Author/s:
Nesic, D;Mohammadi, A;Manzie, C

Title:
A Systematic Approach to Extremum Seeking Based on Parameter Estimation

Date:
2010-01-01

Citation:
Nesic, D., Mohammadi, A. & Manzie, C. (2010). A Systematic Approach to Extremum Seeking Based on Parameter Estimation. Proceedings of 49th IEEE Conference on Decision and Control (CDC), 1, (1), pp.3902-3907. IEEE. https://doi.org/10.1109/CDC.2010.5716937.

A Systematic Approach to Extremum Seeking Based on Parameter Estimation

Dragan Nešić, Alireza Mohammadi and Chris Manzie

Abstract— **We present a systematic approach for design of extremum seeking (ES) controllers for a class of uncertain plants that are parameterized with unknown parameters. First, we present results for static plants and show how it is possible to combine, under certain general conditions, an arbitrary optimization method with an arbitrary parameter estimation method in order to obtain extremum seeking. Our main results also specify how controller needs to be tuned in order to achieve extremum seeking. Then, we consider dynamic plants and separate our results into the stable plant case and unstable plant case. For each of these cases, we present conditions on general plants, controllers, observers, parameter estimators and optimization algorithms that guarantee semi-global practical convergence to the extremum when controller parameters are tuned appropriately. Our results apply to general nonlinear plants with multiple inputs and multiple parameters.**

I. INTRODUCTION

A standing assumption in extremum seeking is that the model of the plant is unknown and that the steady state relationship between reference input signals and plant outputs is such that it contains an extremum, [5]. The goal is to tune the system inputs online so that it operates in the vicinity of this extremum in steady state, [9].

There are two main methods for design of extremum seeking controllers. An adaptive control approach for continuous time systems is pursued in [9], [11], [14], [15] whereas a nonlinear programming approach is proposed in [3] for discrete time systems. Moreover, results in [3] are significant as they show how it is possible to combine an arbitrary nonlinear programming optimization method with a gradient estimator in order to achieve extremum seeking. The power of these results is that they provide a prescriptive framework for extremum controller design and convergence to the extremum is shown under very general conditions.

Results in [12], [13], [10] are derived under subtly different assumptions. Indeed, here the plant is assumed to be parameterized by an unknown parameter and various parameter estimation based techniques are used to achieve extremum seeking. While the parameter is unknown, it is assumed that it is known how the plant model depends on this parameter. This slightly stronger assumption allows a more direct use of classical adaptive control methods in the context of extremum seeking. Nevertheless, the results in [12], [13], [10] are presented for particular classes of plants and particular optimization algorithms are used to achieve extremum seeking. As far as we are aware, a unifying prescriptive framework that would allow us to combine any optimization algorithm with any stable plant (or unstable plant with controller, observer and parameter estimator) that is similar to results in [3] has not been reported in the literature. In our companion paper [6] we propose a similar unifying framework under weaker assumptions; indeed, in [6] we assume that we do not know the reference-to-output map at all and the estimation of derivatives of this map is done directly.

It is the purpose of this paper to propose a prescriptive extremum control design framework reminiscent of [3] for methods based on parameter estimation. The framework we propose provides precise conditions under which we can combine an arbitrary continuous optimization algorithm, such as continuous-gradient or continuous-Newton, with an arbitrary controller, observer and parameter estimator to achieve semi-global practical convergence of the closed loop trajectories to the desired extremum. Moreover, our results prescribe how the controller parameters need to be tuned in order to achieve semi-global practical convergence to the desired extremum. This flexible framework allows us to combine various robust and adaptive stabilization methods with various continuous optimization algorithms to obtain extremum seeking controllers.

The paper is organized as follows. Section II presents preliminaries. The main results of the static plant and dynamic plants are stated in Section III and IV, respectively. Section V demonstrates the application of main results, followed by examples with simulations. Conclusion and future works are presented in Section VI.

II. PRELIMINARIES

The set of real numbers is denoted by $\mathbb{R}$. The continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class of $\mathcal{KL}$ if it is nondecreasing in its first argument, strictly decreasing to zero in its second argument and $\beta(0, t) = 0$ for all $t \geq 0$.

Consider the static mapping $h : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}$ that we denote

$$y = h(\theta, u) \ , \tag{1}$$

where $\theta \in \mathbb{R}^p$ is a fixed unknown parameter vector, $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}$ is the output of the static system. We assume that we know the map $h(\cdot, \cdot)$ but do not know the parameter vector θ. It is assumed that the map $h(\theta, \cdot)$ has a

This work was supported by Australia Research Council Discovery Grant, DP0985388.

