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Giving Daggers to Higher Cats: Generalised Quasi Operads, Astroidal Sets, and a Surface Operad

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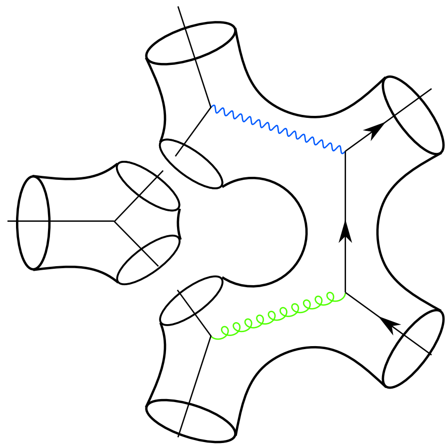
Giving Daggers to Higher Cats

Generalised Quasi Operads, Astroidal Sets, and a Surface Operad



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Abstract

This thesis contains my work on various generalisations of infinity operads, as well as an example inspired by Topological Quantum Field Theories (TQFTs). The main result is a proof of the equivalence between the Segal and strict inner Kan conditions for graphical sets.

There are two different generalisations of infinity operads explored herein. Firstly, there are dagger categories. Inspired by Hilbert spaces, the notion of adjoint is generalised to the category setting. Dagger categories assign to each morphism $f : A \to B$ an adjoint $f^\dagger : B \to A$. One can then consider f to be a morphism between A and B rather than going in any particular direction $A \to B$ or $B \to A$. A cyclic operad is an operad with something akin to an adjoint; the action of the symmetric group interchanges the input objects and the output objects. This thesis contains a theory of quasi cyclic operads, including astroidal sets (presheaves over a category of unrooted trees) and a nerve theorem.

Then, the shape of morphisms can be changed. Operads extend categories by allowing morphisms to have multiple inputs, while infinity operads extend infinity categories by being presheaves over the category of rooted trees rather than the category of paths. This can be further generalised to presheaves over other categories, in particular the category of graphs, to facilitate a connection with surfaces of higher genus. Graphical sets are the cyclic, higher genus analogue of simplicial sets. They are used to represent infinity modular operads. This thesis contains an exploration of each of the four models of infinity categories extended to infinity modular operads, with a focus on quasi modular operads and the inner Kan condition.

Finally, given the applications in TQFTs and Grothendieck-Teichmüller theory, I construct an infinity modular operad of surfaces.

Declaration

The author declares that:

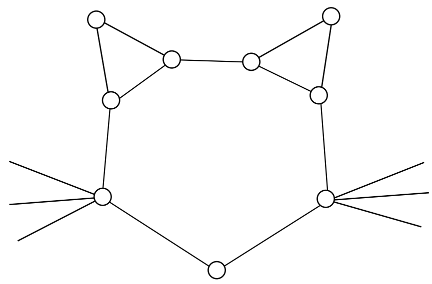
(i) this thesis comprises only the original work of the author towards the degree of Doctor of Philosophy;

(ii) due acknowledgement has been made in the text to all other material used; and

(iii) this thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Signature of the author:

To cat lovers



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Contents

1 Introduction **1**
- 1.1 Structure of this Thesis and Original Work 3
- 1.2 Terminology and Nomenclature 4

2 Categories and Simplicial Sets **7**
- 2.1 Categories 8
- 2.2 Simplicial Sets 10
  - 2.2.1 The Category of Simplices 11
  - 2.2.2 Simplicial Sets 13
- 2.3 Infinity Categories 15
  - 2.3.1 Simplicially Enriched Categories 15
  - 2.3.2 The Inner Kan Condition 16
  - 2.3.3 The Segal Condition 18
  - 2.3.4 Complete Segal Spaces 19

3 Operads and Dendroidal Sets **23**
- 3.1 Operads 23
- 3.2 Dendroidal Sets 27
  - 3.2.1 The Category of Dendrices 27
  - 3.2.2 Dendroidal Sets 39
  - 3.2.3 Boundaries, Horns, and Representables 40
  - 3.2.4 Relationship to Δ 42
- 3.3 Infinity Operads 43
  - 3.3.1 Simplicially Enriched Operads 44
  - 3.3.2 The Dendroidal Inner Kan Condition 44
  - 3.3.3 The Dendroidal Segal Condition 45
  - 3.3.4 Dendroidal Complete Segal Spaces 45

4 Cyclic Operads and Astroidal Sets **47**
- 4.1 Cyclic Operads 48
  - 4.1.1 Dagger Categories 48
  - 4.1.2 Definition of Cyclic Operad 52
  - 4.1.3 Relationship to Dagger Categories 54
  - 4.1.4 Relationship to Operads 56
  - 4.1.5 An Example of Cyclic Operads: Multispans 58
  - 4.1.6 Surface Examples 59
- 4.2 Astroidal Sets 59
  - 4.2.1 The Category of Astrices 60

4.2.2 Astroidal Sets 72
4.2.3 Horns and Representables 74
4.3 Infinity Cyclic Operads 75
4.3.1 Simplicially Enriched 75
4.3.2 The Astroidal Inner Kan Condition 76
4.3.3 The Astroidal Segal Condition 76

5 Modular Operads and Graphical Sets 81
5.1 Modular Operads 82
5.2 Graphical Sets 84
5.2.1 The Category of Graphs 84
5.2.2 Relationships 98
5.2.3 Graphical Sets 101
5.2.4 Faces, boundaries, horns 102
5.3 Infinity Modular Operads 103
5.3.1 Simplicially Enriched Modular Operads 103
5.3.2 The Modular Inner Kan Condition 103
5.3.3 The Modular Segal Condition 104
5.3.4 Modular Complete Segal Spaces 111
5.4 Surfaces 112

6 Conclusion 115

Chapter 1

Introduction

This thesis was motivated, first and foremost, by a fascination with the links between algebra, topology and physics. An overview of these connections can be found in [4]. Initially, questions of the cobordism hypothesis and surface operads kept me occupied, with the aim to find a higher operad of surfaces akin to that of Tillmann [74], and which would be applied to Grothendieck-Teichmüller theory [37]. However, delving into the theory of infinity cyclic and modular operads compelled me to fill in some of the gaps I encountered.

There are two main ways in which algebra and topology (and physics) can be connected via category theory, and this thesis explores specific parts of both. The first, and possibly more obvious, is to use category theory as the bridge between algebra and topology, becoming indispensable as a tool in the study of homotopy theory. Category theory has also been used to form links between other areas of topology and algebra. These include knot theory and Temperly-Lieb algebras, and the idea found in Mathews [54], where information is conveyed by decorated cobordisms.

One particularly important example of category theory being used as a bridge is the notion of a Topological Quantum Field Theory. Given some interactions between subatomic particles, one can draw them in a topological way as a Feynman diagram (see Figure 1.1). However, topologically, these singularities present a problem. String theory removes these by replacing particles by strings (i.e. circles), and their movement through time is represented by cobordisms. Thus, the idea is to associate particles (i.e. vector spaces) to cobordisms. A TQFT is a functor from the category of cobordisms to the category of vector spaces.

Inspired by TQFTs, this thesis explores how to represent the category of cobordisms. Since surfaces may have more than one boundary component, or genus higher than zero, it is more natural to consider these as generalised operads rather than categories. Modular operads were created for a very similar purpose, understanding the combinatorics of Feynman diagrams [29], but Feynman diagrams have a deep connection to surfaces [73]. This is not only relevant to TQFTs but in Grothendiek-Teichmüller theory [37]. Similar work has been done in this area by Tillmann [74]. One of the aims of this thesis was to define a higher operad of surfaces, and understand the connections between modular operads, TQFTs, and surfaces.

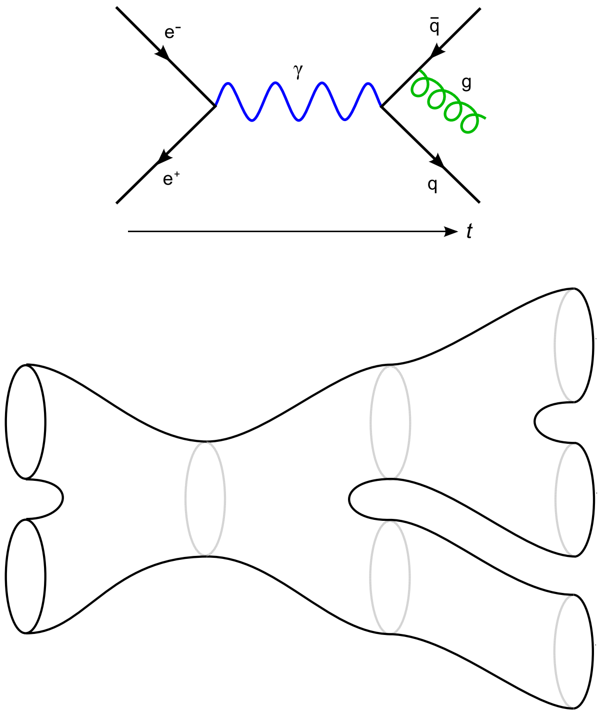


Figure 1.1: An example of a Feynman diagram [40], along with its equivalent cobordism

The other connection between category theory, algebra, and topology, is to consider categories themselves as algebraic objects, and then examine their topology. One example, which also includes connections to logic, is topos theory [51, 49], while another is found in the cobordism hypothesis [3, 48], and yet another in simplicial sets. In this thesis I mainly study generalisations of simplicial sets and infinity categories, and related areas [11, 23]. Infinity categories are those in which composition is not taken to be exact. Simplicial sets are ideal for this, because they can model strings of compositions in a topological way. Each object in the category can be considered to be a point, and each morphism an edge, and the composition of two morphisms $f : A \to B$ and $g : B \to C$ can be considered the face $g \circ f$ between edges f, g, and some $h : A \to C$. Longer compositions of morphisms can be considered as higher dimensional faces, and in this way a simplicial set can be associated to a category. With the aid of the topological structure, one can then consider whether the notion of composition of morphisms must be exact, or whether it may be permitted to be defined up to homotopy. This is the notion of an infinity category.

This notion of weak composition has been extended to operads [57, 58]. Here, compositions of morphisms look more like trees than paths, so infinity operads require a tree-like analogue of simplicial sets. So there are generalisations of categories which can be represented by paths and trees. What about graphs in general? Is there a graphical analogue to simplicial sets, and does there exist a generalisation of infinity categories for which these graphical sets are relevant? This question is partially answered in Hackney, Robertson, and Yau [36, 35], and partly answered in this thesis, with modular operads. Part of the motivation for studying infinity modular operads also comes from TQFTs and cobordisms, because surfaces may have higher genus, and therefore modular operads are important on the quest for a generalised operad of surfaces [70, 74].

Part of what makes graphical sets and modular operads interesting is their non-directional nature. Surfaces, unlike oriented cobordisms, are not necessarily directed. Furthermore, inspired by Hilbert spaces, there is a concept in physics called a dagger category, which is a category in which each morphism has an associated morphism in the opposite direction. This idea can be extended to operads, in the structure known as a cyclic operad. This thesis contains a theory of infinity cyclic operads; developed on the way to infinity modular operads.

The following table illustrates the relationships between categories, modular operads, and other generalised operads.

	Paths	Trees	Graphs
Directed	Categories	Operads	Wheeled properads
Undirected	Dagger categories	Cyclic operads	Modular operads

Wheeled properads are similar to modular operads, but use the category of directed graphs. More information can be found in [33]. The others are covered in this thesis, with most original results in cyclic operads and modular operads.

1.1 Structure of this Thesis and Original Work

This thesis is arranged with each chapter containing the theory of a single type of generalised infinity operad, progressing from categories in Chapter 2 to modular operads in Chapter 5. The operad of surfaces is contained as an example of a modular operad. The genus zero case is an example of a cyclic operad, but because I was able to generalise it successfully I decided to only include the higher genus version.

This thesis begins with a literature review of some models of infinity categories and infinity operads. Chapter 2 contains the definition of a category, a description of the category of simplices, and then an examination of four models for infinity categories. This section also contains some results required for using categories as algebraic tools, but should not be considered an exhausive primer on category theory. The following chapter on operads is similar.

The majority of my time as a student was spent extending this background material to the case of cyclic operads (Chapter 4). Although the literature already contained a definition of a cyclic operad [28], I wrote mine independently for my own ease of understanding. I also include multispans as an original example. Given a category, one can find an associated dagger category of spans of $\mathcal{C}$. Inspired by this, I associate to each category $\mathcal{C}$ an associated cyclic operad of multispans. I then describe the category of astrices, or unrooted trees, which I call Ж for the letter's similarity to a vertex in an unrooted tree. The associated presheaves are called astroidal sets. My main results are in quasi cyclic operads; a quasi cyclic operad is an astroidal set with an inner Kan condition. I include a description of the Segal condition for its similarity to the inner Kan condition, and for its usefulness in the Complete Segal Spaces (CSS) model. Although I do not go into much detail, I include a sketch of how I believe simplicially enriched cyclic operads would be defined. I leave out the astroidal CSS model, because it should be able to be defined from genus zero graphical complete Segal spaces. My main result here is Theorem 4.3.10, and its associated definitions.

Since surfaces have genus, I spent some time exploring the higher genus version of infinity cyclic operads. These are called modular operads, and their associated topological pre-sheaves are called graphical sets. Again, the section on quasi modular operads and the inner Kan condition is original. The Segal condition is not mine [36, 35], but some results are, namely the equivalence of the modular Segal and modular inner Kan conditions (Theorem 5.3.14), which completes the nerve theorem, and Theorem 5.3.15, which states that graphical groups satisfy the Segal condition. For completeness, I include a sketch of what I believe the CSS model should be, as well as a simplicially enriched definition. Although I have not proven their equivalence to the Segal and Kan models, these sketches are original work.

1.2 Terminology and Nomenclature

There are various naming conventions around operads, particularly regarding certain properties and which choices are considered to be the default. The first is whether an operad is symmetric or plane. A symmetric operad has an action of the symmetric group permuting the inputs, whereas a plane operad strictly keeps this ordering. This can be compared to the difference between an abstract tree and a plane tree, where the latter is a particular embedding of a tree in the plane, forcing a particular ordering to its branches. I will use "operad" to refer to a symmetric operad, and "plane operad" otherwise.

Secondly, there is the choice between coloured and monochromatic. A coloured operad is like a multicategory with multiple objects, whereas a monochromatic operad has only one object. I will always mean coloured operad when I say just operad, and will specify monochromatic operad if I mean an operad of just one colour. For historical reasons, the word “colour” is often used in place of the word “object”. In this thesis I will use them interchangably, with a slight bias towards “object” when discussing categories and “colour” when discussing graphs or operads.

In the category case, a category with just one object is also known as a monoid. Likewise, a group has just one object whereas a groupoid has multiple objects. The word “monoid” is an interesting historical artefact. The idea of getting a many-object structure from a one object structure is sometimes called “oidification” (c.f. the word “groupoid”); however, confusingly, the word “monoid” predates this. If one were being obtuse one could rightly call a category a “monoidoid”, and likewise a coloured operad could be called operadoid. (Alternatively, one could call a coloured operad an operoid and a monochromatic operad an oper). In homage to this, I use **Op** to refer to the category of monochromatic operads and **Opd** to refer to the category of coloured operads.

This thesis contains work related to categories, operads, cyclic operads, and modular operads, as well as mentions of similar structures such as props, properads, and wheeled properads. When referring to the general structures that all of these are examples of, I will refer to them, interchangeably, as any of the phrases “structures”, “algebraic structures”, “generalised categories”, or “generalised operads”.

When dealing with various models of infinity generalised operads, I will use the words “higher” and “infinity” interchangably to refer to the general idea, as opposed to specific models like quasi operads or complete Segal spaces. The word “strict” means “not infinity”, with phrases “strict operad” and “strict infinity operad” nearly interchangeable; the former meaning just an operad and the latter meaning a dendroidal set that is isomorphic to the nerve of an operad.

In the chapter on operads, I use the letter Ω to denote multiple things, but I will disambiguate if it is not clear from context. These are:

- The category of rooted trees
- The functor from rooted trees to operads
- The image of this functor, i.e. rooted trees thought of as operads.

The letters Ж and $\mathcal{G}$ are used similarly. I denote the category of trees by Ω because it is entrenched in the literature, despite the fact that Ω is already used for loops spaces. If I were naming this category, I would call it Ψ, because that letter looks like a rooted tree, just as Δ (the category of simplicies) looks like a simplex. On that note, I name the category of unrooted trees Ж, because this Russian letter looks like a star or unrooted tree. This letter is pronounced “zhe”, with “zh” pronounced like the “s” in “leisure” and the “e” in “ten” or “check”. I do, however, use $\mathcal{G}$ to refer to the category of graphs, despite its dissimilarity in shape with a graph (perhaps the letter thorn, Þ, would be more appropriate).

I use dendroidal as the tree analogue of simplicial, as in the literature. I use astroidal as the star analogue, and graphical as the full graph analogue. Note that graphical set was used in Hackney et al. [33] in the context of directed graphs, but I use it here in the context of undirected graphs.

Where it is important, I will use the words “coface” and “codegeneracy” to refer to maps in the categories Δ, Ω, Ж and $\mathcal{G}$, and the words “face” and “degeneracy” to refer to both the arrows in the opposite categories Δ^{op}, Ω^{op}, Ж^{op} and $\mathcal{G}^{op}$, and the induced maps in their associated graphical sets (and similarly for simplicial sets etc.). If the context make it obvious, I will often neglect the “co-” and use “face” and “degeneracy” for both kinds of maps. Similarly, I will use superscripts δ^i, σ^i for coface and codegeneracy maps, and subscripts δ_i, σ_i for face and degeneracy maps.

As an aside, it is interesting to note that, unlike everywhere else in category theory, the word cobordism does not mean “bordism but the arrows are reversed”. The “co” part means together, the “bord” part means boundary, so in this instance the word cobordism comes about because two manifolds are cobordant if together they form the boundary of a third morphism.

Chapter 2

Categories and Simplicial Sets

Categories are a way of talking about certain types of structures found in various areas of mathematics (and also outside mathematics). Saunders and MacLane created categories in 1945 as part of the movement to study the foundations of mathematics [22], looking at notions of equivalence rather than equality, but they are also particularly important in the study of algebraic topology [23, 59, 69, 11]. It is the idea of studying the nature of categories themselves that is explored further in this section. In particular, topology is used to examine the shape of categories and how this can be generalised to quasi categories.

The general structure of this chapter is followed closely in subsequent chapters. The focus is on elements of category theory that will be replicated in the operad case, but there are also included certain categorical preliminaries as may be relevant; however neither this chapter nor this thesis is intended to be an overview of category theory. For that, a good reference is [50].

The first part of this chapter consists of a study of categories themselves, with reference to the path approach to thinking about them. A category consists of objects, with morphisms between them, and a composition function that ensures that the composite of any two compatible morphisms exists. The more usual way of looking at a category is as a multigraph, where the objects are vertices and the morphisms are directed edges, and compositions of morphisms are paths traced through the graph. However, in order to understand operads properly, it is better to first think of categories in a slightly different way. Consider the objects to be edges, or colours of edges, with each vertex an operation taking one object to another. Then composition can be represented by directed paths.

In an ordinary category, each pair of compatible morphisms $f : A \to B$ and $g : B \to C$ will, when composed, be equal to some other morphism $h : A \to C$. That is, we have $g \circ f = h$. However, when thinking about categories topologically, it becomes possible to instead have only $g \circ f \cong h$. That is, composition of morphisms exists only up to homotopy. This map from $g \circ f$ to h is called a 2 morphism (ordinary morphisms are known as 1 morphisms) and the structure is called a 2 category (see [46] for more information). We can also allow compositions of 2 morphisms to be only up to homotopy, and so on, until we arrive at infinity categories.

Section 2.2 explores the category of paths, Δ, which is used to provide a path-like shape to families of sets via presheaves. The way to model this topological structure of categories is found in simplicial sets, although different

Figure 2.1: Given two morphisms $f : a \to b$ and $g : b \to c$, one can imagine composition like this

ways of incorporating simplicial sets will arrive at different definitions of infinity categories. Four such models for infinity categories are discussed in Section 2.3.

2.1 Categories

Categories were born out of the mathematical foundations movement [22], but they hold a prominent place in the study of algebraic topology [23, 59, 69, 11].

A category consists of some objects connected by composable morphisms. Many people think of categories as directed graphs, where the objects are the vertices and the morphisms are edge and paths between vertices. However, in order to facilitate the transition to operads, a better way to think of categories is with the objects being coloured edges, and the morphisms being the vertices. Then composition of morphisms can be given by gluing vertices together along their shared edges to form paths (see Figure 2.1).

Definition 2.1.1 (Category)**.** A category $\mathcal{C}$ consists of:

- A class of objects $\text{ob}(\mathcal{C})$
- For every two objects $A, B \in ob(\mathcal{C})$, a class $\mathcal{C}(A, B)$. The elements of this class are called morphisms, and $f \in \mathcal{C}(A, B)$ is often denoted $f : A \to B$
- A composition function $\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ such that
  - For each object A there is a distinguished morphism $id_A \in \mathcal{C}(A, A)$ which acts as the identity. i.e. for all $f : A \to B$, $id_A \circ f = f \circ id_A = f$
  - This function is associative. i.e. for all $f : A \to B$, $g : B \to C$, $h : C \to D$, $(f \circ g) \circ h = f \circ (g \circ h) = f \circ g \circ h$

Note that, although this definition involves classes for set theoretic reasons, in practice one can often just use sets of objects and morphisms; this describes small categories.

Some examples of categories include the following.

- The category of sets, **Set**, where the objects are sets and the morphisms are functions.
- Any particular group can be considered a category of one object, where the elements of the group are the morphisms and they compose according to the group operation.

- The category of groups, **Grp**, with groups as objects and group homomorphisms as morphisms.

- The category of vector spaces under linear transformation, $\mathbf{Vect_k}$.

- The category **nCob**, with $(n-1)$-dimensional manifolds without boundary as objects and n-dimensional cobordisms between them as morphisms.

When presented with any new mathematical concept, it can be useful to form a category of those objects. It would be useful to form a category of (small) categories, but in order to do this it is necessary to understand what morphisms between categories would look like. These morphisms between categories are known as *functors*.

Definition 2.1.2 (Functor)**.** Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor between them $F : \mathcal{C} \to \mathcal{D}$ is:

- A function $F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$

- A function from morphisms to morphisms, such that, if $f : A \to B$ is a morphism in $\mathcal{C}$, $F(f) : F(A) \to F(B)$ is a morphism in $\mathcal{D}$ satisfying:

  - Identity maps to identity
  - $F(f \circ g) = F(f) \circ F(g)$

Thus, one can form the category of categories, **Cat**, with small categories as objects and functors as morphisms.

A good example of a functor is a TQFT (see Figure 2.2).

Definition 2.1.3 (Topological quantum field theory)**.** A Topological Quantum Field Theory (TQFT), X, is a symmetric monoidal functor

$$X : \mathbf{nCob} \to \mathbf{Vect_k}.$$

Symmetric monoidal functors and symmetric monoidal categories are not defined here (see [50, Chapter XI]); however, one can think of a symmetric monoidal category as a category with some extra tensor structure, and a symmetric monoidal functor is a functor which preserves this structure.

A particular kind of functor that appears here is a presheaf.

Definition 2.1.4 (Presheaf)**.** A presheaf A over a category X is a functor $A : X^{op} \to \mathbf{Set}$, where X^{op} refers to the category X with all the morphism directions reversed.

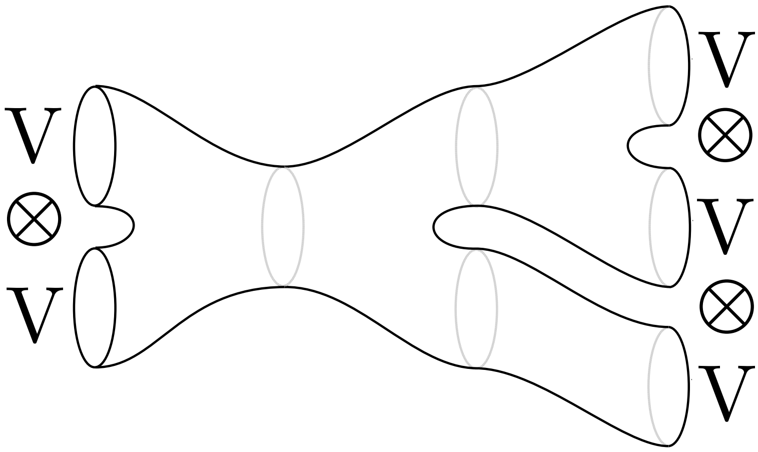


Figure 2.2: An example of part of a TQFT. Here is a morphism in **2cob** from $S^1 \sqcup S^1$ to $S^1 \sqcup S^1 \sqcup S^1$, which is associated to a linear map $V \otimes V \to V \otimes V \otimes V$

Categories are all about relationships, and therefore it is natural to think about a way of transforming between functors.

Definition 2.1.5 (Natural transformation)**.** Let F and G be functors $\mathcal{C} \to \mathcal{D}$. Then a natural transformation $\phi : F \implies G$ is a family of morphisms, one for each object of $\mathcal{C}$, such that, for each morphism $f : x \to y$, the following diagram commutes.

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ \downarrow{\scriptstyle \phi_x} & & \downarrow{\scriptstyle \phi_y} \\ G(x) & \xrightarrow{G(g)} & G(y) \end{array}$$

Natural transformations are particularly useful for relationships between categories of functors, such as TQFTs or presheaf categories.

2.2 Simplicial Sets

The major theme of this thesis is representing algebraic objects topologically, and vice versa. A category itself can be considered to be an algebraic object, which can then be represented topologically. If we think of infinity categories as, in some sense, categories up to homotopy, then it is important to understand the topology of a category, and this can be done through simplicial sets. Simplicial sets, originally a way of understanding classifying spaces of groups, are used as a combinatorial representation of topology. Just as a classifying space assigns a simplicial set (and therefore topological space) to a group, so does the nerve of a category assign a simplicial set to a category [69].

The first step is to define the category of simplices. Then, we can construct a subcategory of **Set** which is "shaped" like the category of simplicies. Each such subcategory is known as a simplicial set. These simplicial sets can then be used to reconstruct categories, via the nerve and realisation, and thence to generalise categories into infinity categories.

2.2.1 The Category of Simplices

Rather than the ordinary definition of simplicies as higher dimensional tetrahedra, they shall be defined here as paths to facilitate comparison with operads. These results can be found in Goerss and Jardine [31]. For an excellent overview of simplicial sets, see Friedman [25]. Paths are simply linear directed graphs. It is not particularly enlightening to formalise the definition of paths here; the images should be sufficient for understanding.

Every path of length n is isomorphic to the path $[0, 1, 2, \ldots, n]$, denoted $[n]$, and referred to as an n-simplex. Just as a graph can be conceived of as itself generating a category, so too can each path. Given any simplex $[n]$, there is a functor i which sends it to its corresponding category $i(n)$. That is, the poset

$$0 \to 1 \to \ldots \to n.$$

In the literature, this distinction is often ignored, because in context the choice of meaning between the simplex $[n]$ or the category $i(n)$ is obvious.

All morphisms in Δ are written as a combination of *coface maps* and *codegeneracies*, denoted δ and σ respectively. The usual notation in the literature is d and s, but δ and σ shall be used instead to match the notation used for operads in future chapters. There are two types of face map, *inner* and *outer*. This distinction is not particular natural in Δ (except occasionally to distinguish δ^0 and δ^n from the rest of the coface maps), but it will become useful when generalising Δ in later chapters.

An inner face map involves contracting an inner edge (Figure 2.3), whereas an outer face map involves removing one of the two outer vertices along with the outer edge attached to it (Figure 2.4). Degeneracy maps, on the other hand, consist of subdividing a particular edge (Figure 2.5). However, these face and degeneracy maps describe the morphisms in Δ^{op}, and when describing morphisms in Δ the terms coface and codegeneracy are used and all arrows are reversed. Note also that superscripts are used for coface and codegeneracy maps, while subscripts are used for face and degeneracy maps.

In the literature, a coface map $[n] \to [n+1]$ corresponding to the face map which removes i would be referred to as δ^i, regardless of whether it is an inner or an outer coface map. However in later chapters the notation used will be slightly different: an inner face map which contracts an edge e would be denoted δ_e, whilst an outer face map which removes a vertex v would be denoted δ_v. Since it is the edges that are labelled with the numbers 0 to n, these notations do match up for inner face maps. However this does not work for outer face maps; these shall be denoted by the edge that was removed, rather than the vertex, for consistency with the usual notation for simplicies. Lastly, a degeneracy map where the edge e is subdivided would be denoted σ^e.

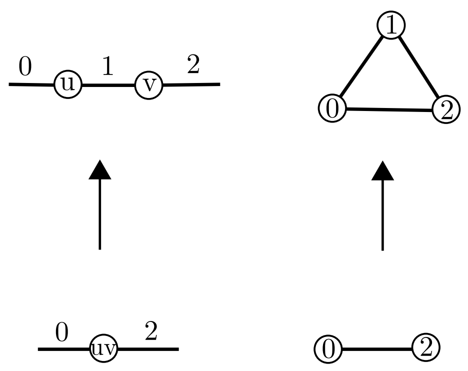


Figure 2.3: Inner coface map. On the left is the operadic conception. There is a face map from top to bottom where the edge labelled 1 is contracted. If the arrow is reversed, it defines a coface map. Compare this with the classical image on the right.

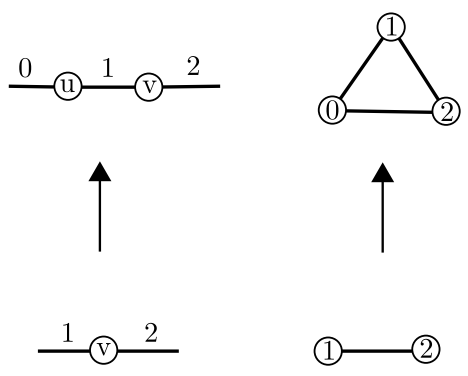


Figure 2.4: Outer coface map. On the left is the operadic conception, where the graph with the vertex u removed is included into the larger graph. Compare with the classical image on the right.

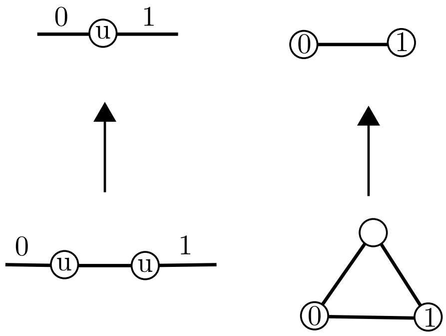


Figure 2.5: A codegeneracy map. We subdivide the edge labelled 1

These coface and codegeneracy maps, δ^i and σ^i, satisfy certain relations, called the simplicial identities, because to show that X is a simplicial set it is sufficient to show that these relations hold.

$$\sigma^j_{q+1}\sigma^i_q = \sigma^i_{q+1}\sigma^{j-1}_q \qquad i < j$$

$$\delta^j_q\delta^i_{q+1} = \delta^i_q\delta^{j+1}_{q+1} \qquad i \leq j$$

$$\delta^j_{q-1}\sigma^i_q = \begin{cases} \sigma^i_{q-1}\delta^{j-1}_{q-2} & i < j \\ 1 & i = j, j+1 \\ \sigma^{i-1}_{q-1}\delta^j_{q-2} & i > j+1 \end{cases}$$

The subscript q refers to the dimension of the simplex, i.e. the length of the path.

Definition 2.2.1 (The category of simplicies, Δ)**.** The simplex category, Δ, consists of directed paths as objects and compositions of coface and codegeneracy maps as morphisms.

Lemma 2.2.2. *Any map of simplicies can be written as a composition first of codegeneracies followed by coface maps.*

The proofs of this lemma is well known and not particularly enlightening, so it is omitted. It can be found in [50, Chapter VII.5, Proposition 2].

2.2.2 Simplicial Sets

A simplicial set is a subcategory of **Set** with the structure of Δ.

Definition 2.2.3 (Simplicial set)**.** A simplicial set is a pre-sheaf $\Delta^{op} \to$ **Set**.

Essentially, there are some sets X_n, one for each n, to which are mapped each n simplex. There are functions between these sets, which are generated by δ_i and σ_j and which satisfy the opposite of the simplicial relations above (Lemma **??**). Since simplicial sets are functors between categories, a morphism of simplicial sets is a natural transformation, with the category of simplicial sets known as **sSet**. Instead of forming a simplicial *set*, a simplicial structure can be imparted to any collection of objects by replacing the category **Set** with another. For example:

Definition 2.2.4 (Simplicial group)**.** A simplicial group is a functor $F : \Delta^{op} \to$ **Grp**. A morphism between simplicial groups is a natural transformation. The category of simplicial groups is **sGrp**.

As stated earlier, moving between categories and simplicial sets is important, and the *nerve* functor facilitates this.

Definition 2.2.5 (Nerve of a category)**.** Let $\mathcal{C}$ be a small category. Then the nerve $n :$ **Cat** $\to$ **sSet** is the functor $n(\mathcal{C}) : \Delta^{op} \to$ **Set** defined by

$$n(\mathcal{C})_j = Fun(i(j), \mathcal{C}),$$

where $Fun(i(j), \mathcal{C})$ is the set of all functors from the category $i(j)$ to the category $\mathcal{C}$.

For clarity, since j is not $[j]$ the poset but rather the category associated to it, $n(\mathcal{C})(j) = Fun(i(j), \mathcal{C})$ could be written instead. This nerve also has a left adjoint, called the *realisation*. The realisation of an object in **sSet** is defined here for clarity.

