cdots \cup C_r$ an equitable partition of Γ with quotient matrix B_Π and characteristic matrix P_Π. Then, each eigenvalue of the quotient graph Γ/Π is also an eigenvalue of Γ. Furthermore, Γ has the following two kinds of eigenvectors:*

(i) The eigenvectors in the column space of P_Π, with corresponding eigenvalues matching those of Γ/Π;

(ii) The eigenvectors orthogonal to the columns of P_Π, that is, those eigenvectors that sum to zero on each cell C_i for $1 \leq i \leq r$.

Chung and Yau [28] developed a generalization of equitable partitions, known as graph coverings, which is applicable to weighed graphs and also serves as a key ingredients in [27].

In the following, we highlight some important applications of equitable partitions as introduced in [58]. Let G be a finite group. In [58] the edge set of a Cayley graph $\mathrm{Cay}(G, S)$ is defined as $\{\{g, sg\} \mid g \in G,\ s \in S\}$, and when composing two permutations $\sigma \circ \tau$ in S_n, the authors in [58] do it from left to right. However, in this paper we define the edge set of $\mathrm{Cay}(G, S)$ as $\{\{g, gs\} \mid g \in G,\ s \in S\}$ and when composing two permutations $\sigma \circ \tau$ in S_n, we do it from right to left. Considering this nonessential difference, we know from [58, Lemma 5] that for any subgroup H of G, the set of right cosets of H in G forms an equitable partition of $\mathrm{Cay}(G, S)$.

Let Ω be a nonempty set, and let G be a group acting on Ω. We say that the action of G on Ω ($|\Omega| \geq s$) is *s-transitive* if for all pairwise distinct $x_1, x_2, \ldots, x_s \in \Omega$ and pairwise distinct $y_1, y_2, \ldots, y_s \in \Omega$, there exists some $g \in G$ such that $x_i^g = y_i$ for $1 \leq i \leq s$. Clearly, an s-transitive action is always t-transitive for any $t < s$. In particular, we say that the action is *transitive* if it is 1-transitive. As usual, we denote by $G_x = \{g \in G \mid x^g = x\}$ the *stabilizer* of $x \in \Omega$ in G. Note that G_x is a subgroup of G.

Now consider the group G as either the symmetric group S_n or the alternating group A_n, both of which act transitively on $[n] = \{1, 2, \ldots, n\}$ for $n \geq 3$. For each fixed $i \in [n]$, by the orbit-stabilizer theorem, we have $|G|/|G_i| = n$. Furthermore, G has the following right coset decomposition:

$$\Pi_i : G = G_i g_{1,i} \cup G_i g_{2,i} \cup \cdots \cup G_i g_{n,i} = G_{1,i} \cup G_{2,i} \cup \cdots \cup G_{n,i}, \tag{2.1}$$

where $g_{j,i}$ is an arbitrary element in G that maps j to i, and

$$G_{j,i} = G_i g_{j,i} = \{g \in G \mid j^g = i\},$$

for all $j \in [n]$. Clearly, $|G_{j,i}| = |G_i| = |G|/n$. Note that the group action in (2.1) is the composition of permutations, which is performed from right to left in this paper.

Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph on G, where G is either S_n or A_n. According to [58, Lemma 5] mentioned above, for each $i \in [n]$, the right coset decomposition Π_i given in (2.1) is an equitable partition of Γ with quotient matrix denoted by $\mathbf{B}_i = (b_{kt})_{n\times n}$, where

$$b_{kt} = |S \cap g_{k,i}^{-1} G_i g_{t,i}| = |S \cap G_{t,k}| \tag{2.2}$$

is exactly the number of elements in S mapping t to k. Since $b_{kt} = |S \cap G_{t,k}|$ is independent on the choice of i, all the equitable partitions Π_i share the same quotient matrix. Therefore, we use $\mathbf{B}$ instead of $\mathbf{B}_i$. Additionally, by counting the edges between $G_{k,i}$ and $G_{t,i}$ in two ways, we obtain $b_{kt} \cdot |G_{k,i}| = b_{tk} \cdot |G_{t,i}|$, which implies that $b_{kt} = b_{tk}$ because $|G_{k,i}| = |G_{t,i}| = |G|/n$. Consequently, $\mathbf{B} = (b_{kt})_{n\times n}$ is symmetric.

The following result provides upper bounds for the eigenvalues of Γ, excluding those of $\mathbf{B}$. The original result, as stated in [58, Theorem 7], applies to any finite group G acting transitively on $[n] = \{1, 2, \ldots, n\}$. Here, we restrict G to be either S_n or A_n.

Lemma 2.14. [58, Theorem 7] *Let G be S_n or A_n, and let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph of G. The right coset decomposition Π_i of G given in (2.1) results in an equitable partition of Γ, and the corresponding quotient matrix $\mathbf{B} = \mathbf{B}_i$ is symmetric and independent of the choice of i. Moreover, if λ is an eigenvalue of Γ which is not an eigenvalue of $\mathbf{B}$, then for each $k \in [n]$, we have*

$$\lambda \le \lambda_2(\mathrm{Cay}(G_k, S \cap G_k)) + \lambda_2(\mathrm{Cay}(G, S \setminus (S \cap G_k))),$$

where G_k is the stabilizer of $k \in [n]$ in G.

Now we relate the above quotient matrix $\mathbf{B}$ with the natural permutation module $M^{(n-1,1)}$ of S_n. Consider the natural action of S_n on $[n] = \{1, 2, \ldots, n\}$. Then S_n acts on

$$\mathbb{C}[n] = \{c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n} \mid c_i \in \mathbb{C} \text{ for all } i\}$$

with the action

$$\sigma(c_1\mathbf{1} + c_2\mathbf{2} + \cdots + c_n\mathbf{n}) = c_1\sigma(\mathbf{1}) + c_2\sigma(\mathbf{2}) + \cdots + c_n\sigma(\mathbf{n})$$

for all $\sigma \in S_n$[2]. Then $\mathbb{C}[n]$ is the so called *natural permutation module* of S_n, denoted by $M^{(n-1,1)}$. Given the standard basis $\{\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\}$ of $\mathbb{C}[n]$, the matrix $X(\sigma)$ of the linear transformation $\sigma \in S_n$ is

$$X(\sigma)_{i,j} = \begin{cases} 1, & \text{if } \sigma(j) = i; \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

---

[2] Here we use bold font to indicate that $\mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}$ are vectors.

This is exactly the matrix $P_{\mathrm{def}}(\sigma)$ of the defining representation of S_n on σ. Consequently, the quotient matrix $\mathbf{B}$ of $\Gamma = \mathrm{Cay}(S_n, S)$ with respect to the right coset decomposition Π_i for any $i \in [n]$ is

$$\mathbf{B} = \sum_{s \in S} X(s) = X(S). \tag{2.4}$$

It is known that the natural permutation module $M^{(n-1,1)}$ (respectively, the defining representation) decomposes into one trivial module $S^{(n)}$ (respectively, trivial representation $\rho_{(n)}$) and one standard module $S^{(n-1,1)}$ (respectively, standard representation $\rho_{(n-1,1)}$). As derived from Lemma 2.13, $\mathrm{spec}(\mathbf{B}) \subset \mathrm{spec}(\Gamma)$, we know that $|S| = \rho_{(n)}(S)$ is the largest eigenvalue of both Γ and $\mathbf{B}$. Hence, we have

$$\lambda_2(\Gamma) \geq \lambda_2(\mathbf{B}) = \lambda_1(\rho_{(n-1,1)}(S)),$$

and that $\mathrm{Cay}(S_n, S)$ has the Aldous property is equivalent to that $\Gamma = \mathrm{Cay}(G, S)$ has the same second largest eigenvalue as the quotient matrix $\mathbf{B}$.

In the following, we introduce the definition of the Schreier coset graph and relate the quotient matrix $\mathbf{B}$ in Lemma 2.14 to the adjacency matrix of a Schreier coset graph. Given a group G, a subgroup H, and a generating subset $S \subseteq G$, the *Schreier coset graph* $\mathrm{Sch}(G, H, S)$ has all right cosets $\{H_i\}$ of H in G as its vertices, and the edges are of the form $(H_i, H_i s)$ for any right coset H_i and any $s \in S$. Given the connected Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ in Lemma 2.14, when H is taken as the stabilizer of $i \in [n]$ in G, the Schreier coset graph $\mathrm{Sch}(G, G_i, S)$ is isomorphic to the quotient graph Γ/Π_i of Γ. Consequently, $\mathbf{B}$ is exactly the adjacency matrix of $\mathrm{Sch}(G, G_i, S)$, and $\mathrm{Cay}(S_n, S)$ has the Aldous property if and only if $\Gamma = \mathrm{Cay}(G, S)$ and $\mathrm{Sch}(G, G_i, S)$ have the same second largest eigenvalue. Finally, note that when S is a generating subset of transpositions in $G = S_n$, the transposition graph $\mathrm{Tra}(S)$, mentioned in Subsection 1.1.3, has the same Laplacian matrix as $\mathrm{Sch}(S_n, G_i, S)$.

Chapter 3

Three families of normal Cayley graphs possessing the Aldous property

3.1 Main results

In this chapter, we present three classes of normal Cayley graphs on S_n that have the Aldous property. Recall that the *support* of a permutation $\sigma \in S_n$ is defined as $\mathrm{supp}(\sigma) = \{i \in [n] \mid \sigma(i) \neq i\}$. For $\emptyset \neq I \subseteq \{2, 3, \ldots, n-1, n\}$ and $2 \leq k \leq n$, set

$$T(n, I) = \{\sigma \in S_n \mid |\mathrm{supp}(\sigma)| \in I\}$$

and

$$T(n, k) = \{\sigma \in S_n \mid 2 \leq |\mathrm{supp}(\sigma)| \leq k\}.$$

The main results in this chapter are as follows.

Theorem 3.1. *There exists a positive integer N such that for every $n \geq N$ and any conjugacy class S of S_n, the normal Cayley graph* $\mathrm{Cay}(S_n, S)$ *has the Aldous property if and only if* $2 \leq |\mathrm{supp}(\sigma)| \leq n-2$ *for some (and hence all) $\sigma \in S$.*

Theorem 3.2. *There exists a positive integer N such that for every $n \geq N$ and any $\emptyset \neq I \subset \{2, 3, \ldots, n-1, n\}$ with $|I \cap \{n-1, n\}| \neq 1$, the normal Cayley graph* $\mathrm{Cay}(S_n, T(n, I))$ *has the Aldous property if and only if $I \cap \{n-1, n\} = \emptyset$.*

Theorem 3.3. *There exists a positive integer N such that for every $n \geq N$ and any $2 \leq k \leq n$, the connected normal Cayley graph* $\mathrm{Cay}(S_n, T(n,k))$ *has the Aldous property.*

It is worth mentioning that the case $2 \leq k \leq n-2$ in Theorem 3.3 is covered by the sufficiency part of Theorem 3.2. We keep the full range $2 \leq k \leq n$ there since from this form one can easily see that Theorem 3.3 is an extension of the "normal" version of Aldous' spectral gap conjecture. In fact, $T(n,2)$ is the conjugacy class of all transpositions of S_n and $\mathrm{Cay}(S_n, T(n,2))$ is the unique normal Cayley graph covered by Aldous' spectral gap conjecture. We have $T(n,2) \subseteq T(n,k)$ for $2 \leq k \leq n$ and therefore $\mathrm{Cay}(S_n, T(n,k))$ is connected indeed.

After the completion of the proof of Theorem 3.3, we realized that the fact that the normal Cayley graph $\mathrm{Cay}(S_n, T(n,n-1))$ has the Aldous property can be derived from [91, Theorem 7.1] (see Remark 3.6 for details). So Theorem 3.3 can be regarded as a corollary of Theorem 3.2 and [91, Theorem 7.1], modulo the trivial case of $\mathrm{Cay}(S_n, T(n,n))$.

In [88, Proposition 2.4], it was proved that, for sufficiently large n and any conjugacy class S of S_n whose elements fix at most one point, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, S)$ is attained by one of the following eight irreducible representations of S_n:

$$\rho_{(n-1,1)},\ \rho_{(n-1,1)'},\ \rho_{(n-2,2)},\ \rho_{(n-2,2)'},\ \rho_{(n-2,1,1)'},\ \rho_{(n-3,3)},\ \rho_{(n-3,2,1)},\ \rho_{(n-4,4)}.$$

The "only if" part of Theorem 3.1 implies that the standard representation $\rho_{(n-1,1)}$ can be removed from this list. Also, for sufficiently large n, a result proved in [57] can be obtained from Theorem 3.1 by taking S to be the conjugacy class of 3-cycles or from Theorem 3.2 by choosing $I = \{3\}$ (see Remark 3.5 for details).

The conditions in Theorems 3.1–3.3 can be stated in terms of the number of fixed points of a permutation. In particular, for $0 \leq k \leq n-2$, $T(n, \{n-k\})$ is the set of permutations of S_n fixing exactly k points, and the normal Cayley graph $\mathrm{Cay}(S_n, T(n, \{n-k\}))$ is exactly the k-point-fixing graph $\mathcal{F}(n,k)$ studied in [71]. In particular, $\mathcal{D}_n := T(n, \{n\})$ is the set of derangements on $[n]$ and $\mathcal{F}(n,0) = \mathrm{Cay}(S_n, \mathcal{D}_n)$ is widely known as the derangement graph on $[n]$. Theorem 3.2 together with some known results from [37, 71, 91] implies the following corollary, which asserts that, for sufficiently large n, $\mathcal{F}(n,0)$ and $\mathcal{F}(n,1)$ are the only graphs among $\mathcal{F}(n,k)$ $(0 \leq k \leq n-2)$ without the Aldous property.

Corollary 3.4. *There exists a positive integer N such that for every $n \geq N$, the k-point-fixing graph $\mathcal{F}(n,k)$ has the Aldous property if and only if $2 \leq k \leq n-2$.*

The integer N in all results above is no more than an integer threshold in [88] which is believed to be as small as 17. In fact, as will be seen in the proofs of Theorems 3.1–3.3 and Corollary 3.4, the integer N in these results is given by $\max\{N_0, 6\}$, $\max\{N_0, 7\}$, $\max\{N_0, 7\}$ and $\max\{N_0, 6\}$, respectively, where N_0 (see Lemma 2.4) is the integer N_1 in [88, Proposition 2.3]. The larger one between this N_1 and the integer N_2 in [88, Proposition 2.4] gives the integer threshold N_0 in [88, Theorem 1.7], and it is conjectured in [88, Conjecture 1.8] that the smallest value of this threshold N_0 is 17. Therefore, if Conjecture 1.8 in [88] is true, then in all our results above N can be set to be 17.

A major tool for our proofs of Theorems 3.1 and 3.2 is Proposition 2.3 in [88] (see Lemma 2.4). In particular, this proposition implies that, for sufficiently large n, the second largest eigenvalue of a connected normal Cayley graph as in Theorem 3.2 satisfying $I \cap \{n-1, n\} = \emptyset$ can be obtained by comparing the two eigenvalues that correspond to the partitions $(n-1, 1)$ and (1^n) of n. As mentioned above, to prove Theorem 3.3 we only need to consider the case when $k = n-1$, and in this case the proof can be reduced to determining the smallest eigenvalue of the complement of $\mathrm{Cay}(S_n, T(n, n-1))$. We will achieve this with the help of an asymptotic upper bound [73] on the irreducible characters of S_n (see Lemma 2.5) and a lower bound [88] on the dimensions of the irreducible representations of S_n (see Lemma 2.6).

The remainder of this chapter is structured as follows. The proofs of Theorems 3.1–3.3 will be given in Sections 3.2–3.4, respectively. The proof of Corollary 3.4 will be given in Section 3.4 as well.

3.2 Proof of Theorem 3.1

Note that $C(S_n, \gamma) \neq \{\mathbf{1}\}$ if and only if $\gamma \neq (1^n)$, which in turn is true if and only if $|\mathrm{supp}(\sigma)| \geq 2$ for any $\sigma \in C(S_n, \gamma)$. Note also that $c_1(\gamma) \geq 2$ if and only if $|\mathrm{supp}(\sigma)| \leq n-2$ for any $\sigma \in C(S_n, \gamma)$. So Theorem 3.1 can be restated as follows: There is a positive integer N such that for every $n \geq N$ and any $(1^n) \neq \gamma \vdash n$, the Cayley graph $\mathrm{Cay}(S_n, C(S_n, \gamma))$ has the Aldous property if and only if $c_1(\gamma) \geq 2$. We prove this statement in the rest of this section.

Proof. Since $\mathrm{Cay}(S_n, C(S_n,\gamma))$ is normal, by Proposition 1.2 its eigenvalue corresponding to the partition $\zeta \vdash n$ is

$$\lambda_\zeta := \sum_{\sigma \in C(S_n,\gamma)} \tilde{\chi}_\zeta(\sigma) = |C(S_n,\gamma)| \cdot \tilde{\chi}_\zeta(\gamma).$$

Note that for any partition $\gamma \neq (1^n)$ of n, the subgroup $\langle C(S_n,\gamma)\rangle$ is either S_n or A_n. Assume first $\langle C(S_n,\gamma)\rangle = S_n$. Then every permutation in $C(S_n,\gamma)$ is odd, and $|C(S_n,\gamma)|$ is a simple eigenvalue of $\mathrm{Cay}(S_n, C(S_n,\gamma))$, which is attained by the trivial representation $\rho_{(n)}$ as $\lambda_{(n)} = |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n)}(\gamma) = |C(S_n,\gamma)|$. The eigenvalue corresponding to the sign representation $\rho_{(1^n)}$ is $\lambda_{(1^n)} = |C(S_n,\gamma)| \cdot \tilde{\chi}_{(1^n)}(\gamma) = -|C(S_n,\gamma)|$. According to the Perron-Frobenius Theorem [56, Theorem 8.4.4], we know that this is the smallest eigenvalue of $\mathrm{Cay}(S_n, C(S_n,\gamma))$. Now assume $\langle C(S_n,\gamma)\rangle = A_n$. Then the largest eigenvalue $|C(S_n,\gamma)|$ of $\mathrm{Cay}(S_n, C(S_n,\gamma))$ has multiplicity 2 and is attained simultaneously by the trivial representation $\rho_{(n)}$ and the sign representation $\rho_{(1^n)}$. Therefore, for any partition $\gamma \neq (1^n)$ of n, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(S_n,\gamma))$ can be expressed as

$$\max_{\substack{\zeta \vdash n \\ \zeta \notin \{(n),(1^n)\}}} \lambda_\zeta = |C(S_n,\gamma)| \max_{\substack{\zeta \vdash n \\ \zeta \notin \{(n),(1^n)\}}} \tilde{\chi}_\zeta(\gamma). \tag{3.1}$$

If $c_1(\gamma) \geq 2$, then by Lemma 2.4 there is a positive integer N_0 such that for every $n \geq N_0$, the maximum on the right-hand side of (3.1) is attained by $\zeta = (n-1,1)$. In other words, if $c_1(\gamma) \geq 2$, then $\mathrm{Cay}(S_n, C(S_n,\gamma))$ has the Aldous property for every $n \geq N_0$.

In the remaining proof we assume that $n \geq 6$ and $(1^n) \neq \gamma \vdash n$ is such that $c_1(\gamma) \leq 1$. We aim to prove that $\mathrm{Cay}(S_n, C(S_n,\gamma))$ does not have the Aldous property in this case.

If $c_1(\gamma) = 1$, then a direct computation using Table 2.1 leads to $\tilde{\chi}_{(n-1,1)}(\gamma) = 0$ and thus $\lambda_{(n-1,1)} = 0$, while the value of (3.1) is at least

$$\begin{aligned}\max\{\lambda_{(n-3,3)},\ \lambda_{(n-3,2,1)}\} &= |C(S_n,\gamma)| \cdot \max\{\tilde{\chi}_{(n-3,3)}(\gamma),\ \tilde{\chi}_{(n-3,2,1)}(\gamma)\} \\ &= |C(S_n,\gamma)| \cdot \max\left\{\frac{6c_3(\gamma)}{n(n-1)(n-5)},\ \frac{3-3c_3(\gamma)}{n(n-2)(n-4)}\right\},\end{aligned}$$

which is positive as $c_3(\gamma) \geq 0$. This means that the maximum on the left-hand side of (3.1) is not attained by $\zeta = (n-1,1)$, and thus $\mathrm{Cay}(S_n, C(S_n,\gamma))$ with $c_1(\gamma) = 1$ does not possess the Aldous property for every $n \geq 6$.

Finally, let us consider the case when $c_1(\gamma) = 0$. In this case, we have $\tilde{\chi}_{(n-1,1)}(\gamma) = -\frac{1}{n-1} < 0$ and hence $\lambda_{(n-1,1)} < 0$. If $\mathrm{sgn}(\gamma) = -1$, then according to Lemma 2.2 the value of (3.1) is at least

$$\lambda_{(n-1,1)'} = |C(S_n, \gamma)| \cdot \tilde{\chi}_{(n-1,1)'}(\gamma) = |C(S_n, \gamma)| \cdot \mathrm{sgn}(\gamma)\tilde{\chi}_{(n-1,1)}(\gamma) > 0.$$

If $\mathrm{sgn}(\gamma) = 1$, then the value of (3.1) is at least

$$\begin{aligned} \max\{\lambda_{(n-2,2)},\ \lambda_{(n-2,1,1)'}\} &= |C(S_n, \gamma)| \cdot \max\left\{\tilde{\chi}_{(n-2,2)}(\gamma),\ \tilde{\chi}_{(n-2,1,1)'}(\gamma)\right\} \\ &= |C(S_n, \gamma)| \cdot \max\left\{\frac{2c_2(\gamma)}{n(n-3)},\ \frac{2-2c_2(\gamma)}{(n-1)(n-2)}\right\}, \end{aligned}$$

which is positive as $c_2(\gamma) \geq 0$. Thus the maximum on the left-hand side of (3.1) is not attained by $\zeta = (n-1, 1)$. In other words, if $c_1(\gamma) = 0$, then $\mathrm{Cay}(S_n, C(S_n, \gamma))$ does not possess the Aldous property for every $n \geq 6$.

To sum up, we have proved that for $n \geq N := \max\{N_0, 6\}$, the normal Cayley graph $\mathrm{Cay}(S_n, C(S_n, \gamma))$ with $(1^n) \neq \gamma \vdash n$ has the Aldous property if and only if $c_1(\gamma) \geq 2$. ☐

3.3 Proof of Theorem 3.2

Proof. We prove the sufficiency and necessity separately.

Sufficiency. Suppose that $I \cap \{n-1, n\} = \emptyset$. Then $\emptyset \neq I \subseteq \{2, 3, \ldots, n-2\}$. By the definition of $T(n, I)$, the subgroup $\langle T(n, I)\rangle$ is either S_n or A_n. Since $\mathrm{Cay}(S_n, T(n, I))$ is normal, by Proposition 1.2 its eigenvalue corresponding to the partition $\zeta \vdash n$ is

$$\lambda_\zeta := \sum_{\sigma \in T(n,I)} \tilde{\chi}_\zeta(\sigma) = \sum_{\substack{\gamma \vdash n \\ n - c_1(\gamma) \in I}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma).$$

Case 1. $\langle T(n, I)\rangle = A_n$.

In this case we have $I = \{3\}$ and $T(n, I) = C\left(S_n, (3, 1^{n-3})\right)$ is the single conjugacy class of all 3-cycles in S_n. By Theorem 3.1 and its proof, there is a positive integer $N_1 := \max\{N_0, 6\}$ such that for every $n \geq N_1$, the Cayley graph $\mathrm{Cay}(S_n, T(n, \{3\}))$ has the Aldous property, where N_0 is the positive integer given in Lemma 2.4.

Case 2. $\langle T(n,I)\rangle = S_n$.

It is clear that the eigenvalue of $\mathrm{Cay}(S_n, T(n,I))$ corresponding to the trivial representation $\rho_{(n)}$ is $\lambda_{(n)} = |T(n,I)|$, which is simple as $\mathrm{Cay}(S_n, T(n,I))$ is connected. The second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,I))$, which is also the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,I))$, is given by

$$\max_{\substack{\zeta\vdash n\\ \zeta\neq(n)}} \lambda_\zeta = \max_{\substack{\zeta\vdash n\\ \zeta\neq(n)}} \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)\in I}} |C(S_n,\gamma)|\cdot\tilde{\chi}_\zeta(\gamma). \tag{3.2}$$

To complete the proof, it suffices to prove that the maximum in (3.2) is attained by $\zeta = (n-1,1)$.

Since I is a nonempty subset of $\{2,3,\dots,n-2\}$, every partition $\gamma\vdash n$ with $n-c_1(\gamma)\in I$ satisfies $c_1(\gamma)\geq 2$. Thus, by Lemma 2.4, there is a positive integer N_0 such that for every $n\geq N_0$ and any $\gamma\vdash n$ with $n-c_1(\gamma)\in I$, we have

$$\max_{\substack{\zeta\vdash n\\ \zeta\notin\{(n),(1^n)\}}} \tilde{\chi}_\zeta(\gamma) = \tilde{\chi}_{(n-1,1)}(\gamma).$$

This implies that, for every $n\geq N_0$,

$$\begin{aligned}
\max_{\substack{\zeta\vdash n\\ \zeta\notin\{(n),(1^n)\}}} \lambda_\zeta &= \max_{\substack{\zeta\vdash n\\ \zeta\notin\{(n),(1^n)\}}} \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)\in I}} |C(S_n,\gamma)|\cdot\tilde{\chi}_\zeta(\gamma)\\
&= \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)\in I}} |C(S_n,\gamma)|\cdot\tilde{\chi}_{(n-1,1)}(\gamma)\\
&= \lambda_{(n-1,1)}.
\end{aligned}$$

Thus, for every $n\geq N_0$, the second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,I))$ given in formula (3.2) equals

$$\max\{\lambda_{(n-1,1)},\ \lambda_{(1^n)}\} = \max_{\zeta\in\{(n-1,1),(1^n)\}} \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)\in I}} |C(S_n,\gamma)|\cdot\tilde{\chi}_\zeta(\gamma). \tag{3.3}$$

In the following we prove that the eigenvalue $\lambda_{(n-1,1)}$ is always greater than or equal to $\lambda_{(1^n)}$ for any $n\geq 7$ and $\emptyset\neq I\subseteq\{2,3,\dots,n-2\}$ with $I\neq\{3\}$. This will be accomplished by giving explicit expressions for $\lambda_{(n-1,1)}$ and $\lambda_{(1^n)}$ and then comparing them on a term-by-term basis.

Let us first calculate the values of $\lambda_{(n-1,1)}$ and $\lambda_{(1^n)}$. When $\zeta = (n-1,1)$, we have $\chi_\zeta(\gamma) = c_1(\gamma) - 1$ and $\chi_\zeta(\mathbf{1}) = n-1$ by Table 2.1. So

$$\begin{aligned}\lambda_{(n-1,1)} &= \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)\in I}} |C(S_n,\gamma)| \cdot \frac{c_1(\gamma)-1}{\chi_\zeta(\mathbf{1})}\\ &= \sum_{t\in I}\frac{n-t-1}{n-1}\cdot \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)=t}} |C(S_n,\gamma)|\\ &= \sum_{t\in I}\frac{n-t-1}{n-1}\cdot\binom{n}{t}\cdot D(t) \qquad (3.4)\\ &> 0, \qquad (3.5)\end{aligned}$$

where $D(t) = t!\cdot\sum_{i=0}^{t}\frac{(-1)^i}{i!}$ is the number of derangements on $[t]$. Note that $D(t) \geq \frac{t!}{3}$ for every integer $t \geq 3$. On the other hand, if $\zeta = (1^n)$, then $\chi_\zeta(\gamma) = \operatorname{sgn}(\gamma)$ and $\chi_\zeta(\mathbf{1}) = 1$. Thus,

$$\begin{aligned}\lambda_{(1^n)} &= \sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)\in I}} |C(S_n,\gamma)|\cdot\operatorname{sgn}(\gamma)\\ &= \sum_{t\in I}\sum_{\substack{\gamma\vdash n\\ n-c_1(\gamma)=t}} |C(S_n,\gamma)|\cdot\operatorname{sgn}(\gamma)\\ &= \sum_{t\in I}\binom{n}{t}(E(t)-O(t))\\ &= \sum_{t\in I}(-1)^{t-1}(t-1)\binom{n}{t}, \qquad (3.6)\end{aligned}$$

where $E(t)$ and $O(t)$ are the numbers of even and odd derangements on $[t]$, respectively, and the last step follows from Lemma 2.3.