D. Nešić is with the Department of Electrical and Electronic Engineering, dnesic@unimelb.edu.au, A. Mohammadi and C. Manzie are with the Department of Mechanical Engineering, a.mohammadi@pgrad.unimelb.edu.au and manziec@unimelb.edu.au University of Melbourne, Parkville, 3010, Victoria, Australia

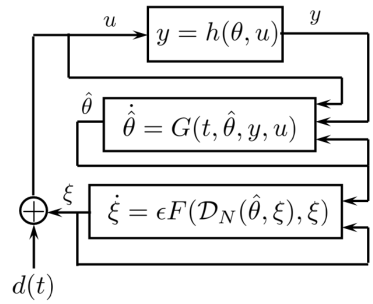

Fig. 1. Adaptive extremum seeking framework

local (or global) extremum. We introduce a vector

$$\mathcal{D}_N(\theta, u) := \begin{pmatrix} h(\theta, u) \\ D^{1,0,\ldots,0} h(\theta, u) \\ \vdots \\ D^{N,\ldots,N} h(\theta, u) \end{pmatrix} ,$$

where $D^{i_1,\ldots,i_m} h(\theta, u) := \frac{\partial^{i_1+\cdots+i_m} h(\theta,u)}{\partial u_1^{i_1} \ldots \partial u_m^{i_m}}$ for $i_m = 0, 1, 2, \ldots, N$ denote the iterated derivatives of h with respect to its input arguments. We will assume that the map $h(\theta, u)$ is smoothly differentiable sufficiently many times.

Suppose that the optimization scheme is of the form:

$$\dot{\xi} = F(\mathcal{D}_N(\theta, \xi), \xi) \tag{2}$$

that we want to generate an extremum seeking scheme.

Assumption 2.1: For the given (but unknown) $\theta \in \mathbb{R}^p$ there exists an equilibrium $\xi^* = \xi^*(\theta)$ of the system (2) which corresponds to the extremum of the map $h(\theta, \cdot)$.

III. Static plant case

We present our first main result in this section for an extremum seeking scheme for the case when the plant is static. Consider the following class of extremum seeking schemes:

$$\begin{aligned} y &= h(\theta, u) , & (3) \\ u &= \xi + d(t) , & (4) \\ \dot{\hat{\theta}} &= G(t, \hat{\theta}, y, u), & (5) \\ \dot{\xi} &= \epsilon\, F(\mathcal{D}_N(\hat{\theta}, \xi), \xi) \qquad \epsilon > 0 & (6) \end{aligned}$$

where ϵ is a controller tuning parameter (typically, a small positive number) that needs to be adjusted. The equation (3) is the static plant model. The equation (4) is the input into the plant where $d(t)$ is a dither signal that is typically chosen so that appropriate parameter convergence can be achieved and ξ comes from the optimization algorithm (6). The optimization algorithm uses the estimated parameter $\hat{\theta}$ that is obtained from the estimator (5). Fig. 1 shows the framework diagram for a static map. By introducing $\tilde{\theta} := \hat{\theta} - \theta$, $\tilde{\xi} := \xi - \xi^*$ and writing the closed loop equations in $\tilde{\theta}, \xi$ coordinates and in the time scale $\tau := \epsilon(t - t_0)$, we obtain the model of the system (5) and (6) in the standard singular perturbation form:

$$\begin{aligned} \epsilon\, \frac{d\tilde{\theta}}{d\tau} &= G\left(t_0 + \frac{\tau}{\epsilon}, \tilde{\theta} + \theta, h\left(\theta, \tilde{\xi} + \xi^* + d\left(t_0 + \frac{\tau}{\epsilon}\right)\right),\right. \\ &\quad \left.\tilde{\xi} + \xi^* + d\left(t_0 + \frac{\tau}{\epsilon}\right)\right) = \tilde{G}\left(t_0 + \frac{\tau}{\epsilon}, \tilde{\theta}, \tilde{\xi}\right) \quad (7) \\ \frac{d\tilde{\xi}}{d\tau} &= F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) = \tilde{F}(\tilde{\theta}, \tilde{\xi}) \quad (8) \end{aligned}$$

Next we use the standard assumptions from singular perturbation techniques to state the following result:

Theorem 3.1: Suppose that the Assumption 2.1 and the following conditions hold:

1) The origin $\tilde{\theta} = 0$ of the boundary layer (fast) system:
$$\dot{\tilde{\theta}} = \tilde{G}(t, \tilde{\theta}, \tilde{\xi}(t_0)) \tag{9}$$
is uniformly globally asymptotically stable (UGAS) uniformly in $\tilde{\xi}(t_0) \in \mathbb{R}^m$ and $t \in \mathbb{R}$.
2) The origin for the reduced (slow) system:
$$\frac{d\tilde{\xi}}{d\tau} = \tilde{F}(0, \tilde{\xi}) \tag{10}$$
is UGAS.

Then, there exist $\beta_1, \beta_2 \in \mathcal{KL}$ such that for any pair of strictly positive real numbers $\Delta > \nu > 0$, there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the following holds:

$$\begin{aligned} |\tilde{\theta}(t)| &\leq \beta_1(|(\tilde{\theta}(t_0), \tilde{\xi}(t_0))|, t - t_0) + \nu & (11) \\ |\tilde{\xi}(t)| &\leq \beta_2\big(|\tilde{\xi}(t_0)|, \epsilon(t - t_0)\big) + \nu \ , & (12) \end{aligned}$$

for all $|(\tilde{\theta}(t_0), \tilde{\xi}(t_0))| \leq \Delta$ and all $t \geq t_0 \geq 0$. In particular, $\limsup_{t\to\infty} |\tilde{\theta}(t)| < \nu$ and $\limsup_{t\to\infty} |\tilde{\xi}(t)| < \nu$.