Definition 2.2.6 (Realisation)**.** Let $X : \Delta^{op} \to$ **Set** be a simplicial set. Then define a functor $\tau :$ **sSet** $\to$ **Cat** as follows:

- The objects ob$(\tau(X))$ are the elements of X_0
- The morphisms between A and B are elements of each X_n, representing the paths of length n from A to B, with composition by concatenation, subject to the equivalence relation generated by the following two relations.

    1. Let $\sigma_i : X_0 \to X_1$ be the map induced by the degeneracy $\sigma^i : [1] \to [0]$. Then for any object $A \in X_0$,

    $$\mathrm{id}_A = \sigma_i(A)$$

    2. Let $\delta_0, \delta_1, \delta_2 : X_2 \to X_1$ be the maps induced by $\delta^0, \delta^1, \delta^2 : [1] \to [2]$. Then

    $$\delta_0(x) \circ \delta_2(x) = \delta_1$$

    for all $x \in X_2$.

Condition 1 ensures that the identity morphism will be the degeneracy of the object. Condition 2 says that if, in a path of two vertices, the first morphism in the path is composed with the second, it is equivalent to contracting the edge between them.

2.3 Infinity Categories

Modelling certain situations involving homotopy can be incredibly difficult and category theory can break down when describing homotopy. Infinity categories come into play to deal with this. They are also useful in physics, especially with regard to the cobordism hypothesis [3, 48], and A_∞ algebras [9].

Infinity categories are categories where composition is weak, and simplicial sets are particularly useful in modelling this. Each object can be considered a point, each morphism an edge. Then strings of morphisms are higher dimensional complexes, with composition being represented by taking faces. In this way, composition up to homotopy may be achieved. For example, in a strict category there may be compatible morphisms f, g, and h such that $g \circ f = h$. In an infinity category, we may instead have f, g, and h as edges of the face $g \circ f$.

There are various ways of representing this connection to simplicial sets, and therefore various notions of infinity category, at least four of which are equivalent. For a discussion of these and their equivalences, see Bergner [7]. These four are:

- Simplicially enriched categories, i.e. categories enriched with simplicial sets [62, 20, 21],
- Quasi categories, which are simplicial sets satisfying an inner Kan condition [9],
- Segal categories, which are simplicial sets satisfying a Segal condition [69],
- Complete Segal spaces, which are simplicial spaces satisfying certain conditions [66].

An overview of these four will be provided because these are the four that are later discussed in the context of cyclic operads and modular operads.

Simplicially enriched categories and quasi categories were introduced first, followed by Segal and finally complete Segal spaces. The main focus is on quasi categories, using the inner Kan condition, but this is similar to the Segal condition. Complete Segal spaces require a knowledge of both conditions, and are particularly useful wherever there is a category without a discrete set of objects; e.g. cobordism categories with a space of objects.

The middle two notions of infinity category both rely on simplicial sets which satisfy a certain condition, either Segal or Kan. The strong versions of these conditions give categories, while the weak versions give infinity categories. For example, the relationship between categories, quasi categories, and simplicial sets is

$$\text{Categories} \subset \text{Quasi categories} \subset \text{Simplicial sets},$$

where categories are simplicial sets which satisfy the strict inner Kan condition and quasi categories are simplicial sets which satisfy the weak inner Kan condition.

2.3.1 Simplicially Enriched Categories

Simplicially enriched categories first appeared in Quillen [62, Chapter II, Section 1], then followed in the context of simplicial sets in Dwyer and Kan [20, 21].

The following definition comes from [75, Definition 2.1.1].

Definition 2.3.1. A simplicially enriched category is a category $\mathcal{C}$ enriched in **sSet**. It can be described explicitly as:

- There is a set of objects, $ob(\mathcal{C})$.
- Given any two objects A and B, the morphisms $\text{Hom}(A, B)$ form a simplicial set.
- Given any three objects A, B, and C, composition of morphisms between them
$$\text{Hom}(A, B) \times \text{Hom}(B, C) \to \text{Hom}(A, C)$$
is a morphism of simplicial sets.
- Given any object $A \in ob(\mathcal{C})$, there is a 0-simplex $id_A \in \text{Hom}(A, A)_0$.

And these are required to be associative and unital.

2.3.2 The Inner Kan Condition

Quasi categories are simplicial sets that satisfy the inner Kan condition. Quasi categories themselves are due to Boardman and Vogt [9], and there is a good reference by Goerss and Jardine [31, Section 1.3], where the following definitions are taken from. The representable of a path $[n]$ in Δ is its associated simplicial set.

Definition 2.3.2 (Representable)**.** Let $[n] \in \Delta$ be a path. Then its representable, $\Delta[n]$, is the simplicial set $\text{Hom}(-, [n])$. That is, for each $i \in \mathbb{N}$,

$$\Delta[n]_i = \text{Hom}([i], [n]).$$

Then, the faces of a representable correspond to the cofaces of the path.

Definition 2.3.3 (Face)**.** Let $n \in \Delta$ be a simplex with coface map $\alpha : m \to n$. Then the α-face of $\Delta[n]$ is the image of the map $\Delta[\alpha] : \Delta[m] \to \Delta[n]$. It is denoted $\partial_\alpha \Delta[n]$. The set of all faces of n is denoted $\text{Faces}(n)$

Finally, the union of all the faces is called the *boundary*. If a face is removed from the boundary, the result is a *horn*, and if this face is an inner face, then the result is an *inner horn*.

Definition 2.3.4 (Inner horn)**.** Let α be an inner coface map, and denote by i the edge which the associated face map contracts. Then

$$\Lambda^i[n] = \bigcup_{\beta \neq \alpha \in \text{Faces}(n)} \partial_\beta \Delta[n]$$

The Kan condition, in essence, states that the horn contains all the information necessary to reconstruct the whole complex, despite missing a face.

Definition 2.3.5 (Inner Kan condition)**.** Consider a simplicial set X, and let $f : \Lambda^k[n] \to X$ be a map from an inner horn to X, with $j : \Lambda^k[n] \rightarrowtail \Delta[n]$ being the inclusion. Then a filler for f is a map $g : \Delta[n] \to X$ such that $f = g \circ j$.

$$\begin{array}{ccc} \Lambda^k[n] & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle j} & \overset{g}{\nearrow} & \\ \Delta[n] & & \end{array}$$

The simplicial set X satisfies the inner Kan condition if every such map of inner horns has a filler.

This is equivalent to the induced map $j^* : \mathbf{sSet}(\Delta[n], X) \to \mathbf{sSet}(\Lambda^k[n], X)$ being surjective for all $0 < k < n$.

Definition 2.3.6 (Quasi category)**.** A simplicial set is said to be a quasi category if it satisfies the inner Kan condition.

Some quasi categories, those where the filler is unique (equivalently, the map j^* is bijective), are said to satisfy the strict inner Kan condition, and due to Theorem 2.3.7, they can be called categories.

Theorem 2.3.7 (Nerve)**.** *A simplicial set X satisfies the strict inner Kan condition if and only if there exists a category $\mathcal{C}$ such that $n(\mathcal{C}) = X$.*

Given the relationship between the inner Kan condition and nerves of categories, it would be natural to try to classify which simplicial sets satisfy the inner Kan condition. The following result (Lemma 2.3.8) uses the possibility of a group structure to help classify simplicial sets [60, Theorem 2.2]. The proof from Weibel [79] is reproduced here. The following lemma is both slightly stronger and slightly weaker than as cited. The original proves that groups satisfy the Kan condition in general, rather than just the inner Kan condition. Additionally, there is nothing in the proof that fails when applied to connected groupoids instead of groups.

Lemma 2.3.8 (8.2.8 in Weibel [79])**.** *If G is a simplicial connected groupoid, then the underlying simplicial set satisfies the inner Kan condition.*

Proof. Assume we have the kth horn in dimension n and we wish to find a filler. So we have $x_i \in G_n$ (where $i \neq k$) and we wish to find some $g \in G_{n+1}$ such that $\delta_i(g) = x_i$ for all $i \neq k$. We assume that our x_i's satisfy $\delta_i x_j = \delta_{j-1} x_i$ for $i < j$, and proceed by defining some g_r and inducting on r. At each stage, we wish to show that $\delta_i(g_r) = x_i$ holds for all $i \leq r$

We begin by setting $g_{-1} = 1 \in G_{n+1}$. Assume for the sake of induction that we have g_{r-1}. If $r = k$, then let $g_r = g_{r-1}$, since x_k is not included so we do not need to add anything extra. If $r \neq k$, let $g_r = g_{r-1}(\sigma_k u)^{-1}$, where $u = x_r^{-1}(\delta_r g_{r-1})$.

Consider $\delta_i(u)$

$$\begin{aligned}\delta_i(u) &= \delta_i(x_r^{-1}(\delta_r g_{r-1}))\\ &= \delta_i(x_r^{-1})\delta_i((\delta_r g_{r-1}))\\ &= (\delta_i x_r)^{-1}\delta_{r-1}\delta_i g_{r-1}\\ &= (\delta_{r-1}x_i)^{-1}\delta_{r-1}x_i\\ &= 1\end{aligned}$$

Now since $\delta_i(u) = 1$, we know that $\delta_i\sigma_k u = \sigma_{k-1}\delta_i u = \sigma_{k-1}1 = 1$. Therefore, we can say that

$$\begin{aligned}\delta_i(g_r) &= (\delta_i g_{r-1})(\delta_i(\sigma_k u))^{-1}\\ &= x_i 1\\ &= x_i\end{aligned}$$

□

2.3.3 The Segal Condition

The Segal condition states that, given just the local information about how the simplicial set is structured, the information about the global structure can be recovered. The Kan condition and the Segal condition both allow the whole to be reconstructed from partial information; however, they are useful in different circumstances. Luckily, these two conditions are equivalent, as seen in the slightly more general nerve theorem below (Theorem 2.3.11). Although the Segal condition is not used much in this section, it is useful for generalised categories and thus it may be useful to compare.

Definition 2.3.9 (Segal core)**.** Let $n > 0$. The Segal core $Sc[n]$ is the subobject of $\Delta[n]$ defined as the union of all the images of maps $\Delta[1] \to \Delta[n]$ corresponding to sub-paths $[2] \to [n]$. Then write

$$Sc[n] = \bigcup \Delta[1].$$

Here, sub-paths $[2] \to [n]$ refer to those paths consisting of two consecutive elements of of the ordered set $[n]$; that is, edges.

Definition 2.3.10 (Segal condition)**.** A simplicial set $X : \Delta^{op} \to \mathbf{Set}$ is said to satisfy the Segal condition if the following map is an isomorphism for all paths $[n]$.

$$\mathrm{Hom}(\Delta[n], X) \cong \mathrm{Hom}(Sc[n], X)$$

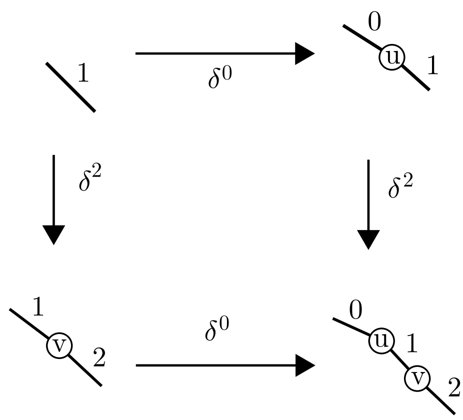
There is an equivalence between the Segal and inner Kan conditions and the nerve of a category. This is the nerve theorem [32, 69].

Theorem 2.3.11 (Nerve)**.** *Let X be a simplicial set. Then the following are equivalent:*

1. *There exists a category $\mathcal{C}$ such that $X = n(\mathcal{C})$*
2. *X satisfies the strict inner Kan condition*
3. *X satisfies the Segal condition*

The proof is omitted, but Example 2.3.12 is illustrative.

Example 2.3.12. The first n for which there exists an inner horn is $n = 2$, with only one inner face, so $\Lambda^1[2]$ is the union of the faces associated to [0] and [2]. But the Segal core is the union of these two outer faces, so $\Lambda^1[2] = Sc[2]$ and therefore $\mathrm{Hom}(\Lambda^1[2], X) \cong \mathrm{Hom}(Sc[2], X)$.



2.3.4 Complete Segal Spaces

Complete Segal spaces were first introduced by Rezk [66]. They are shown to be equivalent to the other models in [42]. A good reference, which this section is based on, is [63]. Morally, a complete Segal space is a two dimensional simplicial set satisfying certain properties.

- Along one dimension it has the *Reedy fibrancy* condition, which makes it act like a topological space.
- Along the other dimension it obeys the *Segal* condition, which means it is like a category
- There is a final condition called the *completeness* condition. Intuitively, it collapses homotopy equivalences to the identity.

In order to define complete Segal spaces, simplicial spaces must first be defined, as well as each of these conditions.

Definition 2.3.13 (Simplicial Space)**.** A simplicial space is a functor $X : \Delta^{op} \to$ **sSet**, where each space X_n is itself a simplicial set

The category of simplicial spaces is denoted **sSpace**. One can picture a simplicial space either as a simplicial set where each set is itself a simplicial set (also known as a space), or one can picture it as $\Delta^{op} \times \Delta^{op} \to$ **Set**, a bisimplicial set X_{mn}, with both a horizontal axis and a vertical one.

$$\begin{array}{ccccccc}
X_{00} & \longleftrightarrow & X_{10} & \longleftrightarrow & X_{20} & \longleftrightarrow & \dots \\
\updownarrow & & \updownarrow & & \updownarrow & & \\
X_{01} & \longleftrightarrow & X_{11} & \longleftrightarrow & X_{21} & \longleftrightarrow & \dots \\
\updownarrow & & \updownarrow & & \updownarrow & & \\
X_{02} & \longleftrightarrow & X_{12} & \longleftrightarrow & X_{22} & \longleftrightarrow & \dots \\
\updownarrow & & \updownarrow & & \updownarrow & & \\
\vdots & & \vdots & & \vdots & & \ddots
\end{array}$$

Note that each column $X_{n\bullet}$ is itself a simplicial set, and one can read across to form a simplicial space. Each double headed arrow stands in for all the face and degeneracy maps between entries. The first property comes from an alternative characterisation of the inner Kan condition, using the language of model categories. Firstly, a couple of terms need to be defined, which can be found in [39].

Definition 2.3.14 (Left lifting property)**.** Let $\mathcal{C}$ be a category. The morphism f has the left lifting property with respect to g if, for every commutative square

$$\begin{array}{ccc}
a & \xrightarrow{u} & c \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
b & \xrightarrow{v} & d
\end{array}$$

there exists a γ such that both triangles commute.

$$\begin{array}{ccc}
a & \xrightarrow{u} & c \\
\downarrow{\scriptstyle f} & \overset{\gamma}{\nearrow} & \downarrow{\scriptstyle g} \\
b & \xrightarrow{v} & d
\end{array}$$

Note that g is also said to have the right lifting property with respect to f, and γ is known as the *lift* or solution to (u, v).

Definition 2.3.15 (Inner Kan fibration)**.** A map of simplicial sets $f : X \to Y$ is an inner kan fibration if it has the left lifting property with respect to any inner horn inclusion.

Now the definition of inner Kan complex (that is, quasi category) can be recovered, and Reedy fibrancy can be defined.

Definition 2.3.16 (Inner Kan complex)**.** A simplicial set is an inner Kan complex if the map $X \to 1$ is an inner kan fibration, where 1 is the terminal object of **sSet**.

Definition 2.3.17 (Reedy fibrant)**.** A simplicial space X is said to be Reedy fibrant if for every path n, the following map of spaces is a Kan fibration:

$$Map_{\mathbf{sSpaces}}(F(n), X) \to Map_{\mathbf{sSpaces}}(\partial F(n), X)$$

Here, F is defined to be the functor from simplicial sets to simplicial spaces which assigns to each simplicial set the constant space. That is, $F(n)_{ij} = \Delta[n]_i$. Then the boundary, $\partial F(n)$, is as it would be for a simplicial set.

Definition 2.3.18 (Segal space)**.** A simplicial space X is a Segal space if:

- It is Reedy fibrant.
- As a simplicial object it satisfies the Segal condition (Definition 2.3.10).

Definition 2.3.19 (X_{hoequ})**.** Given a simplicial set X, the space of homotopy equivalences of X, X_{hoequ}, is defined to be a subspace of X_1 which consists of exactly those components whose points are homotopy equivalences.

Finally, a complete Segal space can be defined.

Definition 2.3.20 (Complete Segal space)**.** A Complete Segal Space (CSS) X is a Segal space which satisfies the completeness condition; that is,

$$s_0 : X_0 \to X_{hoequ}$$

is a weak equivalence.

Chapter 3

Operads and Dendroidal Sets

Recall that the information contained in a category's morphisms and compositions can be encoded in directed paths, with the edges representing the objects and the vertices representing the morphisms between them. In the same way, operads can be thought of as rooted trees; each vertex representing a morphism, each edge representing an object, and multiple inputs allowed on each morphism.

However, this is not how they were historically conceived. Operads arose in the '60s and '70s as a result of work in classifying loop spaces [8, 9, 55] and computing H spaces [72], and, interestingly, were used for dealing with infinity categories [55]. They first appeared in the literature in Boardman and Vogt [8, 9], although the idea had already been circulating [43]. They can be found as multicategories in Lambek [44]. Operads first appear under that name in May [55]. They experienced a renaissance in the '90s for their use in algebras [27, 28, 29, 52]. A particularly good introduction can be found in Markl, Shnider, and Stasheff [53].

Just as the category Δ imparts the path shape of categories onto sets, in the form of simplicial sets, so too does the category of trees impart its shape on sets as dendroidal sets. This category of trees is known as Ω. In Chapter 2 simplicial sets were discussed, being built up from the category Δ. Here, in a similar way, dendroidal sets are used to form the theory of infinity operads.

Some parts, particularly the dendroidal sets section, are based on [56].

3.1 Operads

Operads are similar to categories, except each morphism is allowed to have more than one input. Given a set of colours C, a profile is a finite list $\underline{c} = (c_1, \dots c_n; c_0)$, where each $c_i \in C$

Definition 3.1.1 (Coloured operad). An operad O in a closed symmetric monoidal category $\mathcal{E}$ consists of the following data. Unless otherwise specified, assume all operads are over **Set**.

- A set C, known as the colours (i.e. objects), of the operad.
- For every profile $\underline{c} = (c_1, \dots, c_n; c_0)$, there is an object $\mathcal{O}(\underline{c})$ in $\mathcal{E}$. If $\mathcal{E}$ is **Set**, or the objects of $\mathcal{E}$ have underlying sets, then the elements of this set are referred to as operations.

- For every colour $c \in C$, there is an operation $\eta_c \in \mathcal{O}(c;c)$ which functions like an identity with respect to the composition defined below.
- For each $\mathcal{O}(\underline{c})$, there is a right action of the symmetric group Σ_{n+1} on $\underline{c}$ which fixes c_0.
- For all profiles, and for all $0 < i \leq n$ where $c_0 = d_i$, there is a composition

$$\circ_i : \mathcal{O}(\underline{c}) \otimes \mathcal{O}(\underline{d}) \to \mathcal{O}(d_1, \ldots, d_{i-1}, c_1, \ldots, c_n, d_{i+1}, \ldots, d_m; d_0).$$

For ease of reading there is a slight abuse of notation where profiles resulting from composition may be shortened. For example,

$$\underline{cd}_i = (d_1, \ldots, d_{i-1}, c_1, \ldots, c_n, d_{i+1}, \ldots, d_m; d_0).$$

This composition satisfies the following:

  - Associativity: all diagrams of this form commute

$$\begin{array}{ccc} \mathcal{O}(\underline{c}) \otimes \mathcal{O}(\underline{b}) \otimes \mathcal{O}(\underline{a}) & \xrightarrow{id \otimes \circ_i} & \mathcal{O}(\underline{c}) \otimes \mathcal{O}(\underline{ba}_i) \\ \downarrow{\scriptstyle \circ_j \otimes id} & & \downarrow{\scriptstyle \circ_{i+j-1}} \\ \mathcal{O}(\underline{cb}_j) \otimes \mathcal{O}(\underline{a}) & \xrightarrow{\circ_i} & \mathcal{O}(\underline{cba}_{ji}) \end{array}$$

  - Unitality: for all identities η_{c_i} and operations $\theta \in \mathcal{O}(\underline{c})$, there is

$$\theta \circ_i \eta = \theta = \eta \circ_i \theta$$

  - Equivariance: Let $\theta \in \mathcal{O}(\underline{c})$ and $\theta' \in \mathcal{O}(\underline{d})$. For all $\sigma \in \Sigma_n$ and $\sigma' \in \Sigma_m$, there is

$$(\theta\sigma) \circ_i (\theta'\sigma') = (\theta \circ_{\sigma(i)} \theta')(\sigma' \circ_i \sigma)$$

    where $(\sigma' \circ_i \sigma)$ means inserting σ into the ith place in σ'.

Note that $\mathcal{O}(\underline{c})$ is akin to $\mathcal{C}(A, B)$, and the elements of the underlying set (provided it exists) are referred to as either morphisms or operations.

In the literature, objects satisfying the definition above are sometimes called "coloured operads", and only those with just one colour are called "operads". Here, operads of only one colour are called "monochromatic". In a monochromatic operad, $\mathcal{O}(x, \ldots x; x)$ is often written as $\mathcal{O}(k)$, where x is repeated k times (so there are $k+1$ x's in total).

Definition 3.1.2 (Maps between operads)**.** Let P be a C-coloured operad and Q a D-coloured one, both over the same symmetric monoidal category $\mathcal{E}$. Then a map between P and Q consists of

- A map of colours $f : C \to D$
- Maps $\varphi_{c_1,\ldots c_n;c_0} : P(c_1, \ldots c_n; c_0) \to Q(f(c_1), \ldots f(c_n); f(c_0))$

With the maps being compatible with composition.

With these maps as morphisms, operads form a category, **Opd**. In this thesis, **Op** will denote the category of monochromatic operads and **Opd** will denote the category of colour operads.

Below are some examples of operads.

Example 3.1.3. Given any small category $\mathcal{C}$ one can construct an operad $m(\mathcal{C})$. The colours are the objects of the category. For any profile of length 2, the set of operations $m(\mathcal{C})(B; A)$ is just $\mathrm{Hom}(A, B)$, and for any profile $\underline{c}$ of length ≥ 2, $m(\mathcal{C})(\underline{c}) = \emptyset$.

Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism between two categories. Then $m(f) : m(\mathcal{C}) \to m(\mathcal{D})$ is defined by

- For any colour $c \in ob(m(\mathcal{C}))$, let $m(f)(c) = f(c)$
- For any profile $\underline{c}$,

$$m(f_{\underline{c}}) : m(\mathcal{C})(B; A) \to m(\mathcal{D})(m(B); m(A))$$

This defines a functor $m : \mathbf{Cat} \to \mathbf{Opd}$.

Example 3.1.4. Any symmetric monoidal category $\mathcal{C}$ can also provide an operad $\mathcal{C}$:

- The colours are the objects of the category.
- The operations $\mathcal{C}(B; A)$ are given by the morphisms $\mathrm{Hom}(A, B)$.
- For any profile $\underline{c}$, the operations $\mathcal{C}(\underline{c})$ are given by the set of morphisms

$$c_1 \otimes c_2 \otimes \ldots \otimes c_n \to c_0.$$

Part of the motivation for studying operads comes from their connection to algebras, and from here arise certain well-known examples of monochromatic operads.

Example 3.1.5 (The associative operad, **Ass**)**.** For each k, $\mathbf{Ass}(k) = \Sigma_k$, with the action of the symmetric group on itself in the obvious way.

When the corresponding algebra is defined, one gets an associative algebra.

Definition 3.1.6 (Algebra)**.** Given an operad P in a symmetric monoidal category $\mathcal{E}$, one can define a P algebra. This consists of

- A family $(A_c)_{c \in C}$ of objects in $\mathcal{E}$
- Actions $P(c_1, \ldots c_n; c_0) \otimes A_{c_1} \otimes \ldots A_{c_n} \to A_{c_0}$

The most well known example of an operad of surfaces is the little discs operad. This deals with embedding discs within discs, which could instead be considered as gluing surfaces of genus zero to each other in a particular way (see Figure 3.1).

Definition 3.1.7 (Little discs operad [77])**.** Let D^n be the open unit disc of dimension n. Then for all $k \in \mathbb{N}$, let $E_n(k)$ be the space of embeddings of k disjoint discs into a disc,

$$f : \coprod_k D^n \to D^n$$

where f is a composition of translations and dilations only. Then, the operad of little discs is an operad over **Top** defined as follows.

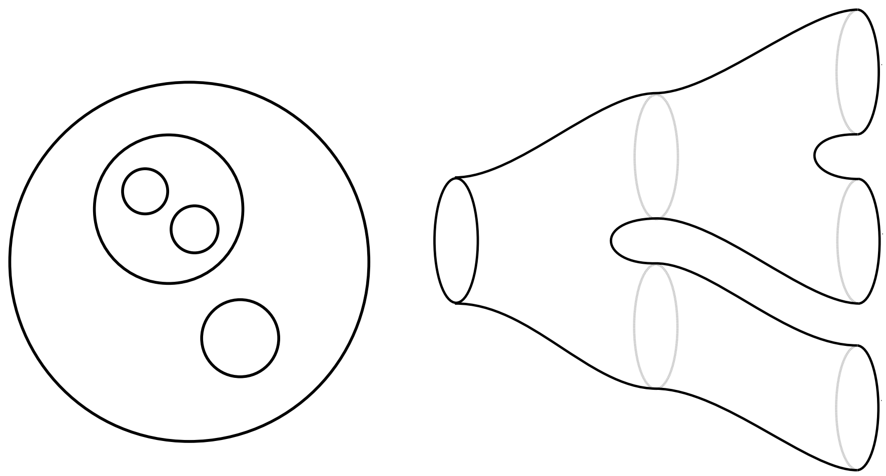


Figure 3.1: What an operation in the operad of little discs might look like. This is an element of the image of $E_2(2) \circ_1 E_2(2) \to E_2(3)$. On the left, the embeddings of disc inside disc are clearly visible. On the right is an alternative view, where the boundaries of the discs correspond to copies of S^1 and the embeddings are represented by cobordisms.

- For each k, the operations $E_n(k) = E_n(D_n, D_n, \dots D_n; D_n)$ are the aforementioned spaces of embeddings.
- The symmetric group acts by permuting the ordering of the discs.
- The compositions are defined by compositions of disjoint unions of maps:

$$\circ(f, g_1, \dots g_k) = \coprod_{n_1+\dots n_k} D^n \xrightarrow{g_1 \sqcup \dots g_k} \coprod_k D^n \xrightarrow{f} D^n$$

Other examples of operads of surfaces would be the mapping class group operad (an original example, defined later in the undirected case for modular operads), and the operad of cobordisms [74].

3.2 Dendroidal Sets

Dendroidal sets are introduced to provide a framework for infinity operads, just as simplicial sets provide one for categories. Dendroidal sets were first introduced by Moerdijk and Weiss [57] to model infinity operads, and further studied in [58, 15, 16, 17]. An excellent reference for dendroidal sets and infinity operads is [56].

Categories are made up of objects, morphisms, and compositions of morphisms. Because morphisms are arrows going from a single object to another, the compositions of morphisms can look like paths traced out between objects. Therefore, simplicial sets are modelled by the category of paths, Δ. However, operations in an operad are arrows from multiple objects to one, so they are better modelled by a category of trees. In particular, operads resemble rooted trees, since morphisms in an operad make a distinction between the input and the output. These relationships can be summarised in the following table, where Ω is the category of rooted trees and **dSet** is the category of presheaves over Ω, known as dendroidal sets.

Cat	Δ	**sSet**
Opd	Ω	**dSet**

In an infinity category, the morphisms can be thought of as faces of a larger composition, and the inner Kan condition essentially says that compositions exist when compatible morphisms exist. The inner Kan condition relies on the notion of faces. An analogue for operads is needed, some notion of faces and degeneracies of trees, and then an inner Kan condition to describe infinity operads. Section 3.2.1 outlines the category of trees, and describes face and degeneracy maps of trees. Then the inner Kan condition can be found in Section 3.3.2.

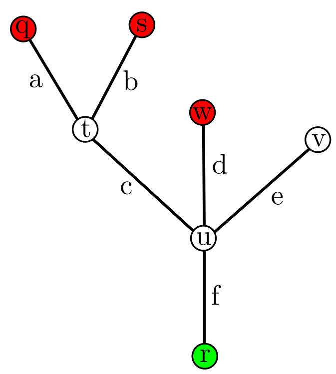
3.2.1 The Category of Dendrices

First, it is necessary to understand the category of dendrices, which imparts structure to dendroidal sets and thence to infinity operads. The particular graph formalism chosen does not matter [5, Propositions 15.2, 15.6, and 15.8]. In the formalism used herein, a graph consists of a set of vertices $V(G)$ and a set of edges $E(G)$, where each edge is a subset of $V(G)$ of size two. Additionally, each edge must be unique and consisting of two different vertices, to avoid multi edges and loops. A vertex is called *outer* if it is incident to only one edge.

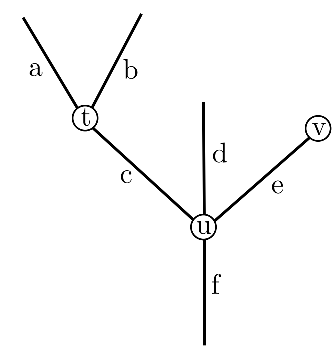
Definition 3.2.1 (Rooted tree)**.** Consider a non-empty, finite, simply connected graph T with a distinguished outer vertex v_o (called the output vertex) and a distiguished set of outer vertices I (called the set of inputs).Then (T, v_o, I) is a rooted tree. Usually, such a tree is simply referred to as T.

The edge connected to the output vertex is called the root, and the edges connected to the input vertices are called the legs. These edges are sometimes called leaves, but to avoid confusion by graph theorists they shall hence be called legs. For now, until astroidal sets and unrooted trees are introduced later, these rooted trees will be referred to as just trees.

Example 3.2.2. In the following tree, the output vertex is r, and the set of inputs is $\{q, s, w\}$. The root is f and the legs are $\{a, b, d\}$ Note that there may be some outer vertices that are neither inputs nor outputs: this is allowed.



By convention, the outer and inner vertices are not drawn.



Note that, as this example is one of the simplest trees that is complex enough to demonstrate these ideas in general, it can be found as an example in other places in the literature too, including [57].

The root induces a directionality on the tree, and by convention trees are drawn so that the root is at one end and the legs (a.k.a. leaves) are at the other. The operations then flow from the legs to the root, so the legs of the tree are the inputs and the root is the output. Likewise, the inputs of each vertex are those edges above, while the output of a vertex is the edge below. As a convention, the output and input vertices of a tree are not drawn, and the word "vertex" refers only to the remaining vertices.

There are some other parts of trees that need defining as these words will be used later on.

Definition 3.2.3 (Inner edge)**.** An inner edge is an edge that is neither a leg nor the root.

Conversely, an *outer edge* is an edge that is not an inner edge.

Definition 3.2.4. Given a vertex v, define

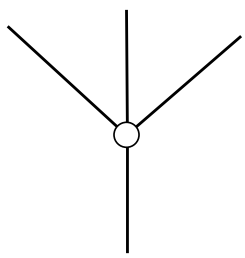
- $in(v)$ = the set of incoming edges.
- $out(v)$ = the outgoing edge.
- The valence of v is the cardinality of $in(v)$
- The output of v is the element of $out(v)$

In Example 3.2.2, $out(t) = \{c\}$, $in(t) = \{a, b\}$, and t has valence 2. The vertex v has no input vertices, so $in(v) = \{\}$ and $out(v) = \{e\}$. Notice that the output and input vertices are not drawn (however, notice that v, despite being an outer vertex, is neither an input nor an output vertex).

The following trees are important examples.

Definition 3.2.5 (Corolla)**.** The n-corolla C_n is the tree that has a single vertex v, with $n-1$ inputs and one output. It can equivalently be written as the $C_{(n-1)+1}$ corolla to emphasise the difference between inputs and outputs.

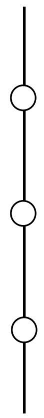
Example 3.2.6. This is the 4-corolla, C_4, also called C_{3+1}.



Sometimes, when talking about a corolla around a particular vertex v, the notation C_v may be used instead.

Definition 3.2.7 (Path)**.** The n-path P_n is a tree which has n vertices (not including the input and output vertices), and each vertex has exactly one input. In reality, it actually has $n+2$ vertices due to the presence of the input and output vertices, but those are suppressed.

Example 3.2.8. This is the 3-path



Example 3.2.9. It is possible to have just a single edge with no vertices. This is the graph η, also called the path of length 0, P_0.

Relationship Between Trees and Operads

Given a tree T, one can define an operad $\Omega(T)$

Definition 3.2.10. Let T be a tree. Then define the corresponding operad $\Omega(T)$ (over **Set**) as follows.

- The colours of $\Omega(T)$ are the edges of T.
- The morphisms are generated by the vertices. For all vertices v with $in(v) = \{c_1, \dots, c_n\}$ and $out(v) = c_0$, and all permutations $\sigma \in \Sigma_n$, let $v_{\sigma(c_1 \dots c_n)} \in \Omega(T)(c_{\sigma(1)}, \dots c_{\sigma(n)}; c_0)$.
- For each colour c, let η_c be the identity. On the tree, this corresponds to the edge c without any vertices.

- The action of the symmetric group is given by

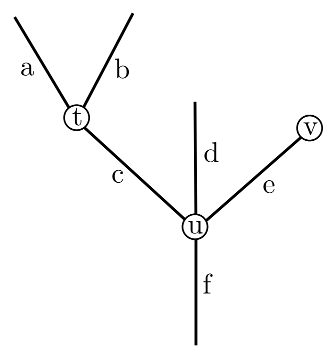
$$\sigma(v_{c_1\dots,c_n}) = v_{\sigma(c_1\dots c_n)} \in \Omega(T)(c_{\sigma(1)},\dots c_{\sigma(n)};c_0).$$

- Two morphisms v and w may be composed at i if the output leg of v is the same edge as the i input leg of w. Call this result $v \circ_i w$.