Now we prove that when $n \geq 7$, for any $\emptyset \neq I \subseteq \{2,3,\ldots,n-2\}$ with $I \neq \{3\}$, the value of formula (3.4) is no less than that of formula (3.6). Define

$$A(t) := \frac{n-t-1}{n-1}\cdot\binom{n}{t}\cdot D(t), \quad B(t) := (-1)^{t-1}(t-1)\binom{n}{t}$$

for $2 \leq t \leq n-2$. By straightforward computations, one can verify that $A(2)-B(2) = n(n-2)$, $A(3)-B(3) = -n(n-2)$, and $A(4)-B(4) = \frac{1}{2}n(n-2)(n-3)(n-4) > n(n-2)$

for $n \geq 7$. For $5 \leq t \leq n-2$, we have

$$\begin{aligned} A(t)-B(t) &= \frac{n-t-1}{n-1} \cdot \binom{n}{t} \cdot D(t) - (-1)^{t-1}(t-1)\binom{n}{t} \\ &\geq \frac{n-t-1}{n-1} \cdot \binom{n}{t} \cdot D(t) - (t-1)\binom{n}{t} \\ &> \frac{n-t-1}{n-1} \cdot \binom{n}{t} \cdot \frac{t!}{3} - (t-1)\binom{n}{t} \\ &= \frac{n-t-1}{n-1} \cdot \binom{n}{t} \cdot \frac{t(t-1)(t-2)!}{3} - (t-1)\binom{n}{t} \\ &\geq \frac{n-t-1}{n-1} \cdot 2t(t-1) \cdot \binom{n}{t} - (t-1)\binom{n}{t} \\ &= \frac{-2t^2+2nt-2t-n+1}{n-1}(t-1)\binom{n}{t}, \end{aligned}$$

where in the second last step we used the fact that $(t-2)! \geq 6$ when $t \geq 5$. Let $f(t) = -2t^2+2nt-2t-n+1$. It can be verified that $f(t) > 0$ for every positive integer t in the interval $[5, n-2]$ and that the minimum value of $f(t)$ in this interval is $f(n-2) = n-3$. So, continuing the estimation above, we obtain

$$\begin{aligned} A(t)-B(t) &> \frac{n-3}{n-1}(t-1)\binom{n}{t} \\ &= \frac{n(n-3)(t-1)}{2} \cdot \frac{(n-2)(n-3)\cdots(n-t+1)}{t(t-1)\cdots 3} \\ &\geq \frac{n(n-3)(t-1)}{2} \\ &\geq 2n(n-3) \\ &> n(n-2). \end{aligned}$$

So far we have proved that if $n \geq 7$, then $A(3)-B(3) = -n(n-2)$ and $A(t)-B(t) \geq n(n-2)$ for any integer t with $3 \neq t \in [2, n-2]$. Therefore, when $n \geq 7$, for any $\emptyset \neq I \subseteq \{2,3,\ldots,n-2\}$ with $I \neq \{3\}$, the value of formula (3.4) is greater than or equal to that of formula (3.6). This implies that the maximum in formula (3.3) is always attained by $\zeta = (n-1,1)$ when $n \geq 7$, and thus the maximum in (3.2) is achieved by $\zeta = (n-1,1)$ for every $n \geq N_2 := \max\{N_0, 7\}$.

To sum up, we have proved that for every $n \geq N := \max\{N_1, N_2\} = \max\{N_0, 7\}$ and any $\emptyset \neq I \subseteq \{2,3,\ldots,n-2\}$, the normal Cayley graph $\mathrm{Cay}(S_n, T(n,I))$ has the Aldous property, where N_0 is the positive integer given in Lemma 2.4. This establishes the sufficiency part of Theorem 3.2.

Necessity. Since $|I \cap \{n-1, n\}| \neq 1$ by our assumption, to establish the necessity it suffices to prove the following statement: There exists a positive integer N such that for every $n \geq N$ and any $\{n-1, n\} \subseteq J \subset \{2, 3, \ldots, n-1, n\}$, $\mathrm{Cay}(S_n, T(n, J))$ does not have the Aldous property.

Now suppose that $\{n-1, n\} \subseteq J \subset \{2, 3, \ldots, n-1, n\}$. Set $I = \{2, 3, \ldots, n-1, n\} \setminus J$. Then $\emptyset \neq I \subseteq \{2, 3, \ldots, n-2\}$ and $\{T(n, J), T(n, I)\}$ is a partition of $S_n \setminus \{\mathbf{1}\}$, where as before $\mathbf{1}$ denotes the identity of S_n. Hence $\mathrm{Cay}(S_n, T(n, J))$ and $\mathrm{Cay}(S_n, T(n, I))$ are complements of each other. It is clear that $\mathrm{Cay}(S_n, T(n, J))$ is connected and its largest eigenvalue $|T(n, J)| = n! - |T(n, I)| - 1$ is achieved by the trivial representation $\rho_{(n)}$ of S_n. According to Proposition 1.2, any other eigenvalue of $\mathrm{Cay}(S_n, T(n, J))$ can be expressed as $\sum_{\sigma \in T(n,J)} \tilde{\chi}_\zeta(\sigma)$ for some $(n) \neq \zeta \vdash n$. Since the complete graph $\mathrm{Cay}(S_n, S_n \setminus \{\mathbf{1}\})$ has eigenvalues $n! - 1$ with multiplicity 1 and -1 with multiplicity $n! - 1$, by Proposition 1.2, we have $\sum_{\sigma \in S_n \setminus \{\mathbf{1}\}} \tilde{\chi}_\zeta(\sigma) = -1$ for any $(n) \neq \zeta \vdash n$. The fact that $\{T(n, J), T(n, I)\}$ is a partition of $S_n \setminus \{\mathbf{1}\}$ enables us to write this as

$$\sum_{\sigma \in T(n,J)} \tilde{\chi}_\zeta(\sigma) + \sum_{\sigma \in T(n,I)} \tilde{\chi}_\zeta(\sigma) = -1 \tag{3.7}$$

for any $(n) \neq \zeta \vdash n$. Note that $\sum_{\sigma \in T(n,I)} \tilde{\chi}_\zeta(\sigma)$ is an eigenvalue of $\mathrm{Cay}(S_n, T(n, I))$.

Case 3. $I \neq \{3\}$.

In this case, $\mathrm{Cay}(S_n, T(n, I))$ is connected and by the sufficiency proved above, there exists a positive integer N such that

$$\max_{\substack{\zeta \vdash n \\ \zeta \neq (n)}} \sum_{\sigma \in T(n,I)} \tilde{\chi}_\zeta(\sigma) = \sum_{\sigma \in T(n,I)} \tilde{\chi}_{(n-1,1)}(\sigma) \tag{3.8}$$

whenever $n \geq N$. Since the sum of the eigenvalues of $\mathrm{Cay}(S_n, T(n, I))$ is 0 and by (3.5) the second largest eigenvalue of $\mathrm{Cay}(S_n, T(n, I))$ as shown on the right-hand side of (3.8) is positive, it follows that $\mathrm{Cay}(S_n, T(n, I))$ has at least three distinct eigenvalues and at least one of them is negative. This together with (3.7) and (3.8) implies that $\mathrm{Cay}(S_n, T(n, J))$ has at least three distinct eigenvalues and the smallest one of them is attained by the standard representation of S_n. Hence $\mathrm{Cay}(S_n, T(n, J))$ does not have the Aldous property when $n \geq N$.

Case 4. $I = \{3\}$.

In this case, $\mathrm{Cay}(S_n, T(n,I))$ is the union of two copies of $\mathrm{Cay}(A_n, T(n,I))$. So the largest eigenvalue $|T(n,I)|$ of $\mathrm{Cay}(S_n, T(n,I))$ has multiplicity 2 and is attained simultaneously by the trivial and sign representations of S_n. Thus, by (3.7), we know that the sign representation attains the smallest eigenvalue of $\mathrm{Cay}(S_n, T(n,J))$, which is $-1-|T(n,I)|$. Moreover, by the sufficiency proved above, whenever $n \geq N$ the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,I))$ is

$$\max_{\substack{\zeta \vdash n \\ \zeta \notin \{(n),(1^n)\}}} \sum_{\sigma \in T(n,I)} \tilde{\chi}_{\zeta}(\sigma) = \sum_{\sigma \in T(n,I)} \tilde{\chi}_{(n-1,1)}(\sigma) > 0. \tag{3.9}$$

Similarly to the case above, we obtain that $\mathrm{Cay}(S_n, T(n,I))$ has at least one negative eigenvalue, say λ, as the sum of its eigenvalues is 0. This together with (3.7) and (3.9) implies that the second smallest eigenvalue of $\mathrm{Cay}(S_n, T(n,J))$ is attained by the standard representation of S_n, which is larger than $-1-|T(n,I)|$ but strictly smaller than $-1-\lambda$. Thus, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,J))$ is not attained by the standard representation of S_n. In other words, $\mathrm{Cay}(S_n, T(n,J))$ does not have the Aldous property when $n \geq N$. ☐

Remark 3.5. In [57, Theorem 3.4], Huang and Huang determined the second largest eigenvalue of the complete alternating group graph $\mathrm{Cay}(A_n, \{(i\ j\ k), (i\ k\ j) \mid 1 \leq i < j < k \leq n\})$. Note that this graph is exactly $\mathrm{Cay}(A_n, T(n,I))$ with $I = \{3\}$. Since $\mathrm{Cay}(S_n, T(n,\{3\}))$ is the union of two copies of $\mathrm{Cay}(A_n, T(n,\{3\}))$, it has the same eigenvalues as the latter but with the multiplicity of each eigenvalue doubled. Thus, for sufficiently large n, we can obtain [57, Theorem 3.4] from the sufficiency part of Theorem 3.2 by choosing $I = \{3\}$ or from Theorem 3.1 by taking S to be the conjugacy class of 3-cycles. In fact, by Theorem 3.1 or 3.2, $\mathrm{Cay}(S_n, T(n,\{3\}))$ has the Aldous property for sufficiently large n. Using this and Table 2.1, we obtain that, for sufficiently large n,

$$\begin{aligned}
\lambda_2(\mathrm{Cay}(A_n, T(n,\{3\}))) &= \lambda_3(\mathrm{Cay}(S_n, T(n,\{3\}))) \\
&= \sum_{\sigma \in T(n,\{3\})} \tilde{\chi}_{(n-1,1)}(\sigma) \\
&= |T(n,\{3\})| \cdot \tilde{\chi}_{(n-1,1)}\big((3, 1^{n-3})\big) \\
&= |T(n,\{3\})| \cdot \frac{n-4}{n-1} \\
&= \frac{1}{3} n(n-2)(n-4),
\end{aligned}$$

which is exactly what is stated in [57, Theorem 3.4].

3.4 Proofs of Theorem 3.3 and Corollary 3.4

We prove Theorem 3.3 first.

Proof. It is clear that for any $2 \leq k \leq n$ we have

$$T(n,k) = \bigcup_{\substack{\gamma \vdash n \\ 2 \leq n - c_1(\gamma) \leq k}} C(S_n, \gamma)$$

and hence $\mathrm{Cay}(S_n, T(n,k))$ is normal. Thus, by Proposition 1.2, the eigenvalues of $\mathrm{Cay}(S_n, T(n,k))$ can be expressed as

$$\sum_{\sigma \in T(n,k)} \tilde{\chi}_\zeta(\sigma) = \sum_{\substack{\gamma \vdash n \\ 2 \leq n - c_1(\gamma) \leq k}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma), \tag{3.10}$$

where $\zeta \vdash n$ is running over all partitions of n. Moreover, $\mathrm{Cay}(S_n, T(n,k))$ is connected as $T(n,2) \subseteq T(n,k)$ and $T(n,2)$ is a generating subset of S_n. So the largest eigenvalue $|T(n,k)|$ of $\mathrm{Cay}(S_n, T(n,k))$ is simple and is attained by the trivial representation $\rho_{(n)}$. Hence (3.10) evaluates to $|T(n,k)|$ when $\zeta = (n)$. Thus, the second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,k))$, which is also the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, T(n,k))$, is equal to

$$\max_{\substack{\zeta \vdash n \\ \zeta \neq (n)}} \sum_{\substack{\gamma \vdash n \\ 2 \leq n - c_1(\gamma) \leq k}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma). \tag{3.11}$$

To prove Theorem 3.3, it suffices to show that the maximum in (3.11) is achieved by $\zeta = (n-1, 1)$.

Case 1. $k = n$, where $n \geq 4$.

In this case, we have $T(n,k) = S_n \setminus \{\mathbf{1}\}$ and $\mathrm{Cay}(S_n, T(n,k))$ is the complete graph, which has two distinct eigenvalues only, namely, $|T(n,k)|$ with multiplicity 1 and -1 with multiplicity $n! - 1$. Thus, for every $(n) \neq \zeta \vdash n$, formula (3.10) evaluates to -1, that is,

$$\sum_{\substack{\gamma \vdash n \\ 2 \leq n - c_1(\gamma) \leq n}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma) = -1 \tag{3.12}$$

for every $(n) \neq \zeta \vdash n$. So the maximum in formula (3.11) also equals -1, which is attained by any $(n) \neq \zeta \vdash n$ and hence by $\zeta = (n-1,1)$ in particular. This means that the result in Theorem 3.3 is true when $k = n$ with $n \geq 4$.

Case 2. $k = n - 1$.

In this case, formula (3.11) becomes

$$\max_{\substack{\zeta \vdash n \\ \zeta \neq (n)}} \sum_{\substack{\gamma \vdash n \\ 2 \leq n - c_1(\gamma) \leq n-1}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma),$$

which equals

$$-1 - \min_{\substack{\zeta \vdash n \\ \zeta \neq (n)}} \sum_{\substack{\gamma \vdash n \\ c_1(\gamma) = 0}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma)$$

due to Eq. (3.12). Thus, to prove that the second largest eigenvalue of $\mathrm{Cay}(S_n, T_{n-1})$ is given by the standard representation of S_n, it suffices to prove that the minimum of

$$\sum_{\substack{\gamma \vdash n \\ c_1(\gamma) = 0}} |C(S_n, \gamma)| \cdot \tilde{\chi}_\zeta(\gamma) \tag{3.13}$$

among all $(n) \neq \zeta \vdash n$ is attained by $\zeta = (n-1,1)$. Note that, for $\zeta = (n-1,1)$ with $n \geq 4$ and any $\gamma \vdash n$ with $c_1(\gamma) = 0$, we have $\tilde{\chi}_\zeta(\gamma) = \frac{c_1(\gamma)-1}{n-1} = -\frac{1}{n-1}$ by Table 2.1. Hence the value of formula (3.13) for $\zeta = (n-1,1)$ is

$$\frac{-1}{n-1} \cdot \sum_{\substack{\gamma \vdash n \\ c_1(\gamma) = 0}} |C(S_n, \gamma)|. \tag{3.14}$$

In the following we will compare this with the value of (3.13) for any $\zeta \vdash n$ with $\zeta \neq (n), (n-1,1)$.

By Lemma 2.5, if γ is a partition of n with $c_1(\gamma) = 0$, then for every $\zeta \vdash n$ we have

$$\chi_\zeta(\gamma) \geq -\chi_\zeta(\mathbf{1})^{\frac{1}{2}+\varepsilon_n}$$

and hence

$$\tilde{\chi}_\zeta(\gamma) \geq -\chi_\zeta(\mathbf{1})^{-\frac{1}{2}+\varepsilon_n}.$$

Moreover, by Lemma 2.6, we have $\chi_\zeta(\mathbf{1}) \geq n^{2.05}$ for any partition ζ of $n \geq 13$ whose Young diagram has at least three boxes outside the first row and at least three boxes outside the first column. Let N_1 be the smallest integer no less than 13 such that $2.05(-\frac{1}{2}+\varepsilon_n) \leq -1$ for all $n \geq N_1$, where ε_n is as in Lemma 2.5. Then

$$\tilde{\chi}_\zeta(\gamma) \geq -\chi_\zeta(\mathbf{1})^{-\frac{1}{2}+\varepsilon_n} \geq -\frac{1}{n} > -\frac{1}{n-1}$$

for any $\zeta \vdash n$ satisfying the conditions in Lemma 2.6. So, when $n \geq N_1$, the values of (3.13) corresponding to these ζ's are greater than (3.14). Therefore, to determine the minimum of (3.13) it remains to consider the following five possibilities for $\zeta \neq (n-1,1)$: $(n-1,1)'$, $(n-2,2)$, $(n-2,2)'$, $(n-2,1,1)$, and $(n-2,1,1)'$. Using Table 2.1 and Lemma 2.2, we obtain that the normalized character $\tilde{\chi}_\zeta(\gamma)$ of $\zeta = (n-1,1)', (n-2,2), (n-2,1,1), (n-2,1,1)'$ on any $\gamma \vdash n$ with $c_1(\gamma) = 0$ is equal to $\frac{-\mathrm{sgn}(\gamma)}{n-1}, \frac{2c_2(\gamma)}{n(n-3)}, \frac{2-2c_2(\gamma)}{(n-1)(n-2)}, \frac{\mathrm{sgn}(\gamma)(2-2c_2(\gamma))}{(n-1)(n-2)}$, respectively. Thus, if $n \geq 4$ and $\zeta = (n-1,1)', (n-2,2), (n-2,1,1), (n-2,1,1)'$, then $\tilde{\chi}_\zeta(\gamma) \geq -\frac{1}{n-1}$ for any $\gamma \vdash n$ with $c_1(\gamma) = 0$. Therefore, the value of (3.13) corresponding to any one of these four ζ's are greater than (3.14) when $n \geq 4$.

Now we assume that $\zeta = (n-2,2)'$ and γ is any partition of n with $c_1(\gamma) = 0$. We aim to prove that the value of (3.13) for this ζ is also greater than (3.14). In fact, by Table 2.1 and Lemma 2.2, we have

$$\tilde{\chi}_\zeta(\gamma) = \mathrm{sgn}(\gamma) \cdot \tilde{\chi}_{(n-2,2)}(\gamma) = \mathrm{sgn}(\gamma) \cdot \frac{2c_2(\gamma)}{n(n-3)}.$$

If $n \geq 4$ is odd, then we have $2c_2(\gamma) \leq n-3$ as $c_1(\gamma) = 0$, which implies that $\tilde{\chi}_\zeta(\gamma) > -\frac{1}{n-1}$. If $n \geq 4$ is even, then $2c_2(\gamma) = n$ or $2c_2(\gamma) \leq n-4$ due to $c_1(\gamma) = 0$. If $2c_2(\gamma) \leq n-4$, then we also have $\tilde{\chi}_\zeta(\gamma) > -\frac{1}{n-1}$. If $2c_2(\gamma) = n$ (that is, $\gamma = (2^{\frac{n}{2}})$), then $\tilde{\chi}_\zeta(\gamma) = \mathrm{sgn}(\gamma) \cdot \frac{1}{n-3}$ is greater than $-\frac{1}{n-1}$ if and only if $\mathrm{sgn}(\gamma) = 1$. Note that when $\gamma = (2^{\frac{n}{2}})$, $\mathrm{sgn}(\gamma) = (-1)^{\frac{n}{2}} = 1$ if and only if $n \equiv 0 \pmod 4$. Thus, for any $n \equiv 0 \pmod 4$, we still have $\tilde{\chi}_\zeta(\gamma) > -\frac{1}{n-1}$ for any $\gamma \vdash n$ with $c_1(\gamma) = 0$. Combining these, we obtain that the value of (3.13) for $\zeta = (n-2,2)'$ is greater than (3.14) whenever $n \geq 4$ and $n \not\equiv 2 \pmod 4$.

Now assume that $n \geq 4$ and $n \equiv 2 \pmod 4$. Then only for $\gamma = (2^{\frac{n}{2}})$ is $\tilde{\chi}_\zeta(\gamma) = -\frac{1}{n-3}$ smaller than $-\frac{1}{n-1}$, and for any other partition γ of n with $c_1(\gamma) = 0$ we still have

$2c_2(\gamma) \leq n-4$ and hence $\tilde{\chi}_\zeta(\gamma) > -\frac{1}{n-1}$. Since $\operatorname{sgn}\left((4, 2^{\frac{n-4}{2}})\right) = 1$, by Table 2.1 we have

$$\tilde{\chi}_\zeta((4, 2^{\frac{n-4}{2}})) = \operatorname{sgn}\big((4, 2^{\frac{n-4}{2}})\big) \cdot \tilde{\chi}_{(n-2,2)}\big((4, 2^{\frac{n-4}{2}})\big) = \frac{n-4}{n(n-3)}.$$

Thus,

$$\begin{aligned}
&\sum_{\substack{\gamma \vdash n \\ c_1(\gamma)=0}} |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n-2,2)'}(\gamma) \\
= &\sum_{\substack{\gamma \vdash n,\ c_1(\gamma)=0 \\ \gamma \notin \{(2^{\frac{n}{2}}),(4,2^{\frac{n-4}{2}})\}}} |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n-2,2)'}(\gamma) + |C(S_n,(2^{\frac{n}{2}}))| \cdot \tilde{\chi}_{(n-2,2)'}((2^{\frac{n}{2}})) \\
&\qquad + |C(S_n,(4,2^{\frac{n-4}{2}}))| \cdot \tilde{\chi}_{(n-2,2)'}((4,2^{\frac{n-4}{2}})) \\
> &\sum_{\substack{\gamma \vdash n,\ c_1(\gamma)=0 \\ \gamma \notin \{(2^{\frac{n}{2}}),(4,2^{\frac{n-4}{2}})\}}} |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n-1,1)}(\gamma) + |C(S_n,(2^{\frac{n}{2}}))| \cdot \tilde{\chi}_{(n-2,2)'}((2^{\frac{n}{2}})) \\
&\qquad + |C(S_n,(4,2^{\frac{n-4}{2}}))| \cdot \tilde{\chi}_{(n-2,2)'}((4,2^{\frac{n-4}{2}})) \\
= &\sum_{\substack{\gamma \vdash n,\ c_1(\gamma)=0 \\ \gamma \notin \{(2^{\frac{n}{2}}),(4,2^{\frac{n-4}{2}})\}}} |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n-1,1)}(\gamma) + |C(S_n,(2^{\frac{n}{2}}))| \cdot \frac{-1}{n-3} \\
&\qquad + |C(S_n,(4,2^{\frac{n-4}{2}}))| \cdot \frac{n-4}{n(n-3)} \\
> &\sum_{\substack{\gamma \vdash n,\ c_1(\gamma)=0 \\ \gamma \notin \{(2^{\frac{n}{2}}),(4,2^{\frac{n-4}{2}})\}}} |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n-1,1)}(\gamma) \\
> &\sum_{\gamma \vdash n,\ c_1(\gamma)=0} |C(S_n,\gamma)| \cdot \tilde{\chi}_{(n-1,1)}(\gamma).
\end{aligned}$$

In the second last step above, the inequality holds as $|C(S_n,(4,2^{\frac{n-4}{2}}))| = \frac{n(n-2)}{4}|C(S_n,(2^{\frac{n}{2}}))|$ and $|C(S_n,(2^{\frac{n}{2}}))| \cdot \frac{-1}{n-3} + |C(S_n,(4,2^{\frac{n-4}{2}}))| \cdot \frac{n-4}{n(n-3)} > 0$. Thus, when $n \geq 4$ and $n \equiv 2 \pmod 4$, the value of (3.13) for $\zeta = (n-2,2)'$ is also greater than (3.14).

To sum up, we have proved that for any $n \geq N_1$ the minimum of (3.13) among all $(n) \neq \zeta \vdash n$ is attained by $\zeta = (n-1,1)$. In other words, the statement in Theorem 3.3 is true for $k = n-1$ whenever $n \geq N_1$.

We claim that N_1 is no more than the integer N_0 in Lemma 2.4. In fact, by the definition of N_1, we know that N_1 is no more than N_3 in [88, Lemma 2.7], which is an integer satisfying $2.05\left(-\frac{1}{2} + \frac{\log 2}{2\log n} + \varepsilon_n\right) \leq -1$ for all $n \geq N_3$. Also, we see from the proof of

[88, Proposition 2.3] that N_0 is no less than N_3 in [88, Lemma 2.7]. Hence $N_1 \leq N_0$ as claimed.

Case 3. $2 \leq k \leq n-2$.

This case is a direct consequence of Theorem 3.2. In fact, letting $I_k = \{2, 3, \ldots, k\}$ for each $2 \leq k \leq n-2$, we have $I_k \subseteq \{2, 3, \ldots, n-2\}$ and $\mathrm{Cay}(S_n, T(n,k)) = \mathrm{Cay}(S_n, T(n, I_k))$ as $T(n,k) = T(n, I_k)$. So, by Theorem 3.2 and its proof, there is a positive integer $N_2 := \max\{N_0, 7\}$ such that for every $n \geq N_2$ and any $2 \leq k \leq n-2$, $\mathrm{Cay}(S_n, T(n,k))$ has the Aldous property, where N_0 is the integer in Lemma 2.4.

In summary, we have proved that for every $n \geq N := \max\{N_1, N_2\} = \max\{N_0, 7\}$ and any $2 \leq k \leq n$, $\mathrm{Cay}(S_n, T(n,k))$ has the Aldous property. $\square$

Remark 3.6. When dealing with the case $k = n-1$ in the proof above, the key was to prove the statement that the minimum of formula (3.13) among all $(n) \neq \zeta \vdash n$ is attained by $\zeta = (n-1, 1)$ when n is sufficiently large. This statement is equivalent to the fact that the smallest eigenvalue of the derangement graph $\mathrm{Cay}(S_n, \mathcal{D}_n)$ is attained by the standard representation of S_n when n is sufficiently large, where $\mathcal{D}_n$ is the set of derangements on $[n]$. After completing the proof of Theorem 3.3, we realized that Renteln had proved a stronger result [91, Theorem 7.1], which asserts that for $n \geq 4$ the smallest eigenvalue of $\mathrm{Cay}(S_n, \mathcal{D}_n)$ is equal to $-\frac{|\mathcal{D}_n|}{n-1}$, settling affirmatively a conjecture of Ku and Wong [72, Conjecture 1], and moreover for $n \geq 5$ this smallest eigenvalue is achieved uniquely by the standard representation of S_n. Renteln proved this result using a recurrence formula [91, Theorem 6.5], while our proof above in the case $k = n-1$ took a different approach.

Finally, we prove Corollary 3.4 with the help of Theorem 3.2 and some results from [37, 71, 91].

Proof. In [37], Deng and Zhang proved that for $n \geq 4$ the second largest eigenvalue of $\mathcal{F}(n, 0)$ is positive and is given by the irreducible representation of S_n corresponding to the partition $(n-2, 2)$ of n. Combining this and the fact [91, Theorem 7.1] that for $n \geq 5$ the smallest eigenvalue of $\mathcal{F}(n, 0)$ is negative and is achieved by $\rho_{(n-1,1)}$ (see Remark 3.6), we know that $\mathcal{F}(n, 0)$ does not have the Aldous property when $n \geq 5$.

In [71], Ku, Lau and Wong proved that for $n \geq 7$ the smallest eigenvalue of $\mathcal{F}(n,1)$ is achieved only by the irreducible representation of S_n corresponding to the partition $(n-2,2)$. In [71, Lemma 3.5], they also proved that for $n \geq 7$ the standard representation of S_n yields the eigenvalue 0 of $\mathcal{F}(n,1)$ while at least one of the partitions (1^n), $(2^2,1^{n-4})$, $(3,1^{n-3})$ produces a positive eigenvalue of $\mathcal{F}(n,1)$ other than $|T(n,\{n-1\})|$. This implies that the second largest eigenvalue of $\mathcal{F}(n,1)$ is not attained by the standard representation of S_n; that is, $\mathcal{F}(n,1)$ does not have the Aldous property when $n \geq 7$.

On the other hand, for $2 \leq k \leq n-2$, we have $\{n-k\} \subseteq \{2,3,\ldots,n-2\}$. Thus, by Theorem 3.2, $\mathcal{F}(n,k) = \mathrm{Cay}(S_n, T(n,\{n-k\}))$ has the Aldous property whenever $n \geq N$, where N is as in Theorem 3.2. □

Chapter 4

Normal Cayley graphs $\mathrm{Cay}(S_n, C(n, I))$ generated by cycles

In this chapter, we investigate the strictly second largest eigenvalue of normal Cayley graphs on symmetric groups generated by cycles. Our goal is to determine necessary and sufficient conditions for these graphs to possess the Aldous property.

4.1 A summary of main results

We study the normal Cayley graphs $\mathrm{Cay}(S_n, C(n, I))$ in this chapter, where $\emptyset \neq I \subseteq \{2, 3, \ldots, n\}$ and

$$C(n, I) = \cup_{k \in I} C(n, k) \tag{4.1}$$

with $C(n, k)$ the set of all k-cycles in S_n. We prove that for any $\emptyset \neq I \subseteq \{2, 3, \ldots, n\}$ the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ can only be achieved by at most four different irreducible representations of S_n, and for some special subsets $I \subseteq \{2, 3, \ldots, n\}$ we obtain further the exact value of this eigenvalue together with its multiplicity (Theorems 4.5, 4.14, 4.21 and 4.25). As a corollary, we give a necessary and sufficient condition for $\mathrm{Cay}(S_n, C(n, I))$ to possess the Aldous property in the case when I contains neither n nor $n - 1$ (Corollary 4.6), and we obtain that this graph does not have the Aldous property whenever $n \in I$ (Corollaries 4.22 and 4.26). As another corollary, we prove the conjecture [81, Conjecture 1.4] (Corollary 4.7) that for any $n \geq 4$ and $2 \leq k \leq n-2$, the normal Cayley graph $\mathrm{Cay}(S_n, C(n,k))$ has the Aldous property, and thus solve an open problem posed by Siemons and Zalesski in [96]. A summary of our main results can be found in Table 4.1, where the third column shows all possible partitions of n which achieve the strictly second largest eigenvalue and the last column indicates the multiplicity of this eigenvalue.

The remainder of this chapter is structured as follows. Since the irreducible characters of $(n-1)$-cycles and n-cycles behave differently from that of cycles with length no more than $n-2$, we divide the family of graphs $\mathrm{Cay}(S_n, C(n,I))$ into four subfamilies and investigate them separately in four sections. More explicitly, in Sections 4.2–4.5 we deal with the case where $I \cap \{n-1, n\} = \emptyset$, $n-1 \in I$ but $n \notin I$, $n \in I$ but $n-1 \notin I$, or $\{n-1, n\} \subseteq I$, respectively. We use tools from the representation and character theory of finite groups together with combinatorial techniques in the proofs of our main results.