Sketch of proof: The proof is omitted since it follows from standard singular perturbation techniques, see [8]. □

Remark 3.1: The condition on UGAS of the boundary layer system may be hard to satisfy in general. Therefore, the estimator design is usually considered for a class of maps that are linearly parameterized with θ.

Remark 3.2: One can easily relax conditions in item 2. of Theorem 3.1 by requiring that there exists a compact set $\mathcal{A} \subset \mathbb{R}^m$ with the following properties:

- $h(\theta, \cdot)$ is constant on $\mathcal{A}$;
- $h(\theta, \cdot)$ achieves its extremum on $\mathcal{A}$;
- $\mathcal{A}$ is UGAS for the continuous optimization scheme (2).

We do not present this result to keep the notation and definitions simpler. See [3] for similar results on nonlinear programming extremum seeking.

Remark 3.3: Note that Theorem 3.1 allows us to combine any continuous optimization method of the form (2) with a parameter estimation scheme (5) in order to achieve extremum seeking. Hence, results of Theorem 3.1 provide a prescriptive framework for extremum seeking controller design that combines an arbitrary optimization method with an arbitrary parameter optimization scheme that satisfy the conditions of the theorem.

Remark 3.4: The conclusions in Theorem 3.1 are quite intuitive. For any given ball B_Δ of initial conditions and any desired accuracy characterized by ν, we can adjust (i.e. reduce) the parameter ϵ so that for all initial conditions in the ball B_Δ we have that:

- The parameter estimate $\hat{\theta}$ converges to the B_ν ball centered at the true value of the parameter θ in time scale t (see equation (11));
- The optimizer state ξ converges in the slow time scale ϵt to the B_ν ball centered at the optimal value ξ^* (see equation (12)).

The following example illustrates our construction and our conditions.

Example 3.1: Consider the following map:

$$y = h(\theta, u) = u^2 + \theta u \ ,$$

where θ is unknown. Let our estimator be given by:

$$\dot{\hat{\theta}} = \sin(t)[y - u^2 - u\hat{\theta}] \ ,$$

which yields error dynamics of the form:

$$\dot{\tilde{\theta}} = -\sin(t) u \tilde{\theta} \ .$$

Note that this dynamics is not UGES uniformly in u and, in particular, when $u = 0$ we do not have parameter convergence. This can be fixed by adding a dither signal to the input and in this case we let $u = \xi + \sin(t)$. The dynamics then become

$$\dot{\tilde{\theta}} = -(\xi \sin(t) + \sin^2(t))\tilde{\theta} \ ,$$

and for any fixed $\xi = \xi(t_0)$, this system is UGES since the function $\xi(t_0)\sin(t) + \sin^2(t)$ satisfies a very strong persistence of excitation condition:

$$\int_{t_0}^{t_0+2\pi} (\xi(t_0)\sin(s) + \sin^2(s))ds = \pi \qquad \forall t_0 \geq 0,$$

that guarantees uniform exponential convergence. Suppose that we want to use a gradient method to search for extremum (introduce $\tilde{\xi} = \xi - \xi^* := \xi + 0.5\theta$):

$$\dot{\tilde{\xi}} = -\epsilon D h(\theta, \xi) = -\epsilon[2\xi + \theta] = -\epsilon 2\tilde{\xi} \ , \qquad u = \xi + \sin(t)$$

Use new time scale $\tau = \epsilon t$ and rewrite the closed loop equations as:

$$\begin{aligned} \epsilon \frac{d\tilde{\theta}}{d\tau} &= -\left((\tilde{\xi} - 0.5\theta) \sin\left(\frac{\tau}{\epsilon}\right) + \sin^2\left(\frac{\tau}{\epsilon}\right) \right) \tilde{\theta} \\ \frac{d\tilde{\xi}}{d\tau} &= -2\tilde{\xi} \end{aligned} \qquad (13)$$

The slow system

$$\frac{d\tilde{\xi}}{d\tau} = -2\tilde{\xi} \ ,$$

is trivially UGES. The fast system was already shown to be UGES, uniformly in $\xi(t_0)$. Hence, the overall scheme yields semi-global convergence to the minimum of $h(\theta, u)$.

IV. Dynamic plant case

In this section, we consider dynamic plants and we state our results for the stable plant case and unstable plant case.

