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AN OUTPUT DEAD BEAT CONTROLLABILITY TEST FOR A CLASS OF ODD POLYNOMIAL SYSTEMS *

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Abstract

An output dead beat controllability test that stops after finitely many rational operations is presented for a class of odd polynomial systems. The test is based on the Gröbner basis method but it needs to be facilitated in general with the QEPCAD symbolic computation package. Geometric properties due to which output dead beat controllability may be lost are identified and analysed. The computational requirements are smaller when compared with the known output dead beat controllability tests.

1 Introduction

Discrete time polynomial systems represent a very large class of nonlinear systems that can model systems ranging from a diesel generator [2] and heat exchanger [6] to effects of drugs [5]. Recently, we used symbolic computation packages, such as Gröbner basis method [4] and QEPCAD [3], to deal with state and output dead beat controllability problems for classes of polynomial systems [9, 10]. In particular, a class of the so called odd polynomial systems was first investigated in [7], where only scalar polynomial systems were considered. More general classes of odd polynomial systems were considered in [9, 8].

Output dead beat controllability of the most general class of odd polynomial was considered in [9]. We showed how it is possible to compute the so called invariant sets and how they can be used to decide whether the system is output dead beat controllable. The computations are carried out using the Gröbner basis method and QEPCAD. However, the output dead beat controllability test

may not terminate after finitely many rational operations, which is not desirable. In [8] we considered a class of second order odd polynomial systems and proposed an output dead beat controllability test. The test is equivalent to a polynomial decomposition and checking whether a set of polynomial divisions is satisfied or not. The importance of the so called strongly invariant sets for output dead beat controllability was shown.

In this paper, we generalise results in [8] to systems of arbitrary order. The class of systems we consider here is a subclass of odd systems considered in [9]. We propose a new output dead beat controllability test which is based on the algorithm for computation of the maximal invariant set given in [9]. It is important to say that the controllability test stops after finite time due to the special structure of odd systems that we consider. The test is based on a repeated use of the Gröbner basis method for computation of all the strongly invariant sets. We explain our method using an illustrative example.

2 Preliminaries

The set of real numbers is denoted as $\mathbb{R}$. Given polynomials $f_1, f_2, \dots, f_s \in \mathbb{R}[x_1, \dots, x_n]$ we denote the ideal generated by the polynomials and their Gröbner basis respectively as $\langle f_1, \dots, f_s \rangle$ and $Gbasis[f_1, \dots, f_n]$ (for more details on algebraic geometry and Gröbner bases see [4]). Real variety defined by these polynomials is denoted as $V(f_1, \dots, f_s) = \{x \in \mathbb{R}^n : f_1 = 0, \dots, f_s = 0\}$. Given two polynomials $f_1, f_2 \in \mathbb{R}[x_1, \dots, x_n]$, we denote their multiplication as $f_1 \cdot f_2$. The composition of two maps f and h is denoted as $f \circ h$.

The class of systems that we consider are given below:

$$y(k+1) = F(y(k), u(k-1), u(k-2), \dots, u(k-n+1), u(k)) \quad (1)$$

where y and u are respectively the output and input of the system and $F(y, v_1, v_2, \dots, v_{n-1}, u)$ is a polynomial in all its arguments. It is assumed that F has rational coefficients and other assumptions on the polynomial F are introduced below. We introduce the state variables

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$x_1(k) = y(k), x_2(k) = u(k-1), \dots, x_n(k) = u(k-n+1)$ and obtain the state and output equations:

$$\begin{aligned} x_1(k+1) &= F(x_1(k), x_2(k), \dots, x_n(k), u(k)) \\ x_2(k+1) &= u(k) \\ x_3(k+1) &= x_2(k) \\ &\dots \\ x_n(k+1) &= x_{n-1}(k) \\ y(k) &= x_1(k) \end{aligned} \quad (2)$$

The polynomial F can be written as:

$$F(x_1, x_2, \dots, x_n, u) = a_m(x_1, \dots, x_n)u^m + \dots + a_0(x_1, \dots, x_n) \quad (3)$$

Assumption 1 m in (3) is an odd integer.