Recall from (4.1) that $C(n,I)$ is the set of all cycles in S_n with lengths in I, where $\emptyset \neq I \subseteq \{2,3,\ldots,n\}$. Since $\mathrm{Cay}(S_n, C(n,I))$ is normal, by Proposition 1.2 we can express its eigenvalues in terms of the irreducible characters of S_n. More specifically, if we denote by λ_ζ^I the eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ corresponding to $\zeta \vdash n$, then

$$\begin{aligned}
\lambda_\zeta^I &= \sum_{\sigma \in C(n,I)} \tilde{\chi}_\zeta(\sigma) \\
&= \sum_{k \in I} |C(n,k)| \cdot \tilde{\chi}_\zeta((k, 1^{n-k})) \\
&= \sum_{k \in I} \binom{n}{k} (k-1)! \cdot \tilde{\chi}_\zeta((k, 1^{n-k})). \qquad (4.2)
\end{aligned}$$

Moreover, the multiplicity of λ_ζ^I is equal to

$$\sum_{\substack{\mu \vdash n \\ \lambda_\mu^I = \lambda_\zeta^I}} \chi_\mu(\mathbf{1})^2. \qquad (4.3)$$

In particular, by (4.2), Table 2.1 and Lemma 2.2, for any $\emptyset \neq I \subseteq \{2,3,\ldots,n\}$, the two eigenvalues of $\mathrm{Cay}(S_n, C(n,I))$ corresponding to the sign and standard representations are

$$\lambda_{(1^n)}^I = \sum_{k \in I} |C(n,k)| \cdot \tilde{\chi}_{(1^n)}((k, 1^{n-k})) = \sum_{k \in I} \binom{n}{k} (k-1)! \cdot (-1)^{k-1} \qquad (4.4)$$

Subfamilies		**Partitions**	**Multiplicity**
$I \cap \{n-1, n\} = \emptyset$	$I \cap N_2 = \emptyset$	$(n-1,1)$ and $(2,1^{n-2})$	$2(n-1)^2$
	$I = \{2,3\}$	$(n-1,1)$ and (1^n)	$(n-1)^2+1$
	$I \neq \{2,3\}$ $I_{\max} \in N_1$ $I \cap N_2 \neq \emptyset$	(1^n)	1
	$I_{\max} \in N_2$	$(n-1,1)$	$(n-1)^2$
$n-1 \in I$, $n \notin I$	$n \in N_2$ $I \cap N_2 \neq \emptyset$	(1^n)	1
	$n \in N_2$ $I \cap N_2 = \emptyset$	$[(n-1,1)$ and $(2,1^{n-2})]$ or $[(n-3,2,1)$ and $(3,2,1^{n-3})]$	Unknown
	$n \in N_1$	$(n-1,1)$, $(n-3,2,1)$, $(2^2,1^{n-4})$ or $(2,1^{n-2})$	Unknown
$n-1 \notin I$, $n \in I$	$n \in N_1$ $I \cap N_2 \neq \emptyset$	(1^n)	1
	$n \in N_1$ $I \cap N_2 = \emptyset$	$(n-2,1^2)$ and $(3,1^{n-3})$	$\frac{(n-1)^2(n-2)^2}{2}$
	$n \in N_2$	$(n-2,1^2)$ or $(2,1^{n-2})$	Unknown
$\{n-1,n\} \subseteq I$	$n \in N_2$	(1^n), $(n-2,1^2)$ or $(2,1^{n-2})$	Unknown
	$n \in N_1$	(1^n), $(n-2,1^2)$, $(3,1^{n-3})$ or $(2^2,1^{n-4})$	Unknown

Table 4.1: Summary of the main results, where $\emptyset \neq I \subseteq \{2,3,\ldots,n-1,n\}$, $I_{\max}$ is the largest number in I, N_1 is the set of odd positive integers, and N_2 is the set of even positive integers

and

$$\lambda^{I}_{(n-1,1)} = \sum_{k\in I} |C(n,k)| \cdot \tilde{\chi}_{(n-1,1)}((k,1^{n-k})) = \sum_{k\in I} \binom{n}{k}(k-1)! \cdot \frac{n-k-1}{n-1}, \tag{4.5}$$

respectively.

4.2 I contains neither $n-1$ nor n

Lemma 4.1. *Suppose $n \geq 8$. Let $\gamma = (k, 1^{n-k})$ be the cycle type of a k-cycle in S_n with $2 \leq k \leq n-3$. For any $\zeta \vdash n$ other than $(n), (1^n), (n-1,1)$ and $(2,1^{n-2})$, we have*

$$\tilde{\chi}_\zeta(\gamma) < \frac{(n-k)(n-k-1)}{n(n-1)}. \tag{4.6}$$

Proof. One can easily verify that the result holds for $n = 8$. Suppose this result holds for some $n-1 \geq 8$. Now we prove (4.6) for $n \geq 9$ and $2 \leq k \leq n-3$. Consider $k = n-3$ first. We list all the partitions of n in Table 4.2 which have border strips with $n-3$ boxes. The dimensions and the characters in Table 4.2 are calculated with the help of the Murnaghan-Nakayama Rule and Hook Length Formula. Also by the Murnaghan-Nakayama Rule, we know that the partitions of n that are not on this list must achieve zero for the normalized character on any $(n-3)$-cycle of S_n. Through simple calculations with the help of Table 4.2, one can verify (4.6) for $k = n-3$.

Now suppose $2 \leq k \leq n-4$. Using Table 2.1 and Lemma 2.2, we can show that (4.6) holds for $\zeta = (n-2,2), (2^2,1^{n-4}), (n-2,1^2), (3,1^{n-3})$. In the following let ζ be any partition of n with at least three boxes outside the first row and at least three boxes outside the first column. Thus, by Branching Rule, we obtain

$$\begin{aligned}
\tilde{\chi}_\zeta(\gamma) &= \frac{\sum_{\zeta^-} \chi_{\zeta^-}(k,1^{n-1-k})}{\sum_{\zeta^-} \chi_{\zeta^-}(\mathbf{1})} \\
&\leq \max_{\zeta^-} \tilde{\chi}_{\zeta^-}((k,1^{n-1-k})) \\
&< \frac{(n-k-1)(n-k-2)}{(n-1)(n-2)} \\
&< \frac{(n-k)(n-k-1)}{n(n-1)},
\end{aligned}$$

$\boldsymbol{\zeta \vdash n}$	$\boldsymbol{d_\zeta = \chi_\zeta(1)}$	$\boldsymbol{\lvert\chi_\zeta(\gamma)\rvert}$
(n) (1^n)	1	1
$(n-1,1)$ $(2,1^{n-2})$	$n-1$	2
$(n-2,1^2)$ $(3,1^{n-3})$	$\frac{(n-1)(n-2)}{2}$	1
$(n-3,2,1)$ $(3,2,1^{n-5})$	$\frac{n(n-2)(n-4)}{3}$	1
$(n-3,3)$ $(2^3,1^{n-6})$	$\frac{n(n-1)(n-5)}{6}$	2
$(n-4,2^2)$ $(3^2,1^{n-6})$	$\frac{n(n-1)(n-4)(n-5)}{12}$	1
$(n-m,3,2,1^{m-5})$	$\frac{n!}{3(n-3)(n-m+1)(n-m-1)(m-1)(m-3)(m-5)!(n-m-3)!}$	2
$(n-m,4,1^{m-4})$	$\frac{n!}{6(n-3)(n-m)(n-m-1)(n-m-2)m(m-4)!(n-m-4)!}$	1
$(n-m,2^3,1^{m-6})$	$\frac{n!}{6(n-3)(n-m+2)(m-2)(m-3)(m-4)(m-6)!(n-m-2)!}$	1

TABLE 4.2: All nonzero characters of irreducible representations of S_n on $\gamma = (n-3,1^3)$

where the penultimate inequality is deduced from our induction hypothesis. We can use this hypothesis because each ζ^- above is a partition of $n-1$ with at least two boxes outside the first row and at least two boxes outside the first column. $\square$

The next lemma shows that on any cycle of S_n with length at most $n-2$ the normalized character of the standard representation is greater than that of those Specht modules not corresponding to $(n), (1^n), (n-1,1)$ or $(2,1^{n-2})$.

Lemma 4.2. *Suppose $n \geq 7$. Let $\gamma = (k, 1^{n-k})$ be the cycle type of a k-cycle in S_n with $2 \leq k \leq n-2$. For any $\zeta \vdash n$ other than $(n), (1^n), (n-1,1)$ and $(2,1^{n-2})$, we have*

$$\tilde{\chi}_\zeta(\gamma) < \tilde{\chi}_{(n-1,1)}(\gamma).$$

Proof. First, suppose $\gamma = (n-2, 1^2)$. Table 4.3 exhibits all the partitions of n which have border strips with $n-2$ boxes and thus achieve nonzero characters on any $(n-2)$-cycle of S_n. Since $n \geq 7$, from Table 4.3 one can see that $\tilde{\chi}_{(n-1,1)}(\gamma) = \frac{1}{n-1}$ and $\tilde{\chi}_{\zeta}(\gamma) < \frac{1}{n-1}$ whenever $\zeta \neq (n), (1^n), (n-1,1), (2,1^{n-2})$.

$\zeta \vdash n$	$d_\zeta = \chi_\zeta(1)$	$\chi_\zeta(\gamma)$
(n)	1	1
(1^n)	1	$(-1)^{n-1}$
$(n-1,1)$	$n-1$	1
$(2,1^{n-2})$	$n-1$	$(-1)^{n-1}$
$(n-2,2)$	$\frac{n(n-3)}{2}$	-1
$(2^2,1^{n-4})$	$\frac{n(n-3)}{2}$	$(-1)^n$
$(n-m,3,1^{m-3})$	$\frac{n!}{2m(n-2)(n-m)(n-m-1)(m-3)!(n-m-3)!}$	$(-1)^m$
$(n-m,2^2,1^{m-4})$	$\frac{n!}{2(m-1)(m-2)(n-2)(n-m+1)(m-4)!(n-m-2)!}$	$(-1)^m$

TABLE 4.3: All nonzero characters of irreducible representations of S_n on $\gamma = (n-2,1^2)$

Now suppose $\gamma = (k, 1^{n-k})$ with $2 \leq k \leq n-3$. One can verify that if $n = 7$ then $\tilde{\chi}_\zeta(\gamma) < \tilde{\chi}_{(6,1)}(\gamma)$ holds for any $\zeta \neq (7), (1^7), (6,1), (2,1^5)$. If $n \geq 8$, then by Lemma 4.1 we have

$$\tilde{\chi}_\zeta(\gamma) < \frac{(n-k)(n-k-1)}{n(n-1)} < \frac{n-k-1}{n-1} = \tilde{\chi}_{(n-1,1)}(\gamma).$$

This completes the proof. □

The next two lemmas compare the eigenvalues $\lambda^I_{(1^n)}$ and $\lambda^I_{(n-1,1)}$ of $\mathrm{Cay}(S_n, C(n,I))$ for any $I \subseteq \{2,3,\ldots,n-1\}$.

Lemma 4.3. *Suppose $n \geq 7$. If $I = \{2,3\}$, then $\lambda^I_{(1^n)} = \lambda^I_{(n-1,1)}$; if $\{2,3\} \neq I \subseteq \{2,3,\ldots,n-1\}$ and the largest number in I is odd, then $\lambda^I_{(1^n)} > \lambda^I_{(n-1,1)}$.*

Proof. If $I = \{2,3\}$, then a straightforward calculation using (4.4) and (4.5) yields $\lambda^I_{(1^n)} = \lambda^I_{(n-1,1)}$.

Now suppose $I \neq \{2,3\}$ and the largest number in I, say, k_0, is odd. If $k_0 = 3$, then $I = \{3\}$ and $\lambda^{\{3\}}_{(1^n)} > \lambda^{\{3\}}_{(n-1,1)}$ by (4.4) and (4.5). It remains to consider the case where

$5 \le k_0 \le n-1$. In this case, by (4.4) and (4.5), we have

$$\begin{aligned}
&\lambda^I_{(1^n)} - \lambda^I_{(n-1,1)} \\
= \ & \binom{n}{k_0}(k_0-1)!\frac{k_0}{n-1} + \sum_{k\in I\setminus\{k_0\}} \binom{n}{k}(k-1)!\left((-1)^{k-1} - \frac{n-k-1}{n-1}\right) \\
\ge \ & \binom{n}{k_0}(k_0-1)!\frac{k_0}{n-1} - \sum_{\substack{2\le k\le k_0-1 \\ k \text{ is even}}} \binom{n}{k}(k-1)!\frac{2n-k-2}{n-1} \\
= \ & n(n-2)!\left(\frac{1}{(n-k_0)!} - \sum_{\substack{2\le k\le k_0-1 \\ k \text{ is even}}} \frac{2n-k-2}{k(n-k)!}\right) \\
= \ & n(n-2)!\left(\frac{1}{(n-5)!} - \sum_{\substack{2\le k\le 5 \\ k \text{ is even}}} \frac{2n-k-2}{k(n-k)!}\right) + \\
& n(n-2)!\left(\sum_{\substack{6\le k\le k_0-1 \\ k \text{ is even}}} \left(\frac{1}{(n-k-1)!} - \frac{1}{(n-k+1)!} - \frac{2n-k-2}{k(n-k)!}\right)\right). \qquad (4.7)
\end{aligned}$$

Since $n \ge 7$, we see that the first part of the lower bound (4.7) is positive. Note that when $k_0 = 5$ the second part of this lower bound valishes. Note also that, for $6 \le k \le n-2$, we have

$$\begin{aligned}
& \frac{1}{(n-k-1)!} - \frac{1}{(n-k+1)!} - \frac{2n-k-2}{k(n-k)!} \\
> \ & \frac{1}{(n-k-1)!} - \frac{k-2}{k(n-k)!} - \frac{2n-k-2}{k(n-k)!} \\
= \ & \frac{1}{(n-k-1)!} - \frac{2n-4}{k(n-k)!} \\
\ge \ & 0.
\end{aligned}$$

Therefore, the second part of the lower bound (4.7) is also positive as required to complete the proof. □

Lemma 4.4. *Suppose $n \ge 7$. If $\emptyset \ne I \subseteq \{2,3,\dots,n-1\}$ and the largest number in I is even, then $\lambda^I_{(n-1,1)} > \lambda^I_{(1^n)}$.*

Proof. Denote by k_0 the largest number in I. By our assumption, k_0 is even. If $k_0 = 2$, then $I = \{2\}$ and $\lambda^{\{2\}}_{(n-1,1)} > \lambda^{\{2\}}_{(1^n)}$ by (4.4) and (4.5).

Now suppose $4 \leq k_0 \leq n-1$. By (4.4) and (4.5), we have

$$\begin{aligned}
& \lambda^I_{(n-1,1)} - \lambda^I_{(1^n)} \\
= \; & \binom{n}{k_0}(k_0-1)!\frac{2n-k_0-2}{n-1} + \sum_{k\in I\setminus\{k_0\}} \binom{n}{k}(k-1)!\left(\frac{n-k-1}{n-1} + (-1)^k\right) \\
\geq \; & \binom{n}{k_0}(k_0-1)!\frac{2n-k_0-2}{n-1} - \sum_{\substack{2\leq k\leq k_0-1 \\ k \text{ is odd}}} \binom{n}{k}(k-1)!\frac{k}{n-1} \\
= \; & n(n-2)!\left(\frac{2n-k_0-2}{k_0(n-k_0)!} - \sum_{\substack{2\leq k\leq k_0-1 \\ k \text{ is odd}}} \frac{1}{(n-k)!}\right) \\
= \; & n(n-2)!\left(\frac{2n-6}{4(n-4)!} - \sum_{\substack{2\leq k\leq 4 \\ k \text{ is odd}}} \frac{1}{(n-k)!}\right) + \\
& n(n-2)! \sum_{\substack{5\leq k\leq k_0-1 \\ k \text{ is odd}}} \left(\frac{2n-k-3}{(k+1)(n-k-1)!} - \frac{2n-k-1}{(k-1)(n-k+1)!} - \frac{1}{(n-k)!}\right) \qquad (4.8)
\end{aligned}$$

Since $n \geq 7$, the first part of the lower bound (4.8) is positive. Note that the second part of (4.8) vanishes when $k_0 = 4$. Note also that, for $5 \leq k \leq n-2$, we have

$$\begin{aligned}
& \frac{2n-k-3}{(k+1)(n-k-1)!} - \frac{2n-k-1}{(k-1)(n-k+1)!} - \frac{1}{(n-k)!} \\
> \; & \frac{2n-k-3}{(k+1)(n-k-1)!} - \frac{2}{(n-k)!} \\
= \; & \frac{(2n-k-3)(n-k) - 2(k+1)}{(k+1)(n-k)!} \\
\geq \; & 0.
\end{aligned}$$

Thus, the second part of the lower bound (4.8) is also positive. This completes the proof. □

The main result in this section is as follows.

Theorem 4.5. *Suppose $n \geq 7$ and $\emptyset \neq I \subseteq \{2, 3, \ldots, n-2\}$. Then the following statements hold:*

(a) *if I only contains odd numbers, then $\mathrm{Cay}(S_n, C(n, I))$ has two connected components and its strictly second largest eigenvalue is attained by $(n-1, 1)$ and $(2, 1^{n-2})$ with multiplicity $2(n-1)^2$;*

(b) $\mathrm{Cay}(S_n, C(n, \{2,3\}))$ *is connected and its second largest eigenvalue is attained by* $(n-1,1)$ *and* (1^n) *with multiplicity* $(n-1)^2+1$*;*

(c) *if* I *contains both even and odd numbers with the largest one odd and at least* 5*, then* $\mathrm{Cay}(S_n, C(n, I))$ *is connected and its second largest eigenvalue is attained uniquely by* (1^n) *with multiplicity* 1*;*

(d) *if the largest number in* I *is even, then* $\mathrm{Cay}(S_n, C(n, I))$ *is connected and its second largest eigenvalue is attained uniquely by* $(n-1,1)$ *with multiplicity* $(n-1)^2$.

Proof. By (4.2) and Lemma 4.2, for any $\zeta \vdash n$ other than $(n), (1^n), (n-1,1)$ and $(2,1^{n-2})$, we have

$$\begin{aligned}\lambda^I_\zeta &= \sum_{k\in I}\binom{n}{k}(k-1)!\cdot\tilde{\chi}_\zeta((k,1^{n-k}))\\ &< \sum_{k\in I}\binom{n}{k}(k-1)!\cdot\tilde{\chi}_{(n-1,1)}((k,1^{n-k}))\\ &= \lambda^I_{(n-1,1)}.\end{aligned}$$

On the other hand, by (4.5) we see that $\lambda^I_{(n-1,1)}$ is strictly smaller than

$$\lambda^I_{(n)} = \sum_{k\in I}\binom{n}{k}(k-1)! = |C(n,I)|,$$

which is the degree of $\mathrm{Cay}(S_n, C(n, I))$. Therefore, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ can only be attained by partitions among $(1^n), (n-1,1)$ and $(2,1^{n-2})$. According to Lemma 2.2, we have

$$\begin{aligned}\lambda^I_{(2,1^{n-2})} &= \sum_{k\in I}\binom{n}{k}(k-1)!\cdot\tilde{\chi}_{(2,1^{n-2})}((k,1^{n-k}))\\ &= \sum_{k\in I}\binom{n}{k}(k-1)!\cdot(-1)^{k-1}\tilde{\chi}_{(n-1,1)}((k,1^{n-k}))\\ &= \sum_{k\in I}\binom{n}{k}(k-1)!\cdot(-1)^{k-1}\frac{n-k-1}{n-1}.\end{aligned}$$

Comparing with (4.5), we obtain that $\lambda^I_{(n-1,1)} \geq \lambda^I_{(2,1^{n-2})}$ and the strict inequality $\lambda^I_{(n-1,1)} > \lambda^I_{(2,1^{n-2})}$ holds if I contains at least one even number.

In the case when I contains only odd numbers, $\mathrm{Cay}(S_n, C(n, I))$ has two connected components each isomorphic to $\mathrm{Cay}(A_n, C(n, I))$ and $\lambda^I_{(n)} = \lambda^I_{(1^n)} > \lambda^I_{(n-1,1)} = \lambda^I_{(2,1^{n-2})}$.

So its strictly second largest eigenvalue is only achieved by $(n-1,1)$ and $(2,1^{n-2})$. Note that both $\rho_{(n-1,1)}$ and $\rho_{(2,1^{n-2})}$ have dimension $n-1$ according to Table 2.1 and Lemma 2.2. We further deduce from (4.3) that the multiplicity of the strictly second largest eigenvalue is $2(n-1)^2$. This proves statement (a). In the other three cases there is at least one even number in I, and hence $\mathrm{Cay}(S_n, C(n,I))$ is connected and $\lambda^I_{(n-1,1)} > \lambda^I_{(2,1^{n-2})}$. So the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be attained by (1^n) or $(n-1,1)$. Combining this with Lemmas 4.3 and 4.4, we obtain (b), (c) and (d), where the multiplicities are calculated directly with the help of equation (4.3). □

Theorem 4.5 implies the following result.

Corollary 4.6. *Suppose $n \geq 7$ and $\emptyset \neq I \subseteq \{2,3,\ldots,n-2\}$. Then $\mathrm{Cay}(S_n, C(n,I))$ has the Aldous property if and only if one of the following conditions holds:*

(a) *$I = \{2,3\}$;*

(b) *I contains only odd numbers;*

(c) *the largest number in I is even.*

The next corollary of Theorem 4.5 confirms Conjecture 1.4 in [81].

Corollary 4.7. *For any $n \geq 4$ and $2 \leq k \leq n-2$, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,k))$ is attained by the standard representation of S_n, and its value is*

$$\lambda^{\{k\}}_{(n-1,1)} = \frac{n-k-1}{n-1}\binom{n}{k}(k-1)!. \tag{4.9}$$

Proof. One can easily verify this result when n is $4, 5$ or 6. Now suppose $n \geq 7$. The statements (a) and (d) in Theorem 4.5 imply that the standard representation $\rho_{(n-1,1)}$ achieves the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,k))$ for odd k and even k, respectively. The value in (4.9) is simply derived from (4.5). □

4.3 I contains $n-1$ but not n

Lemma 4.8. *Suppose $n \geq 8$ is even and $\{n-1\} \subseteq I \subseteq \{2,3,\dots,n-1\}$. Then*

$$\lambda^I_{(1^n)} - \lambda^I_{(n-1,1)} > \frac{n(n-5)}{3}(n-3)!.$$

Proof. Since $n \geq 8$ is even, the largest number $k_0 = n-1$ in I is odd and hence (4.7) can be applied to the current situation. By this inequality, we obtain

$$\begin{aligned}
\lambda^I_{(1^n)} - \lambda^I_{(n-1,1)} &\geq n(n-2)!\left(\frac{1}{(n-5)!} - \sum_{\substack{2\leq k\leq 5\\ k \text{ is even}}} \frac{2n-k-2}{k(n-k)!}\right) + \\
&\quad n(n-2)!\left(\sum_{\substack{6\leq k\leq n-2\\ k \text{ is even}}} \frac{1}{(n-k-1)!} - \frac{1}{(n-k+1)!} - \frac{2n-k-2}{k(n-k)!}\right) \\
&> n(n-2)!\left(\sum_{\substack{k=n-2\\ k \text{ is even}}} \frac{1}{(n-k-1)!} - \frac{1}{(n-k+1)!} - \frac{2n-k-2}{k(n-k)!}\right) \\
&= \frac{n(n-5)}{3}(n-3)!
\end{aligned}$$

as desired. □

Lemma 4.9. *Suppose $n \geq 7$. The following hold:*

(a) *if $k = n-1, n-2$ or $n-3$ and $\zeta = (n-m, 1^m)$ with $3 \leq m \leq n-4$, then*

$$\tilde{\chi}_\zeta((k, 1^{n-k})) = 0;$$

(b) *if $2 \leq k \leq n-4$ and $\zeta = (n-m, 1^m)$ with $4 \leq m \leq n-5$, then*

$$\tilde{\chi}_\zeta((k,1^{n-k})) < \tilde{\chi}_{(n-3,1^3)}((k,1^{n-k})) < \tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})). \tag{4.10}$$

Proof. (a) If $k = n-1, n-2$ or $n-3$, then $\zeta = (n-m, 1^m)$ with $3 \leq m \leq n-4$ does not contain any k-border strip. Hence $\tilde{\chi}_\zeta((k, 1^{n-k})) = 0$ by the Murnaghan-Nakayama Rule.

(b) Suppose $2 \leq k \leq n-4$ and $\zeta = (n-m, 1^m)$, where $n \geq 7$ and $4 \leq m \leq n-5$. According to Table 2.1, we have

$$\tilde{\chi}_{(n-3,1^3)}((k, 1^{n-k})) = \frac{(n-k-1)(n-k-2)(n-k-3) - 6(n-k-1)c_2 + 6c_3}{(n-1)(n-2)(n-3)} \tag{4.11}$$

$$\tilde{\chi}_{(n-2,1^2)}((k, 1^{n-k})) = \frac{(n-k-1)(n-k-2) - 2c_2}{(n-1)(n-2)},$$

where c_i is the number of terms in $(k, 1^{n-k})$ that are equal to i. Using these expressions, one can easily verify the second inequality in (4.10).

It remains to prove the first inequality in (4.10) for $n \geq 9$. (Note that this inequality vanishes when $n = 7$ or 8 as $4 \leq m \leq n-5$ in ζ.) We achieve this by induction on $n \geq 9$. Note from (4.11) and Lemma 2.2 that

$$0 < \tilde{\chi}_{(n-3,1^3)}((k, 1^{n-k})) = |\tilde{\chi}_{(4,1^{n-4})}((k, 1^{n-k}))|. \tag{4.12}$$

It is straightforward to verify that the first inequality in (4.10) holds when $n = 9$, $2 \leq k \leq 5$ and $\zeta = (5, 1^4)$. Assume that $n-1 \geq 9$ and for every $\zeta = (n-1-m, 1^m)$ with $4 \leq m \leq n-6$ and any $2 \leq k \leq n-5$ the following holds:

$$\tilde{\chi}_{\zeta}((k, 1^{n-1-k})) < \tilde{\chi}_{(n-4,1^3)}((k, 1^{n-1-k})) = |\tilde{\chi}_{(4,1^{n-5})}((k, 1^{n-1-k}))|. \tag{4.13}$$

Now let us consider $\zeta = (n-m, 1^m) \vdash n$ with $4 \leq m \leq n-5$ and $2 \leq k \leq n-4$. If $k = n-4$, then for any $\zeta = (n-m, 1^m)$ with $4 \leq m \leq n-5$, the Young diagram of ζ contains no $(n-4)$-border strip. Thus we know from the Murnaghan-Nakayama Rule and inequality (4.12) that for $4 \leq m \leq n-5$,

$$0 = \tilde{\chi}_{(n-m,1^m)}((n-4, 1^4)) < \tilde{\chi}_{(n-3,1^3)}((n-4, 1^4)).$$

If $2 \leq k \leq n-5$, then for any $\zeta = (n-m, 1^m)$ with $4 \leq m \leq n-5$, we apply the Branching Rule to the following normalized character and obtain

$$\begin{aligned}
&\tilde{\chi}_{\zeta}((k,1^{n-k}))\\
=\;& \frac{\sum\limits_{\zeta^-}\chi_{\zeta^-}((k,1^{n-1-k}))}{\sum\limits_{\zeta^-}\chi_{\zeta^-}((1^{n-1}))}\\
=\;& \frac{\chi_{(n-m,1^{m-1})}((k,1^{n-1-k}))+\chi_{(n-m-1,1^m)}((k,1^{n-1-k}))}{\chi_{(n-m,1^{m-1})}((1^{n-1}))+\chi_{(n-m-1,1^m)}((1^{n-1}))}\\
\leq\;& \max\left\{\frac{\chi_{(n-m,1^{m-1})}((k,1^{n-1-k}))}{\chi_{(n-m,1^{m-1})}((1^{n-1}))},\frac{\chi_{(n-m-1,1^m)}((k,1^{n-1-k}))}{\chi_{(n-m-1,1^m)}((1^{n-1}))}\right\}\\
\leq\;& \tilde{\chi}_{(n-4,1^3)}((k,1^{n-1-k})) \qquad (4.14)\\
=\;& \frac{(n-1-k-1)(n-1-k-2)(n-1-k-3)-6(n-1-k-1)c_2+6c_3}{(n-2)(n-3)(n-4)}\\
<\;& \frac{(n-k-1)(n-k-2)(n-k-3)-6(n-k-1)c_2+6c_3}{(n-1)(n-2)(n-3)}\\
=\;& \tilde{\chi}_{(n-3,1^3)}((k,1^{n-k})),
\end{aligned}$$

where (4.14) follows from the induction hypothesis (4.13). □

Lemma 4.10. *Suppose $n \geq 7$ and $n \in I \subseteq \{2,3,\ldots,n\}$. Then the following hold:*

(a) $\max_{1\leq m\leq n-1}\lambda^I_{(n-m,1^m)}$ *can only be attained by $m = 1,2,n-3,n-2$ or $n-1$;*

(b) $\lambda^I_{(n-2,1^2)} \geq \lambda^I_{(n-1,1)}$ *and the equality holds if and only if $I = \{2,3,\ldots,n-2,n\}$ or $I = \{2,3,\ldots,n-1,n\}$;*

(c) *if n is even and $I = \{2,3,\ldots,n-2,n\}$ or $\{2,3,\ldots,n-1,n\}$, then $\lambda^I_{(2,1^{n-2})} > \lambda^I_{(n-1,1)} = \lambda^I_{(n-2,1^2)}$;*

(d) $\lambda^I_{(n-2,1^2)} \geq \lambda^I_{(3,1^{n-3})}$ *and the equality holds if and only if I contains only odd numbers other than $n-1$ and $n-2$.*

Proof. (a) Note from Table 2.1 that $\tilde{\chi}_{(n-2,1^2)}((n-3,1^3)) > 0$ and $\tilde{\chi}_{(n-2,1^2)}((n-2,1^2)) = \tilde{\chi}_{(n-2,1^2)}((n-1,1)) = 0$. Combining the previous lemma with Lemma 2.7, we obtain that for $2 \leq k \leq n-3$ or $k = n$,

$$\tilde{\chi}_{(n-m,1^m)}((k,1^{n-k})) < \tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})), \quad 3 \leq m \leq n-4$$

and for $k = n-1$ or $n-2$,

$$\tilde{\chi}_{(n-m,1^m)}((k,1^{n-k})) = 0, \quad 2 \le m \le n-3.$$

Thus equation (4.2) implies that whenever $n \in I \subseteq \{2,3,\ldots,n\}$ we have $\lambda^I_{(n-m,1^m)} < \lambda^I_{(n-2,1^2)}$ for every $3 \le m \le n-4$, and so the maximum of $\lambda^I_{(n-m,1^m)}$ for $1 \le m \le n-1$ can only be attained by $m = 1, 2, n-3, n-2$ or $n-1$.