Thus, for each profile $\underline{c}$, the set $\Omega(T)(\underline{c})$ consists of both

- vertices with c_0 as output and $c_1,\dots,c_n$ as inputs, and
- compositions of these vertices, each corresponding to a subtree of T.

Example 3.2.11. Consider this tree, T [57]:



The generating morphisms are:

$$\begin{aligned} t_{a,b} &\in T(a,b;c)\\ t_{b,a} &\in T(b,a;c)\\ v &\in T(;e)\\ u_{c,d,e} &\in T(c,d,e;f)\\ u_{c,e,d} &\in T(c,e,d;f)\\ u_{d,c,e} &\in T(d,c,e;f)\\ u_{d,e,c} &\in T(d,e,c;f)\\ u_{e,c,d} &\in T(e,c,d;f)\\ u_{e,d,c} &\in T(e,d,c;f) \end{aligned}$$

There are also compositions (such as $u_{c,d,e} \circ_3 v \in T(c,d;f)$), as well as identities (such as $\eta_c \in T(c;c)$).

Definition 3.2.12 (The category of trees, Ω)**.** Let Ω be the full subcategory of **Opd** consisting of those operads which can be given by $\Omega(T)$ for all trees T.

Thus, trees can be considered as operads, and morphisms of trees are operad morphisms. However, in the next section a graphical representation of these morphisms will be given. If a distinction is necessary, T will refer to the tree and $\Omega(T)$ will refer to the tree as an operad.

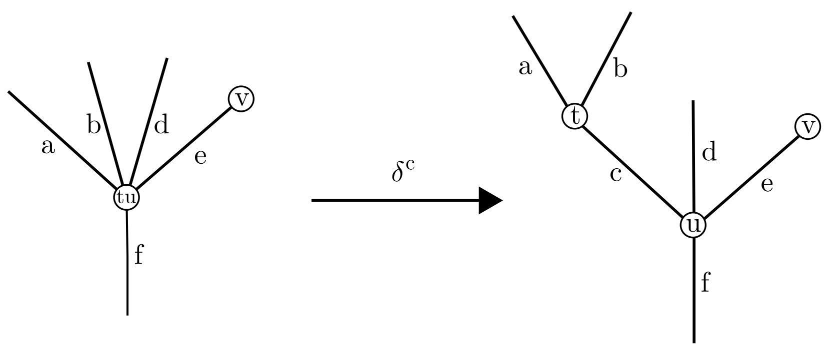
Morphisms of trees

Analogously to the simplicial set category Δ, morphisms in Ω can be compiled from simpler maps, called *elementary maps*. There are two types of coface maps, inner and outer, as well as codegeneracy maps.

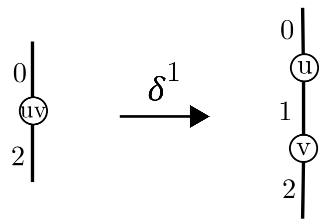
Definition 3.2.13 (Inner coface map)**.** Let T be a tree with an inner edge b, and let T/b be the tree obtained by contracting the edge b of T. There is a map $\delta^b : T/b \to T$ called the inner coface map. Let the two vertices that b is attached to be called x and y, where x is an input vertex of b and y the output vertex of b. The associated vertex in T/b is called xy. This map can be defined precisely when trees are considered as operads:

- It is an inclusion map on the colours (edges),
- The operation v_{xy} maps to the operation $y \circ_b x$,
- All other generating operations (vertices) map to themselves.

Example 3.2.14. An example of an inner coface map.



Example 3.2.15. Any inner coface map between paths, as in the simplicial case.

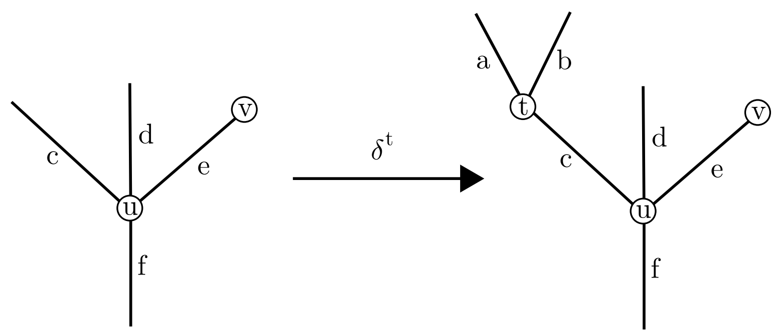


Definition 3.2.16 (Outer coface map)**.** Let T be a tree. Let v be a vertex with exactly one inner edge attached to it. Then let T/v be the tree obtained by deleting v and all the outer edges attached to it. There is an outer coface map $\delta^v : T/v \to T$. As trees, this is defined by the inclusion of T/v into T. As operads, this is defined by:

- For all colours (edges) $e \in E(T/v)$, $e \mapsto e$,
- For all operations (vertices) x in the generating set, $x \mapsto x$.

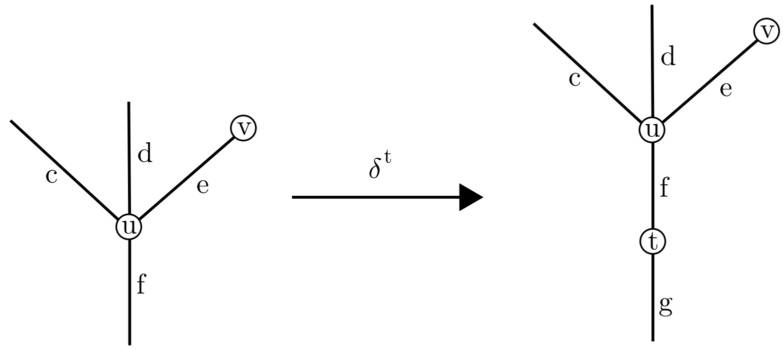
In addition, there is a special case. Let C_n be the corolla (Definition 3.2.5) containing a single vertex v. Then there are n outer coface maps $\delta^v : \eta \to C_n$, one for each edge. If its clear, then δ^v may refer to any of them, but if it is ambiguous then the notation $\delta^{v,e}$ may be used to denote that η maps to the edge e. Since these graphs are labelled, the $\delta^{v,e}$ notation is rarely necessary.

Example 3.2.17. An example of an outer coface map.



Example 3.2.18. Sometimes, the vertex incident to the root may be incident to exactly one inner edge. This is the only situation in which the root may be

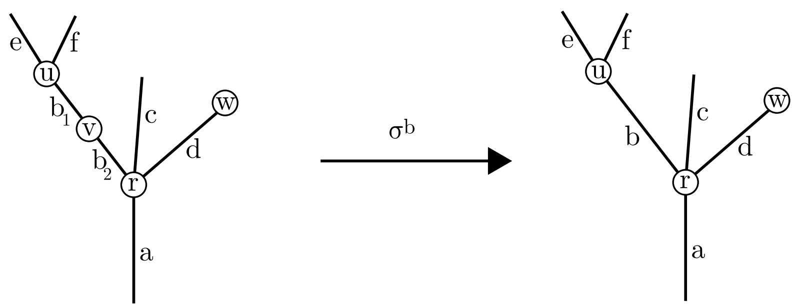
removed. If the vertex incident to the root is incident to more inner edges then when the root is removed there would be multiple edges that are candidates to become the new root, each the output of their attached vertices, but a tree cannot have multiple roots. Nor could these edges be removed, as that would cause the tree to become disconnected.



Definition 3.2.19 (Codegeneracy map)**.** Consider a tree T with some edge b. Then let T_b be the result of subdividing b; that is, splitting the edge b into two edges b_1 and b_2 and adding an extra vertex v which is connected to both copies of b. Then there is a codegeneracy map $\sigma^b : T_b \to T$ defined by:

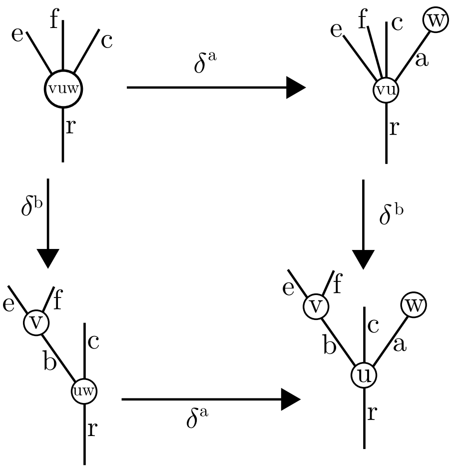
- The colours b_1 and b_2 map to b
- All other colours map to themselves
- The generating operation v maps to the identity id_b
- All other generating operations map to themselves

Example 3.2.20. An example of a codegeneracy map.



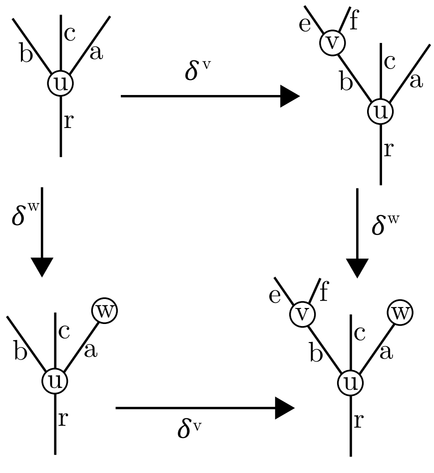
Analogously to simplicial sets, there are relations between these maps. Given two (distinct) inner coface maps δ^a and δ^b, they are related by $(T/a)/b = (T/b)/a$ and:

$$
\begin{array}{ccc}
(T/b)/a & \xrightarrow{\delta^b} & T/a \\
\big\downarrow{\scriptstyle \delta^a} & & \big\downarrow{\scriptstyle \delta^a} \\
T/b & \xrightarrow{\delta^b} & T
\end{array}
$$



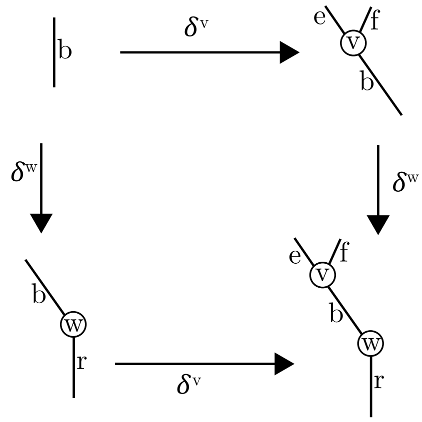
Let T be a tree with at least 3 vertices, and consider two distinct outer coface maps δ^v and δ^w. Then $(T/v)/w = (T/w)/v$ and the diagram commutes:

$$
\begin{array}{ccc}
(T/v)/w & \xrightarrow{\delta^v} & T/w \\
\big\downarrow{\scriptstyle \delta^w} & & \big\downarrow{\scriptstyle \delta^w} \\
T/v & \xrightarrow{\delta^v} & T
\end{array}
$$



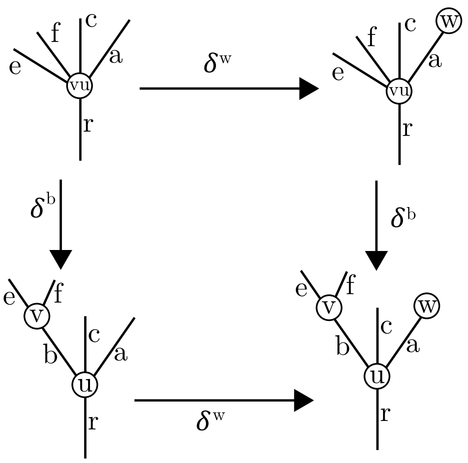
If T is a tree with only 2 vertices, v and w, then (ensuring that the vertex attached to the root has valence 1) the following diagram commutes. If there are less than two vertices in T, then two vertices cannot be removed.

$$\begin{array}{ccc} \eta & \xrightarrow{\delta^v} & T/w \\ \big\downarrow{\scriptstyle \delta^w} & & \big\downarrow{\scriptstyle \delta^w} \\ T/v & \xrightarrow{\delta^v} & T \end{array}$$



Finally, an inner coface map can be combined with an outer coface map. Let δ^v be an outer coface map and δ^e be an inner coface map. If v and e are not adjacent, then $(T/v)/e = (T/e)/v$ and the diagram commutes:

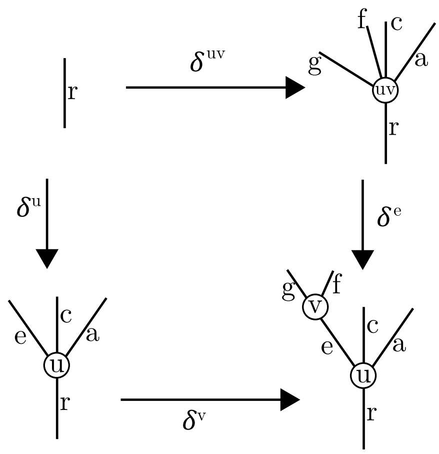
$$\begin{array}{ccc} (T/v)/e & \xrightarrow{\delta^e} & T/v \\ \big\downarrow{\scriptstyle \delta^v} & & \big\downarrow{\scriptstyle \delta^v} \\ T/e & \xrightarrow{\delta^e} & T \end{array}$$



Suppose v and e are adjacent. Then denote the vertex on the other side of e by u. The tree T/e combines v and u into a single vertex, denote this uv. Then

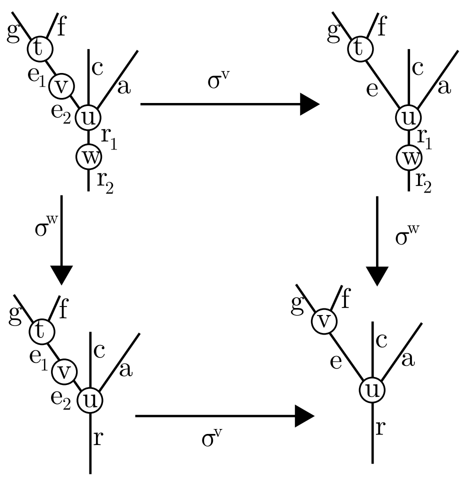
$(T/e)/uv$ exists if and only if $(T/v)/u$ exists, and $(T/e)/uv = (T/v)/u$, in the following commutative diagram

$$\begin{array}{ccc} (T/e)/uv & \xrightarrow{\delta^{uv}} & T/e \\ \downarrow{\scriptstyle \delta^{u}} & & \downarrow{\scriptstyle \delta^{e}} \\ T/v & \xrightarrow{\delta^{v}} & T \end{array}$$



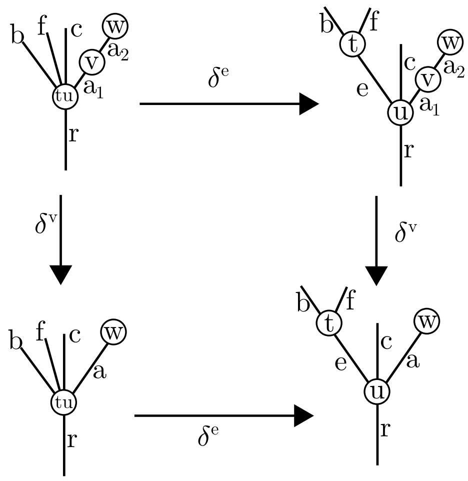
Now consider two degeneracies of T σ^v and σ^w. Then $(T_v)_w = (T_w)_v$ and the following diagram commutes

$$\begin{array}{ccc} T & \xrightarrow{\sigma^{v}} & T_v \\ \downarrow{\scriptstyle \sigma^{w}} & & \downarrow{\scriptstyle \sigma^{w}} \\ T_w & \xrightarrow{\sigma^{v}} & (T_v)_w \end{array}$$

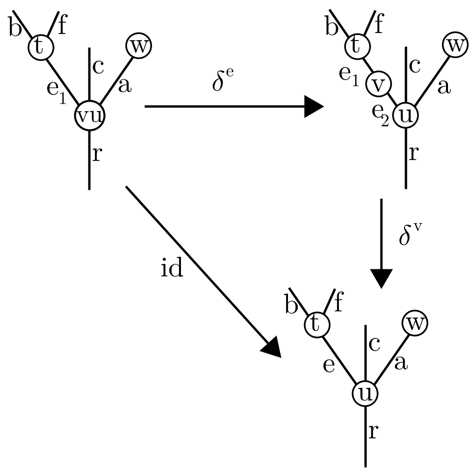


Finally, there are the combined relations. Let $\sigma^v : T \to T_v$ be a codegeneracy and $\delta^e : T/e \to T$ a coface map that does not eliminate v. Then if $\delta^e : T/e_v \to T_v$ is the induced coface map, the following diagram commutes:

$$\begin{array}{ccc} T/e & \xrightarrow{\delta^e} & T \\ \downarrow{\scriptstyle \sigma^v} & & \downarrow{\scriptstyle \sigma^v} \\ T/e_v & \xrightarrow{\delta^e} & T_v \end{array}$$



Otherwise, if δ^e is a coface map induced by either v itself or one of the adjacent edges to v, the following is the identity. $T\backslash v \xrightarrow{\delta^e} T \xrightarrow{\sigma^v} T\backslash v$



These relations can be summarised in the following table. The columns indicate the face we start with and the rows indicate the face operation we do to it. Note that p and q refer to vertices, $\overline{pq}$ the edge between them, and pq the vertex resulting from contracting the edge $\overline{pq}$. Whenever two elementary maps are induced by vertices and edges which are disjoint from each other, they commute. Therefore, in this table, assume they are coincident to each other. That is, involving two vertices with a single edge between them, or two edges incident to the same vertex, or a vertex which is incident to an edge.

inner and inner	$\delta^{\overline{pq}}\delta^{\overline{qr}} = \delta^{\overline{qr}}\delta^{\overline{pq}}$
inner and outer	$\delta^{pq}\delta^{\overline{pq}} = \delta^{q}\delta^{p}$
inner and degeneracy	$\sigma^{\overline{p(qr)}}\delta^{\overline{qr}} = id = \delta^{\overline{qr}}\sigma^{\overline{pq}}$
outer and outer	$\delta^{pq}\delta^{\overline{pq}} = \delta^{q}\delta^{p}$
outer and degeneracy	$\sigma^{\overline{pq}}\delta^{r} = \delta^{r}\sigma^{\overline{pq}}$
degeneracy and degeneracy	$\sigma^{\overline{p(pq)}}\sigma^{\overline{pq}} = \sigma^{\overline{(pq)q}}\sigma^{\overline{pq}}$

Lemma 3.2.21 (Lemma 3.1 in [57])**.** *Any morphism of trees (that is, any operad map $\Omega(S) \to \Omega(T)$, where S and T are trees) can be decomposed into a series of codegeneracy maps followed by a series of coface maps.*

3.2.2 Dendroidal Sets

Dendroidal sets, first appearing in Moerdijk and Weiss [57] are formed in the same way as simplicial sets: presheaves over some category. Here, the category giving the shape of the objects is Ω rather than Δ.

Definition 3.2.22 (Dendroidal Set)**.** A dendroidal set X is a presheaf $X : \Omega^{op} \to \mathbf{Set}$

In other words, a dendroidal set X is:

- For each $T \in \Omega$, a set $X(T)$, denoted X_T. Note that X_T is called the set of dendrices of shape T, and each element is called a dendrex.
- For each morphism $f : S \to T$, a function $X_f : X_T \to X_S$ such that
  - $X_{id:T\to T} = id : X_T \to X_T$,
  - Given two morphisms $R \xrightarrow{\alpha} S \xrightarrow{\beta} T$ in Ω, $X_{(\beta\circ\alpha)} = X_\alpha \circ X_\beta$.

In Ω, all morphisms can be formed from compositions of coface and codegeneracy maps, δ^x and σ^x. These maps then translate into face and degeneracy maps of dendroidal sets, written δ_x and σ_x (rather than X_{δ^x} and X_{σ^x}), and morphisms in dendroidal sets can be formed from compositions of these elementary maps.

Dendroidal sets form a category **dSet** with natural transformations as morphisms.

Definition 3.2.23 (Morphisms)**.** Let $X : \Omega^{op} \to \mathbf{Set}$ and and $Y : \Omega^{op} \to \mathbf{Set}$ be two dendroidal sets. Then a map between them $f : X \to Y$ is given by:

- For each tree $T \in \Omega$, there is a map $f_T : X_T \to Y_T$
- If $\alpha : S \to T$ is a morphism in Ω, then the following diagram commutes

$$\begin{array}{ccc} X_T & \xrightarrow{X_\alpha} & X_S \\ \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f_S} \\ Y_T & \xrightarrow{Y_\alpha} & Y_S \end{array}$$

Again, it is useful to know how dendroidal sets relate to operads, because dendroidal sets form the underlying structure of infinity operads. To this end there exists the following adjunction:

$$\tau_d : \mathbf{dSet} \rightleftarrows \mathbf{Opd} : n_d$$

It is defined as follows.

Definition 3.2.24 (Dendroidal nerve)**.** Let P be an operad over **Set**. Then the dendroidal nerve of P is the dendroidal set given by

$$n_d(P)_T = \mathbf{Opd}(\Omega(T), P)$$

Definition 3.2.25 (Dendroidal realisation)**.** Let X be a dendroidal set (where C_v is the nth corolla). The operad $\tau_d(X)$ is defined via

- The set of colours is the set X_η
- The operations are generated by the elements of $X_{C_{n+1}}$, with τ_d defining a map from elements of each X_T to operations in the operad, under the following relations.
  1. If A is a colour in X_η, and $\sigma^A : C_{1+1} \to \eta$ is a codegeneracy map, then $\sigma_A(A) = id_A \in \tau_d(X)(A; A)$, where $\sigma_A : X_\eta \to X_{C_1}$ is the degeneracy map induced by σ^A.
  2. Let T be a tree with two vertices, v and w. Then if δ_v, δ_w, and $\delta_{\overline{vw}}$ are similarly induced maps,

     $$\delta_w(x) \circ_{\overline{vw}} \delta_v(x) = \delta_{\overline{vw}}(x)$$

     for all $x \in X_T$.

Given a morphism of dendroidal sets $f : X \to Y$, there is a morphism $\tau(f) : \tau(X) \to \tau(Y)$ defined entry-wise on each $\tau(F_T) : \tau(X_T) \to \tau(Y_T)$. This defines the functor τ_d.

Condition 2 is essentially saying that the composition of the morphisms induced by vertices v and w should result in the morphism induced by the contraction of the edge between v and w.

Lemma 3.2.26 (Example 4.2 in [57])**.** *The dendroidal nerve satisfies the following*

- *The dendroidal nerve functor is fully faithful.*
- *The realisation τ_d is left adjoint to the nerve n_d.*

3.2.3 Boundaries, Horns, and Representables

Analogously to the way there is a standard n-simplex, $\Delta[n] \in \mathbf{sSet}$, for every tree T there is a standard dendrex $\Omega[T] \in \mathbf{dSet}$, the representable of T.

Definition 3.2.27. Let $T \in \Omega$ be a tree, then define the dendroidal set $\Omega[T]$ as

$$\Omega[T]_S := \Omega(S, T).$$

Now that the representable has been introduced, there are multiple concepts that share very similar notation, and thus a summary is in order.

- The map Ω maps each tree T to its associated operad $\Omega(T)$. They will often be conflated as T, but $\Omega(T)$ will sometimes be used to emphasise the operadic nature.
- The category Ω refers to both the category of trees with operad maps as morphisms between them, and the full subcategory of **Opd** given by the image of the map Ω.
- Given any tree T, there is an associated dendroidal set $\Omega[T]$ called the representable of T.
- All morphisms in Ω can be formed by compositions of coface and codegeneracy maps. These are written as δ^x and σ^x, where the superscript x refers to the affected edge or vertex.
- Morphisms in Ω^{op} are composed of face and degeneracy maps, and they are written as δ_x and σ_x, with x being a subscript rather than a superscript. They are also written this way when considered as morphisms internal to a dendroidal set.

Lemma 3.2.28 (Section 4.1 in [57])**.** *Let $\alpha : S \to T$ be a map of trees. Then there is an induced map $\Omega[\alpha] : \Omega[S] \to \Omega[T]$ between dendroidal sets.*

By examining the definitions of n_d and $\Omega[T]$, it is clear that $\Omega[T] = n_d(T)$, as would be expected.

Definition 3.2.29 (Face)**.** Let $S, T \in \Omega$ be trees with face map $\alpha : S \to T$. Then the α-face of $\Omega[T]$ is the image of the map $\Omega[\alpha] : \Omega[S] \to \Omega[T]$. It is denoted $\partial_\alpha \Omega[T]$. The set of all faces of T is denoted $\Psi(T)$.

Note that $\Psi(T)$ is the set of images of maps $\Omega[\alpha]$. Each element is denoted $\partial_\alpha \Omega[T]$, but this will occasionally be shortened to $\alpha \in \Psi(T)$.

Definition 3.2.30 (Boundary)**.** The boundary of $\Omega[T]$ is the dendroidal subset which is the union of all possible faces.

$$\partial \Omega[T] = \bigcup_{\alpha \in \Psi(T)} \partial_\alpha \Omega[T]$$

The horn is just the boundary but without one of the faces. The inner horns are those where the face map which is left out is an inner face map, while the rest are outer horns. More formally,

Definition 3.2.31 (Inner horn)**.** Let α be an inner face map which removes the edge e. Then

$$\Lambda^e[T] = \bigcup_{\beta \neq \alpha \in \Psi(T)} \partial_\beta \Omega[T]$$

In a dendroidal set X, an (inner) horn is a map of dendroidal sets $\Lambda^\alpha[T] \to X$.

3.2.4 Relationship to Δ

Note that Δ is a subcategory of Ω consisting only of paths. If dendroidal sets are restricted to just paths, one ought to recover simplicial sets. One way of getting Δ from Ω is via the slice.

Definition 3.2.32 (Slice category)**.** Let $\mathcal{C}$ be a category and X an object in that category. Then $\mathcal{C} \downarrow X$, is the category with:

- $\mathrm{ob}(\mathcal{C} \downarrow X) = \{\mathrm{mor}(-, X)\}$
- $\mathrm{mor}(\mathcal{C} \downarrow X)(A \to X, B \to X)$ are given by commuting diagrams of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow & \downarrow \\ & & X \end{array}$$

Now this can be used to express Δ as a slice of Ω.

Lemma 3.2.33. *There is an equivalence between the category of simplicies and the slice category of trees over η.*

$$\mathbf{\Delta} \cong \mathbf{\Omega} \downarrow \eta$$

Proof. Firstly, note that $i : \mathbf{\Delta} \to \mathbf{\Omega}$ is the inclusion functor sending paths in Δ to paths in Ω [57, Section 3]. Recall that paths in Δ are labelled according to length, keeping in line with the usual notation for simplicial sets. Then $i[0] = \eta$.

The objects of $\mathbf{\Omega} \downarrow i[0]$ are the maps $T \to \eta$. It suffices to show that given any map $T \to \eta$, T must be a path, and that for any path P in Ω, there is a map to η, and that morphisms in the slice category correspond to morphisms of paths in Δ.

Consider a path P_n. It is clear that $\eta = (\sigma)^n P_n$, thus providing a map from any path P to η.

By Lemma 3.2.21, all morphisms in Ω are generated by face maps and degeneracy maps. Call these the generating morphisms. Note that face maps add vertices while degeneracy maps remove them, so the only way to get to η directly through generating morphisms is the degeneracy map $P_1 \to \eta$. Let P_n be a path with an inner edge e and a vertex v connected to exactly one inner edge. Then the possible maps are:

- $\sigma : P_{n+1} \to P_n$
- $\partial_e : P_{n-1} \to P_n$
- $\partial_v : P_{n-1} \to P_n$

By induction, if $g : T \to P_n$ is a generating morphism then T must be a path, so for all $f : T \to \eta$, where f is some composition of generating morphisms, T must be a path.

Each morphism in the slice category is a diagram

$$\begin{array}{ccc} S & \longrightarrow & T \\ & \searrow & \downarrow \\ & & \eta \end{array}$$

where T and S are trees in Ω. But each diagram corresponds to a morphism in Ω between two paths, which by the inclusion $\Delta \hookrightarrow \Omega$ is a morphism in Δ between two paths. $\square$

A similar lemma holds for simplicial sets and dendroidal sets [57].

Lemma 3.2.34.

$$\mathbf{sSet} \cong \mathbf{dSet} \downarrow \Omega[\eta]$$

3.3 Infinity Operads

Infinity operads extend the notion of infinity categories; that is, infinity operads are operads where composition is weak rather than strict. Originally used for the Boardman-Vogt tensor product [57], infinity operads are also important to profinite completions of the little discs operad [10].

Just as in infinity categories, the concept of an operad with weak composition can be implemented in various ways, most involving dendroidal sets. Possibly the most straightforward would be simplicially enriched operads, which I begin with. These are operads where, instead of a set of operations among some objects, there is a space of operations [67]. Then I define quasi operads, which is the main definition I use. Quasi operads are dendroidal sets satisfying a particular condition, the inner Kan condition. The next definition is very similar, dendroidal sets satisfying the Segal condition. Finally, there is dendroidal complete segal spaces, which use both the Kan and Segal conditions.

Quasi operads extend the notion of the simplicial Kan condition; given enough information from face maps one can recover the original tree. That is, the face maps act as substrings of morphisms, or subtrees, and the original tree is like the composition of the full string, and the Kan condition states that this composition exists [58].

The Segal condition, on the other hand, looks at a string of morphisms and says that if the morphisms have a local composition, then the whole thing does too. The Complete Segal Spaces Model uses both Kan and Segal conditions, to show how the space acts both categorically and topologically at once. Although not much attention is devoted to this model, it is included to give a more complete picture of the state of infinity operads [16].

In particular, two of these conceptions of infinity operads are defined to be dendroidal sets satisfying a particular condition (Kan or Segal). These two conditions are the focus of this chapter and later ones. They can be related to each other, and strict operads, via the Nerve Theorem (Theorem 3.3.6). This theorem says that, for a dendroidal set, it is equivalent to say that it satisfies the strict Kan condition, or the strict Segal condition, or is the nerve of an operad. This important theorem is later replicated by me for cyclic operads and modular operads (Theorems 4.3.7 and 5.3.10).

3.3.1 Simplicially Enriched Operads

One way of defining the notion of an "operad up to homotopy" would be to enrich it with some sort of topological structure.

Definition 3.3.1 (Simplicially enriched operad[67]). An operad $\mathcal{O}$ enriched in **sSet** consists of

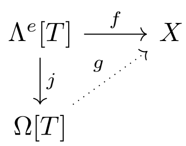
1. A set of colours $\mathcal{C}$
2. For all profiles $(c_1, \dots c_n; c_0)$, a simplicial set $\mathcal{O}(c_1, \dots c_n; c_0)$
3. Maps between operads are maps of simplicial sets which satisfy all the operad conditions.

The idea is that the set of colours on the edges looks like the set of objects of the operad, with morphisms given by corollae and compositions of morphisms by trees.

3.3.2 The Dendroidal Inner Kan Condition

In dendroidal sets, the definition of the Kan condition is similar to that of simplicial sets.

Definition 3.3.2 (Kan condition, Section 7 in [57]). Let X be a dendroidal set, $f : \Lambda^e[T] \to X$ be an inner horn, and let $j : \Lambda^e[T] \rightarrowtail \Omega[T]$ be the inclusion. Then a filler for f is a map $g : \Omega[T] \to X$ such that $f = g \circ j$.



A dendroidal set X is said to satisfy the inner Kan condition if every inner horn has a filler.

To make matters even more complicated, there is both the (weak) inner Kan condition and the strict inner Kan condition. The strict inner Kan condition says that every inner horn has a *unique* filler.

A *quasi operad*, also known as an inner Kan complex, is a dendroidal set for which the inner Kan condition holds. If the strict inner Kan condition holds, then it is a strict operad.

Just like in simplicial sets, the inner Kan condition is equivalent to the following induced map being surjective for all trees T and inner edges e:

$$j^* : \mathbf{dSet}(\Omega[T], X) \to \mathbf{dSet}(\Lambda^e[T], X).$$

If the filler is unique, or j^* is bijective, then X is a strict inner Kan complex and by the Nerve Theorem is thus equivalent to an operad. Theorem 3.3.3 below is the relevant part of the full Nerve theorem, Theorem 3.3.6

Since the definition of quasi operads is based on Kan complexes, it is important to know how these relate to strict operads. Some work towards this was done by Moerdijk and Weiss [58].

Theorem 3.3.3 (Proposition 5.3 in [58]). *A dendroidal set X satisfies the strict inner Kan condition if and only if there exists an operad $\mathcal{O}$ such that $X \cong n(\mathcal{O})$.*

3.3.3 The Dendroidal Segal Condition

In the previous section infinity operads were defined as dendroidal sets which satisfy the Kan condition. In a similar fashion infinity operads can be defined as dendroidal sets satisfying the Segal condition. Instead of the horn, the Segal core is used.

Definition 3.3.4 (Segal core [16]). Let T be a tree with at least one vertex. Recall the corolla C_{n+1} from Example 3.2.5. The Segal core $Sc[T]$ is the subobject of $\Omega[T]$ defined as the union of all the images of maps $\Omega[C_{n+1}] \to \Omega[T]$, where each C_{n+1} is a subtree of T. Note that such a map is completely determined, up to isomorphism, by the vertex v of T in its image. Let $n(v)$ be the number of input edges that v has. Therefore,

$$Sc[T] = \bigcup_{v \in V(T)} \Omega[C_{n(v)}].$$

Intuitively, the Segal core consists of all the vertices, i.e. operations, that make up the tree. Then the Segal condition states that with just this partial information the whole dendroidal set can be reconstructed.