A. Stable plant

We can easily modify the extremum seeking scheme in the previous section to deal with dynamical plants. Consider the following closed loop system with a dynamical plant:

$$\begin{aligned} \dot{x} &= f(\theta, x, u) && (14) \\ y &= Q(\theta, x) \ , && (15) \\ u &= \xi \ , && (16) \\ \dot{\hat{\theta}} &= \epsilon_1 \ G(t, \hat{\theta}, y, \kappa(u,t)) && (17) \\ \dot{\xi} &= \epsilon_1 \ \epsilon_2 \ F(\mathcal{D}_N(\hat{\theta}, \xi), \xi), \qquad \epsilon_1, \ \epsilon_2 > 0, && (18) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the plant state, the scalars $\epsilon_1, \epsilon_2 > 0$ are controller parameters that need to be adjusted and all other variables are the same as before. The map $\kappa(u,t)$ is typically of the form $u + d(t)$ for some dither signal $d(t)$ that may be needed to ensure an appropriate persistence of excitation condition that guarantees convergence of $\tilde{\theta}$. For simplicity, we assume in the following discussion that $\kappa(u, d(t)) = u + d(t)$.

Assumption 4.1: The following equation:

$$0 = f(\theta, x, u) \ ,$$

has a solution $x = \ell(\theta, u)$ and the map:

$$h(\theta, u) := Q(\theta, \ell(\theta, u))$$

has an extremum at $u = \xi^*$.
We will assume that we know the map $h(\cdot, \cdot)$ but we do not know θ. This is a strong assumption since finding the map $\ell(\cdot, \cdot)$ explicitly is hard in general.

Writing the closed loop in coordinates $\tilde{x} = x - \ell(\theta, u), \tilde{\theta} = \hat{\theta} - \theta, \tilde{\xi} = \xi - \xi^*$, we obtain

$$\begin{aligned} \dot{\tilde{x}} &= f(\theta, \tilde{x} + \ell(\theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) \\ &\quad -\epsilon_1 \epsilon_2 \frac{\partial \ell}{\partial u} F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) \\ &=: \widetilde{f}(\tilde{x}, \tilde{\theta}, \tilde{\xi}, \epsilon_1 \epsilon_2) && (19) \\ \dot{\tilde{\theta}} &= \epsilon_1 \ G(t, \tilde{\theta} + \theta, Q(\theta, \tilde{x} + \ell(\theta, \tilde{\xi} + \xi^*)), \tilde{\xi} + \xi^* \\ &\quad + d(t)) =: \epsilon_1 \ \widetilde{G}(t, \tilde{x}, \tilde{\theta}, \tilde{\xi}) && (20) \\ \dot{\tilde{\xi}} &= \epsilon_1 \ \epsilon_2 \ F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) \\ &=: \epsilon_1 \ \epsilon_2 \ \widetilde{F}(\tilde{\theta}, \tilde{\xi}) \ . && (21) \end{aligned}$$

This system is in standard singular perturbation form and it has three time scales (plant is the fastest, then the estimator is middle and the optimization algorithm is the slowest time scale). This is fully consistent with [4] that deals exclusively with this class of systems.

Theorem 4.2: Suppose that the Assumption 4.1 and the following conditions hold:

1) The origin of the fast system:

$$\begin{aligned} \dot{\tilde{x}} &= \widetilde{f}(\tilde{x}, \tilde{\theta}(t_0), \tilde{\xi}(t_0), 0) \\ &= f(\theta, \tilde{x} + \ell(\theta, \tilde{\xi}(t_0) + \xi^*), \tilde{\xi}(t_0) + \xi^*) && (22) \end{aligned}$$

is UGAS, uniformly in $\xi(t_0)$.

2) The origin $\tilde{\theta} = 0$ of the medium system:

$$\frac{d\tilde{\theta}}{d\tau} = \tilde{G}(t_0 + \frac{\tau}{\epsilon_1}, 0, \tilde{\theta}, \tilde{\xi}(t_0)) \quad (23)$$

(in time scale $\tau := \epsilon_1(t - t_0)$) is uniformly globally asymptotically stable (UGAS), uniformly in $\tilde{\xi}(t_0) \in \mathbb{R}^m$.

3) The origin for the slow system:

$$\frac{d\tilde{\xi}}{d\sigma} = \tilde{F}(0, \tilde{\xi}) \quad (24)$$

(in the time scale $\sigma := \epsilon_1\epsilon_2(t - t_0)$) is UGAS.

Then, there exist $\beta_1, \beta_2, \beta_3 \in \mathcal{KL}$ such that for any pair of strictly positive real numbers $\Delta > \nu > 0$, there exists $\epsilon_1^*, \epsilon_2^* > 0$ such that for all $\epsilon_1 \in (0, \epsilon_1^*)$ and $\epsilon_2 \in (0, \epsilon_2^*)$ the following holds:

$$\begin{aligned} |\tilde{x}(t)| &\leq \beta_1(|(\tilde{x}(t_0), \tilde{\theta}(t_0), \tilde{\xi}(t_0))|, t - t_0) + \nu \quad (25) \\ |\tilde{\theta}(t)| &\leq \beta_2(|(\tilde{\theta}(t_0), \tilde{\xi}(t_0))|, \epsilon_1(t - t_0)) + \nu \quad (26) \\ |\tilde{\xi}(t)| &\leq \beta_3\big(|\tilde{\xi}(t_0)|, \epsilon_1\epsilon_2(t - t_0)\big) + \nu \ , \quad (27) \end{aligned}$$

for all $|(\tilde{x}(t_0), \tilde{\theta}(t_0), \tilde{\xi}(t_0))| \leq \Delta$ and all $t \geq t_0 \geq 0$.