(b) We have

$$\begin{aligned}
& \lambda^I_{(n-2,1^2)} - \lambda^I_{(n-1,1)} \\
= \ & \sum_{k \in I \setminus \{n\}} \binom{n}{k}(k-1)!\left(\frac{(n-k-1)(n-k-2)-2c_2}{(n-1)(n-2)} - \frac{n-k-1}{n-1}\right) + 2(n-3)! + (n-2)! \\
\ge \ & -\sum_{2 \le k \le n-2} \binom{n}{k}(k-1)!\frac{k(n-k-1)+2c_2}{(n-1)(n-2)} + 2(n-3)! + (n-2)! \\
= \ & n(n-3)! - \sum_{2 \le k \le n-2} \frac{n(n-3)!}{(n-k)(n-k-2)!} - \frac{n}{n-2} \\
= \ & n(n-3)!\left(1 - \sum_{2 \le k \le n-2} \frac{1}{(n-k)(n-k-2)!}\right) - \frac{n}{n-2} \\
= \ & n(n-3)! \cdot \frac{1}{(n-2)!} - \frac{n}{n-2} \\
= \ & 0.
\end{aligned}$$

Thus $\lambda^I_{(n-2,1^2)} \ge \lambda^I_{(n-1,1)}$ and the equality holds if and only if $I = \{2,3,\ldots,n-2,n\}$ or $I = \{2,3,\ldots,n-1,n\}$.

(c) We have $\lambda^I_{(n-1,1)} = \lambda^I_{(n-2,1^2)}$ from (b). We claim that the partition $(2,1^{n-2})$ yields a larger eigenvalue than $(n-2,1^2)$ and $(n-1,1)$ in this case. In fact,

$$\begin{aligned}
& \lambda^I_{(2,1^{n-2})} - \lambda^I_{(n-1,1)} \\
= \ & \sum_{k=2}^{n-2} \binom{n}{k}(k-1)!(-1)^{k-1}\frac{n-k-1}{n-1} + (n-2)! \\
& -\left(\sum_{k=2}^{n-2} \binom{n}{k}(k-1)!\frac{n-k-1}{n-1} - (n-2)!\right) \\
= \ & -2 \sum_{\substack{2 \le k \le n-2 \\ k \text{ is even}}} \binom{n}{k}(k-1)!\frac{n-k-1}{n-1} + 2(n-2)!
\end{aligned}$$

$$\begin{aligned} &= 2(n-2)!\left(1-\sum_{\substack{2\le k\le n-2\\ k\text{ is even}}}\frac{n}{k(n-k)(n-k-2)!}\right)\\ &> 0. \end{aligned}$$

The last inequality above is deduced from the fact that n is at least 8 and

$$\begin{aligned} \sum_{\substack{2\le k\le n-2\\ k\text{ is even}}}\frac{n}{k(n-k)(n-k-2)!} &= \frac{n}{2(n-2)}+\frac{n}{8(n-4)}+\sum_{\substack{2\le k\le n-6\\ k\text{ is even}}}\frac{n}{k(n-k)(n-k-2)!}\\ &< \frac{n}{2(n-2)}+\frac{n}{8(n-4)}+\sum_{\substack{2\le k\le n-6\\ k\text{ is even}}}\frac{1}{(n-k-2)!}\\ &< \frac{8}{12}+\frac{8}{32}+\frac{2}{4!}\\ &= 1. \end{aligned}$$

(d) Note from Table 2.1 that $\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k}))\ge 0$ for any $2\le k\le n$ and that $\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k}))=0$ if and only if $k=n-1$ or $n-2$. Thus we have $\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k}))\ge\tilde{\chi}_{(3,1^{n-3})}((k,1^{n-k}))=(-1)^{k-1}\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k}))$ for any $2\le k\le n$. This implies that $\lambda^I_{(n-2,1^2)}\ge\lambda^I_{(3,1^{n-3})}$ and the equality holds if and only if I contains only odd numbers other than $n-1$ and $n-2$. □

Lemma 4.11. *Suppose $n\ge 7$. If $2\le k\le n-5$ and $\zeta=(n-m,2,1^{m-2})$ with $4\le m\le n-4$, then*

$$\tilde{\chi}_{\zeta}((k,1^{n-k}))<\tilde{\chi}_{(n-3,2,1)}((k,1^{n-k}))<\tilde{\chi}_{(n-2,2)}((k,1^{n-k})). \tag{4.15}$$

Proof. By Table 2.1, we have

$$\tilde{\chi}_{(n-3,2,1)}((k,1^{n-k}))=\frac{(n-k)(n-k-2)(n-k-4)-3c_3}{n(n-2)(n-4)},$$

$$\tilde{\chi}_{(n-2,2)}((k,1^{n-k}))=\frac{(n-k)(n-k-3)+2c_2}{n(n-3)},$$

where c_i is the number of terms in $(k,1^{n-k})$ which are equal to i. Using these expressions and Lemma 2.2, one can easily verify that

$$|\tilde{\chi}_{(3,2,1^{n-5})}((k,1^{n-k}))|=\tilde{\chi}_{(n-3,2,1)}((k,1^{n-k}))<\tilde{\chi}_{(n-2,2)}((k,1^{n-k}))$$

for $n \geq 7$ and $2 \leq k \leq n-5$. Note that the first inequality in (4.15) vanishes when $n = 7$ as we require $4 \leq m \leq n-4$ in ζ. In the following we prove by induction on $n \geq 8$ that for $2 \leq k \leq n-5$,

$$\tilde{\chi}_{(n-m,2,1^{m-2})}((k,1^{n-k})) < \tilde{\chi}_{(n-3,2,1)}((k,1^{n-k}))$$

holds for $4 \leq m \leq n-4$. One can check that this holds for $n = 8$ and 9. Suppose the above inequality holds for some $n-1 \geq 9$, that is, for $2 \leq k \leq n-6$,

$$\tilde{\chi}_{(n-1-m,2,1^{m-2})}((k,1^{n-1-k})) < \tilde{\chi}_{(n-4,2,1)}((k,1^{n-1-k})) = |\tilde{\chi}_{(3,2,1^{n-6})}((k,1^{n-1-k}))| \tag{4.16}$$

holds for $4 \leq m \leq n-5$.

Now we consider $\tilde{\chi}_{(n-m,2,1^{m-2})}((k,1^{n-k}))$ with $4 \leq m \leq n-4$ and $2 \leq k \leq n-5$. Note that $\tilde{\chi}_{(n-3,2,1)}((k,1^{n-k})) > 0$ for $n \geq 8$ and $2 \leq k \leq n-5$. First, let $k = n-5$. By the Murnaghan-Nakayama Rule we see that if $m \neq 5, n-5$, then $\tilde{\chi}_\zeta((n-5,1^5)) = 0 < \tilde{\chi}_{(n-3,2,1)}((n-5,1^5))$ as there is no $(n-5)$-border strip in ζ. If $m = 5$ or $n-5$, then by a simple computation we still have

$$0 < |\tilde{\chi}_{(n-m,2,1^{m-2})}((n-5,1^5))| < \tilde{\chi}_{(n-3,2,1)}((n-5,1^5)).$$

Thus, for $4 \leq m \leq n-4$,

$$\tilde{\chi}_{(n-m,2,1^{m-2})}((n-5,1^5)) < \tilde{\chi}_{(n-3,2,1)}((n-5,1^5)).$$

Next, let $2 \leq k \leq n-6$. For every $\zeta = (n-m,2,1^{m-2})$ with $4 \leq m \leq n-4$, we have

$$\begin{aligned}
& \tilde{\chi}_\zeta((k,1^{n-k})) \\
= \;& \frac{\sum\limits_{\zeta^-} \chi_{\zeta^-}((k,1^{n-1-k}))}{\sum\limits_{\zeta^-} \chi_{\zeta^-}((1^{n-1}))} \\
= \;& \frac{\chi_{(n-m-1,2,1^{m-2})}((k,1^{n-1-k})) + \chi_{(n-m,2,1^{m-3})}((k,1^{n-1-k})) + \chi_{(n-m,1^{m-1})}((k,1^{n-1-k}))}{\chi_{(n-m-1,2,1^{m-2})}((1^{n-1})) + \chi_{(n-m,2,1^{m-3})}((1^{n-1})) + \chi_{(n-m,1^{m-1})}((1^{n-1}))} \\
\leq \;& \max\left\{\tilde{\chi}_{(n-m-1,2,1^{m-2})}((k,1^{n-1-k})), \tilde{\chi}_{(n-m,2,1^{m-3})}((k,1^{n-1-k})), \tilde{\chi}_{(n-m,1^{m-1})}((k,1^{n-1-k})\right\} \\
\leq \;& \max\left\{\tilde{\chi}_{(n-4,2,1)}((k,1^{n-1-k})), \tilde{\chi}_{(n-4,1^3)}((k,1^{n-1-k})\right\} \qquad (4.17) \\
= \;& \max\left\{\frac{(n-1-k)(n-1-k-2)(n-1-k-4)-3c_3}{(n-1)(n-3)(n-5)},\right.
\end{aligned}$$

$$\begin{aligned} & \left.\frac{(n-k-2)(n-k-3)(n-k-4)-6(n-k-2)c_2+6c_3}{(n-2)(n-3)(n-4)}\right\} \\ < & \frac{(n-k)(n-k-2)(n-k-4)-3c_3}{n(n-2)(n-4)} \\ = & \tilde{\chi}_{(n-3,2,1)}((k,1^{n-k})), \end{aligned}$$

where (4.17) is deduced from the induction hypothesis (4.16) as well as Lemma 4.9. ☐

Lemma 4.12. *Suppose $n \geq 7$. Then for any $\zeta = (n-m, 2, 1^{m-2})$ with $4 \leq m \leq n-4$ the following hold:*

(a) $\tilde{\chi}_\zeta((n-1,1)) < \tilde{\chi}_{(n-3,2,1)}((n-1,1)) < |\tilde{\chi}_{(n-2,2)}((n-1,1))|$;

(b) $\sum_{k\in I}\binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k,1^{n-k})) < \sum_{k\in I}\binom{n}{k}(k-1)!\tilde{\chi}_{(n-3,2,1)}((k,1^{n-k}))$ *for every I with $n-1 \in I \subseteq \{n-4, n-3, n-1\}$.*

Proof. (a) We can obtain the following facts by using Lemma 2.8 and the Hook Length Formula directly: If $n=7$, then $\tilde{\chi}_{(n-3,2,1)}((n-1,1)) < |\tilde{\chi}_{(n-2,2)}((n-1,1))|$; if $n=8$ or 9, then $|\tilde{\chi}_{(n-4,2,1^2)}((n-1,1))| < \tilde{\chi}_{(n-3,2,1)}((n-1,1)) < |\tilde{\chi}_{(n-2,2)}((n-1,1))|$; if $n \geq 10$, then for any $5 \leq m \leq n-5$,

$$\begin{aligned} \tilde{\chi}_{(n-m,2,1^{m-2})}((n-1,1)) & \leq \tilde{\chi}_{(n-5,2,1^3)}((n-1,1)) \\ & < |\tilde{\chi}_{(n-4,2,1^2)}((n-1,1))| \\ & < \tilde{\chi}_{(n-3,2,1)}((n-1,1)) \\ & < |\tilde{\chi}_{(n-2,2)}((n-1,1))|. \end{aligned}$$

(b) One can check that the inequality holds for $n = 8, 9$. Now suppose $n \geq 10$. If $k = n-4$, then for $\zeta = (n-m, 2, 1^{m-2})$ with $3 \leq m \leq n-3$, we have $\tilde{\chi}_\zeta((n-4,1^4)) \neq 0$ if and only if $m = 4$ or $n-4$. Using Table 2.1, we obtain that

$$(-1)^{n-5}\tilde{\chi}_{(4,2,1^{n-6}))}((n-4,1^4)) = \tilde{\chi}_{(n-4,2,1^2))}((n-4,1^4)) < 0 = \tilde{\chi}_{(n-3,2,1))}((n-4,1^4)).$$

If $k = n-3$, then $\tilde{\chi}_\zeta((n-3,1^3)) = 0$ for $\zeta = (n-m, 2, 1^{m-2})$ with $4 \leq m \leq n-4$ and

$$(-1)^{n-4}\cdot\tilde{\chi}_{(3,2,1^{n-5})}((n-3,1^3)) = \tilde{\chi}_{(n-3,2,1)}((n-3,1^3)) = \frac{-3}{n(n-2)(n-4)}.$$

Combining these facts with the first part of this lemma, we have for every $\zeta = (n-m, 2, 1^{m-2})$ with $5 \leq m \leq n-5$ and every I with $n-1 \in I \subseteq \{n-4, n-3, n-1\}$,

$$\sum_{k\in I}\binom{n}{k}(k-1)!\tilde{\chi}_{\zeta}((k,1^{n-k})) \leq \sum_{k\in I}\binom{n}{k}(k-1)!\tilde{\chi}_{(n-5,2,1^3)}((k,1^{n-k})).$$

To complete the proof, it remains to establish the required inequality for $\zeta = (n-m, 2, 1^{m-2})$ with $m = 4, 5$ or $n-4$ and $n-1 \in I \subseteq \{n-4, n-3, n-1\}$. This can be done by straightforward computations with the help of Lemma 2.8 and Table 2.1. $\square$

Remark 4.13. Note from Table 4.3 that $\tilde{\chi}_{(n-m,2,1^{m-2})}((n-2,1^2)) = 0$ for $3 \leq m \leq n-3$ and from Lemma 2.7 that $\tilde{\chi}_{(n-m,2,1^{m-2})}((n)) = 0$ for $2 \leq m \leq n-2$. Combining these with Lemmas 4.11 and 4.12, we obtain that for any $\zeta = (n-m, 2, 1^{m-2})$ with $4 \leq m \leq n-4$ and every I with $n-1 \in I \subseteq \{2, 3, \ldots, n-1, n\}$,

$$\begin{aligned}
\lambda_{\zeta}^{I} &= \sum_{k\in I}\binom{n}{k}(k-1)!\cdot\tilde{\chi}_{\zeta}((k,1^{n-k})) \\
&< \sum_{k\in I}\binom{n}{k}(k-1)!\cdot\tilde{\chi}_{(n-3,2,1)}((k,1^{n-k})) \\
&= \lambda_{(n-3,2,1)}^{I}.
\end{aligned}$$

Thus the maximum of $\lambda_{(n-m,2,1^{m-2})}^{I}$ for $2 \leq m \leq n-2$ can only be attained by $m = 2, 3, n-3, n-2$.

Now we are ready to prove our main result in this section.

Theorem 4.14. *Suppose $n \geq 7$ and $\{n-1\} \subseteq I \subseteq \{2, 3, \ldots, n-1\}$. Then the following statements hold:*

(a) *if n is even and I contains at least one even number, then the second largest eigenvalue of* $\mathrm{Cay}(S_n, C(n, I))$ *is attained uniquely by (1^n), and moreover the multiplicity of this eigenvalue is 1;*

(b) *if n is even and I contains only odd numbers, then the strictly second largest eigenvalue of* $\mathrm{Cay}(S_n, C(n, I))$ *can only be attained by either $(n-1, 1)$ and $(2, 1^{n-2})$ or $(n-3, 2, 1)$ and $(3, 2, 1^{n-5})$;*

(c) *if n is odd, then the second largest eigenvalue of* $\mathrm{Cay}(S_n, C(n, I))$ *is attained by $(n-1, 1)$, $(n-3, 2, 1)$, $(2^2, 1^{n-4}))$ or $(2, 1^{n-2})$.*

Proof. (a) For any $\zeta \vdash n$, we have

$$\begin{aligned}\lambda_\zeta^I &= \sum_{k\in I}\binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k,1^{n-k}))\\ &= \sum_{k\in I\setminus\{n-1\}}\binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k,1^{n-k}))+n(n-2)!\tilde{\chi}_\zeta((n-1,1)).\end{aligned}$$

According to Lemma 2.8, the second term above vanishes unless $\zeta=(n),(1^n)$ or $(n-m,2,1^{m-2})$ with $2\le m\le n-2$. This together with Lemma 4.2 implies that, for $\zeta\ne(n),(1^n),(n-1,1),(2,1^{n-2})$, $(n-m,2,1^{m-2})$ with $2\le m\le n-2$, we have

$$\begin{aligned}\lambda_\zeta^I &= \sum_{k\in I\setminus\{n-1\}}\binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k,1^{n-k}))\\ &< \sum_{k\in I\setminus\{n-1\}}\binom{n}{k}(k-1)!\tilde{\chi}_{(n-1,1)}((k,1^{n-k}))\\ &= \lambda_{(n-1,1)}^I. \end{aligned}\tag{4.18}$$

Since n is even and the largest number $n-1$ in I is odd, Lemma 4.3 implies that $\lambda_{(1^n)}^I>\lambda_{(n-1,1)}^I$. Moreover, as I contains at least one even number smaller than $n-1$, we also have

$$\begin{aligned}\lambda_{(n-1,1)}^I &= \sum_{k\in I}\binom{n}{k}(k-1)!\cdot\frac{n-k-1}{n-1}\\ &> \sum_{k\in I}\binom{n}{k}(k-1)!\cdot(-1)^{k-1}\frac{n-k-1}{n-1}\\ &= \lambda_{(2,1^{n-2})}^I. \end{aligned}\tag{4.19}$$

So the second largest eigenvalue of $\mathrm{Cay}(S_n,C(n,I))$ can only be attained by (1^n) or $(n-m,2,1^{m-2})$ with $2\le m\le n-2$.

In the following we assume $\zeta=(n-m,2,1^{m-2})$ with $2\le m\le n-2$, and we aim to show that ζ does not give the second largest eigenvalue of $\mathrm{Cay}(S_n,C(n,I))$. In fact, by Lemma 4.2,

$$\begin{aligned}\lambda_\zeta^I &= \sum_{k\in I\setminus\{n-1\}}\binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k,1^{n-k}))+n(n-2)!\tilde{\chi}_\zeta((n-1,1))\\ &< \sum_{k\in I\setminus\{n-1\}}\binom{n}{k}(k-1)!\tilde{\chi}_{(n-1,1)}((k,1^{n-k}))+n(n-2)!\tilde{\chi}_\zeta((n-1,1))\\ &= \lambda_{(n-1,1)}^I+n(n-2)!\tilde{\chi}_\zeta((n-1,1)).\end{aligned}$$

On the other hand, by Lemma 2.8,

$$\begin{aligned} n(n-2)!\tilde{\chi}_\zeta((n-1,1)) &= n(n-2)!\frac{\chi_\zeta((n-1,1))}{\chi_\zeta((1^n))} \\ &= n(n-2)!\frac{(-1)^{m-1}(n-1)(n-m)(n-m-2)!m(m-2)!}{n!} \\ &\leq n(n-2)!\frac{3(n-1)(n-3)(n-5)!}{n!} \\ &= 3(n-3)(n-5)!, \end{aligned}$$

where the second last step follows from the fact that n is even and thus the maximum

$$\max_{2\leq m\leq n-2} \frac{(-1)^{m-1}(n-1)(n-m)(n-m-2)!m(m-2)!}{n!}$$

is achieved by $m \in \{3, n-3\}$. Combining these with Lemma 4.8, we obtain

$$\begin{aligned} \lambda^I_{(1^n)} &> \lambda^I_{(n-1,1)} + \frac{n(n-5)}{3}(n-3)! \\ &> \lambda^I_{(n-1,1)} + 3(n-3)(n-5)! \\ &\geq \lambda^I_{(n-1,1)} + n(n-2)!\tilde{\chi}_\zeta((n-1,1)) \\ &> \lambda^I_{(n-m,2,1^{m-2})} \end{aligned}$$

for $2 \leq m \leq n-2$. Therefore, the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ is attained uniquely by (1^n). Moreover, the multiplicity of this eigenvalue is equal to the square of the dimension of $\rho_{(1^n)}$, namely 1.

(b) Since I contains only odd numbers, by Lemma 2.2 we have $\lambda^I_\zeta = \lambda^I_{\zeta'}$ for any $\zeta \vdash n$. According to (4.18), the strictly second largest eigenvalue can only be attained by $(n-1,1), (2,1^{n-2})$ or $(n-m,2,1^{m-2})$ with $2 \leq m \leq n-2$. By a direct computation using Table 2.1, one can verify that for $2 \leq k \leq n-1$ and $m = 2$ or $n-2$, we have $\tilde{\chi}_{(n-m,2,1^{m-2})}((k,1^{n-k})) < \tilde{\chi}_{(n-1,1)}((k,1^{n-k}))$, which implies $\lambda^I_{(n-m,2,1^{m-2})} < \lambda^I_{(n-1,1)} = \lambda^I_{(2,1^{n-2})}$ when $m = 2$ or $n-2$. On the other hand, by Remark 4.13, we have $\lambda^I_{(n-m,2,1^{m-2})} < \lambda^I_{(n-3,2,1)} = \lambda^I_{(3,2,1^{n-5})}$ for $4 \leq m \leq n-4$. The result follows from these inequalities.

(c) Similarly to the proof of (a) above, for $\zeta \neq (n), (1^n), (n-1,1), (2,1^{n-2}), (n-m,2,1^{m-2})$ with $2 \leq m \leq n-2$, we have $\lambda^I_\zeta < \lambda^I_{(n-1,1)}$. Lemma 4.4 implies $\lambda^I_{(n-1,1)} > \lambda^I_{(1^n)}$ when $n \geq 7$ is odd and $n-1$ is the largest number in I. Thus the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be attained by $(n-1,1), (2,1^{n-2})$ or $(n-m,2,1^{m-2})$ with $2 \le m \le n-2$.

According to Remark 4.13, the maximum of $\lambda^I_{(n-m,2,1^{m-2})}$ for $2 \le m \le n-2$ is attained by $m = 2, 3, n-3$ or $n-2$. Furthermore, when $m = 2, n-3$, Lemmas 2.8 and 4.2 imply $\tilde{\chi}_{(n-m,2,1^{m-2})}((n-1,1)) < 0 = \tilde{\chi}_{(n-1,1)}((n-1,1))$ and $\tilde{\chi}_{(n-m,2,1^{m-2})}((k,1^{n-k})) < \tilde{\chi}_{(n-1,1)}((k,1^{n-k}))$ for $2 \le k \le n-2$, respectively. Therefore, for $m = 2, n-3$, we have $\lambda^I_{(n-m,2,1^{m-2})} < \lambda^I_{(n-1,1)}$. Thus the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be attained by $(n-1,1), (2,1^{n-2})$ or $(n-m,2,1^{m-2})$ with $m = 3, n-2$, as desired. □

Remark 4.15. Note that the strictly second largest eigenvalues of $\mathrm{Cay}(S_8, C(8,7))$ and $\mathrm{Cay}(S_8, C(8,\{3,7\}))$ are attained by $(5,2,1)$ and $(3,2,1^3)$, while the strictly second largest eigenvalues of $\mathrm{Cay}(S_8, C(8,\{5,7\}))$ and $\mathrm{Cay}(S_8, C(8,\{3,5,7\}))$ are attained by $(7,1)$ and $(2,1^6)$. This shows that both cases in part (b) of Theorem 4.14 can occur. However, we do not know whether the four partitions in part (b) of Theorem 4.14 can achieve the strictly second largest eigenvalue simultaneously.

Part (a) of Theorem 4.14 implies the following result.

Corollary 4.16. *Suppose $n \ge 8$ is even and $\{n-1\} \subseteq I \subseteq \{2,3,\ldots,n-1\}$. If I contains at least one even number, then* $\mathrm{Cay}(S_n, C(n,I))$ *does not possess the Aldous property.*

More work is required to determine when $\mathrm{Cay}(S_n, C(n,I))$ has the Aldous property under the conditions of parts (b) and (c) of Theorem 4.14.

In (4.19) we saw that $\lambda^I_{(n-1,1)} > \lambda^I_{(2,1^{n-2})}$ whenever $I \subseteq \{2,3\ldots,n-1\}$ contains at least one even number smaller than $n-1$. In general, by Lemma 2.2 and (4.5), for any $I \subseteq \{2,3\ldots,n-1\}$ we have

$$\begin{aligned}
\lambda^I_{(2,1^{n-2})} &= \sum_{k\in I} \binom{n}{k}(k-1)!\tilde{\chi}_{(2,1^{n-2})}((k,1^{n-k})) \\
&= \sum_{k\in I} \binom{n}{k}(k-1)!(-1)^{k-1}\tilde{\chi}_{(n-1,1)}((k,1^{n-k})) \\
&= \sum_{k\in I} \binom{n}{k}(k-1)!(-1)^{k-1}\frac{n-k-1}{n-1}, \\
&\le \lambda^I_{(n-1,1)}.
\end{aligned}$$

We conjecture that the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ in part (c) of Theorem 4.14 can only be attained by $(n-1,1)$, $(2^2,1^{n-4})$ or $(2,1^{n-2})$:

Conjecture 4.17. Suppose $n \geq 7$ is odd and $\{n-1\} \subseteq I \subseteq \{2,3,\ldots,n-1\}$. Then

$$\lambda^I_{(n-3,2,1)} < \max\left\{\lambda^I_{(n-1,1)}, \lambda^I_{(2^2,1^{n-4})}\right\}.$$

4.4 I conatins n but not $n-1$

Lemma 4.18. *Suppose $n \geq 7$ is odd and $\{n\} \subseteq I \subseteq \{2,3,\ldots,n-2,n\}$. Then*

$$\lambda^I_{(1^n)} - \lambda^I_{(n-1,1)} > \frac{n}{2}(n-2)!.$$

Proof. Similarly to the proof of Lemma 4.3, we have

$$\begin{aligned}
\lambda^I_{(1^n)} - \lambda^I_{(n-1,1)} &= n(n-2)! + \sum_{k\in I\setminus\{n\}} \binom{n}{k}(k-1)!\left((-1)^{k-1} - \frac{n-k-1}{n-1}\right) \\
&\geq n(n-2)! - \sum_{\substack{2\leq k\leq n-3\\ k \text{ is even}}} \binom{n}{k}(k-1)!\frac{2n-k-2}{n-1} \\
&= n(n-2)!\left(1 - \sum_{\substack{2\leq k\leq n-3\\ k \text{ is even}}} \frac{2n-k-2}{k(n-k)!}\right) \\
&= n(n-2)!\left(\frac{1}{(n-5)!} - \sum_{\substack{2\leq k\leq 5\\ k \text{ is even}}} \frac{2n-k-2}{k(n-k)!}\right) \\
&\quad + n(n-2)!\left(\sum_{\substack{6\leq k\leq n-3\\ k \text{ is even}}} \frac{1}{(n-k-1)!} - \frac{1}{(n-k+1)!} - \frac{2n-k-2}{k(n-k)!}\right) \\
&\quad + n(n-2)!\left(1 - \frac{1}{2}\right) \\
&> \frac{n}{2}(n-2)!,
\end{aligned}$$

where the last step follows from the fact that the right-hand side of (4.7) is positive when taking $k_0 = n-2$. □

Lemma 4.19. *Suppose $n \geq 8$ is even and $\{n\} \subseteq I \subseteq \{2,3,\ldots,n-2,n\}$. Then $\lambda^I_{(n-1,1)} > \lambda^I_{(1^n)}$.*

Proof. Similarly to the proof of Lemma 4.4, we have

$$
\begin{aligned}
&\lambda^I_{(n-1,1)} - \lambda^I_{(1^n)}\\
&\geq n(n-2)!\left(\frac{n-2}{n} - \sum_{\substack{2\leq k\leq n-3\\ k \text{ is odd}}} \frac{1}{(n-k)!}\right)\\
&= n(n-2)!\left(\frac{2n-6}{4(n-4)!} - \sum_{\substack{2\leq k\leq 4\\ k \text{ is odd}}} \frac{1}{(n-k)!}\right)\\
&+ n(n-2)!\left(\sum_{\substack{5\leq k\leq n-3\\ k \text{ is odd}}} \frac{2n-k-3}{(k+1)(n-k-1)!} - \frac{2n-k-1}{(k-1)(n-k+1)!} - \frac{1}{(n-k)!}\right)\\
&+ n(n-2)!\left(\frac{n-2}{n} - \frac{n}{2(n-2)}\right)\\
&> 0.
\end{aligned}
$$

□

Lemma 4.20. *Suppose $n \geq 9$ and $\{n\} \subseteq I \subseteq \{2,3,\ldots,n-2,n\}$. Then for any $\zeta \neq (2^2,1^{n-4})$, $(n-m,1^m)$ with $0 \leq m \leq n-1$, we have $\lambda^I_\zeta < \lambda^I_{(n-2,1^2)}$.*

Proof. Suppose ζ is any partition of n other than $(2^2,1^{n-4})$ and $(n-m,1^m)$ with $0 \leq m \leq n-1$. By Lemma 4.1, we have $\tilde{\chi}_\zeta((k,1^{n-k})) < \frac{(n-k)(n-k-1)}{n(n-1)}$ for $2 \leq k \leq n-3$. According to Table 4.3, we have $\tilde{\chi}_\zeta((n-2,1^2)) \leq \frac{6}{n(n-1)(n-5)}$. Define

$$
f(n,k) = \begin{cases} \frac{(n-k)(n-k-1)}{n(n-1)}, & \text{if } 2 \leq k \leq n-3;\\ \frac{6}{n(n-1)(n-5)}, & \text{if } k = n-2. \end{cases}
$$

Note that $\tilde{\chi}_\zeta((n)) = 0$ by Lemma 2.7. Therefore,

$$
\lambda^I_\zeta = \sum_{k\in I\setminus\{n\}} \binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k,1^{n-k})) \leq \sum_{k\in I\setminus\{n\}} \binom{n}{k}(k-1)! f(n,k).
$$

One can further verify that $\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})) < f(n,k)$ for $2 \leq k \leq n-2$. Hence

$$
\begin{aligned}
&\lambda^I_{(n-2,1^2)} - \lambda^I_\zeta\\
\geq\ & 2(n-3)! + \sum_{k\in I\setminus\{n\}} \binom{n}{k}(k-1)!\left(\frac{(n-k-1)(n-k-2)-2c_2}{(n-1)(n-2)} - f(n,k)\right)
\end{aligned}
$$

$$\begin{aligned}
&\geq\ 2(n-3)! + \sum_{k=2}^{n-2}\binom{n}{k}(k-1)!\left(\frac{(n-k-1)(n-k-2)-2c_2}{(n-1)(n-2)} - f(n,k)\right)\\
&=\ 2(n-3)! + \sum_{k=2}^{n-3}\binom{n}{k}(k-1)!\left(\frac{(n-k-1)(n-k-2)-2c_2}{(n-1)(n-2)} - \frac{(n-k)(n-k-1)}{n(n-1)}\right)\\
&\quad -\frac{3(n-3)!}{n-5}\\
&=\ 2(n-3)! - \sum_{k=2}^{n-3}\frac{2(n-3)!}{(n-k)(n-k-2)!} - \frac{n}{n-2} - \frac{3(n-3)!}{n-5}\\
&=\ 2(n-3)!\left(1 - \sum_{k=2}^{n-3}\frac{1}{(n-k)(n-k-2)!}\right) - \frac{n}{n-2} - \frac{3(n-3)!}{n-5}\\
&=\ 2(n-3)!\left(\frac{1}{2} + \frac{1}{(n-2)!}\right) - \frac{n}{n-2} - \frac{3(n-3)!}{n-5}\\
&>\ 0.
\end{aligned}$$

This completes the proof. □

The following is the main result in this section.