Definition 3.3.5 (Segal condition [16]). A dendroidal set X satisfies the Segal condition if

$$\mathrm{Hom}(\Omega[T], X) \cong \mathrm{Hom}(Sc[T], X)$$

for all trees T

In other words, the Segal condition can be considered as the statement that

$$X_T \cong X_{C_{v_1}} \times_{X_0} X_{C_{v_2}} \times_{X_0} \cdots X_{c_{v_n}}$$

is a weak equivalence.

The following is a generalisation of Theorem 3.3.3, stated without proof.

Theorem 3.3.6 (Nerve Theorem, Corollary 2.6 in [16]). *Let X be a dendroidal set. Then the following are equivalent:*

- *There exists an operad $\mathcal{O}$ such that $X = n_d(\mathcal{O})$*
- *X satisfies the strict inner Kan condition*
- *X satisfies the strict Segal condition*

3.3.4 Dendroidal Complete Segal Spaces

Another model for infinity categories is the dendroidal version of complete segal spaces. Just like in the simplicial case, a dendroidal complete Segal space is a dendroidal space that satisfies certain conditions.

- As a dendroidal object, the dendroidal space X satisfies the Segal condition. This ensures that it acts something like an operad
- Each X_n satisfies a Reedy fibrancy condition (akin to the Kan condition), ensuring it acts like a topological space

- There is a completeness condition, ensuring that the only homotopy equivalences are the identities.

Defining a dendroidal complete Segal space (dCSS) will first require defining these conditions.

Definition 3.3.7 (Dendroidal space). A dendroidal space is a functor

$$X : \Omega^{op} \to \mathbf{sSet}$$

One could also picture this as a map $\Omega^{op} \times \Delta^{op} \to \mathbf{Set}$. The category of dendroidal spaces is **dSpace**.

Definition 3.3.8 (Reedy fibrant). A dendroidal space X is said to be Reedy fibrant if for every path n, the following map of spaces is a Kan fibration:

$$Map_{\mathbf{dSpaces}}(F(T), X) \to Map_{\mathbf{dSpaces}}(\partial F(T), X)$$

Where F is a functor from dendroidal sets to dendroidal spaces, assigning to each dendroidal set the constant space. That is, $F(T)_{Sj} = \Omega[T]_S$, where S is some tree.

Definition 3.3.9 (Segal space). A dendroidal space X is a dendroidal Segal space if

- As a dendroidal object it satisfies the Segal condition
- It satisfies the Reedy fibrancy condition

Let X_{hoequ} consist of all those homotopy equivalences in each $X_{C_{\underline{c}}}$. Then, recall that $s_0 : X_0 \to X_{hoequ}$ sends each colour a to its identity id_a.

Definition 3.3.10 (Completeness). A dendroidal Segal space X is said to satisfy the completeness condition if $s_0 : X_0 \to X_{hoequ}$ is a weak equivalence.

Therefore a dCSS is a Segal space which satisfies the completeness condition.

Chapter 4

Cyclic Operads and Astroidal Sets

In [28], Getzler and Kapranov introduce the notion of a cyclic operad, as a generalisation of the operations of the cyclic homology of associative algebras. At the time, cyclic operads were important in the study of homotopy theory [55] and topological field theory [26]. However, cyclic operads are also of interest for their connections to physics and dagger categories [2], and, higher cyclic operads especially, as a stepping stone to a modular operad of surfaces [10]. Cyclic operads are useful in this way because surfaces themselves do not necessarily have a direction, and therefore their generalised operad should not either.

This chapter contains mostly new work. Analogously to the operad case, the aim is to describe infinity cyclic operads. For each of the four models of infinity operad, I sketch how they can be extended to cyclic infinity operads, with a focus on quasi cyclic operads.

Cyclic operads are an extension of operads based on non-directionality. Dagger categories are categories where, instead of having a morphism f from A to B, the morphism f is instead considered to be between A and B, with no sense of direction. This is achieved by assigning to each morphism $f : A \to B$ an adjoint $f^\dagger : B \to A$. Extending this to operads, a cyclic operad is one where each morphism has an "adjoint" for each permutation of input and output objects. Because of this, cyclic operads tend to be vastly more complicated than operads: an operad with only a few morphisms may generate a cyclic operad with infinitely many morphisms.

Then, analogously to dendroidal sets and simplicial sets, this chapter develops the theory of astroidal sets. Simplicial and dendroidal sets are founded on certain categories of directed graphs, Δ and Ω, respectively. The category associated to astroidal sets instead consists of *undirected* trees, and it is called Ж, with presheaves over it being known as astroidal sets.

In Section 4.3.2 the inner Kan condition is defined, and its equivalence to the Segal condition proven. The Segal condition was defined in [34], along with a category Ξ that is equivalent to Ж but developed independently. I also provide a definition of a simplicially enriched cyclic operad, since there is an equivalence between quasi operads and simplicially enriched operads (and likewise for categories); however, I have not proven such an equivalence with quasi cyclic operads. During the writing of this thesis, a paper appeared by Drummond-Cole and Hackney [19] which details a model structure for simplicially enriched operads. I have not proven any relationship between my definition and theirs, although theirs appears to differ from mine by the presence of an involution on objects, nor with their model structure for simplicially enriched operads [19, Section 6].

4.1 Cyclic Operads

There are two ways to arrive at cyclic operads from categories. On the one hand, cyclic operads can be formed by taking an operad and extending the action of the symmetric group on the inputs to incorporate the output in addition. On the other hand, one can consider a generalisation of dagger categories (see Section 4.1.1) where each morphism has multiple inputs rather than just one.

	Paths	Trees
Directed	Categories	Operads
Undirected	Dagger categories	Cyclic operads

Section 4.1.5 describes my original example of a cyclic operad. Given a category with pullbacks $\mathcal{C}$, a dagger category $Span(\mathcal{C})$ can be defined, with the same objects and using pullbacks as morphisms. In the same way, a cyclic operad $Multispan(\mathcal{C})$ can be created. Initially, the mapping class group operad was defined in the genus zero case as a cyclic operad, but it was later extended to the higher genus case. It can be found in Section 5.4.

4.1.1 Dagger Categories

Dagger categories are the categorical generalisation of Hilbert spaces and, like Hilbert spaces, they have extensive uses in physics, especially quantum mechanics [2]. The concept of a category with an involution on morphisms was first introduced by Burgin [13]. Dagger categories were further developed by Abramsky and Coecke [1] and Lambek [45], and they were first given the name ‘dagger category’ by Selinger [71].

To motivate dagger categories, recall the definition of a Hilbert space [68, Chapter 4].

Definition 4.1.1 (Hilbert space)**.** A Hilbert space is a vector space equipped with a complete positive definite inner product. In other words, consider a vector space V over $\mathbb{C}$ equipped with two functions $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ and $||\cdot||^2 : V \to \mathbb{C}$ (where $||x||^2 := \langle x, x\rangle$) satisfying:

1. $\langle 0, x\rangle = 0$ and $\langle x, 0\rangle = 0$
2. $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$
3. $\langle cx, y\rangle = \overline{c}\langle x, y\rangle$ and $\langle x, cy\rangle = c\langle x, y\rangle$
4. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
5. $||x||^2 \geq 0$ for all x

6. $||x||^2 = 0$ if $x = 0$

7. Given any infinite sequence (v_i), where each $v_i \in V$, such that

$$\lim_{m,n\to\infty} ||\sum_{i=m}^{m+n} v_i||^2 = 0,$$

there exists a unique sum S such that $\lim_{n\to\infty} ||S - \sum_{i=1}^{n} v_i||^2 = 0$. Then $S = \sum_{i=1}^{\infty} v_i$.

Here $x, y, z \in \mathbb{C}$, and $\overline{x}$ refers to the complex conjugate of x, for all $x \in \mathbb{C}$. This defines a Hilbert space over $\mathbb{C}$.

Definition 4.1.2 (Bounded linear operator)**.** Consider a linear transformation $T : V \to W$ between Hilbert spaces V and W. Then T is a bounded linear operator if there exists some $M > 0$, where $M \in \mathbb{R}$, such that for all $v \in V$, $||Tv||_W \leq M||v||_V$. Here $||\cdot||_W$ and $||\cdot||_V$ refer to the inner products in W and V, respectively.

Definition 4.1.3 (FdHilb). The category of finite dimensional Hilbert spaces, **FdHilb**, is the category with finite dimensional Hilbert spaces as objects, and bounded linear operators as morphisms.

The important thing about this category is the notion of adjoints. Each bounded linear operator, $T : V \to W$, has an adjoint $T^\dagger : W \to V$. It is this notion of adjoint that undergoes a category theoretic generalisation in dagger categories, a category where each morphism has an adjoint that serves the same function as adjoints in Hilbert spaces. The following two definitions are from Selinger [71, Definitions 2.2 and 2.3].

Definition 4.1.4 (Dagger category)**.** A dagger category is a category $\mathcal{C}$ equipped with a functor $\dagger : \mathcal{C}^{op} \to \mathcal{C}$ which is the identity on objects, and such that for all objects $A, B, C \in \mathcal{C}$ and all morphisms $f : A \to B$ and $g : B \to C$

1. $id_A^\dagger = id_A : A \to A$
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$
3. $f^{\dagger\dagger} = f$

Note that here $f^\dagger$ is known as f adjoint, named after the corresponding adjoint in Hilbert spaces. Hilbert spaces have also provided the following terminology for dagger categories.

Definition 4.1.5. A morphism is

- unitary if $f^\dagger = f^{-1}$
- self-adjoint if $f^\dagger = f$
- normal if $f^\dagger f = f f^\dagger$, where $f : X \to X$.

The category of dagger categories, with adjoint-preserving functors as morphisms, will be denoted **Dag**.

One large family of examples of dagger categories is given by groupoids. Given any groupoid G there is a dagger category where each morphism is unitary, because for each morphism f, its inverse f^{-1} can be taken as its adjoint.

Since Hilbert spaces inspired dagger categories, one would hope that they are indeed an example. The following lemma is standard, but a proof is provided as one was unable to be found in the literature.

Lemma 4.1.6. *The category* **FdHilb** *is a dagger category with the adjoint operator defined as follows. Let $A : U \to V$ be a bounded linear operator. Then $A^\dagger : V \to U$ is the adjoint (bounded linear) operator defined such that*

$$\langle A^\dagger v, u\rangle = \langle v, Au\rangle$$

where $u \in U$ and $v \in V$.

Proof. If A is a matrix given by (a_{ij}), then $A^\dagger$ is given by the conjugate transpose, so $(a^\dagger)_{ij} = \overline{a_{ji}}$. It suffices to show that the three properties obeyed by the adjoint are upheld.

1. Let I be the n-dimensional identity matrix. Then $I^T = I$, and all its entries are real so $I^\dagger = I$. Alternatively, $\langle I^\dagger u, v\rangle = \langle u, Iv\rangle = \langle u, v\rangle$

2.
$$\begin{aligned}(A \circ B)^\dagger &= \langle (AB)^\dagger u, v\rangle \\ &= \langle u, (AB)v\rangle \\ &= \langle u, A(Bv)\rangle \\ &= \langle A^\dagger u, Bv\rangle \\ &= \langle B^\dagger(A^\dagger u), v\rangle \\ &= B^\dagger \circ A^\dagger\end{aligned}$$

3. Given a matrix A, the entries of $A^{\dagger\dagger}$ are given by
$$\begin{aligned}((a^\dagger)^\dagger)_{ij} &= \overline{a^\dagger_{ji}} \\ &= \overline{\overline{a_{ij}}} \\ &= a_{ij}\end{aligned}$$

□

Remark 4.1.7. There exists an adjoint pair

$$\mathbf{Cat} \underset{c}{\overset{d}{\rightleftarrows}} \mathbf{Dag}$$

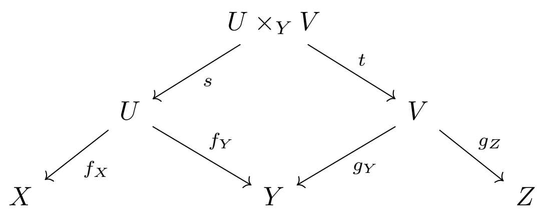
The right adjoint (in the category sense) c forgets the dagger structure. That is, any two adjoint (in the dagger sense) morphisms in $\mathcal{D}$ will both exist in $c(\mathcal{D})$, but their connection to each other, the † function, will not exist.

The left adjoint d is given by freely adding an adjoint (in the dagger sense) for each morphism. Given any category $\mathcal{C}$, the objects of $d(\mathcal{C})$ are the objects of $\mathcal{C}$, and given any morphism $f : A \to B$ in $\mathcal{C}$, $d(\mathcal{C})$ contains both $f : A \to B$ and $f^\dagger : B \to A$, generated according to the definition of dagger category.

There is another way to form a dagger category from a category, thus providing another avenue of examples of dagger categories. This is the *span category*, first introduced by Bénabou [6].

Definition 4.1.8 (Span)**.** Let $\mathcal{C}$ be a category with pullbacks, and consider objects in it, X and Y. Then a span between them is an object Z together with morphisms $f : Z \to X$ and $g : Z \to Y$. Alternatively, a span is a diagram $\cdot \leftarrow \cdot \rightarrow \cdot$.

Definition 4.1.9 (Composition of spans)**.** Let X, Y, Z be objects in a category $\mathcal{C}$ with spans $X \leftarrow U \to Y$ and $Y \leftarrow V \to Z$. Then find a span from X to Z as follows.



First take the pullback $U \times_Y V$, then compose to find a span from X to Z as $(U \times_Y V, f_X \circ s, g_Z \circ t)$.

Definition 4.1.10 (Span category)**.** Let $\mathcal{C}$ be a category with pullbacks. Then $\mathbf{Span}(\mathcal{C})$ is defined as follows.

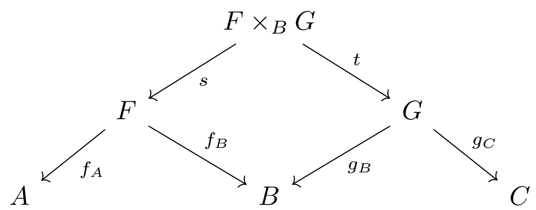
- $ob(\mathbf{Span}(\mathcal{C})) = ob(\mathcal{C})$.
- $mor(A, B) = \{(X, f, g) | f : X \to A, g : X \to B\}$.
- Composition of morphisms is given in Definition 4.1.9.
- $id_A = A \leftarrow A \to A$ where the morphisms are both the identity.

There is also the cospan category, formed in the same way but with the arrows reversed. The category $\mathcal{C}$ is required to have pushouts, the shape of the arrows is $\cdot \to \cdot \leftarrow \cdot$, and composition is given by pushouts.

Lemma 4.1.11. *The span category is a dagger category.*

Proof. Let the adjoint functor $\dagger : \mathbf{Span}(\mathcal{C})^{op} \to \mathbf{Span}(\mathcal{C})$ be defined as the identity on objects, and $(X, f, g) \mapsto (X, g, f)$. Then, for all $A, B, C, F, G \in ob(\mathcal{C})$, the dagger category properties hold.

1. The identity $(A, id, id)^\dagger = (A, id, id)$
2. Let s and t be the morphisms corresponding to the pullback $F \times_B G$. In addition, let (F, f_A, f_B) and (G, g_B, g_C) be defined according to the following diagram.



Then

$$
\begin{aligned}
&((G, g_B, g_C) \circ (F, f_A, f_B))^\dagger \\
=&(F \times_B G, f_A \circ s, g_C \circ t)^\dagger \\
=&(F \times_B G, g_C \circ t, f_A \circ s) \\
&(F, f_A : F \to A, f_B : F \to B)^\dagger \circ (G, g_B : G \to B, g_C : G \to C)^\dagger \\
=&(F, f_B : F \to B, f_A : F \to A) \circ (G, g_C : G \to C, g_B : G \to B) \\
=&(F \times_B G, g_C \circ t, f_A \circ s)
\end{aligned}
$$

3. $(X, f, g)^{\dagger\dagger} = (X, g, f)^\dagger = (X, f, g)$

□

4.1.2 Definition of Cyclic Operad

The first definition of a cyclic operad is due to Getzler and Kapranov [28], with a new axiom added by Van der Laan [76, Section 11]. There is a definition by Obradovic [61] based on a monoidal construction, while Hackney, Robertson, and Yau give a definition based on monads in Section 5.1 of [34]. My definition in equivalent to this last one.

Cheng, Gurski, and Riehl [14] also approach the definition of cyclic operads from the multicategory perspective, but this differs from my definition mainly in the duality of objects. Each object has a dual, which is invoked whenever the action of the symmetric group moves an object from the output to the input or vice versa. There is a choice here, which mirrors a similar one in dagger categories: whether or not the definition should require objects to have a dual. Duals are slightly more general, and a definition of cyclic operads with dualisable objects is also found in [19]. However there are no known examples of cyclic operads which require this extra structure, and in keeping with the classical, monochromatic definition [28], this thesis does not have a dual structure on objects in cyclic operads. Nor were duals of objects included in the definition of dagger categories.

The definition of a cyclic operad and the definition of an operad differ only in that non-cyclic operads have a distinguished output element, often denoted c_0. This definition was arrived at independently, but it is equivalent to that given in [34], which was published during the writing of this thesis.

Definition 4.1.12 (Cyclic operad)**.** A (coloured, symmetric) cyclic operad $\mathcal{C}$ over a closed symmetric monoidal category $\mathcal{E}$ is defined with the following properties.

- A set of objects, $ob(\mathcal{C})$, sometimes called colours.
- For each profile over $ob(\mathcal{C})$, $\underline{c} = (c_1, \ldots, c_n)$, there is an object of $\mathcal{E}$ $\mathcal{C}(c_1, \ldots c_n)$. If $\mathcal{M}$ is **Set**, then this is the set of operations.
- There is a right action of the symmetric group; for any profile $\underline{c}$ and permutation $\sigma \in \Sigma_n$, there is a bijection $\mathcal{C}(c_1, \ldots c_n) \to \mathcal{C}(c_{\sigma(1)}, \ldots c_{\sigma(n)})$.
- There is a composition operation. Given any two operations $\alpha \in \mathcal{C}(\underline{c})$ and $\beta \in \mathcal{C}(\underline{d})$ where the final object in $\underline{c}$ is equal to the ith object in $\underline{d}$, there is a composition $\alpha \circ_i \beta$ which is a map

  $$\mathcal{C}(\underline{c}) \otimes \mathcal{C}(\underline{d}) \to \mathcal{C}(\underline{cd}_i)$$

  Where, for the sake of brevity,

  $$\underline{cd}_i = (d_1, \ldots d_{i-1}, c_1, \ldots c_{m-1}, d_{i+1}, \ldots d_n).$$

  - The composition is associative. That is, all such diagrams commute:

    $$\begin{array}{ccc} \mathcal{C}(\underline{a}) \otimes \mathcal{C}(\underline{b}) \otimes \mathcal{C}(\underline{c}) & \xrightarrow{id \otimes \circ_j} & \mathcal{C}(\underline{a}) \otimes \mathcal{C}(\underline{bc}_j) \\ \Big\downarrow{\scriptstyle \circ_i \otimes id} & & \Big\downarrow{\scriptstyle \circ_{i+j-1}} \\ \mathcal{C}(\underline{ab}_i) \otimes \mathcal{C}(\underline{c}) & \xrightarrow{\circ_j} & \mathcal{C}(\underline{abc}_{ij}) \end{array}$$

    where the lengths of $\underline{a}$, $\underline{b}$, and $\underline{c}$ are ℓ, m, and n, respectively.
  - Composition is unital. That is, for each colour c there exists an identity η_c, such that for all $\theta \in \mathcal{C}(\underline{c})$, $\theta \circ_1 \eta_{c_n} = \theta = \eta_{c_i} \circ_i \theta$.
  - Composition is equivariant. That is, it commutes with the action of the symmetric group. So for any two morphisms $\alpha \in \mathcal{C}(\underline{c})$ and $\beta \in \mathcal{C}(\underline{d})$, and any $\sigma, \tau \in \Sigma_n$,

    $$\alpha \circ_{\sigma(i)} \sigma(\beta) = \sigma'(\alpha \circ_i \beta)$$

    $$\tau(\alpha) \circ_i \beta = \tau'(\alpha \circ_i \beta)$$

    Where $\sigma' \in \Sigma_{n+m}$ refers to the element that acts on

    $$\{d_1, \ldots, d_{i-1}, c_1, \ldots, c_m, d_{i+1}, \ldots, d_n\}$$

    by doing σ on each d_k and the identity on each c_j, and $\tau' \in \Sigma_{n+m}$ does the identity on each d_k and permutes each c_j according to τ.

Then $\mathcal{C}$ is a (coloured symmetric) cyclic operad.

Remark 4.1.13. Inspired by the notation of dagger categories, one could write A^σ to denote $\sigma \in \Sigma$ acting on a morphism A. However, this is non-standard in the cyclic operads literature and will not be used herein.

Note that, unless otherwise specified, all cyclic operads are over **Set**.

In certain applications, it may be advantageous to compose at certain places along the profile, where $c_i = d_j$. Suppose one wishes to form the composition $\alpha \circ_{ij} \beta$, where $\alpha \in \mathcal{C}(\underline{c})$ and $\beta \in \mathcal{C}(\underline{d})$. Then do the following:

1. Apply the action of Σ_n to move c_i to the end, so that the result is $\mathcal{C}(c_1, \ldots c_{i-1}, c_{i+1}, \ldots c_n, c_i)$. Denote this $\sigma_i \in \Sigma_n$.
2. Apply the composition already defined
3. Therefore, define
$$\mathcal{C}(\underline{c}) \circ_{ij} \mathcal{C}(\underline{d}) \to \mathcal{C}(\underline{cd}_{ij})$$
as
$$\mathcal{C}(\underline{c}) \circ_{ij} \mathcal{C}(\underline{d}) = (\mathcal{C}(\sigma_i(\underline{c}))) \circ_j (\mathcal{C}(\underline{d})$$

In order to define the category of cyclic operads, morphisms are required. As with operads, or multicategories, what is wanted is something that looks like a functor but with multiple inputs.

Definition 4.1.14. Let $\mathcal{O}$ and $\mathcal{P}$ be two cyclic operads. A morphism $f : \mathcal{O} \to \mathcal{P}$ consists of the following information:

- A function $f' : ob(\mathcal{O}) \to ob(\mathcal{P})$ that sends the colours of one operad to the set of colours of the other.
- For each profile $\underline{c}$, a map $f_{\underline{c}} : \mathcal{O}(\underline{c}) \to \mathcal{P}(f'(\underline{c}))$ that commutes with the structure maps. i.e.
  - Identity: $f(id_\eta) = id_{f(\eta)}$
  - Symmetric group action: $f(\sigma(v)) = \sigma(f(v))$
  - Composition: $f_{\underline{cd}_i}(v \circ_i u) = f_{\underline{c}}(v) \circ_i f_{\underline{d}}(u)$

  Here $v \in \mathcal{O}(\underline{c})$ and $u \in \mathcal{O}(\underline{d})$

Then the category **CycOpd** can be defined as the category with cyclic operads as objects and morphisms as defined above.

4.1.3 Relationship to Dagger Categories

Given any cyclic operad, one can define a dagger category, and vice versa.

Definition 4.1.15. Given a dagger category $\mathcal{D}$, there exists a cyclic operad $\mathcal{C}$ over **Set** as defined herein. For each profile $\underline{c}$, define

$$\mathcal{C}(\underline{c}) = \begin{cases} \text{Hom}(c_1, c_2) & \text{If the length of } \underline{c} \text{ is } 2 \\ \emptyset & \text{Otherwise} \end{cases}$$

The action of Σ_2 is given by the dagger function.

This defines a functor from the category of dagger categories, **Dag**, to the category of cyclic operads over **Set CycOpd**. It is functorial because the morphisms of **Dag** preserve the dagger structure, and thus in **CycOp** they preserve the cyclic operad structure. Call this functor y : **Dag** $\to$ **CycOpd**, since the image of a dagger category is a cYclic operad.

Given a cyclic operad $\mathcal{C}$, it is possible to define a dagger category by removing all operations of arity higher than two. That is, define g : **CycOp** $\to$ **Dag** as

Definition 4.1.16. Define g : **CycOp** $\to$ **Dag** as follows. Given any cyclic operad $\mathcal{C}$,

- $ob(g(\mathcal{C})) = ob(\mathcal{C})$,
- $mor(A, B) = \mathcal{C}(A, B)$ for all $A, B \in ob(\mathcal{C})$,
- composition is given by composition in the cyclic operad,
- $f^\dagger = \sigma(f)$ for all $f \in \mathcal{C}(A, B)$, where $\sigma \in \Sigma_2$ and $\sigma \neq id$.

Morphisms between cyclic operads become morphisms between dagger categories by dropping all information about operations of arity higher than two. This functor is called g because it maps to daGger categories.

This is in addition to, and distinct from, the way *Span* maps a category with pullbacks to a dagger category. Thus, so far there are the following two diagrams.

$$\begin{array}{ccc}
\mathbf{Cat} & \rightleftarrows & \mathbf{Opd} \\
{\scriptstyle c}\uparrow\downarrow{\scriptstyle d} & & \\
\mathbf{Dag} & \underset{g}{\overset{y}{\rightleftarrows}} & \mathbf{CycOpd}
\end{array}$$

$$\begin{array}{ccc}
\mathbf{Cat} & \underset{\tau}{\overset{n}{\rightleftarrows}} & \mathbf{sSet} \\
{\scriptstyle c}\uparrow\downarrow{\scriptstyle d} & & \\
\mathbf{Dag} & &
\end{array}$$

The functors between **Opd** and **CycOpd** will become clear in Section 4.1.4. These diagrams together form part of a multi-dimensional diagram, with axes along

- cyclic or not,
- multiple inputs or single,
- their associated graphical presheaf (and graphical category, and infinity version).

Remark 4.1.17. The bottom right corner of the second diagram has not been studied, but it would be expected that some notion of dagger simplicial sets would be based around undirected linear graphs.

Remark 4.1.18. There are two ways an operad can be defined from a category. Either by leaving the higher arity operations blank (as in the functor m from Example 3.1.3), or by using a symmetrical monoidal category (as in Example 3.1.4). Above is defined the dagger category equivalent of the functor m, as the functor y. In the same way, since cyclic operads are a multi-input version of dagger categories, it would be expected that some sort of dagger symmetric monoidal category could provide a source of cyclic operads. That is, provided the dagger structure is able to co-exist with the symmetric monoidal structure.

It turns out that some notion of a dagger symmetric multicategory does indeed exist [1, 71], although the exact relationship between dagger symmetric multicategories and cyclic operads has not yet been studied, neither in this thesis nor elsewhere. Nevertheless, a definition is provided here [71, Definition 2.4].

Example 4.1.19 (Dagger symmetric monoidal category)**.** A DSMC is a category with both a dagger structure and a symmetric monoidal structure, such that the dagger preserves the symmetric monoidal structure. That is, the following relations hold:

- $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$
- $\alpha^\dagger = \alpha^{-1} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$, where α is the associator of the symmetric monoidal category (that is, the functor that forces associativity of the monoidal structure).
- Likewise, morphisms $\rho : A \to A \otimes I$ and $\lambda : A \to I \otimes A$ are unitary, where ρ and λ at the morphisms forcing I to be the unit of the monoidal structure.
- Again, $\sigma^\dagger_{A,B} = \sigma^{-1}_{A,B} : B \otimes A \to A \otimes B$, where $\sigma_{A,B}$ is the action of Σ_2 that provides the symmetric structure of the symmetric monoidal category.

4.1.4 Relationship to Operads

In graph theory, one can form a rooted tree from an unrooted tree by defining a root, and vice versa by forgetting the root. Given this relationship, which will be explored one would hope for a similar correspondence between operads and cyclic operads. This is indeed the case, and later it will be seen that this induces a correspondence between astroidal sets and dendroidal sets.

Given any category, a dagger category can be formed by adding in the appropriate $f^\dagger$ for each morphism f, and causing it to satisfy all the conditions of a dagger category (see Remark 4.1.7). In the same way, by adding the appropriate morphisms $\mathcal{O}(c_1, \dots c_0, \dots c_n; c_i)$ for each combination of colours, an operad can become a cyclic operad - i.e. forgetting which element was the output.

Given any dagger category, one can form a category by ignoring the dagger structure. In the same way, given any cyclic operad one can define an operad by removing the action of the symmetric group on one of the colours, declaring this to be the output. The following can be found in [34, Section 1.3, Definition 2.1].

Definition 4.1.20 (Functor from cyclic operads to operads)**.** Define a functor $r : \mathbf{CycOpd} \to \mathbf{Opd}$. Let $\mathcal{C}$ be a cyclic operad. Then

- Colours remain the same: $ob(r(\mathcal{C})) = ob(\mathcal{C})$,
- Let $r(\mathcal{C})(c_1, c_2, \dots, c_{n-1}; c_n) = \mathcal{C}(c_1, c_2, \dots, c_{n-1}, c_n)$
- The symmetric group only acts upon the inputs, rather than the output.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between cyclic operads $\mathcal{C}$ and $\mathcal{D}$. Then

- For colours $c \in r(\mathcal{C})$, $r(F)(c) = F(c)$

- For each morphism $v \in r(\mathcal{C})(c_1, c_2, \ldots, c_{n-1}; c_n)$, let

$$F_{\underline{c}}(v) \in r(\mathcal{D})(F(c_1), F(c_2), \ldots, F(c_{n-1}); F(c_n))$$

- In particular, let $r(F)(\eta_c) = \eta_{r(F)(c)}$. That is, identities are sent to identities.

The functor r preserves all limits, so the adjoint functor theorem [50, Chapter V6] can be applied to give a left adjoint $a : \mathbf{Opd} \to \mathbf{CycOpd}$ which forgets the extra structure. For clarity, a partial definition of this functor a is given below to demonstrate what the functor does on objects of **Opd**.

Definition 4.1.21 (Functor from operads to cyclic operads)**.** Define a functor $a : \mathbf{Opd} \to \mathbf{CycOpd}$. Let $\mathcal{O}$ be an operad, then:

- Colours are the same: $ob(a(\mathcal{O})) = ob(\mathcal{O})$
- Add in all the operations from $\mathcal{O}$, so for all

$$v \in \mathcal{O}(c_1, \ldots c_n; c_0),$$

  let

$$v \in a(\mathcal{O})(c_1, \ldots c_n, c_0)$$

  (and for $\sigma \in \Sigma_n$,

$$\sigma(v) \in \mathcal{O}(c_{\sigma^{-1}(1)}, \ldots c_{\sigma^{-1}(n)}; c_0)$$

  and therefore

$$\sigma(v) \in a(\mathcal{O})(c_{\sigma^{-1}(1)}, \ldots c_{\sigma^{-1}(n)}, c_0)).$$

- Then add all the extra operations required. For instance, if

$$v \in \mathcal{O}(c_1, \ldots c_n; c_0),$$

  then for all $\sigma \in \Sigma_{n+1}$ now add in

$$\sigma(v) \in a(\mathcal{O})(c_{\sigma^{-1}(n+1)}, c_{\sigma^{-1}(1)}, \ldots c_{\sigma^{-1}(n)}).$$

So beginning with an operad, one can add more operations to get a cyclic operad, and then in reverse one can effectively force the last colour to become the root, but now with many more operations than in the beginning. These functors are adjoint, with r the right adjoint to a [34, Corollary 8.2].

These adjunctions can be arranged in the following (non-commuting) diagram.

$$\begin{array}{ccc} \mathbf{Cat} & \underset{\ell}{\overset{m}{\rightleftarrows}} & \mathbf{Opd} \\ c \uparrow \downarrow d & & r \uparrow \downarrow a \\ \mathbf{Dag} & \underset{g}{\overset{y}{\rightleftarrows}} & \mathbf{CycOpd} \end{array}$$

4.1.5 An Example of Cyclic Operads: Multispans

One example of cyclic operads is multi-spans. Given any category with pullbacks $\mathcal{C}$, one can create a dagger category from it, known as $Span(\mathcal{C})$ (see Definition 4.1.8). I have created an equivalent for cyclic operads, which I call multispans.

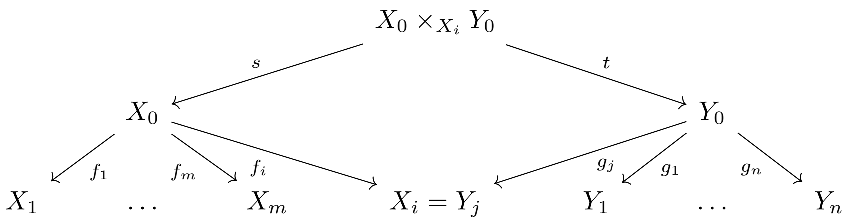
In the same way, multispans provide a way of generating cyclic operads from categories. Recently, Fong provided a connection between cobordisms and the category of cospans [24].

Definition 4.1.22 (Multispan). Let $\mathcal{C}$ be a category with pullbacks, and consider objects in it, $X_1, \dots X_n$. Then a multispan between them is an object X_0 together with morphisms $f_i : X_0 \to X_i$. Denote this by $(X_1, \dots X_n; X_0)$.

Definition 4.1.23 (Composition of multispans). Consider two multispans $(X_1, \dots X_m; X_0)$ and $(Y_1, \dots Y_n; Y_0)$ such that some $X_i = Y_j$. Denote their morphisms by f_i and g_i, respectively. Then a multispan

$$(X_1, \dots \hat{X}_i, \dots X_m, Y_1, \dots \hat{Y}_j, \dots Y_n; X_0 \times_{X_i} Y_0)$$

can be found, where $(X_0 \times_{X_i} Y_0, s, t)$ is the pullback of X_0 and Y_0 over $X_i = Y_j$ and $\hat{X}_i$ and $\hat{Y}_j$ are removed. The internal morphisms will be given by compositions $f_i \circ s$ and $g_j \circ t$.