If the highest coefficient polynomial a_m in (3) is not equal to zero for some $x(0) \in \mathbb{R}^n$, the set of one step reachable outputs from the initial state $x(0)$ is equal to $\mathbb{R}$. If for some $x(0) \in \mathbb{R}^n$ the coefficient a_m vanishes, the set of one step reachable outputs for the system (2) may be much smaller. Therefore, the only states from which it may not be possible to zero the output belong to the real variety V_C defined by

$$V_C = \{x \in \mathbb{R}^n : a_m(x) = 0\} \quad (4)$$

Definition 1 The variety V_C given by (4) is called the **critical variety**.

A sequence of controls of length $p+1$ is denoted as $\mathcal{U}_p = \{u(0), \dots, u(p)\}$ and the state at time $p+1$, emanating from $x(0)$ and under the action $\mathcal{U}_p$ is denoted as $x(p+1, x(0), \mathcal{U}_p)$.

Definition 2 $V_O = \{x \in \mathbb{R}^n : x_1 = 0\}$ is called the zero output variety.

Since the highest exponent of u in the equation (3) is odd and the coefficient $a_m(x)$ is not identically equal to zero, the variety V_C has a dimension lower than n . We use a shorthand notation for (2) $x(k+1) = f(x(k), u(k)), y(k) = h(x(k))$.

Assumption 2 $\forall x \in V_C, \exists u \in \mathbb{R}, f(x, u) = 0$.

Definition 3 The system (2) with Assumption 2 is output dead beat controllable if there exists a positive integer P such that for any initial state $x(0)$ there exists a control sequence U_P such that $h(P+1, x(0), U_P) = 0$.

The polynomials of special form are defined below:

Definition 4 Polynomials of special form are given by:

$$\begin{aligned} f_s &= x_1 - \sum_{i_2 i_3 \dots i_n} b_{i_2 i_3 \dots i_n} x_2^{i_2} x_3^{i_3} \dots x_n^{i_n}, \\ b_{i_2 i_3 \dots i_n} &\in \mathbb{Q}, \forall i_2, i_3, \dots, i_n \end{aligned}$$

and varieties $V_s = \{x : f_s = 0\}$ are called varieties of special form.

Definition 5 The number of varieties of special form that are contained in V_C is denoted by B .

Also, notice that polynomials of special form and the varieties of special form are *irreducible* [4].

Definition 6 The p -step reachable set $V_r^p(x_0)$ from an initial state $x(0)$ is given by:

$$\begin{aligned} V_r^p(x(0)) &= \{\zeta : \zeta = f_{u(p-1)} \circ \dots \circ f_{u(0)}(x(0)), \\ &\quad u(i) \in \mathbb{R}, \forall i = 0, \dots, p-1\} \end{aligned} \quad (5)$$

Consider the polynomial $f_r = f_r(x(n-1), x(0))$ defined as

$$\begin{aligned} f_r &= x_1(n-1) - F_{x_n(n-1)} \circ F_{x_{n-1}(n-1)} \circ \dots \circ F_{x_3(n-1)} \\ &\quad \circ F_{x_2(n-1)}(x(0)) \\ &= x_1(n-1) - \sum_{i_1 i_2 \dots i_n} B_{i_2 i_3 \dots i_n}(x(0)) x_2^{i_2}(n-1) \\ &\quad x_3^{i_3}(n-1) \dots x_n^{i_n}(n-1) \end{aligned} \quad (6)$$

The variety $V(f_r(x(n-1), x(0))) = \{x : f_r = 0\}$ defines the set of states that can be reached from $x(0)$ in $n-1$ time steps, that is $V_r^{n-1}(x(0)) = V(f_r(x(n-1), x(0)))$. The polynomial f_r has special form $\forall x(0) \in \mathbb{R}^n$ and we can conclude the following [4]:

1. $V_r^{n-1}(x(0))$ is an irreducible variety, $\forall x(0) \in \mathbb{R}^n$
2. $\dim V_r^{n-1}(x(0)) = n-1, \forall x(0) \in V_C$
3. $f_r(x(n-1), x(0))$ is an irreducible polynomial in $x(n-1), \forall x(0) \in \mathbb{R}^n$

Let V and W be varieties. We introduce notation:

$$V \rightarrow W \quad (7)$$

to denote that $V_r^{n-1}(x) = W, \forall x \in V$. It should be emphasised that the equation (7) means that the one step reachable set from any initial state in V is *equal to* W . Therefore, by definition $x(0) \rightarrow V_r(x(0))$.

Definition 7 A set $V_{I_j} \subseteq V_C$ is invariant if

$$\forall x \in V_{I_j}, V_r^{n-1}(x) \subseteq V_{I_j} \quad (8)$$

The union of all invariant sets $V_I = \cup_j V_{I_j}$ is called the **maximal invariant set**.

Definition 8 A subset W_{I_j} of the variety V_C is strongly invariant if it is invariant and $\forall x(0) \in W_{I_j}$ there exists an integer $t \geq 0, t = t(x(0))$ and a sequence of controls $\mathcal{U}_t$ which yields $x(t+1, x(0), \mathcal{U}_t) = x(0)$. The union of all strongly invariant sets $W_I = \cup_j W_{I_j}$ is called the **maximal strongly invariant set**.

Definition 9 The number of varieties of special form that are contained in the maximal strongly invariant set W_I of V_C is denoted by L .

Because of Assumption 1, we can split the dead beat control problem into two parts. First we find conditions which guarantee that it is possible to zero the output for any initial state. Once we have zeroed the output, Assumption 1 guarantees that there exists a control sequence which keeps the output at zero for all future time steps.

3 Output dead beat controllability

The special structure of the system (2) yields the particular structure of the set $V_r^{n-1}(x(0))$ which can be used to simplify the output dead beat controllability test.

Indeed, we can prove the following Lemma.

Lemma 1 *The maximal strongly invariant set $W_I \subseteq V_C$ can be decomposed into a finite union of the varieties of special form $W_I = V_{s1} \cup V_{s2} \cup \dots \cup V_{sL}$, $L \leq B$.*

Proof of Lemma 1: The proof is carried out in several steps. First, we prove that at least one variety V_s of special form belongs to the strongly invariant set. Then we show that if two points that belong to a variety of special form V_s have distinct $n-1$ -step reachable sets, then the variety V_s can not be a subset of an invariant set W_I . By induction we prove that the union of varieties of special form is a subset of W_I . Finally, it is shown by contradiction that W_I is equal to the union of varieties of special form.

STEP 1 Consider any initial state $x(0) \in W_I$. From the invariance of W_I it follows that $V_r^{n-1}(x(0)) \subset W_I$. Denote $V_r^{n-1}(x(0))$ as V_{s1} .

STEP 2 V_{s1} is a subset of the strongly invariant set W_I . Notice that if at least one of the coefficients $B_{i_2 i_3 \dots i_n}(x(0))$ is such that its image is an interval when viewed as a function on the variety V_{s1} , then that state $x(0)$ can not belong to an invariant set $W_I \subset V_C$. Indeed, this would imply that infinitely many distinct varieties of special form are contained in V_C , which can not be the case. Hence, because of invariance of W_I we have that states in V_{s1} are mapped to finitely many varieties of special form which are contained in V_C .