Sketch of proof: Local stability analysis of this class of systems was considered in [4] via Lyapunov based arguments. Semi-global version of these results that use trajectory based proofs for systems with two time scales can be found in [2]. Straightforward modifications of the proof in [2] for systems with three time scales lead to conclusions presented in Theorem 4.2. □

Remark 4.1: Note that the reason we have a time scale separation between the $\tilde{x}$ and $\tilde{\theta}$ is because we want to use the steady state map $h(\theta, u)$ for parameter estimation. Note that this is not necessary and it is possible to use directly dynamical equations for x subsystem to estimate θ.

B. Unstable plant

All our results in previous section are for stable plants and this assumption can be relaxed, see [12], [13], [10] for specific plants and optimization scheme. Moreover, results in this section illustrate the point made in Remark 4.1.

Consider an uncertain dynamic plant:

$$\begin{aligned} \dot{x} &= f(\theta, x, u) \quad (28) \\ y &= Q(\theta, x) \ , \quad (29) \end{aligned}$$

where all variables have the same meaning as in the previous section.

Remark 4.2: Here it may be useful to make a distinction between the *measured output* that is used to stabilize the plant by designing the observer, controller and parameter estimator and a *performance output* that we may want to use in extremum seeking.

In this section, we will assume that a controller, parameter estimator and state estimator have been designed and they are of the following form:

$$\begin{aligned} \dot{\hat{x}} &= g(t, \hat{x}, \hat{\theta}, u, y) \\ \dot{\hat{\theta}} &= G(t, \hat{\theta}, \hat{x}, u, y) \quad (30) \\ u &= u(t, \hat{x}, \hat{\theta}, y, r) \ , \end{aligned}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of the state, r is the new "reference" input.

It is convenient to rewrite the above equations by using $e := \hat{x} - x$, $\tilde{\theta} := \hat{\theta} - \theta$ and we obtain:

$$\begin{aligned} \dot{x} &= \widetilde{f}(t, \theta, x, e, \tilde{\theta}, r) \\ \dot{e} &= \widetilde{g}(t, \theta, x, e, \tilde{\theta}, r) \quad (31) \\ \dot{\tilde{\theta}} &= \widetilde{G}(t, \theta, x, e, \tilde{\theta}, r) \ , \end{aligned}$$

By introducing $X := (x, e, \tilde{\theta})$ we can rewrite (31) as follows:

$$\dot{X} = P(t, \theta, X, r) \ . \quad (32)$$

We now assume the following

Assumption 4.3: There exists $\ell : \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^{2n+p}$ such that

$$P(t, \theta, X, r) = 0 \ \Leftrightarrow \ X = \ell(\theta, r) := \begin{pmatrix} \ell_x(\theta, r) \\ \ell_e(\theta, r) \\ \ell_{\tilde{\theta}}(\theta, r) \end{pmatrix}$$

and the map

$$y = h(\theta, r) := Q(\theta, \ell_x(\theta, r))$$

has an extremum (maximum) at $r = \xi^*$.

To state the main result, it is convenient to define $\widetilde{X} := (\tilde{x}, e, \tilde{\theta})$ where $\tilde{x} := x - \ell_x(\theta, r)$ to obtain:

$$\dot{\widetilde{X}} = \widetilde{P}(t, \theta, \widetilde{X}, r) \ . \quad (33)$$

In order to achieve extremum seeking we control the above system with a slow optimization algorithm:

$$\dot{\tilde{\xi}} = \epsilon F(\mathcal{D}_N(\tilde{\theta} + \theta, \tilde{\xi} + \xi^*), \tilde{\xi} + \xi^*) =: \epsilon \widetilde{F}(\tilde{\theta}, \tilde{\xi}) \ ; r = \xi \quad (34)$$

where $r = \xi^*$ is the maximum of the map $h(\theta, \cdot)$ and $\tilde{\xi} := \xi - \xi^*$.

Then, we can state the following result:

Theorem 4.4: Suppose that the Assumption 4.3 and the following conditions hold:

1) The origin $\widetilde{X} = 0$ of the boundary layer (fast) system:

$$\dot{\widetilde{X}} = \widetilde{P}(t, \widetilde{X}, \tilde{\xi}(t_0)) \quad (35)$$

is uniformly globally asymptotically stable (UGAS) uniformly in $\tilde{\xi}(t_0) \in \mathbb{R}^m$ and $t \in \mathbb{R}$.

2) The origin for the reduced (slow) system:

$$\frac{d\tilde{\xi}}{d\tau} = \tilde{F}(0, \tilde{\xi}) \quad (36)$$

is UGAS.

Then, there exist $\beta_1, \beta_2 \in \mathcal{KL}$ such that for any pair of strictly positive real numbers $\Delta > \nu > 0$, there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the following holds:

$$\begin{aligned} |\widetilde{X}(t)| &\leq \beta_1(|(\widetilde{X}(t_0), \tilde{\xi}(t_0))|, t - t_0) + \nu \quad (37) \\ |\tilde{\xi}(t)| &\leq \beta_2\big(|\tilde{\xi}(t_0)|, \epsilon(t - t_0)\big) + \nu \ , \quad (38) \end{aligned}$$

for all $|(\widetilde{X}(t_0), \tilde{\xi}(t_0))| \leq \Delta$ and all $t \geq t_0 \geq 0$.