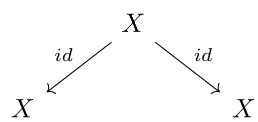


Then

$$(X_1, \dots X_m; X_0)_i \circ_j (Y_1, \dots Y_n; Y_0)$$
$$= (X_1, \dots \hat{X}_i, \dots X_m, Y_1, \dots \hat{Y}_i, \dots Y_n; X_0 \times_{X_i} Y_0)$$

Definition 4.1.24 (Multispan cyclic operad). Let $\mathcal{C}$ be a category with pullbacks. Then **Multispan**$(\mathcal{C})$ is defined as follows:

- The objects, or colours, are $ob(\mathbf{Multispan}(\mathcal{C})) = ob(\mathcal{C})$
- For each profile $\underline{c}$, $\mathcal{C}(\underline{c}) = \{(\underline{c}; c_0) | f_i : c_0 \to c_i, c_0 \in ob(\mathcal{C})\}$
- For any object X, the identity is $(X, X; X)$ where all morphisms are the identity.



- Composition of multispans is as given in Definition 4.1.23.

- The action of the symmetric group permutes the order of the objects in the profile.

The multispan cyclic operad is also known as the multicategory of multispans, in the same way as the category of spans (Definition 4.1.10). In both these cases, there is the extra structure of a dagger category or cyclic operad.

There is also the cospan multicategory, formed in the same way but with the arrows reversed. The category $\mathcal{C}$ is required to have pushouts, the morphisms are according to the diagram $\cdot \to \cdot \leftarrow \cdot$, and composition is given by pushouts.

4.1.6 Surface Examples

The main topological example of a cyclic operad is framed little discs. The definition below is not a cyclic operad, but an equivalence can be found in Budney [12]. The profinite completion of the framed little discs operad is particularly important to the work of Boavida de Brito et al. [10], and the inspiration behind higher cyclic operads [34].

Definition 4.1.25 (Framed little discs operad [77]). Let D^n be the open unit disc. Then for all $k \in \mathbb{N}$, let $fE_n(k)$ be the space of embeddings of k disjoint discs into a disc,

$$f : \coprod_k D^n \to D^n$$

where f is a composition of translations, dilations and rotations. Then, the operad is defined as follows.

- For each k, the operations $fE_n(k)$ are the aforementioned spaces of embeddings.
- The symmetric group acts by permuting the ordering of the discs.
- Then the compositions are defined by compositions of disjoint unions of maps:

$$\circ(f, g_1, \ldots, g_k) = \coprod_{n_1+\ldots+n_k} D^n \xrightarrow{g_1 \sqcup \ldots \sqcup g_k} \coprod_k D^n \xrightarrow{f} D^n$$

- The identity is that element given by the identity translation, dilation, and rotation.

Another topological example of cyclic operads would be the mapping class group operad (Section 5.4) restricted to surfaces of genus zero.

4.2 Astroidal Sets

Astroidal sets are the cyclic operad equivalent of dendroidal sets. Simplicial sets model infinity categories, dendroidal sets model infinity operads, and therefore it would be useful to define astroidal sets on the path to infinity cyclic operads. In this case, the graphical category consists of all those graphs which are unrooted trees. It is so named because each vertex neighbourhood of an unrooted tree resembles a star. This section begins by defining the category of astrices, Ж, which is then used as the pattern for astroidal sets. It then proceeds to the relationship between astroidal sets and cyclic operads, via the astroidal nerve.

4.2.1 The Category of Astrices

In this section, the category Ω is altered to develop the theory of astroidal sets. The category Ω has as objects rooted trees and their associated operads. If the information about the root is removed, the result is Ж, the category of unrooted trees.

Then, the notions of inner face, outer face, and degeneracy maps are extended to the undirected case, and the relations between them are explored.

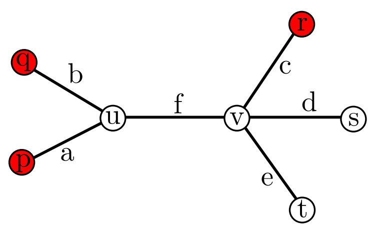
Definition 4.2.1 (Graph)**.** A graph G consists of a set of vertices $V(G)$, along with a set of edges $E(G)$, where each edge $e \in E(G)$ is a set of two distinct vertices, and edges are not repeated.

Given a graph, one can define a topological space where the vertices are points and the edges are intervals. Then, a simply connected graph is one where the associated topological space is simply connected.

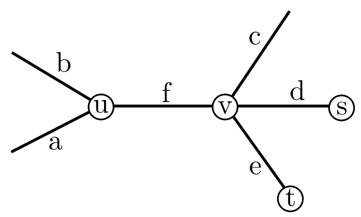
Definition 4.2.2 (Unrooted tree)**.** Let G be a simply connected graph. Let I be a subset of the set of outermost vertices, where the outermost vertices are those connected to only one edge. Then (G, I) is an unrooted tree, usually shortened to just G. As a convention, the input vertices are not drawn, and the word "vertex" refers only to the remaining vertices.

Technically, there is no concept of input and output vertices without direction. However, by analogy with dendroidal sets, this set I is called the set of input vertices.

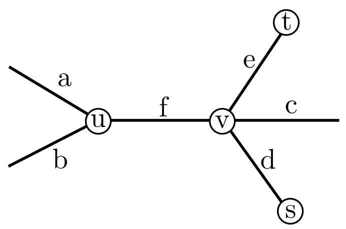
Example 4.2.3. Consider the following tree T:



The set of vertices is $\{p, q, r, s, t, u, v\}$. Technically, according to Definition 4.2.1, each edge is a subset of $V(T)$ consisting of two distinct vertices, so for example the edge f is $\{u, v\}$. However the edges can also be labelled individually as they are in this diagram. The set of inputs is $\{p, q, r\}$ and the leaves are $\{a, b, c\}$. This tree would normally be draw without the input vertices:



How exactly the tree is drawn in the plane is irrelevant. Thus, an equivalent image of the above tree would be



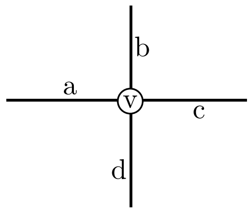
Note that if the particular planar representation of the tree did matter, the result would be something akin to planar operads and planar trees, Ω_p, in Moerdijk and Weiss [57]. These would correspond to the non-symmetric form of cyclic operads, and can be found in [78, Section 5].

The following trees are important examples.

Definition 4.2.4 (Corolla)**.** The n-corolla C_n is the tree that has a single vertex v, with n inputs.

Some sources emphasise the connection to the dendroidal case by writing C_{3+1} rather than C_4. Note that in both the category of rooted trees and unrooted trees C_n always refers to the same underlying graph, that of a single vertex with n edges attached to it, but in the category of rooted trees one of these edges is designated the root. When discussing the corolla around a particular vertex v, the notation C_v may be used.

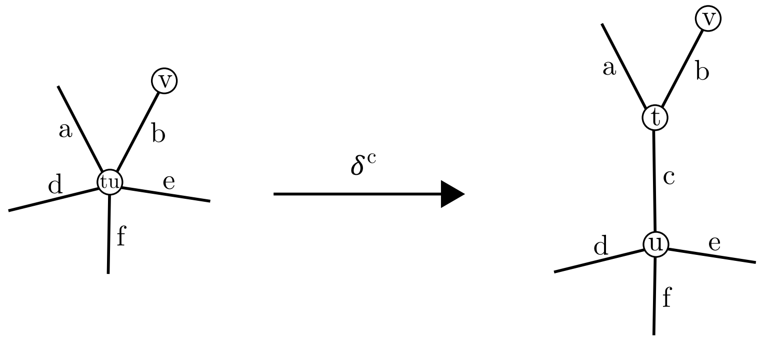
Example 4.2.5. This is the 4-corolla



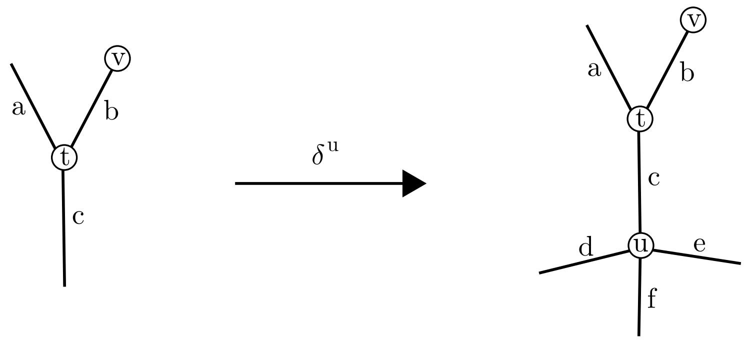
Again, it is possible to have just a single edge with no vertices, referred to as η, or occasionally the path of length zero, P_0.

As with the category of paths and the category of rooted trees, there are certain elementary face and degeneracy maps from which any map of unrooted trees can be built. Note that these are not maps of graphs in the traditional sense, mapping edges to edges and vertices to vertices, but rather maps from one whole graph to another, and morphisms in this category shall be defined as compositions of these maps. There are two types of coface maps, inner and outer, as well as degeneracy maps. In the following, let T be an unrooted tree.

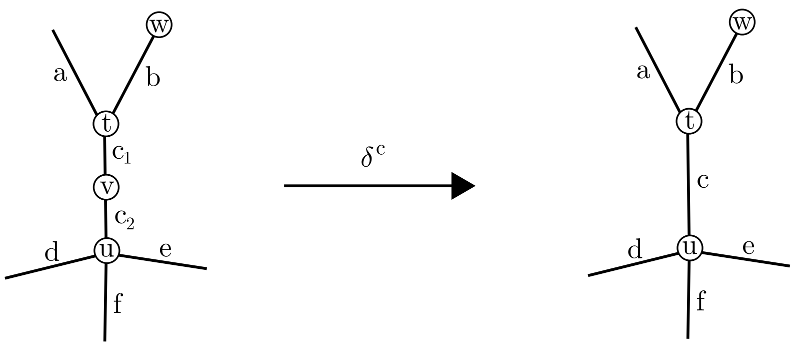
Definition 4.2.6 (Inner coface map)**.** Let c be an inner edge of T. Define T/c as the tree created by contracting the edge c. That is, if x and y are the vertices at either side of c, delete these and the edge c and add a new vertex xy with $nbhd(xy) = (nbhd(x) \cup nbhd(y)) \setminus \{c\}$. Then $\delta^c : T/c \to T$ is the associated inner coface map.



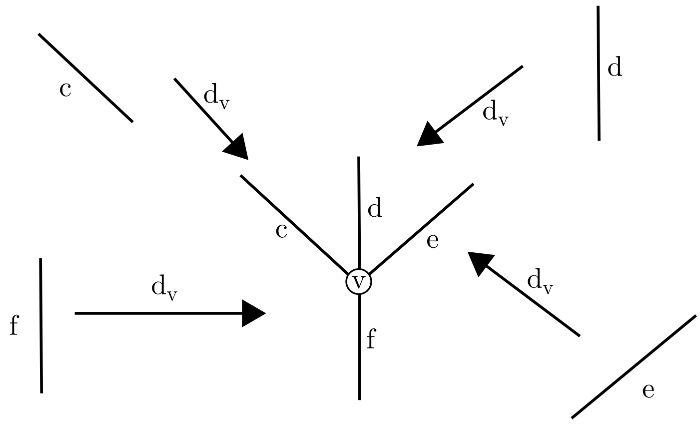
Definition 4.2.7 (Outer coface map)**.** Let u be an outer vertex of T. Then define T/u to be the tree with u and all its leaves deleted. Then $\delta^u : T/u \to T$ is the associated outer coface map.



Definition 4.2.8 (Codegeneracy map)**.** Let c be any edge of T. Then define T_c to be the graph where the edge c has been subdivided with exactly one vertex. Then $\delta^c : T_c \to T$ is the associated codegeneracy map.



Example 4.2.9. The single edge η can be included into an n-leaved corolla C_n in $(n + 1)$ ways.

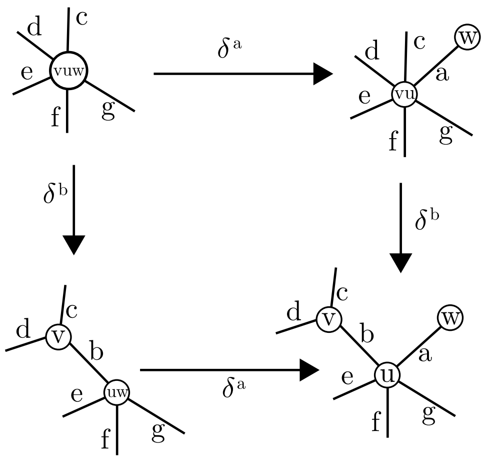


Relations Between Elementary Astroidal Maps

These coface and codegeneracy maps come with relations, just like in the tree and path cases in previous chapters. These relations are chosen due to the equivalences of their resulting graphs.

In fact, the relations between the coface and codegeneracy maps in astroidal sets are in essence identical to those of dendroidal sets, as one would expect given both the functor a : **Opd** $\to$ **CycOpd** and the functor from rooted trees to unrooted trees. Nevertheless, for clarity these relations will be presented again in the astroidal case. The most straightforward relation is between two inner coface maps. They do not interfere with each other, even if they correspond to edges which share a vertex. Given two (distinct) inner coface maps δ^a and δ^b, $(T/a)/b = (T/b)/a$ and the following diagram commutes:

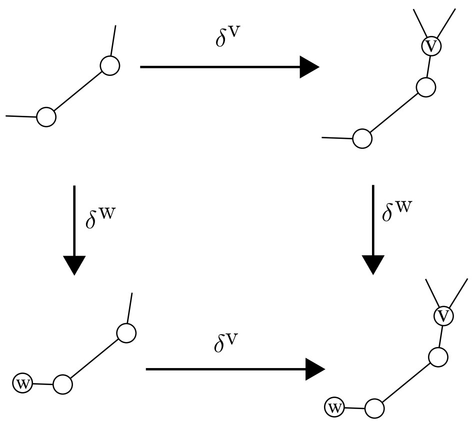
$$\begin{array}{ccc} (T/b)/a & \xrightarrow{\delta^a} & T/a \\ \downarrow{\scriptstyle \delta^b} & & \downarrow{\scriptstyle \delta^b} \\ T/b & \xrightarrow{\delta^a} & T \end{array}$$



Outer vertices are very similar, limited by the size of the graph.

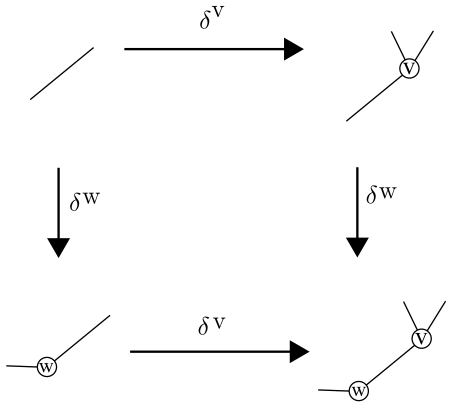
Let T be a tree with at least 3 vertices, and consider two distinct outer coface maps δ^v and δ^w. Then $(T/v)/w = (T/w)/v$ and the diagram commutes:

$$\begin{array}{ccc} (T/v)/w & \xrightarrow{\delta^v} & T/w \\ \downarrow{\scriptstyle \delta^w} & & \downarrow{\scriptstyle \delta^w} \\ T/v & \xrightarrow{\delta^v} & T \end{array}$$



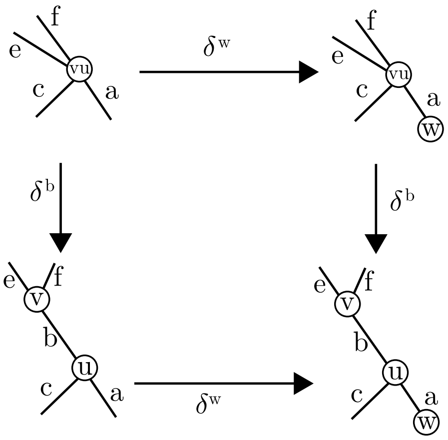
If T is a tree with only two vertices, v and w, then

$$\begin{array}{ccc} \eta & \xrightarrow{\delta^v} & T/w \\ \downarrow{\scriptstyle \delta^w} & & \downarrow{\scriptstyle \delta^w} \\ T/v & \xrightarrow{\delta^v} & T \end{array}$$



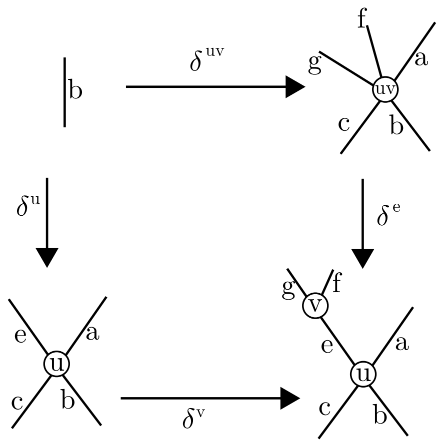
Finally there is the case where an inner coface map is combined with an outer coface map. Let δ^v be an outer coface map and δ^e be an inner coface map. If v and e are not adjacent, then $(T/v)/e = (T/e)/v$ and the diagram commutes:

$$\begin{array}{ccc} (T/v)/e & \xrightarrow{\delta^e} & T/v \\ \downarrow{\scriptstyle \delta^v} & & \downarrow{\scriptstyle \delta^v} \\ T/e & \xrightarrow{\delta^e} & T \end{array}$$



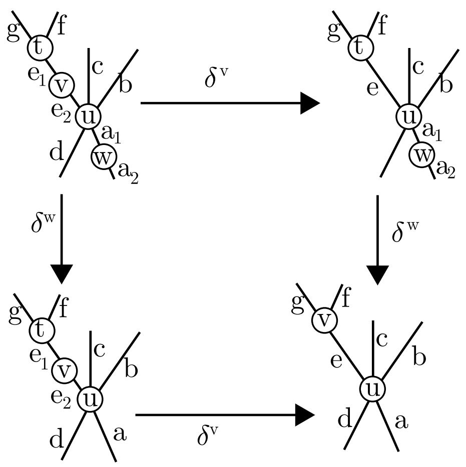
Suppose v and e are adjacent. Then denote the vertex on the other side of e by w. The tree T/e combines v and w into a single vertex, denote this u. Then $(T/e)/u$ exists if and only if $(T/v)/w$ exists, $(T/e)/u = (T/v)/w$, and the following diagram commutes:

$$\begin{array}{ccc} (T/e)/u & \xrightarrow{\delta^u} & T/e \\ \downarrow{\scriptstyle \delta^w} & & \downarrow{\scriptstyle \delta^e} \\ T/v & \xrightarrow{\delta^v} & T \end{array}$$



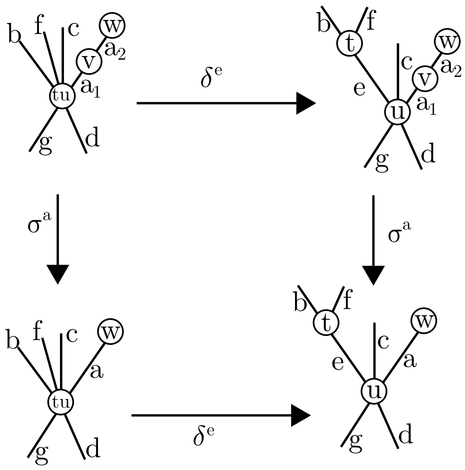
Now consider two degeneracies of T σ^e and σ^f. Then $(T_e)_f = (T_f)_e$ and the following diagram commutes

$$\begin{array}{ccc} (T_e)_f & \xrightarrow{\sigma^e} & T_f \\ \downarrow{\scriptstyle \sigma^f} & & \downarrow{\scriptstyle \sigma^f} \\ T_e & \xrightarrow{\sigma^e} & T \end{array}$$



Finally, there are the combined relations. Let $\sigma^a : T_a \to T$ be a codegeneracy and $\delta^e : T' \to T$ a coface map that does not eliminate a. Then if $\delta^e : T'_a \to T_a$ is the induced coface map, the following diagram commutes:

$$\begin{array}{ccc} T'_e & \xrightarrow{\delta^e} & T_e \\ \downarrow{\scriptstyle \sigma^a} & & \downarrow{\scriptstyle \sigma^a} \\ T' & \xrightarrow{\delta^e} & T \end{array}$$



Consider a codegeneracy map δ^e. Let the resulting vertex be denoted v_e and the resulting edges be x and y. If $\delta : T \to T^e$ is either

- an inner coface map δ/x or δ/y
- an outer coface map δ/v_e

then

$$T \xrightarrow{\delta} T_e \xrightarrow{\sigma^e} T$$

is the identity.

These relations can be arranged in the following table, where $\delta^i\delta^j$ means $\delta^i \circ \delta^j$, i.e. do δ^j first and then δ_i. Recall that p and q denote vertices, $\overline{pq}$ denotes the edge between p and q, and pq denotes the vertex resulting from an inner coface map $\delta^{\overline{pq}}$ or a codegeneracy map $\sigma^{\overline{pq}}$. Whenever two elementary maps are induced by vertices and edges which are disjoint from each other, they commute. Otherwise, in this table, assume they are coincident to each other.

inner and inner	$\delta^{\overline{pq}}\delta^{\overline{qr}} = \delta^{\overline{qr}}\delta^{\overline{pq}}$
inner and outer	$\delta^{pq}\delta^{\overline{pq}} = \delta^q\delta^p$
inner and degeneracy	$\sigma^{\overline{p(qr)}}\delta^{\overline{qr}} = id = \delta^{\overline{qr}}\sigma^{\overline{pq}}$
outer and outer	$\delta^{pq}\delta^{\overline{pq}} = \delta^q\delta^p$
outer and degeneracy	$\sigma^{\overline{pq}}\delta^r = \delta^r\sigma^{\overline{pq}}$
degeneracy and degeneracy	$\sigma^{\overline{p(pq)}}\sigma^{\overline{pq}} = \sigma^{\overline{(pq)q}}\sigma^{\overline{pq}}$

Note that entry for "outer and outer" is identical to that of "inner and outer". For many of these maps, there are certain combinations that do not make sense, because the relevant edges and vertices do not exist. Therefore, the relation listed in each row is the relation which is closest to the description and also makes sense.

Lemma 4.2.10. *Given any sequence of coface and codegeneracy maps, one can find a standard form consisting of a sequence of codegeneracy maps followed by a sequence of coface maps.*

Proof. Consider the above table of relations. Note that if a codegeneracy is coincident to an inner coface map they will annihilate. Otherwise, all other face maps either commute with codegeneracy maps or they are nearly commutative. That is, there will be some relation $\delta\sigma = \sigma\delta$, even if the labels differ. Therefore they can be moved past each other to form a sequence of codegeneracy maps followed by coface maps. □

Finally, the category of unrooted trees can be defined. This category will usually be referred to as Ӂ, but this is somewhat ambiguous, as it refers to both the category of unrooted trees, and the image of this category under the map

$$Ӂ : \textbf{unrootedtrees} \to \textbf{CycOpd}$$

as a subcategory of cyclic operads. As such, in this definition Ӂ will be used, but **unrootedtrees** will be used whenever there is ambiguity (particularly Section 4.2.1).

Definition 4.2.11 (The category of unrooted trees)**.** Let Ӂ be the category with

- The objects of Ӂ are the unrooted trees described in this section (Definition 4.2.2),
- The morphisms of Ӂ are given by compositions of face and degeneracy maps defined in Section 4.2.1, up to the equivalences detailed in Section 4.2.1.

Relationships to other categories

Chapter 2 explored the connections between categories, paths, and simplicial sets, and provided a map from paths to categories. Likewise Chapter 3 provided a map from trees to operads (Definition 3.2.10). This section contains an equivalent map for cyclic operads.

Definition 4.2.12. Given a tree T, a cyclic operad Ӂ(T) can be associated to it.

- The colours of Ӂ(T) are the edges of T.
- The operations are generated by the vertices. Given a vertex v with inputs $\{c_1, \dots, c_n\}$, it generates a morphism for every permutation of inputs:

$$v_{c_1,\dots,c_n} \in Ӂ(T)(c_1, \dots, c_n)$$
$$v_{c_{\sigma(1)},\dots,c_{\sigma(n)}} \in Ӂ(T)(c_{\sigma(1)}, \dots, c_{\sigma(n)}).$$

- Two operations can be composed if they can be composed according to the normal cyclic operad rules. That is, if they share a colour. If two generating operations are composed, this corresponds in the graph to two vertices which share an edge, and the result can be thought of as a subgraph. More generally, any two operations will themselves be subgraphs, and composition of these operations corresponds in the graph to two subgraphs sharing an edge on their boundaries forming a new subgraph that is both of them glued together along this edge.

- There is an identity $\eta_c \in Ж(T)(c, c)$ for each colour c. This corresponds to the edge c. If a subgraph containing the edge c on its boundary is composed with that edge c, the result is the original subgraph.

- This composition is associative by definition, since the operations are generated according to definition of cyclic operad. This makes sense, since for any compatible operations x, y, z, $x \circ_i (y \circ_j z)$ and $(x \circ_i y) \circ_j z$ both correspond in the graph to the same subgraph, and whether they are glued along the edge i or j first does not matter.

All morphisms in the category of trees are compositions of inner coface, outer coface, and degeneracy maps, so it suffices to show what the map Ж does to each of these.

Inner coface maps Let $\delta^e : T/e \to T$ be an inner coface map which contracts an edge e between vertices v and w. Denote this resulting vertex vw. Then consider the map $Ж(\delta^e) : Ж(T/e) \to Ж(T)$. The edges (colours) of $Ж(T/e)$ map injectively to their counterparts in $Ж(T)$. Then, for all vertices except vw, let the operations similarly map to their counterparts. Finally, for each $vw_{c_1,\ldots,c_n}$, let

$$vw_{\underline{cd}_e} \mapsto v_{\underline{c}} \circ_e w_{\underline{d}}$$

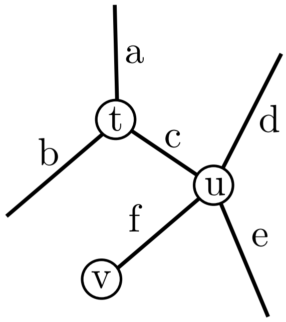
where the order of edges in $\underline{cd}_e$ determines the order in each of $\underline{c}$ and $\underline{d}$.

Outer coface maps Let $\delta^v : T/v \to T$ be an outer coface map with T/v being the tree that removes an outer vertex v. Then $Ж(\delta^v)$ is the inclusion.

Codegeneracy maps Let $\sigma^e : T_e \to T$ be a codegeneracy map, with T_e being the tree that subdivides edge e into two edges x and y, with vertex v in between them. Again, the associated operad map $Ж(\sigma^e)$ is the identity on most vertices and edges, except for x, y, v_{xy}, and v_{yx}. Both $x, y \in ob(Ж(T_e))$ map to $e \in ob(Ж(T)$. And both v_{xy} and v_{yx} map to $\eta_e \in Ж(T_e)(e, e)$.

Note that the category Ж will refer to both the category of unrooted trees with coface and codegeneracy maps, and the subcategory of **CycOpd** given by the image of the map Ж. Where disambiguation is necessary, $Ж(T)$ will refer to the cyclic operad associated with the tree T.

Now that trees can be treated as cyclic operads, the maps a : **Opd** $\to$ **CycOpd** and r : **CycOpd** $\to$ **Opd** can be applied to them. This can be illustrated with an example.



This tree, T, can be considered either as a rooted tree, and therefore an operad, or an unrooted tree, and therefore a cyclic operad. In either case, the objects of T are $\{a, b, c, d, e\}$. As a rooted tree, any of its outer edges could be designated the root, so for this example let e be the root. Then the operations are generated by

$$\begin{aligned} t_{ab} &\in \Omega(T)(a, b; c) \\ t_{ba} &\in \Omega(T)(a, b; c) \\ u_{cdf} &\in \Omega(T)(c, d, f; e) \\ u_{cfd} &\in \Omega(T)(c, f, d; e) \\ &\dots \\ u_{fdc} &\in \Omega(T)(f, d, c; e) \\ v &\in \Omega(T)(; f) \end{aligned}$$

and some examples of compositions of these operations are $t_{ab} \circ_c u_{dfc}$, $v \circ_f u_{cdf}$. However, if T is considered to be an unrooted tree, then there are immensely more operations. The generating operations will look similar:

$$\begin{aligned} t_{abc} &\in Ж(T)(a, b, c) \\ t_{acb} &\in Ж(T)(a, c, b) \\ &\dots \\ u_{cdef} &\in Ж(T)(c, d, e, f) \\ &\dots \\ v &\in Ж(T)(f) \end{aligned}$$

and the equivalents of the operad compositions will exist. However, many more compositions will exist here that do not exist in the operad case; for example, $v \circ_f v$ and $t_{abc} \circ_c t_{bac}$.

Now, consider T as a rooted tree. Then $a(\Omega(T))$ will be exactly $Ж(T)$ as written above. However, $r(Ж(T)$ will have many more morphisms than simply going back to $\Omega(T)$. One useful mental image is to think of $r(\text{Ж}(T)$ as containing all possible ways of finding a root for this tree.

4.2.2 Astroidal Sets

An astroidal set is a collection of sets patterned after the category of astrices. Astroidal sets shall be defined first, then their relationship to cyclic operads.

Definition 4.2.13 (Astroidal Set)**.** An astroidal set is a presheaf $X : \text{Ж}^{op} \to \mathbf{Set}$.

In other words, an astroidal set X consists of the following information.

- For each $T \in \text{Ж}$, a set $X(T)$, denoted X_T. Note that X_T is called the set of astrices of shape T,
- For each morphism $f : S \to T$, a function $X_f : X_T \to X_S$,
- With $X_{id:T\to T} = id : X_T \to X_T$,
- Given two morphisms $R \xrightarrow{\beta} S \xrightarrow{\alpha} T$ in Ж, $X_{(\alpha\circ\beta)} = X_\beta \circ X_\alpha$.

Definition 4.2.14 (Morphisms)**.** Let $X : \text{Ж}^{op} \to \mathbf{Set}$ and and $Y : \text{Ж}^{op} \to \mathbf{Set}$ be two astroidal sets. Then a map between them $f : X \to Y$ is a natural transformation. In other words,

- For each tree $T \in \text{Ж}$, there is a map $f_T : X_T \to Y_T$,
- If $\alpha : S \to T$ is a morphism in Ж, then the following diagram commutes.

$$\begin{array}{ccc} X_t & \xrightarrow{X_\alpha} & X_S \\ \downarrow{\scriptstyle f_T} & & \downarrow{\scriptstyle f_S} \\ Y_T & \xrightarrow{Y_\alpha} & Y_s \end{array}$$

The category of astroidal sets is defined in the expected way.

Definition 4.2.15 (Astroidal sets)**.** The category of astroidal sets, **aSet** is defined to be the category consisting of $ob(\mathbf{aSet}) = \{\text{Ж}^{op} \to \mathbf{Set}\}$ with natural transformations as morphisms, with natural transformations composing in the usual way.

Definition 4.2.16 (Astroidal subset)**.** Let X and Y be astroidal sets such that

- For every tree $T \in \text{Ж}$, $Y_T \subseteq X_T$
- The inclusions $Y_T \hookrightarrow X_T$ together form a morphism of astroidal sets.

Then Y is an astroidal subset of X.

Now that astroidal sets have been defined, their connection to cyclic operads can be explored. This relationship is important, because infinity cyclic operads will eventually be defined in terms of dendroidal sets, and it is important to understand the relationship between cyclic operads and infinity cyclic operads.

This information can be provided by the astroidal nerve, $n_a : \mathbf{CycOpd} \to \mathbf{aSet}$. The aim is to arrive at an adjunction

$$\mathbf{aSet} \rightleftarrows \mathbf{CycOpd}.$$

The following definitions can be found in Hackney et al. [34, Definition 5.8].

Definition 4.2.17 (Astroidal nerve)**.** Let $\mathcal{C}$ be a cyclic operad. Then the astroidal nerve of $\mathcal{C}$ is the astroidal set given by

$$n_a(\mathcal{C})_T = \mathbf{CycOpd}(\text{Ж}(T), \mathcal{C})$$

As in the simplicial and dendroidal cases, there is a left adjoint to the nerve, called the *astroidal realisation*.

Definition 4.2.18 (Astroidal realisation)**.** Let X be an astroidal set. Then $\tau_a(X)$ is the cyclic operad given by:

- The colours are $ob(\tau_a(X)) = X_\eta$
- The operations are generated by the elements of X_{C_n}, closed under the actions of Σ_n and the compositions. Note that here C_n refers to the unrooted corolla.
- The face maps induce relations between these operations.
  1. If A is a colour in X_η, and $\sigma^A : C_2 \to \eta$, then $\sigma_A(A) = id_A \in \tau_a(X)(A; A)$ is required, where $\sigma_A : X_\eta \to X_{C_1}$ is the map induced by σ^A
  2. Let T be the tree with two vertices, v and w. Then if δ_v, δ_w, and $\delta_{\overline{vw}}$ are similarly induced maps, one requires

     $$\delta_w(x) \circ_{\overline{vw}} \delta_v(x) = \delta_{\overline{vw}}(x)$$

     for all $x \in X_T$.

Given a morphism of astroidal sets $f : X \to Y$, there is a morphism $\tau_a(f) : \tau_a(X) \to \tau_a(Y)$ defined entry-wise on each $\tau_a(F_T) : \tau_a(X_T) \to \tau_a(Y_T)$.