Theorem 4.21. *Suppose $n \geq 7$ and $\{n\} \subseteq I \subseteq \{2, 3, \ldots, n-2, n\}$. Then the following statements hold:*

(a) *if n is odd and I contains at least one even number, then the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ is attained uniquely by (1^n), and moreover the multiplicity of this eigenvalue is 1;*

(b) *if n is odd and I only contains odd numbers, then the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ is attained by $(n-2, 1^2)$ and $(3, 1^{n-3})$, and moreover the multiplicity of this eigenvalue is $\frac{(n-1)^2(n-2)^2}{2}$;*

(c) *if n is even, then the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ is attained by $(n-2, 1^2)$ or $(2, 1^{n-2})$.*

Proof. (a) For any $\zeta \vdash n$, we have

$$\lambda_\zeta^I = \sum_{k\in I\setminus\{n\}}\binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k, 1^{n-k})) + (n-1)!\tilde{\chi}_\zeta((n)).$$

According to Lemma 2.7, if $\zeta \neq (n-m, 1^m)$ with $0 \leq m \leq n-1$, then $\tilde{\chi}_\zeta((n)) = 0$. Thus, by Lemma 4.2, for any $\zeta \neq (n-m, 1^m)$ with $0 \leq m \leq n-1$,

$$\begin{aligned}
\lambda_\zeta^I &= \sum_{k \in I \setminus \{n\}} \binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k, 1^{n-k})) \\
&< \sum_{k \in I \setminus \{n\}} \binom{n}{k}(k-1)!\tilde{\chi}_{(n-1,1)}((k, 1^{n-k})) \\
&= \lambda_{(n-1,1)}^I - (n-1)!\tilde{\chi}_{(n-1,1)}((n)) \\
&= \lambda_{(n-1,1)}^I + (n-2)!. \qquad (4.20)
\end{aligned}$$

Now suppose $\zeta = (n-m, 1^m)$ with $2 \leq m \leq n-2$. Since $n \geq 7$ is odd, $\{n\} \subset I \subseteq \{2, 3, \ldots, n-2, n\}$ and I contains at least one even number less than $n-2$, by Lemmas 2.7 and 4.2 we obtain

$$\begin{aligned}
\lambda_\zeta^I &= \sum_{k \in I \setminus \{n\}} \binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k, 1^{n-k})) + (n-1)!\frac{\chi_\zeta((n))}{\chi_\zeta(\mathbf{1})} \\
&= \sum_{k \in I \setminus \{n\}} \binom{n}{k}(k-1)!\tilde{\chi}_\zeta((k, 1^{n-k})) + (-1)^m(n-m-1)!m! \\
&< \sum_{k \in I \setminus \{n\}} \binom{n}{k}(k-1)!\tilde{\chi}_{(n-1,1)}((k, 1^{n-k})) + 2(n-3)! \\
&= \lambda_{(n-1,1)}^I + (n-2)! + 2(n-3)! \\
&= \lambda_{(n-1,1)}^I + n(n-3)! \\
&< \lambda_{(n-1,1)}^I + 2(n-2)!. \qquad (4.21)
\end{aligned}$$

On the other hand, by Lemma 4.18 we have

$$\lambda_{(1^n)}^I - \lambda_{(n-1,1)}^I > \frac{n}{2}(n-2)!. \qquad (4.22)$$

It follows from (4.20), (4.21) and (4.22) that (1^n) is the unique partition of n whose corresponding Specht module achieves the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$. By (4.3), the multiplicity of this eigenvalue is equal to the square of the degree of the sign representation $\rho_{(1^n)}$, which is equal to 1.

(b) One can easily verify the result for $n = 7$. Now suppose $n \geq 9$. As there are only odd numbers in I, $\mathrm{Cay}(S_n, C(n, I))$ has exactly two connected components and $\lambda_\zeta^I = \lambda_{\zeta'}^I$ for any $\zeta \vdash n$. Thus the largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ is attained by (n) and

(1^n). Since n is odd, one can verify that the function $f(n,k)$ defined in the proof of Lemma 4.20 also satisfies $f(n,k) \geq \tilde{\chi}_{(2^2,1^{n-4})}((k,1^{n-k}))$ for $2 \leq k \leq n-2$. Thus the result in Lemma 4.20 actually applies to any $\zeta \neq (n-m,1^m)$ with $0 \leq m \leq n-1$, that is, $\lambda^I_\zeta < \lambda^I_{(n-2,1^2)}$. Since I contains only odd numbers, we have $I \neq \{2,3,\ldots,n-2,n\}$. So by parts (a) and (b) of Lemma 4.10, we get $\lambda^I_{(n-2,1^2)} > \lambda^I_{(n-1,1)} = \lambda^I_{(2,1^{n-2})}$ and the maximum of $\lambda^I_{(n-m,1^m)}$ for $1 \leq m \leq n-2$ can only be attained by $m = 2, n-3$. That is, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ is attained by $(n-2,1^2)$ and $(3,1^{n-3})$. As the dimensions of $\rho_{(n-2,1^2)}$ and $\rho_{(3,1^{n-3})}$ are both $\frac{(n-1)(n-2)}{2}$, the multiplicity of this eigenvalue is $\frac{(n-1)^2(n-2)^2}{2}$ by equation (4.3).

(c) One can verify that the result is true for $n = 8$. Now suppose $n \geq 10$. First, by Lemma 4.20, for any $\zeta \neq (2^2,1^{n-4})$, $(n-m,1^m)$ with $0 \leq m \leq n-1$, we have $\lambda^I_\zeta < \lambda^I_{(n-2,1^2)}$. Second, by parts (a) and (b) of Lemma 4.10, the maximum of $\lambda^I_{(n-m,1^m)}$ for $1 \leq m \leq n-1$ can only be attained by $m = 1, 2, n-3, n-2$ or $n-1$, and $\lambda^I_{(n-2,1^2)} > \lambda^I_{(3,1^{n-3})}$ as I contains the even number n. Thirdly, by Lemma 4.19 and parts (b) and (c) of Lemma 4.10, we have $\lambda^I_{(1^n)} < \lambda^I_{(n-1,1)} < \max\left\{\lambda^I_{(n-2,1^2)}, \lambda^I_{(2,1^{n-2})}\right\}$. Therefore, the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be attained by $(n-2,1^2)$, $(2,1^{n-2})$ and $(2^2,1^{n-4})$. We now show that we can rule out $(2^2,1^{n-4})$. In fact, since n is even, we have $\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})) > \tilde{\chi}_{(2^2,1^{n-4})}((k,1^{n-k}))$ for every $2 \leq k \leq n$ except $k = n-2$. Thus,

$$\begin{aligned}
& \lambda^I_{(n-2,1^2)} - \lambda^I_{(2^2,1^{n-4})} \\
= \; & \sum_{k \in I} \binom{n}{k} (k-1)! \left(\frac{(n-k-1)(n-k-2) - 2c_2}{(n-1)(n-2)} + (-1)^k \frac{(n-k)(n-k-3) + 2c_2}{n(n-3)} \right) \\
\geq \; & \sum_{k \in \{n-2,n\}} \binom{n}{k} (k-1)! \left(\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})) - \tilde{\chi}_{(2^2,1^{n-4})}((k,1^{n-k})) \right) \\
= \; & 2(n-3)! - (n-1)(n-4)! \\
> \; & 0, \qquad (4.23)
\end{aligned}$$

from which the desired result follows. ☐

The following is an immediate corollary of Theorem 4.21.

Corollary 4.22. *Suppose $n \geq 7$ and $\{n\} \subseteq I \subseteq \{2,3,\ldots,n-2,n\}$. Then $\mathrm{Cay}(S_n, C(n,I))$ does not possess the Aldous property.*

4.5 I contains both $n-1$ and n

Lemma 4.23. *Suppose $n \geq 7$ is odd and $\{n-1, n\} \subseteq I \subseteq \{2, 3, \ldots, n-1, n\}$. Then the following statements hold:*

(a) *if the largest number in $I \setminus \{n-1, n\}$ is odd and $I \neq \{2, 3, n-1, n\}$, then $\lambda^I_{(1^n)} > \lambda^I_{(n-1,1)}$;*

(b) *if the largest number in $I \setminus \{n-1, n\}$ is even, then $\lambda^I_{(n-1,1)} > \lambda^I_{(1^n)}$;*

(c) *if $I = \{2, 3, n-1, n\}$ or $\{n-1, n\}$, then $\lambda^I_{(n-1,1)} = \lambda^I_{(1^n)}$.*

Proof. Note that

$$\begin{aligned}
\lambda^I_{(1^n)} - \lambda^I_{(n-1,1)} &= \sum_{k \in I} \binom{n}{k} (k-1)! \left((-1)^{k-1} - \frac{n-k-1}{n-1} \right) \\
&= \sum_{k \in I \setminus \{n-1,n\}} \binom{n}{k} (k-1)! \left((-1)^{k-1} - \frac{n-k-1}{n-1} \right) \\
&= \lambda^{I \setminus \{n-1,n\}}_{(1^n)} - \lambda^{I \setminus \{n-1,n\}}_{(n-1,1)}.
\end{aligned}$$

Thus, if $I = \{n-1, n\}$, then $\lambda^I_{(1^n)} = \lambda^I_{(n-1,1)}$, and if $I \neq \{n-1, n\}$, then we obtain the desired results by applying Lemmas 4.3 and 4.4 directly to $I \setminus \{n-1, n\} \neq \emptyset$. ☐

Lemma 4.24. *Suppose $n \geq 7$ and $\{n, n-1\} \subseteq I \subseteq \{2, 3, \ldots, n\}$. If n is even, then $\lambda^I_{(n-2,1^2)} > \lambda^I_{(n-m,2,1^{m-2})}$ for $m = 2, 3, n-3, n-2$; if n is odd, then $\lambda^I_{(n-2,1^2)} > \lambda^I_{(n-m,2,1^{m-2})}$ for $m = 2, 3, n-3$.*

Proof. With the help of Table 2.1 one can verify that for $m = 3, n-3$ we have

$$\sum_{k=n-1}^{n} \binom{n}{k} (k-1)! \tilde{\chi}_{(n-2,1^2)}((k, 1^{n-k})) > \sum_{k=n-1}^{n} \binom{n}{k} (k-1)! \tilde{\chi}_{(n-m,2,1^{m-2})}((k, 1^{n-k}))$$

and

$$\tilde{\chi}_{(n-2,1^2)}((k, 1^{n-k})) \geq \tilde{\chi}_{(n-m,2,1^{m-2})}((k, 1^{n-k})) \quad \text{for any } 2 \leq k \leq n-2.$$

Since $\{n-1, n\} \subseteq I \subseteq \{2, 3, \ldots, n\}$, it follows that $\lambda^I_{(n-2,1^2)} > \lambda^I_{(n-m,2,1^{m-2})}$ for $m = 3, n-3$.

We have $\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})) > \tilde{\chi}_{(n-2,2)}((k,1^{n-k}))$ for $3 \leq k \leq n$. We also have $\sum_{k\in\{2,n\}}\binom{n}{k}(k-1)!\tilde{\chi}_{(n-2,1^2)}((k,1^{n-k})) > \sum_{k\in\{2,n\}}\binom{n}{k}(k-1)!\tilde{\chi}_{(n-2,2)}((k,1^{n-k}))$. Thus $\lambda^I_{(n-2,1^2)} > \lambda^I_{(n-2,2)}$ whenever $n \in I \subseteq \{2,3,\ldots,n\}$. Inequality (4.23) implies that $\lambda^I_{(n-2,1^2)} > \lambda^I_{(2^2,1^{n-4})}$ whenever n is even and $n \in I \subseteq \{2,3,\ldots,n\}$. □

The main result in this section is as follows.

Theorem 4.25. *Suppose $n \geq 7$ and $\{n, n-1\} \subseteq I \subseteq \{2,3,\ldots,n\}$. Then the following statements hold:*

(a) *if n is even, then the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be attained by $(1^n), (n-2,1^2)$ or $(2,1^{n-2})$;*

(b) *if n is odd, then the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be achieved by $(1^n), (n-2,1^2), (3,1^{n-3})$ or $(2^2,1^{n-4})$.*

Proof. (a) One can verify that the result is true for $n = 8$. Now suppose $n \geq 10$ and n is even. According to Lemmas 2.8 and 4.20, if $\zeta \neq (n-m,1^m), (n-m,2,1^{m-2})$, then $\lambda^I_\zeta = \lambda^{I\setminus\{n-1\}}_\zeta < \lambda^{I\setminus\{n-1\}}_{(n-2,1^2)} = \lambda^I_{(n-2,1^2)}$. On the other hand, by Lemma 4.10 we have $\lambda^I_{(n-1,1)} < \max\left\{\lambda^I_{(n-2,1^2)}, \lambda^I_{(2,1^{n-2})}\right\}$ and $\lambda^I_\zeta < \lambda^I_{(n-2,1^2)}$ for $\zeta = (n-m,1^m)$ with $3 \leq m \leq n-3$. Hence $\lambda^I_\zeta < \max\left\{\lambda^I_{(n-2,1^2)}, \lambda^I_{(2,1^{n-2})}\right\}$ for any $\zeta \neq (n), (1^n), (n-2,1^2), (2,1^{n-2})$, $(n-m,2,1^{m-2})$ with $2 \leq m \leq n-2$.

Since $n-1 \in I \subseteq \{2,3,\ldots,n-1,n\}$, by Remark 4.13 the maximum of $\lambda^I_{(n-m,2,1^{m-2})}$ for $2 \leq m \leq n-2$ can only be attained by $m = 2,3,n-3$ or $n-2$. Moreover, Lemma 4.24 implies that $\lambda^I_{(n-m,2,1^{m-2})} < \lambda^I_{(n-2,1^2)}$ for $m = 2,3,n-3,n-2$. Therefore, the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,I))$ can only be attained by $(1^n), (n-2,1^2)$ or $(2,1^{n-2})$.

(b) One can easily verify this result for $n = 7$. Now suppose $n \geq 9$ and n is odd. Similarly to the proof of part (a) above, one can prove that $\lambda^I_\zeta = \lambda^{I\setminus\{n-1\}}_\zeta < \lambda^{I\setminus\{n-1\}}_{(n-2,1^2)} = \lambda^I_{(n-2,1^2)}$ for any ζ other than $(n-m,1^m)$ and $(n-m,2,1^{m-2})$. By Lemma 4.10, we have $\lambda^I_{(n-1,1)} \leq \lambda^I_{(n-2,1^2)}$ and $\lambda^I_{(n-m,1^m)} < \lambda^I_{(n-2,1^2)}$ for $3 \leq m \leq n-4$. Note that, if $\lambda^I_{(n-1,1)} = \lambda^I_{(n-2,1^2)}$, then $I = \{2,3,\ldots,n\}$ and thus $\lambda^I_{(1^n)} > \lambda^I_{(n-1,1)}$ by Lemma 4.23. A straightforward computation shows that $\lambda^I_{(2,1^{n-2})} \leq \lambda^I_{(n-1,1)} < \max\left\{\lambda^I_{(n-2,1^2)}, \lambda^I_{(1^n)}\right\}$. Finally, by Lemma 4.24 we have $\lambda^I_{(n-2,1^2)} > \lambda^I_{(n-m,2,1^{m-2})}$ for $m = 2,3,n-3$. Combining all these with Remark 4.13, we obtain the desired result. □

Theorem 4.25 implies the following result.

Corollary 4.26. *Suppose* $n \geq 7$ *and* $\{n, n-1\} \subseteq I \subseteq \{2, 3, \dots, n\}$*. Then* $\mathrm{Cay}(S_n, C(n, I))$ *does not have the Aldous property.*

We conjecture that the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, I))$ in part (b) of Theorem 4.25 can only be achieved by $(1^n), (n-2, 1^2)$ or $(2^2, 1^{n-4})$:

Conjecture 4.27. Suppose $n \geq 7$ is odd and $\{n-1, n\} \subseteq I \subseteq \{2, 3, \dots, n\}$. Then

$$\lambda^I_{(3,1^{n-3})} < \max\left\{\lambda^I_{(1^n)}, \lambda^I_{(n-2,1^2)}, \lambda^I_{(2^2,1^{n-4})}\right\}.$$

Note that, by Lemma 4.10, we already know that $\lambda^I_{(3,1^{n-3})} \leq \lambda^I_{(n-2,1^2)}$ and the equality holds if and only if I contains only odd numbers other than $n-1$ and $n-2$.

Chapter 5

Nonnormal Cayley graphs on symmetric groups generated by cycles

5.1 Introduction

Suppose r, k, n are three integers satisfying $1 \leq r < k < n$. In [96], Siemons and Zalesski defined $H := C(n,k;r)$, a subset of $C(n,k)$, to be the set of k-cycles of S_n moving every point from 1 to r, and they determined all the eigenvalues of $H^+ := \sum_{h \in H} h \in \mathbb{C}S_n$ on the natural permutation module $M^{(n-1,1)}$ of S_n, among which the second largest one is

$$\mu_2(n,k;r) = (k-2)!\binom{n-r}{k-r}\frac{1}{n-r}\left((k-1)(n-k) - \frac{(k-r-1)(k-r)}{n-r-1}\right).$$

Then $\mu_2(n,k;r)$ provides a lower bound for the strictly second largest eigenvalue of the nonnormal Cayley graph $\mathrm{Cay}(S_n, C(n,k;r))$ (see [96, Theorem 1.3]). Siemons and Zalesski [96] then conjectured that $\mu_2(n,k;r)$ is exactly the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,k;r))$ for all triplets (n,k,r) satisfying $1 \leq r < k < n$. Since the natural permutation module $M^{(n-1,1)}$ decomposes into one trivial representation and one standard representation, $\mu_2(n,k;r)$ actually is the largest eigenvalue of H^+ on the standard representation of S_n. Thus their conjecture can be restated in the following form.

Conjecture 5.1 ([96])**.** Suppose $n \geq 5$ and $1 \leq r < k < n$. The strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,k;r))$ is $\mu_2(n,k;r)$, attained by the standard representation of S_n. In other words, $\mathrm{Cay}(S_n, C(n,k;r))$ has the Aldous property.

The case $k = r+1$ with $2 \leq r \leq n-2$ of this conjecture has been confirmed in [96, Theorem 1.4]. A recent result in [57] also indicates that $\alpha_2(\mathrm{Cay}(S_n, C(n,3;2))) = \mu_2(n,3;2)$. Combining these with the very early work in [45] about $\lambda_2(\mathrm{Cay}(S_n, C(n,2;1))) = n-2$, we have the following theorem.

Theorem 5.2. ([45], [57] and [96, Theorem 1.4]) *Suppose* $n \geq 5$ *and* $2 \leq k \leq n-1$*. The strictly second largest eigenvalue of* $\mathrm{Cay}(S_n, C(n,k;k-1))$ *is* $(k-1)!(n-k)$*, attained by the standard representation of* S_n*. In other words,* $\mathrm{Cay}(S_n, C(n,k;k-1))$ *has the Aldous property.*

Further evidence supporting Conjecture 5.1 can be found in [57], where one can deduce that $\alpha_2(\mathrm{Cay}(S_n, C(n,3;1))) = n^2 - 5n + 5$, which is equal to $\mu_2(n,3;1)$. Combining this with Theorem 5.2, we see that Conjecture 5.1 is true for $k = 2, 3$. In this chapter we solve Conjecture 5.1 in the general case where $4 \leq k \leq n-1$. We prove this conjecture is almost always true and determine exactly when the conjecture is not true. The following is a summary of our main results.

Theorem 5.3. *Suppose* $n \geq 5$*,* $k \geq 4$ *and* $1 \leq r < k \leq n-1$*. Then* $\mathrm{Cay}(S_n, C(n,k;r))$ *has the Aldous property except for the following cases:*

(1) $n = 6$*,* $k = 5$ *and* $r = 1$*;*

(2) $n \geq 5$ *is odd,* $k = n-1$ *and* $1 \leq r < \frac{n}{2}$*.*

The overall method used in this chapter is induction. One induction base is the case $r = k-1$ and the Aldous property of $\mathrm{Cay}(S_n, C(n,k;k-1))$ as seen in Theorem 5.2. We then build another induction base in Section 5.3, dealing with the case $k = n-1$. As we can see from Theorem 5.3, there are some exceptions in the case $k = n-1$. Thus in Section 5.4 we first build some additional induction base in the case $k = n-2$ and then we do induction on both $n-k$ and $n-r$ to show that $\mathrm{Cay}(S_n, C(n,k;r))$ has the Aldous property whenever $4 \leq k \leq n-2$. Here this induction method is similar to that used in [27] for dealing with the reversal graphs. We decompose $\mathrm{Cay}(S_n, C(n,k;r))$ into one

$\mathrm{Cay}(S_n, C(n,k;r+1))$ and n copies of $\mathrm{Cay}(S_{n-1}, C(n-1,k;r))$. Thanks to a recurrence relation for $\mu_2(n,k;r)$ (see (5.9)), we can finally prove in Theorems 5.22 and 5.24 that $\alpha_2(\mathrm{Cay}(S_n, C(n,k;r))) = \mu_2(n,k;r)$ for any $4 \leq k \leq n-2$ and $1 \leq r < k$.

The most subtle case $k = n-1$ will be handled in Section 5.3, where the approach used is to decompose the nonnormal connection set $C(n,n-1;r)$ into some mutually disjoint subsets $P_i := C(n,n-1) \cap G_i$ with $r+1 \leq i \leq n$. Here G_i is the stabilizer of $i \in [n]$ in S_n. This kind of decomposition works well when r is large. For small r, we decompose $C(n,n-1;r)$ in another way by deleting $P_i := C(n,n-1) \cap G_i$ with $1 \leq i \leq r$ from $C(n,n-1)$. Now each P_i is normal not in S_n but in some subgroup of S_n which is isomorphic to S_{n-1}. With these decompositions, we can apply Branching Rule and use irreducible characters of S_{n-1} to express the eigenvalues of each irreducible representation of S_n on every $P_i^+ = \sum_{h \in P_i} h \in \mathbb{C}S_n$. Then by Weyl Inequalities, we can make use of these eigenvalues to bound that of each irreducible representation of S_n on $C(n,n-1;r)$ and finally identify the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-1;r))$.

The main results in Section 5.3 are Theorems 5.12 and 5.14, which are summarized in the following Table 5.1. The third column of this table shows the partitions of n with their corresponding irreducible representations attaining the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-1;r))$ and the last column demonstrates the multiplicity of that eigenvalue. Note that the standard representation of S_n just corresponds to the partiton $(n-1,1)$. Thus from Table 5.1 we know exactly when $\mathrm{Cay}(S_n, C(n,n-1;r))$ has the Aldous property. As byproducts of Theorems 5.12 and 5.14, we also determined the (second) smallest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-1;r))$ in Corollaries 5.13 and 5.17.

Theorem 5.3 will follow from Theorems 5.12, 5.14, 5.22 and 5.24 with no effort.

5.2 Preliminaries

Recall that in Section 2.3, we use G to denote either S_n or A_n, and G_i to denote the stabilizer of $i \in [n]$ in G. We use $G_{j,i}$ to denote the set of permutations in G that send j to i. Then we have the right coset decomposition of G_i:

$$\Pi_i : G = G_{1,i} \cup G_{2,i} \cup \cdots \cup G_{n,i},$$

$\mathbf{C(n, n-1; r)}$ with $\mathbf{n \geq 7}$		**Partitions**	**Multiplicity**
n is odd	$r = 1$	$(2, 1^{n-2})$	$n-1$
	$2 \leq r < \frac{n}{2}$	$(2, 1^{n-2})$	$(n-1)(r-1)$
	$\frac{n}{2} < r \leq n-2$	$(n-1, 1)$	$(n-1)(n-r-1)$
n is even	$r = 1, 2$	$(n-1, 1)$, $(2, 1^{n-2})$ $((n-2, 1^2)$, $(3, 1^{n-3})$)	Unknown
	$3 \leq r \leq n-2$	$(n-1, 1)$, $(2, 1^{n-2})$	$2(n-1)(n-r-1)$

TABLE 5.1: Summary of Theorems 5.12 and 5.14. The partitions enclosed in parentheses are the potential candidates for attaining the strictly second largest eigenvalue.

as given in (2.1). Now we apply Lemma 2.14 to $\mathrm{Cay}(S_n, C(n,k;r))$ in the following manner.

Lemma 5.4. [58, Theorem 7] *Let $H = C(n,k;r)$ and $\Gamma = \mathrm{Cay}(S_n, H)$, where $n \geq 5$ and $1 \leq r < k < n$. The right coset decomposition Π_i of S_n given in (2.1) leads to an equitable partition of Γ, and the corresponding quotient matrix $\mathbf{B}$ is symmetric and independent on the choice of $i \in [n]$. Moreover, if λ is an eigenvalue of Γ which is not an eigenvalue of $\mathbf{B}$, then, for each $j \in [n]$, we have*

$$\lambda \leqslant \lambda_2 \left(\mathrm{Cay}\left(G_j, H \cap G_j\right)\right) + \lambda_2 \left(\mathrm{Cay}\left(S_n, H \backslash \left(H \cap G_j\right)\right)\right), \tag{5.1}$$

where G_j is the stabilizer of $j \in [n]$ in S_n.

Remark 5.5. (i) The quotient matrix $\mathbf{B}$ of Π_i for any $i \in [n]$ is exactly the permutation matrix of H^+ arising from the natural permutation module $M^{(n-1,1)}$ of S_n. Thus as the second largest eigenvalue of H^+ on $M^{(n-1,1)}$, $\mu_2(n,k;r)$ is exactly $\lambda_2(\mathbf{B})$. By Young's Rule, the natural permutation module $M^{(n-1,1)}$ of S_n decomposes into one trivial module $S^{(n)}$ and one $S^{(n-1,1)}$. Thus the spectrum of H^+ on $M^{(n-1,1)}$ is the union of the spectra of H^+ on $S^{(n)}$ and $S^{(n-1,1)}$. Clearly, H^+ acting on $S^{(n)}$ gives the largest eigenvalue $|H|$ of $\mathrm{Cay}(S_n, H)$. Then the spectrum of H^+ on $S^{(n-1,1)}$ is obtained by removing the largest eigenvalue $|H|$ from the spectrum of H^+ on $M^{(n-1,1)}$. In particular, the second largest eigenvalue $\mu_2(n,k;r)$ of H^+ on

$M^{(n-1,1)}$, which is also $\lambda_2(\mathbf{B})$, is exactly the largest eigenvalue of H^+ on $S^{(n-1,1)}$, that is, $\alpha_1(\rho_{(n-1,1)}(H))$.

(ii) When $j \leq r$, the definition of $H = C(n,k;r)$ implies that $H \cap G_j = \emptyset$, and thus the right hand side of (5.1) is just $\lambda_2(\mathrm{Cay}(S_n, C(n,k;r)))$, which is a trivial upper bound for λ. When making use of (5.1), we shall take j from $[n] \setminus [r]$ and mostly we just let $j = n$.

The following two lemmas on the spectrum of $\rho_{(n-1,1)}(C(n,k;r))$ is a direct corollary of Theorem 5.2, Lemma 5.3 and Lemma 3.2 in [96].