Sketch of proof: This proof follows directly from singular perturbation arguments, see [2]. □

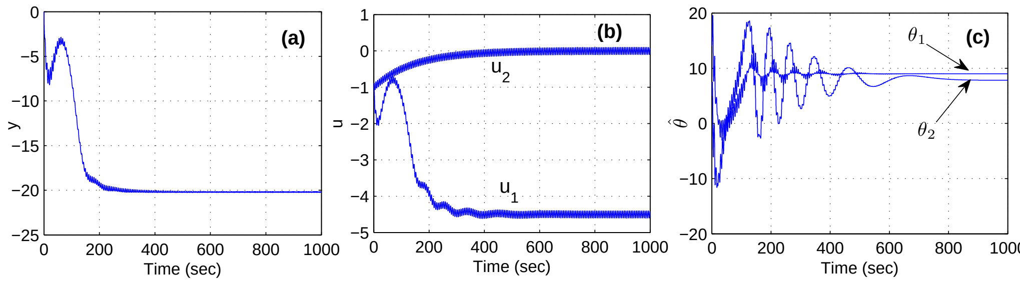

Fig. 2. Static plant with nonlinear, linearly parameterized output: (a) performance output; (b) control inputs; (c) parameters estimate.

Remark 4.3: Note also that while [12], [13], [10] do not state their results at this level of generality, they do require appropriate persistence of excitation that guarantees UGAS of (33).

Remark 4.4: Note that all our results can be restated so that instead of UGAS requirements in our assumptions we assume UAS with a bounded domain of attraction. All our results still hold but the conclusions are then weaker as semi-global stability can not be achieved and regional stability holds instead.

V. Applications of Main Results

In this section, to illustrate the generality of the framework, we will consider particular type of plant, optimization scheme, and parameter estimator.

A. Static linearly parameterized plants

Consider a class of static plants that are linearly parameterized in θ:

$$y = h_0(u) + h_1(u)^T\theta \ , \tag{39}$$

where $\theta \in \mathbb{R}^p$, $h_0 : \mathbb{R} \to \mathbb{R}$ and $h_1 : \mathbb{R} \to \mathbb{R}^p$.

Then, multiplication of a column vector dither signal $d_1 \in \mathbb{R}^p$ with discrepancy between the real output and the estimated output $(y - \hat{y})$ gives the following error dynamics of parameter estimation:

$$\dot{\tilde{\theta}} = -d_1(t)h_1(u)^T\tilde{\theta} \ , \tag{40}$$

where $\tilde{\theta} = \hat{\theta} - \theta$. Introducing $u = d_2(t) + \xi(t_0)$ and defining $H(t,u) := d_1(t)h_1(u)^T$, we have the closed loop error dynamics as:

$$\dot{\tilde{\theta}} = -H(t,\xi(t_0))\tilde{\theta} \ . \tag{41}$$

Based on the results of [1], the stability properties of above system are stated in the following proposition:

Proposition 5.1: The origin of closed loop system (41) is UGAS uniformly in $\xi(t_0)$ if $H(.,\xi(t_0))$ is persistently exciting, i.e., there exists $T > 0$ and $\mu > 0$,

$$\int_{t_0}^{t_0+T} |H(t,\xi(t_0))\tilde{\theta}| \mathrm{d}\tau \geq \mu|\tilde{\theta}|, \quad \forall \tilde{\theta} \neq 0, \ \forall t_0 \in \mathbb{R}. \tag{42}$$

Then we can present following corollary to Theorem 3.1:

Corollary 5.1: If the conditions of Proposition 5.1 hold uniformly in $\xi(t_0)$, and the origin of the optimization system (6) is UGAS, then, the overall system that consists of (39), (41) and (6) is SPA stable uniformly in small ϵ.

Example 5.1: Consider the static mapping

$$y = u_1^2 + \theta_1 u_1 + \theta_2 u_2^2, \tag{43}$$

with the dither signals chosen as $d_1(t) = [\sin(t) \quad \cos(t)]^T$. Since the $H(t,\xi_1(t_0),\xi_2(t_0))$ satisfies PE condition (42) with $T = 2\pi$ and $\mu = 10^{-5}$, the origin of parameter estimation dynamics is UGAS, uniformly in $\xi_1(t_0)$ and $\xi_2(t_0)$.

Next, we use the Newton method as optimization scheme

$$\dot{\xi} = -\epsilon \frac{Dh(\xi,\theta)}{D^2h(\xi,\theta)}, \quad u = d_2(t) + \xi. \tag{44}$$

where $d_2(t) = [\sin(t) \quad \sin(t)]^T$.