Lemma 4.2.19. *The astroidal nerve functor is full but not faithful.*

The proof of this can be found in Hackney et al. [34, Theorem 6.7].

Lemma 4.2.20. *The left adjoint defined above is indeed a left adjoint.*

Proof. Firstly, consider what occurs on a corolla C_v, where the neighbourhood of v is the profile $c_1, \dots c_n$. We wish to show that

$$\mathrm{Hom}_{\mathbf{CycOpd}}(\tau_a(X_{C_v}), \mathcal{O}) \cong \mathrm{Hom}_{\mathbf{aSet}}(X_{C_v}, n_a(\mathcal{O})),$$

where X is an astroidal set and $\mathcal{O}$ is a cyclic operad. The maps on the right-hand side $X_{C_v} \to X_{C_v}$ commute with face maps, and the maps on the left-hand side $\tau_a(X_{C_v})$ commute with composition, so composition is induced by face maps and therefore this is sufficient.

□

Relationship with Dendroidal Sets

Given an astroidal set X, a dendroidal set can be defined by following the functors in the following diagram, along with their adjoints. Call this function $\hat{a} : \mathbf{aSet} \to \mathbf{dSet}$.

$$\begin{array}{ccc} \mathbf{CycOpd} & \underset{a}{\overset{r}{\rightleftarrows}} & \mathbf{Opd} \\ \tau_a \big\uparrow \big\downarrow n_a & & \tau_d \big\uparrow \big\downarrow n_d \\ \mathbf{aSet} & \underset{\hat{a}}{\overset{\hat{r}}{\rightleftarrows}} & \mathbf{dSet} \end{array}$$

Likewise, there is a reverse functor $\hat{r} : \mathbf{dSet} \to \mathbf{aSet}$.

4.2.3 Horns and Representables

Analogously to the way there is a standard n-simplex, $\Delta[n]$, for every tree there is a standard astrex.

Definition 4.2.21 (Standard T astrex)**.** Let $T \in \text{Ж}$ be a tree. Then the standard T-astrex is the representable presheaf $\text{Ж}[T] = \text{Ж}(-, T)$. At each tree $S \in \text{Ж}$, $\text{Ж}[T]_S = \text{Ж}(S, T)$.

It is obvious from the definitions (in particular, note Definition 4.2.12)that $n_a(\text{Ж}(T)) = \text{Ж}[T]$, as one would hope. This is clearly a functor $\text{Ж}^{op} \to \mathbf{Set}$, and therefore is a astroidal set.

Lemma 4.2.22. *Let $\alpha : S \to T$ be a map of unrooted trees. Then there is an induced map $\text{Ж}[\alpha] : \text{Ж}[S] \to \text{Ж}[T]$ of dendroidal sets.*

Here, the map is defined according to this diagram, where f is some map in $\mathrm{Hom}(R, S)$, and the image of this map is $\alpha \circ f$

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & T \\ f \big\uparrow & \nearrow_{\alpha \circ f} & \\ R & & \end{array}$$

Definition 4.2.23 (Face)**.** Let $T \in \text{Ж}$ be a tree with face map $\alpha : S \to T$. Then the α-face of $\text{Ж}[T]$ is the image of the map $\text{Ж}[\alpha] : \text{Ж}[S] \to \text{Ж}[T]$. It is denoted $\partial_\alpha \text{Ж}[T]$.

When describing a face induced by a particular edge or vertex, that is where $\alpha = \delta^e$, this will often be written as simply ∂_e. By this stage, the reader may have become confused by the plethora of ∂'s and δ's, so here is a reminder (assuming, for illustrative purposes, contracting a particular edge e).

1. Coface maps in the category of trees, Ж, are represented by δ^e
2. Face maps in an astroidal set are represented by δ_e
3. Faces, as defined above, are represented by ∂_e

Definition 4.2.24 (Boundary)**.** The astroidal subset which is the union of all possible faces.

$$\partial Ж[T] = \bigcup_{\alpha \in faces(T)} \partial_\alpha Ж[T]$$

Note that each face is the image of some map $Ж[S] \to Ж[T]$, or in other words $\mathrm{Hom}(-, S) \to \mathrm{Hom}(-, T)$. So each face is the set of maps into T which can be decomposed into maps $\cdot \to S \to T$. Thus $\partial Ж[T] \hookrightarrow Ж[T]$.

Then a horn is like a boundary, but with one face excluded. In particular, an inner horn is one where the face which is left out is an inner face. More formally,

Definition 4.2.25 (Inner horn)**.** Let α be an inner face map which removes the edge e. Then

$$\Lambda^e[T] = \bigcup_{\alpha \neq e \in faces(T)} \partial_\alpha Ж[T]$$

In an astroidal set X, an (inner) horn is a map of dendroidal sets $\Lambda^\alpha[T] \to X$.

Again, each face is the image of some map $\mathrm{Hom}(-, S) \to \mathrm{Hom}(-, T)$, so there is an inclusion $j : \Lambda^e[T] \to Ж[T]$. More formally, let $g \in \Lambda^e[T]$. Then g is in at least one of $\partial_\alpha Ж[T]$, so for some $\alpha : S \to T$, $g = \alpha \circ f$, and thus $g \in \mathrm{Hom}(-, T) = Ж[T]$. So j is indeed an inclusion $\Lambda^e[T] \to Ж[T]$.

4.3 Infinity Cyclic Operads

The notion of a cyclic operad is important for several relationships it has with surfaces, and in each of these cases it is in fact higher cyclic operads which are needed (and higher modular operads). These include the cobordism hypothesis [3, 48], profinite completions of the framed little discs operad [10], and others [74].

In this section, I cover four different versions of infinity cyclic operads, corresponding to the same four as in the previous chapters. I begin with the simplest to define, simplicially enriched cyclic operad, albeit with no proof of equivalence to the others. Then, I define quasi cyclic operads as inner Kan complexes. This is entirely my own work, and the main focus of this section. The Segal condition already exists in the literature [34], so I include it for comparison with the inner Kan condition. I do not define astroidal complete Segal spaces, because one should be able to take modular complete Segal spaces of genus zero. I also include a proof that astroidal groups satisfy the inner Kan condition.

4.3.1 Simplicially Enriched

The following definition was originally conjectured based on similar definitions in the simplicial and dendroidal cases [20, 21, 67]. A variant appears in the recent preprint by Drummond-Cole and Hackney [19, Section 6].

Definition 4.3.1 (Enriched cyclic operad)**.** A cyclic operad $\mathcal{C}$ enriched in **sSet** consists of

1. A set of colours $ob(\mathcal{C})$
2. For all profiles $c_1, \ldots c_n$, a simplicial set $\mathcal{C}(c_1, \ldots c_n)$

3. There is a right action of the symmetric group, also a map of simplicial sets
4. Composition is given by maps of simplicial sets. It is associative, unital, and equivariant.
5. Maps of cyclic operads which satisfy the conditions that maps between simplicial sets should obey, and the simplicial set structure should commute with the cyclic operad structure.

Conjecture 4.3.2. *There is an equivalence between simplicially enriched cyclic operads, and quasi cyclic operads (see Definition 4.3.3).*

4.3.2 The Astroidal Inner Kan Condition

The definition of Kan condition for astroidal sets is similar to that of dendroidal sets.

Definition 4.3.3 (Inner Kan complex)**.** Let X be an astroidal set, $f : \Lambda^k[T] \to X$ be an inner horn, and let $j : \Lambda^k[T] \rightarrowtail \text{ЖK}[T]$ be the inclusion. Then a filler for f is a map $g : \text{ЖK}[n] \to X$ such that $f = g \circ j$.

$$\begin{array}{ccc} \Lambda^k[T] & \xrightarrow{f} & X \\ \big\downarrow{\scriptstyle j} & \nearrow_{g} & \\ \text{ЖK}[T] & & \end{array}$$

The astroidal set X is said to be an inner Kan complex if every inner horn has a filler.

Then, an astroidal set satisfying the Kan condition is an inner Kan complex, otherwise known as a quasi cyclic operad. If the filler is unique then it is a strict inner Kan complex; that is, there is a corresponding cyclic operad (see Theorem 4.3.7).

4.3.3 The Astroidal Segal Condition

The Segal condition for astroidal sets shall be defined analogously to the Segal condition in dendroidal sets. A simplicially enriched version is studied in Hackney et al. [34, Definitions 1.34 and 8.8].

Definition 4.3.4 (Segal core)**.** Let T be a tree with at least one vertex. Recall the corolla C_n. The Segal core $Sc[T]$ is the sub-object of $\text{ЖK}[T]$ defined as the union of all the images of maps $\text{ЖK}[C_n] \to \text{ЖK}[T]$ corresponding to sub-trees of shape $C_n \to T$. Note that such a map is completely determined, up to isomorphism, by the vertex v of T in its image. Let $n(v)$ be the number of input edges that v has. Therefore, one can write

$$Sc[T] = \bigcup_{v \in V(T)} \text{ЖK}[C_{n(v)}].$$

Then the Segal condition can be defined as follows.

Definition 4.3.5. An astroidal set X satisfies the Segal condition if for every tree T the map

$$\mathrm{Hom}(Sc[T], X) \to \mathrm{Hom}(\text{Ж}[T], X)$$

is a bijection.

There is an alternative characterisation of the Segal condition as follows.

Proposition 4.3.6. *An astroidal set X satisfies the Segal condition if and only if for every tree T there is a bijection*

$$X_T \cong X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}}.$$

Proof. Consider the following three equivalences (given by the Yoneda Lemma)

$$X_T \cong \mathrm{Hom}(\text{Ж}[T], X)$$
$$X_{C_v} \cong \mathrm{Hom}(\text{Ж}[C_v], X)$$
$$X_\eta \cong \mathrm{Hom}(\text{Ж}[\eta], X)$$

as well as the pullback $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}}$. It suffices to show that $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}} \cong \mathrm{Hom}(Sc[T], X)$.

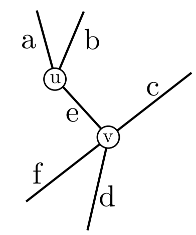
By Definition 4.3.4, the Segal core is the union of the images of the maps $\text{Ж}[C_n] \to \text{Ж}[T]$ corresponding to sub-trees of shape $C_n \to T$. But these maps will agree whenever two corollae share an edge. Likewise, the pullback $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}}$ corresponds to maps $X_{C_v} \cong \mathrm{Hom}(\text{Ж}[C_v], X)$ which agree over X_η. □

Theorem 4.3.7 (Nerve theorem)**.** *Let $\mathcal{A}$ be an astroidal set. Then the following are equivalent:*

1. *There exists a cyclic operad $\mathcal{C}$ such that $\mathcal{A} \cong n_a(\mathcal{C})$*
2. *$\mathcal{A}$ satisfies the Segal condition*
3. *$\mathcal{A}$ satisfies the strict inner Kan condition*

The proof of the equivalence of 1 and 2 can be found in Hackney et al. [34, Theorem 6.7]. The following (Lemmas 4.3.9 and 4.3.10) proves the equivalence of 2 and 3. To give some insight, the base step of the induction will be provided separately, as an example.

Example 4.3.8. The smallest tree containing at least three face maps, at least one being an inner face map, is the barbell:



The three face maps are δ_v, δ_u, and δ_e, of which only δ_e is an inner face map. Therefore, there is only one inner horn,

$$\Lambda^e[T] = \partial_u Ж[T] \cup \partial_v Ж[T].$$

The result of ∂_u is the corolla C_v, and likewise ∂_v results in C_u. Therefore, the inner horn is

$$\Lambda^e[T] = Ж[C_v] \cup Ж[C_u].$$

This is the Segal core, so, in this instance,

$$\mathrm{Hom}(\Lambda^e[G], X) \cong \mathrm{Hom}(Sc[G], X).$$

Usually, the Segal core is a subset of the horn, and therefore $\mathrm{Hom}(\Lambda^e[G], X)$ will be a subset of $\mathrm{Hom}(Sc[G], X)$. However, in this case, they are in bijection.

Lemma 4.3.9. *Let T be an unrooted tree with some inner edge e, and let X be a graphical set. If*

$$\mathrm{Hom}(\Lambda^e[T], X) \cong \mathrm{Hom}(Ж[T], X)$$

then

$$\mathrm{Hom}(Sc[T], X) \cong \mathrm{Hom}(Ж[T], X).$$

Proof. This shall be proven via strong induction on the number of vertices in T. Omitting those graphs with fewer than two vertices, the barbell graph shall form the base case, explored in Example 4.3.8.

Given a map $f \in \mathrm{Hom}(Sc[T], X)$, we shall find a map $f_\alpha \in \mathrm{Hom}(Sc[T/\alpha], X)$ for each face in the horn. Then, by the induction, we associate to it some function $h_\alpha \in \mathrm{Hom}(Ж[T/\alpha], X)$. Finally, note that $\Lambda^e[T] = \bigcup_{\alpha \neq e} \partial_\alpha Ж[T]$, so $h \in \mathrm{Hom}(\Lambda^e[T], X)$ can be defined from the collection of maps h_α, and then the Kan condition gives a map $g \in \mathrm{Hom}(Ж[T], X)$.

Firstly, define each f_α. If α is an outer face map then $Sc[T/\alpha]$ includes into $Sc[T]$, since $Sc[T/\alpha]$ merely excludes the corolla associated to α. Otherwise, α is an inner face map. Let $\overline{vw}$ denote the edge which is contracted, between the vertices v and w, and let vw denote the edge formed from the combination of v and w. For all other vertices, their associated corollas will be found in both $Sc[T]$ and $Sc[T/\alpha]$. To complete the definition of f_α, it is necessary to define $f_\alpha(Ж[C_{vw}]$. But by the inductive hypothesis $\mathrm{Hom}(Ж[C_{vw}], X) \cong \mathrm{Hom}(Sc[C_{vw}], X)$, and $Sc[C_{vw}] = Ж[C_v] \cup Ж[C_w]$. Let

$$h(\partial_\alpha Ж[T]) = h_\alpha(Ж[T/\alpha]),$$

noting that we have $\partial_\alpha Ж[T] = Ж[T/\alpha]$.

It remains to be shown that this is well defined, and that it is indeed a bijection. Let us consider well defined first. Given two faces ∂_α and ∂_β, we require that

$$h(\partial_\alpha Ж[T]) = h(\partial_\beta Ж[T])$$

when restricted to the intersection $\partial_\alpha Ж[T] \cap \partial_\beta Ж[T]$. But this follows from the astroidal identities (Section 4.2.1) and the nature of astroidal sets. In most cases, $\delta^\alpha \delta^\beta = \delta^\beta \delta^\alpha$, so the aforementioned intersection is in bijection with the set $\partial_\alpha \partial_\beta Ж[T]$.

However, if we consider an outer face map and an inner face map, corresponding to an inner edge incident to an outer vertex, then this conception of

$\partial_\alpha\partial_\beta$Ж$[T]$ does not entirely make sense. Likewise with two outer face maps where one of them is done on a vertex that is not an outer vertex until the other face map is done. In both of these cases, there is still an intersection involved, but we can't simply commute the two face maps. Let our vertices be p and q, connected via an edge r. Then, due to the graphical identities, we know that $\partial_q\partial_p$Ж$[T] = \partial_{pq}\partial_r$Ж$[T]$, where pq is the outer vertex resulting from contracting the edge r. This intersection must be well defined too, because the graphical set satisfies these graphical identities. (There are no faces ∂_q and ∂_{pq} of Ж$[T]$, so we need not worry about them.) For some examples, see Section 4.2.1.

Now to ensure that there is a bijection between the sets Hom(Ж$[T], X$) and $\mathrm{Hom}(Sc[T], X)$. Call the map defined above

$$\varphi : \mathrm{Hom}(Sc[T], X) \to \mathrm{Hom}(\text{Ж}[T], X).$$

Given any map in Hom(Ж$[T], X$), it is clear that a map in $\mathrm{Hom}(Sc[T], X)$ can be defined via a restriction. Call this map

$$\rho : \mathrm{Hom}(\text{Ж}[T], X) \to \mathrm{Hom}(Sc[T], X).$$

So it must be shown that $\varphi \circ \rho = id = \rho \circ \varphi$. But, the graphical set relationship between $Sc[T]$, $\Lambda^e[T]$, and Ж$[T]$ ensures that, as we go back and forth along ρ and φ

□

Lemma 4.3.10. *Let X be an astroidal set. If X satisfies the astroidal Segal condition then it satisfies the strict inner astroidal Kan condition.*

Proof. This can be shown directly. Let X be an astroidal set that satisfies the Segal condition, T be a tree with inner edge e, and let $h \in \mathrm{Hom}(\Lambda^e[T], X)$. Each vertex of T can be found in at least one face, so h gives a map $f : Sc[T] \to X$. By the Segal condition, f provides a map g : Ж$[T] \to X$. Therefore we have

$$Hom(Sc[T], X) \subset Hom(\Lambda^e[T], X) \subset Hom(\text{Ж}[T], X).$$

The Segal condition states that $Hom(Sc[T], X) \cong Hom(\text{Ж}[T], X)$ is a bijection, so we must therefore also have a bijection $\mathrm{Hom}(\Lambda^e[T], X) \cong \mathrm{Hom}(\text{Ж}[T], X)$, giving us the Kan condition □

Chapter 5

Modular Operads and Graphical Sets

Modular operads, the higher genus version of operads, are intimately connected to Feynman diagrams and TQFTs, and thus my study of them was inevitable. They were first used to aid in Feynman diagram calculations [29], but their connection to surfaces has become apparent in [73, 30, 37, 24, 74]. Modular operads are particularly well suited to studying surfaces because, unlike operads, they take genus into account. The relationship between Feynman diagrams and modular operads is known, as is the relationship between Feynman diagrams and TQFTs; this thesis, particularly Section 5.4, aims to shed light on the relationship between TQFTs and modular operads. This shall be conducted first through quasi modular operads, then by defining a modular operad of mapping class groups of surfaces.

Additionally, part of my interest in modular operads stems from the prospect of defining their infinity counterpart as presheaves over some graphical category, as a generalisation of dendroidal sets and simplicial sets. Just as categories are path shaped and operads are tree shaped, modular operads fill in the graph shaped column of this table.

	Paths	Trees	Graphs
Directed	Categories	Operads	Wheeled properads
Undirected	Dagger categories	Cyclic operads	Modular operads

As seen in the above table, there are both directed and undirected forms of higher genus operads, generalising both operads and cyclic operads. Wheeled properads are properads in which the morphisms may have loops, with properads being operads in which the morphisms may have both multiple inputs and multiple outputs. Wheeled properads and properads have previously been studied by Hackney et al. [33]. On the other hand, modular operads are the undirected version of wheeled properads, first introduced by Getzler and Kapranov [29]. Modular operads are sometimes known as Compact Symmetric Multicategories, although usually there is a slight difference in that CSMs contain an involution on objects [41].

This section shall focus on modular operads. As with infinity cyclic operads, infinity modular operads are constructed with the aid of presheaves over some category of graphs. Rather than just unrooted trees, as in the cyclic operads case, this category shall consist of graphs in general.

5.1 Modular Operads

These were introduced by Getzler and Kapranov [29] and further studied by Hackney et al. [36] and Raynor [65]. There is also a definition by Hinich and Vaintrob [38, Definition 3.3.1], which uses a set of inputs rather than a list, removing the need for a symmetric group action. The following definition is based on [65]. If an involution is added to the set of objects, the result is something equivalent to Compact Symmetric Multicategories [41].

Definition 5.1.1 (Modular operad)**.** A coloured modular operad, $\mathcal{M}$, over a symmetric monoidal category $\mathcal{E}$, is defined by

- A set of objects $ob(\mathcal{M})$
- For each profile $\underline{c}$, there is an object of $\mathcal{E}$, $\mathcal{M}(\underline{c})$. If $\mathcal{E}$ is **Set**, then each $\mathcal{M}(\underline{c})$ is a set of operations.
- There is a right action of the symmetric group. That is, for any profile $\underline{c} = \{c_{i_1}, \dots c_{i_n}\}$ and permutation $\sigma \in \Sigma_n$, there is a bijection $\mathcal{M}(c_1, \dots, c_n) \to \mathcal{M}(c_{\sigma(1)}, \dots, c_{\sigma(n)})$
- There is a composition. Let $\underline{U} = \{c_{i_1}, \dots c_{i_r}\} = \{d_{j_1}, \dots d_{j_r}\}$ be a sub-profile of $\underline{c}$ (equal to some other sub-profile of $\underline{d}$). Then for $\theta \in \mathcal{M}(\underline{c})$ and $\theta' \in \mathcal{M}(\underline{d})$,
$$\theta \circ_U \theta' \mapsto \theta''$$
where $\theta'' \in \mathcal{M}(c_1, \dots \hat{c_{i_1}}, \dots \hat{c_{i_r}}, \dots c_m, d_1, \dots \hat{d_{j_1}}, \dots \hat{d_{j_r}}, \dots d_n)$. Note that $\hat{c_{i_j}}$ is used to denote removing c_{i_j}, and that the removed c_{i_j}'s may be interlaced with the non-removed objects. For ease of reading, let
$$\underline{cd}_U = c_1, \dots \hat{c_{i_1}}, \dots \hat{c_{i_r}}, \dots c_m, d_1, \dots \hat{d_{j_1}}, \dots \hat{d_{j_r}}, \dots d_n.$$
- There is a contraction. Let $\underline{c}$ be a profile in which at least one object is repeated, so $c_i = c_j$. Then there is a contraction, $\zeta_{ij} : \mathcal{M}(\underline{c}) \to \mathcal{M}(c_1, \dots, \hat{c}_i, \dots, \hat{c}_j, \dots, c_n)$.

Satisfying these axioms:

1. Composition is associative. That is, for all profiles $\underline{a}$, $\underline{b}$, and $\underline{c}$:
$$\begin{array}{ccc} \mathcal{M}(\underline{a}) \otimes \mathcal{M}(\underline{b}) \otimes \mathcal{M}(\underline{c}) & \xrightarrow{\circ_V} & \mathcal{M}(\underline{a}) \otimes \mathcal{M}(\underline{bc}_V) \\ \downarrow{\circ_U} & & \downarrow{\circ_U} \\ \mathcal{M}(\underline{ab}_U) \otimes \mathcal{M}(\underline{c}) & \xrightarrow{\circ_V} & \mathcal{M}(\underline{abc}_{UV}) \end{array}$$
where the lengths of $\underline{a}$, $\underline{b}$, and $\underline{c}$ are ℓ, m, and n, respectively.
2. Composition is unital. That is, for each colour c there exists an identity η_c, such that for all $\theta \in \mathcal{M}(\underline{c})$, $\theta \circ_i \eta = \theta = \eta \circ_i \theta$.

3. Composition is equivariant. That is, it commutes with the action of the symmetric group. So for any two morphisms $\alpha \in \mathcal{M}(\underline{c})$ and $\beta \in \mathcal{M}(\underline{d})$, where $\underline{c}$ is of length m and $\underline{d}$ is of length n, and any $\sigma \in \Sigma_n, \tau \in \Sigma_m$,

$$\alpha \circ_{\sigma(U)} \sigma(\beta) = \sigma'(\alpha \circ_U \beta)$$

$$\tau(\alpha) \circ_U \beta = \tau'(\alpha \circ_i \beta)$$

Where $\sigma' \in \Sigma_{n+m-2|u|}$ refers to the element that acts on $\underline{cd}_U$ by doing σ on each d_k and the identity on each c_j, and $\tau' \in \Sigma_{n+m-2|u|}$ does the identity on each d_k and permutes each c_j according to τ.

4. If $\underline{c}$ is a profile in which at least two objects are repeated, so $c_i = c_j$ and $c_k = c_\ell$, then:

$$\begin{array}{ccc} \mathcal{M}(\underline{c}) & \xrightarrow{\zeta_{ij}} & \mathcal{M}(c_1, \ldots, \hat{c_i}, \ldots, \hat{c_j}, \ldots, c_n) \\ \downarrow{\scriptstyle \zeta_{k\ell}} & & \downarrow{\scriptstyle \zeta_{k\ell}} \\ \mathcal{M}(c_1, \ldots, \hat{c_k}, \ldots, \hat{c_\ell}, \ldots, c_n) & \xrightarrow{\zeta_{ij}} & \mathcal{M}(c_1, \ldots, \hat{c_i}, \ldots, \hat{c_j}, \ldots, \hat{c_k}, \ldots, \hat{c_\ell}, \ldots, c_n) \end{array}$$

Note that the profile $(c_1, \ldots, \hat{c_i}, \ldots, \hat{c_j}, \ldots, c_n)$ may equivalently be referred to as $\underline{c} \setminus c_i, c_j$.

5. The contraction commutes with the composition. That is, if $\underline{c}$ is a profile such that some $c_i = c_j$, and neither c_i nor c_j is in U, then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}(\underline{c}) \otimes \mathcal{M}(\underline{d}) & \xrightarrow{\zeta_{ij}} & \mathcal{M}(\underline{c} \setminus c_i, c_j) \otimes \mathcal{M}(\underline{d}) \\ \downarrow{\scriptstyle \circ_U} & & \downarrow{\scriptstyle \circ_U} \\ \mathcal{M}(\underline{cd}_U) & \xrightarrow{\zeta_{ij}} & \mathcal{M}(\underline{cd}_U \setminus c_i, c_j) \end{array}$$

6. Parallel gluing of distinct elements. Let $\underline{c}$ and $\underline{d}$ be two profiles which share U. Let a and b be two colours in U, and let $U \setminus i$ denote the subprofile of U without i. Then the following diagram commutes.

$$\begin{array}{ccc} \mathcal{M}(\underline{c}) \otimes \mathcal{M}(\underline{d}) & \xrightarrow{\circ_{U \setminus a}} & \mathcal{M}(\underline{cd}_{U \setminus a}) \\ \downarrow{\scriptstyle \circ_{U \setminus b}} & & \downarrow{\scriptstyle \zeta_a} \\ \mathcal{M}(\underline{cd}_{U \setminus b}) & \xrightarrow{\zeta_b} & \mathcal{M}(\underline{cd}_U) \end{array}$$

where ζ_a refers to the map which contracts the two instances of a, and likewise with ζ_b.

Just as operations in an operad can be pictured as corollae, with composition by glueing into trees, operations in a modular operad can be pictured as corollae with loops. Composition involves glueing edges together, but unlike operads, these operations may be glued into higher genus graphs, rather than just trees. To compose two operations, glue some legs of one to similarly coloured legs of the other. The contraction operation represents gluing together two legs of the same colour attached to the same operation.

Note that axioms 5 and 6 of Definition 5.1.1 are both referring to the composition commuting with contraction. In 5, it is a local contraction that is not influenced by the composition, whereas in 6 it is two different contractions that become essentially equivalent by the composition which occurs.

Morphisms between modular operads can be defined in the same way as for other operadic generalisations in this thesis, and then the category of modular operads can be defined.

Definition 5.1.2 (Category of Modular operads, **ModOpd**)**.** A morphism $F : \mathcal{O} \to \mathcal{P}$ consists of the following information.

- A function $F : ob(\mathcal{O}) \to ob(\mathcal{P})$
- For each profile $\underline{c}$, a function

$$F_{\underline{c}} : \mathcal{O}(c_1, c_2, \dots c_n) \to \mathcal{P}_g(F(c_1), F(c_2), \dots F(c_n))$$

- Such that identities, symmetric group actions, composition, and contraction are all preserved.

Then this is the category of modular operads **ModOpd**

5.2 Graphical Sets

Simplicial sets and dendroidal sets both have an indexing category of a type of graph, paths in the case of simplicial sets and trees in the case of dendroidal sets. Graphical sets take this concept and use an underlying category of all graphs.

5.2.1 The Category of Graphs

There are multiple ways of defining a graph. The following definition of a graph is different to that used in the previous chapter (Definition 4.2.1), and is used so as to match the usual definition in the modular operads literature. For an examination of the various definitions and proofs of their equivalence, see [5, Proposition 15.8]. The following definitions, leading up to and including $\mathcal{G}$, can be found in [36, Section 1].

Definition 5.2.1 (Graph)**.** A graph consists of sets $E, H, V \subseteq \mathcal{F}$, where $\mathcal{F}$ is some infinite set.

$$i \circlearrowright E \xleftarrow{s} H \xrightarrow{t} V$$

such that s is a monomorphism, and i is a fixed point free involution.

For those who think in terms of a more topological definition of graph, the letters these sets are denoted by above provide some clue as to the relationship between these definitions. The set V is the set of vertices, and the set E is the set of directed edges, or half edges, with an involution sending half edges to their opposite direction counterparts. An edge would then be defined as a set $\{a, i(a)\}$, where a is some half edge.

The set H contains all those half edges which point towards a vertex. The map t sends each to its target vertex, which the map s includes into the set of all directed edges, E. The internal edges are those connected to a vertex on both sides, that is, those of the form $\{sd, sd'\}$, where $d, d' \in H$. The boundary, $\partial(G)$, is the set $E \setminus sH$.

Note that each graph can be considered a functor from the diagram category of the form

$$i \circlearrowright \cdot \xleftarrow{s} \cdot \longrightarrow \cdot$$

to the category of finite sets. Note that this is a diagram of three objects, with a fixed point involution i and a monomorphism s.

Note that Definition 5.2.1 does not adequately capture the existence of the graph S^1, consisting of a single edge looped into itself, and no vertices. This graph is appended to the set defined above. Also, assume all graphs are connected.

Graphical maps, which form the morphisms between graphs, shall be defined shortly. This is not the only way to form morphisms between graphs: Raynor has a less general definition in [65], but results here should transfer across. In the following four definitions let G and G' be graphs, where G is

$$i \circlearrowright E \xleftarrow{s} H \xrightarrow{t} V$$

and G' is

$$i' \circlearrowright E' \xleftarrow{s'} H' \xrightarrow{t'} V'$$

Definition 5.2.2 (Étale)**.** A natural transformation $G \to G'$ is an étale map if the right hand square is a pullback:

$$\begin{array}{ccccc} i \circlearrowright E & \xleftarrow{s} & H & \xrightarrow{t} & V \\ \downarrow & & \downarrow & & \downarrow \\ i' \circlearrowright E' & \xleftarrow{s'} & H' & \xrightarrow{t'} & V' \end{array}$$

Definition 5.2.3 (Embedding)**.** An embedding $G \to G'$ is an étale map for which $V \to V'$ is a monomorphism.

Definition 5.2.4 (Emb)**.** Given a graph G, let $\widetilde{Emb}(G)$ be the collection of embeddings of graphs into G. Then $Emb(G)$ is the result of modding out by isomorphisms.

Note that $Emb(G)$ can be considered as the collection of graphs that can be embedded into G

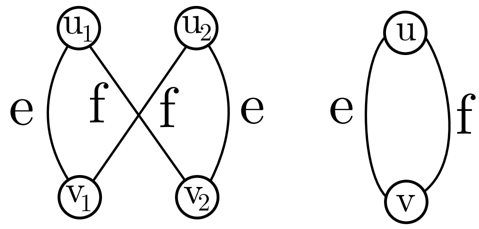


Figure 5.1: This map is an example of a map between graphs which is étale but not an embedding. It sends u_i to u, v_i to v, and both e's and f's to e and f

Definition 5.2.5 (Vertex sum ς)**.** Let $f : G \to G'$ be an embedding. Then there is a corresponding element in the free commutative monoid on V,

$$\sum_{v \in V} f(v) \in \mathbb{N}V'.$$

Denote this map by $\varsigma : Emb(G) \to \mathbb{N}V$

In the above definition, note that $\varsigma(f) = \sum_{v \in W} v$, where W is a subset of the vertices in V'. If $Y \subseteq X$, then $\sum_{v \in Y} v \leq \sum_{v \in X} v$.

Given any embedding, note that, due to the étale nature of the map, the $H \to H'$ will also be a monomorphism. Not all étale maps will be embeddings, as seen in Example 5.1. Now that these definitions have been established, the definition of graphical map is presented.

Definition 5.2.6 (Graphical map)**.** A graphical map $\varphi : G \to G'$ consists of:

- A map of involutive sets $\varphi_0 : E \to E'$, i.e. a map which respects the involutions,
- A function $\varphi_1 : V \to Emb(G')$,

such that

1. This inequality holds:

$$\sum_{v \in V} \varsigma(\varphi_1(v)) \leq \sum_{w \in V'} w,$$

2. There is a unique bijection making this diagram commute:

$$\begin{array}{ccc} nb(v) & \xrightarrow{i} & E \\ \downarrow{\cong} & & \downarrow{\varphi_0} \\ \partial(\varphi_1(v)) & \xrightarrow{inj} & E' \end{array}$$

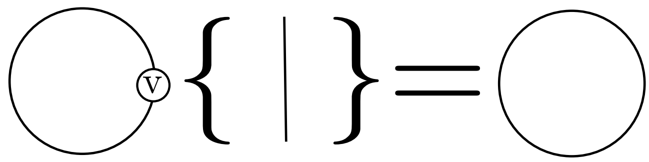


Figure 5.2: This map is not allowed by condition 3

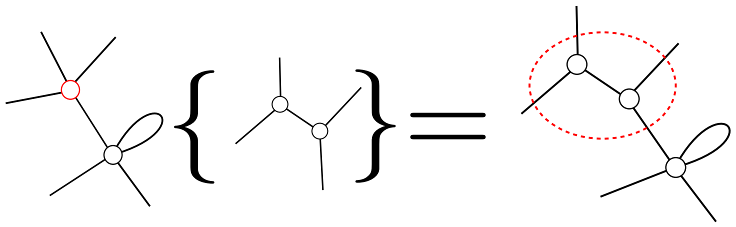


Figure 5.3: The graph in brackets in inserted into the red vertex to form the new graph.