Lemma 5.6. *Let $H = C(n,k;r)$, where $n \geq 5$ and $2 \leq r < k < n$. Then the distinct eigenvalues of H^+ on the standard representation of S_n are*

$$\begin{aligned}
\alpha_1(\rho_{(n-1,1)}(H)) &= (k-2)!\binom{n-r}{k-r}\frac{1}{n-r}\left((k-1)(n-k) - \frac{(k-r-1)(k-r)}{n-r-1}\right),\\
\alpha_2(\rho_{(n-1,1)}(H)) &= (k-2)!\binom{n-r}{k-r}\left(\frac{r(n-k)}{n-r} - 1\right), \text{ and}\\
\alpha_3(\rho_{(n-1,1)}(H)) &= -(k-2)!\binom{n-r}{k-r}.
\end{aligned}$$

Lemma 5.7. *Let $H = C(n,k;1)$, where $n \geq 5$ and $2 \leq k < n$. Then the distinct eigenvalues of H^+ on the standard representation of S_n are*

$$\begin{aligned}
\alpha_1(\rho_{(n-1,1)}(H)) &= (k-2)!\binom{n-1}{k-1}\frac{1}{n-1}\left((k-1)(n-k) - \frac{(k-2)(k-1)}{n-2}\right), \text{ and}\\
\alpha_2(\rho_{(n-1,1)}(H)) &= -(k-2)!\binom{n-2}{k-2}.
\end{aligned}$$

The following lemma gives the multiplicities of the eigenvalues of $\rho_{(n-1,1)}(C(n,n-1;r))$. The proof here is similar to that of Lemma 6.1 in [96].

Lemma 5.8. *Let $n \geq 5$ and $H = C(n,n-1;r)$. When $2 \leq r \leq n-2$, the spectrum of H^+ on the standard representation of S_n are*

$$\mathrm{Spec}(\rho_{(n-1,1)}(H)) = \begin{pmatrix} r(n-3)! & (2r-n)(n-3)! & (r-n)(n-3)! \\ n-r-1 & 1 & r-1 \end{pmatrix}.$$

When $r = 1$, the spectrum of H^+ on the standard representation of S_n are

$$\mathrm{Spec}(\rho_{(n-1,1)}(H)) = \begin{pmatrix} (n-3)! & -(n-2)! \\ n-2 & 1 \end{pmatrix}.$$

Proof. Suppose $2 \leq r \leq n-2$. From [96, Theorem 5.2], we obtain that all the eigenvalues of H^+ on the natural permutation module $M^{(n-1,1)}$ are $|H|$, $r(n-3)!$, $(2r-n)(n-3)!$ and $(r-n)(n-3)!$. Clearly, the largest eigenvalue $|H|$ is simple. Lemma 3.2 in [96] indicates that the multiplicities $x,\ y,\ z$ of the other three eigenvalues are such that $\{x, y, z\} = \{1,\ n-r-1,\ r-1\}$. For each $\sigma \in S_n$, the trace of σ on $M^{(n-1,1)}$ is the number of fixed points of σ. Thus the trace of H^+ on $M^{(n-1,1)}$ is $|H|$. Then we have

$$|H| + x \cdot r(n-3)! + y \cdot (2r-n)(n-3)! + z \cdot (r-n)(n-3)! = |H|.$$

If we take $x = n-r-1$, $y = 1$ and $z = r-1$, then the quality holds. In addition, the equality fails for any other choice of $x,\ y,\ z$ such that $\{x, y, z\} = \{1,\ n-r-1,\ r-1\}$.

The proof for the case $r = 1$ is similar. □

Finally, we cite the following lemma, which indicates that $\mathrm{Cay}(S_n, C(n,k;r))$ is connected if k is even and has two connected components each isomorphic to $\mathrm{Cay}(A_n, C(n,k;r))$ if k is odd.

Lemma 5.9. [96, Lemma 5.1] *Let $1 \leq r < k \leq n$ and let X be the smallest subgroup of S_n containing $C(n,k;r)$. Then $X = S_n$ if k is even, and $X = A_n$ if k is odd.*

5.3 The case when all cycles have length $n-1$ and move each of $1, \ldots, r$

For every $i \in [n]$, the symmetric group $\mathrm{Sym}_{[n]\setminus\{i\}}$ is isomorphic to S_{n-1} under the isomorphism $f_i : \mathrm{Sym}_{[n]\setminus\{i\}} \to S_{n-1},\ g \mapsto (n\ i)g(n\ i)$, and f_i sends

$$P_i := C(n, n-1) \cap G_i$$

to $C(n-1, n-1)$. Thus for any $\zeta \vdash n$, we have $\rho_\zeta(C(n-1, n-1)) = M_i \rho_\zeta(P_i) M_i^{-1}$ with $M_i = \rho_\zeta((n\ i))$. This indicates that $\rho_\zeta(P_1)$, $\rho_\zeta(P_2)$, $\ldots$, $\rho_\zeta(P_n)$ share the same spectrum with $\rho_\zeta(C(n-1,n-1))$ for any fixed $\zeta \vdash n$. Now we use Branching Rule to calculate the eigenvalues of $\rho_\zeta(C(n-1,n-1))$ for every irreducible representation ρ_ζ of S_n except for the trivial representation $\rho_{(n)}$ and the sign representation $\rho_{(1^n)}$. The results are recorded in the following lemma.

Lemma 5.10. *Let $H = C(n-1,n-1)$, where $n \geq 7$, and let $\zeta \vdash n$ be such that $\zeta \neq (n)$ or (1^n).*

(a) *If ζ is not a hook or a near hook, then all the eigenvalues of $\rho_\zeta(H)$ are 0.*

(b) *If $\zeta = (n-m, 1^m)$ is a hook with $1 \leq m \leq n-2$, then the distinct eigenvalues of $\rho_\zeta(H)$ are $(-1)^m m!(n-2-m)!$ and $(-1)^{m-1}(m-1)!(n-1-m)!$.*

(c) *If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$, then the distinct eigenvalues of $\rho_\zeta(H)$ are $(-1)^{m-1}(m-1)!(n-1-m)!$ and 0.*

Proof. Since $H = C(n-1,n-1)$ is normal in S_{n-1}, we obtain by Branching Rule $\rho_\zeta(H) = \bigoplus_{\zeta^-} \rho_{\zeta^-}(H)$, and by Schur's Lemma each $\rho_{\zeta^-}(H)$ is a scalar matrix. We then derive from the definition of normalized characters and the fact that they are class functions that $\rho_{\zeta^-}(H) = |H| \cdot \tilde{\chi}_{\zeta^-}(\sigma) \cdot \mathcal{I}_m$, where σ is any $(n-1)$-cycle of S_{n-1} and $\mathcal{I}_m$ is the identity matrix with dimension $m = \dim \rho_{\zeta^-}$. Thus the distinct eigenvalues of $\rho_\zeta(H)$ are $\{|H| \cdot \tilde{\chi}_{\zeta^-}(\sigma)\}_{\zeta^-}$, where $\tilde{\chi}_{\zeta^-}(\sigma)$ can be calculated by Lemma 2.7.

(a) When ζ is not a hook or a near hook, ζ^- is never a hook. From Lemma 2.7, the normalized character of ζ^- on σ is 0. Then the eigenvalues of $\rho_\zeta(H)$ are all 0's.

(b) If $\zeta = (n-m, 1^m)$ is a hook with $1 \leq m \leq n-2$, then $\zeta^- = (n-1-m, 1^m)$ or $(n-m, 1^{m-1})$. By Lemma 2.7, we have $|H| \cdot \tilde{\chi}_{(n-1-m,1^m)}(\sigma) = (-1)^m m!(n-2-m)!$ and $|H| \cdot \tilde{\chi}_{(n-m,1^{m-1})}(\sigma) = (-1)^{m-1}(m-1)!(n-1-m)!$. Thus the distinct eigenvalues of $\rho_\zeta(H)$ are $(-1)^m m!(n-2-m)!$ and $(-1)^{m-1}(m-1)!(n-1-m)!$.

(c) If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$, then $\zeta^- = (n-m, 1^{m-1})$, $(n-m-1, 2, 1^{m-2})$ if $m \leq n-3$, or $(n-m, 2, 1^{m-3})$ if $m \geq 3$. The last two values of ζ^- attain 0 characters on σ by Lemma 2.7 and $|H| \cdot \tilde{\chi}_{(n-m,1^{m-1})}(\sigma) = (-1)^{m-1}(m-1)!(n-1-m)!$. Thus the distinct eigenvalues of $\rho_\zeta(H)$ are $(-1)^{m-1}(m-1)!(n-1-m)!$ and 0.

This completes the proof. □

Lemma 5.11. *Let* $H = C(n, n-1; r)$, *where* $n \geq 7$ *and* $r \in \{1, 2, \dots n-2\}$, *and let* $\zeta \vdash n$ *be such that* $\zeta \neq (n)$, $(n-1, 1)$, $(2, 1^{n-2})$ *or* (1^n).

(a) *If* n *is odd and* $1 \leq r \leq 2$, *then* $\lambda_1(\rho_\zeta(H)) \leq 2(n-2)(n-4)!$.

(b) *If* n *is odd and* $3 \leq r < n/2$, *then* $\lambda_1(\rho_\zeta(H)) \leq r(n-3)!$.

(c) *If* n *is odd and* $n/2 < r \leq n-2$, *then* $\lambda_1(\rho_\zeta(H)) \leq (n-r)(n-3)!$.

(d) *If* n *is even and* $1 \leq r \leq 2$, *then* $\lambda_1(\rho_\zeta(H)) \leq r(n-3)!$; *moreover, if in addition* $\zeta \neq (n-2, 1^2)$ *or* $(3, 1^{n-3})$, *then* $\lambda_1(\rho_\zeta(H)) < r(n-3)!$.

(e) *If* n *is even and* $3 \leq r \leq n-2$, *then* $\lambda_1(\rho_\zeta(H)) \leq 2(n-r)(n-4)!$.

Proof. Recall $P_i = C(n, n-1) \cap G_i$ for every $i \in [n]$. Then P_i is closed under inverse and conjugation in $\mathrm{Sym}_{[n]\setminus\{i\}}$ and these P_i's are mutually disjoint. For any $\zeta \vdash n$ and any $i \in [n]$, the matrix $\rho_\zeta(P_i)$ is symmetric, guaranteed by Lemma 1.3, and shares the same eigenvalues with $\rho_\zeta(C(n-1, n-1))$ by the analysis above Lemma 5.10.

We decompose H into $n-r$ parts as $H = \dot\bigcup_{i=r+1}^{n} P_i$. Then for any $\zeta \vdash n$ we have $\rho_\zeta(H) = \sum_{i=r+1}^{n} \rho_\zeta(P_i)$ and by Weyl Inequalities, the following bound for $\lambda_1(\rho_\zeta(H))$ holds:

$$\lambda_1(\rho_\zeta(H)) \leq (n-r)\lambda_1(\rho_\zeta(C(n-1, n-1))). \tag{5.2}$$

Now substitute the results of Lemma 5.10 into (5.2).

- If ζ is not a hook or a near hook, then $\lambda_1(\rho_\zeta(H)) \leq 0$.
- If $\zeta = (n-m, 1^m)$ with $2 \leq m \leq n-3$ and m is even, then

$$\lambda_1(\rho_\zeta(H)) \leq (n-r)m!(n-2-m)! \leq \begin{cases} (n-r)(n-3)!, & \text{if } n \text{ is odd;} \\ 2(n-r)(n-4)!, & \text{if } n \text{ is even.} \end{cases}$$

- If $\zeta = (n-m, 1^m)$ with $2 \leq m \leq n-3$ and m is odd, then

$$\lambda_1(\rho_\zeta(H)) \leq (n-r)(m-1)!(n-1-m)! \leq 2(n-r)(n-4)!.$$

- If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$ and m is even, then $\lambda_1(\rho_\zeta(H)) \leq 0$.

- If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$ and m is odd, then

$$\lambda_1(\rho_\zeta(H)) \leq (n-r)(m-1)!(n-1-m)! \leq \begin{cases} (n-r)(n-3)!, & \text{if } n \text{ is odd;} \\ 2(n-r)(n-4)!, & \text{if } n \text{ is even.} \end{cases}$$

To sum up, when n is odd, the largest eigenvalue of $\rho_\zeta(H)$ is no larger than $(n-r)(n-3)!$; when n is even, the largest eigenvalue of $\rho_\zeta(H)$ is no larger than $2(n-r)(n-4)!$. This completes the proof of (c) and (e).

We now express H as $H = C(n, n-1) \setminus (\dot\cup_{i=1}^r P_i)$. Then for any $\zeta \vdash n$, we have $\rho_\zeta(H) = \rho_\zeta(C(n, n-1)) - \sum_{i=1}^i \rho_\zeta(P_i)$. By Weyl Inequalities, we have the following bound for $\lambda_1(\rho_\zeta(H))$:

$$\lambda_1(\rho_\zeta(H)) \leq \lambda_1(\rho_\zeta(C(n, n-1))) - r \cdot \lambda_{min}(\rho_\zeta(C(n-1, n-1))). \tag{5.3}$$

As $C(n, n-1)$ is a conjugacy class of S_n, the matrix $\rho_\zeta(C(n, n-1))$ is a scalar matrix and the unique eigenvalue of $\rho_\zeta(C(n, n-1))$ is given by $|C(n, n-1)| \cdot \tilde\chi_\zeta(\sigma)$, where σ is any element in $C(n, n-1)$ and $\tilde\chi_\zeta(\sigma)$ can be calculated by Lemma 2.8. Substituting Lemma 5.10 into (5.3) gives us the following results:

- Suppose ζ is not a hook or a near hook. The eigenvalues of $\rho_\zeta(C(n, n-1))$ and $\rho_\zeta(C(n-1, n-1))$ are all 0. Thus from (5.3) we derive that $\lambda_1(\rho_\zeta(H)) \leq 0$.

- If $\zeta = (n-m, 1^m)$ with $2 \leq m \leq n-3$ and m is even, then all eigenvalues of $\rho_\zeta(C(n, n-1))$ are 0 and thus $\lambda_1(\rho_\zeta(H)) \leq r \cdot (m-1)!(n-1-m)! \leq r(n-3)!$. If in addition $\zeta \neq (n-2, 1^2)$, then $\lambda_1(\rho_\zeta(H)) < r(n-3)!$.

- If $\zeta = (n-m, 1^m)$ with $2 \leq m \leq n-3$ and m is odd, then we have

  $$\lambda_1(\rho_\zeta(H)) \leq r \cdot m!(n-2-m)! \leq \begin{cases} 2r(n-4)!, & \text{if } n \text{ is odd;} \\ r(n-3)!, & \text{if } n \text{ is even.} \end{cases}$$

  Here if in addition n is even and $\zeta \neq (3, 1^{n-3})$, then $\lambda_1(\rho_\zeta(H)) < r(n-3)!$.

- If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$ and m is even, then the unique eigenvalue of $\rho_\zeta(C(n, n-1))$ is $-(n-m)(n-2-m)!m(m-2)!$ and thus $\lambda_1(\rho_\zeta(H)) \leq$

$-(n-m)(n-2-m)!m(m-2)!+r\cdot(m-1)!(n-1-m)! < (r-1)\cdot(m-1)!(n-1-m)! < (r-1)(n-3)!$.

- If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$ and m is odd, then the unique eigenvalue of $\rho_\zeta(C(n,n-1))$ is $(n-m)(n-2-m)!m(m-2)!$ and thus

$$\lambda_1(\rho_\zeta(H)) \leq (n-m)(n-2-m)!m(m-2) \leq \begin{cases} 2(n-2)(n-4)!, & \text{if } n \text{ is odd;} \\ 3(n-3)(n-5)!, & \text{if } n \text{ is even.} \end{cases}$$

If n is odd and $1 \leq r \leq 2$, then

$$\lambda_1(\rho_\zeta) \leq \min\{(n-r)(n-3)!, 2(n-2)(n-4)!\} = 2(n-2)(n-4)!.$$

If n is odd and $3 \leq r < n/2$, then

$$\lambda_1(\rho_\zeta) \leq \min\{(n-r)(n-3)!, r(n-3)!\} = r(n-3)!.$$

If n is even and $1 \leq r \leq 2$, then

$$\lambda_1(\rho_\zeta) \leq \min\{2(n-r)(n-4)!, r(n-3)!\} = r(n-3)!;$$

moreover, if in addition $\zeta \neq (n-2, 1^2)$ or $(3, 1^{n-3})$, then $\lambda_1(\rho_\zeta) < r(n-3)!$. $\square$

Theorem 5.12. *Let* $\Gamma = \mathrm{Cay}(S_n, C(n, n-1; r))$ *with* $n \geq 5$ *odd and* $r \in \{1, 2, \ldots n-2\}$. *Then* Γ *is connected and bipartite with* $\lambda_2(\Gamma) = \alpha_2(\Gamma)$ *such that the following statements hold:*

(a) *If* $n = 5$ *and* $r = 1$, *then* $\lambda_2(\Gamma) = 6$ *with multiplicity* 14, *attained exactly by* $(2, 1^3)$ *and* $(2, 2, 1)$.

(b) *If* $n \geq 7$ *and* $r = 1$, *then* $\lambda_2(\Gamma) = (n-2)!$ *with multiplicity* $n-1$, *attained uniquely by* $(2, 1^{n-2})$.

(c) *If* $2 \leq r < n/2$, *then* $\lambda_2(\Gamma) = (n-r)(n-3)!$ *with multiplicity* $(n-1)(r-1)$, *attained uniquely by* $(2, 1^{n-2})$.

(d) *If* $n/2 < r < n-1$, *then* $\lambda_2(\Gamma) = r(n-3)!$ *with multiplicity* $(n-1)(n-r-1)$, *attained uniquely by* $(n-1, 1)$.

Proof. Let $H = C(n, n-1; r)$ with $1 \leq r \leq n-2$. Since n is odd, all permutations in H are odd and thus $\Gamma = \mathrm{Cay}(S_n, H)$ is a connected regular bipartite graph with $\lambda_2(\Gamma) = \alpha_2(\Gamma)$. When $n = 5$ and $1 \leq r \leq n-2$, we verify $\lambda_2(\Gamma)$ by computation in Magma [17]. For the remainder of this proof, suppose $n \geq 7$.

It is clear that if $\zeta = (n)$, then matrix $\rho_\zeta(H)$ has only one eigenvalue, which is $|H| = (n-r)(n-2)!$; if $\zeta = (1^n)$, then ρ_ζ is the sign representation and sends H to $-|H|$. With Γ a connected regular bipartite graph, we know that $|H|$ and $-|H|$ are the simple largest and smallest eigenvalues of Γ, respectively.

By Lemma 5.8, we get $\lambda_1(\rho_{(n-1,1)}(H)) = r(n-3)!$ with multiplicity $a := n-r-1$ and

$$\lambda_{\min}(\rho_{(n-1,1)}(H)) = \begin{cases} -(n-r)(n-3)!, & \text{if } 2 \leq r \leq n-2; \\ -(n-2)!, & \text{if } r = 1 \end{cases}$$

with multiplicity

$$b := \begin{cases} r-1, & \text{if } 2 \leq r \leq n-2; \\ 1, & \text{if } r = 1. \end{cases}$$

Since Lemma 2.2 implies that $\rho_{(2,1^{n-2})}$ only differs from $\rho_{(n-1,1)}$ at odd permutations by the sign, we have $\rho_{(2,1^{n-2})}(H) = -\rho_{(n-1,1)}(H)$, which implies

$$\lambda_1(\rho_{(2,1^{n-2})}(H)) = \begin{cases} (n-r)(n-3)!, & \text{if } 2 \leq r \leq n-2; \\ (n-2)!, & \text{if } r = 1 \end{cases}$$

with multiplicity b and $\lambda_{\min}(\rho_{(2,1^{n-2})}(H)) = -r(n-3)!$ with multiplicity a.

First suppose $r = 1$ or 2. From Lemma 5.11 (a) we know that if $\zeta \vdash n$ and $\zeta \neq (n), (1^n), (n-1,1)$ or $(2,1^{n-2})$, then $\lambda_1(\rho_\zeta(H)) \leq 2(n-2)(n-4)!$, which is strictly smaller than $(n-2)!$. Thus $\lambda_2(\Gamma) = (n-2)!$, attained uniquely by $(2,1^{n-2})$.

Next suppose $3 \leq r < n/2$. We have verified in Lemma 5.11 (b) that if $\zeta \vdash n$ and $\zeta \neq (n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), then $\lambda_1(\rho_\zeta(H)) \leq r(n-3)!$, which is strictly smaller than $(n-r)(n-3)!$. Thus $\lambda_2(\Gamma) = (n-r)(n-3)!$, attained uniquely by $(2,1^{n-2})$.

Now suppose $n/2 < r \leq n-2$. We have verified in Lemma 5.11 (c) that if $\zeta \vdash n$ and $\zeta \neq (n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), then $\lambda_1(\rho_\zeta(H)) \leq (n-r)(n-3)!$, which is strictly smaller than $r(n-3)!$. Thus $\lambda_2(\Gamma) = r(n-3)!$, attained uniquely by $(n-1,1)$.

Finally, we apply Proposition 1.1 to derive the conclusion on the multiplicity of $\lambda_2(\Gamma)$. If $r = 1$, then $\lambda_2(\Gamma) = (n-2)!$ is attained uniquely by $(2, 1^{n-2})$ and has multiplicity $d_{(2,1^{n-2})} \cdot b = n-1$. If $2 \leq r < n/2$, then $\lambda_2(\Gamma) = (n-r)(n-3)!$ is attained uniquely by $(2, 1^{n-2})$ and has multiplicity $d_{(2,1^{n-2})} \cdot b = (n-1)(r-1)$. If $n/2 < r \leq n-2$, then $\lambda_2(\Gamma) = r(n-3)!$ is attained uniquely by $(n-1, 1)$ and has multiplicity $d_{(n-1,1)} \cdot a = (n-1)(n-r-1)$. □

The following corollary is about the two smallest eigenvalues of $\mathrm{Cay}(S_n, C(n, n-1; r))$ with odd n.

Corollary 5.13. *Let $\Gamma = \mathrm{Cay}(S_n, C(n, n-1; r))$ with $n \geq 5$ odd and $r \in \{1, 2, \ldots, n-2\}$. The smallest eigenvalue of Γ is $-(n-r)(n-2)!$, which is simple and attained uniquely by (1^n).*

(a) *If $n = 5$ and $r = 1$, then the second smallest eigenvalue of Γ is -6 with multiplicity 14, attained exactly by $(4, 1)$ and $(3, 2)$.*

(b) *If $n \geq 7$ and $r = 1$, then the second smallest eigenvalue of Γ is $-(n-2)!$ with multiplicity $n-1$, attained uniquely by $(n-1, 1)$.*

(c) *If $2 \leq r < n/2$, then the second smallest eigenvalue of Γ is $-(n-r)(n-3)!$ with multiplicity $(n-1)(r-1)$, attained uniquely by $(n-1, 1)$.*

(d) *If $n/2 < r < n-1$, then the second smallest eigenvalue of Γ is $-r(n-3)!$ with multiplicity $(n-1)(n-r-1)$, attained uniquely by $(2, 1^{n-2})$.*

Proof. Note that Γ is a connected bipartite graph when n is odd. Thus the spectrum of Γ is symmetric about 0, that is, if λ is an eigenvalue of Γ with multiplicity m_λ, then $-\lambda$ is also an eigenvalue of Γ with multiplicity m_λ. Since n is odd, all the permutations in $C(n, n-1; r)$ are odd and thus by Lemma 2.2, $\rho_{\zeta'}(H) = -\rho_\zeta(H)$ for any $\zeta \vdash n$. Hence λ is an eigenvalue of $\rho_\zeta(H)$ with multiplicity m_ζ^λ if and only if $-\lambda$ is an eigenvalue of $\rho_{\zeta'}(H)$ with multiplicity m_ζ^λ. Note also that ρ_ζ and $\rho_{\zeta'}$ have the same dimension by Lemma 2.1. Now this corollary follows from Theorem 5.12 and Proposition 1.1. □

Theorem 5.14. *Let $\Gamma = \mathrm{Cay}(S_n, C(n, n-1; r))$ with $n \geq 6$ even and $r \in \{1, 2, \ldots, n-2\}$.*

(a) *If* $n=6$ *and* $r=1$*, then* $\alpha_2(\Gamma)=9$ *with multiplicity* 160*, attained uniquely by* $(3,2,1)$.

(b) *If* $(n,r)\neq(6,1)$*, then* $\alpha_2(\Gamma)=r(n-3)!$*. Moreover, the following statements hold:*

  (b.1) *If* $r\leq 2$*, then* $\alpha_2(\Gamma)$ *is attained by* $(n-1,1)$ *and* $(2,1^{n-2})$*, and can only be attained by* $(n-1,1)$*,* $(2,1^{n-2})$*,* $(n-2,1^2)$ *and* $(3,1^{n-3})$.

  (b.2) *If* $3\leq r\leq n-2$*, then* $\alpha_2(\Gamma)$ *has multiplicity* $2(n-1)(n-r-1)$ *and is attained exactly by* $(n-1,1)$ *and* $(2,1^{n-2})$.

Proof. Since n is even, all permutations in $H=C(n,n-1;r)$ are even. The graph $\Gamma=\mathrm{Cay}(S_n,H)$ has two connected components and both are isomorphic to $\mathrm{Cay}(A_n,H)$. Thus the degree $|H|=(n-r)(n-2)!$ is the largest eigenvalue of Γ with multiplicity 2, attained by $\rho_{(n)}$ and $\rho_{(1^n)}$ simultaneously. When $n=6$ and $1\leq r\leq 4$, we verify $\alpha_2(\Gamma)$ via computation in Magma [17]. Now suppose $n\geq 8$.

We write $a=n-r-1$. By Lemma 5.8, we get $\lambda_1(\rho_{(n-1,1)}(H))=r(n-3)!$ with multiplicity a. Since Lemma 2.2 implies that $\rho_{(2,1^{n-2})}(H)=\rho_{(n-1,1)}(H)$, we have $\lambda_1(\rho_{(2,1^{n-2})}(H))=\lambda_1(\rho_{(n-1,1)}(H))=r(n-3)!$ with multiplicity a.

First suppose $1\leq r\leq 2$. According to Lemma 5.11 (d), if $\zeta\vdash n$ is such that $\zeta\neq(n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), then the eigenvalue $\lambda_1(\rho_\zeta(H))\leq r(n-3)!$, and if in addition $\zeta\neq(n-2,1^2)$ or $(3,1^{n-3})$, then $\lambda_1(\rho_\zeta(H))<r(n-3)!$. Thus we concolude that $\alpha_2(\Gamma)=r(n-3)!$, which is attained by $(n-1,1)$ and $(2,1^{n-2})$, and can only be attained by $(n-1,1)$, $(2,1^{n-2})$, $(n-2,1^2)$ and $(3,1^{n-3})$.

Next suppose $3\leq r\leq n-2$. Lemma 5.11 (e) implies that, when $\zeta\vdash n$ such that $\zeta\neq(n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), the eigenvalue $\lambda_1(\rho_\zeta(H))\leq 2(n-r)(n-4)!<r(n-3)!$. Thus we conclude that $\alpha_2(\Gamma)=r(n-3)!$, which is attained exactly by $(n-1,1)$ and $(2,1^{n-2})$, and that its multiplicity equals $d_{(n-1,1)}\cdot a+d_{(2,1^{n-2})}\cdot a=2(n-1)(n-r-1)$ by Proposition 1.1. □

Remark 5.15. In item (b.1) of the above theorem, we cannot rule out $(n-2,1^2)$ and $(3,1^{n-3})$ for attaining $\alpha_2(\mathrm{Cay}(S_n,C(n,n-1;r)))$. For example, the strictly second largest eigenvalue of $\mathrm{Cay}(S_6,C(6,5;2))$ is 12 with multiplicity 50, attained exactly by $(5,1)$, $(4,1^2)$,$(3,1^3)$ and $(2,1^4)$; for $\mathrm{Cay}(S_8,C(8,7;r))$ with $r=1$ or 2, its strictly second largest eigenvalue is attained exactly by $(7,1)$, $(6,1^2)$, $(3,1^5)$ and $(2,1^6)$. However, we also cannot confirm that $(n-2,1^2)$ and $(3,1^{n-3})$ always attain $\alpha_2(\mathrm{Cay}(S_n,C(n,n-1;r)))$ in the case of (b.1).

Lemma 5.16. *Let* $H=C(n,n-1;r)$*, where* $n\geq 7$ *and* $r\in\{1,2,\dots n-2\}$*, and let* $\zeta\vdash n$ *be such that* $\zeta\neq(n)$*,* $(n-1,1)$*,* $(2,1^{n-2})$ *or* (1^n)*. The following bounds for* $\lambda_{\min}(\rho_\zeta(H))$ *hold:*

(a) *If* n *is odd and* $1\leq r\leq 2$*, then* $\lambda_{\min}(\rho_\zeta(H))\geq -2(n-2)(n-4)!$*.*

(b) *If* n *is odd and* $3\leq r<n/2$*, then* $\lambda_{\min}(\rho_\zeta(H))\geq -r(n-3)!$*.*

(c) *If* n *is odd and* $n/2\leq r\leq n-2$*, then* $\lambda_{\min}(\rho_\zeta(H))\geq -(n-r)(n-3)!$*.*

(d) *If* n *is even and* $1\leq r\leq n-3$*, then* $\lambda_{\min}(\rho_\zeta(H))\geq -2(n-2)(n-4)!$*.*

(e) *If* n *is even and* $r=n-2$*, then* $\lambda_{\min}(\rho_\zeta(H))\geq -(n-r)(n-3)!$*; moreover, if in addition* $\zeta\neq(n-2,1^2)$*,* $(3,1^{n-3})$*,* $(n-2,2)$ *or* $(2,2,1^{n-4})$*, then* $\lambda_{\min}(\rho_\zeta(H))>-(n-r)(n-3)!$*.*

Proof. Recall $P_i=C(n,n-1)\cap G_i$ for every $i\in[n]$, and that for any $\zeta\vdash n$ and any $i\in[n]$, the matrix $\rho_\zeta(P_i)$ is symmetric and has the same eigenvalues as $\rho_\zeta(C(n-1,n-1))$.