Suppose now that if $x(0) \in V_{s1}$, then either $V_r^{n-1}(x(0)) = V_{s2}$ or $V_r^{n-1}(x(0)) = V_{s3}$ where $V_{s2} \neq V_{s3}$. From the structure of (6) we see that $V_r^{n-1}(\hat{x}(0)) \neq V_r^{n-1}(\hat{x}(0))$ if and only if there exists $i_1^* i_2^* \dots i_n^*$ such that $B_{i_1^* i_2^* \dots i_n^*}(\hat{x}(0)) \neq B_{i_1^* i_2^* \dots i_n^*}(\hat{x}(0))$. Assume that:

$$\forall x(0) \in V_{s1}, \quad B_{i_1^* i_2^* \dots i_n^*}(x(0)) = b^1 \text{ or } B_{i_1^* i_2^* \dots i_n^*}(x(0)) = b_2$$

with $b_1 \neq b_2$. Consider now the polynomials $B_{i_1^* i_2^* \dots i_n^*}(\zeta) - b^1$ and $B_{i_1^* i_2^* \dots i_n^*}(\zeta) - b^2$ where $\zeta \in V_{s1}$. By construction, these polynomials are not identically equal to zero on V_{s1} but their product is:

$$(B_{i_1^* i_2^* \dots i_n^*}(\zeta) - b^1)(B_{i_1^* i_2^* \dots i_n^*}(\zeta) - b^2) \equiv 0, \quad \forall \zeta \in V_{s1}$$

This, however, contradicts the irreducibility of V_{s1} [4, pg. 216]. By contradiction, we have that $B_{i_1 i_2 \dots i_n}(x(0)) =$

const., $\forall i_1, i_2, \dots, i_n, \forall x(0) \in V_{s1}$. So $V_r^{n-1}(x(0)) = V_{sk}$, $\forall x(0) \in V_{s1}$ where $V_{sk} \subset V_C$ and we use the notation $V_{s1} \rightarrow V_{sk}$.

STEP 3 Because of invariance of W_I , all initial states in V_{s1} are mapped to a variety of special form which is a subset of V_C . Note that V_C can contain only finitely many varieties of special form V_{si} , $i = 1, 2, \dots, B$. Thus, there exists $i = 1, 2, \dots, B$ such that V_{s1} is mapped to V_{si} . If $i = 1$, then V_{s1} is a strongly invariant set. If not, assume that $i = 2$. Because of invariance, there exists $i = 1, 2, \dots, B$ such that V_{s2} is mapped to V_{si} . If $i = 1$ or 2 we have constructed a strongly invariant set $V_{s1} \cup V_{s2}$. If not, assume $i = 3$, etc. Therefore, we have $V_{s1} \cup \dots \cup V_{sL} \subset W_I$, $L \leq B$.

STEP 4 Suppose that the strongly invariant set can be decomposed as $W_I = V_{s1} \cup \dots \cup V_{sL} \cup S$, where $S \not\subset \cup_i V_{si}$. Any point in S is mapped to one of V_{si} , $i = 1, 2, \dots, L$ because of invariance of W_I but the points of S can not be reached from V_{si} . If the set S were not empty, W_I would not be strongly invariant. Q.E.D.

Using arguments very similar to the proof of Lemma 1, we can prove the following Lemmas.

Lemma 2 *Every invariant set must contain a strongly invariant set.*

Proof of Lemma 2: Suppose that $V_I \subseteq V_C$ is an invariant set and that it does not contain any strongly invariant subsets. If $x(0) \in V_I$ then because of invariance of V_I we have that $V_r^{n-1}(x(0)) \subset V_I$ and we can denote it as V_{s1} . Notice that there may be at most B varieties of special form contained in V_I . Using the property proved in Step 2 of previous Lemma, we have that $V_{s1} \rightarrow V_{si}$, $i = 1, 2, \dots, B$. However, since we assumed that there are no strongly invariant sets in V_I , we must have that $i \neq 1$. Therefore, $V_{s1} \rightarrow V_{si}$, $i = 2, \dots, B$, and we can assume $i = 2$. Using the same argument we have that $V_{s2} \rightarrow V_{si}$, $i = 3, \dots, B$ and we can assume that $i = 3$, etc. After $B-1$ steps we obtain that $V_{sB} \rightarrow V_{si}$, $i = 1, 2, \dots, B$ because of invariance of V_I but this contradicts the assumption that there are no strongly invariant sets contained in V_I . The contradiction completes the proof. Q.E.D.