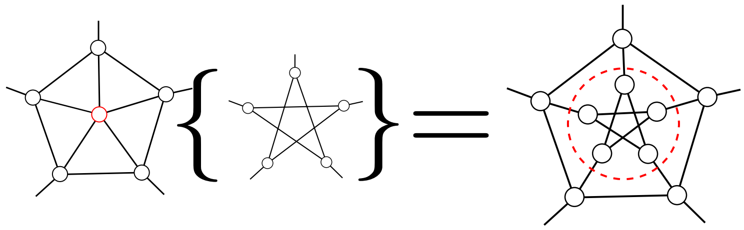


Figure 5.4: A non-planar example.

Here the top map i is the restriction of the involution on A and the bottom map is an injection. Note that $nb(v)$ refers to the neighbourhood of v, and $\partial(\varphi_1(v))$ refers to the boundary of the graph given by the image of the embedding $\varphi_1(v)$.

3. If $\partial G = \emptyset$, then there exists a v such that $\varphi_1(v)$ is not an edge (see Figure 5.2).

The function φ_1 is essentially saying that for each vertex v in G, a graph H_v is assigned to it, which is then embedded into G to get G'. In other words, one can select a vertex $v \in V(G)$ of degree n, and a graph H with n legs, and insert H in place of v to arrive at G' (see Figure 5.3). This is written as $G\{H_v\} = G'$.

The bijection is stating that the legs of H_v correspond to the legs of the vertex v. There is a slight overload of notation going on. The expression $\partial(\phi_1(v))$ actually refers to the boundary of the image of the embedding into G', and the commutative diagram is saying that the boundary of the graph H_v inserted into the vertex v should match the boundary of v.

Then the inequality is stating that there should be fewer vertices in G than in G'. The final requirement ensures that there is no map from the loop on one vertex to the graph consisting of a loop with no vertices. Figures 5.2.1 and 5.3 show two examples of graphical maps, while Figure 5.1 is a non-example.

Graphical maps can be written down in multiple ways. Firstly, in terms of two maps, one from edges to edges and the other from vertices to embeddings, as in Definition 5.2.6. Secondly, in terms of coface and codegeneracy maps, which shall be detailed shortly. Thirdly, as graph substitutions. A graph substitution, $G' = G\{H_v\}$, presents G' as the result of replacing each vertex v in G with a graph H_v, where the boundary of H_v is compatible with the neighbourhood of v. According to the following lemma [36, Prop 1.38], each graphical map can be written as a graph substitution followed by an inclusion.

Lemma 5.2.7. *Let $\phi : G \to G'$ be a graphical map, and denote by $\phi_v : H_v \hookrightarrow G'$ each $\phi_1(v) \in Emb(G')$. Then there is an embedding $G\{H_v\} \hookrightarrow G'$ which factors through each ϕ_v.*

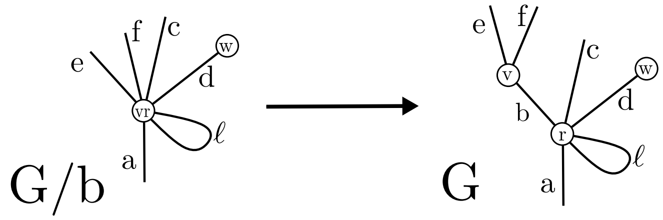
Definition 5.2.8 (The category of graphs, $\mathcal{G}$)**.** Let $\mathcal{G}$ be the category with graphs as objects and graphical maps as morphisms, and with composition of graphical maps inherited from composition of graph substitutions.

Remark 5.2.9. There is a way to get the topological realisation of a graph, and then to define properties such as connectedness and genus. Although, it is possible these properties will be able to be defined without reference to the topological realisation.

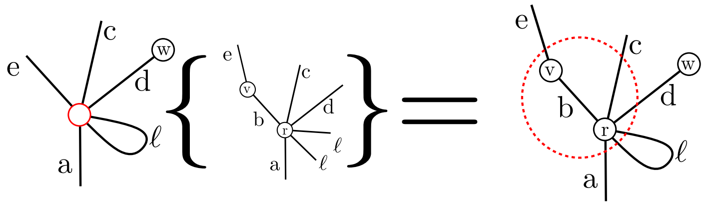
Face and Degeneracy Maps

In the categories Δ, Ω, and Ж, it was possible to decompose the morphisms into faces and degeneracies. The morphisms in this category can also be generated by some elementary morphisms, which can be arranged into face maps and then degeneracy maps (Lemma 5.2.16). First, these face and degeneracy maps must be defined. There are two types of coface maps, as well as degeneracy maps. In the following, let T be an unrooted tree.

Definition 5.2.10 (Inner coface map)**.** Let b be an inner edge of T. Define T/b as the tree created by contracting the edge b. That is, if x and y are the vertices at either side of b, delete these and the edge b and add a new vertex xy with $nbhd(xy) = (nbhd(x) \cup nbhd(y)) \setminus \{b\}$. Then $\delta^b : T/b \to T$ is the associated inner coface map.



And the same example as a graphical map, where a barbell graph is inserted into the appropriate vertex.

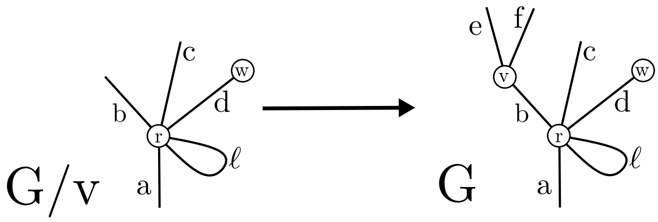


That is, $\delta^b : G/b \to G$ is defined with δ_0^b the identity on all edges in G/b and δ_1^b given by

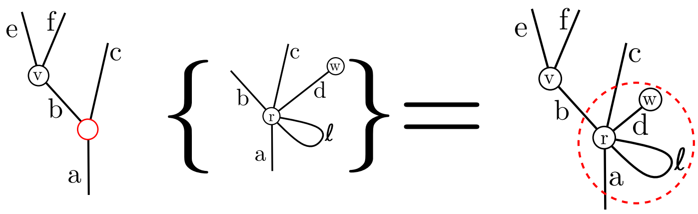
$$
\begin{aligned}
w &\mapsto C_w \\
vr &\mapsto B
\end{aligned}
$$

where B is the barbell graph given in the braces in the image. The two properties of the graphical maps definition hold because the total vertices in the embeddings matches the total vertices in G, and each border matches where it attaches to its neighbouring vertices.

Definition 5.2.11 (Outer coface map)**.** Let v be an outer vertex of G. Then define G/v to be the graph with v and all its leaves deleted. Then $\delta^v : G/v \to G$ is the associated outer coface map.



As a graphical map, the graph G/v is inserted in place of a vertex in an appropriate barbell graph. This is the opposite way around to inner coface maps when pictured:

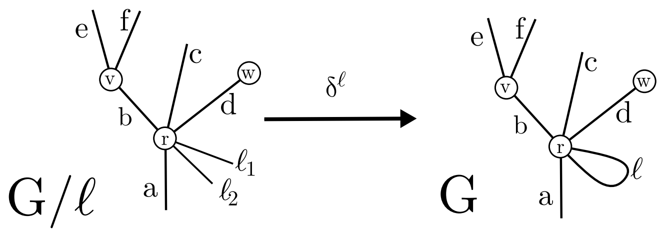


but it can still be defined as a graphical map, with δ_0^v the identity on edges and δ_1^v the identity on vertices. That is, δ_1^v sends each vertex in G/v to its associated corolla C_v as an embedding in G.

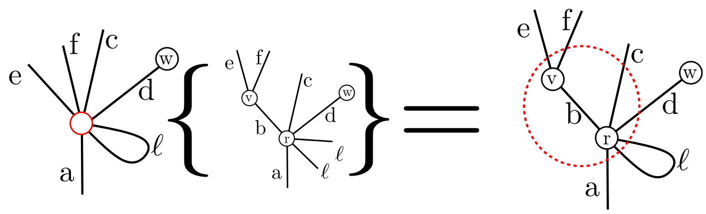
Remark 5.2.12. One important thing to note is that outer vertices are those which have at most one non-leg edge connected to them.

Definition 5.2.13 (Cosnip)**.** Note that this is a type of outer coface map.

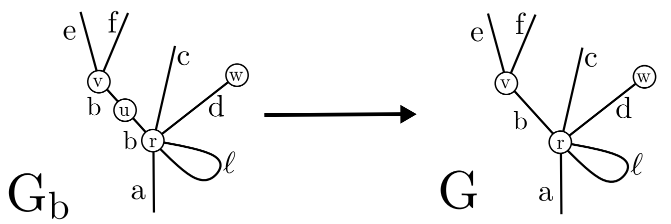
Let G be a graph with a loop ℓ (i.e. an edge from a vertex to itself), and define G/ℓ to be the graph where ℓ has been "snipped" to form two edges. Then $\delta^\ell : G/\ell \to G$ is the associated cosnip map



And as a graphical map, δ_0^ℓ would be surjective on edges, with $\delta_0^\ell(\ell_1) = \delta_0^\ell(\ell_2) = \ell$ and δ_1^ℓ the identity on vertices.

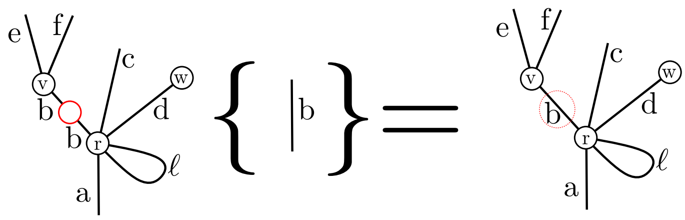


Definition 5.2.14 (Codegeneracy map)**.** Let b be any edge of G. Then define G_b to be the graph where the edge b has been subdivided with exactly one vertex. Then $\sigma^b : G_b \to G$ is the associated degeneracy map.



And as a graphical map σ_0^b maps edges to edges according to the labelling, noting that b in G is mapped to by two edges, while σ_1^b is given by

$$\begin{aligned} r &\mapsto C_r \\ u &\mapsto \eta_b \\ v &\mapsto C_v \\ w &\mapsto C_w \end{aligned}$$



As is evident above, any of these elementary maps can be written in terms of graphical maps. The other direction is proven in Lemma 5.2.16, after the relations are given.

Relations between Elementary Graphical Maps

To make working with this category easier, it is useful to have some relations between these face and degeneracy maps, especially for comparison with other graphical categories like $\mathbf{\Delta}$.

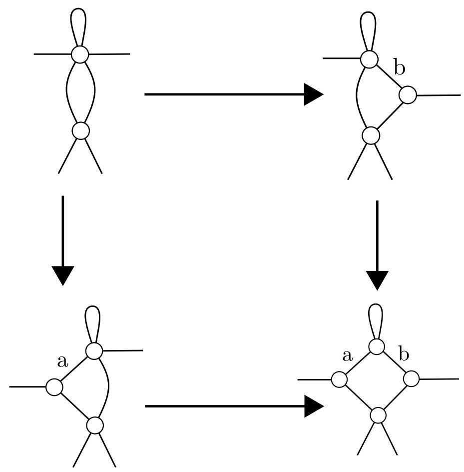
Firstly, all the relations from astroidal sets are obeyed. For clarity, some of them are listed below, with some illustrative examples.

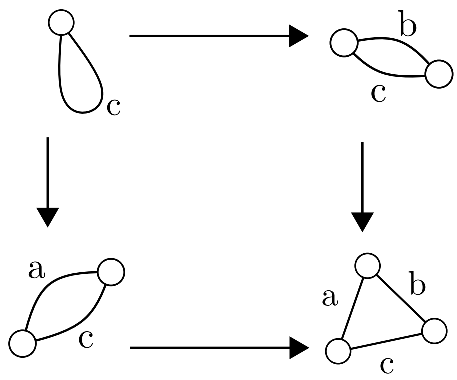
Inner coface maps Given two (distinct) inner face maps δ^a and δ^b, $(G/a)/b = (G/b)/a$ and the following diagram commutes:

$$\begin{array}{ccc} (G/b)/a & \xrightarrow{\delta^b} & G/a \\ \downarrow{\scriptstyle \delta^a} & & \downarrow{\scriptstyle \delta^a} \\ G/b & \xrightarrow{\delta^b} & G \end{array}$$

In the language of Definition 5.2.6, both $\delta_0^a \circ \delta_0^b$ and $\delta_0^b \circ \delta_0^a$ will be the same inclusion on edges, with a and b not in the image. Let a_1 and a_2 denote the vertices on each end of the edge a, and likewise let b_1 and b_2 be the vertices either end of b. Then both $\delta_0^a \circ \delta_0^b$ and $\delta_0^b \circ \delta_0^a$ will be the map $V(G/a/b) \to Emb(G)$ that sends the composite vertices a_1a_2 and b_1b_2 to their associated barbell graphs.

It is easy to see that this works even in the higher genus case, with loops around vertices, or with cycles of length greater than or equal to 2.

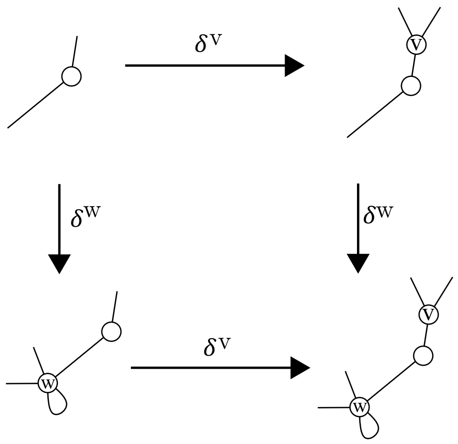




In the case of a cycle of length 2, that is, two edges between the same two vertices, once one is contracted it becomes impossible to contract the other. Thus, if a and b are both edges between vertices u and v, $\delta^a = \delta^b$.

Outer coface maps Let G be a tree with at least 3 vertices, and consider two distinct outer face maps δ^v and δ^w. Then $(G/v)/w = (G/w)/v$ and the diagram commutes:

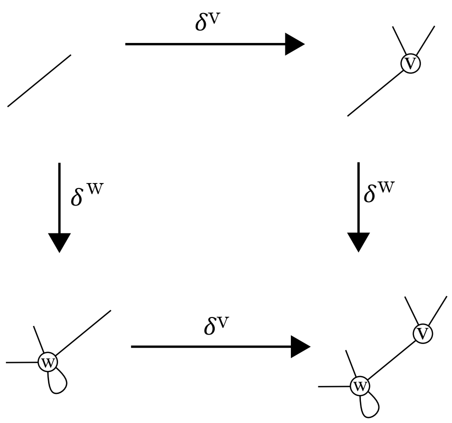
$$\begin{array}{ccc} (T/v)/w & \xrightarrow{\delta^v} & T/w \\ \downarrow{\scriptstyle \delta^w} & & \downarrow{\scriptstyle \delta^w} \\ T/v & \xrightarrow{\delta^v} & T \end{array}$$



As with inner coface maps, both $\delta^v \circ \delta^w$ and $\delta^w \circ \delta_v$ will result in the same graphical map, where $(\delta^v \circ \delta^w)_0$ includes on edges (leaving out those outer edges attached to v and w) and $(\delta^v \circ \delta^w)_1$ sends each vertex to its associated corolla embedding (essentially, the identity on vertices).

If G is a graph with only two vertices, v and w, this still works provided that there is only one edge between them.

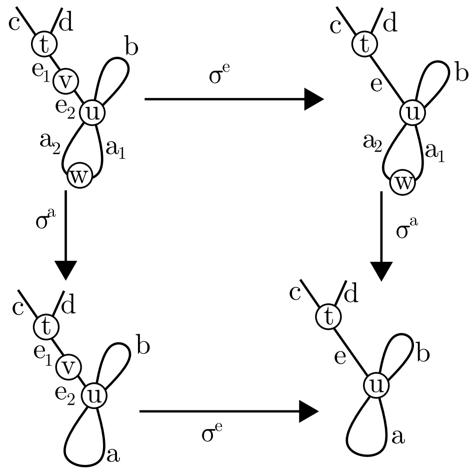
$$\begin{array}{ccc} \eta & \xrightarrow{\delta^v} & G/w \\ \downarrow{\scriptstyle \delta^w} & & \downarrow{\scriptstyle \delta^w} \\ G/v & \xrightarrow{\delta^v} & G \end{array}$$



Degeneracies Now consider two codegeneracy maps of G σ^e and σ^a. Then $(G_e)_a = (G_a)_e$ and the following diagram commutes

$$\begin{array}{ccc} (G_e)_a & \xrightarrow{\sigma^e} & G_a \\ \big\downarrow{\scriptstyle \sigma^a} & & \big\downarrow{\scriptstyle \sigma^a} \\ G_e & \xrightarrow{\sigma^e} & G \end{array}$$

This describes the map that sends e_1 and e_2 to e, a_1 and a_2 to a, and all other edges to themselves. All vertices are sent to their associated corollas, except for those extra ones (denoted v and w in the following image) which are sent to η.

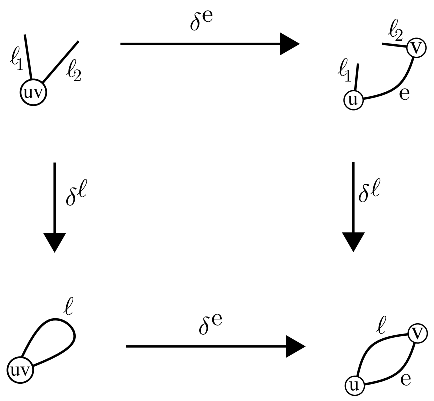


Snip Although the cosnip map is a type of outer face map, it acts sufficiently differently to deserve its own treatment. Firstly, two cosnips commute, as does a snip with an inner coface map. Both of these compositions can be easily written down as a graphical map, as was done in the previous paragraphs. Given an inner edge e and a loop ℓ,

$$\begin{array}{ccc} (G/\ell)/e & \xrightarrow{\delta^e} & G/\ell \\ \big\downarrow{\scriptstyle \delta^\ell} & & \big\downarrow{\scriptstyle \delta^\ell} \\ G/e & \xrightarrow{\delta^e} & G \end{array}$$

The only interesting part is when the loop is as a result of an inner coface map. In that case ℓ_1 and ℓ_2 are both mapped to ℓ, and the vertex resulting from the inner face map e is mapped to the barbell containing e as an inner edge, and all other vertices and edges are mapped to themselves.

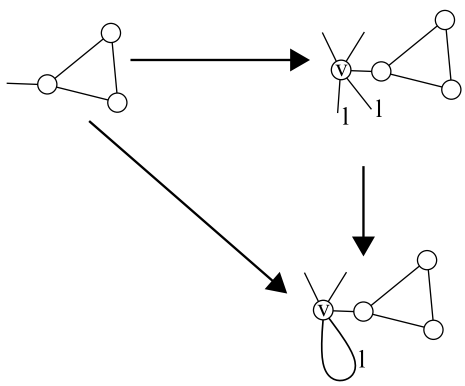
$$\begin{array}{ccc} (G/\ell)/e & \xrightarrow{\delta^e} & G/\ell \\ \big\downarrow{\scriptstyle \delta^\ell} & & \big\downarrow{\scriptstyle f} \\ G/e & \xrightarrow{\delta^e} & G \end{array}$$



Given an outer vertex and a disjoint loop, the outer coface map commutes with the snip. However, when the loop is attached to the outer vertex it becomes a little more complicated. Let v be an outer vertex containing a loop ℓ. Then

$$\begin{array}{ccc} G/v & \xrightarrow{\delta^v} & G/\ell \\ & \searrow^{\delta^v} & \downarrow{\scriptstyle \delta^\ell} \\ & & G \end{array}$$

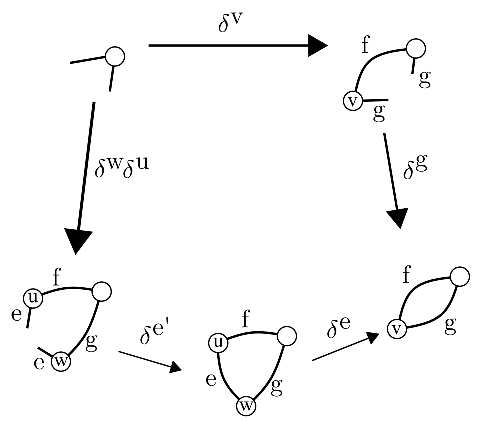
This graphical map is the inclusion on both the edges and the vertices, so even though there is a difference between G/ℓ and G, this difference lies on the vertex v and its associated outer edges, none of which are in the image of this graphical map.



There are also a couple of relations between the snip and an outer face map. Let v be an outer vertex connected to two inner edges f and g. If v is split into two vertices by an inner coface map, let these be labelled u and w, and the relevant inner edge be labelled e. Then

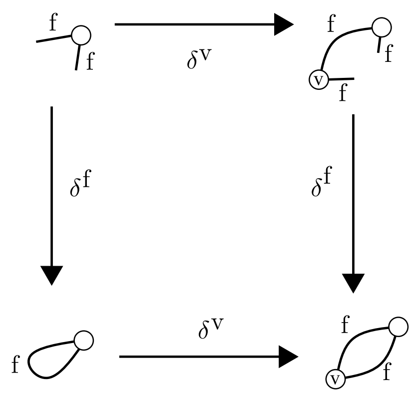
$$\delta^g \delta^v = \delta^e \delta^{e'} \delta^u \delta^w.$$

Where, to disambiguate, the $\delta^{e'}$ refers to cosnipping the edge e and δ^e refers to contracting the edge e.



This is because δ^g and $\delta^e\delta^{e'}$ only affect those edges and vertices which are not in the image of δ^v and $\delta^w\delta^u$.

Then, one may wonder whether there are any tricks related to the following image, but as is clear below this only occurs when certain edges are the same colour, and these maps commute anyway. Again, the composite map is the inclusion, and the differences are happening on a vertex that is not in the image.



For ease of access, these relations will once again be arranged in a table. The heading on the right indicates which map is done first, and then the left column says which map is done second. Those relations between vertices and edges which are disjoint will be omitted, as they always commute, so in this table assume the face and degeneracy maps are always coincident. Additionally, the elementary maps labelled in the columns are applied first, followed by the rows.

	inner
inner	$\delta^{\overline{pq}}\delta^{\overline{qr}} = \delta^{\overline{qr}}\delta^{\overline{pq}}$
outer	$\delta^{pq}\delta^{\overline{pq}} = \delta^{q}\delta^{p}$
degeneracy	$\sigma^{\overline{p(qr)}}\delta^{\overline{qr}} = id = \delta^{\overline{qr}}\sigma^{\overline{pq}}$
snip	$\delta^{\overline{pq}}\delta^{\overline{(pq)(pq)}} = f\delta^{\overline{pq}}$
	outer
inner	$\delta^{\overline{pq}}\delta^{p}$ undefined
outer	$\delta^{q}\delta^{p} = \delta^{pq}\delta^{\overline{pq}}$
degeneracy	$\sigma^{p}\delta^{q} = \delta^{q}\sigma^{\overline{pq}}$
snip	$\delta^{\overline{pp}}\delta^{p} = \delta^{p}$
	degeneracy
inner	$\delta^{\overline{p(pq)}}\sigma^{\overline{pq}} = id = \sigma^{\overline{r(pq)}}\delta^{\overline{pq}}$
outer	$\sigma^{p}\delta^{q} = \delta^{q}\sigma^{\overline{pq}}$
degeneracy	$\sigma^{\overline{p(pq)}}\sigma^{\overline{pq}} = \sigma^{\overline{(pq)q}}\sigma^{\overline{pq}}$
snip	undefined or commutes

If an edge has been removed by face map, then to snip it is undefined. If there is a vertex of degree 2, and one incident edge is snipped, then the other map cannot be, for this would make the graph disconnected. Otherwise, the snip commutes with itself. In addition, there is another relation between a snip coincident to an outer face map. If an outer vertex v is connected to two inner edges f and g, and v is split into two vertices by an inner coface map, let these be labelled u and w, and the relevant inner edge be labelled e. Then

$$\delta^{f}\delta^{v} = \delta^{e}\delta^{e'}\delta^{u}\delta^{w}.$$

Lemma 5.2.15. *Given any sequence of coface and codegeneracy maps, one can find a standard form consisting of a sequence of codegeneracy maps followed by a sequence of coface maps.*

Proof. Consider the above table of relations. Note that if a codegeneracy is coincident to an inner coface map they will annihilate. Otherwise, all other maps commute with codegeneracy maps. Therefore they can be moved past each other to form a sequence of codegeneracy maps followed by coface maps. □

One important thing to note is that morphisms, when written in terms of coface and codegeneracy maps, are slightly different in form to morphisms written in terms of graphical maps. The former usually take the form of a sequence of maps $f^a : G' \to G$, where a represents the vertex or edge to be removed or added. The latter take the form $G' = GH_v$ or $G' = HG_v$, where the H contains the information about which vertex or edge is to be modified. Theorem 5.2.16 gives a proof of their equivalence. Note that an alternative proof of this theorem can also be found in [36, Theorem 2.7].

Theorem 5.2.16. *Any morphism of graphs $G \to G''$ can be decomposed into a composition of codegeneracy maps followed by a composition of coface maps.*

Proof. By Lemma 5.2.7 each graphical map can be written as a series of graph substitutions followed by an inclusion. Denote this $G \to G' \to G''$. By Lemma 5.2.15 a series of coface and codegeneracy maps can be converted to a standard form, so it suffices to show that each graph substitution and inclusion can be decomposed into a series of coface and codegenercy maps.

We will proceed by induction on the number of vertices in G''. Firstly, consider a graphical map $G \to \eta$. The only inclusion is $\eta \hookrightarrow \eta$ itself, and the only graph substitutions available are codegeneracies. This can be written as a series of graph substitutions followed by an inclusion.

Consider a single graph substitution into a vertex v, $G'_v = G\{H_v\}$, where H_v has fewer vertices than G''. By the inductive hypothesis, we can write H_v as a series of inner coface and codegeneracy maps applied to the corolla C_v. Then, because these inner coface and codegeneracy maps only involve that part of G' which is contained in H_v, we can move them outside and produce the following.

$$\begin{aligned} G'_v &= G\{H_v\} \\ &= G\{\delta^1 \dots \delta^m \sigma^1 \dots \sigma^n C_v\} \\ &= \delta^1 \dots \delta^m \sigma^1 \dots \sigma^n G\{C_v\} \\ &= \delta^1 \dots \delta^m \sigma^1 \dots \sigma^n G \end{aligned}$$

This process can be repeated for each subsequent vertex and associated graph substitution, with the composition of these maps being a map $G \to G'$

Then, there is the inclusion $G' \to G''$. This can be written as a graph substitution $G'' = H\{G'\}$, where H is the graph obtained by replacing G' in G'' with its associated corolla. That is, the corolla which is identical on its boundary. Again, the graph H can be built up from a series of coface and codegeneracy maps, starting from C_v and without interfering with that vertex. Then

$$\begin{aligned} G' &= H\{G_v\} \\ &= \delta^1 \dots \delta^m \sigma^1 \dots \sigma^n C_v\{G\} \\ &= \delta^1 \dots \delta^m \sigma^1 \dots \sigma^n G \end{aligned}$$

□

5.2.2 Relationships

Up until now, $\mathcal{G}$ has referred to the category of graphs. However, as in previous chapters, a functor from graphs to modular operads will shortly be defined. To match the literature, $\mathcal{G}$ will be used to refer to the category of graphs, the functor from the category of graphs to the category of modular operads, as well as the image of this map. For clarity, however, G shall refer to a graph and $\mathcal{G}(G)$ will refer to its associated modular operad, and in the next section the category of graphs will be referred to as **Graph**

Definition 5.2.17. There is a functor $\mathcal{G}$: **Graph** $\to$ **ModOpd** from the category of graphs to the category of modular operads. Let G be a graph in **Graph**. Then $\mathcal{G}(G)$ is defined as:

- The colours of $\mathcal{G}(G)$ are the edges of G.

- The operations are generated by the vertices. Each vertex v with boundary $\{c_1, \ldots, c_n\}$, generates a morphism for every permutation of boundary edges. That is, for each $\sigma \in \Sigma_n$,

$$v_{c_{\sigma(1)}, \ldots, c_{\sigma(n)}} \in \mathcal{G}(G)(c_1, \ldots, c_n).$$

  In addition, there is a unit $\eta_e \in \mathcal{G}(G)$ for each edge e.

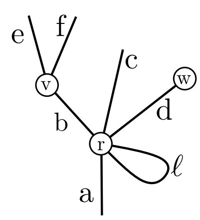
- Two operations may be composed if they share at least one colour in common. This corresponds in the graph to two subgraphs sharing an edge on the boundary of both. If w and v are composed, then $w \circ v$ represents the subgraph containing both w and v, including the edge between them.

- This composition is unital. Each unit η_e may be composed with any operation containing e as one of its colours. If v is a subgraph containing the edge e, then $v \circ \eta_e$ represents v glued to e, which is thus left unchanged. Similarly for $\eta_e \circ v$.

- The composition is associative. The order in which edges in the graph are glued together does not matter.

- A contraction may be performed on a vertex with two edges that are the same edge; that is, if it has a loop.

Example 5.2.18. In the following graph, G,



the modular operad is defined as follows.

- The set of colours is $\{a, b, c, d, e, f, \ell\}$.

- The operations are:

$$\begin{aligned}
v_{e,f,b} &\in \mathcal{G}(G)(e,f,b)\\
v_{e,b,f} &\in \mathcal{G}(G)(e,b,f)\\
&\vdots\\
v_{b,e,f} &\in \mathcal{G}(G)(b,e,f)\\
r_{a,b,c,d,\ell,\ell} &\in \mathcal{G}(G)(a,b,c,d,\ell,\ell)\\
r_{a,b,c,\ell,d,\ell} &\in \mathcal{G}(G)(a,b,c,\ell,d,\ell)\\
&\vdots\\
r_{d,\ell,c,\ell,b,a} &\in \mathcal{G}(G)(d,\ell,c,\ell,b,a)\\
r_{a,b,c,d} &\in \mathcal{G}(G)(a,b,c,d)\\
r_{a,b,c,d} &\in \mathcal{G}(G)(a,b,c,d)\\
&\vdots\\
r_{d,c,b,a} &\in \mathcal{G}(G)(d,c,b,a)\\
w_d &\in \mathcal{G}(G)(d)
\end{aligned}$$

  Along with all permutations of $v \circ_b r$, $r \circ_d w$, and $v \circ_b r \circ_d w$.

- There is a contraction, $\zeta(r_{a,b,c,d,\ell,\ell}) = r_{a,b,c,d}$, along with all permutations thereof.

In Chapter 4, a category of unrooted trees was defined, along with various functors showing the relationship between ꓘK and other graphical categories. Now that $\mathcal{G}$ has been defined, a functor from ꓘK to $\mathcal{G}$ can also be defined, since $\mathcal{G}$ contains ꓘK as a subcategory consisting only of trees.

Lemma 5.2.19. *The category* ꓘK *is the subcategory of* $\mathcal{G}$ *consisting of those graphs of genus* 0

Proof. We shall define an inclusion functor $h :$ ꓘK $\to \mathcal{G}$. Each tree (T, I) in ꓘK can be regarded as a graph in $\mathcal{G}$.

- For each edge in $E(T)$, let there be two half edges in the set of half edges of $h(T, I)$, one for each direction, with the involution connecting them.

- The set of vertices of $h(T, I)$ consists of $V(T) \setminus I$.

- The set of half edges of $h(T, I)$ consists of all those half edges that point to a vertex.

Then, given a morphism of trees $f : S \to T$, a morphism of graphs can be defined. Each morphism of trees is defined as a series of coface and codegeneracy maps (Definition 4.2.11), which then correspond to coface and codegeneracy maps in $\mathcal{G}$, each of which can be written as a graphical map (see Section 5.2.1). □

5.2.3 Graphical Sets

Just like in previous cases, the aim is to define presheaves over some category, in this case the category of graphs, $\mathcal{G}$. Infinity modular operads can be derived from there. Since these are presheaves over the category of graphs, they shall be called graphical sets. Various slightly different conceptions of graphical sets have been previously studied by Hackney et al. [36], Joyal and Kock [41], and Raynor [64, 65]. In this section, the graphical category is based on the definition given in Hackney et al. [36]; although, when conceiving of morphisms between graphs, the emphasis here is on face and degeneracy maps, and the table of relations is new.

Definition 5.2.20 (Graphical set)**.** A graphical set X is a presheaf $X : \mathcal{G}^{op} \to \mathbf{Set}$

Denote by X_G the set which is the image of a graph G under X. The category of graphical sets with natural transformations between them is denoted **gSet**.

Remark 5.2.21. In the same way that simplicial sets can be extended to simplicial objects, in the modular case there are also maps $\mathcal{G}^{op} \to \mathcal{C}$ where $\mathcal{C}$ is a cartesian monoidal category. In particular, graphical groups will be used in this thesis, presheaves $\mathcal{G}^{op} \to \mathbf{Grp}$.

As with previous chapters, the relationship between modular operads and graphical sets is important, and to this end there is a nerve defined in Hackney et al. [35].

Definition 5.2.22 (Modular nerve)**.** The nerve of a modular operad $\mathcal{M}$ is a functor $n_g : \mathbf{ModOpd} \to \mathbf{gSet}$. This is given by, at each graph G,

$$n_g(\mathcal{M})_G = \mathbf{ModOpd}(\mathcal{G}(G), \mathcal{M})$$

Then, defined here is the modular realisation, a left adjoint to the nerve.