We apply Weyl Inequalities to two different decompositions of H. The first decomposition is $H=\dot\bigcup_{i=r+1}^{n}P_i$. Applying Weyl Inequalities, we obtain:

$$\lambda_{\min}(\rho_\zeta(H))\geq (n-r)\lambda_{\min}(\rho_\zeta(C(n-1,n-1))). \tag{5.4}$$

Now we substitute Lemma 5.10 to (5.4).

- If ζ is not a hook or a near hook, then $\lambda_{\min}(\rho_\zeta(H))\geq 0$.
- If $\zeta=(n-m,1^m)$ with $2\leq m\leq n-3$ and m is even, then $\lambda_{\min}(\rho_\zeta(H))\geq -(n-r)(m-1)!(n-1-m)!\geq -(n-r)(n-3)!$, and if in addition $\zeta\neq(n-2,1^2)$, then $\lambda_{\min}(\rho_\zeta(H))>-(n-r)(n-3)!$.
- If $\zeta=(n-m,1^m)$ with $2\leq m\leq n-3$ and m is odd, then

$$\lambda_{\min}(\rho_\zeta(H))\geq -(n-r)m!(n-2-m)!\geq\begin{cases}-2(n-r)(n-4)!, & \text{if } n \text{ is odd;}\\ -(n-r)(n-3)!, & \text{if } n \text{ is even.}\end{cases}$$

When n is even, if in addition $\zeta \neq (3,1^{n-3})$, then $\lambda_{\min}(\rho_\zeta(H)) > -(n-r)(n-3)!$.

- If $\zeta = (n-m,2,1^{m-2})$ with $2 \le m \le n-2$ and m is even, then $\lambda_{\min}(\rho_\zeta(H)) \ge -(n-r)(m-1)!(n-1-m)! \ge -(n-r)(n-3)!$. When n is even, if in addition $\zeta \neq (n-2,2)$ or $(2,2,1^{n-4})$, then $\lambda_{\min}(\rho_\zeta(H)) > -(n-r)(n-3)!$.
- If $\zeta = (n-m,2,1^{m-2})$ with $2 \le m \le n-2$ and m is odd, then $\lambda_{\min}(\rho_\zeta(H)) \ge 0$.

To sum up, we have $\lambda_{\min}(\rho_\zeta(H)) \ge -(n-r)(n-3)!$. For even n, if in addition $\zeta \neq (n-2,1^2)$, $(3,1^{n-3})$, $(n-2,2)$ or $(2,2,1^{n-4})$, then $\lambda_{\min}(\rho_\zeta(H)) > -(n-r)(n-3)!$.

The second decomposition of H is $H = C(n,n-1) \setminus (\dot\cup_{i=1}^{r} P_i)$. This implies $\rho_\zeta(H) = \rho_\zeta(C(n,n-1)) - \sum_{i=1}^{i} \rho_\zeta(P_i)$ for any $\zeta \vdash n$. By Weyl Inequalities, we have:

$$\lambda_{\min}(\rho_\zeta(H)) \ge \lambda_{\min}(\rho_\zeta(C(n,n-1))) - r \cdot \lambda_1(\rho_\zeta(C(n-1,n-1))). \tag{5.5}$$

The unique eigenvalue of $\rho_\zeta(C(n,n-1))$ is given by $|C(n,n-1)| \cdot \tilde\chi_\zeta(\sigma)$, where σ is any element in $C(n,n-1)$ and $\tilde\chi_\zeta(\sigma)$ can be calculated by Lemma 2.8. Substituting Lemma 5.10 into (5.5) gives us the following results:

- If ζ is not a hook or a near hook, then the eigenvalues of $\rho_\zeta(C(n,n-1))$ and $\rho_\zeta(C(n-1,n-1))$ are all 0, and thus by (5.5) we derive $\lambda_{\min}(\rho_\zeta) \ge 0$.
- If $\zeta = (n-m,1^m)$ with $2 \le m \le n-3$ and m is even, then all eigenvalues of $\rho_\zeta(C(n,n-1))$ are 0 and thus

$$\lambda_{\min}(\rho_\zeta(H)) \ge -r \cdot m!(n-2-m)! \ge \begin{cases} -r(n-3)!, & \text{if } n \text{ is odd;} \\ -2r(n-4)!, & \text{if } n \text{ is even.} \end{cases}$$

- If $\zeta = (n-m,1^m)$ with $2 \le m \le n-3$ and m is odd, then $\lambda_{\min}(\rho_\zeta(H)) \ge -r \cdot (m-1)!(n-1-m)! \ge -2r(n-4)!$.
- If $\zeta = (n-m,2,1^{m-2})$ with $2 \le m \le n-2$ and m is even, then the unique eigenvalue of $\rho_\zeta(C(n,n-1))$ is $-(n-m)(n-2-m)!m(m-2)!$ and $\lambda_{\min}(\rho_\zeta(H)) \ge -(n-m)(n-2-m)!m(m-2)! \ge -2(n-2)(n-4)!$.

- If $\zeta = (n-m, 2, 1^{m-2})$ with $2 \leq m \leq n-2$ and m is odd, then the unique eigenvalue of $\rho_\zeta(C(n,n-1))$ is $(n-m)(n-2-m)!m(m-2)!$ and

$$\begin{aligned}\lambda_{\min}(\rho_\zeta(H)) &\geq (n-m)(n-2-m)!m(m-2)! - r\cdot(m-1)!(n-1-m)!\\ &> -(r-1)\cdot(m-1)!(n-1-m)!\\ &\geq \begin{cases} -(r-1)(n-3)!, & \text{if } n \text{ is odd;}\\ -2(r-1)(n-4)!, & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

To sum up, if n is odd and $1 \leq r \leq 2$, then $\lambda_{\min}(\rho_\zeta(H)) \geq -2(n-2)(n-4)!$; if n is odd and $3 \leq r \leq n-2$, then $\lambda_{\min}(\rho_\zeta(H)) \geq -r(n-3)!$; if n is even, then $\lambda_{\min}(\rho_\zeta(H)) \geq -2(n-2)(n-4)!$.

Combining the bounds we built via the two decompositions, we have the following results. If n is odd and $1 \leq r \leq 2$, then

$$\lambda_{\min}(\rho_\zeta(H)) \geq \max\{-(n-r)(n-3)!, -2(n-2)(n-4)!\} = -2(n-2)(n-4)!.$$

If n is odd and $3 \leq r < n/2$, then

$$\lambda_{\min}(\rho_\zeta(H)) \geq \max\{-(n-r)(n-3)!, -r(n-3)!\} = -r(n-3)!.$$

If n is odd and $n/2 < r \leq n-2$, then

$$\lambda_{\min}(\rho_\zeta(H)) \geq \max\{-(n-r)(n-3)!, -r(n-3)!\} = -(n-r)(n-3)!.$$

If n is even and $1 \leq r \leq n-3$, then

$$\lambda_{\min}(\rho_\zeta(H)) \geq \max\{-(n-r)(n-3)!, -2(n-2)(n-4)!\} = -2(n-2)(n-4)!.$$

If n is even and $r = n-2$, then

$$\lambda_{\min}(\rho_\zeta(H)) \geq \max\{-(n-r)(n-3)!, -2(n-2)(n-4)!\} = -(n-r)(n-3)!,$$

and if in addition $\zeta \neq (n-2, 1^2)$, $(3, 1^{n-3})$, $(n-2, 2)$ or $(2, 2, 1^{n-4})$, then

$$\lambda_{\min}(\rho_\zeta(H)) > -(n-r)(n-3)!. \qquad \square$$

Corollary 5.17. *Let* $\Gamma = \mathrm{Cay}(S_n, C(n,n-1;r))$ *with* $n \geq 6$ *even and* $r \in \{1,2,\dots,n-2\}$.

(a) *If* $r=1$, *then the smallest eigenvalue of* Γ *is* $-(n-2)!$ *with multiplicity* $2(n-1)$, *attained exactly by* $(n-1,1)$ *and* $(2,1^{n-2})$.

(b) *If* $2 \leq r \leq n-3$, *then the smallest eigenvalue of* Γ *is* $-(n-r)(n-3)!$ *with multiplicity* $2(n-1)(r-1)$, *attained exactly by* $(n-1,1)$ *and* $(2,1^{n-2})$.

(c) *If* $r=n-2$, *then the smallest eigenvalue of* Γ *is* $-(n-r)(n-3)!$, *which is attained by* $(n-1,1)$ *and* $(2,1^{n-2})$ *and can only be attained possibly by* $(n-1,1)$, $(2,1^{n-2})$, $(n-2,1^2)$, $(3,1^{n-3})$, $(n-2,2)$ *and* $(2,2,1^{n-4})$.

Proof. Since n is even, all permutations in $H = C(n,n-1;r)$ are even. The largest eigenvalue of Γ is $|H| = (n-r)(n-2)!$ with multiplicity 2, attained by (n) and (1^n) simultaneously. When $n=6$ and $1 \leq r \leq 4$, we verify the samllest eigenvalue of Γ via computation in Magma [17]. Now suppose $n \geq 8$.

By Lemma 5.8 and Lemma 2.2, we have

$$\lambda_{\min}(\rho_{(n-1,1)}(H)) = \lambda_{\min}(\rho_{(2,1^{n-2})}(H)) = \begin{cases} -(n-r)(n-3)!, & \text{if } 2 \leq r \leq n-2; \\ -(n-2)!, & \text{if } r=1 \end{cases}$$

with multiplicity

$$b := \begin{cases} r-1, & \text{if } 2 \leq r \leq n-2; \\ 1, & \text{if } r=1. \end{cases}$$

First suppose $r=1$. We find from Lemma 5.16 (d) that for any $\zeta \vdash n$ and $\zeta \neq (n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), we have the smallest eigenvalue $\lambda_{\min}(\rho_\zeta(H)) \geq -2(n-2)(n-4)!$, which is strictly larger than $-(n-2)!$. Thus the smallest eigenvalue of Γ is $-(n-2)!$, which is attained exactly by $(n-1,1)$ and $(2,1^{n-2})$ with multiplicity $d_{(n-1,1)} \cdot b + d_{(2,1^{n-2})} \cdot b = 2(n-1)$.

Next suppose $2 \leq r \leq n-3$. We find from Lemma 5.16 (d) that for any $\zeta \vdash n$ and $\zeta \neq (n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), we have the smallest eigenvalue $\lambda_{\min}(\rho_\zeta(H)) \geq -2(n-2)(n-4)! > -(n-r)(n-3)!$. Thus the smallest eigenvalue of Γ is $-(n-r)(n-$

$3)!$, attained exactly by $(n-1,1)$ and $(2,1^{n-2})$. The multiplicity of this eigenvalue is $d_{(n-1,1)} \cdot b + d_{(2,1^{n-2})} \cdot b = 2(n-1)(r-1)$.

Finally, suppose $r = n-2$. We find from Lemma 5.16 (e) that for any $\zeta \vdash n$ and $\zeta \neq (n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n), we have the smallest eigenvalue $\lambda_{\min}(\rho_\zeta(H)) \geq -(n-r)(n-3)!$. Thus the smallest eigenvalue of Γ is $-(n-r)(n-3)!$, which is attained by $(n-1,1)$ and $(2,1^{n-2})$, and can only be attained possibly by $(n-1,1)$, $(2,1^{n-2})$, $(n-2,1^2)$, $(3,1^{n-3})$, $(n-2,2)$ and $(2,2,1^{n-4})$. ☐

Remark 5.18. Corollary 5.13 can also be derived based on Lemma 5.16. Moreover, when $n=6$, the smallest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-1;n-2))$ is -12 with multiplicity 68, attained exactly by $(n-1,1)$, $(2,1^{n-2})$, $(n-2,1^2)$, $(3,1^{n-3})$, $(n-2,2)$ and $(2,2,1^{n-4})$.

5.4 The case when all cycles have length $k \leq n-2$ and move each of $1,\ldots,r$

In this section, we prove in Theorems 5.22 and 5.24 that $\mathrm{Cay}(S_n, C(n,k;r))$ has the Aldous property whenever $4 \leq k \leq n-2$ and $1 \leq r \leq k-1$. The cases of $k=2$ and 3 have already been solved in [45] and [57], respectively. Before going straight to proving Theorems 5.22 and 5.24, we first solve the case of even $k = n-2$, which acts as the base for induction in the proof of Theorem 5.22.

Lemma 5.19. *Suppose that $n \geq 8$ is even and $\zeta \vdash n$ such that $\zeta \neq (n)$, $(n-1,1)$, $(2,1^{n-2})$ or (1^n). Let $H = C(n,n-2;1)$. If $\zeta = (n-2,2)$ or $(n-m,1^m)$ with $2 \leq m \leq n-3$, then $\lambda_1(\rho_\zeta(H)) \leq 0$; otherwise $\lambda_1(\rho_\zeta(H)) \leq (n-1)(n-4)!$.*

Proof. Note that $H = C(n,n-2) \setminus (C(n,n-2) \cap G_1)$, where G_1 is the stabilizer of 1 in S_n. Similar to the analysis at the start of Section 5.3, the map f_1 sends $C(n,n-2) \cap G_1$ to $C(n-1,n-2)$ and thus for any $\zeta \vdash n$, the spectrum of $\rho_\zeta(C(n,n-2) \cap G_1)$ is the same as that of $\rho_\zeta(C(n-1,n-2))$. By Lemma 1.3 and Weyl Inequalities, we deduce

$$\lambda_1(\rho_\zeta(H)) \leq \lambda_1(\rho_\zeta(C(n,n-2))) - \lambda_{\min}(\rho_\zeta(C(n-1,n-2))). \qquad (5.6)$$

The set $C(n,n-2)$ is the conjugacy class of $(n-2)$-cycles in S_n of size $\binom{n}{2}(n-3)!$. Hence by Schur's Lemma, $\rho_\zeta(C(n,n-2))$ is a scalar matrix with the unique eigenvalue

$|C(n, n-2)| \cdot \tilde{\chi}_\zeta(\sigma)$, where σ is any $(n-2)$-cycle of S_n. Then we can refer to Table 4.3 for all nonzero characters of $(n-2)$-cycles in S_n.

As for $C(n-1, n-2)$, it is the conjugacy class of $(n-2)$-cycles in S_{n-1}. By Branching Rule, for any $\zeta \vdash n$ we have $\rho_\zeta(C(n-1, n-2)) = \oplus_{\zeta^-} \rho_{\zeta^-}(C(n-1, n-2))$. Now each $\rho_{\zeta^-}(C(n-1, n-2))$ is a scalar matrix and we can get its unique eigenvalue by Lemma 2.8.

- If $\zeta = (n-2, 2)$, then we see from Table 4.3 that the unique eigenvalue of $\rho_\zeta(C(n, n-2))$ is $-(n-1)(n-4)!$. The eigenvalues of $\rho_{\zeta^-}(C(n-1, n-2))$ are 0 and $-2(n-3)(n-5)!$ by Lemma 2.8. Thus it follows from (5.6) that $\lambda_1(\rho_\zeta(H)) \leq -(n-1)(n-4)! + 2(n-3)(n-5)! < 0$.

- If $\zeta = (2, 2, 1^{n-4})$, then we have $\lambda_1(\rho_\zeta(C(n, n-2))) = -\lambda_{\min}(\rho_{(n-2,2)}(C(n, n-2))) = (n-1)(n-4)!$, as $\rho_\zeta(C(n, n-2)) = -\rho_{(n-2,2)}(C(n, n-2))$ by Lemma 2.2. Similarly, the eigenvalues of $\rho_{\zeta^-}(C(n-1, n-2))$ are 0 and $2(n-3)(n-5)!$. Thus $\lambda_1(\rho_\zeta(H)) \leq (n-1)(n-4)!$ follows from (5.6).

- If $\zeta = (n-m, 2, 1^{m-2})$ with $3 \leq m \leq n-3$, then we deduce from Table 4.3 that $\lambda_1(\rho_\zeta(C(n, n-2))) = 0$. In this case, ζ^- has the form $(n-1-s, 1^s)$ with $2 \leq s \leq n-4$ or $(n-1-t, 2, 1^{t-2})$ with $2 \leq t \leq n-3$. The eigenvalues of $\rho_{\zeta^-}(C(n-1, n-2))$ are 0 and $(-1)^{t-1}(n-1-t)(n-3-t)!t(t-2)!$ with $2 \leq t \leq n-3$. Thus by (5.6) we deduce $\lambda_1(\rho_\zeta(H)) \leq 2(n-3)(n-5)!$.

- If $\zeta = (n-m, 3, 1^{m-3})$ with $3 \leq m \leq n-3$, then $\lambda_1(\rho_\zeta(C(n, n-2))) = (-1)^m m(n-m)(n-m-1)(m-3)!(n-m-3)!$ by Table 4.3. In this case, $\zeta^- = (n-1-m, 3, 1^{m-3})$ if $n \leq n-4$, $(n-m, 3, 1^{m-4})$ if $n \geq 4$, or $(n-m, 2, 1^{m-3})$. By Lemma 2.8, the eigenvalues of $\rho_{\zeta^-}(C(n-1, n-2))$ are 0 or $(-1)^m(n-m)(n-2-m)!(m-1)(m-3)!$. When m is even, $\lambda_1(\rho_\zeta(H)) \leq m(n-m)(n-m-1)(m-3)!(n-m-3)! \leq 4(n-4)(n-5)(n-7)!$. When m is odd, $\lambda_1(\rho_\zeta(H)) \leq -m(n-m)(n-m-1)(m-3)!(n-m-3)! + (n-m)(n-2-m)!(m-1)(m-3)! < 0$. Overall, we have $\lambda_1(\rho_\zeta(H)) \leq 4(n-4)(n-5)(n-7)!$.

- If $\zeta = (n-m, 2, 2, 1^{m-4})$ with $4 \leq m \leq n-2$, then we have $\lambda_1(\rho_\zeta(C(n, n-2))) = (-1)^m(m-1)(m-2)(n-m+1)(m-4)!(n-m-2)!$. In this case, $\zeta^- = (n-1-m, 2, 2, 1^{m-4})$ if $m \leq n-3$, $(n-m, 2, 2, 1^{m-5})$ if $m \geq 5$, or $(n-m, 2, 1^{m-3})$. By Lemma 2.8, the eigenvalues of $\rho_{\zeta^-}(C(n-1, n-2))$ are 0 or $(-1)^m(n-m)(n-m-2)!(m-1)(m-3)!$. When m is even, $\lambda_1(\rho_\zeta(H)) \leq (m-1)(m-2)(n-m+1)(m-4)!(n-m-2)! \leq 3(n-3)(n-4)(n-6)!$. When m is odd, $\lambda_1(\rho_\zeta(H)) \leq -(m-1)(m-2)(n-m+1)(m-4)!(n-m-2)! + (n-m)(n-m-2)!(m-1)(m-3)! < 0$. Overall, we have $\lambda_1(\rho_\zeta(H)) \leq 3(n-3)(n-4)(n-6)!$.

- For any other $\zeta \vdash n$, Table 4.3 and Lemma 2.8 imply that all the eigenvalues of $\rho_\zeta(C(n,n-2))$ and $\rho_{\zeta^-}(C(n-1,n-2))$ are 0, and thus $\lambda_1(\rho_\zeta(H)) \leq 0$ by (5.6).

To sum up, if $\zeta = (n-2,2)$ or $(n-m,1^m)$ with $2 \leq m \leq n-3$, then $\lambda_1(\rho_\zeta(C(n,n-2;1))) \leq 0$; otherwise $\lambda_1(\rho_\zeta(C(n,n-2;1))) \leq (n-1)(n-4)!$. ☐

Lemma 5.20. *Suppose that $n \geq 8$ is an even integer. The second largest eigenvalue of $\Gamma = \mathrm{Cay}(S_n, C(n,n-2;1))$ is attained uniquely by $(n-1,1)$. In particular, Γ has the Aldous property.*

Proof. Let $H = C(n,n-2;1)$. If $\zeta = (n)$, then the unique eigenvalue of $\rho_\zeta(H)$ is $|H| = \binom{n-1}{2}(n-3)!$, which is the largest eigenvalue of Γ. Similarly, $\zeta = (1^n)$ gives the smallest eigenvalue $-|H| = -\binom{n-1}{2}(n-3)!$ of Γ. By Lemmas 2.2 and 5.7, we have $\lambda_1(\rho_{(n-1,1)}(H)) = n/2(n-3)!$ and $\lambda_1(\rho_{(2,1^{n-2})}(H)) = -\lambda_{\min}(\rho_{(n-1,1)}(H)) = (n-2)!/2$. When $\zeta \neq (n)$, (1^n), $(n-1,1)$ or $(2,1^{n-2})$, we have seen from Lemma 5.19 that $\lambda_1(\rho_\zeta(C(n,n-2;1))) \leq (n-1)(n-4)! < n/2(n-3)!$. Thus the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-2;1))$ is $n/2(n-3)!$, attained uniquely by $(n-1,1)$. ☐

Lemma 5.21. *Suppose that $n \geq 8$ is even and $2 \leq r < (n-1)/2$. The second largest eigenvalue of $\Gamma = \mathrm{Cay}(S_n, C(n,n-2;r))$ is attained uniquely by $(n-1,1)$. In particular, Γ has the Aldous property.*

Proof. Let $H = C(n,n-2;r)$. If $\zeta = (n)$, then $\lambda_1(\rho_\zeta(H)) = |H| = \binom{n-r}{2}(n-3)!$ is the largest eigenvalue of Γ. If $\zeta = (1^n)$, then $\lambda_1(\rho_\zeta(H)) = -|H| = -\binom{n-r}{2}(n-3)!$ is the smallest eigenvalue of Γ. We then derive from Lemmas 2.2 and 5.6 that

$$\begin{aligned}
\lambda_1(\rho_{(n-1,1)}(H)) &= \frac{1}{2}(n-4)!((n-r-1)(n+r-3) + (n-r-3)) \\
&= \frac{1}{2}(n-4)!(n^2 - 3n - r^2 + r)
\end{aligned}$$

and

$$\begin{aligned}
\lambda_1(\rho_{(2,1^{n-2})}(H)) &= -\lambda_{\min}(\rho_{(n-1,1)}(H)) \\
&= \frac{1}{2}(n-4)!(n-r)(n-r-1) \\
&= \frac{1}{2}(n-4)!(n^2-2rn-n+r^2+r) \\
&< \frac{1}{2}(n-4)!(n^2-3n-r^2+r).
\end{aligned}$$

Now that we have verified $\lambda_1(\rho_\zeta(H))$ for $\zeta = (n)$, (1^n), $(n-1,1)$ and $(2,1^{n-2})$, we assume in the following that $\zeta \vdash n$ such that $\zeta \neq (n)$, (1^n), $(n-1,1)$ or $(2,1^{n-2})$.

Notice that $C(n,n-2;i) = C(n,n-2;i-1) \setminus (C(n,n-2;i-1) \cap G_i)$ for any $2 \leq i \leq r$. By a similar analysis as at the beginning of Section 5.3, we know that the spectrum of $\rho_\zeta(C(n,n-2;i-1) \cap G_i)$ is the same as that of $\rho_\zeta(C(n-1,n-2;i-1))$ for any $\zeta \vdash n$. Thus by Weyl Inequalities, we have for any $2 \leq i \leq r$,

$$\lambda_1(\rho_\zeta(C(n,n-2;i))) \leq \lambda_1(\rho_\zeta(C(n,n-2;i-1))) - \lambda_{\min}(\rho_\zeta(C(n-1,n-2;i-1))),$$

and thus

$$\begin{aligned}
\lambda_1(\rho_\zeta(H)) &\leq \lambda_1(\rho_\zeta(C(n,n-2;r-1))) - \lambda_{\min}(\rho_\zeta(C(n-1,n-2;r-1))) \\
&\leq \lambda_1(\rho_\zeta(C(n,n-2;r-2))) - \sum_{j=r-2}^{r-1} \lambda_{\min}(\rho_\zeta(C(n-1,n-2;j))) \\
&\leq \lambda_1(\rho_\zeta(C(n,n-2;1))) - \sum_{j=1}^{r-1} \lambda_{\min}(\rho_\zeta(C(n-1,n-2;j))).
\end{aligned}$$

Branching Rule implies that for $1 \leq j \leq r-1$,

$$\rho_\zeta(C(n-1,n-2;j)) = \bigoplus_{\zeta^-} \rho_{\zeta^-}(C(n-1,n-2;j)),$$

and so $\lambda_{\min}(\rho_\zeta(C(n-1,n-2;j))) = \min_{\zeta^-} \lambda_{\min}(\rho_{\zeta^-}(C(n-1,n-2;j)))$. We then have the following inequality:

$$\lambda_1(\rho_\zeta(H)) \leq \lambda_1(\rho_\zeta(C(n,n-2;1))) - \sum_{j=1}^{r-1} \min_{\zeta^-} \lambda_{\min}(\rho_{\zeta^-}(C(n-1,n-2;j))). \tag{5.7}$$

Now we make use of Lemma 5.19 to estimate $\lambda_1(\rho_\zeta(C(n,n-2;1)))$ and Lemma 5.16 to bound $\lambda_{\min}(\rho_{\zeta^-}(C(n-1,n-2;j)))$ for $1 \le j \le r-1 < (n-3)/2$.

- If $\zeta = (n-2,2)$ or $(n-m,1^m)$ with $2 \le m \le n-3$, then by Lemma 5.19, $\lambda_1(\rho_\zeta(C(n,n-2;1))) \le 0$. In this case, $\zeta^- = (n-3,2)$ or has the form $(n-1-s,1^s)$ with $1 \le s \le n-3$. Lemmas 2.2 and 5.8 give

  $$\lambda_{\min}(\rho_{(n-2,1)}(C(n-1,n-2;j))) = \begin{cases} -(n-3)!, & \text{if } j=1; \\ -(n-1-j)(n-4)!, & \text{if } 2 \le j \le r-1; \end{cases}$$

  $$\lambda_{\min}(\rho_{(2,1^{n-3})}(C(n-1,n-2;j))) = -\lambda_1(\rho_{(n-2,1)}(C(n-1,n-2;j))) = -j(n-4)!.$$

  Parts (a) and (b) in Lemma 5.16 imply that when $\zeta^- = (n-3,2)$ or $\zeta^- = (n-1-s,1^s)$ with $2 \le s \le n-4$,

  $$\lambda_{\min}(\rho_{\zeta^-}(C(n-1,n-2;j))) \ge \begin{cases} -2(n-3)(n-5)!, & \text{if } 1 \le j \le 2; \\ -j(n-4)!, & \text{if } 3 \le j \le r-1. \end{cases}$$

  Therefore, $\lambda_{\min}(\rho_{(n-2,1)}(C(n-1,n-2;j))) = \min_{\zeta^-} \lambda_{\min}(\rho_{\zeta^-}(C(n-1,n-2;j)))$ for every $j \in \{1,2,\ldots,r-1\}$, and by (5.7)

  $$\begin{aligned} \lambda_1(\rho_\zeta(C(n,n-2;r))) &\le (n-3)! + (n-4)! \sum_{j=2}^{r-1}(n-1-j) \\ &= \frac{1}{2}(n-4)!(2nr - r^2 - r - 2n) \\ &< \frac{1}{2}(n-4)!(n^2 - 3n - r^2 + r). \end{aligned}$$

- For any other ζ, we know from Lemma 5.19 that $\lambda_1(\rho_\zeta(C(n,n-2;1))) \le (n-1)(n-4)!$. In this case, $\zeta^- \ne (n-1)$, $(n-2,1)$ or (1^{n-1}). We have seen that

  $$\lambda_{\min}(\rho_{(2,1^{n-3})}(C(n-1,n-2;j))) = -j(n-4)!.$$

We also know from parts (a) and (b) of Lemma 5.16 that when $\zeta^- \neq (n-1)$, $(n-2,1)$, $(2,1^{n-3})$ or (1^{n-1}),

$$\lambda_{\min}(\rho_{\zeta^-}(C(n-1,n-2;j))) \geq \begin{cases} -2(n-3)(n-5)!, & \text{if } 1 \leq j \leq 2; \\ -j(n-4)!, & \text{if } 3 \leq j \leq r-1. \end{cases}$$

Note that $-2(n-3)(n-5) \leq -j(n-4)!$ for $1 \leq j \leq 2$. Hence we get by (5.7)

$$\begin{aligned} \lambda_1(\rho_\zeta(C(n,n-2;r))) &\leq (n-1)(n-4)! + 4(n-3)(n-5)! + (n-4)! \sum_{j=3}^{r-1} j \\ &< \frac{1}{2}(n-4)!(n^2-3n-r^2+r). \end{aligned}$$

To sum up, the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-2;r))$ is attained uniquely by $(n-1,1)$. □

Recall from Lemma 5.4 and its remark that the eigenvalue

$$\mu_2(n,k;r) = (k-2)!\binom{n-r}{k-r}\frac{1}{n-r}\left((k-1)(n-k) - \frac{(k-r-1)(k-r)}{n-r-1}\right) \tag{5.8}$$

of $\mathrm{Cay}(S_n, C(n,k;r))$ discovered in [96, Theorem 1.3] is precisely $\lambda_2(\mathbf{B})$ with $\mathbf{B}$ the quotient matrix of the equitable partition Π_i of $\mathrm{Cay}(S_n, C(n,k;r))$, and that $\mu_2(n,k;r)$ is also the largest eigenvalue of the standard representation $\rho_{(n-1,1)}$ of S_n on $C(n,k;r)$, that is, $\alpha_1(\rho_{(n-1,1)}(C(n,k;r)))$. Consequently, $\mu_2(n,k;r)$ provides a lower bound for $\alpha_2(\mathrm{Cay}(S_n, C(n,k;r)))$. One can directly verify from (5.8) the recurrence relation

$$\mu_2(n,k;r) = \mu_2(n-1,k;r) + \mu_2(n,k;r+1) \tag{5.9}$$

for any $1 \leq r \leq k-2$ and $k \leq n-2$.