With introducing $\tilde{\xi}_1 = \xi_1 + 0.5\theta_1$ and $\tilde{\xi}_2 = \xi_2$, gives

$$\dot{\tilde{\xi}} = \begin{pmatrix} -\epsilon_1(\xi_1 + 0.5\theta_1) \\ -\epsilon_2 \ \xi_2 \end{pmatrix} = \begin{pmatrix} -\epsilon_1 \ \tilde{\xi}_1 \\ -\epsilon_2 \ \tilde{\xi}_2 \end{pmatrix}, \tag{45}$$

which is trivially UGES. Hence, the conditions of Corollary 5.1 hold, and the plant (43) converges to its extremum.

For the simulation, the true values of the unknown parameters are chosen as $\theta_1 = 9$ and $\theta_2 = 8$. The initial values used are $u_1(0) = u_2(0) = -1$, $\hat{\theta}_1(0) = 5$, $\hat{\theta}_2(0) = 13$ while the extremum values are $u_1^* = -4.5$, $u_2^* = 0$ and $y^* = -20.25$. The simulation results are shown in Fig. 2.

B. A class of stable linear systems with output nonlinearities

Consider a class of linear systems of the form

$$\dot{x} = Ax + Bu \tag{46}$$

$$y = \tilde{h}_0(x,u) + \theta\tilde{h}_1(x,u), \tag{47}$$

where A is Hurwitz. By solving

$$0 = Ax + bu \implies x = l(u) := -A^{-1}Bu$$

we obtain a steady state map of the form:

$$y = \tilde{h}_0(l(u),u) + \theta\tilde{h}_1(l(u),u) = h_0(u) + \theta h_1(u) \ ,$$

This is in the form of system (39). Also, we consider parameter estimation and optimization dynamics as

$$\dot{\tilde{\theta}} = -\epsilon_1 d_1(t)h_1(u)^T\tilde{\theta} \tag{48}$$

$$\dot{\tilde{\xi}} = -\epsilon_1\epsilon_2 Dh(u,\theta). \tag{49}$$

It is worth emphasizing that this gradient method can be replaced by Newton method or any other optimization scheme

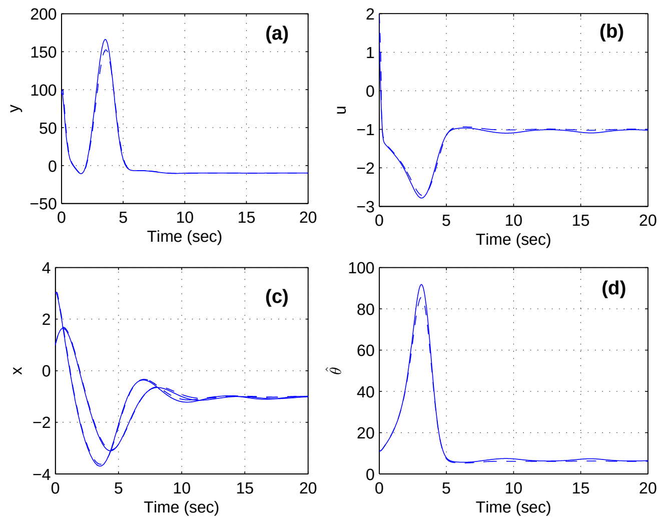

Fig. 3. Dynamic plant with global stable gradient (solid) and Newton (dashed) methods: (a) performance output; (b) control input; (c) state; (d) parameter estimate.

that satisfy our assumptions. Now, we can state the following obvious corollary to Theorem 4.2:

Corollary 5.2: Consider system (46)-(49) and assume that parameter estimator dynamics (48) is UGAS, uniformly in $\tilde{\xi}(t_0)$ and optimization scheme (49) are UGAS. Then, the overall system is SPA stable uniformly in small ϵ_1 and ϵ_2.

Example 5.2: Consider the following linear plant with nonlinear, linearly parameterized output which is similar to the problem addressed in [13]:

$$\dot{x}_1 = -x_2 + u, \quad \dot{x}_2 = x_1 - x_2, \quad y = h(x, \theta), \tag{50}$$

where$h(x,\theta) = (x_1^4 + x_2^2) + \theta(x_1 - 1)$. Stable equilibrium of this plant is at $x_1 = x_2 = u$, which leads to performance function $h(u,\theta) = (u^4 + u^2) + \theta(u - 1)$. Therefore, the parameter estimation dynamics is:

$$\dot{\tilde{\theta}}_1 = -\epsilon_1 \sin(t)(u-1)\tilde{\theta}, \tag{51}$$

which satisfies PE condition (42) with $T = 2\pi$ and $\mu = 1$ for all $u = \xi(t_0) + \sin(t)$. Hence, based on Proposition 5.1 we can conclude that the estimation error dynamics is UGAS, uniformly in $\xi(t_0)$.