Definition 5.2.23 (Modular realisation)**.** The aim is to define $\tau_g : \mathbf{gSet} \to \mathbf{ModOpd}$. Let X be a graphical set. Then a modular operad $\tau_g(X)$ can be defined from it as follows:

- The colours, $ob(\tau_g(X))$, are the set X_η
- The operations are generated by $\tau_g(X)(\underline{c}) = X_{C_{\underline{c}}}$, where $C_{\underline{c}}$ is the corolla with edges coloured by $\underline{c}$.
- The face maps induce relations between these operations.
  1. If A is a colour in X_η, and $\sigma : C_1 \to \eta$, then $\sigma^A(A) = id_A \in \tau_d(X)(A; A)$ is required, where $\sigma_A : X_\eta \to X_{C_1}$ is the map induced by σ^A.
  2. Let T be the barbell; i.e. the tree with two vertices v and w. Then if δ_v, δ_w, and $\delta_{\overline{vw}}$ are similarly induced maps, require

$$\delta_w(x) \circ_{\overline{vw}} \delta_v(x) = \delta_{\overline{vw}}(x)$$

     for all $x \in X_T$.

Note that the barbell graph is somewhat special, as it is the simplest graph on two vertices, and its role in this definition is gluing two edges together.

Lemma 5.2.24. *The realisation τ_g is left adjoint to the nerve n_g.*

Proof. The category $\mathcal{G}$ is small, and **ModOpd** has all small colimits. Therefore, since the Yoneda embedding is full and faithful, the left adjoint is the left Kan extension of the inclusion $\mathcal{G} \hookrightarrow$ **ModOpd** along the Yoneda embedding [50, Chapter X]. □

Given a graph $G \in \mathcal{G}$, the representable $\mathcal{G}[G]$ can be defined in a similar way to Definition 4.2.21. Note the similarity in notation with $\mathcal{G}(G)$, the modular operad associated to a graph G.

Definition 5.2.25. Let $G \in \mathcal{G}$ be a graph. Then the representable presheaf is $\mathcal{G}[G] = \mathcal{G}(-, G)$. At each graph $H \in \mathcal{G}$, $\mathcal{G}[G]_H = \mathcal{G}(H, G)$.

Note that the nerve of a graph does not always correspond to the representable of a graph (Remark 3.2 in [35]), because the connection between morphisms in the category of graphs $\mathcal{G}$ and morphisms in the category of modular operads **ModOpd** is not straightforward. As an example, consider the loop with one vertex, K. This graph has two directed edges, a and b, in opposing directions. Consider the nerve of a graph G at the point K. Then

$$n_g(G)_K = \mathbf{ModOpd}(K, G)$$

and

$$\mathcal{G}[G]_K = \mathcal{G}(K, G)$$

But the former allows maps sending the directed edge a to any directed edge $e \in G$ and the lone vertex of K to the edge e, whereas the latter does not, so the result is $\mathcal{G}[G]_K \subsetneq n_g(G)_K$.

5.2.4 Faces, boundaries, horns

Faces, boundaries, and horns are primarily defined here for their used in the Kan condition. However, their simplicial analogues have other applications, and it is therefore anticipated that these may too.

Definition 5.2.26 (Face)**.** Let $G \in \mathcal{G}$ be a graph, and let $\alpha : H \to G$ be a face map. Then the α-face of $\mathcal{G}[G]$ is the image of the map $\mathcal{G}[\alpha] : \mathcal{G}[H] \to \mathcal{G}[G]$. It is denoted $\partial_\alpha \mathcal{G}[G]$.

Definition 5.2.27 (Boundary)**.** The graphical subset which is the union of all possible faces.

$$\partial \mathcal{G}[G] = \bigcup_\alpha \partial_\alpha \mathcal{G}[G]$$

Definition 5.2.28 (Inner horn)**.** Let δ_e be an inner face map. Then the horn Λ_e is the graphical subset which is the union of all possible faces except ∂_e:

$$\Lambda_e[G] = \bigcup_{\alpha \neq e} \partial_\alpha \mathcal{G}[G].$$

5.3 Infinity Modular Operads

Infinity modular operads are useful in the definition of an infinity modular operad of surfaces. My definition is found in Section 5.4, but similar ideas can be found in [37, 74].

There are various models of infinity categories and infinity operads. Therefore, there should be corresponding models for infinity modular operads. The idea of simplicially enriched modular operads is included mainly for completeness. This definition does not appear to exist in the literature, so, while not the focus, a sketch of what it should be is included. However, details are provided for the inner Kan condition, where this original definition is important to my later theorems. Then the Segal condition is described, found in [36, 35], and an original proof of the equivalence between the Segal condition and the inner Kan condition for modular operads is provided. Again, for completeness, a sketch of a conjectured definition for modular complete Segal spaces is included.

5.3.1 Simplicially Enriched Modular Operads

There does not appear to be a definition of a simplicially enriched modular operad in the literature. If there were, it would be something like the following, based on the similar definitions for infinity categories and infinity operads [20, 21, 67]. This is the higher genus version of Definition 4.3.1.

Definition 5.3.1. A modular operad $\mathcal{M}$ is called simplicially enriched if

- each set of operations $\mathcal{M}(\underline{c})$ is a simplicial set,
- composition (and likewise contraction)

$$\mathcal{M}(\underline{c}) \times \mathcal{M}(\underline{d}) \to \mathcal{M}(c_1, \dots \hat{c_{i_1}}, \dots \hat{c_{i_r}}, \dots c_m, d_1, \dots \hat{d_{j_k}}, \dots d_n)$$

  is a map of simplicial sets, where $\{c_{i_1}, \dots c_{i_r}\} = \{d_{j_1}, \dots d_{j_r}\}$.

A proper definition which accords with other definitions of infinity modular operads may require some other conditions, but exactly what they are is as yet unknown.

Conjecture 5.3.2. *There is an equivalence between simplicially enriched modular operads, and quasi modular operads (see Definition 5.3.3).*

5.3.2 The Modular Inner Kan Condition

Given its relevance in quasi operads and quasi categories, it is natural to examine what a Kan condition of modular operads should be. This definition appears to be the correct one, since, as shown in Theorem 5.3.14, it is equivalent to the Segal condition.

Definition 5.3.3 (Inner Kan condition)**.** A graphical set X is said to satisfy the inner Kan condition if, for every graph G and inner horn $\Lambda^e[G]$, the diagram

$$\begin{array}{ccc} \Lambda^e[G] & \longrightarrow & X \\ \big\downarrow & \overset{\exists!}{\nearrow} & \\ \mathcal{G}[G] & & \end{array}$$

admits a filler

A quasi modular operad, therefore, can be defined in a similar way to a quasi category or a quasi operad.

Definition 5.3.4 (Quasi modular operad)**.** A quasi modular operad is a graphical set X that satisfies the inner Kan condition.

The reason for this definition becomes clear with Theorem 5.3.5, showing the link between quasi modular operads and modular operads. If the filler in Definition 5.3.3 is unique, then a graphical set is said to satisfy the *strict* Kan condition, and has a corresponding modular operad.

Theorem 5.3.5. *A graphical set X satisfies the strict inner Kan condition if and only if there is a modular operad $\mathcal{M}$ such that $n_g(\mathcal{M}) = X$*

The proof of this theorem will be deferred to Section 5.3.3, in order to make use of the equivalence between the strict inner Kan condition and the Segal condition.

5.3.3 The Modular Segal Condition

As seen in Theorems 2.3.11, 3.3.6, and 4.3.7, there is a Segal condition and a nerve theorem for each of simplicial, dendroidal, and astroidal sets. Likewise, there is a Segal condition for graphical sets. This is introduced by Hackney et al. [36], with the nerve theorem detailed in [35]. This section is included to compare with the inner Kan condition (an original proof), to provide some detail for Complete Segal Spaces, and as a basis for my proof that graphical groups satisfy the Segal condition.

Definition 5.3.6 (Segal Core)**.** Let G be a graph with a least one vertex. Then the Segal core $Sc[G]$ is the subset consisting of the union of corollae:

$$Sc[G] = \bigcup_{v \in V(G)} \mathcal{G}[C_{\underline{c}(v)}]$$

Where $\underline{c}(v)$ is the profile associated to the vertex v, and $C_{\underline{c}(v)}$ is the corolla associated to this vertex. Sometimes the notation C_v will be used to mean the same thing.

Thus, one can write down a Segal condition that is similar to the alternative formulation of the Kan condition

Definition 5.3.7 (Segal condition)**.** A graphical set X satisfies the Segal condition if, for all graphs $G \in \mathcal{G}$, there is a bijection

$$Hom(\mathcal{G}[G], X) \cong Hom(Sc[G], X)$$

(induced by the Segal core inclusion $Sc[G] \hookrightarrow \mathcal{G}[G]$)

As in previous chapters, there is an alternative characterisation of the Segal condition.

Proposition 5.3.8. *A graphical set X satisfies the Segal condition if and only if for every graph G there is a bijection*

$$X_G \cong X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}}.$$

Proof. Consider the following three equivalences (given by the Yoneda Lemma)

$$X_G \cong \mathrm{Hom}(\mathcal{G}[G], X)$$
$$X_{C_v} \cong \mathrm{Hom}(\mathcal{G}[C_v], X)$$
$$X_\eta \cong \mathrm{Hom}(\mathcal{G}[\eta], X)$$

as well as the pullback $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}}$. It suffices to show that $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}} \cong \mathrm{Hom}(Sc[G], X)$.

By Definition 4.3.4, the Segal core is the union of the images of the maps $\mathcal{G}[C_n] \to \mathcal{G}[G]$ corresponding to sub-trees of shape $C_v \to G$. But these maps will agree whenever two corollae share an edge. Likewise, the pullback $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \ldots \times_{X_\eta} X_{C_{v_n}}$ corresponds to maps $X_{C_v} \cong \mathrm{Hom}(\mathcal{G}[C_v], X)$ which agree over X_η. □

An example is provided to illuminate these concepts.

Example 5.3.9. Let G be the tree containing two vertices, v and w, joined by an edge e, where v has a different number of legs than w. Consider the graphical set $\mathcal{G}[G]$. The Segal map shall be examined for the graph G in relation to this graphical set. Note that, for most graphs H unrelated to G, the set $\mathcal{G}[G]_H$ will be empty (the main exceptions being degeneracy maps), and thus uninteresting for the purpose of example.

Note the following square.

$$\begin{array}{ccc} G & \longleftarrow & C_v \\ \uparrow & & \uparrow \\ C_w & \longleftarrow & \eta \end{array}$$

For any graph H, there are many choices for the map $\eta \hookrightarrow H$, one for each edge in H. However, in the above diagram the only way for it to commute is to chose the maps $\eta \hookrightarrow C_v$ and $\eta \hookrightarrow C_w$ where η is mapped to their edge in common, e.

Now, consider the components of the Segal condition:

$$\mathcal{G}[G]_\eta = \mathrm{Hom}(\eta, G)$$
$$\mathcal{G}[G]_{C_v} = \mathrm{Hom}(C_v, G)$$
$$\mathcal{G}[G]_{C_w} = \mathrm{Hom}(C_w, G)$$

The first set has already been discussed. For the other two, there is only one option, the outer coface maps $\delta^w : C_v \to G$ and $\delta^v : C_w \to G$.

Although in theory there may be multiple choices of map $\mathcal{G}[G]_{C_v} \to \mathcal{G}[G]_\eta$ and $\mathcal{G}[G]_{C_w} \to \mathcal{G}[G]_\eta$, the way that G is connected means that only the map induced by the above commutative diagram can be used.

Thus, the pullback

$$\begin{array}{ccc} \mathcal{G}[G]_{C_v} \times_{\mathcal{G}[G]_\eta} \mathcal{G}[G]_{C_w} & \longrightarrow & \mathcal{G}[G]_{C_v} \\ \downarrow & & \downarrow \\ \mathcal{G}[G]_{C_w} & \longrightarrow & \mathcal{G}[G]_\eta \end{array}$$

contains only one map, the map which sends $C_v \to C_v$ and $C_w \to C_w$, up to their equivalence on the edge e. But this describes the only map in $\mathcal{G}[G]_G$, $G \to G$.

As in the simplicial and dendroidal cases, a nerve theorem is important to have, to be able to compare graphical sets with modular operads.

Theorem 5.3.10. *Let X be a graphical set $X : \mathcal{G}^{op} \to$ **Set**. Then the following are equivalent:*

1. *There exists a modular operad $\mathcal{M}$ such that $n_g(\mathcal{M}) = X$*
2. *X satisfies the (strict) inner Kan condition*
3. *X satisfies the Segal condition*

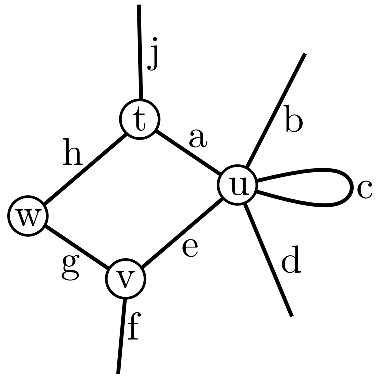
The equivalence between 1 and 3 can be found in [35, Theorem 3.6]. The proof of the equivalence between 2 and 3 is original and is found below. The simply connected case can be found in the section on astroidal sets (Theorem 4.3.10). Then, this theorem is extended to general graphs via the concept of spanning trees. The core of this proof can be found in Lemma 5.3.12, and it all comes together in Theorem 5.3.14.

The definition of spanning tree used herein is slightly different to the standard definition in graph theory.

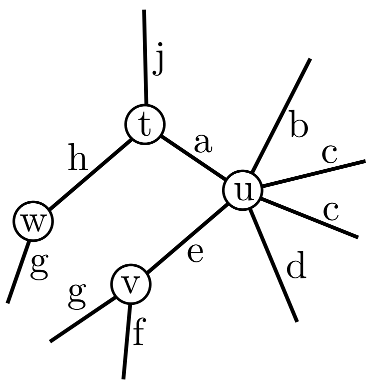
Definition 5.3.11 (Spanning tree)**.** Let G be a graph. Then a spanning tree of G, T_G, is a graph which

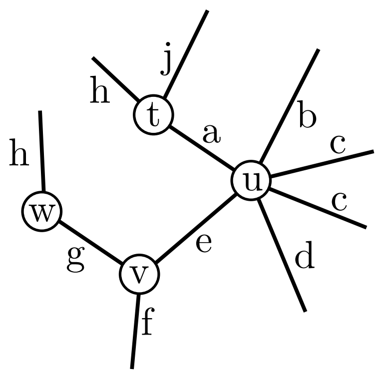
- is a subgraph of G
- is connected
- has genus 0
- includes each vertex of G, and the corolla associated to each vertex (that is, the set of edges connected to each vertex) is identical in both G and T_G.

Of particular note is that the standard definition of spanning graph does not have half edges, and so if the fourth property is true then all edges of the graph will be included and T_G will be equal to G. However, in this definition of graph half edges *are* allowed, and so spanning trees correspond to breaking edges rather than removing them entirely. Also, for any particular graph, note that there may be more than one spanning tree. For example, consider this graph, G.



It has four different spanning trees, each corresponding to breaking one edge in the cycle. Two of them are:





Note that, in each case, the loop (edge c) must also be broken. Additionally, only one edge in the cycle may be removed, or the resulting tree will not be connected. For each broken edge, both sides have been labelled by the label of the original edge; however, it is important to note that each edge consists of two half edges, and the two halves will each be one of these half edges.

Also, note that the Segal cores of a graph and its spanning trees will be identical.

Proposition 5.3.12. *Let X be a graphical set, and consider a graph G. Let T_i be a collection of spanning trees of G such that each edge in G is included in at least one spanning tree. If each of these spanning trees satisfy*

$$\mathrm{Hom}(Sc[T_i], X) \cong \mathrm{Hom}(\mathcal{G}[T_i], X),$$

then

$$\mathrm{Hom}(Sc[G], X) \cong \mathrm{Hom}(\mathcal{G}[G], X)$$

Proof. We wish to construct a bijection $\mathrm{Hom}(Sc[G], X) \cong \mathrm{Hom}(\mathcal{G}[G], X)$. Given a map $f : Sc[G] \to X$, we wish to find a map $h : \mathcal{G}[G] \to X$, then show that $f \mapsto h$ is a bijection.

Firstly, consider any particular spanning tree T_i. Since $Sc[G] \cong Sc[T_i]$, let $f_i : Sc[T_i] \to X$ be defined by $f_i(x) = f(x)$. Then, by our hypothesis, we can construct a map $h_i : \mathcal{G}[T_i] \to X$. Then, each T_i can be reached from G by a series of cosnip maps. We also know that the cosnip is the only face or degeneracy map which changes genus. So we get $\mathcal{G}[G] \to \mathcal{G}[T_i]$ via a series of snips. These shall be denoted d_i in this proof.

From this collection of maps h_i and d_i, h shall be constructed as $h = \bigcup h_i \circ d_i$. It remains to be shown that this map is well defined, and a bijection. Let this map $f \mapsto h$ be denoted by ψ. The following diagram is illustrative.

$$\begin{array}{ccc}
\mathcal{G}[G] & \overset{h}{\searrow} & \\
\downarrow{\scriptstyle d_i} & & \\
\mathcal{G}[T_i] & \xrightarrow{h_i} & X \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} \\
Sc[T_i] & \xrightarrow{f_i} & X \\
\downarrow{\scriptstyle =} & \overset{f}{\nearrow} & \\
Sc[G] & &
\end{array}$$

This map is indeed well-defined. Consider two spanning trees T_j and T_k. We wish to compare $h = h_j \circ d_j$ and $h' = h_k \circ d_k$. But the coface and codegeneracy maps in $\mathcal{G}$ are associated with the face and degeneracy maps in X according to the graphical relations. That is, if $x \in \mathcal{G}[G]_S$, then $x : S \to G$ maps to some $d_j(x)$, which is x restricted to the spanning tree T_j. Because each map $\mathcal{G}[T_i] \to X$ is induced by the coface maps d^i, h and h' agree on vertices and their neighbouring half-edges. Then, the graphical relations ensure that these maps agree on edges between two different corollas.

Now, consider a map $h \in \mathrm{Hom}(\mathcal{G}[G], X)$. From h, a map $f \in \mathrm{Hom}(Sc[G], X)$ can be defined by restricting h to each corolla $\mathcal{G}[C_v]$. This provides a map ψ^{-1}. Then $\psi \circ \psi^{-1} = \psi^{-1} \circ \psi = id$ because the information at each corolla is preserved. Thus, the map $\psi : \mathrm{Hom}(Sc[G], X) \to \mathrm{Hom}(\mathcal{G}[G], X)$ is a bijection.

□

Proposition 5.3.13. *Let X be a graphical set that satisfies the graphical Segal condition. Then it also satisfies the strict inner graphical Kan condition.*

Proof. Let X be a graphical set that satisfies the Segal condition.

Firstly, consider the following series of inclusions (for all graphs G and all inner faces e)

$$Sc[G] \subset \Lambda^e[G] \subset \mathcal{G}[G]$$

An element of $Sc[G]_H$ is an element of $\mathcal{G}[C_v]_H$, where C_v is the corolla associated to a vertex v of G. That is, a graphical map $H \to C_v$. But such a graphical map partially defines a graphical map $H \to \partial_\alpha \mathcal{G}[G]$, because each corolla is contained in at least one face of G. Therefore, $Sc[G] \subset \Lambda^e[G]$. Likewise, each graphical map in the inner horn partially defines a graphical map on the whole graph, so $\Lambda^e[G] \subset \mathcal{G}[G]$.

Therefore, if we have a map $Sc[G] \to X$, and thus by the Segal condition one $\mathcal{G}[G] \to X$, we must also have one $\Lambda^e[G] \to X$. Therefore, we have

$$Hom(Sc[G], X) \subset Hom(\Lambda^e[G], X) \subset Hom(\mathcal{G}[G], X).$$

The Segal condition states that $Hom(Sc[G], X) \cong Hom(\mathcal{G}[G], X)$ is a bijection, so we must therefore also have a bijection $\Lambda^e[G] \cong \mathcal{G}[G]$, giving us the Kan condition. □

Theorem 5.3.14. *Let X be a graphical set. Then X satisfies the graphical Segal condition if and only if X satisfies the strict inner graphical Kan condition.*

Proof. The proof that Segal implies Kan is found in Proposition 5.3.13. Now assume that X satisfies the Kan condition. We shall prove that

$$\begin{aligned} &\mathrm{Hom}(\Lambda^e[G], X) \cong \mathrm{Hom}(\mathcal{G}[G], X) \\ \implies &\mathrm{Hom}(Sc[G], X) \cong \mathrm{Hom}(\mathcal{G}[G], X) \end{aligned}$$

for all graphs G. Firstly, the simply connected case can be found in Theorem 4.3.10.

Then, assume G is a higher genus graph, and consider a spanning tree T_G. Since $\mathrm{Hom}(\Lambda^e[G], X) \cong \mathrm{Hom}(\mathcal{G}[G], X)$, we know that $\mathrm{Hom}(\Lambda^e[T_G], X) \cong \mathrm{Hom}(\mathcal{G}[T_G], X)$ for all spanning trees T_G, unless T_G would remove the edge e. In this case, $\Lambda^e[T_G], X)$ is undefined. However, a spanning tree may only remove the inner edge e if it exists as part of some cycle, but that implies that there must be some other edge in the cycle, f, and the spanning tree(s) that remove f will include e. Note that, if e is part of a cycle, then there may be a spanning tree T_G which is formed by removing e. In this case, $\Lambda^e[T_G]$ cannot be defined; however, since e is part of a cycle there exists some other edge e'. By Lemma 4.3.10, we know that, for each spanning tree, $\mathrm{Hom}(Sc[T_G], X) \cong \mathrm{Hom}(\mathcal{G}[T_G], X)$, and thus by Lemma 5.3.12 we have $\mathrm{Hom}(Sc[G], X) \cong \mathrm{Hom}(\mathcal{G}[G], X)$, as required. ☐

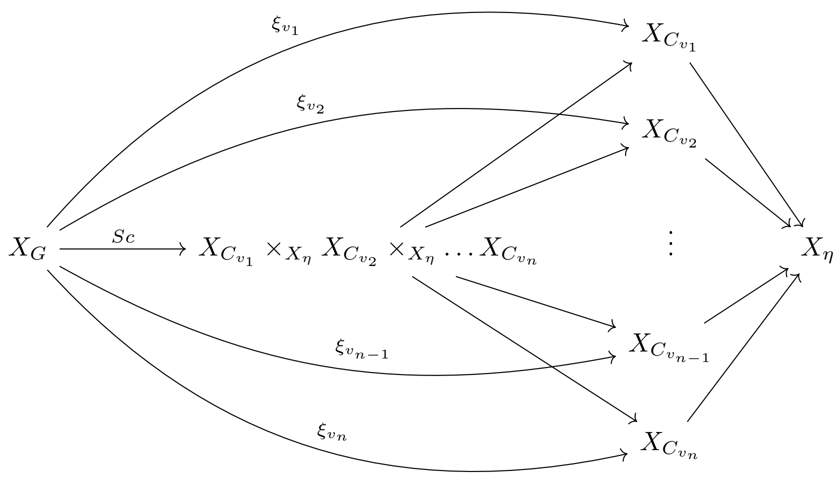
As promised, here is the proof that modular groups satisfy the Segal condition. Just as graphical sets can be defined as functors $\mathcal{G}^{op} \to \mathbf{Set}$, graphical groups can be defined as functors $\mathcal{G}^{op} \to \mathbf{Grp}$. In the following lemma, note that each map $\xi_v : \mathcal{G}[G] \to \mathcal{G}[C_v]$ refers to a series of outer face maps resulting in just the corolla remaining.

Theorem 5.3.15. *Let G be a graphical group. If, for all graphs G,*

$$\bigcap_{v \in V(G)} \ker(\xi_v) = \{1\},$$

then the underlying graphical set satisfies the Segal condition.

Proof. Consider the following diagram, where the maps $X_{C_{v_i}} \to X_\eta$ are induced by $\eta \to C_{v_i}$ (Example 5.3.9 is illustrative).



We wish to show that the Segal map Sc is a bijection. This map is obviously surjective, since $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \dots X_{C_{v_n}}$ is a subset of X_G.

Now to show injective. Assume for the sake of contradiction that there is some element $a \in ker(Sc)$ which is not the identity. We know that $X_{C_{v_1}} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \dots X_{C_{v_n}}$ is the limit so we know that for all i, $\xi_{v_i}(a) = id$. So $a \in \bigcap_{v \in V(G)} \ker(\xi_v)$. But our hypothesis states that the identity is the only element in this set, a contradiction. Therefore our function is injective. □

This also, by Theorem 5.3.14, means that graphical groups satisfy the Kan condition.

5.3.4 Modular Complete Segal Spaces

One model which has proven useful in the study of infinity categories and operads is that of complete Segal spaces. A sketch of what that should look like for the modular operads case is outlined here, since this does not appear to be in the literature.

In the simplicial case, there is something like the Kan condition making the space act as a topological space should, and the Segal condition making the space act as a category should. In the graphical case, the simplicial Kan condition is retained, and the simplicial Segal condition is swapped out for the graphical Segal condition.

Definition 5.3.16 (Graphical Space)**.** A graphical space X is a presheaf

$$X : \mathcal{G}^{op} \to \mathbf{sSet}$$

The first condition, the generalisation of the Kan condition, is borrowed heavily from Rasekh [63], but applied to graphical spaces instead.

Definition 5.3.17 (Reedy fibrant)**.** A graphical space X is said to be Reedy fibrant if for every graph G, the following map of spaces is a Kan fibration:

$$Map_{\mathbf{gSpaces}}(F(G), X) \to Map_{\mathbf{gSpaces}}(\partial F(G), X)$$

Here, one would expect F to be some map $\mathcal{G}$ to **gSpace**, which sends each graph G to a graphical space via its representable $\mathcal{G}[G]$. As with simplicial spaces and dendroidal spaces, the canonical way to get from **gSet** to **gSpace** should be by assigning to each set the constant simplicial space. That is, $F(G)_i = \mathcal{G}[G]$. Then, the graphical space must satisfy the Segal condition, in the same way as any graphical object satisfies the Segal condition. Recall:

Definition 5.3.18 (Segal)**.** A Reedy fibrant graphical space X is said to be Segal if the Segal map:

$$X_G \to X_{C_1} \times_{X_\eta} X_{C_{v_2}} \times_{X_\eta} \dots X_{C_{v_m}}$$

is a weak homotopy equivalence for all graphs G.

A graphical Segal space is a graphical space which is both Reedy fibrant and Segal. All that is left is to define the completeness condition. First, recall that X_η is the space of objects and for each profile $\underline{c}$ there is a space of maps $X_{C_{\underline{c}}}$. Also note that there is a map $s_0 : X_\eta \hookrightarrow X_{C_2}$ which takes each object to its identity morphism. Here C_2 refers to a corolla of two legs of the same colour.

Definition 5.3.19 (Graphical Complete Segal Space)**.** Let X be a graphical Segal space. Then for each $X_{C_{\underline{c}}}$, there is a subset $X_{hoequiv}$ consisting of all the homotopy equivalences. Then X is a graphical complete Segal space if the map $s_0 : X_\eta \to X_{hoequiv}$ is an equivalence.

Note that this is the same as saying that there are no non-identity homotopy equivalences. Morally, this functor sends all invertible morphisms to identities, turning a category into a reduced version of itself. A groupoid would end up a discrete collection of objects.

5.4 Surfaces

Surfaces can have non-zero genus, and so the kind of surface operad that would make the most sense would be a modular operad. Hopefully, this would be useful in the study of TQFTs.

However, one thing to note is that TQFTs are not cyclic. That is, they are modelled by oriented cobordisms with an "in" boundary and an "out" boundary. Thus, perhaps TQFTs would be better modelled by a non-cyclic version of modular operads (wheeled properads). Or maybe modular operads will bring some new insight into the nature of TQFTs, given that there may be physical reasons for cobordisms to be unoriented. This section contains an original definition for a quasi modular operad inspired by mapping class groups of surfaces. It shall be defined via a graphical set of mapping class groups. Firstly, recall the definition of the mapping class group of a surface Σ_g^n of genus g with n boundary components.

Definition 5.4.1 (Mapping Class Group)**.** Let Σ_g^n be a surface of genus g with n boundary components. Then the mapping class group $MCG(\Sigma_g^n)$ is the group of isotopy preserving diffeomorphisms of a surface.

Given a surface Σ_g^n, the associated mapping class group is also denoted S_g^n.

The elements of the mapping class group can be described as compositions of *Dehn twists*. The following information was first published by Dehn [18], and again independently by Lickorish [47].

Definition 5.4.2 (Dehn twist)**.** Let C be a simple closed curve on a surface Σ, and let A be a neighbourhood of C forming an annulus. Then a Dehn twist is a homeomorphism $d : \Sigma \to \Sigma$ which is the identity on $\Sigma \setminus A$ and is twisted by an angle of 2π within A. That is, if $A = [0,1] \times S^1$, where S^1 is the circle, then

$$d(t, e^{i\theta}) = (t, e^{i(\theta - 2\pi t)}).$$

Theorem 5.4.3 (Dehn-Lickorish theorem)**.** *Let Σ be a closed connected orientable surface. Then the mapping class group of Σ is generated by Dehn twists on Σ.*

As an aside, a category can be formed by taking S_g^n as objects and with morphisms being diffeomorphisms and inclusions between the underlying surfaces.

The MCG quasi modular operad will be defined first as a graphical group, and thus with an underlying graphical set, and then will be shown that it satisfies the Kan condition.

Definition 5.4.4 (MCG graphical group)**.** The graphical set $S : \mathcal{G}^{op} \to \mathbf{Grp}$ will be defined. To each graph G, of n leaves and genus g, a mapping class group of a surface will be associated, according to the rule:

- Each vertex of degree d, as well as its neighbouring half-edges, is associated to the surface Σ_0^d, with d labelled boundary circles, one for each edge.
- An edge between two vertices is recorded by glueing the corresponding boundary components of those vertices together. (If this was an edge going from a vertex to itself, this will form a handle.)

Thus, after the appropriated gluings, the result is the surface Σ_g^n. Each graph G maps to the associated mapping class group of the surface, so $S_G = S_g^{n+1}$.

Then a group homomorphism will be associated to each elementary graphical map, as follows:

- The identity is sent to the identity group homomorphism.
- To an inner face map δ_e will be associated the group homomorphism corresponding to composing each diffeomorphism in the mapping class group with a Dehn twist around the place on the surface corresponding to the edge in the graph.
- To an outer face map $\delta_v : G \to G/v$ will be associated the group homomorphism induced by the inclusion of $S(G/v)$ into $S(G)$. This includes cosnip maps.
- To a degeneracy map will be associated the group homomorphism corresponding to composing each differomorphism in the mapping class group with an inverse Dehn twist around the place on the surface corresponding to the edge in the graph.

Lemma 5.4.5. *This is indeed a graphical group.*

Proof. We have already defined above where the objects and morphisms go, and it is clear that any morphism $f : X \to Y$ is mapped to a morphism $S_X \to S_Y$. It remains to be shown that these obey the relationships required of functorhood - that is, that identity and composition are preserved. But identity is also given in the definition.

Composition of functions is also reasonably clear. Each function is either concatenating a Dehn twist, concatenating an inverse Dehn twist, or including the mapping class group into a larger surface. It should be clear that one can add Dehn twists (and their inverses) only if they are acting upon the parts of the surface that already exist, just as face and degeneracy maps can only be enacted on a graph which has the relevant edge or vertex.

A composition of some number of degeneracy and inner face maps will result in concatenating Dehn twists around their relevant places. If the inner face and degeneracy maps are disjoint, then the Dehn twists will be able to commute. If there is a degeneracy map followed by an inner face map coincident to it, they will cancel out, just like what happens with a Dehn twist and its inverse.

You may wonder, what if they are the other way around? If there is an inner face map first, then it is impossible to do a degeneracy on that edge, so you can only do the Dehn twist but not the inverse. But the graphical set defined above does not have to include the homomorphism corresponding to the inverse Dehn twist. It only includes those homomophisms we choose to include, and we only choose to include those which are relevant to the category of graphs.

Just as two disjoint face and degeneracy maps will commute, concatenating Dehn twists commutes with function inclusion, provided the Dehn twists are away from each other and are not around the parts of the surface that do not exist before the inclusion.

And there may be combinations of inner face maps or degeneracy maps, followed by some number of outer face maps which result in the loss of the edges affected by the inner face and degeneracy maps. But this corresponds to doing first some inclusions, and then adding the relevant Dehn twists. And in neither case could this be done in the reverse order. □

Then, by Theorem 5.3.15, since this graphical set has an underlying graphical group, it satisfies the inner Kan condition and is thus a quasi-modular operad.

Chapter 6

Conclusion

I began by looking for a cyclic operad of surfaces, with the secret goal of finding some sort of higher genus operad. But as I progressed in the background reading, it became apparent that there were plenty of gaps to be filled in the literature on generalised operads. During the course of my thesis, it came to light that some notions of a higher genus operad already existed [41, 33], with a couple of papers accepted for publication during my candidature [35, 36].

I believe I have succeeded at defining quasi cyclic operads and quasi modular operads, and proving the equivalence of the Kan condition and the Segal condition. I did indeed define a modular operad of surfaces too, the mapping class group operad. This is also notable in that it was defined in terms of graphical sets, rather than from modular operads.

There is still plenty to do. In the near future, I am hoping to flesh out modular complete Segal spaces and then publish this, along with the quasi model for modular operads. In particular, I would like to show the equivalence of the four definitions of infinity cyclic operad, as well as in the modular operads case.

Then there is still the surface operad to explore, and the connections between various surface inspired generalised operads. I would like to examine the relationship between the mapping class group operad, the nerve of the modular operad of cobordisms, and the surface operad from Tillmann [74]. Then, I would like to see how these relate to TQFTs. There is also a need to consider the profinite completion of a surface operad and how that relates to Grothendiek - Teichmüller theory [10].

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