Lemma 3 *Every state in $V_C - V_I$ can be transferred to $\mathbb{R}^n - V_C$ in finite time.*

Proof of Lemma 3: The Lemma follows trivially from the definition of the maximal invariant set V_I . Q.E.D.

Lemma 4 *Every state in $V_I - W_I$ is transferred to a strongly invariant set W_I in finite time.*

Proof of Lemma 4: Assume that $W_I \neq \emptyset$. We assume that there exists a state $x(0) \in V_I - W_I$ such that $x(k, x(0), \mathcal{U}_{k-1}) \in V_I - W_I, \forall k, \forall \mathcal{U}_{k-1}$. In this case it follows that the set $V_I - W_I$ contains an invariant subset

and from Lemma 2 it follows that $V_I - W_I$ contains a strongly invariant subset. Hence, we have that W_I is not the maximal strongly invariant set. The contradiction completes the proof. Q.E.D.

We can combine Lemmas 1, 2, 3 and 4, to obtain the following result:

Theorem 1 *The odd polynomial system (2) is output DB controllable if and only if either $W_I = \emptyset$ or every variety of special form contained in the maximal strongly invariant set W_I intersects the zero output variety V_O .*

Proof of Theorem 1:

Necessity: Suppose that there exists a variety of special form V_s contained in the maximal strongly invariant set which is such that its intersection with V_O is empty. If the variety V_s is a strongly invariant set itself then there is no control sequence which transfers any initial state in V_s to V_O . If V_s is a subset of a larger strongly invariant set W_I^* and $V_s \cap V_O = \emptyset$ then $W_I^* \cap V_O = \emptyset$ because of Assumption 1 and the same argument applies.

Sufficiency: We partition the whole state space $\mathbb{R}^n = (V_C - V_I) \cup (V_I - W_I) \cup W_I \cup (\mathbb{R}^n - V_C)$ and consider what happens on each of the subsets. If $x(0) \in \mathbb{R}^n - V_C$ we can zero the output in one step. If $x(0) \in V_C - V_I$, according to Lemma 3, it follows that the initial state can be transferred to $\mathbb{R}^n - V_C$ in finite time and consequently to V_O . Consider $x(0) \in V_I - W_I$. From Lemma 4 it follows that $x(0)$ is transferred to W_I in finite time. Since all irreducible components of W_I intersect V_O and because of Assumption 2 it follows that any state in V_I can be transferred to V_O in finite time. Because of Assumption 2 we conclude that the system is output dead beat controllable. Q.E.D.

The following Corollaries may help us to reduce computations even more.

Corollary 1 *If $\dim V_C = \dim V(a_m) < n - 1$ the system is output dead beat controllable.*

Proof: Since $\dim V_r^{n-1}(x(0)) = n - 1, \forall x(0) \in \mathbb{R}^n$, it follows that $V_r^{n-1}(x(0)) \not\subset V_C, \forall x(0) \in V_C$. Thus, we need at most n steps to map any initial state to V_O . Q.E.D.

It is possible to use the method based on the affine Hilbert polynomial (see the last chapter of [4]) in order to check the dimension of the variety V_C .

Corollary 2 *If V_C does not contain varieties of special form, that is a_m does not contain irreducible polynomials of special form, the system (2) is output dead beat controllable.*

Proof: From properties $V_r^{n-1}(x(0))$ it follows that $V_r^{n-1}(x(0))$ can not be a subset of $V_C, \forall x(0) \in V_C$. Q.E.D.

Corollary 3 *Suppose that there are B varieties of special form $V_{s_i} = V(f_{s_i})$ contained in V_C . The system (2) is output dead beat controllable if $V_O \cap V_{s_i} \neq \emptyset, \forall i = 1, 2, \dots, B$.*

Comment 1 It is important to notice that Theorem 1 provides conditions for output controllability to the hyper-plane $x_1 = 0$. If we want to check output controllability to some other point $x_1 = y^*, y^* \neq 0$ then all irreducible components (varieties) V_{s_i} of the maximal strongly invariant set W_I should intersect the hyper-plane $x_1 = y^*$.