Next, we consider two different methods for optimization, gradient and Newton method. The gradient method with defining $\tilde{\xi} := \xi - \xi^*$, which $u = \xi^*$ is the extremum of the map $h(\theta, u)$, is as following:

$$\dot{\tilde{\xi}} = -\epsilon_1\epsilon_2 Dh(u,\theta) = -4\epsilon_1\epsilon_2\left(\tilde{\xi}^3 + 3\xi^*\tilde{\xi}^2 + (3{\xi^*}^2 + 0.5)\tilde{\xi}\right) \tag{52}$$

and the Newton method is:

$$\dot{\tilde{\xi}} = -\epsilon_1\epsilon_2 \frac{Dh(u,\theta)}{D^2h(u,\theta)} = -\epsilon_1\epsilon_2 \frac{\tilde{\xi}^3 + 3\xi^*\tilde{\xi}^2 + (3{\xi^*}^2 + 0.5)\tilde{\xi}}{3(\tilde{\xi} + \xi^*)^2 + 0.5}. \tag{53}$$

To show UGAS of the systems (52) and (53), we define a Lyapunov candidate function $V = \frac{1}{2}\tilde{\xi}^2$. The time derivative of this function for the both of gradient and Newton methods are negative at $\mathbb{R} - \{0\}$ and equal to zero at $\tilde{\xi} = 0$. Hence, according to Theorem 4.1 in [8], the origins of systems (52) and (53) are UGAS. Thus, conditions of Corollary 5.2 hold, and the overall system converges to its extremum in $u = \xi^*$.

We now simulate this scheme, where initial conditions are $x(0) = [3, 1]^T$, $u(0) = \xi(0) = 2$ and $\hat{\theta}(0) = 11$, for both optimization algorithms. The simulation results can be found in Fig. 3, where the performance output approaches its minimum at $u^* = -1$ with minimum value $y^* = -10$ and the state x_1 and x_2 converge to the optimum values $x_1^* = x_2^* = -1$. Also, the parameter estimate converges to the actual value of $\theta = 6$.

VI. CONCLUSIONS

A systematic approach is proposed to design extremum seeking controllers for a class of uncertain plants which are parameterized with unknown parameters. Some future research will include the application of our last theorem in model reference adaptive controllers with unknown parameters, which has many different applications.

REFERENCES

[1] A. Loria, E. Panteley and A. Teel, "δ-Persistency of excitation: a necessary and sufficient condition for uniform attractivity", 41st IEEE Conference on Decision and Control, Las Vegas, NV, USA, 2002.

[2] A. R. Teel, L. Moreau and D. Nešić, "A unified framework for input-to-state stability in systems with two time scales", IEEE Transactions on Automatic Control, **48**, No.9, pp. 1526–1544, 2003.

[3] A. R. Teel and D. I. Popović, "Solving smooth and nonsmooth extremum seeking problems by methods of nonlinear programming," *Proc. of the American Contr. Conf.*, June 2001, Washington, DC, 2001.

[4] C. A. Desoer and S.M. Shahruz, "Stability of nonlinear systems with three time scales", *Circ. Sys. Sig. Proc.*, vol. 5 No. 4, 1986, pp. 449–464.

[5] D. Nešić, "Extremum seeking control: convergence analysis", *Europ. J. Contr.*, vol. 15, No. 3-4, 2009, pp. 331-347.

[6] D. Nešić, Y. Tan, W. H. Moase and C. Manzie, "A unifying approach to extremum seeking: adaptive schemes based on derivatives estimation", to appear in the *IEEE Conference on Decision and Control*, 2010.

[7] E. Panteley, A. Loria and A.R. Teel, "Relaxed persistency of excitation for uniform asymptotic stability", *IEEE Trans. on Automat. Contr.*, vol. 46, No. 12, 2001, pp. 1874-1886.

[8] H. K. Khalil, *Nonlinear Systems*, Third edition Prentice Hall, Upper Saddle River, New Jersey, 2002.

[9] K. B. Ariyur and M. Krstić, *Real-Time Optimization by Extremum-Seeking Control*, Wiley-Interscience, A John Wiley & Sons, Inc., Publication, 2003.

[10] M. Guay and T. Zhang, "Adaptive extremum seeking control of nonlinear dynamic systems with parametric uncertainties", *Automatica* **39**, pp. 1283-1293, 2003.

[11] M. Krstić and H. H. Wang, "Stability of extremum seeking feedback for general nonlinear dynamic systems" *Automatica*, **36** pp. 595–601, 2000.

[12] V. Adetola and M. Guay, "Guaranteed Parameter Convergence for Extremum-Seeking Control of Nonlinear Systems," Automatica 43, 1, 105-110, 2007.

[13] V. Adetola and M. Guay, "Adaptive output feedback extremum seeking receding horizon control of linear systems," Journal of Process Control 16, 5, 521-533, 2006.

[14] Y. Tan, D. Nešić and I.M.Y. Mareels, "On non-local stability properties of extremum seeking control", *Automatica*, vol. 42 (2006), pp. 889–903.

[15] Y. Tan, D. Nešić, I.M.Y. Mareels and A. Astolfi, "On global extremum seeking in the presence of local extrema", Automatica 45 (2009), pp. 245–251.

[16] Y. Tan, D. Nešić and I.M.Y. Mareels, "On the choice of dither in extremum seeking systems: A case study", Automatica, vol. 44 (2008), pp. 1446–1450.