According to Lemma 5.9, when k is even, $H = C(n,k;r)$ generates S_n. In this case, for $r \in \{1,2,\dots,k-2\}$ and $j \in [n] \setminus [r]$, the three graphs $\mathrm{Cay}(S_n, H)$, $\mathrm{Cay}(S_n, H \setminus (H \cap G_j))$ and $\mathrm{Cay}(G_j, H \cap G_j)$ are all connected, and the strictly second largest eigenvalue of each of them is exactly their second largest eigenvalue. Now we make use of Lemma 5.4 to confirm the Aldous property of $\mathrm{Cay}(S_n, C(n,k;r))$ when k is even and $k \leq n-2$.

Theorem 5.22. *If k is even and $4 \leq k \leq n-2$, then* $\mathrm{Cay}(S_n, C(n,k;r))$ *has the Aldous property for every $r \in \{1,2,\dots,k-1\}$.*

Proof. Theorem 5.2 indicates that $\mathrm{Cay}(S_n, C(n,k;r))$ with $r = k-1$ has the Aldous property, and thus $\lambda_2(\mathrm{Cay}(S_n, C(n,k;k-1))) = \mu_2(n,k;k-1)$.

First assume $k = n-2$. We verify via computation in MAGMA [17] that $(5,1)$ is the unique partition attaining the second largest eigenvalue of $\mathrm{Cay}(S_6, C(6,4;r))$ for $r = 1$, 2 and 3. Thus this theorem is true when $k = 4$ and $1 \leq r \leq 3$. Lemmas 5.20 and 5.21 show that when $k \geq 6$ and $1 \leq r < (n-1)/2$, the conclusion of this theorem still holds. Now suppose $k \geq 6$ and $(n-1)/2 \leq r \leq k-1 = n-3$, and we prove the theorem by induction on $n-r$. When $n-r = 3$, that is, $r = n-3$, we know from Theorem 5.2 that $\lambda_2(\mathrm{Cay}(S_n, C(n,n-2;n-3))) = \mu_2(n,n-2;n-3)$. Suppose $\lambda_2(\mathrm{Cay}(S_n, C(n,n-2;n-t))) = \mu_2(n,n-2;n-t)$ for some $t \in \{3,4,\ldots,(n-2)/2\}$. Let $n-r = t+1$, that is, $r = n-t-1 \in \{n/2, (n+2)/2, \ldots, n-4\}$, and let $H = C(n,n-2;r)$. By Lemma 5.4, if λ is an eigenvalue of $\mathrm{Cay}(S_n, H)$ which is not an eigenvalue of $\mathbf{B}$, then

$$\lambda \leq \lambda_2\left(\mathrm{Cay}\left(G_n, H \cap G_n\right)\right) + \lambda_2\left(\mathrm{Cay}\left(S_n, H \backslash \left(H \cap G_n\right)\right)\right). \tag{5.10}$$

Here $\mathrm{Cay}(G_n, H \cap G_n)$ is isomorphic to $\mathrm{Cay}(S_{n-1}, C(n-1,n-2;r))$ and $\mathrm{Cay}(S_n, H\backslash(H \cap G_n))$ is isomorphic to $\mathrm{Cay}(S_n, C(n,n-2;r+1))$. By the induction hypothesis, we have $\lambda_2(\mathrm{Cay}(S_n, C(n,n-2;r+1))) = \mu_2(n,n-2;r+1)$. As $(n-1)/2 < r \leq n-4$, Theorem 5.12 (d) indicates that the second largest eigenvalue of $\mathrm{Cay}(S_{n-1}, C(n-1,n-2;r))$ is attained uniquely by $(n-1,1)$, and thus $\lambda_2\left(\mathrm{Cay}\left(G_n, H \cap G_n\right)\right) = \mu_2(n-1,n-2;r)$. Then the right hand side of (5.10) is $\mu_2(n-1,n-2;r) + \mu_2(n,n-2;r+1)$, which is exactly $\mu_2(n,n-2;r)$ by the recurrence relation (5.9). Thus (5.10) gives $\lambda \leq \mu_2(n,n-2;r)$. Combining this with [96, Theorem 1.3], which states that $\mu_2(n,n-2;r)$ is a lower bound for $\lambda_2(\mathrm{Cay}(S_n, C(n,n-2;r)))$, we conclude that $\mu_2(n,n-2;r)$ is the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,n-2;r))$. Therefore, $\mathrm{Cay}(S_n, C(n,n-2;r))$ has the Aldous property for every $r \in \{1,2,\ldots,n-3\}$.

Now we conclude the proof by induction on $n-k$. Suppose the conclusion holds when $n-k = i-1$ for some $i \in \{3,4,\ldots,n-4\}$. Next let $n-k = i$. Since Theorem 5.2 states that $\lambda_2(\mathrm{Cay}(S_n, C(n,n-i;n-i-1))) = \mu_2(n,n-i;n-i-1)$, we assume $r \in \{1,2,\ldots,n-i-2\}$ and let $H = C(n,n-i;r)$. By Lemma 5.4, if λ is an eigenvalue of $\mathrm{Cay}(S_n, H)$ which is not an eigenvalue of $\mathbf{B}$, then

$$\lambda \leq \lambda_2(\mathrm{Cay}(S_{n-1}, C(n-1,n-i;r))) + \lambda_2(\mathrm{Cay}(S_n, C(n,n-i;r+1))).$$

As the conclusion of this theorem holds when $n-k=i-1$, we have $\lambda_2(\mathrm{Cay}(S_{n-1}, C(n-1, n-i; r)) = \mu_2(n-1, n-i; r)$. Then based on the recurrence relation (5.9) and [96, Theorem 1.3], the equality $\lambda_2(\mathrm{Cay}(S_n, C(n, n-i; r+1))) = \mu_2(n, n-i; r+1)$ implies $\lambda_2(\mathrm{Cay}(S_n, H)) = \mu_2(n, n-i; r)$. Hence we conclude that $\mathrm{Cay}(S_n, C(n, n-i; r))$ has the Aldous property for every $r \in \{1, 2, \ldots, n-i-1\}$.

We finally arrive at that when k is even and $4 \leq k \leq n-2$, for any $1 \leq r \leq k-1$, the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n,k;r))$ equals $\mu_2(n,k;r)$, which is exactly $\alpha_1(\rho_{(n-1,1)}(C(n,k;r)))$. In particular, $\mathrm{Cay}(S_n, C(n,k;r))$ has the Aldous property. ☐

When k is odd, $H = C(n,k;r)$ only generates A_n according to Lemma 5.9. If $1 \leq r \leq k-2$ and $j \in [n] \setminus [r]$, then the three graphs $\mathrm{Cay}(S_n, H)$, $\mathrm{Cay}(S_n, H \setminus (H \cap G_j))$ and $\mathrm{Cay}(G_j, H \cap G_j)$ all have two isomorphic components and their largest eigenvalues all have multiplicity 2. In this case, $\mathrm{Cay}(S_n, H)$ has the Aldous property if and only if

$$\lambda_3(\mathrm{Cay}(S_n, C(n,k;r))) = \mu_2(n,k;r) \quad \text{for } 1 \leq r < k \leq n-2. \tag{5.11}$$

Thus when k is odd, Lemma 5.4 is not strong enough for proving (5.11). Here we have two ways to fix this situation. The first way is to make the alternating group A_n as the underlying group and apply [58, Theorem 7] (see Lemma 2.14) to $\mathrm{Cay}(A_n, C(n,k;r))$. Since $\lambda_2(\mathrm{Cay}(A_n, C(n,k;r))) = \lambda_3(\mathrm{Cay}(S_n, C(n,k;r)))$, to prove (5.11) we only need to show $\lambda_2(\mathrm{Cay}(A_n, C(n,k;r))) = \mu_2(n,k;r)$. The second way, which we adopt here, is that with some additional discussion, we can get a stronger version of Lemma 5.4, which still has S_n as the underlying group. The key point is that when k is odd, all permutations in $H = C(n,k;r)$ are even and thus $\rho_\zeta(H) = \rho_{\zeta'}(H)$ for any $\zeta \vdash n$. From the other point of view, if λ_0 is an eigenvalue of $\mathrm{Cay}(S_n, H)$ with f an λ_0-eigenvector, then $g := f \cdot \mathrm{sgn}$ is also an λ_0-eigenvector of $\mathrm{Cay}(S_n, H)$. In fact, for any $\sigma \in S_n$,

$$\begin{aligned} Ag(\sigma) &= \sum_{h \in H} g(\sigma h) = \sum_{h \in H} f(\sigma h) \cdot \mathrm{sgn}(\sigma h) = \sum_{h \in H} f(\sigma h) \cdot \mathrm{sgn}(\sigma) \\ &= \mathrm{sgn}(\sigma) \cdot \sum_{h \in H} f(\sigma h) = \mathrm{sgn}(\sigma) \cdot \lambda_0 f(\sigma) = \lambda_0 g(\sigma), \end{aligned}$$

where A denotes the adjacency matrix of $\mathrm{Cay}(S_n, H)$. Here we use the facts that $\mathrm{sgn}(h) = 1$ for every $h \in H$ and $Af(\sigma) = \sum_{h \in H} f(\sigma h) = \lambda_0 f(\sigma)$ for any $\sigma \in S_n$. In particular, the all one vector $\mathbf{1}$ and the vector $\mathbf{sgn}$ with its σ-entry $\mathrm{sgn}(\sigma)$ are the two eigenvectors of the largest eigenvalue $|H|$ of $\mathrm{Cay}(S_n, H)$. For $i \in [n]$, let $\mathbf{P}_{\Pi_i}$ denote the *characteristic matrix* of the partition Π_i of $\mathrm{Cay}(S_n, H)$ given in (2.1), which is the $n! \times n$ matrix with columns the characteristic vectors of Π_i. If λ is an eigenvalue of $\mathrm{Cay}(S_n, H)$ which is not an eigenvalue of $\mathbf{B}$ and f is a λ-eigenvector of $\mathrm{Cay}(S_n, H)$, then f is not only orthogonal to the column space of $\mathbf{P}_{\Pi_i}$ for every $i \in [n]$ but also orthogonal to the column space of $\tilde{\mathbf{P}}_{\Pi_i}$, which is obtained by multiplying each $(\sigma, G_{j,i})$-entry of $\mathbf{P}_{\Pi_i}$ by $\mathrm{sgn}(\sigma)$. With these additional conditions on f, we deduce the following lemma by applying [58, Theorem 7] (see Lemma 2.14) to $\mathrm{Cay}(S_n, C(n,k;r))$ for odd k.

Lemma 5.23. *Let $H = C(n,k;r)$ and $\Gamma = \mathrm{Cay}(S_n, H)$, where $n \geq 5$ and $1 \leq r < k < n$. The right coset decomposition Π_i of S_n given in (2.1) leads to an equitable partition of Γ, and the corresponding quotient matrix $\mathbf{B} = \mathbf{B}_{\Pi_i}$ is symmetric and independent of the choice of $i \in [n]$. Moreover, if k is odd and λ is an eigenvalue of Γ which is not an eigenvalue of $\mathbf{B}$, then, for each $j \in [n]$, we have*

$$\lambda \leqslant \lambda_3\left(\mathrm{Cay}\left(G_j, H \cap G_j\right)\right) + \lambda_3\left(\mathrm{Cay}\left(S_n, H \backslash \left(H \cap G_j\right)\right)\right),$$

where G_j is the stabilizer of j in S_n.

Using Lemma 5.23, the proof of the following theorem is similar to that of Theorem 5.22.

Theorem 5.24. *If k is odd and $5 \leq k \leq n-2$, then $\mathrm{Cay}(S_n, C(n,k;r))$ has the Aldous property for every $r \in \{1, 2, \ldots, k-1\}$.*

Proof. Theorem 5.2 indicates that $\mathrm{Cay}(S_n, C(n,k;r))$ with $r = k-1$ has the Aldous property, and thus $\lambda_3(\mathrm{Cay}(S_n, C(n,k;k-1))) = \mu_2(n,k;k-1)$.

First assume $k = n-2$. We verify via computation in Magma [17] that the strictly second largest eigenvalue of $\mathrm{Cay}(S_7, C(7,5;r))$ with $r = 1$, 2, 3 and 4 is attained exactly by $(6,1)$ and $(2,1^5)$. Thus this theorem is true when $k = 5$ and $1 \leq r \leq 4$. Now suppose $k \geq 7$ and $1 \leq r \leq k-1 = n-3$. We prove the conclusion of this theorem by induction on $n-r$. When $n-r = 3$, that is, $r = n-3$, we know from Theorem 5.2 that $\lambda_3(\mathrm{Cay}(S_n, C(n,n-2;n-3))) = \mu_2(n,n-2;n-3)$. Suppose $\lambda_3(\mathrm{Cay}(S_n, C(n,n-2;n-t))) = \mu_2(n,n-2;n-t)$ for some $t \in \{3,4,\ldots,n-2\}$. Let $n-r = t+1$, that is, $r = n-t-1 \in \{1,2,\ldots,n-4\}$, and let $H = C(n,n-2;r)$. By Lemma 5.23, if λ is an eigenvalue of $\mathrm{Cay}(S_n, H)$ which is not an eigenvalue of $\mathbf{B}$, then

$$\lambda \leq \lambda_3\left(\mathrm{Cay}\left(G_n, H\cap G_n\right)\right) + \lambda_3\left(\mathrm{Cay}\left(S_n, H\backslash\left(H\cap G_n\right)\right)\right). \tag{5.12}$$

Here $\mathrm{Cay}(G_n, H\cap G_n)$ is isomorphic to $\mathrm{Cay}(S_{n-1}, C(n-1, n-2; r))$ and $\mathrm{Cay}(S_n, H\setminus (H\cap G_n))$ is isomorphic to $\mathrm{Cay}(S_n, C(n, n-2; r+1))$. By the induction hypothesis, we have $\lambda_3(\mathrm{Cay}(S_n, C(n, n-2; r+1))) = \mu_2(n, n-2; r+1)$. Theorem 5.14 (b) indicates that the strictly second largest eigenvalue of $\mathrm{Cay}(S_{n-1}, C(n-1, n-2; r))$ is attained by $(n-2, 1)$, and thus $\lambda_3\left(\mathrm{Cay}\left(G_n, H\cap G_n\right)\right) = \mu_2(n-1, n-2; r)$. Then the right hand side of (5.12) is $\mu_2(n-1, n-2; r) + \mu_2(n, n-2; r+1)$, which is exactly $\mu_2(n, n-2; r)$ by the recurrence relation (5.9). Thus (5.12) gives $\lambda \leq \mu_2(n, n-2; r)$. Combining this with [96, Theorem 1.3], which states that $\mu_2(n, n-2; r)$ is a lower bound for $\lambda_3(\mathrm{Cay}(S_n, C(n, n-2; r)))$, we conclude that $\mu_2(n, n-2; r)$ is the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, n-2; r))$. Therefore, $\mathrm{Cay}(S_n, C(n, n-2; r))$ has the Aldous property for every $r \in \{1, 2, \dots, n-3\}$.

Now we conclude the proof by induction on $n-k$. Suppose the conclusion holds when $n-k = i-1$ for some $i \in \{3, 4, \dots, n-4\}$. Next let $n-k = i$. Since Theorem 5.2 states that $\lambda_3(\mathrm{Cay}(S_n, C(n, n-i; n-i-1))) = \mu_2(n, n-i; n-i-1)$, we assume $r \in \{1, 2, \dots, n-i-2\}$ and let $H = C(n, n-i; r)$. By Lemma 5.4, if λ is an eigenvalue of $\mathrm{Cay}(S_n, H)$ other than that of $\mathbf{B}$, then

$$\lambda \leq \lambda_3(\mathrm{Cay}(S_{n-1}, C(n-1, n-i; r))) + \lambda_3(\mathrm{Cay}(S_n, C(n, n-i; r+1))).$$

As this theorem holds when $n-k = i-1$, we have $\lambda_3(\mathrm{Cay}(S_{n-1}, C(n-1, n-i; r)) = \mu_2(n-1, n-i; r)$. Then based on the recurrence relation (5.9) and [96, Theorem 1.3], the equality $\lambda_3(\mathrm{Cay}(S_n, C(n, n-i; r+1))) = \mu_2(n, n-i; r+1)$ implies $\lambda_3(\mathrm{Cay}(S_n, H)) = \mu_2(n, n-i; r)$. Hence we conclude that $\mathrm{Cay}(S_n, C(n, n-i; r))$ has the Aldous property for every $r \in \{1, 2, \dots, n-i-1\}$.

We finally arrive at that when k is odd and $5 \leq k \leq n-2$, for any $1 \leq r \leq k-1$, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, k; r))$ equals $\mu_2(n, k; r)$, which is exactly $\alpha_1(\rho_{(n-1,1)}(C(n, k; r)))$. In particular, $\mathrm{Cay}(S_n, C(n, k; r))$ has the Aldous property. □

Chapter 6

Concluding remarks and open problems

This chapter summarizes the pending tasks from Chapters 3–5 and outlines several open problems related to Aldous' spectral gap conjecture, offering potential avenues for future research in this field.

6.1 Unsolved problems from our research

Recall that in Chapter 3, we define for any $\emptyset \neq I \subseteq \{2, 3, \ldots, n-1, n\}$ the normal Cayley graph $\mathrm{Cay}(S_n, T(n, I))$ with $T(n, I) = \{\sigma \in S_n \mid |\mathrm{supp}(\sigma)| \in I\}$. Theorems 3.2 and 3.3 and Corollary 3.4 together imply that, as far as the Aldous property of $\mathrm{Cay}(S_n, T(n, I))$ for sufficiently large n is concerned, the only unsettled case is the one in which $|I| \geq 2$, $|I \cap \{n-1, n\}| = 1$ and $I \neq \{2, 3, \ldots, n-1\}$. Our attempt to solve this case suggest that the subcases where $I \cap \{n-1, n\} = \{n-1\}$ and $I \cap \{n-1, n\} = \{n\}$, respectively, may need separate treatments as they behave differently. So we propose the following two problems separately.

Problem 6.1. [74, Problem 1.1] Give a necessary and sufficient condition under which $\mathrm{Cay}(S_n, T(n, I))$ with $\{n-1\} \subset I \subset \{2, 3, \ldots, n-2, n-1\}$ has the Aldous property for sufficiently large n.

Problem 6.2. [74, Problem 1.2] Give a necessary and sufficient condition under which $\mathrm{Cay}(S_n, T(n, I))$ with $\{n\} \subset I \subseteq \{2, 3, \ldots, n-2, n\}$ has the Aldous property for sufficiently large n.

In Chapter 4, we examine the normal Cayley graphs on S_n ($n \geq 7$) generated by cycles, denoted as $\mathrm{Cay}(S_n, C(n, I))$, where $C(n, I)$ represents the set of cycles in S_n with lengths in I. According to Corollary 4.6, when I contains neither $n-1$ nor n, we can determine precisely when $\mathrm{Cay}(S_n, C(n, I))$ processes the Aldous property. Additionally, Corollaries 4.22 and 4.26 indicate that $\mathrm{Cay}(S_n, C(n, I))$ does not have the Aldous property whenever $n \in I$. Thus we propose the following problem.

Problem 6.3. Give a necessary and sufficient condition for $\mathrm{Cay}(S_n, C(n, I))$ with $\{n-1\} \subseteq I \subseteq \{2, 3, \ldots, n-1\}$ to have the Aldous property when $n \geq 7$.

Note that Corollary 4.16 partially addresses Problem 6.3. It demonstrates that when n is even and the set $\{n-1\} \subseteq I \subseteq \{2, 3, \ldots, n-1\}$ contains at least one even number, $\mathrm{Cay}(S_n, C(n, I))$ does not exhibit the Aldous property.

Recall that for $1 \leq r < k < n$, the set $C(n, k; r)$ consists of k-cycles of S_n that move every point from 1 to r. We finally propose the following conjecture on the strictly second largest eigenvalue of the nonnormal Cayley graph $\mathrm{Cay}(S_n, C(n, k; r))$ with $1 \leq r < k \leq n-2$, which is stronger than Theorems 5.22 and 5.24.

Conjecture 6.4. [75, Conjecture 4.7] Suppose that $n \geq 5$ and $1 \leq r < k \leq n-2$. When k is even, the second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, k; r))$ is attained uniquely by the standard representation $\rho_{(n-1,1)}$. When k is odd, the strictly second largest eigenvalue of $\mathrm{Cay}(S_n, C(n, k; r))$ is attained exactly by $\rho_{(n-1,1)}$ and $\rho_{(2,1^{n-2})}$.

The proofs of Theorems 5.22 and 5.24 rely on Lemmas 5.4 and 5.23, and we are only able to show that if λ is an eigenvalue of $\mathrm{Cay}(S_n, C(n, k; r))$ which is not an eigenvalue of $\mathbf{B}$, then

$$\lambda \leq \mu_2(n, k; r). \tag{6.1}$$

However, the above conjecture requires (6.1) to be a strict inequality, that is, $\lambda < \mu_2(n, k; r)$. Note that Lemmas 5.20 and 5.21 state that this conjecture is true for $\mathrm{Cay}(S_n, C(n, n-2; r))$ with n even and $1 \leq r \leq (n-1)/2$.

6.2 Open problems

We conclude this thesis by mentioning several significant problems and conjectures related to Aldous' spectral gap conjecture that merit attention.

Recall that Aldous' spectral gap conjecture states that for any weighted graph Γ, the two continuous-time Markov Chains on Γ, the interchange process and the random walk, have the same spectral gap. This conjecture was confirmed by Caputo, Liggett, and Richthammer [20] (see Theorem 1.5). Caputo [3] then conjectured that Aldous' spectral gap conjecture should be extended to larger classes of continuous-time Markov chains by replacing graphs with hypergraphs. A hypergraph $\tilde{\Gamma}$ with vertex set $V = \{1, 2, \dots, n\}$ has a collection of distinct subsets of V as edges. For each $A \subset \{1, 2, \dots, n\}$, assign to A a weight $\alpha_A \geq 0$ such that A is an edge of $\tilde{\Gamma}$ if and only if $\alpha_A > 0$. Let $S_{n,A}$ be the set of permutations of S_n such that the support of each $\sigma \in S_{n,A}$ is contained in A, that is,

$$S_{n,A} = \{\sigma \in S_n \mid \operatorname{supp}(\sigma) \subseteq A\}.$$

The *α-shuffle process* is a continuous-time Markov chain with state space S_n, where a state is an assignment of n distinct particles to the n vertices of $\tilde{\Gamma}$, ensuring that each vertex is occupied by one particle. In one transition, for each A independently, at rate α_A pick a uniformly random permutation in $S_{n,A}$ and move around accordingly the particles at the vertices in A. When following just one particle in the α-shuffle process, we get the random walk with state space V and transition rates

$$c_\alpha(i, j) = \sum_{A \subset V : i,j \in A} \frac{\alpha_A}{|A|}.$$

Caputo's conjecture [3] states that for any weighted hypergraph $\tilde{\Gamma}$, the α-shuffle process on $\tilde{\Gamma}$ and the random walk on V with rates $c_\alpha(\cdot, \cdot)$ have the same spectral gap. Refer to [89] for the solution of the "simple" case of Caputo's conjecture. For an equivalent form of Caputo's conjecture, see [23, The α-shuffles Conjecture]. We present this conjecture from the perspective of algebraic graph theory.

Conjecture 6.5. Let $n \geq 3$ be a positive integer. For each $A \subset \{1, 2, \dots, n\}$, let $\alpha_A \geq 0$ and let $\omega = \sum_{A \subset \{1,2,\dots,n\}} \alpha_A S_{n,A}^{+} \in \mathbb{C}S_n$. If $\operatorname{supp}(\omega)$ generates S_n, then the second largest eigenvalue of $\operatorname{Cay}(S_n, \omega)$ is achieved by the standard representation $\rho_{(n-1,1)}$ of S_n.

As a generalization of Aldous' spectral gap conjecture, G. Alon, G. Kozma, and D. Puder [6] confirmed the following case of Caputo's conjecture.

Theorem 6.6. [6, Theorem 1.3] *Let* $\tilde{\Gamma}$ *be a weighted hypergraph with vertex set* V *and nonnegative weights. If, for a fixed subset* B *of* V*, every edge* A *of* $\tilde{\Gamma}$ *satisfies* $B \subseteq A$ *and* $|A \setminus B| \leq 2$*, then the* α*-shuffle process on* $\tilde{\Gamma}$ *and the corresponding random walk have the same spectral gap.*

In Chapter 2, we show that $\Gamma = \mathrm{Cay}(S_n, S)$ has Aldous property if and only if Γ and $\mathrm{Sch}(S_n, G_i, S)$ have the same strictly second largest eigenvalue, where G_i is the stabilizer of $i \in [n]$ in S_n. Here $\mathrm{Sch}(S_n, G_i, S)$ is isomorphic to the quotient graph Γ/Π_i of Γ, where Π_i is the equitable partition of Γ formed by right cosets of G_i. Thus $\mathrm{Sch}(S_n, G_i, S)$ depicts the action of S_n on $[n]$ with respect to S. In [88], the authors consider the Schreier coset graph $\mathrm{Sch}(S_n, S_{n-k}, S)$ depicting the action of S_n on $[n]_k$, the set of k-tuples of distinct elements from $[n]$, for some fixed k. Let $\rho_{[n]_k}(\sigma)$ be the permutation matrix depicting the action of $\sigma \in S_n$ on $[n]_k$. For some $\omega = \sum_{\sigma \in S_n} \omega_\sigma \sigma \in \mathbb{R}S_n$, denote

$$\rho_{[n]_k}(\omega) = \sum_{\sigma \in S_n} \omega_\sigma \rho_{[n]_k}(\sigma), \quad \rho^{\mathrm{sgn}}_{[n]_k}(\omega) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\omega_\sigma \rho_{[n]_k}(\sigma).$$

Let $\alpha_2(k, \omega)$ (respectively, $\alpha_2^{\mathrm{sgn}}(k, \omega)$) be the strictly second largest eigenvalue of $\rho_{[n]_k}(\omega)$ (respectively, $\rho^{\mathrm{sgn}}_{[n]_k}(\omega)$). Note that the representation $\rho_{[n]_k}$ decomposes into all irreducible representations ρ_ζ of S_n that have at most k boxes outside the first row of $\zeta \vdash n$ (with some multiplicities). Similarly, $\rho^{\mathrm{sgn}}_{[n]_k}$ decomposes into all irreducible representations ρ_ζ of S_n that have at most k boxes outside the first column of $\zeta \vdash n$. Parzanchevski and Puder [88, Theorem 1.9] proved in the following theorem that no finite set of irreducible representations are sufficient to capture $\alpha_2(\mathrm{Cay}(S_n, \omega))$, where ω is any nonnegative normal element in $\mathbb{R}S_n$.

Theorem 6.7. [88, Theorem 1.9] *For every* $k \geq 1$ *and every large enough* n*, there is a nonnegative normal element* $\omega \in \mathbb{R}S_n$ *such that*

$$\max\{\alpha_2(k, \omega), \alpha_2^{\mathrm{sgn}}(k, \omega)\} < \alpha_2(\mathrm{Cay}(S_n, \omega)).$$

The following conjecture was raised by Gady Kozma and Doron Puder.

Conjecture 6.8. [88, Conjecture 1.11] There is an integer $k \geq 4$ and a universal constant $0 < c < 1$ such that for large enough n and for every symmetric nonnegative $\omega \in \mathbb{R}S_n$,

$$|\omega| - \alpha_2(\mathrm{Cay}(S_n, \omega)) \geq c \cdot [|\omega| - \max\{\alpha_2(k, \omega), \alpha_2^{\mathrm{sgn}}(k, \omega)\}].$$

This conjecture, if true, would imply that random pairs of permutations in S_n generate a uniform family of expanders, addressing a long-standing open question [79, Problem 2.28]. Additionally, it would confirm a conjecture by Babai [10, Conjecture 1.7], which states that for any generating set S of A_n, the diameter of $\mathrm{Cay}(A_n, S)$ is bounded by some n^c, where c is a universal constant. For details, see Section 4 in [88].

For any $\sigma \in S_n$, there exists a unique partition $[n] = I_1 \cup I_2 \cup \cdots \cup I_m$ such that $\sigma(I_i) = I_i$ and each I_i consists of consecutive numbers in $[n]$. More specifically, for each I_i, there exists a pair of numbers $a \leq b$ such that $I_i = \{a, a+1, \ldots, b\}$. When this partition has m parts, we say σ is an *m-reducible permutation*. In [34, 35], Dai introduced the *Full-Flag Johnson graph* $FJ(n, r)$ $(r < n)$ as a variant of Johnson graphs and investigated its combinatorial properties. He showed that $FJ(n, r)$ is isomorphic to $\mathrm{Cay}(S_n, RP^{(r)})$, where $RP^{(r)}$ is the set of $(n-r)$-reducible permutations of S_n. Note that $RP^1 = \{(i, i+1) \mid 1 \leq i \leq n-1\}$. In [35], Dai calculated all eigenvalues of the quotient graph $FJ(n, 1)/\Pi_i$ of $FJ(n, 1) = \mathrm{Cay}(S_n, RP^{(1)})$ with respect to the equitable partition Π_i defined in (2.1). He further conjectured that $FJ(n, 1)$ and $FJ(n, 1)/\Pi_i$ have the same second-largest eigenvalue, which follows immediately from Aldous' spectral gap conjecture.

Huang et al. proposed one conjecture and one problem regarding $FJ(n, r)$ for $2 \leq r \leq n-1$, formulated in different but equivalent forms.

Conjecture 6.9. [58, Conjecture 23] For $n \geq 4$, the Full-Flag Johnson graph $FJ(n, 2)$ has the Aldous property.

Problem 6.10. [58, Problem 25] For $3 \leq r \leq n-1$, does the Full-Flag Johnson graph $FJ(n, r)$ always possess the Aldous property?

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