Comment 2 The objects that are described in this paper, that is invariant and strongly invariant sets, play an important role in the characterisation of other controllability properties for class of polynomial systems. Indeed, we may lose state dead beat or complete controllability if there exist some invariant sets. Moreover, invariant sets are important for controllability properties of even systems as well. The below given example illustrates our claim. Consider the even system:

$$y(k+1) = (y(k) - u^3(k-1))u^4(k) + u^3(k)$$

Obviously there is a strongly invariant set $W_I = \{(y, v) : y - v^3 = 0\}$. Notice that the output can be zeroed from the states that belong to the strongly invariant set. Moreover, the system is output and state dead beat controllable. However, it is not completely controllable since any state that belongs to the strongly invariant set W_I can not be mapped outside the set. A number of other examples can be found where we lose one of the controllability properties whereas some might still be preserved.

Comment 3 Even systems may exhibit other forms of invariance. Indeed, the system:

$$\begin{aligned} x_1(k+1) &= u(k) \\ x_2(k+1) &= x_2(k) - x_1^2(k) + u^2(k) \end{aligned}$$

has one invariant variety $V(x_2 - x_1^2)$ and two invariant semi algebraic sets $I_1 = \{x \in \mathbb{R}^2 : x_2 < x_1^2\}$ and $I_2 = \{x \in \mathbb{R}^2 : x_2 > x_1^2\}$.

One way to check the existence of semi-algebraic invariant sets is to use the methodology described in [9] which is based on the QEPCAD algorithm. The test for the existence of invariant semi-algebraic sets is more computationally expensive and this is one of the main hindrances to a more elaborate investigation of controllability properties of even systems.

4 Output dead beat controllability test

So far we have considered what happens geometrically, whereas an algebraic test is needed to check the conditions of Theorem 1. Before we state the test we need the result from [9]:

ALGORITHM 1: Given a polynomial system $x(k+1) = f(x(k), u(k))$ and a variety $V(a_m) = \{x : a_m(x) = 0\}$, we can determine the maximal invariant sub set of $V(a_m)$ using the following algorithm:

1. Input: $a_m(x)$, $f(x, u)$
2. $G_0 = \{a_m\}$
3. $k=0$
4. $k=k+1$
5. Compute $a_m \circ f_{u_{k-1}} \circ \dots \circ f_{u_0}(x)$.
6. Compute the reduced Gröbner basis G_k which is the basis of polynomial coefficients in x of the compositions $a_m \circ f_{u_j} \circ \dots \circ f_{u_0}(x)$, $j = 0, 1, \dots, k-1$ when they are regarded as a polynomials in $u_0, \dots, u_{k-1}$
7. If $G_k = G_{k+1}$ stop the computation and G_k defines the maximal invariant set V_I . If $G_k \neq G_{k+1}$ go to 4.

The algorithm stops after a finite number of steps. It is essential in the above algorithm that we compute the reduced Gröbner bases G_k since they are unique for a given monomial ordering, which enables us to compare ideals by comparing their reduced Gröbner bases [4, 9, 10].

From Lemma 1 and definition of strongly invariant sets, we can deduce the following method to check output dead beat controllability of systems (2).

TEST:

1. Check Assumptions 1 and 2. Assumption 2 is checked using QEPCAD.
2. Decompose the polynomial $a_m \in Q[x_1, \dots, x_n]$ into irreducible polynomials (using eg. the command “factor” in Maple) and identify all polynomials that have special form. Denote this set as $\Sigma_1 = \{f_{s1}, f_{s2}, \dots, f_{sB}\}$.
3. (a) Check whether any of the varieties $V(f_{si})$, $i = 1, 2, \dots, B$ is invariant using Algorithm 1. Denote the set of all polynomials f_{si} that yield invariant varieties as Σ_1^I . Obviously $\Sigma_1^I \subseteq \Sigma_1$, and find the set $\Sigma_2 = \Sigma_1 - \Sigma_1^I$.
 (b) If $\Sigma_2 \neq \emptyset$, find all products $f_{sj} \cdot f_{sk}$, $f_{sj}, f_{sk} \in \Sigma_2$, and check the invariance of all varieties $V(f_{sj} \cdot f_{sk})$ using the Gröbner basis method. The set of all polynomials for which varieties $V(f_{sj} \cdot f_{sk})$ are invariant is denoted as Σ_2^I . Obviously, $\Sigma_2^I \subseteq \Sigma_2$. Define a new set $\Sigma_3 = \Sigma_2 - \Sigma_2^I$, etc.
 (c) If $\Sigma_B \neq \emptyset$ find the product $f_{s1} \cdot \dots \cdot f_{sB}$ and check the invariance of the variety $V(f_{s1} \cdot \dots \cdot f_{sB})$ using the Gröbner basis method. If the variety is invariant then $\Sigma_B^I = \Sigma_1$. Otherwise, $\Sigma_B^I = \emptyset$. Find the set $\Sigma^I = \cup_{i=1}^B \Sigma_i^I$. The maximal strongly invariant set is then

$$W_I = V\left(\prod_{f_{si} \in \Sigma^I} f_{si}\right)$$

4. Check whether $V_O \cap V(f_{si}) \neq \emptyset, \forall f_{si} \in \Sigma^I$ using QEPCAD. If this is true system is output dead beat controllable and vice versa.

Comment 4 It is very important to notice that the above given output DB controllability test stops after a finite number of operations. This was not the case with the systems considered in papers [9, 10]. We emphasize that the structure of the class of systems (2) guarantees that the above given test stops in finite time.

Comment 5 Notice that the above given test gives us the partition of the strongly invariant set into strongly invariant subsets. In other words, we find all strongly invariant sets.

5 Example

Example 1 Consider the system:

$$\begin{aligned} y(k+1) = & (y^2(k) - 2y(k)u^2(k-1)u^2(k-2) - 3y(k) \\ & + u^4(k-1)u^4(k-2) + 3u^2(k-1)u^2(k-2) \\ & + 2)u(k)^3 + u(k)^2u(k-1)^2 - y(k) + u(k-1)^2 \\ & u(k-2)^2 + 3 \end{aligned}$$

Introduce the state variables $x_1(k) = y(k)$, $x_2(k) = u(k-1)$ and $x_3(k) = u(k-2)$ we obtain the state space model:

$$\begin{aligned} x_1(k+1) = & (x_1^2(k) - 2x_1(k)x_2^2(k)x_3^2(k) - 3x_1(k) \\ & + x_2^4(k)x_3^4(k) + 3x_2^2(k)x_3^2(k) + 2)u(k)^3 \\ & + u(k)^2x_2^2(k) - x_1(k) + x_2^2(k)x_3^2(k) + 3 \\ x_2(k+1) = & u(k) \\ x_3(k+1) = & x_2(k) \\ y(k) = & x_1(k) \end{aligned}$$

We are going to use the method based on the Gröbner basis algorithm for this example.

Step 1: It is easy to see that Assumptions 1 and 2 are satisfied.

Step 2: Using the command “factor” in Maple for the polynomial $x_1^2 - 2x_1x_2^2x_3^2 - 3x_1 + x_2^4x_3^4 + 3x_2^2x_3^2 + 2$ we find that the only two polynomials of special form are $f_{s1} = x_1 - x_2^2x_3^2 - 1$ and $f_{s2} = x_1 - x_2^2x_3^2 - 2$. In other words, $V_{s1} = V(f_{s1}) \subset V_C$ and $V_{s2} = V(f_{s2}) \subset V_C$. Step 3: We check whether the variety V_{s1} is invariant:

$$\begin{aligned} f_{s1} = & x_1 - x_2^2x_3^2 - 1 \\ f_{s1} \circ f_u(x) = & (x_1 - x_2^2x_3^2 - 1)(x_1 - x_2^2x_3^2 - 2)u^3 - x_1 \\ & + x_2^2x_3^2 + 2 \\ G_0 = & \{x_1 - x_2^2x_3^2 - 1\} \\ G_1 = & Gbasis[x_1 - x_2^2x_3^2 - 1, (x_1 - x_2^2x_3^2 - 1) \\ & (x_1 - x_2^2x_3^2 - 2), -x_1 + x_2^2x_3^2 + 2] \\ = & \{1\} \end{aligned}$$

and since $G_2 = \{1\}$ it follows that V_{s1} is not invariant. Similarly, we have for variety V_{s2} :

$$\begin{aligned} f_{s2} &= x_1 - x_2^2 x_3^2 - 2 \\ f_{s2} \circ f_u(x) &= (x_1 - x_2^2 x_3^2 - 1)(x_1 - x_2^2 x_3^2 - 2)u^3 - x_1 \\ &\quad + x_2^2 x_3^2 + 1 \\ G_0 &= \{x_1 - x_2^2 x_3^2 - 2\} \\ G_1 &= Gbasis[x_1 - x_2^2 x_3^2 - 2, (x_1 - x_2^2 x_3^2 - 1) \\ &\quad (x_1 - x_2^2 x_3^2 - 2), -x_1 + x_2^2 x_3^2 + 1] \\ &= \{1\} \end{aligned}$$

Therefore, V_{s2} is not invariant. Consider now the variety $V(f_{s1} \cdot f_{s2})$. We obtain:

$$\begin{aligned} f_{s1} \cdot f_{s2} &= (x_1 - x_2^2 x_3^2 - 1)(x_1 - x_2^2 x_3^2 - 2) \\ (f_{s1} \cdot f_{s2}) \circ f_u(x) &= [(x_1 - x_2^2 x_3^2 - 1)(x_1 - x_2^2 x_3^2 - 2)u^3 \\ &\quad - x_1 + x_2^2 x_3^2 + 2] \\ &\quad [(x_1 - x_2^2 x_3^2 - 1)(x_1 - x_2^2 x_3^2 - 2)u^3 \\ &\quad - x_1 + x_2^2 x_3^2 + 1] \\ G_0 &= \{(x_1 - x_2^2 x_3^2 - 1)(x_1 - x_2^2 x_3^2 - 2)\} \\ G_1 &= Gbasis[(x_1 - x_2^2 x_3^2 - 1)^2(x_1 \\ &\quad - x_2^2 x_3^2 - 2)^2, (x_1 - x_2^2 x_3^2 - 1) \\ &\quad (x_1 - x_2^2 x_3^2 - 2)^2, (x_1 - x_2^2 x_3^2 - 1)^2 \\ &\quad (x_1 - x_2^2 x_3^2 - 2), (x_1 - x_2^2 x_3^2 - 1) \\ &\quad (x_1 - x_2^2 x_3^2 - 2)] \\ &= \{(x_1 - x_2^2 x_3^2 - 1)(x_1 - x_2^2 x_3^2 - 2)\} \end{aligned}$$

Since $G_0 = G_1$, we conclude that the variety $V(f_{s1} \cdot f_{s2})$ is invariant. It is not difficult to see that we actually have that $V_{s1} \rightarrow V_{s2} \rightarrow V_{s1} \rightarrow \dots$

Step 4: We do not need to use QEPCAD in this case. Indeed, since $x_2^2 x_3^2 = -K$, $K = 1, 2$ have no real solutions in x_2, x_3 , we conclude that $V_{s1} \cap V_O = \emptyset$ and $V_{s2} \cap V_O = \emptyset$ and the systems is not output dead beat controllable.

6 Conclusion

We have presented an output dead beat controllability test for a class of odd polynomial systems of arbitrary order. The test repeatedly uses Algorithm 1 which was first presented in [9] and is based on the Gröbner basis method. These results are a generalisation of results for a class of second order odd polynomial systems that were investigated in [8]. The output controllability test presented in this paper is different from the test presented in [8] and either can be used to check output dead beat controllability of odd systems considered in [8].

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